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Multipliers for Hermite Expansions

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1. Introduction

The aim of this paper is to prove certain multiplier theorems for the Hermite series. Given a function μ defined on the set of positive integers we can define, at least formally, the operator T_μ by the prescription

$$T_\mu f(x) = \sum_{\alpha \geq 0} \mu(2|\alpha| + n) f^\wedge(\alpha)\Phi_\alpha(x) \tag{1.1}$$

whenever f has the Hermite expansion

$$f(x) = \sum_{\alpha \geq 0} f^\wedge(\alpha)\Phi_\alpha(x) \tag{1.2}$$

We want to find conditions on the function μ so that the operator T_μ is bounded on L^p, for all p, $1 < p < \infty$. Clearly the boundedness of the function μ is a necessary condition which is also sufficient when $p = 2$. But for p different from 2 some more conditions are needed to ensure the boundedness. The classical Marcinkiewicz multiplier theorem for the Fourier series asserts the following. If f has the Fourier series expansion $f(\theta) = \sum a_k e^{ik\theta}$ and if (μ_k) is a bounded sequence of complex numbers satisfying the condition

$$\sup_j \sum_{2^j \leq k \leq 2^{j+1}} |\mu_k - \mu_{k+1}| \leqslant C \tag{1.3}$$

then the following inequality holds for $1 < p < \infty$.

$$\|\sum \mu_k a_k e^{ik\theta}\|_p \leqslant C \|\sum a_k e^{ik\theta}\|_p \tag{1.4}$$

Our aim in this paper is to generalize this result to the Hermite expansions.

A version of the Marcinkiewicz multiplier theorem for the Spherical Harmonic expansions was proved by Bonami-Clerc [1] and Strichartz [17]. Bonami-Clerc used the arguments of Muckenhoupt-Stein [9] together with the Cesaro summability results. On the other hand Strichartz used the method employed by Stein [15] in his proof of Hormander-Mihlin multiplier theorem for Fourier integrals. To state their results let us introduce the following finite difference operators. These operators are defined inductively as follows:

$$\Delta\mu(N) = \mu(N+1) - \mu(N)$$

and for $k \geqslant 1$, they are defined by

$$\Delta^{k+1}\mu(N) = \Delta^k\mu(N+1) - \Delta^k\mu(N).$$

The following is the Marcinkiewicz multiplier theorem for the Spherical Harmonic expansions.

Let be (μ_k) a bounded sequence of complex numbers satisfying the condition

$$\sup_j 2^{j(N-1)} \sum_{2^j \leq k \leq 2^{j+1}} |\Delta^N \mu_k| \leqslant C \tag{1.5}$$

where N is the smallest integer greater than $n/2$. Then we have the inequality for $1 < p < \infty$,

$$\| \sum \mu_k H_k f \|_p \leqslant C \| \sum H_k f \|_p \tag{1.6}$$

where $H_k f$ is the orthogonal projection of f into the k-th eigenspace.

In [8] G. Mauceri studied Marcinkiewicz multiplier theorm for the Hermite expansions. His conditions on μ involve finite difference operators of order $(n+1)$. If we use the summability results proved in [19] and [20] we can greatly improve Mauceri's result. The arguments of Bonami and Clerc can be used together with the summability results to prove the following result.

Assume that the function μ satisfies the condition

$$\sup_j 2^{j(k-1)} \sum_{2^j \leq N \leq 2^{j+1}} |\Delta^k \mu(N)| \leqslant C$$

where $k = [(3n-2)/6] + 2$. Then the operator T_μ is bounded on L^p, $1 < p < \infty$.

This result is already an improvement over Mauceri's result when $n > 1$. As we noted before, in the case of Spherical Harmonics and Fourier series the number k of finite differences entering the conditions is the smallest integer bigger than $n/2$. We will show that this is true in the case of Hermite series also. The following is our version of the Marcinkiewicz multiplier theorem for Hermite series.

Theorem 1. *Assume that the function μ satisfies the conditions*

$$|\Delta^r \mu(N)| \leqslant CN^{-r} \quad for \quad r = 0, 1, \ldots, k,$$

where $k > n/2$. Then the operator T_μ is bounded on L^p, $1 < p < \infty$.

Observe that our conditions on μ are just like the Hormander-Mihlin conditions. The proof depends on some boundedness properties of g and g^* functions. We introduce and study these functions in section 2. So much for the Marcinkiewicz multiplier theorem. Another multiplier theorem we are interested in is given by the function

$$\mu(|\nu|) = (2|\nu| + n)^{-\alpha} e^{(2|\nu| + n)it}. \tag{1.7}$$

This defines the operator $T_t(\alpha)$ given by

$$T_t(\alpha)f(x) = \sum_{\nu \geq 0} (2|\nu| + n)^{-\alpha} e^{(2|\nu| + n)it} f^\wedge(\nu)\Phi_\nu(x) \tag{1.8}$$

This function μ does not satisfy the conditions of Theorem 1 unless $\alpha > n$ and so we cannot apply the Marcinkiewicz multiplier theorm. Fortunately, the kernel of this operator can be calculated explicitly and studied by other means. This operator behaves more or less like the operator given by convolution with the oscillating kernel $|x|^{-\alpha} e^{ix.x}$. Such operators have been studied by many authors, see e.g. [12], and [13]. For these operators we prove the following theorem.

Theorem 2. *When $\alpha = n|1/p\text{-}1/2|$, $1 < p < \infty$, the operators $T_t(\alpha)$ are bounded on L^p, i.e.,*

$$\| T_t(\alpha) f \|_p \leqslant C \| f \|_p. \tag{1.8}$$

When $p = 1$ and $\alpha = n/2$, $T_t(\alpha)$ is bounded from H^1 into L^1, where H^1 is the Hardy space.

$$\| T_t(\alpha) f \|_1 \leqslant C \| f \|_{H^1}. \tag{1.9}$$

This theorem extends the classical Hardy-Littlewood theorem for the Fourier transform. Recall that Hardy-Littlewood inequalities state the following.

$$\int |f^\wedge(x)|^p |x|^{n(p-2)} \, dx \leqslant C \int |f(x)|^p \, dx, \quad \text{for} \quad 1 < p \leqslant 2 \tag{1.10}$$

$$\int |f^\wedge(x)|^p \, dx \leqslant C \int |f(x)|^p |x|^{n(p-2)} \, dx, \quad \text{for} \quad p \geqslant 2 \tag{1.11}$$

$$\int |f^\wedge(x)| |x|^{-n} \, dx \leqslant C \| f \|_{H^1}. \tag{1.12}$$

These inequalities were first proved by Hardy and Littlewood [5] in 1926. For an easy proof see Sadosky [11]. The first two inequalities follow easily if we apply Marcinkiewicz interpolation theorem to the operators f going to $f^\wedge(x)|x|^n$ in the space $L^p(|x|^{-2n}\,dx)$. The third inequality can be proved using the atomic theory of H^1 spaces.

We can rewrite the above inequalities in the following way. Consider the fractional powers of the Laplacian defined as follows.

$$\{(-\Delta)^{-\alpha}f\}^\wedge(x) = |x|^{-2\alpha}f^\wedge(x).$$

If we let $T(\alpha)f = (-\Delta)^{-\alpha}f$, then the above inequalities take the following form with $\alpha = n|1/p - 1/2|$.

$$\|\{T(\alpha)f\}^\wedge\|_p \leqslant C\|f\|_p, \quad \text{for} \quad 1 < p \leqslant 2 \tag{1.13}$$

$$\|f\|_p \leqslant C\|\{T(\alpha)f\}^\wedge\|_p, \quad \text{for} \quad p \geqslant 2 \tag{1.14}$$

$$\|\{T(\alpha)f\}^\wedge\|_1 \leqslant C\|f\|_{H^1} \tag{1.15}$$

Theorem 2 gives inequalities of this type for the operator $(-\Delta + |x|^2)^{-\alpha}$. Observe that

$$(-\Delta + |x|^2)^{-\alpha}f(x) = \sum_{\nu \geq 0} (2|\nu| + n)^{-\alpha}f^\wedge(\nu)\Phi_\nu(x) \tag{1.16}$$

Let F stand for the Fourier transform. Since F commutes with the operator $(-\Delta + |x|^2)^{-\alpha}$ and $F\{\Phi_\nu(x)\} = i^{|\nu|}\Phi_\nu(x)$ we have the following formula.

$$F(-\Delta + |x|^2)^{-\alpha}f(x) = e^{-in\pi/4} \sum_{\nu \geq 0} (2|\nu| + n)^{-\alpha}e^{(2|\nu| + n)i\pi/4}f^\wedge(\nu)\Phi_\nu(x) \tag{1.17}$$

Thus we get the Hardy-Littlewood inequalities for the operator $(-\Delta + |x|^2)^{-\alpha}$. The inequalities of Theorem 2 have another application to the solutions of the Schrodinger equation $-i\partial_t u(x, t) = (-\Delta + |x|^2)u(x, t)$. Let $u(x, t)$ denote the solution of the initial value problem

$$-i\partial_t u(x, t) = (-\Delta + |x|^2)u(x, t), \qquad u(x, 0) = f(x) \tag{1.18}$$

The solution of this problem has the following expansion in terms of the Hermite functions.

$$u(x, t) = \sum_{\nu \geq 0} e^{(2|\nu| + n)it}f^\wedge(\nu)\Phi_\nu(x) \tag{1.19}$$

We like to know if any inequality of the type $\|u(x, t)\|_p \leqslant C(t)\|f\|_p$ holds. But this is too much to ask for. Indeed, $u(x, t)$ is nothing but a fractional power of the Fourier transform of f and as we know, Fourier transform, and for that matter any fractional power of that, cannot map L^p into itself unless $p = 2$.

Therefore, following Sjostrand [14], we define the Riesz means

$$G_\tau(\alpha)f(x) = \alpha\tau^{-\alpha}\int_0^\tau (\tau - t)^{\alpha-1}u(x,t)\,dt$$

and ask the question, «For what values of α the operators $G_\tau(\alpha)$ will be bounded on L^p?». As we will see, this boils down to the study of the operators $T_t(\alpha)$. Using the Hardy-Littlewood inequalities we can prove the following theorem.

Theorem 3. *If $\alpha \geqslant n|1/p - 1/2|$, then the operators $G_\tau(\alpha)$ are bounded on L^p.*

$$\|G_\tau(\alpha)f\|_p \leqslant C(\tau)\|f\|_p, \qquad 1 < p < \infty. \tag{1.20}$$

When $p = 1$ and $\alpha = n/2$, $G_\tau(\alpha)$ is bounded from H^1 into L^1.

This paper is organised as follows. In the next section we study the Littlewood-Paley-Stein g functions. To fix the ideas we first consider the one dimensional case and then indicate how we prove the results in the general case. In section 3, we prove the Marcinkiewicz multiplier theorem. In section 4, we prove Hardy-Littlewood inqualities and study the Riesz means for the solutions of the Schrodinger equation in the last section.

This paper forms one part of my Princeton University thesis written under the guidance of Prof. E. M. Stein. The amount of help I got from him and the real interest he showed in the progress of this work cannot be exaggerated. I would like to thank him for everything. I am also grateful to Dr. Chris Sogge for turning my attention towards Marcinkiewicz multiplier theorem.

2. Littlewood-Paley-Stein Theory of g Functions

The g functions are defined in [16] in the more general context of semigroups of operators satisfying certain conditions. Here we are interested in the Hermite semigroup H^t. For $t > 0$ these operators are defined by

$$H^t f(x) = \sum e^{-Nt} f^\wedge(n)\varphi_n(x)$$

where $N = 2n + 1$ as usual and they have the kernel

$$K_t(x,y) = \sum e^{-Nt}\varphi_n(x)\varphi_n(y).$$

In view of the Mehler's formula K_t is given by

$$K_t(x,y) = (\sinh 2t)^{-1/2} e^{\varphi(t)}$$

where

$$\varphi(t) = -1/2(x^2 + y^2)\coth 2t + xy \operatorname{cosech} 2t.$$

It is easy to see that H^t forms a semigroup of operators satisfying all the conditions except the last one listed in [16]. We can now define the g function by setting

$$g(f,x)^2 = \int_0^\infty t|\partial_t H^t f(x)|^2\, dt.$$

Since the Hermite semigroup fails to satisfy the condition $H^t 1 = 1$, the general theory developed in [16] cannot be applied. But in view of the explicit form of the kernel $K_t(x,y)$ we can prove the following theorem without much difficulty.

Theorem 2.1. *With some constants C_1 and C_2 we have the following inequality*

$$C_1 \| f \|_p \leqslant \| g(f) \|_p \leqslant C_2 \| f \|_p, \qquad 1 < p < \infty.$$

PROOF. The L^2 boundedness of the g function is easy. Since

$$\partial_t H^t f(x) = -\sum e^{-Nt} N f^\wedge(n) \phi_n(x)$$

$$\int g(f,x)^2\, dx = \int_0^\infty t\, dt \int |\partial_t H^t f(x)|^2\, dx.$$

But it is immediate that

$$\int |\partial_t H^t f(x)|^2\, dx = \sum e^{-2Nt} N^2 |f^\wedge(n)|^2$$

and hence we get

$$\| g(f) \|_2^2 = \sum_{n=0}^{\infty} \int_0^\infty t e^{-2Nt} N^2\, dt |f^\wedge(n)|^2$$

which is equal to $1/4 \sum |f^\wedge(n)|^2 = 1/4 \| f \|^2$. This proves the L^2 boundedness. We will now prove that $g(f)$ is weak type (1, 1). That will prove the inequality $\| g(f) \|_p \leqslant C_2 \| f \|_p$ and the deduction of the inequality in the other direction is routine.

In proving the weak type (1,1) inequality we closely follow Stein [15]. We consider g as a Hilbert space valued singular integral operator. To be precise, g is a singular integral operator whose kernel $\partial_t K_t(x,y)$ is taking values in the Hilbert space $L^2(\mathbb{R}^+, t\, dt)$. Since g is already known to be bounded on L^2 we need to check the following condition on K_t.

$$\text{(2.1)} \qquad \int_{|x-y^*| \geq 2|y-y^*|} \| \partial_t K_t(x,y) - \partial_t K_t(x,y^*) \|\, dx \leqslant C$$

where $\| \cdot \|$ is the norm of the Hilbert space $L^2(\mathbb{R}^+, t\, dt)$. Once this condition is checked we can invoke Theorem 5.1 in [15] to get the weak type estimate. The condition (2.1) is checked using the following estimate on the kernel $\partial_t K_t(x,y)$.

Lemma 2.1. $|\partial_y \partial_t K_t(x,y)| \leqslant Ct^{-3/2}|x-y|^{-1}(1+t^{-1/2}|x-y|)^{-2}.$

PROOF. The function φ can be written as

$$\varphi(t) = -(x-y)^2/(2\sinh 2t) - \tanh t\,(x^2+y^2)/2.$$

Since $\partial_y \partial_t K_t(x,y)$ is going to have many terms we indicate how to estimate one typical term viz.

$$J = (\sinh 2t)^{-7/2}\cosh 2t\,(x-y)^3 e^{\varphi(t)}.$$

First let us assume that $0<t<1$ so that $\sinh 2t = O(t)$ and $\cosh 2t = O(1)$. Then it is clear that we have the estimates

$$|J| \leqslant Ct^{-3/2}|x-y|^{-1} \quad \text{and} \quad |J| \leqslant Ct^{-1/2}|x-y|^{-3}$$

hence

$$|J| \leqslant Ct^{-3/2}|x-y|^{-1}(1+t^{-1/2}|x-y|)^{-2}.$$

The other terms are estimated similarly. Getting the estimates when t is greater than 1 is similar. In fact, we can get better estimates since $\sinh 2t = O(e^t)$ and $\cosh 2t = O(e^t)$ when $t \geqslant 1$. The details are omitted. This completes the proof of the Lemma.

Now it is easy to see how the condition follows from the Lemma. First we see that

$$\|\partial_t \partial_y K_t(x,y)\|^2 \leqslant C|x-y|^{-2}\int_0^\infty t^{-2}(1+t^{-1/2}|x-y|)^{-4}\,dt$$

which is less than or equal to $|x-y|^{-4}$. Now an application of the mean value theorem shows that

$$\begin{aligned}(2.2)\quad \int_{|x-y^*|\geq 2|y-y^*|} &\|\partial_t K_t(x,y) - \partial_t K_t(x,y^*)\|\,dx \\ &\leqslant C\int_{|x-y^*|\geq 2|y-y^*|}\|\partial_y\partial_t K_t(x,y_0)\|\,|y-y^*|\,dx \\ &\leqslant C\int_{|x-y^*|\geq 2|y-y^*|}|x-y_0|^{-2}|y-y^*|\,dx\end{aligned}$$

where y_0 lies between y and y^*. Since $|x-y_0| \geqslant 1/2|x-y^*|$ the condition (2.1) is verified.

To prove the Marcinkiewicz multiplier theorem, we have to introduce some more auxiliary functions. For any integer $k>1$, we can define the functions

$$g_k(f,x)^2 = \int_0^\infty t^{2k-1}|\partial_t^k H^t f(x)|^2\,dt.$$

Then it is an easy matter to prove that $g(f,x) \leqslant Cg_{k+1}(f,x)$. Indeed, by setting $u(x,t) = H^t f(x)$ we see that all t derivatives of $u(x,t)$ tend to zero as t goes to infinity. Therefore, writing

$$\partial_t^k u(x,t) = \int_t^\infty \partial_s^{k+1} u(x,s) s^k s^{-k}\, ds$$

we get the estimate

$$|\partial_t^k u(x,t)|^2 \leqslant \left\{\int_t^\infty |\partial_s^{k+1} u(x,s)|^2 s^{2k}\, ds\right\}\left\{\int_1^\infty s^{-2k}\, ds\right\}$$

Thus $g_k(f,x) \leqslant A_k g_{k+1}(f,x)$ and the claim is proved by induction. Another function we need is the g^* function which is defined by

$$g^*(f,x)^2 = \int_{-\infty}^{+\infty}\int_0^\infty t^{1/2}(1 + t^{-1/2}|x-y|)^{-2}|\partial_t H^t f(y)|^2\, dy\, dt.$$

The basic result about g^* which we are going to use is the following theorem. The proof is easy and for the sake of completeness we sketch it here.

Theorem 2.2. $\|g^*(f)\|_p \leqslant C\|f\|_p$, *for* $2 < p < \infty$.

PROOF. Let Ψ be a nonnegative function. We claim

$$\int g^*(f,x)^2 \Psi(x)\, dx \leqslant C \int g(f,x)^2 \Lambda\Psi(x)\, dx$$

where Λ is the Hardy-Littlewood maximal function. This is an easy consequence of the fact that

$$\sup_{t>0} \int t^{-1/2}(1 + t^{-1/2}|x-y|)^{-2}\Psi(y)\, dy \leqslant C\Lambda\Psi(x).$$

Since Λ is bounded on L^p, $1 < p < \infty$, an application of Hölder's inequality proves the theorem.

3. Marcinkiewicz Multiplier Theorem

We will prove that $g(F,x) \leqslant Cg^*(f,x)$ where $F(x) = T_\mu f(x)$. Then in view of theorems 2.1 and 2.2 it will follow that $\|F\|_p \leqslant C\|f\|_p$. Again, we need only to prove the inequality $g_2(F,x) \leqslant Cg^*(f,x)$. To prove this we introduce the function M.

$$M(t,x,y) = \sum e^{-Nt}\mu(n)\varphi_n(x)\varphi_n(y) \tag{3.1}$$

If we let $u(x,t) = H^t f(x)$ and $U(x,t) = H^t F(x)$, then we can write

$$U(x,t+s) = \int u(y,t) M(s,x,y)\, dy. \tag{3.2}$$

Differentiating the above expression with respect to s and t and then setting $t = s$ we get

$$\partial_t^2 U(x, 2t) = \int \partial_t u(y, t) \partial_t M(t, x, y)\, dy \tag{3.3}$$

Now we need to translate the hypothesis on the function μ into properties of M. This is done in the next lemma. For technical reasons we assume that $\mu(0) = 0$ without losing any generality.

Lemma 3.1. *Assume that μ satisfies the condition $k|\Delta\mu(k)| \leqslant C$. Then we have*

$$|\partial_t M(t, x, y)| \leqslant Ct^{-3/2} \tag{3.4}$$

$$\int |x - y|^2 |\partial_t M(t, x, y)|^2\, dy \leqslant Ct^{-3/2}. \tag{3.5}$$

Assuming the lemma for a moment we will first prove the inequality $g_2(F, x) \leqslant Cg^*(f, x)$.

$$\begin{aligned} \partial_t^2 U(x, 2t) &= \int_{|x-y| \leq t^{1/2}} \partial_t u(y, t) \partial_t M(t, x, y)\, dy \\ &\quad + \int_{|x-y| > t^{1/2}} \partial_t u(y, t) \partial_t M(t, x, y)\, dy \\ &= A_t(x) + B_t(x). \end{aligned} \tag{3.6}$$

Applying Schwarz inequality and using (3.4) we see that

$$\begin{aligned} |A_t(x)|^2 &\leqslant \int_{|x-y| \leq t^{1/2}} |\partial_t u(y, t)|^2\, dy \int_{|x-y| \leq t^{1/2}} |\partial_t M(t, x, y)|^2\, dy \\ &\leqslant Ct^{-5/2} \int (1 + t^{-1/2}|x - y|)^{-2} |\partial_t u(y, t)|^2\, dy. \end{aligned} \tag{3.7}$$

Another application of Schwarz inequality to $B_t(x)$ gives

$$\begin{aligned} |B_t(x)|^2 &\leqslant \int_{|x-y| > t^{1/2}} |x - y|^{-2} |\partial_t u(y, t)|^2\, dy \\ &\quad \cdot \int_{|x-y| > t^{1/2}} |x - y|^2 |\partial_t M(t, x, y)|^2\, dy \end{aligned} \tag{3.8}$$

In view of the estimate (3.5) the above becomes

$$|B_t(x)|^2 \leqslant Ct^{-5/2} \int (1 + t^{-1/2}|x - y|)^{-2} |\partial_t u(y, t)|^2\, dy \tag{3.9}$$

Thus we have

$$|\partial_t^2 U(x, 2t)|^2 \leqslant Ct^{-5/2} \int (1 + t^{-1/2}|x - y|)^{-2} |\partial_t u(y, t)|^2\, dy$$

and hence

$$g_2(F,x)^2 \leqslant C\int_{t>0}\int t^{1/2}(1+t^{-1/2}|x-y|)^{-2}|\partial_t u(y,t)|^2\,dy\,dt \leqslant Cg^*(f,x)^2. \tag{3.10}$$

Let us now prove the Lemma 3.1. (3.4) is a simple consequence of the boundedness of μ.

$$|\partial_t M(t,x,y)|^2 = |\sum e^{-Nt}N\mu(n)\varphi_n(x)\varphi_n(y)|^2 \leqslant \{\sum e^{-Nt}N\varphi_n(x)^2\}\{\sum e^{-Nt}N\varphi_n(y)^2\}. \tag{3.11}$$

Since

$$\sum e^{-Nt}N\varphi_n(x)^2 = -\partial_t\{\sum e^{-Nt}\varphi_n(x)^2\} = -\partial_t\{(\sinh 2t)^{-1/2}\exp(-x^2\tanh t)\},$$

(3.4) follows immediately. To prove (3.5) we use the following recursion formula (see [18]).

$$2x\varphi_n(x) = \{2(n+1)\}^{1/2}\varphi_{n+1}(x) + (2n)^{1/2}\varphi_{n-1}(x). \tag{3.12}$$

Let us introduce the operators A and B defined by $A = -d/dx + x$ and $B = -d/dy + y$. These operators have the following effect on the Hermite functions:

$$A\varphi_n(x) = (2(n+1))^{1/2}\varphi_{n+1}(x) \quad\text{and}\quad B\varphi_n(y) = (2(n+1))^{1/2}\varphi_{n+1}(y).$$

We use the recursion formula to calculate $2(x-y)\partial_t M(t,x,y)$. An easy calculation using the recursion formula and the action of A and B show that

$$2(x-y)\partial_t M(t,x,y) = (B-A)\{\sum \Delta\Psi(n)\varphi_n(y)\varphi_n(x)\} \tag{3.13}$$

where $\Psi(n) = e^{-Nt}N\mu(n)$. Applying the Leibnitz rule for the finite differences, a typical term will be of the form $(B-A)\{\sum e^{-(N+2)t}(N+2)\,\Delta\mu(n)\varphi_n(x)\varphi_n(y)\}$. Since $(B-A)$ brings down a factor of $(2n+2)^{1/2}$ the square of the L^2 norm of this series is bounded by

$$\sum e^{-2(N+2)t}(N+2)^2|\Delta\mu(n)|^2(N+1)\varphi_n(x)^2.$$

By the hypothesis on μ the term $(N+2)^2|\Delta\mu(n)|^2$ is bounded independent of n and hence the above sum is dominated by a constant times $\sum e^{-Nt}N\phi_n(x)^2$ which is bounded by $Ct^{-3/2}$. Similar estimates hold for all other terms and this completes the proof of the Lemma.

Let us now consider the n dimensional case. The n dimensional Hermite semigroup H^t is defined by means of the kernel $K_t(x,y)$ which is given by

$$K_t(x,y) = \sum e^{-(2|\alpha|+n)t}\Phi_\alpha(x)\Phi_\alpha(y)$$

where Φ_α are the n dimensional Hermite functions. In view of the Mehler's formula we have

$$K_t(x, y) = (\sinh 2t)^{-n/2} \exp\{\Phi(t)\}$$

where

$$\Phi(t) = -1/2(|x|^2 + |y|^2)\coth 2t + x \cdot y \operatorname{cosech} 2t.$$

Denoting the differentiation with respect to y_j by ∂_j the following estimates can be obtained just like the one dimensional case.

$$|\partial_t \partial_j K_t(x, y)| \leqslant C t^{-n/2-1} |x - y|^{-1} (1 + t^{-1/2}|x - y|)^{-n-1} \tag{3.14}$$

for $j = 1, 2, \ldots, n$. If we define the g and g_k functions as in the one dimensional case, then in view of the above estimate it is easily seen that Theorem 2.1 holds true in the n dimensional case also. We also have the relation

$$g(f, x) \leqslant A_k g_{k+1}(f, x)$$

between g and g_k. We need one more auxiliary function which is the n dimensional version of the g^* function. For $k > 0$ we define g_k^* by

$$(g_k^*(f, x))^2 = \int \int_0^\infty t^{(2-n)/2} (1 + t^{-1/2}|x - y|)^{-2k} |\partial_t H^t f(y)|^2 \, dy \, dt.$$

For $k > n/2$ the function $(1 + |x - y|)^{-2k}$ belongs to L^1 and hence it is easy to prove Theorem 2.2 for the g_k^* function i.e. we have the inequality

$$\|g_k^*(f)\|_p \leqslant C \|f\|_p,$$

provided $k > n/2$. As in the one dimensional case we set $F(x) = T_\mu f(x)$ and will prove that

$$g_{k+1}(F, x) \leqslant C g_k^*(f, x)$$

where $k > n/2$ is an integer. This will then prove the multiplier theorem. We start by defining

$$M(t, x, y) = \sum e^{-(2|\alpha| + n)t} \mu(2|\alpha| + n) \Phi_\alpha(x) \Phi_\alpha(y). \tag{3.15}$$

The following Lemma translates the hypothesis on μ into properties of $M(t, x, y)$.

Lemma 3.2. *Assume the function μ satisfies the hypothesis of Theorem 1. Then we have*

$$|\partial_t^k M(t, x, y)| \leqslant C t^{-n/2-k} \tag{3.16}$$

$$\int |x - y|^{2k} |\partial_t^k M(t, x, y)|^2 \, dy \leqslant C t^{-n/2-k}. \tag{3.17}$$

PROOF. The estimate (3.16) follows from the boundedness of μ as in the one dimensional case. The other estimate is a consequence of the following estimates:

(3.18) $\int |(x-y)^\beta \partial_t^k M(t,x,y)|^2\,dy \leqslant Ct^{-n/2-k}$, for all β with $|\beta| = k$.

To prove these estimates we have to introduce some more notation. Consider the following operators A_j and B_j defined by

$$A_j = -d/dx_j + x_j \quad \text{and} \quad B_j = -d/dy_j + y_j.$$

These operators have the following effect on Φ_α:

$$A_j\Phi_\alpha(x) = \{2(\alpha_j+1)\}^{1/2}\Phi_{\alpha+e^j}(x)$$

and

$$B_j\Phi_\alpha(y) = \{2(\alpha_j+1)\}^{1/2}\Phi_{\alpha+e^j}(y)$$

where e^j is the j-th co-ordinate vector. Given a series

$$M(t,x,y) = \sum \Psi(|\alpha|)\Phi_\alpha(x)\Phi_\alpha(y)$$

we denote by $\Delta^r M(t,x,y)$ the series defined by

$$\Delta^r M(t,x,y) = \sum \Delta^r\Psi(|\alpha|)\Phi_\alpha(x)\Phi_\alpha(y)$$

where $\Delta^r\Psi$ is the finite difference of order r of Ψ. For technical convenience we assume that $\Psi(|\alpha|) = 0$ for all α with $|\alpha| \leqslant k$. Let us first calculate $2(x_j - y_j)M(t,x,y)$. Proceeding as in the one dimensional case we obtain

$$2(x_j-y_j)M(t,x,y) = (B_j - A_j)\,\Delta M(t,x,y).$$

Now it is clear how to proceed further. Iteration of the above procedure produces

$$(*) \qquad 2^m(x_j-y_j)^m M(t,x,y) = \sum C_{rs}(B_j-A_j)^r\,\Delta^s M(t,x,y)$$

where the sum is extended over all r and s satisfying the conditions $2s - r = m$, $s \leqslant m$.

The proof of (*) is by induction. As we have seen the result is true for $m = 1$. Assuming the result for m, we will now consider

$$(3.19) \qquad 2^{m+1}(x_j-y_j)^{m+1}M(t,x,y) = \sum C_{rs}2(x_j-y_j)(B_j-A_j)^r\,\Delta^s M(t,x,y)$$

Let us write

$$(x_j-y_j)(B_j-A_j)^r = [(x_j-y_j),(B_j-A_j)^r] + (B_j-A_j)^r(x_j-y_j)$$

and calculate the commutator $[(x_j - y_j), (B_j - A_j)^r]$. It is easily seen that

$$[(x_j - y_j), (A_j - B_j)] = 2I,$$

where I is the identity operator. Now we claim that

$$[(x_j - y_j), (B_j - A_j)^r] = -2r(B_j - A_j)^{r-1}.$$

We prove the claim by induction. Suppose we have $[T, S] = 2I$ and $[T, S^r] = 2rS^{r-1}$.

$$\begin{aligned}(3.20)\qquad [T, S^{r+1}] &= (TS^r - S^rT + S^rT)S - S(S^rT - TS^r + TS^r)\\ &= 4rS^r + S^r(TS - ST + ST) - (ST - TS + TS)S^r\\ &= 4(r+1)S^r - [T, S^{r+1}]\end{aligned}$$

so that $[T, S^{r+1}] = 2(r+1)S^r$ and this proves the claim. Thus we have the equations

$$(3.21)\qquad 2(x_j - y_j)(B_j - A_j)^r = -2r(B_j - A_j)^{r-1} + (B_j - A_j)^r 2(x_j - y_j)$$

$$\begin{aligned}(3.22)\quad & 2^{m+1}(x_j - y_j)^{m+1}M(t, x, y)\\ &= \sum C_{rs}\{-2r(B_j - A_j)^{r-1} + (B_j - A_j)^r 2(x_j - y_j)\}\, \Delta^s M(t, x, y)\end{aligned}$$

which equals to $\sum A_{rs}(B_j - A_j)^r \Delta^s M(t, x, y)$ with the conditions $2s - r = m + 1$, $s \leqslant m + 1$. This proves the equation $(*)$.

Since the operator $(x_j - y_j)$ commutes with $(A_i - B_i)$ for i different from j, repeated application of $(*)$ produces the following result

$$(3.23)\qquad (x - y)^\beta M(t, x, y) = \sum C_{\gamma\delta}(B - A)^\gamma \Delta^{|\delta|} M(t, x, y)$$

where $2\delta_j - \gamma_j = \beta_j$, $\delta_j \leqslant \beta_j$ and $(B - A)^\gamma$ stands for the product $\Pi(B_j - A_j)^{\gamma_j}$. Now we can complete the proof. Since

$$\partial_t^k M(t, x, y) = (-1)^k \sum e^{-(2|\alpha| + n)t}(2|\alpha| + n)^k \mu(2|\alpha| + n)\Phi_\alpha(x)\Phi_\alpha(y),$$

the above result (3.23) applied to $\partial_t^k M(t, x, y)$ gives

$$(3.24)\qquad (x - y)^\beta \partial_t^k M(t, x, y) = \sum C_{\gamma\delta}(B - A)^\gamma \Delta^{|\delta|} M_0(t, x, y)$$

where

$$M_0(t, x, y) = \sum \Psi(|\alpha|)\Phi_\alpha(x)\Phi_\alpha(y)$$

with

$$\Psi(|\alpha|) = (-1)^k e^{-(2|\alpha| + n)t}(2|\alpha| + n)^k \mu(2|\alpha| + n).$$

If we expand $(B - A)^\gamma$ and apply Leibnitz rule for finite differences, we see that a typical term in the sum (3.24) is of the form

$$\sum e^{-(2|\alpha|+n)t}(2|\alpha|+n)^k \Delta^{|\delta|}\mu(2|\alpha|+n) A^\sigma \Phi_\alpha(x) B^\tau \Phi_\alpha(y) \tag{3.25}$$

where $2|\delta| - |\tau| - |\sigma| = k$. Recalling the definition of the operators A and B we see that the square of the L^2 norm of the above sum is dominated by

$$\sum e^{-2(2|\alpha|+n)t}(2|\alpha|+n)^{2k+|\sigma|+|\tau|}|\Delta^{|\delta|}\mu(2|\alpha|+n)|^2|\Phi_\alpha(x)|^2. \tag{3.26}$$

Since

$$|\Delta^{|\delta|}\mu(2|\alpha|+n)|^2 \leqslant C(2|\alpha|+n)^{-2|\delta|} \quad \text{and} \quad 2|\delta| - |\tau| - |\sigma| = k,$$

the above sum is dominated by a constant times

$$\sum e^{-(2|\alpha|+n)t}(2|\alpha|+n)^k|\Phi_\alpha(x)|^2$$

which is bounded by $Ct^{-n/2-k}$. All other terms are similarly estimated. This completes the proof of Lemma 3.2.

Having proved the Lemma, Theorem 1 is proved just like the one dimensional version. We write

$$H^{t+s}F(x) = \int M(s,x,y)H^t f(y)\,dy.$$

Taking k derivatives with respect to s and one derivative with respect to t and then putting $t = s$, we get the expression

$$\partial_t^{k+1}H^{2t}F(x) = \int \partial_t^k M(t,x,y)\partial_t H^t f(y)\,dy. \tag{3.27}$$

In view of the Lemma we get $g_{k+1}(F,x) \leqslant Cg_k^*(f,x)$ and this completes the proof.

4. Hardy-Littlewood Inequalities for $(-\Delta + |x|^2)$

We prove Theorem 2 when $n = 1$. There is absolutely no change in the proof for the general case. We first prove the inequality (1.9). The operators $T_t(\alpha)$ are all bounded on L^2. The other inequality (1.8) is then proved by interpolating between the L^2 result and the inequality (1.9). Then following interpolation theorem due Fefferman-Stein [3] is the one we are going to apply.

Theorem 4.1 (Fefferman-Stein). *Suppose S_z is an analytic family of operators satisfying*

(i) $$\|S_{iy}f\|_1 \leqslant A_0(y)\|f\|_{H^1}$$

(ii) $$\|S_{1+iy}f\|_2 \leqslant A_1(y)\|f\|_2$$

for all y, $-\infty < y < \infty$. *Assume that* $A_j(y)$ *satisfies the condition* $\log A_j(y) \leqslant c_j \exp\{d_j|y|\}$, $c_j > 0$ *and* $0 < d_j < \pi$. *If* $1/p = 1 - t/2$, $0 < t \leqslant 1$, *then* $\|S_t f\|_p \leqslant A\|f\|_p$.

The inequality (1.9) is proved using the atomic theory of H^p spaces. We say that a function φ is a p-atom if there is a ball B in $\mathbb{R}^n$ such that φ has the following properties.

$$\text{supp}(\varphi) \subset B, \quad \|\varphi\|_\infty \leqslant |B|^{-1/p} \tag{4.1}$$

$$\int \varphi(x)P(x)\,dx = 0 \tag{4.2}$$

for all polynomials of degree less than or equal to $k = n(1/p - 1)$. If f belong to $H^p(\mathbb{R}^n)$ it can be shown that there exists a sequence of p-atoms (φ_j) and a sequence of complex numbers (λ_j) such that $f = \sum \lambda_j \varphi_j$ in the sense of distributions and $\left(\sum|\lambda_j|^p\right)^{1/p} \leqslant C_p\|f\|_{H^p}$. Conversely, if f has the form $f = \sum \lambda_j \varphi_j$, then f belongs to $H^p(\mathbb{R}^n)$ and $\|f\|_{H^p} \leqslant C_p\left(\sum|\lambda_j|^p\right)^{1/p}$.

With these preliminaries consider the operator $T_t(\alpha)$ which is defined by

$$T_t(\alpha) = \sum (2n+1)^{-\alpha} e^{(2n+1)it} f^\wedge(n)\varphi_n(x) \tag{4.3}$$

The operator $T_t(\alpha)$ has the following kernel

$$K_t(x,y) = \sum (2n+1)^{-\alpha} e^{(2n+1)it}\varphi_n(y)\varphi_n(x). \tag{4.4}$$

We can write this kernel as

$$K_t(x,y) = 1/\Gamma(\alpha)\int_{t>0} \lambda^{\alpha-1} K_t^*(x,y,\lambda)\,d\lambda$$

where we have set

$$K_t^*(x,y,\lambda) = \sum e^{(2n+1)(-\lambda+it)}\varphi_n(y)\varphi_n(x). \tag{4.5}$$

In view of Mehler's formula we have with $r = e^{-2\lambda}$ the following expression

$$K_t^*(x,y,\lambda) = ce^{-(\lambda-it)}(1 - r^2e^{-4it})^{-1/2}\exp\{B_r(t,x,y)\} \tag{4.6}$$

where

$$B_r(t,x,y) = (1 - r^2e^{-4it})^{-1}\{-1/2(x^2+y^2)(1 + r^2e^{-4it}) + 2xyre^{-2it}\}.$$

To prove that $T_t(\alpha)$ is bounded on L^p it is enough to show that

$$\int |K_t(x,y)|\,dx \leqslant C,$$

with a C independent of y.

It is easy to calculate the L^1 norm of $K_t(x,y)$. Let

$$C_r(t,x,y) = (1 - r^2 e^{-4it})^{-1/2} \exp\{B_r(t,x,y)\}.$$

Then an easy calculation shows that

$$\begin{aligned}(4.7)\quad & |C_r(t,x,y)| \\ &= a^{1/2} \exp\{-1/2a^2(1-r^4)(x^2+y^2) + 2rxya^2(1-r^2)\cos 2t\}\end{aligned}$$

where

$$a^2 = \{(1-r^2)^2 + 4r^2\sin^2 2t\}^{-1}.$$

Letting $b^2 = a^2(1-r^4)$ and $c = 2r\cos 2t(1+r^2)^{-1}$ we have

$$(4.8)\quad |C_r(t,x,y)| = a^{1/2}\exp\{-1/2b^2(x-cy)^2\}\exp\{-1/2b^2(1-c^2)y^2\}.$$

It is easily seen that $b^2(1-c^2) = (1-r^2)/(1+r^2)$. Thus we have

$$\begin{aligned}(4.9)\quad & \int |C_r(t,x,y)|\,dx \\ &= a^{1/2}\exp\{-1/2(1-r^2)/(1+r^2)y^2\}\int \exp\{-1/2b^2(x-cy)^2\}\,dx\end{aligned}$$

which is equal to

$$Aa^{-1/2}(1-r^4)^{-1/2}\exp\{-1/2(1-r^2)/(1+r^2)y^2\}.$$

Therefore, we have

$$\begin{aligned}(4.10)\quad & \int |K_t(x,y)|\,dx \\ &\leqslant A\int_{\lambda>0}\lambda^{\alpha-1}e^{-\lambda}(1-r^2)^{-1/2}\{(1-r^2)^2 + 4r^2\sin^2 2t\}^{1/4}\,d\lambda.\end{aligned}$$

From this we see that when $\sin 2t = 0$, the kernel $K_t(x,y)$ is integrable for all $\alpha > 0$ but when $\sin 2t$ is not 0 the kernel is integrable only if $\alpha > 1/2$. Since we are interested in the case $\alpha = 1/2$ and $t = \pi/4$, we have to study the operators by other means. That is why we need the atomic theory of the Hardy spaces.

Suppressing α, let us considet the operator $T_t(1/2) = T_t$. The kernel $K_t(x,y)$ of this operator is given by

$$(4.11)\quad K_t(x,y) = \sum (2n+1)^{-1/2} e^{(2n+1)it}\varphi_n(y)\varphi_n(x)$$

Since

$$K_t^*(x,y,\lambda) = \sum e^{(2n+1)(-\lambda+it)}\varphi_n(y)\varphi_n(x)$$

we can write the kernel $K_t(x,y)$ as

$$K_t(x,y) = c\int_0^\infty \lambda^{-1/2} K_t^*(x,y,\lambda)\, d\lambda.$$

A simple calculation shows that

$$K_t^*(x,y,\lambda) = c\{\sinh 2(\lambda - it)\}^{-1/2} e^{-A_t(x,y,\lambda)} e^{iB_t(x,y,\lambda)} \tag{4.12}$$

where $A_t(x,y,\lambda)$ and $B_t(x,y,\lambda)$ are given by the following equations.

$$\begin{aligned} 2A_t(x,y,\lambda) = {} & (\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sinh 2\lambda)\{\cos 2t(x-y)^2 \\ & + (\cosh 2\lambda - \cos 2t)(x^2+y^2)\} \end{aligned} \tag{4.13}$$

$$\begin{aligned} 2B_t(x,y,\lambda) = {} & -(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sin 2t)\{\cosh 2\lambda(x-y)^2 \\ & - (\cosh 2\lambda - \cos 2t)(x^2+y^2)\}. \end{aligned} \tag{4.14}$$

First consider the integral taken from 1 to infinity. Since $\sinh 2\lambda$ behaves like $e^{2\lambda}$ for $\lambda \geqslant 1$, it can be easily checked that the integral

$$\int_1^\infty \lambda^{-1/2}\{\sinh 2(\lambda - it)\}^{-1/2} e^{iB_t(x,y,\lambda)} e^{-A_t(x,y,\lambda)}\, d\lambda$$

defines a nice L^1 kernel and hence the operator corresponding to this kernel is bounded on L^p, for all p, $1 \leqslant p \leqslant \infty$. So we can very well assume that the kernel of the operator T_t is given by

$$K_t(x,y) = \int_0^1 \lambda^{-1/2}\{\sinh 2(\lambda - it)\}^{-1/2} e^{-A_t(x,y,\lambda)} e^{iB_t(x,y,\lambda)}\, d\lambda$$

In view of the atomic decomposition, T_t will be bounded from H^1 into L^1 once we prove the following proposition.

Proposition 4.1. $\int |T_t f(x)|\, dx \leqslant C$ *whenever f is an* 1-*atom.*

In proving this proposition we closely follow Phong and Stein. In [10], they studied the boundedness of the operator T whose kernel is of the form $K(x-y)e^{iB(x,y)}$ where K is a Calderón-Zygmund kernel and B is a non-degenerate bilinear form. To prove the proposition we need certain estimates for the kernel $K_t(x,y)$. These estimates are proved in the next lemma. Let us write $K_t(x,y,\lambda) = \{\sinh 2(\lambda - it)\}^{-1} e^{-A_t(x,y,\lambda)}$.

Lemma 4.1. *We have the following inequalities.*

$$\left| \int_0^1 \lambda^{-1/2} \partial_y K_t(x,y,\lambda) e^{iB_t(x,y,\lambda)}\, d\lambda \right| \leqslant C|x-y|^{-2}$$

$$\left| \int_0^1 \lambda^{-1/2} K_t(x,y,\lambda) \partial_y \{e^{iB_t(x,y,\lambda)}\}\, d\lambda \right| \leqslant C(\sin 2t)^{-3/2}$$

$$\left| \int_0^1 \lambda^{-1/2} \partial_\lambda e^{iB_t(x,y,\lambda)} \lambda K_t(x,y,\lambda)\, d\lambda \right| \leqslant C|x-y|^{-3}.$$

PROOF. The proof of this lemma is elementary. We prove it when $0 < t \leqslant \pi/8$. The proof of the lemma for other intervals of t is similar. First we calculate that

$$(4.15) \qquad |\sinh 2(\lambda - it)|^2 = c(\sinh^2 2\lambda + \sin^2 2t)$$

Observe that for $0 \leqslant \lambda \leqslant 1$, $(\sinh^2 2\lambda + \sin^2 2t)$ behaves like $(\lambda^2 + \sin^2 2t)$. Let us prove the first inequality of the lemma. $\partial_y K_t(x, y, \lambda)$ has two terms. We will estimate only the contribution of

$$(4.16) \qquad J = \{\sinh 2(\lambda - it)\}^{-1/2}(\sinh^2 2\lambda + \sin^2 2t)^{-1}$$
$$(\sinh 2\lambda)(\cos 2t)(x - y)e^{-A_t(x,y,\lambda)}$$

Since $0 < t \leqslant \pi/8$, $\cos 2t \geqslant 2^{-1/2}$. We consider two cases. First assume that $t \leqslant \lambda$. In this case $(\sinh^2 2\lambda + \sin^2 2t)$ behaves like λ^2. Therefore,

$$(4.17) \qquad |J| \leqslant C(x - y)\lambda^{-3/2} \exp\{-c\lambda^{-1}(x - y)^2\}.$$

Integrating this agains $\lambda^{-1/2}$ we have

$$(x - y)\int_0^1 \lambda^{-2} e^{-c\lambda^{-1}(x-y)^2}\, d\lambda = (x - y)\int_1^\infty e^{-c\lambda(x-y)^2}\, d\lambda$$
$$\leqslant (x - y)\int_0^\infty \lambda^{1/2} e^{-c\lambda(x-y)^2}\, d\lambda.$$

This gives the estimate $C|x - y|^{-2}$. Next assume that $t > \lambda$. In this case $(\sinh^2 2\lambda + \sin^2 2t)$ behaves like t^2. Therefore,

$$|J| \leqslant C(x - y)\lambda t^{-5/2} \exp\{-c\lambda t^{-2}(x - y)^2\}.$$

This gives the integral

$$(x - y)t^{-5/2}\int_0^1 \lambda^{1/2} e^{-c\lambda t^{-2}(x-y)^2}\, d\lambda \leqslant (x - y)t^{-5/2}\int_0^\infty \lambda^{3/2-1} e^{-c\lambda t^{-2}(x-y)^2}\, d\lambda$$

which is bounded by $Ct^{1/2}|x - y|^{-2}$. This proves the first inequality when $0 < t \leqslant \pi/8$. If t is in the neighbourhood of $\pi/4$ we can use

$$(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sinh 2\lambda)(\cosh 2\lambda - \cos 2t)(x^2 + y^2)$$

in place of

$$(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\sinh 2\lambda)\cos 2t(x - y)^2$$

since $(\cosh 2\lambda - \cos 2t) \geqslant 1 - \cos 2t \geqslant 2\sin^2 t \geqslant c$. This completes the proof of the first inequality.

The proof of the second inequality is similar. Unfortunately the estimate we get is not uniform in t. We believe that a uniform estimate is possible though we are not able to prove it now. Again we will be having two terms. Consider the term

$$G = K_t(x, y, \lambda)(\sin 2t)(\sinh^2 2\lambda + \sin^2 2t)^{-1}(\cosh 2\lambda)(x - y).$$

Since $c \leqslant (\sinh^2 2\lambda + \sin^2 2t)^{-1} \leqslant (\sin^2 2t)^{-1}$, G in modulus is bounded by

$$|G| \leqslant C(x - y)(\sin 2t)^{-3/2} \exp\{-c\lambda(x - y)^2\}. \tag{4.18}$$

Integrating this against $\lambda^{-1/2}$ proves the desired inequality. The other term is estimated similarly. The proof of the third inequality follows along similar lines. Differentiation with respect to λ brings down a factor of λ and hence we get $|x - y|^{-3}$. Hence the lemma.

Having proved the required estimates, we can now prove Proposition 4.1. Assume that f is an 1-atom supported in $|x - y^*| \leqslant \delta$ i.e., f satisfies the following two conditions.

(i) $$\|f\|_\infty \leqslant \delta^{-1}$$

(ii) $$\int f(x)\, dx = 0.$$

Let Q_δ denote the ball of radius δ centered at y^* and let CQ_d stand for the complement of Q_δ. We write $F(x) = T_t f(x)$ as a sum of three functions, $F = F_1 + F_2 + F_3$ where $F_1(x) = F(x)$ on $Q_{2\delta}$, 0 elsewhere; $F_2(x) = F(x)$ on $CQ_{2\delta} \cap Q_{\delta^{-1}}$, 0 elsewhere and $F_3(x) = F(x)$ on $CQ_{2\delta} \cap CQ_{\delta^{-1}}$, 0 elsewhere. We note that $F_2 = 0$ when $\delta \geqslant 2^{-1/2}$.

To study F_1, we apply the L^2 theory of T_t. From the definition, it is clear that T_t is bounded on L^2. Therefore

$$\begin{aligned}\int |F_1(x)|\, dx &\leqslant |Q_{2\delta}|^{1/2}\left\{\int |F_1(x)|^2\, dx\right\}^{1/2}\\ &\leqslant \delta^{1/2}\left\{\int |T_t f(x)|^2\, dx\right\}^{1/2}\\ &\leqslant C\delta^{1/2}\left\{\int |f(x)|^2\, dx\right\}^{1/2}.\end{aligned}$$

Since

$$\left\{\int |f(x)|^2\, dx\right\}^{1/2} \leqslant C\delta^{-1/2}$$

we get $\int |F_1(x)|\, dx \leqslant C$.

To study F_2 we write the kernel $K_t(x, y)$ in the following way.

$$K_t(x, y) = E_t(x, y) + G_t(x, y) + H_t(x, y)$$

where

$$E_t(x, y) = \int_0^1 \lambda^{-1/2}\{K_t(x, y, \lambda) - K_t(x, y^*, \lambda)\} e^{iB_t(x, y, \lambda)}\, d\lambda$$

$$G_t(x, y) = \int_0^1 \lambda^{-1/2} K_t(x, y^*, \lambda)\{e^{iB_t(x, y, \lambda)} - e^{iB_t(x, y^*, \lambda)}\}\, d\lambda$$

Since f has mean value 0, the kernel $H_t(x,y)$ does not contribute anything. Observe that when x is in $\mathbf{C}Q_{2\delta}$ and y is in Q_δ we have $|x-y^*| \geqslant 2|y-y^*|$. In view of Lemma 4.1 we have the estimates

$$|E_t(x,y)| \leqslant C|y-y^*|\,|x-y^*|^{-2} \tag{4.19}$$

$$|G_t(x,y)| \leqslant C|y-y^*|. \tag{4.20}$$

Therefore,

$$|F_2(x)| \leqslant C\delta\{1+|x-y^*|^{-2}\}.$$

Since $F_2(x)$ is supported in $2\delta \leqslant |x-y^*| \leqslant \delta^{-1}$, we get

$$\int |F_2(x)|\,dx \leqslant C\delta\left\{\int_{|x-y^*|\leq 1/\delta} dx + \int_{|x-y^*|\geq 2\delta} |x-y^*|^{-2}\,dx\right\} \leqslant C.$$

Finally, we consider F_3. We write the kernel as the sum of the following three terms.

$$E_t(x,y) = \int_0^1 \lambda^{-1/2}\{K_t(x,y,\lambda) - K_t(x,y^*,\lambda)\}e^{iB_t(x,y,\lambda)}\,d\lambda$$
$$G_t(x,y) = \int_0^1 \lambda^{-1/2}K_t(x,y^*,\lambda)\{e^{iB_t(x,y,\lambda)} - e^{iB_t(x,y,0)}\}\,d\lambda$$
$$H_t(x,y) = \int_0^1 \lambda^{-1/2}K_t(x,y^*,\lambda)e^{iB_t(x,y,0)}\,d\lambda$$

Using Lemma 4.1 we get the following estimates.

$$|E_t(x,y)| \leqslant C|y-y^*|\,|x-y^*|^{-2} \tag{4.21}$$

$$|G_t(x,y)| \leqslant C|x-y^*|^{-3}. \tag{4.22}$$

These estimates will imply that

$$|E_tf(x)| \leqslant C\delta|x-y^*|^{-2} \quad \text{and} \quad |G_tf(x)| \leqslant C|x-y^*|^{-3}.$$

Therefore,

$$\begin{aligned}\int_{|x-y^*|\geq \delta^{-1}} |E_tf(x)|\,dx &\leqslant C\delta\left\{\int_{|x-y^*|\geq\delta^{-1}} |x-y^*|^{-2}\,dx\right\}\\ &\leqslant C\delta\left\{\int_{|x-y^*|\geq 2\delta} |x-y^*|^{-2}\,dx\right\}\\ &\leqslant C.\end{aligned}$$

When $\delta \leqslant 1$ we get

$$\int_{|x-y^*|\geq\delta^{-1}} |G_tf(x)|\,dx \leqslant C\left\{\int_{|x-y^*|\geq\delta^{-1}} |x-y^*|^{-3}\,dx\right\} \leqslant C\delta^2 \leqslant C$$

and when $\delta > 1$ we have

$$\int_{|x-y^*|\geq 2\delta} |G_tf(x)|\,dx \leqslant C\left\{\int_{|x-y^*|\geq 2\delta} |x-y^*|^{-3}\,dx\right\} \leqslant C\delta^{-2} \leqslant C.$$

This takes care of the terms $E_tf(x)$ and $G_tf(x)$. Finally $H_tf(x)$ is given by

$$H_tf(x) = F_tf(x)\left\{\int_0^1 \lambda^{-1/2} K_t(x, y^*, \lambda)\, d\lambda\right\} = F_tf(x)g_t(x)$$

where

$$F_tf(x) = \int e^{iB_t(x,y,0)} f(y)\, dy.$$

By Plancherel's theorem $\|F_tf\|_2 = \|f\|_2$ and we also have $|g_t(x)| \leqslant C|x - y^*|^{-1}$. Hence by Schwarz inequality we obtain

$$\int_{|x-y^*|\geq\delta^{-1}} |H_tf(x)|\, dx \leqslant C\left\{\int |F_tf(x)|^2\, dx\right\}^{1/2}\left\{\int_{|x-y^*|\geq\delta^{-1}} |x - y^*|^{-2}\, dx\right\}^{1/2}$$
$$\leqslant C$$

since $\|f\|_2 \leqslant \delta^{-1/2}$. This completes the proof of the Proposition 4.1 and hence proves that the operator T_t maps H^1 boundedly into L^1.

We can now prove Theorem 2. Consider the analytic family S_z of operators defined by $S_z = T_t\{(1 - z)/2\}$. Then clearly $S_{1+iy} = T_t(-iy/2)$ is bounded on L^2. Also $S_{iy} = T_t\{(1 - iy)/2)$ and for this operator we can prove that it maps H^1 boundedly into L^1 by repeating the proof of Proposition 4.1. By applying the interpolation theorem of Fefferman and Stein we get the theorem for $1 < p \leqslant 2$. For $p \geqslant 2$, the theorem follows from duality.

5. An Application to the Solutions of the Schrodinger Equation

Consider the solution $u(x, t)$ of the initial value problem for the Schrodinger equation

$$-i\partial_t u(x, t) = (-\Delta + |x|^2)u(x, t), \qquad u(x, 0) = f(x) \tag{5.1}$$

where f is a nice function, say f belongs to the Schwartz class $S(\mathbb{R}^n)$. The solution $u(x, t)$ has the Hermite expansion

$$u(x, t) = \sum e^{(2|\alpha| + n)it} f^\wedge(\alpha)\Phi_\alpha(x)$$

where $f^\wedge(\alpha)$ are the Hermite coefficients of the function f. $f \to u$ defines an operator $F(t)$ which, in view of the above expansion, is given by a kernel as follows:

$$F(t)f(x) = \pi^{-n/2} \int (\sin 2t)^{-n/2} e^{i\varphi(t,x,y)} f(y)\, dy \tag{5.2}$$

where

$$\varphi(t, x, y) = -x \cdot y \operatorname{cosec} 2t + 1/2(|x|^2 + |y|^2)\cot 2t.$$

Clearly, $F(t)$ defines a unitary operator on $L^2(\mathbb{R}^n)$. It also has the group property $F(t)F(s) = F(t+s)$, $F(0) =$ identity. Furthermore, when t takes values in the set $\{\pi/k: k = 1, 2, 3, \ldots\}$ $F(t)$ is a fractional power of F, the Fourier transform. For example, $F(\pi/4) = F$ and $F(\pi/8) = F^{1/2}$. Because of this and the group property the operators $F(t)$ cannot be bounded on L^p. For otherwise the Fourier transform has to be bounded on L^p which, of course, is not true.

A similar situation occurs when we consider the Schrodinger equation without the potential. Let $H(t)$ be the solution operator for the equation $-i\partial_t u(x,t) = -\Delta u(x,t)$, $u(x,0) = f(x)$. Formally this operator can be defined by $(H(t)f)^\wedge(\xi) = \exp\{it|\xi|^2\}f^\wedge(\xi)$. This operator is unitary on $L^2(\mathbb{R}^n)$. But as shown in Hormander [6] and Littman et al. [7], it fails to be bounded on L^p when p is different from 2. In this connection Sjostrand [14] considered the Riesz means of the operator $H(t)$ defined by

$$G_\tau(\alpha)f(x) = \alpha\tau^{-\alpha}\int_0^\tau (\tau - t)^{\alpha-1}H(t)f(x)\,dt$$

and proved that the Riesz means $G_\tau(\alpha)$ are bounded on L^p if and only if $\alpha > n|1/p - 1/2|$. So it is natural to ask the same question with regard to the Riesz means $G_\tau(\alpha)$ of the operators $F(t)$. It turns out that Theorem 3 is true. For the sake of simplicity we treat the one dimensional case.

We write down the Hermite expansion of $F(t)f(x)$ for a smooth function f.

$$F(t)f(x) = \sum e^{(2n+1)it}f^\wedge(n)\varphi_n(x) \tag{5.3}$$

When f is in the Schwartz class the series converges uniformly. Let g be the inverse Fourier transform of the function $h(t)$ defined to be $(1-t)^{\alpha-1}$ for $0 < t < 1$ and 0 elsewhere. Multiplying the above series by $(\alpha/\tau)h(t/\tau)$ and integrating with respect to t we obtain

$$G_\tau(\alpha)f(x) = \sum g\{(2n+1)\tau\}f^\wedge(n)\varphi_n(x). \tag{5.4}$$

The function $g(t)$ can be calculated explicitly. We can write

$$g(t) = \int_{-\infty}^1 (1-s)^{\alpha-1}e^{its}\,ds - \int_{-\infty}^0 (1-s)^{\alpha-1}e^{its}\,ds.$$

We have the following formula for the first integral as proved in Gelfand-Shilov [4], p. 171.

$$g_1(t) = c_\alpha t^{-\alpha}e^{it}, \qquad c_\alpha = \Gamma(\alpha)e^{-i\alpha\pi/2}. \tag{5.5}$$

An integration by parts will show that the second integral is equal to $d_\alpha t^{-1} + g_2(t)$ where the function $g_2(t) = O(t^{-2})$ at infinity.

Thus $G_\tau(\alpha)$ is a sum of three operators, $G_\tau(\alpha)f = T_\tau(\alpha)f + T_0(1)f + Tf$ where

$$Tf(x) = \sum g_2\{(2n+1)\tau\}f^\wedge(n)\varphi_n(x). \tag{5.6}$$

Since $|g_2(2n+1)| \leqslant C(2n+1)^{-2}$ and $\|\varphi_n\|_\infty \|\varphi_n\|_1 \leqslant C(2n+1)^{1/6}$, we see that T is given by an L^1 kernel and so T is bounded on L^p, for all p, $1 \leqslant p \leqslant \infty$. The operator $T_0(1)$ is clearly bounded on L^p, $1 \leqslant p \leqslant \infty$ for the same reason as we have noticed in the previous section. Finally we can apply Theorem 2 to the operators $T_\tau(\alpha)$. For $\alpha \geqslant |1/p - 1/2|$, these operators are bounded on L^p, for $1 < p < \infty$ and when $p = 1$ and $\alpha = 1/2$ it is bounded from H^1 into L^1. This proves Theorem 3.

References

[1] Bonami, A. and Clerc, J. L. Somes de Cesaro et multiplicateurs des developpements en harmonics spheriques, *Trans. Amer. Math. Soc.*, **183**(1973), 223-263.

[2] Coifman, R. and Weiss, G. Extensions of Hardy spaces and their uses in analysis, *Bull. Amer. Math. Soc.*, **83**(1977), 569-645.

[3] Fefferman, C. and Stein, E. M. H^p spaces of several variables, *Acta Math.,* **129**(1972), 137-193.

[4] Gelfand, I. M. and Shilov, G. E. Generalised functions, Vol. I, Academic Press, N. Y., (1964).

[5] Hardy, G. H. and Littlewood, J. E. Some new properties of Fourier constants, *Math. Ann.* **97**(1926), 159-209.

[6] Hormander, L. Estimates for translation invariant operators in L^p spaces, *Acta Math.*, **104**(1960), 93-140.

[7] Littman, W. et al. The non-existence of L^p estimates for certain translation invariant operators, *Studia Math.* **30** (1968), 219-229.

[8] Maureci, G. The Weyl transform and bounded operators on $L^p(\mathbb{R}^n)$, *J. Funct. Anal.* **39**(1980), 408-429.

[9] Muckenhoupt, B. and Stein, E. M. Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118**(1965), 17-92.

[10] Phong, D. H. and Stein, E. M. Hilbert integrals, Singular integrals and Radon transforms I, *Acta Math.* **157**(1986), 99-157.

[11] Sadosky, C. Interpolation of operators and Singular integrals, Marcel Dekker, New York, Basel, 1979.

[12] Sjolin, P. L^p estimates for strongly singular convolution operators in $\mathbb{R}^n$, *Ark. Math.* **14**(1976), 59-64.

[13] Sjolin, P. An H^p inequality for strongly singular integrals, *Math. Z.* **165**(1979), 231-238.

[14] Sjostrand, S. On the Riesz means of the solutions of the Schrodinger equation, *Ann. Scoula. Norm. Sup. Pisa,* **3**(1970), 331-348.

[15] Stein, E. M. Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, N. J. (1971).

[16] Stein, E. M. Topics in harmonic analysis related to Littlewood-Paley theory, *Ann. Math. Studies*, no. 63, Princeton Univ. Press, Princeton, N. J., 1971.

[17] Strichartz, R. S. Multipliers for spherical harmonic expansions. *Trans. Amer. Math. Soc.* **167**(1972), 115-124.

[18] Szego, G. Orthogonal polynomials, *Amer. Math. Soc. Colloq. Publi.*, vol. 23, Amer. Math. Soc. Providence, R. I.

[19] Thangavelu, S. Summability of Hermite expansions I, submitted to *Trans. Amer. Math. Soc.*
[20] Thangvelu, S. Summability of Hermite expansions II, submitted to *Trans. Amer. Math. Soc.*

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Weak Type Endpoint Bounds for Bochner-Riesz Multipliers

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The Bochner-Riesz multipliers are defined for testing functions f on $\mathbb{R}^n$ by

$$(T_\lambda f)^\wedge(\xi) = (1 - |\xi|^2)_+^\lambda \hat{f}(\xi).$$

Questions concerning the convergence or multiple Fourier series have led to the study of their L^p boundedness. It is conjectured that for $n > 1$, for all exponents $p \in (1, 2(n-1)/n)$, T_λ is bounded on L^p for all

$$\lambda > \lambda(p) = n(p^{-1} - 2^{-1}) - 2^{-1} > 0.$$

What is known is that the conjecture holds for the full range of exponents in dimension two [1], and for the smaller range $1 < p \leqslant 2(n+1)/(n+3)$ for all $n \geqslant 3$. Moreover it is very easy to see that T_λ is unbounded for all $\lambda \leqslant \lambda(p)$; it suffices to compute the associated convolution kernel and to examine its action on the characteristic function of the unit ball. Nevertheless there is a positive result at the critical value $\lambda(p)$, at least for a certain range of exponents:

Theorem. *For all $n \geqslant 2$ and $1 < p < 2(n+1)/(n+3)$, $T_{\lambda(p)}$ is of weak type (p,p).*

Temporarily define

$$(T_\lambda^r f)^\wedge(\xi) = (1 - |r^{-1}\xi|^2)_+^\lambda \hat{f}(\xi).$$

Corollary. *For all* $n \geqslant 2$, $1 < p < 2(n+1)/(n+3)$ *and* $f \in L^p(R^n)$, $T_\lambda^r f \to f$ *in measure as* $r \to \infty$.

The result for $p = 1$ was recently proved in [4]. Our proof involves an application of the method of [4], a slight refinement of estimates already known on L^{p_0}, where $p_0 \equiv 2(n+1)/(n+3)$, and an interpolation between L^1 and L^{p_0}.

To begin fix $p \in (1, p_0)$. Write $p^{-1} = \theta \cdot 1 + (1-\theta)p_0^{-1}$, where $0 < \theta < 1$. Fix $\lambda = \lambda(p) = n(p^{-1} - 2^{-1}) - 2^{-1}$, and set $m(\xi) = (1 - |\xi|^2)_+^\lambda$. Let $f \in L^p$ and $\alpha > 0$ be arbitrary. In order to estimate the measure of the set where $|T_\lambda f| > \alpha$, apply the Calderón-Zygmund decomposition to f^p at height α^p to obtain $f = g + b$ where $\|g\|_p \leqslant C\|f\|_p$, $\|g\|_\infty \leqslant C\alpha$, and $b = \Sigma_Q b_Q$ where each b_Q is supported on a dyadic cube Q,

$$\int |b_Q|^p \leqslant \alpha^p |Q|,$$

the cubes Q have pairwise disjoint interiors, and

$$\Sigma_Q |Q| \leqslant C\alpha^{-p}\|f\|_p^p.$$

Since T_λ is bounded on L^2,

$$|\{x: |T_\lambda g(x)| > \alpha/2\}| \leqslant C\alpha^{-2}\|g\|_2^2 \leqslant C\alpha^{-p}\|f\|_p^p.$$

Let E be the union of the doubles of the cubes Q. Then

$$|E| \leqslant C\alpha^{-p}\|f\|_p^p,$$

so it suffices to show that

$$|\{x \notin E: |T_\lambda b(x)| > \alpha/2\}| \leqslant C\alpha^{-p}\|b\|_p^p.$$

This will follow by Chebychev's inequality from

$$\|T_\lambda b\|_{L^2(\mathbb{R}^n \setminus E)}^2 \leqslant C\alpha^{2-p}\|b\|_p^p. \tag{1}$$

Fix $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$, radial and supported in $\{|x| \leqslant 1\}$ and satisfying $\varphi_0(x) \equiv 1$ for $|x| \leqslant 1/4$ and

$$\int (\partial^k \hat{\varphi}_0 / \partial \xi_1^k)(\xi) \cdot (\xi_1)_+^\lambda \, d\xi = 0 \tag{2}$$

for $k = 0, 1$. Let

$$\varphi_j(x) = \varphi_0(2^{-j}x) \quad \text{and} \quad \psi_j = \varphi_j - \varphi_{j-1}.$$

For $j > 0$ let

$$K_j(x) = \psi_j(x)\check{m}(x),$$

and let $K_0 = \varphi_0 \cdot \check{m}$, so that $\check{m} = \Sigma K_j$.

For $0 \leqslant i \in \mathbb{Z}$ let $B_i = \Sigma b_Q$, the sum being taken over all Q with sidelength 2^i when $i > 0$ and sidelength less than or equal to one when $i = 0$. The contribution of B_0 turns out to be relatively easy to treat, so we shall ignore it until the end of the argument and concentrate instead on $\Sigma_{i>0} B_i$. Note that if Q has sidelength 2^i, then for all $j \leqslant i$, $b_Q * K_j$ is supported on the double of Q, hence on E. Consequently for all $x \notin E$,

$$T_\lambda(\Sigma_{i>0} B_i)(x) = \Sigma_{i>0} B_i * (\Sigma_{j>i} K_j)(x) = \Sigma_{s>0} \Sigma_{j>s} B_{j-s} * K_j(x).$$

Hence (1) is a consequence of

$$\|\Sigma_{j>s} B_{j-s} * K_j\|^2_{L^2(\mathbb{R}^n)} \leqslant C 2^{-\epsilon s} \alpha^{2-p} \|b\|_p^p \tag{3}$$

for all $s \in \mathbb{Z}^+$, for some $\epsilon > 0$.

Fix linear functions $l_1, l_2 : \mathbb{C} \to \mathbb{C}$ such that $Re(l_1(z)) \equiv p$ when $Re(z) = 1$, $\equiv p \cdot p_0^{-1}$ when $Re(z) = 0$, and $Re(l_2(z)) \equiv n(p^{-1} - 1)$ when $Re(z) = 1$ and $\equiv n(p^{-1} - p_1^{-1})$ when $Re(z) = 0$. Then $l_1(\theta) = 1$ and $l_2(\theta) = 0$. Define $B_{i,z}(x) = [B_i(x)]^{l_1(z)}$, interpreted as is customary in the standard proof of the Riesz-Thörin interpolation theorem. Define $K_{j,z}(x) = 2^{jl_2(z)} K_j(x)$. Then (3) follows by interpolation between the two endpoint estimates

$$\|\Sigma_{j>s} B_{j-s,z} * K_{j,z}\|_2^2 \leqslant C 2^{-\epsilon s} \alpha^p \|b\|_p^p \tag{4}$$

when $Re(z) = 1$ and

$$\|\Sigma_{j>s} B_{j-s,z} * K_{j,z}\|_2^2 \leqslant C \alpha^{p(2p_0^{-1} - 1)} \|b\|_p^p \tag{5}$$

when $Re(z) = 0$.

To justify (4) consider any collection $\{A_j : j > 0\}$ of functions satisfying

$$\int_Q |A_j| \leqslant C \alpha^p |Q|$$

for all cubes Q in R^n of sidelength 2^j. Consider further any collection of kernels

$$H_j(x) = \Phi(x) h_j(x)$$

where

$$\Phi(x) = \cos(2\pi|x| - \pi(n-1)/4)$$

and each h_j is supported in $\{2^{j-3} \leqslant |x| \leqslant c_2 2^j\}$ and satisfies

$$\|h_j\|_\infty + 2^j \|\nabla h_j\|_\infty \leqslant 2^{-nj}.$$

It is proved in [4] that

$$\|\Sigma_{j>s} A_{j-s} * H_j\|_2^2 \leqslant C 2^{-\epsilon s} \alpha^p \|\Sigma |A_j|\|_1 \tag{4'}$$

for a certain $\epsilon > 0$. This is done by first, for technical reasons, introducing a finite partition of unity $\{\eta_\beta\}$ on $\mathbb{R}^n \setminus \{0\}$ with each η_β homogeneous of degree zero and supported in some cone $\{x: \langle x, v_\beta\rangle > \delta|x|\}$ for some $\delta > 0$ and $v_\beta \in S^{n-1}$. (4′) follows from the variant of itself defined by replacing each H_j by $J_j = H_j \cdot \eta_\beta$, for then one may sum over β. This modified (4′) is an easy consequence of the estimates

$$|J_j * \tilde{J}_j(x)| \leqslant C2^{-nj}(1 + |x|)^{-\mu}$$

and

$$\|J_i * \tilde{J}_j\|_\infty \leqslant 2^{-nj}2^{-\mu i} \quad \text{for all} \quad 0 < i < j - 3,$$

where $\tilde{J}_j(x) \equiv J_j(-x)$ and $\mu = (n-1)/2$; these are not difficult to verify by direct computation using the stated properties of $\{H_j\}$.

When $Re(z) = 1$, $A_j = B_{j,z}$ and $H_j = K_{j,z}$ have all these properties (H_j does, by the known asymptotics for Bessel functions). Therefore we consider (4) to be proved and concentrate on (5). For a single term $B_{j-s,z} * K_{j,z}$, it turns out that the desired bound follows at once from the estimates in [7]; the technical manipulations which follow are designed to enable us to pass from bounds for these individual terms to a bound for the entire sum.

Let $m_j = \hat{K}_j = m * \hat{\psi}_j$ (for $j > 0$).

Lemma 1.

(6) $\|\partial^\alpha m_j / \partial\xi^\alpha\|_\infty \leqslant C_\alpha 2^{j|\alpha|}2^{-j\lambda}$ *for all multi-indices* α.

(7) $|m_j(\xi)| + 2^{-j}|\nabla m_j(\xi)| \leqslant C_M 2^{-jM}$ *for all* M *and all* $\xi \notin [1/2, 3/2]$.

(8) $|m_j(\xi)| + 2^{-j}|\nabla m_j(\xi)| \leqslant C_M 2^{-j\lambda}(1 + 2^j|1 - |\xi||)^{-M}$ *for all* $|\xi| \in [1/2, 3/2]$.

(9) *There exists* $\delta > 0$ *such that*

$$|m_j(\xi)| + 2^{-j}|\nabla m_j(\xi)| \leqslant C2^{-j\lambda} \max(2^j|1 - |\xi||, 2^{-j\delta})$$

for all $|\xi| \in [1 - 2^{-j}, 1 + 2^{-j}]$.

The conclusions are all totally routine bounds for $m_j = m * \hat{\psi}_j$ except for (9), which relies on the technical condition (2). To obtain the bound in (9) for $m_j(\xi)$, observe that since $|m_j(\xi)| \leqslant C2^{-j\lambda}$ when $|\xi| = 1 \pm 2^{-j}$ by (8), and since $\|\nabla m_j\|_\infty \leqslant C2^{j(1-\lambda)}$, it suffices by the fundamental theorem of calculus to prove (9) for $|\xi| = 1$. Both m and $\hat{\psi}_j$ are radial, so we may take $\xi = \xi_0 = (1, 0, \ldots, 0)$.

$$(m * \hat{\psi}_j)(\xi_0) = \int \hat{\psi}_j(\xi_0 - \zeta) \cdot [(1 - |\zeta|^2)_+^\lambda - |2(1 - \zeta_1)_+^\lambda|]\, d\zeta,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots)$, since the term subtracted is actually zero by (2) (with $k = 0$). The function $\hat{\psi}_j(\xi_0 - \bullet)$ is essentially supported on a ball of radius 2^{-j}

centered at ξ_0. On this ball

$$|[1-|\zeta|^2]_+^\lambda - [2(1-\zeta_1)_+]^\lambda| \leqslant C2^{-2j\lambda};$$

the best way to see this is to introduce new coordinates centered at ξ_0 and rescaled by a factor of 2^j. In such coordinates the boundary of the unit ball becomes almost flat as $j\to\infty$, producing an extra factor of $2^{-j\lambda}$. Hence (9) holds for m_j; we omit the precise details. ∇m_j may be estimated in the same way, using (2) with $k=1$.

Lemma 2. *There exist positive radial functions* $\{\eta_j : j>0\}$ *such that* $\Sigma\eta_j^2 \in L^\infty$ *and the multipliers* $n_j = m_j/\eta_j$ *satisfy* (7) *and* (8).

Indeed, define $\eta_j(\xi)=1$ if $|\xi| = 1\pm 2^{-j}$, $=2^{-j\delta/2}$ if $|\xi|=1$, where δ is the exponent in (9), and interpolate smoothly for intermediate values of $|\xi|$. Proceed similarly for $|\xi| \notin [1-2^{-j}, 1+2^{-j}]$.

We may now deduce (5). Suppose that $Re(z)=0$.

$$\begin{aligned}\|\Sigma_{j>s}B_{j-s,z}*K_{j,z}\|_2^2 &= \int |\Sigma\hat{B}_{j-s,z}(\xi)\cdot 2^{jl_2(z)}n_j(\xi)\eta_j(\xi)|^2\,d\xi \\ &\leqslant \int (\Sigma\eta_j(\xi)^2)(\Sigma|\hat{B}_{j-s,z}(\xi)\cdot 2^{l_2(z)}n_j(\xi)|^2)\,d\xi \\ &\leqslant C\int \Sigma|\hat{B}_{j-s,z}\cdot 2^{jl_2(z)}n_j(\xi)|^2\,d\xi \\ &= \Sigma\|B_{j-s,z}*2^{jl_2(z)}\check{n}_j\|_2^2 = \Sigma\|B_{j-s,z}*2^{jn(p^{-1}-p_0^{-1})}\check{n}_j\|_2^2.\end{aligned}$$

Therefore it suffices to show that for all $F\in L^{p_0}(\mathbb{R}^n)$ satisfying

$$\int_Q |F|^{p_0} \leqslant \beta|Q| \tag{10}$$

for all cubes Q of sidelength 2^j, we have

$$\|F*2^{jn(p^{-1}-p_0^{-1})}n_j\|_2^2 \leqslant C\beta^{2p_0^{-1}-1}\|F\|_{p_0}^{p_0}, \tag{11}$$

for $B_{j-s,z}$ satisfies (10) uniformly for all $s\in\mathbb{Z}^-$, $z\in i\mathbb{R}$, with $\beta=\alpha^p$. Set $L_j = 2^{jn(p^{-1}-p_0^{-1})}\check{n}_j\varphi_j$, and for all $i>j$ set $L_i = 2^{jn(p^{-1}-p_0^{-1})}\check{n}_j\psi_i$. We will prove that there exists $\epsilon>0$ such that for *all* $F\in L^{p_0}$ and all $i\geqslant j$,

$$\|F*L_i\|_2^2 \leqslant C2^{-\epsilon(i-j)}2^{-ni(2p_0^{-1}-1)}\|F\|_{p_0}^2. \tag{12}$$

Since L_i is supported on $\{|x|\leqslant 2^i\}$, it follows at once that

$$\|F*L_i\|_2^2 \leqslant C2^{-\epsilon(i-j)}\beta^{2p_0^{-1}-1}\|F\|_{p_0}^{p_0}$$

for all $F\in L^{p_0}$ satisfying (10). Summing over i gives (11).

Finally (12) is a straightforward consequence of the L^2 restriction theorem of Tomas and Stein, as in [7]. For if $I = [1/2, 3/2]$ and $B = \{|\xi| \notin I\}$, then

$$\|F * L_i\|_2^2 = \int_B |\hat{F}(\xi)|^2 |\hat{L}_i(\xi)|^2 \, d\xi + \int_I \left(\int_{S^{n-1}} |\hat{F}(r\theta)|^2 \, d\theta \right) \cdot |\hat{L}_i(r)|^2 r^{n-1} \, dr$$

where we have written $\hat{L}_i(r)$ for $\hat{L}_i(\xi)$ when $|\xi| = r$, recalling that $\hat{L}_i$ is radial. For $\xi \in B$,

$$\begin{aligned} |\hat{L}_i(\xi)| &= 2^{jn(p^{-1} - p_0^{-1})} \cdot |n_j * \hat{\psi}_i(\xi)| \quad (\text{or } \hat{\varphi}_j \text{ when } i = j) \\ &\leqslant C_M 2^{-(i-j)} 2^{-Mj} (1 + |\xi|)^{-M} \end{aligned}$$

for all $M < \infty$, by the bounds (7) and (8) for n_j and its gradient, and routine estimation. Hence the Hausdorff-Young inequality gives

$$\begin{aligned} \int_B |\hat{F}(\xi)|^2 |\hat{L}_i(\xi)|^2 \, d\xi &\leqslant C_M 2^{-2(i-j)} 2^{-Mj} \|F\|_{p_0}^2 \\ &= 2^{-\epsilon(i-j)} 2^{-ni(2p_0^{-1} - 1)} 2^{j(-M + (2p_0^{-1} - 1))} \|F\|_{p_0}^2 \end{aligned}$$

where $\epsilon = 2 - n(2p_0^{-1} - 1) = 2/(n+1) > 0$. Thus the desired bound follows as soon as $M \geqslant 2p_0^{-1} - 1$. On the other hand for $r \in I$ we have

$$\int_{S^{n-1}} |\hat{F}(r\theta)|^2 \, d\theta \leqslant C \|F\|_{p_0}^2$$

by the restriction theorem. Hence

$$\int_{\mathbb{R}^n \setminus B} |\hat{F}(\xi)|^2 |\hat{L}_i(\xi)|^2 \, d\xi \leqslant C \|F\|_{p_0}^2 \int_I |\hat{L}_i(r)|^2 \, dr.$$

It follows from (7), (8) and routine computation that for $r \in I$,

$$|\hat{L}_i(r)| \leqslant C_M 2^{jn(p^{-1} - p_0^{-1})} 2^{-j\lambda} (1 + 2^j |1 - |\xi||)^{-M} \cdot 2^{j-i}$$

for all $M < \infty$. Hence

$$\begin{aligned} \int_I |\hat{L}_i(r)|^2 \, dr &\leqslant 2^{2jn(p^{-1} - p_0^{-1})} 2^{-2j\lambda} \cdot 2^{-j} \cdot 2^{-2(i-j)} \\ &= 2^{-in(2p_0^{-1} - 1)} 2^{-\epsilon(i-j)} \end{aligned}$$

where again $\epsilon = 2/(n+1)$. This concludes the proof of (5).

Only the contribution of B_0 remains to be treated. Again form the analytic functions $B_{0,z}$ and $K_{j,z}$ as above. When $Re(z) = 0$ it follows from the L^2 restriction theorem that

$$\|B_{0,z} * K_{j,z}\|_2^2 \leqslant C\alpha^{p(2p_0^{-1} - 1)} \|B_0\|_p^p$$

as above; now it is not necessary to introduce the η_j and n_j, so the proof is straightforward. On the other hand it is shown in [4] that when $Re(z) = 1$,

$$\|B_{0,z} * K_{j,z}\|_2^2 \leqslant C2^{-j(n-1)/2}\alpha^p \|B_{0,z}\|_1.$$

Since the right-hand side is equal to $C2^{-j(n-1)/2}\alpha^p \|B_0\|_p^p$, interpolation gives $\|B_{0,z} * K_{j,z}\|_2^2 \leqslant C2^{-j\theta(n-1)/2}\alpha^{2-p}\|B_0\|_p^p$. So

$$\begin{aligned}\|B_0 * \Sigma K_j\|_2 &\leqslant \Sigma \|B_0 * K_j\|_2 \\ &\leqslant C\alpha^{(2-p)/2}\|B_0\|_p^{p/2}\Sigma 2^{-j\theta(n-1)/4} \\ &\leqslant C[\alpha^{2-p}\|B_0\|_p^p]^{1/2}.\end{aligned}$$

Remark. In dimension $n = 2$ T_λ is known [1] to be bounded on L^p for all $\lambda > \lambda(p)$, for all $p \leqslant 4/3$, but our proof applies only in the smaller range $p < 6/5$. It remains an open question whether weak type endpoint results hold in the full range of exponents, even in dimension two. In [2] this has been shown to be the case for radial functions.

References

[1] Carleson, L. and Sjölin, P. Oscillatory integrals and a multiplier problem for the disc, *Studia Math.* **44**(1972), 287-299.
[2] Chanillo, S. and Muckenhoupt, B. Weak type estimates for Bochner-Riesz spherical summation multipliers, to appear in *Trans. Amer. Math. Soc.*
[3] Christ, M. On almost everywhere convergence of Bochner-Riesz means in higher dimensions, *Trans. Amer. Math. Soc.* **95**(1985), 16-20.
[4] —, Weak type (1,1) bounds for rough operators, to appear in *Annals of Math.*
[5] Córdoba, A. A note on Bochner-Riesz operators, *Duke Math. J.* **46**(1979), 505-511.
[6] Fefferman, C. Inequalities for strongly singular convolution operators, *Acta Math.* **124**(1970), 9-36.
[7] —, A note on spherical summation multipliers, *Israel J. Math.* **15**(1973), 44-52.

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Total Curvature of Non-Differentiable Curves

Gustavo Corach and Horacio Porta

Introduction

This paper deals with the total curvature of curves γ in euclidean space. It is defined as the supremum of the expressions $\sum \alpha_i$ where α_i = angle formed by succesive chords C_i, C_{i+1} determined by a partition of the parameter interval, and we denote it by $T(\gamma)$. Notice that when γ has a curvature k then $T(\gamma) = \int k\, ds$, where ds is the element of arc-length (see comments at the end of section 2).

Our aim is to study curves for which $T(\gamma) < +\infty$ without a priori conditions regarding smoothness of γ: in this sense, the paper is more «real variables» than «differential geometry». This approach has been used by Borsuk [2], Fáry [3] and Milnor [7] among others in their study of knots, and some results below are extensions or improvements of their findings. In particular, Proposition 4.5 below (whose proof was communicated to us by A. P. Calderón) generalizes a statement by Fáry (third paragraph on *p.* 130 of [3]; see also [1]).

Furthermore, the hypothesis $T < +\infty$ in conjunction with an interior cone condition was used by McGowan and Porta (see [6]) as a substitute for convexity to extend Paul Levy's integral representation to distances in the plane which are not norms. This notion also appears in Finsler spaces (see Rund [8]), at least in the general form given in section 6 below, and in isoperimetric problems (see Bandle [0]); however differentiability or rectifiability is usually required. Finally we mention the following result of Gleason (see [4]): if γ_n

is the longest polygon with n vertices all on a curve γ, then the correspondi lengths L_n and L satisfy

$$\lim n^2(L - L_n) = (1/24)\left(\int k^{2/3}\, ds\right)^3.$$

Since

$$\lim T(\gamma_n) = \int k\, ds$$

we may ask for other relations involving T and length and also for t geometric significance of other moments of the curvature (for the second, s Weinèr [12]).

The main results obtained are the following: under the hypothesis $T(\gamma) < +$ we pove that γ has one-sided tangents everywhere (Theorem 2.3) which coi cide at all but countable many points (Corollary 3.8). Furthermore, γ can decomposed into finitely many graphs of Lipschitz functions (Propositi 3.9) and, if τ denotes the Gauss map of γ defined by $\tau(t)$ = right unit tange vector at $\gamma(t)$, then $T(\gamma)$ = length of τ considered as a curve in the unit sphe S under the geodesic distance (the distance is relevant because τ is disco tinuous in general).

The last two sections are devoted to the non-Hilbert case and to the rel tions among total curvature, rectifiability, bounded variation and the like

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1. Preliminary Remarks

1.1. In the sequel we often consider angles formed by elements of a Hilbe space H. If U, V are non-zero elements of H we define ang(U, V) by

$$\cos \operatorname{ang}(U, V) = \langle U, V\rangle / \|U\|\ \|V\|,$$

where we require that $0 \leqslant \operatorname{ang}(U, V) \leqslant \pi$. It is clear that ang is a continuo function from $(H - \{0\}) \times (H - \{0\})$ into $[0, \pi]$, and that it verifies

$$\operatorname{ang}(U, V) + \operatorname{ang}(V, W) \geqslant \operatorname{ang}(U, W) \tag{1.1a}$$

whenever U, V, W are non-zero. This *angle triangle inequality* has the usu consequences (like its iterated form $\Sigma \operatorname{ang}(U_i, U_{i+1}) \geqslant \operatorname{ang}(U_1, U_n)$, f example).

When restricted to the unit sphere S of H, ang is a distance. Furthermor

$$\operatorname{ang}(U, V) \geqslant \|U - V\| \geqslant (1 - d^2/6)\operatorname{ang}(U, V), \tag{1.1b}$$

when $\| U \| = \| V \| = 1$ and $d = \| U - V \|$. Occasionally UV is used as an abbreviation of ang (U, V).

1.2. Let $G(u, v)$ be an interval function, i.e., a real valued function defined for $a \leqslant u \leqslant v \leqslant b$. We denote by $G'(c) = \lim G(u, v)$ taken when $u < c < v$ and $u, v \to c$. Admitedly, this limit may not exist; if G is monotonic, G' exists almost everywhere (see [5], page 94).

1.3. Suppose now that X is a metric space, with distance d and let σ be a (not necessarily continuous) function $\sigma: [a, b] \to X$. The total variation $\operatorname{Sup} \Sigma d(\sigma(t_i), \sigma(t_{i+1}))$ is called the *length* of σ. When the length is finite, we say that σ is *rectifiable*. This notion appears below in two different settings: when X is a Hilbert space H with the norm distance (in which case the length of σ is denoted by $l(\sigma)$) and when X is the unit sphere S of H and $d =$ ang (and then we use $l_S(\sigma)$ for the length of σ).

We remark without proof that $l(\sigma) \leqslant l_S(\sigma)$ for all $\sigma: [a, b] \to S$ with equality when σ is continuous (the proof uses 1.1*b*).

By a «curve» in H we mean a continuous simple curve defined by a parametrization $\gamma: [a, b] \to H$; therefore γ is a homeomorphism from $[a, b]$ onto its image, and $l(\gamma)$ is the length of the curve.

The following notation will be used throughout: if U, V are distinct vectors, $C(U, V) = (V - U)/\| V - U \|$. If $\gamma(t)$, $a \leqslant t \leqslant b$ is a curve, and $a \leqslant u \leqslant v \leqslant b$, then $C(u, v) = C(\gamma(u), \gamma(v))$, so that $C(u, v)$ is the normalized chord from $\gamma(u)$ to $\gamma(v)$. Also, the curve $\bar{\gamma}$ is a *shortcut* of the curve γ if $\bar{\gamma}(t) = \gamma(t)$ for $a \leqslant t \leqslant u$ and $v \leqslant t \leqslant a$ while $\bar{\gamma}(t)$, $u \leqslant t \leqslant v$, coincides with the straight line segment joining $\gamma(u)$ and $\gamma(v)$.

2. Total Curvature of Curves in Hilbert Space

Suppose that $\gamma(t)$, $a \leqslant t \leqslant b$ is a curve in H, Hilbert space.

Let $\Pi = \{c_0, c_1, c_2, \ldots, c_{n+1}\}$ satisfy $a \leqslant c_0 < c_1 < c_2 < \cdots < c_{n+1} \leqslant b$ (a «partition» in $[a, b]$). We set $T(\Pi) = T\{c_0, c_1, \ldots, c_{n+1}\} = \Sigma \alpha_i$ where $\alpha_i = \operatorname{ang}(C(c_{i-1}, c_i), C(c_i, c_{i+1}))$. If the particular curve under consideration has to be identified, we write $T(\gamma; \Pi)$, etc.

Suppose now that I is a interval (of any kind) contained in $[a, b]$, with end-points $u < v$. We set $T(I) = T(u, v) = \operatorname{Sup} T(\Pi)$, where Π ranges over all partitions satisfying $u \leqslant c_0 < c_1 < \cdots < c_{n+1} \leqslant v$. We repeat that this definition does not distinguish between $I = [u, v]$, $I = [u, v)$, etc. Just as above we write $T(\gamma; u, v)$ when necessary.

2.1. Definition. *The curve γ has finite total curvature when $T(a, b) < +\infty$. In this case $T(a, b)$ is called the total curvature of γ.*

This terminology is justified at the end of this section.

We list below a few properties of the interval function T and indicate of the proofs.

2.2*a*. *Positivity*: $T \geqslant 0$.

2.2*b*. *Monotonicity with respect to shortcuts*: if $\bar{\gamma}$ is a shortcut of γ $T(\bar{\gamma}) \leqslant T(\gamma)$. (cf. [7], Cor. 1.2).

PROOF. Consider the family of partitions Π_0 having u and v as adjace tition points, where u, v have the same meaning as in (1.3). $\operatorname{Sup} T(\gamma; \Pi_0) \leqslant \operatorname{Sup} T(\gamma; \Pi) = T(\gamma)$. But for $\bar{\gamma}$ the partitions Π_0 are j good as all partitions since adding new points between u and v do change the value of $T(\bar{\gamma}; \Pi_0)$. Hence $\operatorname{Sup} T(\bar{\gamma}; \Pi_0) = T(\bar{\gamma})$ and 2.2*b* fo

Observe that this implies the following «bang-bang» principle. If the γ is a polygonal line with vertices $P_0, P_1, \ldots, P_{n+1}$, then the total cur of γ is the smallest among the curves passing throught $P_0, P_1, \ldots, P$ that order. In other words: «least twisted = shortest».

2.2*c*. *Superadditivity with respect to intervals*: if (u_j, v_j) are disjoint sul vals of (u, v), then $\Sigma T(u_j, v_j) \leqslant T(u, v)$. In particular, T is *monotonic*

2.2*d*. *Invariance under parameter changes*: if γ and γ_1 are parametriz of the same curve, then $T(\gamma) = T(\gamma_1)$.

The following theorem is the key result of this section.

2.3. Theorem. *Let $\gamma(t)$, $a \leqslant t \leqslant b$, be a curve with finite total curv Then for each $a \leqslant c < b$ the limit.*

2.3*a*. $$T^+(c) = \lim\nolimits_{c \leq u < v,\, v \to c} C(u, v)$$

exists in the following sense: for each $\epsilon > 0$ there exists $\delta > 0$ suc $\| C(u, v) - T^+(c) \| < \epsilon$ whenever $c \leqslant u < v \leqslant c + \delta$. A similar statemen for

2.3*b*. $$T^-(c) = \lim\nolimits_{u < v \leq c,\, u \to c} C(u, v)$$

when $a < c \leqslant b$. In particular, $T^+(c)$ and $T^-(c)$ are the right and le tangent vector to γ at $\gamma(c)$, respectively.

PROOF. First step: consider a sequence $t_1 > u_1 > t_2 > u_2 > \cdots > c$ with and form the series

$$s = \Sigma \operatorname{ang}(C(u_k, t_k), C(u_{k+1}, t_{k+1})).$$

If we abbreviate $\Pi_n = \{u_n, t_n, u_{n-1}, t_{n-1}, \ldots, u_1, t_1)$ then the partial sums s_n of s satisfy $s_n \leqslant T(\Pi_n)$, since the k^{th} term of s is majorated by

$$w_k = T\{u_{k+1}, t_{k+1}, u_k, t_k\} \quad \text{and} \quad T(\Pi_n) = \sum_{1 \leq k \leq n-1} w_k.$$

Thus $s_n \leqslant T(a, b)$ and s is a convergent series.

On the other hand, for $k > j$:

$$\text{ang}\,(C(u_k, t_k), C(u_j, t_j)) \leqslant s_k - s_{j-1}$$

and therefore we have

$$\lim_{k,j\to\infty} \text{ang}\,(C(u_k, t_k), C(u_j, t_j)) = 0.$$

This means that $\{C(u_n, t_n)\}$ is a Cauchy sequence for the ang distance whence, by (1.1b), it is also a Cauchy sequence in the norm. Therefore there exists the limit $V = \lim C(u_n, t_n)$.

This limit is independent of the particular sequence (u_n, t_n): if (u'_n, t'_n) is a second such sequence with $V' = \lim C(u'_n, t'_n)$ we can thin out both of them to obtain subsequences (denoted by the same symbols) satisfying

$$t_1 > u_1 > t'_1 > u'_1 > t_2 > u_2 > t'_2 > u'_2 > \cdots$$

But this combined sequence is again convergent, which can only happen if $V = V'$.

Second step: Suppose only that $c < u_n < t_n$ with $t_n \to c$. Any subsequence of $C(u_n, t_n)$ has itself a subsequence with limit V for, discarding enough terms, we can obtain the alternation $t_1 > u_1 > t_2 > u_2 > \cdots$ and the argument of the first step applies. But then the whole sequence $C(u_n, t_n)$ converges to V.

Third step: Let now $c \leqslant u_n < t_n$ with $t_n \to c$. Choose $u_n < u'_n < t_n$ such that $\|C(u_n, t_n) - C(u'_n, t_n)\| < 1/n$. Then, $C(u'_n, t_n)$ being convergent to V by the second step, we also have $\lim C(u_n, t_n) = V$ as claimed.

We complete the definition of T^+ and T^- setting

$$T^+(b) = T^-(b),$$
$$T^-(a) = T^+(a).$$

Then:

2.4. Corollary. *The functions $T^+ : [a, b] \to S$, $T^- : [a, b] \to S$ have the following properties*:

2.4*a*. *T^+ is right continuous and T^- is left continuous.*

2.4*b*. $$T^+(c) \to T^-(c_0) \quad \textit{when} \quad c \to c_0, \qquad c < c_0.$$
$$T^-(c) \to T^+(c_0) \quad \textit{when} \quad c \to c_0, \qquad c > c_0.$$

2.4*c*. *T^+ and T^- are rectifiable for the* ang *distance on S, and*

$$l_S(T^+) \leqslant T(\gamma),$$
$$l_S(T^-) \leqslant T(\gamma).$$

Note. The inequalities in 2.4*c* are equalities (see 4.5*b*).

PROOF. Let $c_n \to c_0$, $c_n \geqslant c_0$. For $\epsilon > 0$ we have

$$\| C(c_n, c_n + 1/j) - T^+(c_0) \| \leqslant \epsilon,$$

for n, j large enough. Taking limits as $j \to \infty$ we get from 2.3

$$\| T^+(c_n) - T^+(c_0) \| \leqslant \epsilon$$

for the same values of n, so that 2.4*a* is proved for T^+. The proof for T^- is similar.

Assume now that $c_n \to c_0$, $c_n < c_0$ and let $\epsilon > 0$. Then by 2.3*b* there exists N_ϵ such that for $j > 1/(c_0 - c_n)$ and $n \geqslant N_\epsilon$ we have

$$\| C(c_n, c_n + 1/j) - T^-(c_0) \| \leqslant \epsilon.$$

Taking the limit as $j \to \infty$ we get $\| T^+(c_n) - T^-(c_0) \| \leqslant \epsilon$ which proves the first part of 2.4*b*. The second part is similar.

Finally, if $\epsilon > 0$ and $a = t_0 < t_1 < \cdots < t_n = b$, we can find $\Pi: t_0 < t'_0 < t_1 < t'_1 < \cdots$ with

$$\operatorname{ang}(T^+(t_i), C(t_i, t'_i)) \leqslant \epsilon/n.$$

Then

$$\sum \operatorname{ang}(T^+(t_i), T^+(t_{i+1})) \leqslant \operatorname{ang}(C(t_i, t'_i), C(t_{i+1}, t'_{i+1})) + 2\epsilon \leqslant T(\epsilon) + 2\epsilon$$

and therefore $l_S(T^+) \leqslant T(\gamma)$ as claimed. The proof for T^- is similar.

Remark. We close this section with a sketch of the proof that $T(a, b)$ is the «total curvature» of the curve γ when γ has a curvature $k = dT/dS$, where $T =$ unit tangent vector $= d\gamma/ds$ and s is arclength. For a complete proof see [7], Theorem 2.2. For simplicity we assume that γ is a curve in $\mathbb{R}^3$ parametrized by arclength.

Setting cartesian coordinates in convenient way we have (see [10], Vol. 1, Chapter 1):

2.5. $$\gamma(s) = se_1 + (1/2)s^2ke_2 + \text{terms of higher order in } s$$

where $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Here k denotes the curvature of γ at $s = 0$.

Let now $\epsilon > 0$ and $K > 1$, and pick $\eta > 0$ so that $0 < \eta < \epsilon$, that 2.5 is valid in the interval $0 < s < \eta$, and that $\|e_1 - d\gamma/ds\| < \epsilon$ for $0 < s < \eta$. Let $0 = s_0 < s_1 < \cdots < s_n = \eta$ be chosen so that $\|\gamma(s_j) - \gamma(s_{j-1})\| = \delta$ is independent of j. It is easy to see that $\delta \leqslant s_j - s_{j-1} \leqslant (1 - \epsilon)\delta$. Also, if η is small enough, for $C_j = C(s_j, s_{j+1})$ we have

$$\|C_{j-1} - C_j\| \leqslant \operatorname{ang}(C_{j-1}, C_j) \leqslant K\|C_{j-1} - C_j\|.$$

Then

$$T(\Pi) = \sum \operatorname{ang}(C_{j-1}, C_j) \leqslant K \sum \|C_{j-1} - C_j\|.$$

On the other hand $C_{j-1} = (\gamma(s_j) - \gamma(s_{j-1}))/\delta$ and therefore

$$\|C_{j-1} - C_j\| = \|2\gamma(s_j) - \gamma(s_{j+1}) - \gamma(s_{j-1})\|/\delta.$$

Using 2.5 we obtain

$$2\gamma(s_j) - \gamma(s_{j+1}) - \gamma(s_{j-1}) = (2s_j - s_{j+1} - s_{j-1})e_1 + (k/2)(2s_j^2 - s_{j+1}^2 - s_{j-1}^2)e_2 + h\cdot o\cdot t.$$

($h \cdot o \cdot t$ = higher order terms) so that, from $s_j - s_{j-1} = \delta + h \cdot o \cdot t$ we conclude that

$$\|2\gamma(s_j) - \gamma(s_{j+1}) - \gamma(s_{j-1})\| = k\delta^2 + h \cdot o \cdot t.$$

Thus

$$T(\Pi) \leqslant K\eta\delta k + h \cdot o \cdot t = Kk\eta + h \cdot o \cdot t$$

because $n\delta = \eta + h \cdot o \cdot t$. Therefore $T(0, \eta) \leqslant Kk\eta$ and $K > 1$ being arbitrary we get $dT/ds \leqslant k$. It follows that $T(a, b) \leqslant \int k\, ds$.

The converse inequality also holds since, using 2.4c, we get

$$\int k\, ds = \int \left\| \frac{dT}{ds} \right\| ds = l(T) = l_S(T) \leqslant T(a, b).$$

Therefore

$$T(a, b) = \int k\, ds, \tag{2.6}$$

and this justifies the terminology «total curvature» used for $T(a, b)$.

3. Further Properties of the Total Curvature

The following result is easy to obtain:

3.1. *For $U, V, W \in H - \{0\}$ we have*

$$\operatorname{ang}(U, V) + \operatorname{ang}(V, W) = \operatorname{ang}(U, W)$$

whenever $U = -W$ or $V = rU + sW$ for some $r, s \geqslant 0$.

It corresponds to the fact that in any triangle an exterior angle is the sum of the non-adjacent interior angles.

The next result is a corollary of 3.1. Consider distinct vectors $V_0, V_1, \ldots, V_{n+1}$ in H and let $V'_0 = V_0$, $V'_1 = V_1, \ldots, V'_{j-1} = V_{j-1}$, $V'_j = V_{j+1}, \ldots, V'_n = V_{n+1}$. Denote $D_i = V_{i+1} - V_i$, $D'_i = V'_{i+1} - V'_i$, $\eta_i = \operatorname{ang}(D_i, D_{i-1})$, $\eta'_i = \operatorname{ang}(D'_i, D'_{i-1})$. Then

3.2. $$\sum \eta_i \geqslant \sum \eta'_i.$$

We leave the special cases $j = 1$ and $j = n$ to the reader and prove 3.2 under the assumption $1 < j < n$. After cancellation of like terms, we get that 3.2 amounts to

3.2*a*. $$\eta_{j-1} + \eta_j + \eta_{j+1} \geqslant \eta'_{j-1} + \eta'_j.$$

Using $D'_{j-2} = D_{j-2}$, $D'_{j-1} = D_{j-1} + D_j$, $D'_j = D_{j+1}$ we obtain from 3.1

$$\eta_j = \operatorname{ang}(D_{j-1}, D'_{j-1}) + \operatorname{ang}(D'_{j-1}, D_j).$$

On the other hand, by the angle triangle inequality 1.1*a* we get

$$\eta_{j-1} + \operatorname{ang}(D_{j-1}, D'_{j-1}) \geqslant \eta'_{j-1}$$

$$\eta_{j+1} + \operatorname{ang}(D'_{j-1}, D_j) \geqslant \eta'_j$$

so that, adding up the last three relations, we get 3.2*a*.

In the sequel the following property, which sharpens 1.1*b*, is used several times:

3.3. *When restricted to the unit sphere $S = \{\|U\| = 1\}$ the function* ang *is a distance equivalent to the norm distance, since*

$$\|U - V\| \leqslant \operatorname{ang}(U, V) \leqslant (\pi/2)\|U - V\|.$$

With the aid of 3.1, 3.2 and 3.3 we can obtain the following additional properties of the function T for a curve γ in a Hilbert space H.

3.4*a*. *Monotonicity with respect to partitions*: if Π_2 is a refinement of Π_1, then $T(\Pi_2) \leqslant T(\Pi_1)$. In particular $T(\Pi) = \lim T(\Pi)$. (see [2], pp. 254-256 or [7], Lemma 1.1).

PROOF. If suffices to consider the case where $\Pi_2 = \{c_0, c_1, \ldots, c_{n+1}\}$ and $\Pi_1 = \{c_0, c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{n+1}\}$ and then apply 3.2 with $V_i = \gamma(c_i)$.

3.4*b*. *Total curvature of polygonal lines*: if γ is a polygonal line with vertices at $\gamma(c_0)$, $\gamma(c_1), \ldots$, then $T(\gamma) = T(\Pi_0)$ where $\Pi_0 = \{c_0, c_1, \ldots\}$.

PROOF. Using 3.4a we have $T(\gamma) = \lim T(\Pi)$; but $T(\Pi) = T(\Pi_0)$ for any Π finer than Π_0.

3.4*c*. *Lower continuity with respect to intervals*: if $I_1 \supset I_2 \supset \cdots$ are intervals contained in $[a, b]$ with $\cap I_n = \varnothing$ and γ has finite total curvature on $[a, b]$ then $T(I_n) \to 0$.

PROOF. It is clear from the hypothesis that $\cap \bar{I}_n$ consists of exactly one point, say r. Also denote a_n, b_n the left and right endpoints of I_n. Since $r \notin \cap I_n$ we have $r \notin I_n$ for n large. This implies that $r = a_n$ for all $n \geqslant N$ or $r = b_n$ for all $n \geqslant N$. Consider the first case, the other being similar, and assume by contradiction that $T(I_n) > k > 0$ for all $n \geqslant N$. Let $\Pi_1 = \{c_0 = r, c_1, \ldots, c_{n+1} = b_N\}$ be a partition such that $T(\Pi_1) > k$. Replacing $c_0 = r$ by $e_1 = r + \epsilon$ we get a new partition Π_1'; using the continuity of ang (see 3.3) we may assume that $T(\Pi_1') > k$ by choosing ϵ small enough. Then setting $d_1 = b_N$ we get $T(e_1, d_1) > k$.

Since $b_n \to r$ we have $r < b_m < e_1$ for some m and repeating the argument we conclude that $T(e_2, d_2) > k$ for appropiate $r < e_2 < d_2 = b_m < e_1$. Continuing in this way we obtain disjoint intervals $J_n = (e_n, d_n)$ with $T(J_n) > k$ which contradicts $T(a, b) < +\infty$ in view of 2.2c.

3.4*d*. *Upper continuity with respect to intervals*: If $I_1 \subset I_2 \subset \cdots$ are intervals contained in $[a, b]$ and $I = \cup I_n$, then $T(I_n) \to T(I)$.

PROOF. Denote by a_n, b_n the endpoints of I_n. For $\epsilon > 0$ let $\Pi = \{a, c_1, c_2, \ldots, c_k, b\}$ be a partition of $[a, b]$ with $T(\Pi) > T(a, b) - \epsilon$. Using again the continuity of ang we see that $T(\Pi_n) > T(a, b) - \epsilon$, where $\Pi_n = \{a_n, c_1, \ldots, c_k, b_n\}$ and n is large enough. But then

$$\lim T(I_n) \geqslant \lim T(\Pi_n) \geqslant T(a, b) - \epsilon$$

and the result follows since $\epsilon > 0$ is arbitrary.

3.4*e*. If $I_1 = (u_1, v_1)$, $I_2 = (u_2, v_2)$ are contained in (a, b) then

$$\operatorname{ang}(C(u_1, v_1), C(u_2, v_2)) \leqslant T(a, b).$$

The following property of T is a valuable tool for the sequel.

3.5. Proposition (the addition formula). *Let γ be a curve with finite total curvature and let $a \leqslant u < c < v \leqslant b$. Then*

3.5*a*. $T(u, c) + \operatorname{ang}(T^-(c), T^+(c)) + T(c, v) = T(u, v)$.

PROOF. Let Π_n, $n = 1, 2, \ldots$ be a sequence of partitions in $[u, v]$ such that $T(\Pi_n) \to T(u, v)$. By 3.4*a* the convergence is preserved if we add partition points to Π_n so that we may assume that $c \in \Pi_n$ and that $\Pi'_n = \{c_j \in \Pi_n; c_j < c\}$ and $\Pi''_n = \{c_j \in \Pi_n; c_j < c\}$ satisfy $T(\Pi'_n) \to T(u, c)$ and $T(\Pi''_n) \to T(c, v)$. Abbreviate now α_j = angle formed by the chords $C(c_{j-1}, c_j)$ and $C(c_j, c_{j+1})$ and use β for the α_i corresponding to $c_i = c$. Then

$$T(\Pi_n) = \sum \alpha_j,$$

$$T(\Pi'_n) = \sum \{\alpha_j; c_j < c\}$$

and

$$T(\Pi''_n) = \sum \{\alpha_j; c_j > c\},$$

and therefore

$$T(\Pi_n) = T(\Pi'_n) + \beta + T(\Pi''_n).$$

Taking limits we get the desired formula as an application of 2.3.

Using the notation

$$T'(c) = \lim T(a_n, b_n), \qquad a_n < c < b_n \qquad b_n - a_n \to 0$$

introduced in 1.2, we have

3.6. Corollary. *For any $c \in (a, b)$,*

3.6*a*. $T'(c) = \operatorname{ang}(T^-(c), T^+(c))$.

PROOF. Write

$$T(a_n, b_n) = T(a_n, c) + \operatorname{ang}(T^-(c), T^+(c)) + T(c, b_n).$$

Taking limits and using 3.4*c* we get 3.6*a*.

Of course the addition formula 3.5*a* holds more generally in the form

3.7. $$T(a_0, a_{n+1}) = \sum_{i=0}^{n} T(a_i, a_{i+1}) + \sum_{i=1}^{n} T'(a_i)$$

for $a \leqslant a_0 < a_1 < \cdots < a_{n+1} \leqslant b$.

3.8. Corollary. *If γ has finite total curvature, then*

(*a*) *The inequality $T'(c) \neq 0$ can happen only for countably many values $s_1, s_2, \ldots$ of c and the series $\sum T'(s_k)$ is convergent.*
(*b*) *The curve γ has a unique tangent at all but countably many points.*

PROOF. From 3.7 we obtain

$$\sum T'(a_i) \leqslant T(a, b) < +\infty$$

for any choice of $a_0 < a_1 < \cdots < a_{n+1}$ and this suffices to obtain 3.8*a*. Using 3.6*a* we see that 3.8*b* follows from 3.8*a*.

Property 3.8*b* can be sharpened in the following way:

3.9. Proposition. *If γ is a curve with finite total curvature, its graph splits in a finite number of graphs of Lipschitz functions. In particular the curve is rectifiable.*

PROOF. Observe first that given any $m > 0$ there is a partition $a = c_0 < c_1 < \cdots < c_{n+1} = b$ such that $T(c_i, c_{i+1}) < m$ for all i. In fact, there exist only finitely many t with $T'(t) \geqslant m$. Label them $t_1, t_2, \ldots, t_k$. For each u interior to an interval $J = [t_i, t_{i+1}]$ we have $T'(u) < m$ and therefore there is a neighborhood $(u - \epsilon, u + \epsilon)$ with $T(u - \epsilon, u + \epsilon) < m$. Also there exist $t_i < t_i'$ and $t_{i+1}' < t_{i+1}$ with $T(t_i, t_i') < m$ and $T(t_{i+1}', t_{i+1}) < m$, by 3.4*c*. Hence by compactness we obtain a partition

$$t_i = u_0 < u_1 < \cdots < u_l = t_{i+1} \quad \text{with} \quad T(u_i, u_{i+1}) < m.$$

This can be repeated for all $J = [t_i, t_{i+1}]$ to obtain the desired partition $c_0, \ldots, c_{n+1}$.

Suppose that $0 < m \leqslant \pi$. Then on each $I = [c_i, c_{i+1}]$ we have $T(I) < \pi$ and therefore $T'(c) < \pi$ for c interior to I. This means that $W(c) = T^+(c) + T^-(c)$ satisfies $W(c) \neq 0$ at all interior points.

Let now T^+ and T^- be arbitrary unit vectors with $W = T^+ + T^- \neq 0$. Denote by $L = \{\alpha W; \alpha \text{ real}\}$ the line generated by W and by $L^\perp$ the orthogonal complement of L, $L^\perp = \{U \in H; \langle U, W \rangle = 0\}$. Suppose that $X = \|X\| V$ with V a unit vector satisfying $\|T^+ - V\| \leqslant \|W\|/4$, and write the decomposition $X = hW + Y$ (with h real, $Y \in L^\perp$) induced by $H = L \oplus L^\perp$.

Under these conditions we claim that

3.9*a*. $$\| Y \| \leqslant 2h.$$

In fact,

3.9*b*. $$\langle X, W \rangle = \langle hW, W \rangle = h \| W \|^2$$

and

$$\begin{aligned} \langle X, W \rangle &= \|X\| \langle V, W \rangle \\ &= \|X\| (\langle T^+, W \rangle - \langle T^+ - V, W \rangle) \\ &\geqslant \|X\| (\langle T^+, W \rangle - \| W \|^2/4). \end{aligned}$$

But

$$\langle T^+, W \rangle = 1 + \langle T^+, T^- \rangle = (1/2) \| T^+ + T^- \|^2 = (1/2) \| W \|$$

so that

3.9*c*. $$\langle X, W \rangle \geqslant \|X\| \; \| W \|^2/2.$$

Combining 3.9*b* and 3.9*c* we get $\|X\| \leqslant 2h$; then a fortiori $\| Y \|$ ⋮
claimed.

Fix now c interior to $J = [c_i, c_{i+1}]$ and apply this to the case T^+ =
$T^- = T^-(c)$, $X = \gamma(t) - \gamma(s)$, $V = C(s, t)$. Certainly the hy
$\| T^+ - V \| \leqslant \| W \|/4$ holds if $c \leqslant s < t \leqslant c + \epsilon$ and ϵ is small (by 2.⁚
write $\gamma(u) = h(u)W + Y(u)$, then $X = (h(t) - h(s))W + Y(t) - Y$
therefore from 3.9*a* we get, for $c \leqslant s < t \leqslant c + \epsilon$:

3.9*d*. $$\| Y(t) - Y(s) \| \leqslant 2(h(t) - h(s)).$$

In particular $h(s) \leqslant h(t)$. However $h(s) = h(t)$ implies $Y(s) = Y(t)$ fr
and then $\gamma(s) = \gamma(t)$, impossible. Thus $u \to h(u)$ is a strictly monoto
tion for $c \leqslant u \leqslant c + \epsilon$. This allows us to use $x = h(u)$ as a new variable
from $x = h(x)$ to $x_0 + \eta = h(c + \epsilon)$. Set $Z(x) = Y(u)$ when $x = h(u)$.
$x = h(s)$, $y = h(t)$ we have $\| Z(x) - Z(y) \| \leqslant 2 \| x - y \|$ and therefore
is a Lipschitz function from $[x_0, x_0 + \eta]$ into $L^{\perp}$. The equality

$$xW + Z(x) = h(u)W + Y(u) = \gamma(u)$$

shows that the curve $\gamma(t)$, $c \leqslant t \leqslant c + \epsilon$ is the graph of Z.

A similar reasoning yields the definition of Z on the interval $[x_0 - \eta$
glueing both halves together we conclude that the curve is the gra
Lipschitz function on an interval with $\gamma(c)$ corresponding to an interi

The special case when c is an end point of J is handled in the same way taking T^+ and T^- both equal to the one-sided tangent available.

Finally, a compactness argument yields a desired decomposition.

3.10 Corollary. *A curve with finite total curvature can be parametrized by $\gamma(t)$, $a \leqslant t \leqslant b$ in such a way that the right and left derivatives of γ exist at all t and neither of them vanishes. Further these derivatives coincide except at countable many values of t.*

4. Associated Functions

From this section on, all curves (unless specified) will be assumed to have finite total curvature.

4.1. Definition. *For $a \leqslant u < v \leqslant b$ set*

4.1*a*. $E(u, v) = \operatorname{ang}(C(u, v), T^+(u)) + \operatorname{ang}(C(u, v), T^-(v))$.

4.1*b*. $\Xi(u, v) = T(u, v) - E(u, v)$.

In order to study these functions we begin with a lemma about partitions.

4.2 Lemma. *For any partition $\Pi = \{u, u_1, \ldots, u_n, v\}$ of $[u, v]$ we have*

$$T(u, v) \geqslant \operatorname{ang}(T^+(u), C(u, u_1)) + T(\Pi) + \operatorname{ang}(T^-(v), C(u_n, v)).$$

PROOF. Pick $u < u' < u_1$ and $u_n < v' < v$. Then

$$T(u, v) \geqslant T\{u, u', u_1, u_2, \ldots, u_n, v', v\}.$$

But this last number is the sum of

$$x = \operatorname{ang}(C(u, u'), C(u', u_1)) + \operatorname{ang}(C(u_n, v'), C(v', v))$$

and $z = T(u', u_1, u_2, \ldots, u_n, v)$, so that taking limits as $u' \to u$ and $v' \to v$ we get that x and z approach, respectively,

$$\operatorname{ang}(T^+(u), C(u, u_1)) + \operatorname{ang}(C(u_n, v), T^-(v))$$

and $T(\Pi)$, which proves the lemma.

4.3. Proposition. *The function Ξ has the following properties:*

4.3*b*. $$\sum \Xi(u_j, v_j) \leqslant \Xi(u, v)$$

for any system of disjoint intervals (u_j, v_j) *contained in* (u, v). *In part is monotonic.*

4.3*c*. $\Xi'(c) = 0$ *for all* c.

PROOF. (*a*) Taking the trivial partition $\Pi = \{u, v\}$ we obtain from

$$T(u, v) \geqslant \operatorname{ang}(T^+(u), C(u, v) + \operatorname{ang}(T^-(v), C(u, v))$$

so that $T \geqslant E$ which means that $\Xi \geqslant 0$.

(*b*) It suffices to prove that

4.3*d*. $$\Xi(u, v) \geqslant \Xi(u, c) + \Xi(c, v), \quad \text{for} \quad u < c < v.$$

Denote

$U = T^+(u)$, $V = T^-(v)$, $C^- = T^-(c)$, $C^+ = T^+(c)$, $L = C(u, c)$, $R =$

and abbreviate $UL = \operatorname{ang}(U, L)$, $LC^- = \operatorname{ang}(L, C^-)$, etc.; by defini

$$\Xi(u, v) = T(u, v) - (WU + WV)$$
$$\Xi(u, c) = T(u, c) - (LU + LC^-)$$
$$\Xi(c, v) = T(c, v) - (RC^- - RV).$$

Using the addition formula 3.7 we get

$$\Xi(u, v) = \Xi(u, c) + \Xi(c, v) + \Delta$$

where

4.3*e*. $$\Delta = LU + LC' + RC^+ + RV + T'(c) - WU - WV$$

and therefore the desired inequality 4.3*d* is equivalent to $\Delta \geqslant 0$.

Observing that $T'(c) = C^-C^+$, from the triangle inequality 1.1 w

4.3*f* $$LC^- + RC^- + T'(c) \geqslant LR$$

and from the fact that $\gamma(u)$, $\gamma(c)$ and $\gamma(v)$ are the vertices of a triangle clude (see the figure 1 in the following page), that $LR = WL + WR$ from 4.3*e* and 4.3*f* we obtain (using again the triangle inequality):

$$\Delta \geqslant LU + RV + WL + WR - WU - WV'$$
$$= (LU + WL - WU) + (RV + WR - WV)$$

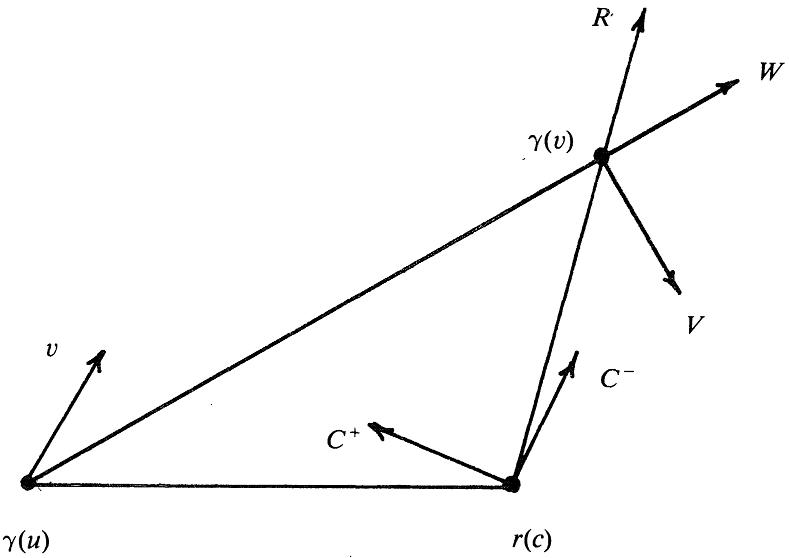

Figure 1

(*c*) First observe that if B is a limit of chords $C(c - \epsilon_n, c + \delta_n)$, then

$$B = pT^-(c) + qT^+(c)$$

for some $p, q \geqslant 0$. In fact, write

$$C(c - \epsilon_n, c + \delta_n) = p_n C(c - \epsilon_n, c) + q_n C(c, c + \delta_n)$$

with

$$p_n = \|\gamma(c) - \gamma(c - \epsilon_n)\| / \|\gamma(c + \delta_n) - \gamma(c - \epsilon_n)\|$$
$$q_n = \|\gamma(c + \delta_n) - \gamma(c)\| / \|\gamma(c + \delta_n) - \gamma(c - \epsilon_n)\| .$$

Taking limits we get $p = \lim p_n \geqslant 0$, $q = \lim q_n \geqslant 0$. This means that

$$BC^- + BC^+ = C^-C^+ = T'(c).$$

On the other hand, $T^+(c - \epsilon_n) \to C^-$ by 2.4*a*, whence

$$VC^- = \lim \operatorname{ang}(C(c - \epsilon_n, c + \delta_n), T^+(c - \epsilon_n));$$

similarly

$$VC^+ = \lim \operatorname{ang}(C(c - \epsilon_n, c + \delta_n), T^-(c + \delta_n)).$$

This of course gives lim ...

Another function associated to a curve is the *Gauss map* τ: $[a, b] \to S$ (where S is the sphere $\|X\| = 1$) defined by $\tau(t) = T^{+}(t)$, $a \leqslant t < b$, and $\tau(b) = T^{-}(b)$. In general, τ is discontinuous.

Recall that the length $l_S(\tau)$ is defined by

$$l_S(\tau) = l_S(a, b) = \operatorname{Sup} \sum \operatorname{ang}(\tau(x_i), \tau(x_{i+1})),$$

the supremum taken over all partitions.

Observe that we have

4.4. $$l_S(a, b) = l_S(a, c) + T'(c) + l_S(c, b),$$

for $a < c < b$, so that «arc length» is additive only if the partition point is a point of continuity of τ. For the same reason, the length of τ, considered as a map in H, will be equal to $l_S(\tau)$ only if τ is continuous everywhere.

The main property of τ is given in 4.7, which requires the following proposition.

4.5. Proposition. *Let f: $[a, b] \to L$ (L a Hilbert space) be a Lipschitz function with right (resp. left) derivative $f'_{+}(t)$ (resp. $f'_{-}(t)$) at all $a \leqslant t < b$ (resp. $a < t \leqslant b$), and let γ: $[a, b] \to R \oplus L$ be the graph of f,* i.e., $\gamma(t) = (t, f(t))$. *Suppose that f'_{+} is a function of bounded variation on $[a, b)$ with $l(f'_{+}) < 1$. Then*

4.5*a*. *γ has a finite total curvature*;

4.5*b*. $T(\gamma) = l_S(\tau)$.

PROOF. The hypotheses on f imply that γ is absolutely continuous with $\gamma'_{+} = (1, f'_{+})$ of bounded variation. Hence, for $a \leqslant u \leqslant v \leqslant b$ we have

$$\gamma(v) - \gamma(u) = \int_u^v \gamma'_{+}(t)\, dt = \int_u^v \tau(t) \|\gamma'_{+}(t)\|\, dt.$$

Now the integrand $\tau\|\gamma'_{+}\| = \gamma'_{+}$ is a function of bounded variation, hence Riemann integrable, and therefore $\gamma(v) - \gamma(u)$ is the limit of Riemann sums

$$R = \sum \tau(t_i) \|\gamma'_{+}(t_i)\|\, \Delta_i t$$

$(\Delta_i t = t_i - t_{i-1})$. On the other hand, $R = h \sum a_i \tau(t_i)$ with $a_i = \|\gamma'_{+}(t_i)\|\, \Delta_i t/h$ and $h = \sum \|\gamma'_{+}(t_j)\|\, \Delta_j t$. Taking limits we get that $(\gamma(v) - \gamma(u))/l(\gamma)$ is the limit of convex combinations of points of the form $\tau(t)$, $u \leqslant t \leqslant v$. In particular, $C(u, v)$ belongs to the closed convex cone spanned by $\{\tau(t); u \leqslant t \leqslant v\}$.

We need the following lemma (whose proof is given below):

4.6. Lemma. *Let $N \subset S$ be a subset with diameter strictly less than $\pi/2$ for the* ang *distance, and let M denote the intersection of S with the closed convex cone spanned by N. Pick U, V in N and denote by g the function defined on S by* $g(X) = \operatorname{ang}(U, X) + \operatorname{ang}(X, V)$. *Then* $\operatorname{Sup}_M g = \operatorname{Sup}_N g$.

Let now $\Pi = \{t_0, t_1, \ldots, t_m\}$, $t_0 = a$, $t_m = b$ be a partition of $[a, b]$ and let $\epsilon > 0$.

Observe that, for $a \leqslant s, t \leqslant b$,

$$\operatorname{ang}(\tau(t), \tau(s)) \leqslant (\pi/2)\|\tau(t) - \tau(s)\| \leqslant (\pi/2)\|\gamma'_+(t) - \gamma'_+(s)\| \leqslant (\pi/2)l(\gamma'_+) = (\pi/2)l(f'_+) < \pi/2$$

so that the set $\{\pi(t); a \leqslant t \leqslant b\}$ has diameter less than $\pi/2$ for the ang distance. Thus, the lemma applies with $U = \tau(t_i)$, $V = \tau(t_{i+1})$ and $N = \{\tau(t), t_i \leqslant t \leqslant t_{i+1}\}$. Clearly there is $u_i \in [t_i, t_{i+1}]$ with $\operatorname{Sup}_N g \leqslant g(\tau(u_i)) + \epsilon/m$, and therefore (by the lemma),

$$\operatorname{ang}(\tau(t_i), X) + \operatorname{ang}(X, \tau(t_{i+1})) \leqslant \operatorname{ang}(\tau(t_i), \tau(u_i)) + \operatorname{ang}(\tau(u_i), \tau(t_{i+1})) + \epsilon/m$$

for all X in M. In particular, as proved above, this inequality holds for $X = C(t_i, t_{i+1})$.

Abbreviating $C(t_i, t_j) = C_{ij}$, $\tau(t_i) = \tau_i$ and $\operatorname{ang}(X, Y) = XY$, we get

$$\begin{aligned} T(\Pi) &= C_{01}C_{12} + C_{12}C_{23} + \cdots \\ &\leqslant \tau_0 C_{01} + C_{01}\tau_1 + \tau_1 C_{12} + C_{12}\tau_2 + \tau_2 C_{23} + \cdots \\ &\leqslant \tau_0\tau(u_0) + \tau(u_0)\tau_1 + \tau_1\tau(u_1) + \tau(u_1)\tau_2 + \cdots + \epsilon \\ &\leqslant l_S(\tau) + \epsilon, \end{aligned}$$

whence $T(\gamma) \leqslant l_S(\tau)$ which proves 4.5*a*. The converse inequality was proved in 2.4*c*, so that $T(\gamma) = l_S(\tau)$, as claimed in 4.5*b*.

PROOF OF 4.6. It suffices to prove that if F is a spherical polygon with vertices $P_1, P_2, \ldots, P_m$ in N (each side is a maximum circle segment) with diameter of F strictly less than $\pi/2$ (for the ang distance), then

4.6*a*. $$g(X) \leqslant \max\{g(P_1), g(P_2), \ldots, g(P_m)\}.$$

for X inside F.

Observe that g is the sum of functions of the form $g_1(X) = \operatorname{ang}(X, U)$. It is not hard to see that 4.6*a* follows if we prove that g_1 is convex, i.e.,

4.6*b*. $$g_1(X(\lambda\eta + (1-\lambda)\xi)) \leqslant \lambda g_1(X(\eta)) + (1-\lambda)g_1(X(\xi))$$

where $X(\theta)$ runs on a maximum circle. We will assume that coordinates ar set on a three-dimensional subspace containing U, $X(\theta)$, so that

$$X(\theta) = (\cos\theta, \operatorname{sen}\theta, 0), \qquad U = (v, 0, w), \qquad 0 \leqslant v \leqslant 1 \qquad 0 \leqslant w \leqslant 1.$$

Then, letting $\alpha(\theta) = g_1(X(\theta))$, we have $0 \leqslant \alpha < \pi/2$ and $\cos\alpha(\theta) = v\cos\theta$ Differentiating,

$$\frac{d^2\alpha}{d\theta^2}(\theta) = \operatorname{cotan}\alpha(1 - v^2\operatorname{sen}^2\theta/\operatorname{sen}^2\alpha)$$

$$= \operatorname{cotan}\alpha(1 - v^2)/\operatorname{sen}^2\alpha \geqslant 0$$

and this implies 4.6*b*, which completes the proof of the lemma.

Observe that the same proof applies to the case of

$$g(X) = \operatorname{ang}(X, U_1) + \cdots + \operatorname{ang}(X, U_n),$$

and that the U_i need not belong to N as long as $N \cup \{U_1, \ldots, U_n\}$ ha diameter less than $\pi/2$.

4.7. Corollary. *For a curve to have finite total curvature it is necessary an sufficient that it can be parametrized as $\gamma(t)$, $a \leqslant t \leqslant b$, in such a way tha*

4.7*a*. *$\gamma(t)$ is a Lipschitz function;*

4.7*b*. *$\gamma'_+ = d^+\gamma/dt$ exists and satisfies $\|\gamma'_+(t)\| \geqslant 1$ for all t;*

4.7*c*. *γ'_+ is rectifiable.*

4.8. Corollary. *Any curve γ with finite total curvature satisfies $T(\gamma) = l_S(\tau$*

For the proofs, combine 2.4*c*, 3.7, 3.9, 3.10 , 4.4 and 4.6.

We close this section by indicating a measure-theoretic interpretation of 7 Recall that any non-decreasing function $g: [a, b] \to \mathbb{R}$ determines a positiv regular Borel measure μ by means of the Lebesgue-Stieltjes integral, whic satisfies $\mu[u, v] = g(v^+) - g(u^-)$, $\mu[u, v) = g(v^-) - g(u^-)$, etc. Takin $g(t) = T(a, t)$, it follows from 3.4*c*, 3.4*d* and 3.5 that g is non-decreasing an left-continuous so that:

4.9. Proposition. *Let $\gamma(t)$, $a \leqslant t \leqslant b$, be a curve with finite total curvature Then there is a unique positive real Borel measure μ on $[a, b]$ such that for an*

PROOF. Let $a < c < c + h < b$. Then from 3.5

$$T(a, c) + T'(x) + T(c, c + h) = T(a, c + h).$$

Letting $h \to 0^+$ and using 3.4c we obtain $T(a, c) + T'(x) = T(a, c^+)$ c $+ T'(c) = g(c^+)$. Since $g(c^-) = g(c)$ we get $T'(c) = g(c^+) - g(c^-)$ proves that $\mu(\{c\}) = T'(c)$. On the other hand, take $a < u < v < b$. T

$$\begin{aligned} T(u, v) = T(a, v) - T(a, u) - T'(u) &= g(v) - g(u) - g(u^+) + g(u) \\ &= g(v) - g(u^+) = \mu(u, v), \end{aligned}$$

as claimed. The case $a = u$ is similar.

We have no answer for the following question: (*a*) which measures μ a in this way? (*b*) what is the measure-theoretic interpretation of Ξ and

5. Total Curvature of Plane Curves

In this section we assume that H is the plane $\mathbb{R}^2$.

5.1. Theorem. *Let $\gamma(t)$, $a \leqslant t \leqslant b$ be a plane curve. Then the followin ditions are equivalent:*

5.1.1. *γ has finite total curvature.*

5.1.2. *γ is the union of finitely many graphs of real functions f wi properties:*

(*a*) *f has a right derivative f'_+ everywhere,*
(*b*) *f'_+ is a function of bounded variation.*

5.1.3. *γ is the union of finitely many graphs of functions which a ferences of Lipschitz convex functions.*

PROOF. The equivalence between 5.1.1 and 5.1.2 is a special case c (2) implies (3): A theorem-of-the-mean-like argument shows that und hypothesis of 5.1.2 f satisfies: for $x < y$ there exists ξ and η between x with

$$f'_+(\eta) \leqslant (f(y) - f(x))/(y - x) \leqslant f'_+(\xi).$$

But then, f'_+ being (of bounded variation, hence) bounded we concluc f is a Lipschitz function. Therefore

$$f(z) = f(a) + \int_a^s f'_+(t)\,dt.$$

Write now $f'_+ = h - g$ with $h \geqslant 0$, $g \geqslant 0$ and h, g non-decreasing. Then $f = H - G$ where G, H (the indefinite integrals of g and h) are convex Lipschitz functions.

(3) implies (1): If f is the difference of two Lipschitz convex functions then f'_+ exists at all points and, being the difference of two non-decreasing functions, it has bounded variation.

It is clear that in any Hilbert space $T \equiv 0$ characterizes straight lines. For plane curves the next result gives and interpretation of $\Xi \equiv 0$.

5.2. Proposition. *The plane curve $\gamma(t)$ with finite total curvature is convex on the interval $a \leqslant t \leqslant b$ if and only if $\Xi(a, b) = 0$.*

PROOF. Convex plane curves can be characterized by the following property: whenever $u < u' < v' < v$, the line segment $[\gamma(u), \gamma(v)]$ and $[\gamma(u'), \gamma(v')]$ do not cross each other, i.e., either they are disjoint or the first contains the second.

Suppose now that γ is not convex and that u, u', v and v' have been chosen so that the segments do cross (see the figure 2).

Denote also the following angles (not all drawn) by the indicated letters

$$\alpha = \text{ang}\,(T^+(u), C(u, v))$$
$$\alpha' = \text{ang}\,(T^+(u), C(u, u'))$$
$$\delta = \text{ang}\,(C(u, u'), C(u, v))$$

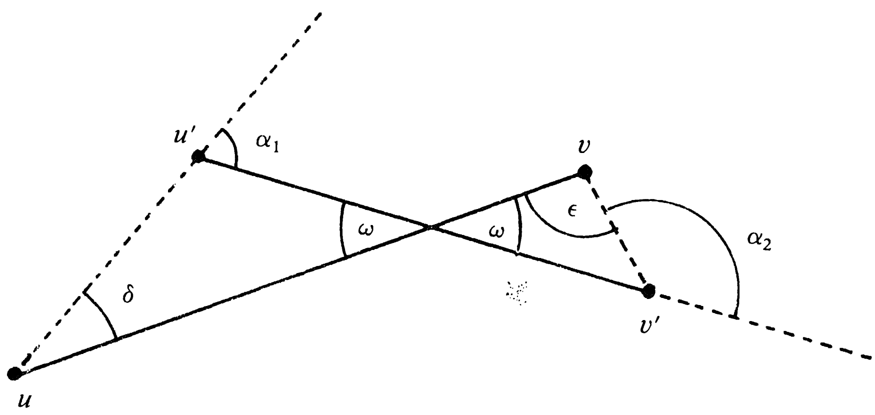

Figure 2

$$\beta = \text{ang}\,(T^-(v), C(u, v))$$
$$\beta' = \text{ang}\,(T^-(v), C(v', v))$$
$$\epsilon = \text{ang}\,(C(v', v), C(u, v)).$$

Using the angle triangle inequality 1.1.*a* we obtain $\alpha' \geqslant \alpha - \delta$ and $\beta' \geqslant \beta - \epsilon$. Now, according to 4.2

$$\begin{aligned} T(u, v) &\geqslant \alpha' + \alpha_1 + \alpha_2 + \beta' \\ &\geqslant \alpha - \delta + \alpha_1 + \alpha_2 + \beta - \epsilon = \alpha + \omega + \omega + \beta, \end{aligned}$$

so that $\Xi(u, v) \geqslant 2\omega > 0$. This shows that γ is convex when Ξ is zero.

The converse is easy since all polygonal lines inscribed in a convex curve satisfy $\alpha' + \beta' = \alpha_1 + \alpha_1 + \cdots + \alpha_n$,

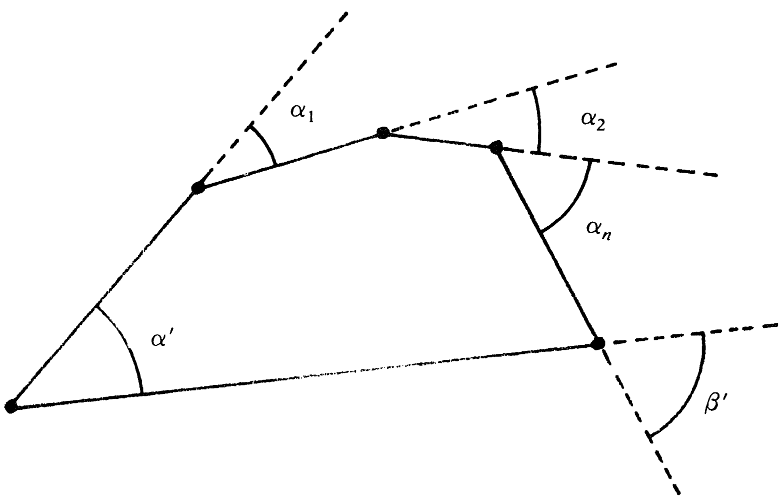

Figure 3

and taking limits we get $T = E$, or $\Xi = 0$.

In view of 5.1.3 and 5.2 it may be true that functions in general euclidean spaces whose graphs have finite total curvature can be written as differences of functions with $\Xi = 0$; we know no proof of this.

6. The non-Hilbert Case

Let X be a real Banach space, S the unit sphere $\|x\| = 1$ and δ the *geodesic distance* on S: $\delta(x, y) = \inf l_S(\sigma)$ where σ ranges over all continuous curves in

S joining x and y and l_S denotes the length of the curve σ,

$$l_S(\sigma) = \operatorname{Sup} \sum \|\sigma(t_{i+1}) - \sigma(t_i)\|.$$

It is well known that ([6]):

$$\|x - y\| \leqslant \delta(x, y) \leqslant 2\|x - y\|$$

for any pair $x, y \in S$.

Of course, in a Hilbert space $\delta(x, y) = \operatorname{ang}(x, y)$ and the inequalities are trivial consequences of 3.3. This suggests that we define, for a curv X and a partition $\Pi = \{t_0, t_1, \ldots\}$, the number $T(\Pi)$ by $T(\Pi) = \sum \delta(C_{i-}$ where $C_j = C(t_j, t_{j+1})$ and, as above,

$$C(u, v) = (\gamma(v) - \gamma(u))/\|\gamma(v) - \gamma(u)\| \in S.$$

In the same way, we set $T(\gamma) = \operatorname{Sup} T(\Pi)$, and define curves of *finite cur* by the property $T(\gamma) < +\infty$. With these definitions, all the results in Sec hold true without changing their proofs.

For the results in Section 3 the situation is different. In fact, the are on 3.1, 3.2 and 3.3. Now 3.3 is valid in any Banach space with $\pi/2$ re by 2, as observed above, which does not affect the use it is made of 3. on. Also, 3.2 is a corollary of 3.1. Thus, only 3.1 has to be checked f validity of all results in Section 3.

It turns out however that 3.1 holds for some Banach spaces and do hold for others (see below), so it may be thought of as too restrictive is not the case if the monotonicity of T with respect to partitions (3.4*a*) i sidered a natural condition. In fact we have:

6.1. Proposition. *Let X be a Banach space. Then the following pro are equivalent*:

6.1.1. *For any curve γ in X, $T(\Pi)$ increases when more partition poin added to* Π.

6.1.2. *The equality $\delta(U, V) + \delta(V, W) = \delta(U, W)$ holds for any U, V S with $U = -W$ or $V = pU + qW$ for some $p, q \geqslant 0$.*

6.1.3. *For any U, V in with $U + V \neq 0$, the function*

$$\sigma(t) = ((1 - t)U + tV)/\|(1 - t)U + tV\|$$

defines a curve with minimal length joining U and V, i.e., $l_S(\sigma) = \delta($

PROOF. First let us see that 6.1.1 is equivalent to

6.1.4. *For any U, W, V, R and Z in S satisfying $V = pR + qZ$ for son* $p, q \geqslant 0$, *the inequality*

6.1.5. $$\delta(U, V) + \delta(V, W) \leqslant \delta(U, R) + \delta(R, Z) + \delta(Z, W)$$

holds.

Assume 6.1.1 and consider a curve γ satisfying $\gamma(t_0) = 0$, $\gamma(t_1) = U$, $\gamma($ $= U + dpR$, $\gamma(t_3) = U + dV$, $\gamma(t_4) = U + dV + eW$ for a partition $\Pi = \{$ $t_1, t_2, t_3, t_4\}$ of its domain, where $0 < d \leqslant 1$ and $0 < e \leqslant 1$ are convenient chosen to avoid selfintersections (we are also assuming that $V \neq R$ and $V \neq$ since in either case 6.1.5 follows from the triangle inequality). If $\Pi_1 = \{t_0,$ $t_3, t_4\}$ then $T(\Pi_1)$ and $T(\Pi)$ are equal, respectively, to the left and right ha side terms of 6.1.5.

Conversely, if 6.1.4 holds it is easy to see that for any curve and any pa of partitions Π_1 and Π with Π having one more point than Π_1 we ha $T(\Pi_1) \leqslant T(\Pi)$. An induction argument finishes the proof.

Setting $R = U$ and $Z = W$ in 6.1.5 and using the triangle inequality we g 6.1.2; conversely, from 6.1.2 we obtain $\delta(R, Z) = \delta(R, V) + \delta(V, Z)$ a using twice the triangle inequality we get 6.1.5. This shows that 6.1.2 a 6.1.4 are equivalent.

We prove that 6.1.3 implies 6.1.2: let

$$\sigma(t) = ((1 - t)U + tW)/\|(1 - t)U + tW\|$$

and denote by σ_1 and σ_2 the restrictions of σ that join U to V and V to respectively. We have, using 6.1.3,

$$\delta(U, W) = l_S(\sigma) = l_S(\sigma_1) + l_S(\sigma_2) = \delta(U, V) + \delta(V, W).$$

To prove the converse, pick a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such th (with σ as above):

$$l_S(\sigma) - \epsilon \leqslant \sum \|\sigma(t_{i+1}) - \sigma(t_i)\|.$$

From 6.1.2,

$$\sum \|\sigma(t_{i+1}) - \sigma(t_i)\| \leqslant \sum \delta(\sigma(t_i), \sigma(t_{i+1})) = \delta(\sigma(0), \sigma(1))$$
$$= \delta(U, V)$$

so that, ϵ being arbitrary, $l_S(\sigma) \leqslant \delta(U, V)$ whence $l_S(\sigma) = \delta(U, V)$ and 6.1 follows. Thus 6.1.1, 6.1.3 and 6.1.4 are all equivalent.

6.2. Theorem. *All properties of T stated in Sections 2 through 5 are va in Banach spaces having the equivalent properties of* 6.1.

6.3. *Remark.* Hilbert spaces and two-dimensional Banach spaces have tl equivalent properties of 6.1. It is possible that these are the only ones, but v know no proof of this fact. In support of this observe that such spaces ha\ the following property, not hard to obtain from 6.1.3: all curves of the for: $S \cap V$, where V is a two-dimensional subspace of X, have the *same* length. S(also [9] for related notions. Finally we observe that $\mathbb{R}^3$ with the nor: $(x^2 + y^2)^{1/2} + |z|$ is a Banach space where 6.1.1 (and then also 6.1.2 and 6.1.. fails.

7. Related Concept

Let γ be a plane curve given in polar coordinates by $\gamma(\theta) = r(\theta)(\cos\theta, \sin 6$ $0 \leqslant \theta \leqslant \alpha < 2\pi$, where $r(\theta) > 0$ is a continuous function.

7.1. Proposition. *Consider the following properties*:

(*a*) *There exists $s > 0$ such that for each $0 \leqslant \theta \leqslant \alpha$ and each point z in t plane satisfying $\|z\| \leqslant s$, the line segment joining z to $\gamma(\theta)$ meets t curve only at $\gamma(\theta)$ («interior cone condition»).*
(*a'*) *$r(\theta)$ is a Lipschitz function.*
(*b*) *$r(\theta)$ is a function of bounded variation.*
(*b'*) *γ is a rectifiable curve.*
(*c*) *γ has finite total curvature.*

Then:

7.1.1. (*a*) *and* (*a'*) *are equivalent.*

7.1.2. (*b*) *and* (*b'*) *are equivalent.*

7.1.3. (*a*) *implies* (*b*) *and* (*c*) *implies* (*b*).

7.1.4. *All other implications fail in general.*

PROOF. (*a'*) $\Rightarrow$ (*a*) is proved in [6], 7.1, and (*a*) $\Rightarrow$ (*a'*) is proved in [11]; tl settles 7.1.1.

Assume now that γ is rectifiable. Then $r(\theta)\cos\theta$ and $r(\theta)\sin\theta$ are functio of bounded variation, which is equivalent to $r = (r^2\cos^2\theta + r^2 \operatorname{sen}^2\theta)^{1/2}$ bei: of bounded variation (use $r \geqslant \min r > 0$ and the differentiability of squa root away from 0). The converse is just as easy, so that 7.1.2 is proved.

Next (*a'*) $\Rightarrow$ (*b*) and (*c*) $\Rightarrow$ (*b'*) (see 3.9) so that 7.1.3 follows.

We finish the proof with three examples: first, let γ_1 be the curve whos graph is the cusp $|x|^{1/2} + y^{1/2} = 1$, $-1 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Next, let γ_2 be th curve described by the figure 4

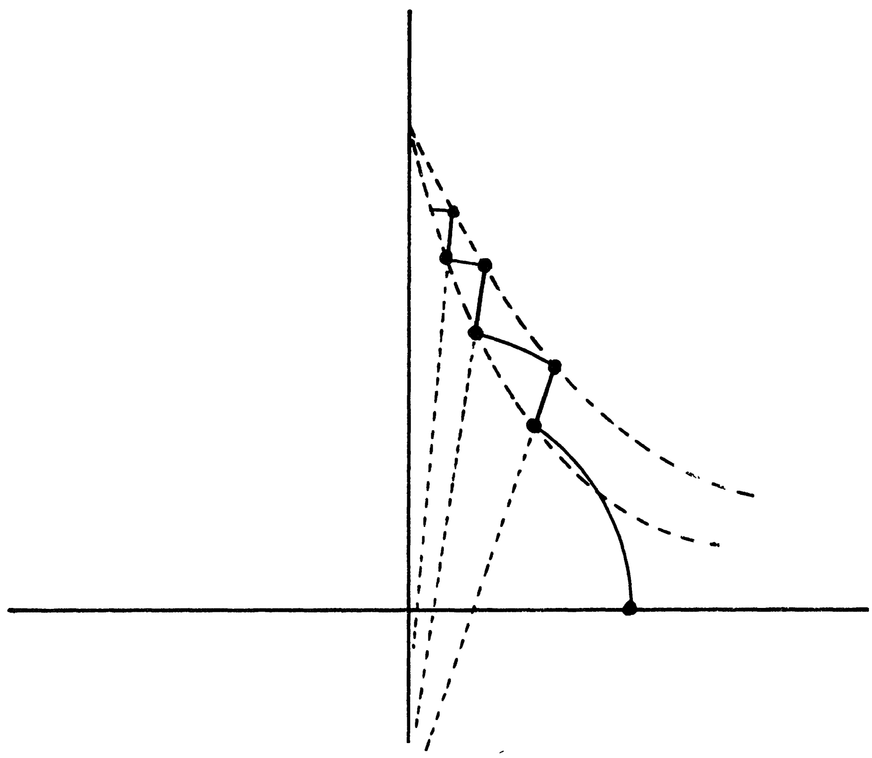

Figura 4

Here r increases from $\theta = 0$ to $\theta = \pi/2$. Finally, let $P_1, P_2, P_3, \ldots$ be sequence on $x^2 + y^2 = 1$ converging orderly to $(0, 1)$, and let γ_3 be the curv obtained by joining P_j to P_{j+1} with the broken line formed by tangents to th circle $x^2 + y^2 = 1/4$ (see the figure 5 in the next page).

It is not hard to see that (*b*) holds for γ_1, but (*a'*) and (*c*) fail; (*c*) holds fc γ_2 but (*a*) fails, and (*a*) holds for γ_3 but (*c*) fails. This completes the proof c 7.1.

Remark. Observe that 7.1 improves the statement (*a*) + (*c*) $\Rightarrow$ (*a'*) $\Rightarrow$ ((proved in [6].

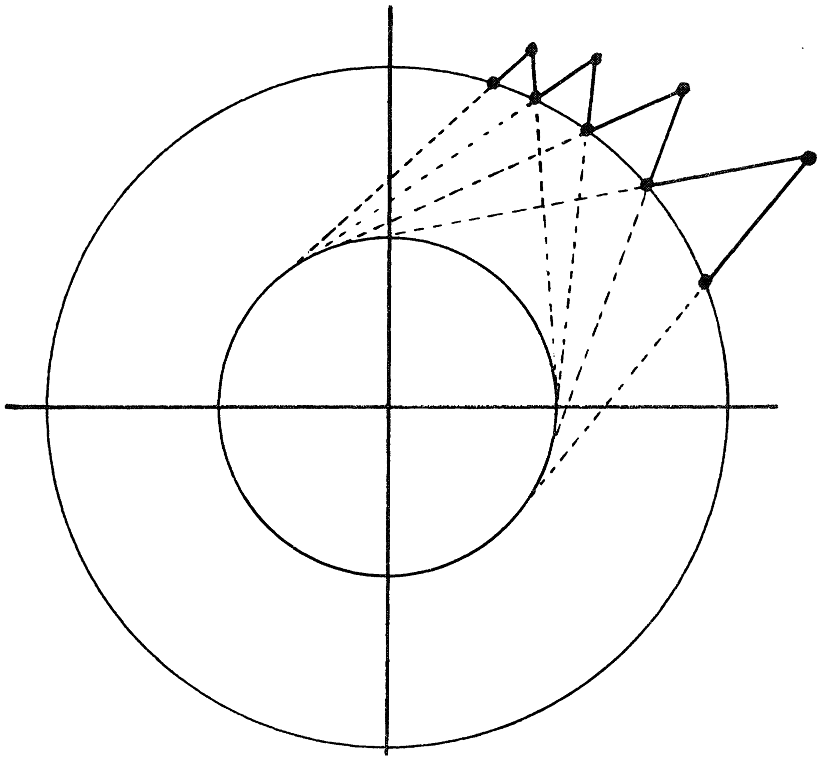

Figure 5

References

[0] Bandle, Catherine Isoperimetric inequalities and applications, Pitman, 198(
[1] Berg, I. David An estimate on the total curvature of a geodesic in Euclide 3-space-with-boundary, *Geometriae Dedicata.* **13**(1982), 1-6.
[2] Borsuk, Karol Sur la courbure totale des courbes fermées, *Ann. Soc. P* *Math.* **20**(1948), 251-265.
[3] Fáry, István Sur la courbure totale d'une courbe faisant un noeud, *Bull. S* *Math. France* **77**(1949), 128-138.
[4] Gleason, Andrew A curvature formula, *Amer. J. Math.,* **101**(1979), 86-93.
[5] Goffman, Casper Real Functions, Prindle, Webber & Schmidt, 1953.
[6] McGowan, John and Porta, Horacio On representations of distance functio in the plane. *Seminar on Functional Analysis, Holomorphy and Approximati theory*, Rio de Janeiro, North Holland Publ., 1983.
[7] Milnor, John, W. On the total curvature of knots, *Ann. of Math.* **52**(195 248-257.
[8] Rund, Hanno The differential geometric of Finsler spaces, Springer, 1959.
[9] Schäffer, Juan J. The geometry of spheres in normed spaces, Marcel Dekk 1976.

[10] Spivak, Michael Differential geometry, Publish or Perish, 1970.
[11] Toranzos, Fausto A. Radial functions of convex and star-shaped bodies, *Math. Monthly*, **74**(1967), 278-280.
[12] Weiner, Joel An inequality involving the length, curvature and torsions of a curve in euclidean n-space, *Pacific J. Math.*, **74**(1978), 531-534.

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Hankel Forms and the Fock Space

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Abstract

We consider Hankel forms on the Hilbert space of analytic functions square integrable with respect to a given measure on a domain in $\mathbb{C}^n$. Under rather restrictive hypotheses, essentially implying «homogeneity» of the set-up, we obtain necessary and sufficient conditions for boundedness, compactness and belonging to Schatten classes S_p, $p \geqslant 1$, for Hankel forms (analogues of the theorems of Nehari, Hartman and Peller). There are several conceivable notions of «symbol»; choosing the appropriate one, these conditions are expressed in terms of the symbol of the form belonging to certain weighted L^p-spaces.

Our theory applies in particular to the Fock spaces (defined by a Gaussian measure in $\mathbb{C}^n$). For the corresponding L^p-spaces we obtain also a lot of other results: interpolation (pointwise, abstract), approximation, decomposition etc. We also briefly treat Bergman spaces.

A specific feature of our theory is that it is «gauge invariant». (A gauge transformation is the simultaneous replacement of functions f by $f\phi$ and $d\mu$ by $|\phi|^{-2}\,d\mu$, where ϕ is a given (non-vanishing) function). For instance, in the Fock case, an interesting alternative interpretation of the results is obtained if we pass to the measure $\exp(-y^2)\,dx\,dy$. In this context we introduce some new function spaces E_p, which are Fourier, and even Mehler invariant.

0. Introduction

0.1. Background. By a *Hankel form* we will in this paper informally refer to any (continuous) bilinear form H defined on a Hilbert space $\mathcal{H}$ of analytic functions (usually consisting of (all) functions square integrable with respect to a given measure μ; cf. *infra* §0.2) such that its value $H(f, g)$ for any $f, g \in \mathcal{H}$ only depends on the product $f \cdot g$. In particular, one then has the functional equation

$$H(\phi f, g) = H(f, \phi g)$$

where ϕ is any (analytic) multiplier on $\mathcal{H}$.

Example 0.1. In the case of the usual Hardy class $\mathcal{H} = H^2(\mathbb{T})$ ($\mathbb{T}$ = unit circle) the Hankel for $H_b^{\mathbb{T}}$ with symbol b is defined by

$$H_b^{\mathbb{T}}(f, g) = \frac{1}{2\pi} \int_{\mathbb{T}} \bar{b} f g |dz| = \langle fg, b \rangle_{H^2(\mathbb{T})}$$

In the canonical basis $\{z^j\}_{j \geq 0}$ it is given by the *Hankel matrix* $(\hat{\bar{b}}(i + j))_{i,j \geq 0}$.

For the (classical) theory of Hankel forms in this case, highlighted by a number of agenda such as the issue of

finite rank (Kronecker)
boundedness (Nehari)
compactness (Hartman)
belonging to Schatten-von Neumann class (Peller),

we refer to, Sarason (1978), Power (1980, 1982*a*), Nikol'skiĭ (1985, 1986), Nikol'skiĭ and Peller (1987?).

Usually, though, one formulates the results for *operators*, not forms. With the form H one can associate the Hankel operator $\tilde{H}$ defined by

$$H(f, g) = \langle f, \tilde{H} g \rangle. \tag{0.1}$$

Notice that $\tilde{H}$ is an anti-linear operator in $\mathcal{H}$. To get a linear operator one combines $\tilde{H}$ with a conjugation; e.g. on $\mathbb{T}$ one usually considers $f \mapsto \overline{\tilde{H} f}$ with the range $\overline{H^2(\mathbb{T})}$, or a variant with range $H^2_-(\mathbb{T})$.

For various reasons we prefer to work with bilinear forms instead. For instance, this «zwanglos» suggests the extension of our theory to the *multilinear* case (§5).

An easy extension of the $H^2(\mathbb{T})$-theory concerns the space $B_s^2(D)$ ($s < 0$; D unit disc, $\partial D = \mathbb{T}$) defined by the condition

$$\frac{1}{\pi} \int_D |f(z)|^2 (1 - |z|^2)^a \, dm(z) < \infty \qquad (a = -1 - 2s > -1),$$

where the letter B may at will be read as Bergman on Besov (see Peller (1982), Peetre (1983, 1984, 1985) and, for an extension to the case of the unit ball in several complex variables, Ahlmann (1984) and Burbea (1986)); one also writes $A^{a,2}(D)$ for the same spaces. Actually, already on this level the theory bifurcates according to (speaking of (linear) operators) whether one wants the range to be $\overline{B_s^2(D)}$ or $B_s^2(D)^\perp$. Here we will only be concerned with the first alternative. (The study of Hankel operators of the second species —which do *not* correspond to Hankel forms in our sense —was initiated only recently by Axler (1985) and then further pursued in Arazy, Fisher and Peetre (1986)).

See also the works of Luecking (1985) and Zhu (1985) for *Toeplitz* operators in Bergman space. (Some remarks in the case of general (homogeneous) domains are further made in Arazy and Upmeier (1985)).

As a formal limiting case ($a \to -1$) of the spaces $B_s^2(D)$ one recaptures the previous Hardy class $H^2(\mathbb{T})$ (the normalized 2-dimensional measure $(a+1) \cdot (1-|z|^2)^a \, dm(z)/\pi$ over D tends to the 1-dimensional measure $|dz|/2\pi$ concentrated on $\mathbb{T}$).

Another limiting case ($a \to \infty$) deals with the Fock space $F_\alpha^2(\mathbb{C})$ ($\alpha > 0$) defined by the condition

$$\frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} \, dm(z) < \infty.$$

(If one writes the definition for the B-spaces for a concentric disc of radius R then the weight factor becomes $[1-|z|^2/R^2]^a$. If now $a = \alpha R^2$ and $R \to \infty$ we formally get the weight $e^{-\alpha|z|^2}$).

The number α plays a role similar to Planck's constant in physics.

Remark 0.1. Besides Fock, other names occasionally are attached to this spaces, viz. Bargmann-Segal, Fisher and possibly others. The same is true for Bergman spaces (see e.g. Dzhrbashyan (1983)), so perhaps a more appropriate appelation, without digging too deeply into the history of the subject, would have been spaces of Bargmann-Besov-Bergman-Dzhrbashyan-Fisher-Fock-Segal type.

Toeplitz operators in Fock space are considered in Berger and Coburn (1985), (1986?) and Berger, Coburn and Zhu (1985).

0.2. Main Results (General Theory, §§1-6, 14). The aim of the present work is to develop a theory of Hankel forms over quite general (in practise «homogeneous») domains, which comprises both the Bergman and the Fock case (the other limiting case of the Hardy class being *excluded*) and this in any number of dimensions (a few results for the Fock space being formally valid also in the physically most interesting case of dimension ∞, see §7). As there is in general no boundary (and no Besov spaces) one has to proceed differently then before. Note that *potentially* our theory is applicable to a much broader range (including arbitrary symmetric domains and various limiting cases).

More precisely, we consider the following set-up. Let Ω be a domain in $\mathbb{C}^n$ and as in the beginning of §0.1, let $\mathcal{H}$ be a Hilbert space of analytic functions now defined on Ω. If ξ is a positive measure on Ω, which we, for simplicity, assume to be absolutely continuous with respect to the Lebesgue measure m on Ω, we say that a Hankel form H defined on $\mathcal{H}$ has *symbol b with respect to ξ* if (with a convenient interpretation of the integral, if the latter is not absolutely convergent; cf. §6)

$$H(f, g) = \int_\Omega \bar{b} f g \, d\xi \qquad (f, g \in \mathcal{H}),$$

notation:

$$H = H_b^\xi.$$

The point is that a form may have several (interesting) symbols with respect to different measures and to some extent our theory is about the interplay between various symbols.

In most of the discussion we *fix* once and for all one such measure (fulfilling the assumption V0 stated in Section 1) and take $\mathcal{H} = A^2(\mu)$, the subspace of $L^2(\mu)$ consisting of all square integrable (with respect to μ), analytic functions on Ω. Clearly $A^2(\mu)$ is a Hilbert space with a reproducing kernel denoted by $K(z, w)$ or $K_w(z)$. We let P denote the orthogonal («Bergman») projection of $L^2(\mu)$ onto $A^2(\mu)$ and we further set

$$\omega(z) = \frac{1}{K(z, z)}.$$

It then turns out to be advantageous to take symbols not with respect to μ, but with the *associated* measure ν defined as

$$d\nu = \omega(z)\, d\mu = \frac{d\mu}{K(z, z)}.$$

We will in the sequel use the notations

$$\Gamma_b = H_b^\nu, \qquad H_b = H_b^\mu.$$

We further let L and Q denote the reproducing kernel in the Hilbert space $A^2(\nu)$ and the corresponding projection, respectively.

Remark 0.2. For a general measure we similarly have the Hilbert space $A^2(\xi)$ with a reproducing kernel K^ξ and the projection P^ξ. We will use these concepts only for $\xi = \mu$ or ν, where μ and ν are as above. We summarize the special notations used for these cases in the form of a table. (The notation A^2_ω will be explained in (0.3) below).

The Hankel forms always act in $A^2(\mu)$.

Measure	*Hilbert space*	*Kernel*	*Projection*	*Hankel form*
ξ (general)	$A^2(\xi)$	K^ξ	P^ξ	H_b^ξ
μ (fixed)	$A^2(\mu)$	K	P	H_b
ν (associated)	A_ω^2	L	Q	Γ_b

We occasionally write $\Gamma(b)$ for Γ_b, etc.

To get a reasonable theory one has to introduce some supplementary assumptions V1, V2 and V3 (see §3). The most severe of these is V2 which amonts to requiring that

$$L(z, w) = \varkappa K(z, w)^2 \tag{0.2}$$

where $\varkappa$ is a constant.

Before stating our main result (*infra*) we need one more concept, the natural scale of *weighted* L^p and A^p-spaces pertinent to our situation. We say that f is in L_ω^p iff

$$\int_\Omega |f|^p \omega^{p-2}\, d\nu < \infty \tag{0.3}$$

(f is in L_ω^∞ if and only if ωf is essentially bounded on Ω), and let A_ω^p be the subspace of L_ω^p consisting of analytic functions in L_ω^p. Let a_ω^∞ be the closure of A_ω^2 in the A_ω^∞-metric.

We can now announce:

Scholium 0.1. *Under the assumptions* V0 – V3 *the following is true.*

(*a*) Γ_b *is bounded* (*in* $A^2(\mu)$) *if and only if* $Qb \in A_\omega^\infty$.
(*b*) Γ_b *is compact if and only if* $Qb \in a_\omega^\infty$.
(*c*) Γ_b *is in* S_p, *where* $1 < p < \infty$, *if and only if* $Qb \in A_\omega^p$.

The Schatten-von Neumann classes of bilinear forms S_p (where in general $0 < p \leqslant \infty$) are discussed in Sub-Section 0.3. Some other comments are in order.

Comment 0.1. From this it is in principle easy to get results for general symbols, because H_b^ξ has the symbol $b\, d\xi/d\nu$ with respect to ν, $H_b^\xi = \Gamma_{b d\xi/d\nu}$. This is discussed in §6. Notice also that $H_b^\xi = H_{P^\xi b}^\xi$, so that in many cases it is natural to confine oneself to analytic symbols.

Comment 0.2. We expect part (*c*) of the Scholium to be true also in the range $0 < p < 1$ but this we have not been able to show.

Comment 0.3. The proofs can be found in §4, where also other results can be found, especially pertaining to the «Hankel projection». A crucial step is however taken already in §3, where the boundedness of the projection Q in L^p_ω, $1 \leqslant p \leqslant \infty$, is proved.

Our assumptions, in particular the crucial hypothesis V2, are fulfilled in all cases when the situation admits sufficiently many «automorphisms». This can in principle be found in the literature, but of course not in the Hankel context. We refer especially to Selberg (1957), Stoll (1977) and Inoue (1982). In particular, our theory applies in the B-case (the group is the Möbius group PSU (1,1)), see §§12-13, and in the F-case (the group is now its «contraction», the Heisenberg group).

We do not know of any other cases than homogeneous domains with highly symmetrical measures when the assumption V2 *is fulfilled.*

However, there is a deeper reason for the appearance of the strange looking hypothesis as condition V2 relating the square of the kernel K to the kernel L: Namely, that the whole set-up admits certain «supersymmetries», here termed *gauge transformations*. Let us briefly indicate what this is about.

Consider, quite generally, a closed subspace $\mathcal{H}$ of $L^2(\Omega, \mu)$, where Ω is some space equipped with a positive measure μ. We argue that we get an essentially equivalent theory if we simultaneously replace f by ϕf and μ by $|\phi|^{-2}\mu$, where ϕ is any non-vanishing (measurable) function. This is gauge transformation or change of gauge. The point is that one should work only with gauge invariant quantities. (A related point of view can be found e.g. in the works of Berezin (see e.g. Berezin (1975) for a start), but also elsewhere). Especially in our case (confining ourselves to analytic ϕ's), the («given») kernel K transforms according to the rule $K(z, w) \to \phi(z)\overline{\phi(w)}K(z, w)$, where as the («associated») kernel L experiences the change $L(z, w) \to \phi(z)^2\overline{\phi(w)^2}L(z, w)$ (see §§1 and 3). Thus V2 is a gauge invariant condition. Similarly, our preference for the Hankel operator Γ_b with symbol taken with respect to the measure ν (and not μ, as would seem natural at the first glance) is explained by the fact that Γ_b is gauge invariant (with the symbol transforming $b \to \phi^2 b$).

Note also that the measure λ defined as

$$d\lambda(z) = K(z, z)\, d\mu(z)$$

has a gauge invariant meaning; in all group theoretic cases it reduces to the usual invariant measure, in the very special case of the unit disc thus to a constant multiple of the Poincaré measure $(1 - |z|^2)^{-2}\, dm(z)$.

Finally, let us mention that we also prove a very general Kronecker theorem (concerning the structure of finite rank Hankel forms). This is basically an excercise in commutative algebra (sic!) and has little to do with the rest of the paper so it has been relegated to the end of the paper, more or less as an appendix (§14).

0.3. Schatten-von Neumann classes of bilinear forms. The Schatten-von Neumann classes (or trace ideals) S_p, $0 < p < \infty$, of (bounded) operators in Hilbert space have been studied extensively (see e.g. McCarthy (1967), Gohberg and Krein (1965), Simon (1979) and, as far as interpolation goes, Bergh and Löfström (1976)). To define the same classes for bilinear forms there are several (equivalent) avenues.

(*a*) Via operators (cf. Peetre (1985)). If H is a bilinear form on $\mathcal{H}_1 \times \mathcal{H}_2$, then $\tilde{H}$ defined by

$$\tilde{H}(g): f \to H(f, g) \tag{0.4}$$

is a linear operator from $\mathcal{H}_2$ into $\mathcal{H}_1^*$. (The natural, anti-linear, identification of $\mathcal{H}_1^*$ and $\mathcal{H}_1$ yields the anti-linear operator from $\mathcal{H}_2$ into $\mathcal{H}_1$ defined by (0.1)). We say that H is in S_p if and only if $\tilde{H}$ is in S_p, i.e. if and only if the positive operator $(\tilde{H}^*\tilde{H})^{p/2}$ has finite trace. We define S_∞ to be the space of all bounded bilinear forms (operators). (Some authors prefer to let S_∞ denote the compact operators).

One can also associate with H a linear operator $\tilde{\tilde{H}}$: $\mathcal{H}_2 \to \mathcal{H}_1$ doing the same job, but not in a canonical way. Indeed, if J: $\mathcal{H}_1 \to \mathcal{H}_1$ is any conjugation on $\mathcal{H}_1$ (J is antilinear with $J^2 = Id$), then J defines a *linear* isometry of $\mathcal{H}_1^*$ onto $\mathcal{H}_1$ (which we also denote by J) and we can take $\tilde{\tilde{H}} = J \circ \tilde{H}$. Notice in particular that $\tilde{\tilde{H}}^*\tilde{\tilde{H}} = \tilde{H}^*\tilde{H}$ independently of J. (If $\{h_{ij}\}$ is the matrix of H with respect to some orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$, then this operator has the matrix $\{b_{ik}\}$ with $b_{ik} = \sum h_{ji}\overline{h_{jk}}$).

Remark 0.3. For some spaces there is a natural choice of J, e.g. if $\mathcal{H}_1 = \bar{\mathcal{H}}_1$ (say $\mathcal{H}_1 = L^2(\mu)$), $Jf = \bar{f}$ and if $\mathcal{H}_1$ is a suitable Hilbert space of analytic functions in the unit disc (or the complex plane) $Jf(z) = \overline{f(\bar{z})}$.

(*b*) Directly using *s*-numbers (Schmidt, approximation). Put

$$s_n(H) = \inf \|H - F\|_\infty \tag{0.5}$$

where $\|\cdot\|_\infty$ is the supremum norm and F runs through the set of all forms of finite rank $\leqslant n$. We say that H is in S_p if and only if $(s_n(H))_{n=0}^\infty \in l_p$, $0 < p \leqslant \infty$. (Note that H is compact if and only if $(s_n(H))_{n=0}^\infty \in c_0$).

0.4. Hankel forms of class S_2 (Hilbert-Schmidt). To give the reader at least a feel what it all is about we now briefly outline a direct treatment of the S_2 theory.

The Hankel form $\Gamma(L_z)$ with symbol L_z with respect to ν is

$$(f, g) \to \int \bar{L}_z fg \, d\nu = fg(z) = f(z)g(z) = \langle f, K_z \rangle \langle g, K_z \rangle \tag{0.6}$$

(This is a continuous form of rank 1, and belongs thus to every S_p). Thus

$$\langle \Gamma(L_z), \Gamma(L_w) \rangle_{S_2} = \langle K_w, K_z \rangle \langle K_w, K_z \rangle = K(z, w)^2.$$

If now

$$L(z, w) = \varkappa K(z, w)^2, \qquad \varkappa > 0,$$

it follows that

$$\varkappa \langle \Gamma(L_z), \Gamma(L_w) \rangle = L(z, w) = \langle L_w, L_z \rangle, \tag{0.7}$$

whence $b \to \varkappa^{1/2}\Gamma_b$ is an anti-linear isometry of $A^2(\nu)$ into S_2. Conversely, if $b \to \varkappa^{1/2}\Gamma_b$ is an isometry of $A^2(\nu)$ into S_2 for some $\varkappa$, then the argument above shows that

$$L(z, w) = \langle L_w, L_z \rangle = \varkappa K(z, w)^2.$$

This is closely related to the criterion by Aronszajn (1950), Theorem 8 II, p. 361, for $L = K^2$.

0.5. Contents. Results for Fock and Bergman spaces (§§7-13). Again for the benefit of the reader we pass to a more detailed description of the contents of the individual divisions, including an explicit mention of the main results in the Fock and Bergman cases.

Section 1 sets forth some basic material connected with Hilbert spaces with a reproducing kernel (for a more detailed treatment we refer to Aronszajn (1950)).

In the analytic case we also state the basic assumption V0 (p. 74).

In Section 2 we study the reproducing kernels when there are sufficiently many automorphisms. The main result is Theorem 2.1 proving the aforementioned condition V2 in such cases.

In Section 3 we introduce the assumptions V1-V4 (pp. 80-81) and we study the «Bergman» projection Q, especially establishing its boundedness in the full scale L^p_ω, $1 \leqslant p \leqslant \infty$ (Theorem 3.1). This result has a number of important corollaries (Cor. 3.1-3.8).

Section 4 is devoted to the study of Hankel forms in the general context of the assumption V0-V3 and we establish in particular all the results which above were summarized in Scholium 0.1.

In §5 the extension to the multilinear case is briefly treated. As far as we know, no theory is yet developed for S_p-classes of multilinear forms. Here we propose to define S_p, $1 < p < \infty$, using interpolation between S_1 and S_∞; in the two latter cases the definition is unambiguous.

§6 gives various complements to the previous discussion (§§1-5). In particular we discuss a more general definition of symbols (hitherto the defining integral was taken to be absolutely convergent) and consider also symbols with respect to a general measure ξ.

We also establish the minimality of A^1_ω in a certain sense, and prove a weak factorization result for A^1_ω.

In §7 we begin the study of Fock space proper. It then turns out to be natural to study the whole family of measures

$$d\mu_\alpha = (\alpha/\pi)^n e^{-\alpha|z|^2}\, dm(z) \qquad (\alpha > 0),$$

on $\mathbb{C}^n$, letting L^p_α $(1 \leqslant p \leqslant \infty)$ to the space of measurable functions f such that $f(z)e^{-\alpha|z|^2/2} \in L^p(m)$ and F^p_α be its analytic subspace, denoting the correspondingly projection by P_α. More precisely, we consider the action of Hankel forms on some fixed Hilbert space F^2_α but take symbols with respect to an arbitrary measure $d\mu_\beta$. The main result is Theorem 7.5 (= an almost immediate convergence of the results in §§1-6 in the «abstract» case; cf. Scholium 0.2 *infra*).

We turn also the reader's attention to Theorem 7.8, which gives an exact result (not just a norm equivalence), and thus is potentially susceptible to an extension to infinitely many variables. This is however only for the special powers $p = 2$ and $p = 4$ and why this is so is a tantalizing question we do not quite understand.

In §8 we go on studying the spaces F^p_α and especially establish decomposition approximation and interpolation (pointwise, not abstract interpolation!) descriptions.

We interrupt at this junctune the exposition by the collecting the results for Hankel forms on the Fock space F^2_α as a Scholium (for those who like many equivalent conditions). For simplicity we state them in terms of the symbol taken with respect to the measure $d\mu_{2\alpha}$ which corresponds to the associated measure $d\nu$ in the general case (§4), when $d\mu = d\mu_\alpha$. We thus consider the Hankel form $H^{2\alpha}_b$ given by

$$H^{2\alpha}_b(f, g) = \int_{\mathbb{C}^n} \bar{b} f g \, d\mu_{2\alpha}$$

acting in the Hilbert space F^2_α. Let k^α_w be the normalized reproducing kernel in F^2_α, viz.

$$k^\alpha_w(z) = e^{\alpha\langle z, w\rangle - \alpha|w|^2/2},$$

and use the notation $k^{2\alpha}_w$ in the same sense.

Scholium 0.2. *The following are equivalent for* $1 \leqslant p \leqslant \infty$ *and any entire function* b.

(i) $H^{2\alpha}_b \in S_p$.

(ii) $H_b^{2\alpha}(f,g) = \sum \lambda_i \langle f, k_i^\alpha \rangle \langle g, k_i^\alpha \rangle$ *where* $\{\lambda_i\} \in l^p$ *(and* $k_i^\alpha = k_{w_i}^\alpha$ *for a suitable sequence* $\{w_i\}$, *separation condition etc.).*

(iii) $b = \sum \lambda_i k_i^{2\alpha}$ *with* $\{\lambda_i\} \in l^p$ *(same qualifications for* $k_i^{2\alpha}$*).*

(iv) $b \in P_{2\alpha}(L_{2\alpha}^p)$.

(v) $\{d_N\}_0^\infty \in l^p$ *where* $d_N = \inf\{\|b-g\|_{F_{2\alpha}^\infty} : g \in P_N\}$, P_N *being the set of linear combinations* $\sum_{j=1}^N a_j k_j^{2\alpha}$ *of length N.*

(vi) $b \in F_{2\alpha}^p$.

In §9 we first investigate for which values of the parameters involved the projection P_α is bounded as a map from L_β^p into F_γ^p. (Answer: The n and s condition is $\alpha^2/\gamma > 2\alpha - \beta$). This improves on an old result of Sjögren's (1976), who was interested when P_α maps $L^p(\mu_\alpha)$ into $L^q(\mu_\alpha)$. (Answer: $q < 4/p$, or $p = q = 2$). It is also connected with a duality result (Theorem 9.2):

$$(F_\beta^p)^* \cong F_{\alpha^2/\beta}^{p'},$$

in the duality induced by the inner product in F_α^2. We also study the (complex) interpolation of the 2-parameter family F_α^p (Theorem 9.3). It is somewhat surprising but at second thought quite understandable that the parameter α interpolates «logarithmically» (Theorem 9.4):

$$[F_{\alpha_0}^{p_0}, F_{\alpha_1}^{p_1}]_\theta = F_\alpha^p \quad \text{if} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \alpha = \alpha_0^{1-\theta}\alpha_1^\theta.$$

It is an interesting (*open*) question to determine the spaces which arise by *real* interpolation from this scale. This can for *p fixed* be rephrased as a problem about spectral analysis for the dilation operator $D_\delta : f(z) \to f(\delta z)$ $(0 < \delta < 1)$ in the space F_α^p.

Example 0.2. Let

$$f(z) = \sum_{n=0}^\infty \frac{(az)^n}{n!(1+t\delta^n)}$$

where a is complex, $t > 0$ and δ fixed, $0 < \delta < 1$. Is it true that

$$\iint_{\mathbf{C}} |f(z)| e^{-|z|^2/2}\, dm(z) = 0(e^{|a|^2/2}),$$

with a constant *independent* of t? If this were the case we could prove that D_δ is a «positive» operator so the usual Grisvard type machinery can be set at work (see e.g. Triebel (1977), Section 1.14).

§10 is likewise devoted to Fock space and treats various left-overs from the previous sections.

In §11 we treat Fock space in a different gauge (from the group representation point of view this is something half way in between the Bargman-Segal representatition thriving on $F^2_\alpha(\mathbb{C}^n)$ and the Heisenberg representation acting in $L^2(\mathbb{R}^n)$). In this connection we are led to introduce some new function (distribution) spaces E_p, whose definition formally reminds of the use of Besov spaces, only that the convolution parameter enters in an additive way ($f \in E_p \Leftrightarrow \phi_y * f(\cdot) \in L^p(L^p)$ where $\hat{\phi}_y(\xi) = \hat{\phi}(\xi + y)$ and ϕ is a «test» function) and have the conspicuous property of being *Fourier*, and even *Mehler invariant*. Indeed, it turns out that they are special cases of more general spaces known as *modulation spaces* and studied by Feichtinger (see, e.g. Feichtinger (1981*a*), (1981*b*) and the discussion in remark 11.3).

The following two sections (§§12 and 13) are devoted to B space theory. In §12 we spell out our results in the case of weighted Bergman spaces on the complex unit ball (the «Rudin» ball). In §13 again we make changes of variables and gauge and consider the case of the upper half plane, but only for $n = 1$. (This is really a pity, for the case $n > 1$ when one thus has a Siegel domain of the second kind (a generalized upper half plane) should be susceptible to potentially interesting considerations. Cf. Gindikin (1964)).

Finally, as already recorded at the end of §0.2, we give in §14 our general Kronecker result.

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Note. (added Jan. 1988). In two loose appendices (written in the spring of 1987), for which the middle author alone is responsible, we indicate some further developments after the main body of the paper was completed (June 1986).

1. Reproducing Kernels

In this section we collect some elementary, presumably well-known results which will be used later. We begin with a very general setting, see Aronszajn (1950).

Let $\mathcal{H}$ be a Hilbert space of functions on some set Ω such that the point evaluations $f \to f(z)$ are continuous linear functionals on $\mathcal{H}$ for all $z \in \Omega$. Then there exist unique functions $K_z \in \mathcal{H}$, $z \in \Omega$, such that

$$f(z) = \langle f, K_z \rangle, \qquad f \in \mathcal{H} \quad \text{and} \quad z \in \Omega, \tag{1.1}$$

and we define the reproducing kernel as the function

$$K(z, w) = K_w(z), \qquad (z, w) \in \Omega^2. \tag{1.2}$$

The definitions (1.1) and (1.2) yield (for $z, w \in \Omega$)

$$K(z, w) = K_w(z) = \langle K_w, K_z \rangle. \tag{1.3}$$

Consequently,

$$K(w, z) = \overline{K(z, w)} \tag{1.4}$$

$$K(z, z) = \|K_z\|^2 \geqslant 0 \tag{1.5}$$

$$|K(z, w)|^2 \leqslant K(z, z)K(w, w) \tag{1.6}$$

$$|f(z)| \leqslant \|f\| \, \|K_z\| = K(z, z)^{1/2} \|f\|. \tag{1.7}$$

Furthermore,

$$K(z, z) = 0 \Leftrightarrow K_z = 0 \Leftrightarrow f(z) = 0 \quad \text{for every} \quad f \in \mathcal{H}. \tag{1.8}$$

If $\{\phi_\alpha\}$ is an ON-basis in $\mathcal{H}$, then

$$\begin{aligned} K(z, w) = K_w(z) &= \sum_\alpha \langle K_w, \phi_\alpha \rangle \phi_\alpha(z) \\ &= \sum_\alpha \overline{\langle \phi_\alpha, K_w \rangle} \phi_\alpha(z) \\ &= \sum_\alpha \phi_\alpha(z) \overline{\phi_\alpha(w)}. \end{aligned}$$

(The sums converge absolutely).

Finally we note that the linear span of $\{K_z\}$ is dense in $\mathcal{H}$, because no non-zero function is orthogonal to every K_z.

We next impose additional structures on Ω.

Continuity. If Ω is a topological space and every function in $\mathcal{H}$ is continous, then $K(w, z)$ is separately continuous (because of $K_z \in \mathcal{H}$ and (1.4)), but not necessarily continuous. (Counterexamples are easily constructed, but we leave that to the reader).

Proposition 1.1. *If every function in $\mathcal{H}$ is continuous, then the following are equivalent.*

(i) $(z, w) \to K(z, w)$ *is continuous*;
(ii) $z \to K(z, z)$ *is continuous*;
(iii) $z \to K_z$ *is continuous* (*mapping* Ω *into* $\mathcal{H}$).

PROOF. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Fix z. If $w \to z$ then $K(w, w) \to K(z, z)$ by (ii) and $K(w, z) = K_z(w) \to K_z(z) = K(z, z)$ because $K_z \in \mathcal{H}$.

Hence, using (1.3),

$$\begin{aligned} \|K_w - K_z\|^2 &= \langle K_w, K_w \rangle + \langle K_z, K_z \rangle - 2\,\mathrm{Re}\,\langle K_z, K_w \rangle \\ &= K(w, w) + K(z, z) - 2\,\mathrm{Re}\,K(w, z) \\ &\to 0. \end{aligned} \tag{1.10}$$

(iii) $\Rightarrow$ (i). Immediate by (1.3). □

L^2-spaces. In the remainder of this section we assume that μ is a measure on Ω and that $\mathcal{H}$ is a closed subspace of $L^2(\mu)$ such that the point evaluations are continuous on $\mathcal{H}$. (Note that the functions in $\mathcal{H}$ thus are defined everywhere although functions in $L^2(\mu)$ are defined only a.e.).

Let P denote the orthogonal projection $L^2(\mu) \to \mathcal{H}$. Then, if $f \in L^2(\mu)$ and $z \in \Omega$, by $K_z \in \mathcal{H}$, (1.2) and (1.4),

$$\begin{aligned} Pf(z) = \langle Pf, K_z \rangle = \langle f, K_z \rangle = \int_\Omega f(w)\overline{K_z(w)}\, d\mu(w) = \\ = \int_\Omega K(z, w) f(w)\, d\mu(w). \end{aligned} \tag{1.11}$$

Change of gauge. Let ϕ be a non-zero measurable function on Ω and consider the map

$$f \to \phi f, \qquad \mu \to |\phi|^{-2}\mu, \tag{1.12}$$

which maps $L^2(\mu)$ isometrically onto $\phi L^2(\mu) = L^2(|\phi|^{-2}\mu)$ and $\mathcal{H}$ onto the subspace $\phi\mathcal{H} = \{f : \phi^{-1} f \in \mathcal{H}\}$ of $L^2(|\phi|^{-2}\mu)$.

This map, which we call a change of gauge, obviously gives an isomorphic theory. It will later be important to see how various entities transform.

Proposition 1.2. *The reproducing kernel for $\phi\mathcal{H}$ is $\phi(z)\overline{\phi(w)}K(z, w)$.*

PROOF. E.g. by (1.9), since $\{\phi\phi_\alpha\}$ is an ON-basis in $\phi\mathcal{H}$. □

Corollary 1.1. *The measure $K(z, z)\, d\mu(z)$ is invariant under all changes of gauge.*

PROOF. A change of gauge transforms $d\mu(z) \to |\phi(z)|^{-2}\, d\mu(z)$ by definition. □

Change of variables. Let Ψ be a bijection of Ω onto Ω'. Then Ψ maps μ onto $\mu \circ \Psi^{-1}$, and the map $f \to f \circ \Psi^{-1}$ maps $\mathcal{H}$ isometrically onto

$$\mathcal{H} \circ \Psi^{-1} \subset L^2(\mu) \circ \Psi^{-1} = L^2(\mu \circ \Psi^{-1}).$$

Proposition 1.3. *The reproducing kernel for* $\mathcal{H} \circ \Psi^{-1}$ *is* $K(\Psi^{-1}(z), \Psi^{-1}(w))$, $z, w \in \Omega'$. □

This triviality will be useful in conjunction with a simultaneous change of gauge in the next section.

Analytic functions. In the remainder of the paper (except in §14), we make the following assumptions, for future references denoted V0.

V0: *Ω is a connected open subset of $\mathbb{C}^n$ and μ is an absolutely continuous measure on Ω with continuous, strictly positive Radon-Nikodym derivative $d\mu/dm$ (m is the Lebesgue measure).*

Our basic Hilbert space is the space $A^2(\mu) = \mathcal{H}(\Omega) \cap L^2(\mu)$, i.e. the space of square integrable analytic functions. ($\mathcal{H}(\Omega)$ is the Frechet space of all analytic functions in Ω. It is easily seen that $A^2(\mu)$ is a closed subspace of $L^2(\mu)$ and that point evaluations are continuous; in fact, the embedding $A^2(\mu) \to \mathcal{H}(\Omega)$ is continuous). We let K denote the reproducing kernel in $A^2(\mu)$; all previous considerations of this section apply. (In the special case $\mu = m$, K is known as the Bergman kernel (in Ω)).

We will henceforth only consider analytic changes of gauge and analytic changes of variables, and note that they preserve our setting; e.g. if ϕ is analytic and non-zero, then $\phi A^2(\mu) = A^2(|\phi|^{-2}\mu)$.

Proposition 1.4. *$K(z, w)$ is continuous on $\Omega \times \Omega$, analytic in z and anti-analytic in w.*

PROOF. $K(z, w) = K_w(z)$ is analytic in z because $K_w \in A^2(\mu)$. By (1.4), $K(z, w)$ then is anti-analytic in w. Hence $K(z, \bar{w})$ is analytic in each variable on $\Omega \times \bar{\Omega}$ and thus, by Hartogs' theorem, analytic, in particular continuous. □

Corollary 1.2. *Proposition* 1.1 *yields that $z \to K_z$ is a continuous map of Ω into $A^2(\mu)$.* □

We next prove that K is determined by its restriction to the diagonal and the properties above.

Proposition 1.5. *Suppose that $J(z, w)$ is analytic in z and anti-analytic in w on $\Omega \times \Omega$ and that $J(z, z) = K(z, z)$, $z \in \Omega$. Then $J(z, w) = K(z, w)$.*

PROOF. We may assume that $0 \in \Omega$. The function $f(z, w) = J(z, \bar{w}) - K(z, \bar{w})$ is analytic, and $f(z, \bar{z}) = 0$ in a neighborhood of 0. Hence $f = 0$, see e.g. Bochner and Martin (1948), Chapter II, Theorem 7. □

2. Symmetries

Let $\Omega \subset \mathbb{C}^n$ and μ be as in V0 (see above). In §§3-6, we will impose further restrictions on Ω and μ. These restrictions seem very restrictive and we guess that the theory developed there only covers very special cases. The purposes of the present section is to show that at least highly symmetric cases, such as the Fock and Bergman spaces, are covered. Our main results extend results by Selberg (1957), Stoll (1977) and Inoue (1982). A general reference to the theory of automorphism groups is Narasimhan (1971), Chapters 5 and 9.

Let Aut (Ω) denote the group of analytic bijections of Ω onto itself. This group is too large for our purposes, while the subgroup of maps that leave μ invariant is too small (and has the further defect of not being gauge invariant). Instead, we study the subgroup of maps that leave μ invariant modulo an analytic change of gauge.

Definitions. $G(\mu)$ is the set of all $\gamma \in \mathrm{Aut}(\Omega)$ such that, for some analytic function ϕ on Ω,

$$\mu \circ \gamma^{-1} = |\phi|^2 \mu. \tag{2.1}$$

(Cf. (1.12). Since necessarily $\phi \neq 0$, we may here replace ϕ^{-1} by ϕ).

$$G^*(\mu) = \{(\gamma, \phi) \in \mathrm{Aut}(\Omega) \times \mathcal{H}(\Omega) \colon \mu \circ \gamma^{-1} = |\phi|^2 \mu\}.$$

$G^*(\mu)$ is a group with the natural group law

$$(\gamma, \phi) \circ (\delta, \psi) = (\gamma \circ \delta, \phi \cdot (\psi \circ \gamma^{-1}));$$

$G(\mu)$ is a subgroup of Aut (Ω) and a quotient group of $G^*(\mu)$.

Remark 2.1. ϕ is determined by (2.1) up to a unimodular constant. Hence $G^*(\mu)$ is an extension of $G(\mu)$ by $\mathbb{T}$. A unitary representation of $G^*(\mu)$ in $A^2(\mu)$ is defined by

$$R_{(\gamma, \phi)} f(z) = \phi(z) f(\gamma^{-1}(z)). \tag{2.2}$$

Remark 2.2. For the Fock spaces (§§7-11), $G(\mu)$ is strictly smaller than Aut (Ω), while $G(\mu) = \mathrm{Aut}(\Omega)$ for the Bergman spaces (§§12-13), and for any domain Ω when μ is the Lebesgue measure (let ϕ in (2.1) be the Jacobian of γ^{-1}).

Proposition 2.1. *If* $(\gamma, \phi) \in G^*(\mu)$, *then*

$$K(\gamma^{-1}(z), \gamma^{-1}(w)) = \phi(z)^{-1} \overline{\phi(w)}^{-1} K(z, w), \qquad z, w \in \Omega. \tag{2.3}$$

Proof. Immediate by propositions 1.2 and 1.3, since the change of gauge induced by ϕ^{-1} and the change of variables induced by γ map $A^2(\mu)$ onto the same space, and thus they transform K into the same kernel. □

Corollary 2.1. *The measure $K(z, z)\, d\mu(z)$ is invariant for all $\gamma \in G(\mu)$.* □

Corollary 2.2. $|K(z, w)|^2/K(z, z)K(w, w)$ *is a $G(\mu)$-invariant function of $(z, w) \in \Omega \times \Omega$.* □

Transitivity. We say that $G(\mu)$ is transitive if for every $z, w \in \Omega$ there exists $\gamma \in G(\mu)$ with $\gamma(z) = w$.

Lemma 2.1. *If $G(\mu)$ is transitive and $A^2(\mu) \neq \{0\}$, then $K(z, z) \neq 0$ for all $z \in \Omega$.*

Proof. Otherwise, by Proposition 2.1, $K(z, z) = 0$ for every $z \in \Omega$, which contradicts (1.7). □

Theorem 2.1. *Suppose that $G(\mu)$ is transitive and $A^2(\mu) \neq \{0\}$. Let r be an integer and let ν be the measure $K(z, z)^{-r}\, d\mu(z)$. Denote the reproducing kernel for $A^2(\nu)$ by L. Then, for some constant $c_r \geqslant 0$,*

$$L(z, w) = c_r K(z, w)^{r+1}. \tag{2.4}$$

In other words, if f is analytic and $f \in L^2(K(z, z)^{-r}\, d\mu)$,

$$f(z) = c_r \int_\Omega f(w) \frac{K(z, w)^{r+1}}{K(w, w)^r}\, d\mu(w). \tag{2.5}$$

Furthermore, $G(\nu) \supset G(\mu)$.

Proof. Let $\gamma \in G(\mu)$ and choose ϕ such that (2.1) holds. Then, using (2.3)

$$\frac{d\nu \circ \gamma^{-1}}{d\nu}(z) = \frac{K(\gamma^{-1}(z), \gamma^{-1}(z))^{-r}}{K(z, z)^{-r}} \frac{d\mu \circ \gamma^{-1}}{d\mu}(z)$$
$$= |\phi|^{2r} |\phi|^2 = |\phi^{r+1}|^2.$$

Since ϕ^{r+1} is analytic, γ belongs to $G(\nu)$. Corollary 2.1 now shows that γ preserves the measures $K(z, z)\, d\mu(z)$ and $L(z, z)\, d\nu(z)$ and thus the Radon-Nikodym derivative

$$\frac{L(z, z)\, d\nu(z)}{K(z, z)\, d\mu(z)} = L(z, z) K(z, z)^{-r-1}.$$

Hence this function is left invariant by every $\gamma \in G(\mu)$, and, since $G(\mu)$ is transitive, it has to be a constant, c_r say, i.e.

$$L(z, z) = c_r K(z, z)^{r+1}.$$

The proof is completed by Proposition 1.5. □

Remark 2.3. If Ω is simply connected, the theorem holds for any real number r. In particular, $K(z, w)^r$ is well-defined unless $c_r = 0$, i.e. unless $A^2(\nu) = \{0\}$. (The Lu Qi-keng conjecture states that $K(z, w) \neq 0$ for any simply connected domain (with the Lebesgue measure), cf. Lu Qi-keng (1966), Skwarczynski (1969)).†

Remark 2.4. A similar argument shows that the group of μ-invariant automorphisms (i.e. those with $\phi \equiv 1$ in (2.1)) is transitive only in trivial cases. ($K(z, z)$ has to be constant, whence $K(z, w)$ is constant and $A^2(\mu) = \{0\}$ or $\mathbb{C}$. We do not know whether $A^2(\mu) = \mathbb{C}$ actually is possible). Note also the related fact (valid without any assumptions on $G(\mu)$) that $A^2(K(z, z)\, d\mu) = 0$ or $\mathbb{C}$, the latter case occuring if and only if $A^2(\mu)$ has finite dimension. (Sketch of proof. It follows from (1.9) that if $f \in A^2(K(z, z)\, d\mu)$, then $M_f: g \to fg$ defines a Hilbert-Schmidt operator in $A^2(\mu)$. Thus, the spectrum of M_f is discrete which implies that f is constant).

Isotropy. Define, for $z \in \Omega$, $G(\mu)_z = \{\gamma \in G(\mu): \gamma(z) = z\}$. In this subsection we assume that $G(\mu)_z$ is large enough, more precisely:

There exists a compact group H with $H \subset G(\mu)_z$ such that $(\gamma, z) \to \gamma(z)$ is continuous $H \times \Omega \to \Omega$ and that the only H-invariant analytic functions on Ω are the constant functions. (2.6)

We let $d\gamma$ denote the normalized Haar measure on H.

Lemma 2.2. *Assume that* $z \in \Omega$ *is such that* (2.6) *holds. Then, for any* $f \in \mathcal{H}(\Omega)$ *and* $w \in \Omega$,

$$\int_H f(\gamma(w))\, d\gamma = f(z).$$

PROOF. The integral defines an analytic H-invariant function of w, and is thus independent of w. Choosing $w = z$ we obtain

† *Added May* 1987. After the above was written we have been told that the Lu-Gineng conjecture has been settled by Harold Boas.

$$\int f(\gamma(z))\, d\gamma = f(z). \qquad \square$$

The next lemma may be compared to the Lu Qi-keng conjecture in Remark 2.3.

Lemma 2.3. *Suppose that* $z \in \Omega$ *and that* $K(z,z) \neq 0$ *and that* (2.6) *holds. Then* $K(z,w) \neq 0$ *for all* $w \in \Omega$.

PROOF. Suppose on the contrary that $K(w,z) = 0$ for some w. By proposition 2.1,

$$K_z(\gamma(w)) = K(\gamma(w), z) = K(\gamma(w), \gamma(z)) = 0 \quad \text{for all} \quad \gamma \in H \subset \Gamma(\mu)_z.$$

Lemma 2.2 with $f = K_z$ yields

$$K(z,z) = K_z(z) = \int K_z(\gamma(w))\, d\gamma = 0,$$

a contradiction. $\square$

We may now extend the reproducing formula to functions outside $A^2(\mu)$.

Theorem 2.2. *Suppose that* $z \in \Omega$ *is such that* (2.6) *holds and* $K(z,z) \neq 0$. *If* f *is an analytic function such that*

$$\int_\Omega |K(z,w) f(w)|\, d\mu(w) < \infty,$$

then

$$\int_\Omega K(z,w) f(w)\, d\mu(w) = f(z).$$

PROOF. Corollary 2.2 implies that $|K(z,w)|^2/K(w,w)$ is a $G(\mu)_z$-invariant function of w. This and Corollary 2.1 imply that $|K(z,w)|^2\, d\mu(w)$ is a $G(\mu)_z$-invariant measure. Consequently, if $g = f/K_z$ (which is analytic by Lemma 2.3), then for any $\gamma \in H \subset G(\mu)_z$,

$$\int K(z,w) f(w)\, d\mu(w) = \int g(w) K(w,z) K(z,w)\, d\mu(w) = \int g(\gamma(w)) |K(z,w)|^2\, d\mu(w).$$

Integrating over H, we obtain by Fubini's theorem, Lemma 2.2 and (1.4)-(1.5),

$$\begin{aligned}
\int_\Omega K(z,w) f(w)\, d\mu(w) &= \int_H \int_\Omega g(\gamma(w)) |K(z,w)|^2\, d\mu(w)\, d\gamma \\
&= \int_\Omega \int_H g(\gamma(w))\, d\gamma |K(z,w)|^2\, d\mu(w) \\
&= \int g(z) |K_z(w)|^2\, d\mu(w) \\
&= g(z) \|K_z\|^2 \\
&= g(z) K(z,z) \\
&= f(z). \quad \square
\end{aligned}$$

Proper actions and invariant measures. We say that a topological group $G \subset \mathrm{Aut}(\Omega)$ acts properly on Ω if the action $\gamma(z)$ is continuous $G \times \Omega \to \Omega$ and the map $G \times \Omega \to \Omega \times \Omega$, $(\gamma, z) \to (\gamma(z), z)$ is proper. If G acts properly, then its topology coincides with the compact-open topology. $\mathrm{Aut}(\Omega)$ with the compact-open topology is a topological group, but is does not always act properly.

A related question concerns the existence of G-invariant metrics (defining the usual topology) on Ω. In fact, if such a metric exists and G is a closed subgroup of $\mathrm{Aut}(\Omega)$, then G acts properly, see van Dantzig and van der Waerden (1928) and Kaup (1967).

Now assume that

$$K(z, z) \neq 0 \quad \text{for every} \quad z \in \Omega.$$

Then the Bergman (pseudo)metric (with respect to μ) is defined as the Riemannian (pseudo) metric with the infinitesimal from

$$ds^2 = \sum_{ij} \frac{\partial^2 \log K(z, z)}{\partial z_i \partial \bar{z}_i} dz_i \overline{dz_j}, \tag{2.7}$$

cf. Bergman (1950), Chapter IX.3.

The form (2.7) is positive semidefinite (the proof of Kobayashi (1959), Theorem 3.1, holds verbatim in our situation too) and is positive definite if and only if

$$\{\operatorname{grad} f(z) : f \in A^2(\mu) \text{ and } f(z) = 0\} = \mathbb{C}^n. \tag{2.8}$$

For example, if $\int (1 + |z|^2)\, d\mu < \infty$, then all affine functions belong to $A^2(\mu)$, whence (2.8) is satisfied for every $z \in \Omega$ and the Bergman metric is a metric.

Furthermore the form (2.7) is invariant under (analytic) changes of coordinates and changes of gauge; hence (2.7) and the Bergman metric are $G(\mu)$-invariant.

We are now prepared to show that (1.7) can be improved to

$$f(z) = o(K(z, z)^{1/2})$$

in some cases, cf. Kobayashi (1959), Section 9.

Theorem 2.3. *Assume that $G(\mu)$ is transitive and that* (2.8) *holds for some* (*and thus all*) $z \in \Omega$. *Then, for every $f \in A^2(\mu)$,*

$$f(z)/K(z, z)^{1/2} \in C_0(\Omega). \tag{2.9}$$

PROOF. It suffices to prove (2.9) when $f = K_w$, $w \in \Omega$, because of (1.7) and the fact that these functions span a dense subspace of $A^2(\mu)$. Assume thus, in order to achieve a contradiction, that $w \in \Omega$ and that (2.9) fails for $f = K_w$. Then there exists a sequence $\{z_n\} \subset \Omega$ that is not relatively compact such that $\inf_n K_w(z_n)/K(z_n, z_n)^{1/2} > 0$. Thus, for some $\delta > 0$ and every n,

$$|K(z_n, w)| > 2\delta K(w, w)^{1/2} K(z_n, z_n)^{1/2}. \tag{2.10}$$

Let $k_z(w) = K_z(w)/\|K_z\| = K(w, z)/K(z, z)^{1/2}$. Then $\|k_z\| = 1$ and

$$\langle k_z, k_w \rangle = \frac{\langle K_z, K_w \rangle}{\|K_z\| \, \|K_w\|} = \frac{K(w, z)}{(K(z, z)K(w, w))^{1/2}}.$$

Let $A = \{z \in \Omega: \|k_z - k_w\| < \delta\}$. Since $z \to k_z$ is continuous, A is open. Choose $\gamma_n \in G(\mu)$ such that $\gamma_n w = z_n$. Then, if $z \in A$, using Corollary 2.2,

$$|\langle k_w, k_{\gamma_n^{-1} z} \rangle| = |\langle k_{\gamma_n w}, k_z \rangle| \geqslant |\langle k_{z_n}, k_w \rangle| - \|k_z - k_w\| > 2\delta - \delta = \delta.$$

Consequently, by Corollary 2.1, for every n,

$$\int_{\gamma_n^{-1}A} |k_w(z)|^2 \, d\mu(z) = \int_{\gamma_n^{-1}A} |\langle k_w, k_z \rangle|^2 K(z, z) \, d\mu(z) > \int_A \delta^2 K(z, z) \, d\mu(z) > 0.$$

Now, let B be a compact subset of Ω such that

$$\int_{\Omega \setminus B} |k_w|^2 \, d\mu < \delta^2 \int_A K(z, z) \, d\mu(z).$$

Then $B \cap \gamma_n^{-1} A \neq \varnothing$ for every n. Since the Bergman metric is $G(\mu)$-invariant, and $G(\mu)$ is a closed subgroup of $\operatorname{Aut}(\Omega)$, $G(\mu)$ acts properly by the result referred to above. Hence $\{\gamma: \gamma A \cap B \neq 0\}$ is compact, whence $\{\gamma_n\}$ and $\{z_n\} = \{\gamma_n w\}$ are relatively compact, a contradiction. □

Remark 2.5. The assumptions of Theorem 2.3 imply also that $G(\mu)$ is a real Lie group, and Ω thus a homogenous space, cf. e.g. Kobayashi (1959).

3. The Bergman Projection

We assume that Ω and μ satisfy the basic condition V0 in Section 1 and furthermore:

V1: If $z \in \Omega$ then $f(z) \neq 0$ for some $f \in A^2(\mu)$.

Equivalently, $K(z, z) > 0$ for $z \in \Omega$.

We introduce, as in §0.2, additional notations and assumptions which will be used in the remainder of the paper (except Section 14).

Definitions. λ and ν are the measures given by

$$d\lambda(z) = K(z, z)\, d\mu(z), \tag{3.1}$$

$$d\nu(z) = K(z, z)^{-1}\, d\mu(z). \tag{3.2}$$

$L(z, w)$ is the reproducing kernel for $A^2(\nu)$. Q is the projection $L^2(\nu) \to A^2(\nu)$. (K and P denote as before the corresponding objects for $A^2(\mu)$). ω is the function

$$\omega(z) = K(z, z)^{-1}. \tag{3.3}$$

L^p_ω, $1 \leqslant p \leqslant \infty$, is the weighted L^p-space $\{f\colon \omega f \in L^p(\lambda)\}$ with the obvious norm, and A^p_ω is the subspace of analytic functions.

Note that λ is the invariant measure of Corollaries 1.1 and 2.1, and that $L^1_\omega = L^1(\mu)$ and $L^2_\omega = L^2(\nu)$, whence $A^2_\omega = A^2(\nu)$.

We wish to stress that the spaces L^p_ω are the natural L^p-spaces to consider in our setting, and not the differently weighted spaces $L^p(\nu)$. (For example, the results for L^p_ω in this section do not hold for $L^p(\nu)$, see Section 9).

It is easily seen that under the (analytic) change of gauge (1.12), $\nu \to |\phi|^{-4}\nu$, $L(z, w) \to \phi(z)^2\overline{\phi(w)}^2 L(z, w)$, $\omega \to |\phi|^{-2}\omega$ and $L^p_\omega \to \phi^2 L^p_\omega$, $A^p_\omega \to \phi^2 A^p_\omega$. Hence the transformation $f \to \phi^2 f$ («of weight 2») operates on L^p_ω and A^p_ω (in particular, on $A^2(\nu)$).

We make two additional basic assumptions. Presumably, the first is very restrictive while the second is more technical. Both assumptions are gauge invariant.

V2: $L(z, w) = \varkappa(K(z, w))^2$ for some constant $\varkappa \geqslant 0$.

V3: If f is analytic on Ω and

$$\int |L(z, w) f(w)|\, d\nu(w) < \infty$$

for every z, then

$$\int L(z, w) f(w)\, d\nu(w) = f(z), \qquad z \in \Omega. \tag{3.4}$$

At a few places we need a further assumption.

V4: If $f \in A^2(\mu)$, then $f(z)/K(z, z)^{1/2} \in C_0(\Omega)$.

(It suffices that this holds when $f = K_w$, $w \in \Omega$, because of (1.7) and density. Hence V4 is equivalent to $|K(z, w)|^2/K(z, z)K(w, w) \in C_0(\Omega)$ for every fixed w).

We will always let $\varkappa$ denote the constant in V2; it will appear in various norm estimates.

If V0 and V1 hold, then $K_z^2 \in A^2(\nu)$ because, by (1.6) and (1.5),

$$\begin{aligned}\int |K_z|^4\, d\nu &\leqslant \int |K_z(w)|^2 K(z,z)K(w,w)\, d\nu(w)\\ &= K(z,z)\int |K_z(w)|^2\, d\mu(w)\\ &= K(z,z)^2 < \infty\end{aligned} \tag{3.5}$$

Hence (1.8) implies that $L(z,z) > 0$ and thus, if V2 too holds, $\varkappa > 0$. In fact, by (1.5) and (3.5), then

$$L(z,z) = \|L_z\|^2_{L^2(\nu)} = \varkappa^2 \int |K_z|^4\, d\nu \leqslant \varkappa^2 K(z,z)^2 = \varkappa L(z,z),$$

and thus $\varkappa \geqslant 1$ (with equality iff $A^2(\mu)$ is one-dimensional).

Remark 3.1. An inspection of the proofs below shows that in most places we could replace V2 by the weaker $L_z \in A^2(\mu) \hat{\otimes} A^2(\mu)$ with norm bounded by $\varkappa K(z,z)$ for each z. However, we do not know of any example that satisfies this condition but not V2. (Cf. the Appendices, written much later).

We collect the main results of Section 2.

Proposition 3.1. *Suppose that* V0 *holds and* $A^2(\mu) \neq \{0\}$. *Suppose further that* $G(\mu)$ *is transitive and that* (2.6) *holds for some* $z \in \Omega$. *Then* V1, V2 *and* V3 *hold. If furthermore* (2.8) *holds for some* $z \in \Omega$, *then* V4 *holds too.*

PROOF. V1 and V2 follow by Lemma 2.1 and Theorem 2.1. Since $G(\mu)$ is transitive, (2.6) holds for every z. Since $G(\mu) \subset G(\nu)$ by Theorem 2.1 and, as was shown above, $L(z,z) > 0$, V3 follows by Theorem 2.2 applied to ν and L. V4 follows from (2.8) by Theorem 2.3. □

This proposition gives us the only non-trivial examples satisfying V0 – V3 that we know. After these preliminaries, we show that the «Bergman» projection Q can be extended to L^p_ω for any $p \in [1,\infty]$. Note that this contrasts to the classical case of $H^2(\mathbb{T})$, where the analytic projection is a bounded operator in L^p for $1 < p < \infty$, but not for $p = 1$ or $p = \infty$. Recall (cf. (1.11)) that if $f \in L^2_\omega = L^2(\nu)$,

$$Qf(z) = \int_\Omega L(z,w) f(w)\, d\nu(w). \tag{3.6}$$

We use this formula to extend the domain of Q.

Theorem 3.1. *Suppose that* V0 – V3 *hold. Then*

(*a*) (3.6) *defines* Q *as a bounded linear operator* $L^1_\omega + L^\infty_\omega \to A^\infty_\omega$,

(*b*) Q *is a bounded linear projection of* L^p_ω *onto* A^p_ω *for every* p, $1 \leqslant p \leqslant \infty$.

PROOF. If $f \in L^1_\omega$, then, by V2 and (1.6),

$$\begin{aligned}\int |L(z,w)f(w)|\,d\nu(w) &= \varkappa \int |K(z,w)^2 f(w)|\omega(w)^2\,d\lambda(w)\\ &\leqslant \varkappa \int K(z,z)|f(w)|\omega(w)\,d\lambda(w)\\ &= \varkappa K(z,z)\|f\|_{L^1_\omega}.\end{aligned} \tag{3.7}$$

Hence $Qf(z)$ is well defined and $Qf \in L^\infty_\omega$. Next we observe that, by V2 and (1.5),

$$\begin{aligned}\int |L(z,w)|\omega(w)\,d\lambda(w) &= \varkappa \int |K(w,z)|^2\omega(w)\,d\lambda(w)\\ &= \varkappa \int |K_z|^2\,d\mu = \varkappa K(z,z)\end{aligned} \tag{3.8}$$

Hence, if $f \in L^\infty_\omega$,

$$\begin{aligned}\int |L(z,w)f(w)|\,d\nu(w) &= \int |f(w)\omega(w)|\,|L(z,w)|\omega(w)\,d\lambda(w)\\ &\leqslant \|f\|_{L^\infty_\omega}\varkappa K(z,z),\end{aligned} \tag{3.9}$$

whence $Qf \in L^\infty_\omega$.

It follows that Q maps $L^1_\omega + L^\infty_\omega$ into L^∞_ω. Furthermore, by Morera's theorem (using Fubini's theorem and the estimates (3.7) and (3.9)), Qf is analytic if $f \in L^1_\omega + L^\infty_\omega$, i.e. $Qf \in A^\infty_\omega$. This proves (*a*). We have proved in (3.9) that $Q\colon L^\infty_\omega \to L^\infty_\omega$. Dually, if $f \in L^1_\omega$, then by (3.8),

$$\begin{aligned}\|Qf\|_{L^1_\omega} &\leqslant \iint |L(z,w)f(w)|\,d\nu(w)\omega(z)\,d\lambda(z)\\ &= \iint |L(w,z)|\omega(z)\,d\lambda(z)|f(w)|\,d\nu(w)\\ &= \varkappa \int |f(w)|K(w,w)\,d\nu(w)\\ &= \varkappa\|f\|_{L^1_\omega}.\end{aligned} \tag{3.10}$$

Hence $Q\colon L^1_\omega \to L^1_\omega$.

By interpolation, $Q\colon L^p_\omega \to L^p_\omega$ for every $p \in [1,\infty]$. Since Qf is analytic for any $f \in L^p_\omega \subset L^1_\omega + L^\infty_\omega$ by (*a*), $Q\colon L^p_\omega \to A^p_\omega$. Finally, V3, (3.7) and (3.9) show that if $f \in L^1_\omega + L^\infty_\omega$ is analytic (in particular, if $f \in A^p_\omega$ for some $p \in [1,\infty]$), then $Qf = f$. □

Remark 3.2. The proof shows that the norm of Q as an operator in L^p_ω, $1 \leqslant p \leqslant \infty$, is at most $\varkappa$. It is easily seen that this norm equals $\varkappa$ for $p = 1$ and $p = \infty$. On the other hand, when $p = 2$ the norm is 1. Interpolation yields better estimates for $p \neq 1, 2, \infty$, but these estimates are presumably not sharp. The norm is strictly greater that 1 for any $p \neq 2$ (unless $A^2(\mu)$ is one-dimensional), since otherwise a result of Strichartz (1986) would imply that the norm would equal 1 for all p, $1 \leqslant p \leqslant \infty$, which contradicts the fact that the norm for $p = 1$ is $\varkappa > 1$.

Corollary 3.1. *If* $1 \leqslant p \leqslant q \leqslant \infty$, *then* $A^p_\omega \subset A^q_\omega$.

PROOF. If $f \in A^p_\omega$ then $f = Qf \in L^p_\omega \cap L^\infty_\omega \subset L^q_\omega$. □

Corollary 3.2. *The spaces* A^p_ω, $1 \leqslant p \leqslant \infty$, *interpolate as expected for the real and complex methods*:

$$[A^{p_0}_\omega, A^{p_1}_\omega]_\theta = (A^{p_0}_\omega, A^{p_1}_\omega)_{\theta p_\theta} = A^{p_\theta}_\omega, \quad \textit{where} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \quad \square$$

It is obvious that, if $1 \leqslant p < \infty$, $(L^p_\omega)^* = L^{p'}_\omega$ $(1/p + 1/p' = 1$ as usual) with the pairing $\langle \omega f, \omega g \rangle_\lambda = \langle f, g \rangle_\nu$.

Corollary 3.3. *Q is self-adjoint in the sense that if* $f \in L^p_\omega$ *and* $g \in L^{p'}_\omega$, $1 \leqslant p \leqslant \infty$, *then*

$$\langle Qf, g \rangle_\nu = \langle f, Qg \rangle_\nu. \tag{3.11}$$

PROOF. By (1.4) and Fubini's theorem, justified by (3.7) and (3.9). □

Corollary 3.4. *If* $1 \leqslant p < \infty$, *then* $(A^p_\omega)^* \cong A^{p'}_\omega$ *with the pairing* $\langle \quad \rangle_\nu$. □

Corollary 3.5. *The linear span of* $\{L_z\}$ *is a dense subspace of* A^p_ω *for every* p, $1 \leqslant p < \infty$.

PROOF. $L_z \in A^1_\omega \subset A^p_\omega$ by (3.8) and Corollary 3.1. If $g \in (A^p_\omega)^* = A^{p'}_\omega$ is orthogonal to every L_z, then $g(z) = Qg(z) = \langle g, L_z \rangle = 0$ for every z. □

Corollaries 3.4 and 3.5 fail for $p = \infty$, but we have the following substitute. Define a^∞_ω as the closed linear span of $\{L_z\}$ in A^∞_ω.

Corollary 3.6. *If* $1 \leqslant p < \infty$, *then* $A^p_\omega \subset a^\infty_\omega$ *densely.* $(a^\infty_\omega)^* \cong A^1_\omega$ *with the pairing* $\langle \quad \rangle_\nu$.

PROOF. The first assertion follows by Corollaries 3.1 and 3.5. Thus, if $\chi \in (a^\infty_\omega)^*$ there exists $g \in (A^2_\omega)^* = A^2_\omega$ such that $\chi(f) = \langle f, g \rangle_\nu$ for every $f \in A^2_\omega \subset a^\infty_\omega$. Hence, if $f \in L^2_\omega \cap L^\infty_\omega$, by Theorem 3.1 and Corollary 3.3,

$$\langle f, g \rangle_\nu = \langle f, Qg \rangle_\nu = \langle Qf, g \rangle_\nu = \chi(Qf)$$

and

$$\left|\int_\Omega \bar{f} g\, d\nu\right| = |\langle f, g\rangle_\nu| \leqslant \|\chi\| \, \|Qf\|_{A^\infty_\omega} \leqslant C\|f\|_{L^\infty_\omega}.$$

This implies $g \in L^1_\omega$, i.e. $g \in A^1_\omega$. The rest is easy. □

Corollary 3.7. *A^p_ω is reflexive when $1 < p < \infty$. $(a^\infty_\omega)^{**} = A^\infty_\omega$. Thus A^1_ω and A^∞_ω are reflexive iff $a^\infty_\omega = A^\infty_\omega$.* □

Corollary 3.8. *Suppose that also* V4 *holds. Then*

$$a^\infty_\omega = \{f \in \mathcal{H}(\Omega) : \omega f \in C_o(\Omega)\}.$$

PROOF. $A^\infty_\omega \cap \omega^{-1} C_o(\Omega)$ is a closed subspace of A^∞_ω, which by assumption contains every $L_z = \varkappa K_z^2$ and thus a^∞_ω. On the other hand, if $f \in A^\infty_\omega \cap \omega^{-1} C_o(\Omega)$, let $f_j = \chi_{K_j} f$, where $\{K_j\}$ is an increasing sequence of compact subsets of Ω with $\bigcup_1^\infty \operatorname{int}(K_j) = \Omega$. Then $f_j \in L^2_\omega \cap L^\infty_\omega$ and $f_j \to f$ in L^∞_ω, whence $Qf_j \in A^2_\omega \subset a^\infty_\omega$ and $Qf_j \to Qf = f$ in A^∞_ω. □

4. Hankel Forms

We assume throughout this section that the conditions V0 – V3 are satisfied. We continue to use the notations introduced in §§0 and 3.

As explained in the introduction, we will in this section study Hankel forms on $A^2(\mu)$ with symbols taken with respect to ν, i.e.

$$\Gamma_b(f, g) = \langle fg, b\rangle_\nu = \int_\Omega \bar{b} f g\, d\nu, \tag{4.1}$$

where $f, g \in A^2(\mu)$.

Theorem 4.1. *Let $b \in L^1_\omega + L^\infty_\omega$. Then Γ_b is a bounded bilinear form on $A^2(\mu)$, $\Gamma_b = \Gamma_{Qb}$ and*

$$\varkappa^{-1}\|Qb\|_{A^\infty_\omega} \leqslant \|\Gamma_b\| \leqslant \|Q_b\|_{A^\infty_\omega} \tag{4.2}$$

PROOF. By Hölder's inequality,

$$\|fg\|_{L^1_\omega} = \|fg\|_{L^1(\mu)} \leqslant \|f\|_{A^2(\mu)} \|g\|_{A^2(\mu)} \tag{4.3}$$

Thus $fg \in A^1_\omega \subset L^1_\omega \cap L^\infty_\omega$ (Corollary 3.1) which proves the first assertion, and

$$\|\Gamma_b\| \leqslant \|b\|_{L^\infty_\omega}. \tag{4.4}$$

By Corollary 3.3 and Theorem 3.1,

$$\Gamma_{Qb}(f,g) = \langle fg, Qb\rangle_\nu = \langle Q(fg), b\rangle_\nu = \langle fg, b\rangle_\nu = \Gamma_b(f,g),$$

which proves the second assertion, and

$$\|\Gamma_b\| = \|\Gamma_{Qb}\| \leqslant \|Qb\|_{A^\infty_\omega}.$$

Finally, if $z \in \Omega$,

$$\begin{aligned} |\varkappa^{-1}Qb(z)| &= |\varkappa^{-1}\langle b, L_z\rangle_\nu| = |\langle b, K_z^2\rangle_\nu| \\ &= |\Gamma_b(K_z, K_z)| \leqslant \|\Gamma_b\| \cdot \|K_z\|^2 \\ &= \|\Gamma_b\| K(z,z). \quad \square \end{aligned}$$

We proceed to the Schatten-von Neumann theory. We define an anti-linear operator $\Gamma: L^1_\omega + L^\infty_\omega \to S_\infty$ by $\Gamma(b) = \Gamma_b$.

Theorem 4.2. *If* $1 \leqslant p \leqslant \infty$ *and* $b \in L^p_\omega$, *then* $\Gamma_b \in S_p$ *and* $\|\Gamma_b\|_{S_p} \leqslant \|b\|_{L^p_\omega}$.

PROOF. If suffices to prove the result for $p = 1$, since the general case then follows by (4.4) and interpolation. Thus, assume that $b \in L^1_\omega$. The Banach space valued integral $\int b(z)L_z\,d\nu(z)$ then converges in $A^2(\nu) = A^2_\omega$, because the integrand is measurable (recall that $z \to L_z$ is continuous by Corollary 1.2 applied to ν) and

$$\begin{aligned} \int |b(z)|\, \|L_z\|_{A^2(\nu)}\, d\nu &= \int |b(z)| L(z,z)^{1/2}\, d\nu \\ &= \int \varkappa^{1/2} |b(z)| \omega(z)^{-1}\, d\nu \\ &= \varkappa^{1/2} \|b\|_{L^1_\omega} < \infty. \end{aligned} \tag{4.5}$$

Evaluating the integral pointwise by (1.2) and (3.6), we obtain

$$Qb = \int_\Omega b(z) L_z\, d\nu(z). \tag{4.6}$$

Since, by Theorem 4.1, Γ is a bounded anti-linear operator:

$$A^2_\omega \subset L^1_\omega + L^\infty_\omega \to S_\infty,$$

this yields

$$\Gamma(b) = \Gamma(Qb) = \int_\Omega \overline{b(z)}\, \Gamma(L_z)\, d\nu(z) \tag{4.7}$$

with the integral convergent in S_∞.

However,

$$\Gamma(L_z)(f,g) = \langle fg, L_z\rangle_\nu = fg(z) = \langle f, K_z\rangle_\mu \langle g, K_z\rangle_\mu. \tag{4.8}$$

Thus $\Gamma(L_z)$ is a bilinear form of rank 1, and

$$\|\Gamma(L_z)\|_{S_1} = \|\Gamma(L_z)\|_{S_\infty} = \|K_z\|^2_{A^2(\mu)} = K(z,z). \tag{4.9}$$

Thus

$$\int \|\overline{b(z)}\Gamma(L_z)\|_{S_1}\, d\nu = \int |b(z)|K(z,z)\, d\nu = \|b\|_{L^1_\omega} < \infty. \tag{4.10}$$

Furthermore, since $\Gamma(L_z) - \Gamma(L_w)$ has rank at most 2,

$$\|\Gamma(L_z) - \Gamma(L_w)\|_{S_1} \leqslant 2\|\Gamma(L_z) - \Gamma(L_w)\|_{S_\infty} \leqslant C\|L_z - L_w\|_{A^2(\nu)} \to 0$$

as $z \to w$, whence $z \to \Gamma(L_z)$ is a continuous map of Ω into S_1 and $z \to \overline{b(z)}\Gamma(L_z)$ is a measurable map into S_1. Consequently the integral (4.7) converges in S_1 as well and $\Gamma(b) \in S_1$ with norm bounded by (4.10). □

Next we define, for every bounded bilinear form T on $A^2(\mu)$,

$$\Gamma^*(T)(z) = \overline{T(K_z, K_z)}. \tag{4.11}$$

Cf. (for operators) Aronszajn (1950) and Berezin (1975).

Theorem 4.3. Γ^* *is a bounded anti-linear mapping of* S_∞ *into* A^∞_ω *that maps* S_p *into* A^p_ω *with*

$$\|\Gamma^*(T)\|_{A^p_\omega} \leqslant \|T\|_{S_p}, \qquad 1 \leqslant p \leqslant \infty. \tag{4.12}$$

PROOF. Fix $w \in \Omega$. Since $f \to T(f, K_w)$ is a bounded linear form on $A^2(\mu)$, there exists $g \in A^2(\mu)$ such that $T(f, K_w) = \langle f, g\rangle$. Thus

$$\overline{T(K_z, K_w)} = \overline{\langle K_z, g\rangle} = \langle g, K_z\rangle = g(z)$$

is analytic in z. By symmetry, $\overline{T(K_z, K_w)}$ is analytic in w too, whence it is analytic in (z, w) by Hartogs's theorem. In particular, $\Gamma^*(T)(z)$ is analytic.

It remains to prove that $\|\Gamma^*(T)\|_{L^p_\omega} \leqslant \|T\|_{S_p}$. By interpolation, it suffices to consider $p = \infty$ and $p = 1$. The case $p = \infty$ follows by

$$|\Gamma^* T(z)| \leqslant \|T\|_{S_\infty} \|K_z\|^2_{A^2(\mu)} = \|T\|_{S_\infty} K(z,z) = \|T\|_{S_\infty} \omega(z)^{-1}.$$

Next, if T is of rank 1, say $T(f,g) = \langle f, \phi\rangle_\mu \langle g, \psi\rangle_\mu$, then

$$\Gamma^* T(z) = \overline{\langle K_z, \phi\rangle_\mu}\,\overline{\langle K_z, \psi\rangle_\mu} = \phi(z)\psi(z) \tag{4.13}$$

and thus by Hölder's inequality

$$\|\Gamma^* T\|_{L^1_\omega} = \|\phi\psi\|_{L^1(\mu)} \leqslant \|\phi\|_{A^2(\mu)} \|\psi\|_{A^2(\mu)} = \|T\|_{S_1}.$$

Since S_1 is spanned by forms of rank 1, the case $p = 1$ follows. □

Next we prove that, as our notation suggests, Γ^* is the adjoint of Γ. (Recall that the operators are anti-linear which explains the form of (4.14)).

Theorem 4.4. *If* $1 \leqslant p \leqslant \infty$ *and* $1/p + 1/p' = 1$, *then*

$$\langle T, \Gamma_b \rangle = \langle b, \Gamma^* T \rangle_\nu, \qquad T \in S_{p'}, \qquad b \in L^p_\omega \tag{4.14}$$

PROOF. We study two cases separately. If $1 < p \leqslant \infty$, then forms of finite rank are dense in $S_{p'}$. Since both scalar products in (4.14) are bounded bilinear forms on $S_{p'} \times L^p_\omega$ (by Theorems 4.2 and 4.3), it suffices to prove (4.14) when T has rank one, say $T(f, g) = \langle f, \phi \rangle_\mu \langle g, \psi \rangle_\mu$. In this case $\Gamma^* T = \phi\psi$ by (4.13), and

$$\langle T, \Gamma_b \rangle = \overline{\Gamma_b(\phi, \psi)} = \overline{\langle \phi\psi, b \rangle_\nu} = \langle b, \phi\psi \rangle_\nu = \langle b, \Gamma^* T \rangle_\nu.$$

If $p = 1$ we use the representation (4.7)

$$\Gamma_b = \int \overline{b(z)} \Gamma(L_z)\, d\nu,$$

which converges in S_1 by the proof of Theorem 4.2. Since

$$\Gamma(L_z)(f, g) = \langle f, K_z \rangle_\mu \langle g, K_z \rangle_\mu$$

by (4.8),

$$\langle T, \Gamma(L_z) \rangle = T(K_z, K_z) = \overline{\Gamma^* T(z)},$$

and

$$\langle T, \Gamma_b \rangle = \int b(z) \langle T, \Gamma(L_z) \rangle\, d\nu = \int b(z) \overline{\Gamma^* T(z)}\, d\nu = \langle b, \Gamma^* T \rangle_\nu. \quad \square$$

We proceed to study $\Gamma^*\Gamma$ and $\Gamma\Gamma^*$.

Theorem 4.5. $\Gamma^*\Gamma(b) = \varkappa^{-1} Q b$ *for every* $b \in L^1_\omega + L^\infty_\omega$.

PROOF.

$$\Gamma^*\Gamma(b)(z) = \overline{\Gamma_b(K_z, K_z)} = \overline{\langle K_z^2, b \rangle_\nu} = \langle b, \varkappa^{-1} L_z \rangle = \varkappa^{-1} Q b(z). \quad \square \tag{4.15}$$

Theorems 4.1, 4.2, 4.3 and 4.5 yield one of our main results.

Theorem 4.6. *Let* $b \in L^1_\omega + L^\infty_\omega$ *and* $1 \leqslant p \leqslant \infty$. *Then* $\Gamma_b \in S_p$ *if and only if* $Qb \in A^p_\omega$. □

Theorem 4.7. *Let* $1 \leqslant p \leqslant \infty$. *Then* Γ *is an anti-linear isomorphism mapping* A^p_ω *onto the set of Hankel forms in* S_p. *The inverse is given by* $\varkappa\Gamma^*$. *If* $p = 2$ *then* $\sqrt{\varkappa}\,\Gamma$ *is an anti-linear isometry.*

PROOF. The first assertions follow immediately. That $\varkappa^{1/2}\Gamma$ is an isometry was proved in the introduction (0.7), and follows also by Theorems 4.4 and 4.5. □

Remark 4.1. The proof of Theorem 4.6 yields the estimates

$$\|\Gamma_b\|_{S_p} \leqslant \|Q_b\|_{A^p_\omega} \leqslant \varkappa\|\Gamma_b\|_{S_p}, \qquad 1 \leqslant p \leqslant \infty,$$

but Theorem 4.7 shows that improved estimates can be obtained for $1 < p < \infty$ by interpolating with the case $p = 2$.

Theorem 4.5 yields $\varkappa\Gamma\Gamma^*\Gamma = \Gamma Q = \Gamma$ and $\varkappa\Gamma^*\Gamma\Gamma = \Gamma^*$. The results above now give the following results on the Hankel projection.

Theorem 4.8. $\varkappa\Gamma\Gamma^*$ *is a linear projection of* S_∞ *onto the subspace of Hankel forms.* $\varkappa\Gamma\Gamma^*$ *is bounded on* S_p, $1 \leqslant p \leqslant \infty$, *and* $\langle \varkappa\Gamma\Gamma^* S, T\rangle = \langle S, \varkappa\Gamma\Gamma^* T\rangle$ *for* $S \in S_p$, $T \in S_{p'}$, $1/p + 1/p' = 1$. *In particular, the restriction of* $\varkappa\Gamma\Gamma^*$ *to* S_2 *is the orthogonal projection onto the space of Hilbert-Schmidt Hankel forms.* □

Remark 4.2. This contrasts to the classical case $H^2(\mathbb{T})$, where the Hankel projection is bounded when $1 < p < \infty$, but not at the endpoints, see e.g. Peller (1980).

Corresponding results for compactness are easily obtained using the fact that the space of compact forms equals the closed hull of S_2 in S_∞ together with Corollary 3.6.

Theorem 4.9. Γ_b *is compact if and only if* $Qb \in a^\infty_\omega$. Γ *maps* a^∞_ω *onto the set of compact Hankel forms. The Hankel projection* $\varkappa\Gamma\Gamma^*$ *maps compact forms to compact Hankel forms.* □

5. Multilinear Hankel Forms

The theory above for bilinear Hankel forms is easily generalized to multilinear forms. We will here sketch this generalization omitting most of the details.

Let $m > 2$ be an integer ($m = 2$ gives the results of the preceding sections) and define the measure $d\nu_m = \omega^m \, d\lambda$ and, for $f_1, \ldots, f_m \in A^2(\mu)$ and b a suitable function on Ω,

$$\Gamma_b(f_1, \ldots, f_m) = \int_\Omega \bar{b} f_1 \cdot \ldots \cdot f_m \, d\nu_m. \tag{5.1}$$

The weight ω^m in the definition of ν_m makes the expression (5.1) gauge invariant, with b transforming as $b \to \phi^m b$ («weight m») under an analytic change of gauge (1.12).

Let L_m denote the reproducing kernel in $A^2(\nu_m)$. We assume throughout this section that V0 and V1 hold, that $L_m(z, w) = \varkappa_m K(z, w)^m$ for some constant $\varkappa_m$, and that

$$\int L_m(z, w) f(w) \, d\nu_m(w) = f(z), \qquad z \in \Omega \tag{5.2}$$

for every analytic function f such that the left-hand side is defined for all z. (The natural generalizations of V2 and V3 to the present situation). Note that these conditions are satisfied, for every m, whenever V0 holds, $A^2(\mu) \neq \{0\}$, $G(\mu)$ is transitive and (2.6) holds for some (and thus all) $z \in \Omega$. (Because the proof of Proposition 3.1 extends immediately, using Theorem 2.1 with $r = m - 1$).

Define Q_m by

$$Q_m f(z) = \int L_m(z, w) f(w) \, d\nu_m(w)$$

and let

$$L^p_{\omega^{m/2}} = \{ f \colon \omega^{m/2} f \in L^p(\lambda) \}, \qquad A^p_{\omega^{m/2}} = L^p_{\omega^{m/2}} \cap \mathcal{H}(\Omega).$$

Note that if $f \in A^2(\mu)$, then

$$\| \omega^{1/2} f \|_{L^2(\lambda)} = \| f \|_{A^2(\mu)}$$

and, by (1.7),

$$\| \omega^{1/2} f \|_{L^\infty(\lambda)} \leqslant \| f \|_{A^2(\mu)}.$$

Hence also

$$\| \omega^{1/2} f \|_{L^m(\lambda)} \leqslant \| f \|_{A^2(\mu)}$$

and, by Hölder's inequality,

$$\| \omega^{m/2} f_1 \cdot \ldots \cdot f_m \|_{L^1(\lambda)} \leqslant \| f_1 \|_{A^2(\mu)} \cdot \ldots \cdot \| f_m \|_{A^2(\mu)} \tag{5.3}$$

$$\| \omega^{m/2} f_1 \cdot \ldots \cdot f_m \|_{L^\infty(\lambda)} \leqslant \| f_1 \|_{A^2(\mu)} \cdot \ldots \cdot \| f_m \|_{A^2(\mu)}. \tag{5.4}$$

Consequently,

$$f_1 \cdot \ldots \cdot f_m \in A^1_{\omega m/2} \cap A^\infty_{\omega m/2} \quad \text{if} \quad f_1, \ldots, f_m \in A^2(\mu).$$

In particular,

$$L_{m,z} = \varkappa_m K_z^m \in A^1_{\omega m/2} \cap A^\infty_{\omega m/2}.$$

It is now easily seen, as in Section 3, that Q_m is a projection of $L^p_{\omega m/2}$ onto $A^p_{\omega m/2}$ for $1 \leqslant p \leqslant \infty$, and that the analogues of Corollaries 3.1-3.8 hold. Equipped with these results, we proceed to study the multilinear Hankel form defined by (5.1). Theorem 4.1 extends easily.

Theorem 5.1 *Let $b \in L^1_{\omega m/2} + L^\infty_{\omega m/2}$. Then Γ_b is a bounded multilinear form on $A^2(\mu)$, $\Gamma_b = \Gamma_{Q_m b}$ and*

$$\varkappa_m^{-1} \| Q_m b \|_{A^\infty_{\omega m/2}} \leqslant \| \Gamma_b \| \leqslant \| Q_m b \|_{A^\infty_{\omega m/2}} \tag{5.5}$$

PROOF. We prove (5.5) and leave the rest to the reader. We may assume that $b = Q_m b$. Then, if $\|f_1\|_{A^2(\mu)}, \ldots, \|f_m\|_{A^2(\mu)} \leqslant 1$, (5.3) yields

$$|\Gamma_b(f_1, \ldots, f_m)| = \left| \int \omega^{m/2} \bar{b} \omega^{m/2} f_1 \ldots f_m \, d\lambda \right| \leqslant \| \omega^{m/2} \bar{b} \|_{L^\infty(\lambda)} = \| b \|_{A^\infty_{\omega m/2}}$$

which proves the right inequality. The left inequality follows by

$$\begin{aligned} |Q_m b(z)| &= \left| \int \bar{b} L_{m,z} \, d\nu_m \right| = \left| \varkappa_m \int \bar{b} K_z^m d\nu_m \right| = \varkappa_m |\Gamma_b(K_z, \ldots, K_z)| \\ &\leqslant \varkappa_m \| \Gamma_b \| \, \| K_z \|^m_{A^2(\mu)} \\ &= \varkappa_m \| \Gamma_b \| \omega(z)^{-m/2}. \quad \square \end{aligned}$$

Also the S_p-results of Section 4 extend. However, as far as we know, no theory is so far developed for S_p-classes of multilinear forms on a Hilbert space $\mathcal{H}$. Hence we confine ourselves to the case $p = 1, 2, \infty$.

Let S_∞ be the space of all bounded multilinear forms on $\mathcal{H} \times \cdots \times \mathcal{H}$. Let S_1 be the space of nuclear forms, i.e.

$$\left\{ (x_1, \ldots, x_m) \to \sum_{j=1}^{\infty} a_j \prod_{i=1}^{m} \langle x_i, y_{ij} \rangle : \sum_j |a_j| \prod_i \| y_{ij} \|_{\mathcal{H}} < \infty \right\};$$

S_1 is the m-fold projective tensor product $\mathcal{H} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}$ (identyfying $\mathcal{H}$ and its dual).

Let S_2 be the space of Hilbert-Schmidt forms; S_2 is the Hilbert tensor product $\mathcal{H} \hat{\otimes}_2 \cdots \hat{\otimes}_2 \mathcal{H}$.

It follows that $S_1 \subset S_2 \subset S_\infty$, and $S_1^* \cong S_\infty$.

Furthermore,

$$[S_1, S_\infty]_{1/2} = (S_1, S_\infty)_{1/2,2} = S_2 .$$

Remark 5.1. This is an instance of the general principle that interpolation between a space and its dual (by one of these two interpolation methods) gives a Hilbert space. (We call this a «principle», not a «theorem», because it is not yet proved in a complete generality, see Janson (1986)).

Let $\Gamma(b) = \Gamma_b$. $\Gamma(L_{m,z})$ is a multilinear form of rank 1 and it follows as in Theorem 4.2 that $b \in L^1_{\omega m/2} \to \Gamma_b \in S_1$. For the converse we define for any multilinear form T on $A^2(\mu)$,

$$\Gamma^* T(z) = \overline{T(K_z, \ldots, K_z)}$$

(cf. (4.11)) and obtain as in Theorem 4.3, using (5.3), that Γ^* maps S_∞ into $A^\infty_{\omega m/2}$ and S_1 into $A^1_{\omega m/2}$. Furthermore, $\varkappa_m \Gamma^* \Gamma = Q_m$. Hence we obtain the extension of Theorem 4.6.

Theorem 5.2. *If* $p = 1, 2, \infty$, *then*

$$\Gamma_b \in S_p \quad \textit{if and only if} \quad Q_m b \in A^p_{\omega m/2}. \quad \square$$

It is also easy to treat the case S_2 directly as in §0.4.

Remark 5.2. If we *define* S_p for $1 < p < \infty$ by (real or complex) interpolation between S_1 and S_∞, Theorem 5.2 holds for every $p \geqslant 1$. Indeed, this seems to be the only reasonable definition one can think of if one wants to carry over the usual theorems on Hankel forms (operators) to the multilinear situation (see Peetre (1985)).

6. Miscellaneous Complements

We assume that V0 – V3 hold.

6.1. More general symbols. In §4 we assumed for technical reasons that $b \in L^1_\omega + L^\infty_\omega$, which made all occurring integrals finite. In the next subsection we have to consider more general symbols, which may be done as follows. Recalling that the linear span of $\{K_z\}$ is dense in $A^2(\mu)$, we say that Γ_b exists when

$$\int |b K_z K_w| \, d\nu < \infty \quad \text{for all} \quad z, w,$$

and

$$\left| \int \bar{b} f g \, d\nu \right| \leqslant C \|f\| \, \|g\| \quad \text{for all} \quad f, g \in \operatorname{span}\{K_z\}. \tag{6.1}$$

Then Γ_b is defined on $A^2(\mu) \times A^2(\mu)$ by continuity. The assumption (6.1) implies that Qb is well-defined and

$$Qb(z) = \int b\bar{L}_z \, d\nu = \varkappa \int b\bar{K}_z^2 \, d\nu = \varkappa \Gamma^* \Gamma_b \in A_\omega^\infty$$

by Theorem 4.3, whence the preceding theory applies to Γ_{Qb}.

If we assume that $\Gamma_b = \Gamma_{Qb}$ for all b such that (6.1) holds, then the results of §4 can be carried over to this enlarged classs of symbols; in particular, it follows that $\Gamma_b \in S_p$ if and only if $Qb \in A_\omega^p$.

An alternative formulation of this assumption is:

If (6.1) holds and

$$\int \bar{b} K_z^2 \, d\nu = 0 \quad \text{for every } z$$

then

$$\int \bar{b} K_z K_w \, d\nu = 0 \quad \text{for all} \quad z, w \tag{6.2}$$

To see this equivalence, notice that

$$\int (\bar{b} - \overline{Qb}) K_z^2 \, d\nu = 0.$$

Thus, by (6.2),

$$\int \bar{b} K_z K_w \, d\nu = \int \overline{Qb} K_z K_w \, d\nu,$$

that is, $\Gamma_b = \Gamma_{Qb}$.

Unfortunately, we have not been able to prove (6.2) in general, but it is easily verified in the examples in §§7-13. Note that (6.2) is gauge invariant.

6.2. Symbols with respect to other measures. The time has come to treat Hankel forms with symbols with respect to general absolutely continuous measures. We recall the notations, cf. §0.2,

$$H_b^\xi(f, g) = \int \bar{b} f g \, d\xi \tag{6.3}$$

and, as a special case,

$$H_b = H_b^\mu. \tag{6.4}$$

More precisely, we say that H_b^ξ exists if $\int \bar{b} f g \, d\xi$ is absolutely convergent and defines a bounded form for $f, g \in \operatorname{span}\{K_z\}$.

If follows from the definition that

$$H_b^\xi = \Gamma\left(b \frac{d\xi}{d\nu}\right) \tag{6.5}$$

where the two sides are defined (in the sense of this section) for the same set of b. Hence the previous results for Γ can be transfered. The condition (6.2) is equivalent to

If

$$\left|\int \bar{b}fg\,d\xi\right| \leqslant C\|f\|\,\|g\|, \qquad f,g \in \operatorname{span}\{K_z\}, \quad \text{and} \quad \int \bar{b}K_z^2\,d\xi = 0 \tag{6.6}$$

for every z, then

$$\int \bar{b}K_zK_w\,d\xi = 0$$

for all z, w.

The discussion above and Theorem 4.6 and 4.9 yield:

Corollary 6.1. *Suppose that* (6.6) *holds. Then*

$$H_b^\xi \in S_p \quad \textit{if and only if} \quad Q\left(b\frac{d\xi}{d\nu}\right) \in A_\omega^p, \qquad 1 \leqslant p \leqslant \infty.$$

$$H_b^\xi \quad \textit{is compact if and only if} \quad Q\left(b\frac{d\xi}{d\nu}\right) \in a_\omega^p.$$

Note that, for $\xi = \mu(H_b^\xi = H_b)$, $Q(b \cdot d\mu/d\nu) = Q(\omega^{-1}b)$.

It remains to identify $Q(b \cdot d\xi/d\nu)$, in particular for analytic symbols b. Here the general theory fails us (even for $\xi = \mu$), and this has to be done by a separate analysis in each case (because $b \mapsto Q(b \cdot d\mu/d\nu) = Q(b(z)K(z,z))$ is *not* gauge-invariant, cf. §§7 and 11). We observe nevertheless the formula

$$Q\left(b\frac{d\xi}{d\nu}\right)(z) = \int b\frac{d\xi}{d\nu}\bar{L}_z\,d\nu = \int b\,\bar{L}_z\,d\xi \tag{6.7}$$

and that $b \mapsto Q(b \cdot d\xi/d\nu)$ formally is the adjoint of the (possibly unbounded) identity map $A^2(\nu) \to A^2(\xi)$ because, for $f \in A^2(\nu)$,

$$\left\langle Q\left(b\frac{d\xi}{d\nu}\right), f\right\rangle_\nu = \left\langle b\frac{d\xi}{d\nu}, f\right\rangle_\nu = \langle b, f\rangle_\xi. \tag{6.8}$$

6.3. Minimal and maximal invariant spaces. The representation $f = \int f(z)\,L_z\,d\nu$ expresses any function in A_ω^1 as a continuous linear combination of $\{L_z\}$. There is also a discrete counterpart.

Theorem 6.1. $f \in A_\omega^1$ *if and only if* $f = \sum_1^\infty a_i\omega(z_i)L_{z_i}$ *for some sequences* $\{z_i\} \subset \Omega$ *and* $\{a_i\} \in l^1$. $\|f\|_{A_\omega^1}$ *is equivalent to the infimum of* $\sum |a_i|$ *extended over all such representations.*

PROOF. By (3.8),

$$\|\omega(z)L_z\|_{A^1_\omega} = \omega(z)\varkappa K(z,z) = \varkappa \quad \text{for every} \quad z \in \Omega.$$

Hence we may define a linear operator $T: l^1(\Omega) \to A^1_\omega$ (with norm $\varkappa$) by

$$T\{a_z\} = \sum_{z\in\Omega} a_z \omega(z) L_z.$$

The adjoint T^* maps $A^\infty_\omega \cong (A^1_\omega)^*$ (cf. Corollary 3.4) into $l^\infty(\Omega) \cong (l^1(\Omega))^*$. Let $g \in A^\infty_\omega$. Since

$$\langle g, L_z\rangle_\nu = Qg(z) = g(z),$$

it is easily seen that $T^*g = \{\omega(z)g(z)\}$, and thus $\|T^*g\|_{l^\infty(\Omega)} = \|g\|_{A^\infty_\omega}$.

Consequently T^* is an isomorphism into, and T is onto. □

In other terms, A^1_ω is the smallest Banach space that contains all L_z with norms bounded by some constant times $K(z,z)$.

It is easily seen that $G^*(\mu)$ acts isometrically in each A^p_ω by the action

$$R_{(\gamma,\phi)}f(z) = \phi(z)^2 f(\gamma^{-1}(z)) \tag{6.9}$$

(Cf. (2.2) and recall that A^p_ω transforms with weight 2). Using (2.3), it follows easily that

$$R_{(\gamma,\phi)}L_w(z) = \phi(z)^2 L(\gamma^{-1}(z), w) = \overline{\phi(\gamma(w))}^{-2} L(z, \gamma(w))$$

and

$$R_{(\gamma,\phi)}(\omega(w)L_w) = \omega(w)\overline{\phi(\gamma(w))}^{-2} L_{\gamma(w)} = \operatorname{sign}\phi(\gamma(w))^2 \omega(\gamma(w)) L_{\gamma(w)}.$$

Hence, if G is a transitive subgroup of $G(\mu)$ and G^* is the corresponding subgroup of $G^*(\mu)$, the theorem above shows that A^1_ω is the smallest G^*-invariant (under the action (6.9)) Banach space that contains some L_z.

Dually, A^∞_ω is the largest G^*-invariant Banach space of analytic functions in Ω admitting continuous evaluation at some point. This follows because, if $|f(z_0)| \leqslant C\|f\|$ for every function in the space, $(\gamma,\phi) \in G^*$ implies by (2.3) and (6.5)

$$\begin{aligned}
\omega(\gamma^{-1}(z_0))|f(\gamma^{-1}(z_0))| &= |\phi(z_0)|^2 \omega(z_0)|f(\gamma^{-1}(z_0))| \\
&= \omega(z_0)|R_{(\gamma,\phi)}f(z_0)| \\
&\leqslant C\omega(z_0)\|R_{(\gamma,\phi)}f\| \\
&= C\omega(z_0)\|f\|.
\end{aligned}$$

We will not pursue the investigation of invariant spaces here, but refer to the surveys Arazy and Fisher (1984) and Peetre (1984), (1985).

6.4. Factorization.

Theorem 6.2. *$f \in A^1_\omega$ if and only if $f = \sum_1^\infty a_i g_i h_i$ for some sequences $\{g_i\}$ and $\{h_i\}$ in the unit ball of $A^2(\mu)$ and $\{a_i\} \in l^1$. $\|f\|_{A^1_\omega}$ is equivalent to the infimum of $\sum |a_i|$ over all such representations.*

PROOF. Hölder's inequality yields

$$\left\| \sum_1^\infty a_i g_i h_i \right\|_{A^1_\omega} \leqslant \sum |a_i| \, \|g_i\|_{A^2(\mu)} \|h_i\|_{A^2(\mu)}.$$

The existence of representations follows by duality as in the proof of Theorem 6.1, or alternatively, from Theorem 1 by taking $g_i = h_i = K_{z_i}/\|K_{z_i}\|$ and replacing a_i by $\varkappa a_i$ (because then

$$\varkappa g_i h_i = \varkappa K_{z_i}^2 / \|K_{z_i}\|^2 = \omega(z_i) L_{z_i}). \quad \square$$

This is a so-called weak factorization. We do not know whether a similar strong factorization is valid in general, i.e. whether each $f \in A^1_\omega$ can be factorized as gh with g and h in $A^2(\mu)$ and $\|g\| \, \|h\| \leqslant C\|f\|_{A^1_\omega}$. For the special case of Bergman spaces in the disc, Horowitz (1977) proved strong factorization.

6.5. Another S_p criterion. Let $k_z = K_z / \|K_z\|$ be the normalized reproducing kernels. If T is any bounded linear operator of $A^2(\mu)$ into a Hilbert space $\mathcal{H}_1$, then if $\{e_\alpha\}$ is an ON-basis in $\mathcal{H}_1$,

$$\begin{aligned} \int \|Tk_z\|^2 \, d\lambda(z) &= \int \|TK_z\|^2 \, d\mu(z) = \int \sum_\alpha |\langle TK_z, e_\alpha\rangle|^2 \, d\mu(z) \\ &= \sum_\alpha \int |\langle K_z, T^*e_\alpha\rangle|^2 \, d\mu(z) = \sum_\alpha \int |T^*e_\alpha(z)|^2 \, d\mu(z) \\ &= \sum \|T^*e_\alpha\|^2 = \|T^*\|_{S_2}^2 \\ &= \|T\|_{S_2}^2 \end{aligned} \tag{6.10}$$

It follows, by interpolation with $p = \infty$, that

$$\| \, \|Tk_z\|_{\mathcal{H}_1} \|_{L^p(\lambda)} \leqslant \|T\|_{S_p}, \qquad 2 \leqslant p \leqslant \infty \tag{6.11}$$

and, by duality,

$$\| \, \|Tk_z\|_{\mathcal{H}_1} \|_{L^p(\lambda)} \geqslant \|T\|_{S_p}, \qquad 1 \leqslant p \leqslant 2. \tag{6.12}$$

For Hankel operators there exist converses to (6.11) and (6.12) (within constants). Let $\tilde{\Gamma}_b$ be the operator corresponding to the form Γ_b as in (0.1). We may assume that $b \in A_\omega^\infty$. We begin with the case $p \geqslant 2$, where there are no problems.

Theorem 6.3. *If* $2 \leqslant p \leqslant \infty$, *then*

$$\tilde{\Gamma}_b \in S_p \quad \textit{if and only if} \quad \|\tilde{\Gamma}_b k_z\| \in L^p(\lambda). \tag{6.13}$$

PROOF.

$$\|\tilde{\Gamma}_b k_z\| \geqslant |\langle \tilde{\Gamma}_b k_z, k_z \rangle| = |\Gamma_b(k_z, k_z)| = \omega(z)|\Gamma_b(K_z, K_z)| = \varkappa^{-1}\omega(z)|b(z)|,$$

cf. (4.15). Thus, by Theorem 4.2,

$$\|\tilde{\Gamma}_b\|_{S_p} = \|\Gamma_b\|_{S_p} \leqslant \|\omega b\|_{L^p(\lambda)} \leqslant \varkappa \|\, \|\tilde{\Gamma}_b k_z\| \,\|_{L^p(\lambda)}. \quad \square$$

The converse for $p < 2$ only holds in some cases, however.

Theorem 6.4. *Suppose that*

$$\sup_{w \in \Omega} \int \frac{|K(z, w)|}{(K(z, z)K(w, w))^{1/2}} \, d\lambda(z) < \infty. \tag{6.14}$$

Then, for every $1 \leqslant p \leqslant \infty$,

$$\tilde{\Gamma}_b \in S_p \quad \textit{if and only if} \quad \|\tilde{\Gamma}_b k_z\| \in L^p(\lambda). \tag{6.15}$$

Conversely, if (6.15) *holds for* $p = 1$, *then* (6.14) *holds.*

PROOF. By Theorem 6.3 and Theorem 4.6, it suffices to prove that if (6.14) holds, $b \in A_\omega^p$ implies $\|\Gamma_b k_z\| \in L^p(\lambda)$. By interpolation we may assume $p = 1$, and by Theorem 6.1 this implication is equivalent to

$$\sup \{ \|\Gamma_b k_z\|_{L^1(\lambda)} : b = \omega(w) L_w \} < \infty. \tag{6.16}$$

A simple calculation shows that (6.16) is the same as (6.14). $\square$

If $G(\mu)$ is transitive, then, by Corollaries 2.1 and 2.2 the integral in (6.14) is independent of w, whence it is sufficient that it is finite for some w. Hence (take $w = 0$) (6.14) and (6.15) hold for the Fock space (§§7-11), and for the Bergman spaces (§§12, 13) with parameter $\gamma > n - 1$, but (6.14) does not hold for Bergman spaces with $\gamma \leqslant n - 1$. (Presumably, (6.13) holds for some $p < 2$ even in the latter case; more research is needed).

The corresponding result for $H^2(\mathbb{T})$ and $p = \infty$ is given by Bonsall (1984); it is equivalent to an oscillation condition.

7. Fock Space

The general theory will now be applied to the Fock space. Let, as in the introduction, $\Omega = \mathbb{C}^n$ ($n = 1, 2, \ldots$ will be fixed in the sequel) and, for $\alpha > 0$,

$$d\mu_\alpha = (\alpha/\pi)^n e^{-\alpha|z|^2} dm. \tag{7.1}$$

We define F^2_α (Fock space) as the Hilbert space $A^2(\mu_\alpha)$.

More generally, let L^p_α be the space of measurable functions f on $\mathbb{C}^n$ such that $f(z)e^{-\alpha|z|^2/2} \in L^p(m)$, and let F^p_α be the subspace of entire functions. (We normalize the norms so that $\|1\| = 1$. In any case, the results below in general hold only up to equivalence of norms).

Remark 7.1. Note that L^p_α is not the same as $L^p(\mu_\alpha)$ unless $p = 2$; in fact, $L^p(\mu_\alpha) = L^p_{2\alpha/p}$. The parametrization L^p_α is, as we will see, very natural. We return to $L^p(\mu_\alpha)$ is Section 9.

Remark 7.2. In our analysis it is natural to consider the whole scale of spaces F^p_α at this time. The parameter α which plays something like the rôle of Planck's constant, is of course devoid of intrinsic interest. Notice that the dilation $f \mapsto f((\beta/\alpha)^{1/2} z)$ maps F^p_α into F^p_β isometrically. This is exploited several times below.

Whenever necessary, we add a subscript α to the notation. Thus

$$\langle f, g \rangle_\alpha = \int f\bar{g}\, d\mu_\alpha,$$

K_α is the reproducing kernel in F^2_α, etc.

It is easy to see that $\{z^\gamma\}$, where γ ranges over all multi-indices, is an orthogonal basis in F^2_α and that $\|z^\gamma\|^2_\alpha = \alpha^{-|\gamma|}\gamma!$. Hence, by (1.9),

$$K_\alpha(z, w) = \sum z^\gamma \bar{w}^\gamma \alpha^\gamma/\gamma! = e^{\alpha\langle z, w\rangle}. \tag{7.2}$$

($\langle z, w\rangle = \sum_1^n z_i \bar{w}_i$ is the scalar product in $\mathbb{C}^n$).

It is easy to see that, for each $w \in \mathbb{C}^n$, the mapping $C_\alpha(w)$ defined by

$$C_\alpha(w)f(z) = f(z - w)e^{\alpha\langle z, w\rangle - \alpha|w|^2/2} \tag{7.3}$$

is an isometry of F^p_α (and L^p_α) onto itself, $1 \leqslant p \leqslant \infty$. Further,

$$C_\alpha(w_1 + w_2) = C_\alpha(w_1)C_\alpha(w_2)e^{i\alpha \operatorname{Im}\langle w_1, w_2\rangle}. \tag{7.4}$$

Hence $(w, t) \to e^{i\alpha t}C_\alpha(w)$ is a unitary representation of the Heisenberg group in F^2_α. (Recall that the Heisenberg group is $\mathbb{C}^n \times \mathbb{R}$ with the group law $(z, t) \circ (w, s) = (z + w, t + s - \operatorname{Im}\langle z, w\rangle)$).

In the notation of Section 2, $G(\mu_\alpha)$ contains the group of translations of $\mathbb{C}^n$, and the corresponding subgroup of $G^*(\mu_\alpha)$ is essentially the Heisenberg group. (It is the quotient group $\mathbb{C}^n \times \mathbb{T} \cong (\mathbb{C}^n \times \mathbb{R})/2\pi\mathbb{Z}$). In particular, $G(\mu_\alpha)$ is transitive. Furthermore, $G(\mu_\alpha)$ obviously contains the group $U(n)$ of linear isometries, which satisfies (2.6) for $z = 0$.

Proposition 3.1 shows that V0 – V4 holds, so our theory is applicable.

Let us identify the notations in §3. $\mu = \mu_\alpha$ and $K = K_\alpha$ are given above. Hence

$$d\lambda = e^{\alpha|z|^2}\, d\mu_\alpha = (\alpha/\pi)^n\, dm,$$

a constant multiple of the Lebesgue measure, and

$$d\nu = e^{-\alpha|z|^2} d\mu_\alpha = (\alpha/\pi)^n e^{-2\alpha|z|^2}\, dm = 2^{-n}\, d\mu_{2\alpha}. \tag{7.5}$$

Thus, e.g. by Proposition 1.2, the reproducing kernel for ν is

$$L = 2^n K_{2\alpha} = 2^n K^2_\alpha \tag{7.6}$$

which gives a direct proof of V2 and shows that $\varkappa = 2^n$. Q is the orthogonal projection onto

$$A^2(\mu) = F^2_{2\mu}; \tag{7.7}$$

hence $Q = P_{2\alpha}$. By (7.2), $w(z) = e^{-\alpha|z|^2}$, and thus $L^p_\omega = L^p_{2\alpha}$ and $A^p_\omega = F^p_{2\alpha}$. If we write $f^\infty_\alpha = \{f \in \mathcal{H}(\mathbb{C}^n): f(z) = o(e^{\alpha|z|^2/2})$ as $|z| \to \infty\}$, then $a^\infty_\omega = f^\infty_{2\alpha}$ by Corollary 3.8.

Thus translating, and replacing α by $\alpha/2$, the results of Section 3 yield the following for every $\alpha > 0$.

Theorem 7.1. *P_α, defined by*

$$P_\alpha f(z) = \int e^{\alpha\langle z, w\rangle} f(w)\, d\mu_\alpha(w), \tag{7.8}$$

is a bounded self-adjoint projection of

$$L^p_\alpha \quad \text{onto} \quad F^p_\alpha, \qquad 1 \leqslant p \leqslant \infty. \quad \square$$

Theorem 7.2. *If $1 \leqslant p \leqslant q \leqslant \infty$, then*

$$F^p_\alpha \subset F^q_\alpha \subset f^\infty_\alpha \subset F^\infty_\alpha.$$

This first and second inclusions have dense ranges. $\square$

Theorem 7.3. *If* $1 \leqslant p_0 \leqslant p_1 \leqslant \infty$ *and* $0 \leqslant \theta \leqslant 1$,

$$[F_\alpha^{p_0}, F^{p_1}]_\theta = (F_\alpha^{p_0}, F_\alpha^{p_1})_{\theta p_\theta} = F_\alpha^{p_\theta},$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. □

Theorem 7.4. $(F_\alpha^p)^* \cong F_\alpha^{p'}$, $1 \leqslant p < \infty$, *and* $(f_\alpha^\infty)^* \cong F_\alpha^1$ *with the pairing* $\langle\ \rangle_\alpha$. □

Since $e^{\alpha z^2/2} \in F_\alpha^\infty \backslash f_\alpha^\infty$, the spaces F_α^1, F_α^∞ and f_α^∞ are not reflexive. It is also easily seen that $F_\alpha^p \neq F_\alpha^q$ when $p \neq q$.

We turn to Hankel forms in F_α^2. Let H_b^β, $\beta > 0$, denote the Hankel form with symbol b with respect to μ_β (i.e. H_b^ξ with $\xi = \mu_\beta$ in out general notation);

$$H_b^\beta(f, g) = \int \bar{b} f g \, d\mu_\beta \tag{7.9}$$

(suitably interpreted).

Thus $H_b = H_b^\alpha$ and, by (7.5), $\Gamma_b = 2^{-n} H_b^{2\alpha}$. Note that (6.2) and (6.6) hold because $K_z K_w = K^2_{(z+w)/2}$. Furthermore, by (6.7), (7.6), (7.2) and (7.8),

$$\begin{aligned} Q\left(b \frac{d\mu_\beta}{d\nu}\right)(z) &= \int b \bar{L}_z \, d\mu_\beta = \int b(w) 2^n e^{2\alpha\langle z, w\rangle} \, d\mu_\beta(w) \\ &= 2^n \int b(w) K_\beta\left(\frac{2\alpha}{\beta} z, w\right) d\mu_\beta(w) \\ &= 2^n P_\beta b\left(\frac{2\alpha}{\beta} z\right). \end{aligned} \tag{7.10}$$

Thus, if we restrict attention to analytic b,

$$Q\left(b \frac{d\mu_\beta}{d\nu}\right) \in A_\omega^p = F_{2\alpha}^p \Leftrightarrow b\left(\frac{2\alpha}{\beta} z\right) \in F_{2\alpha}^p \Leftrightarrow b \in F^p_{\beta^2/2\alpha}. \tag{7.11}$$

Consequently, the results of §§4 and 6 yield

Theorem 7.5. *Suppose that b is an entire function on $\mathbb{C}^n$ and $\alpha > 0$, $\beta > 0$, $1 \leqslant p \leqslant \infty$. Then*

(*a*) *$H_b^\beta \in S_p(F_\alpha^2)$ if and only if $b \in F^p_{\beta^2/2\alpha}$, i.e. if and only if $b(z) e^{-(\beta^2/4\alpha)|z|^2} \in L^p(dm)$. The respective norms are equivalent within constants.*

(*b*) *H_b^β is compact if and only if $b \in f^\infty_{\beta^2/2\alpha}$, i.e. if and only if $b(z) = o(e^{(\beta^2/4\alpha)|z|^2})$.*

(*c*) *The Hankel projection is bounded in every S_p.*

In particular, $\Gamma_b \in S_p$ if and only if $b \in F_{2\alpha}^p$, and $H_b \in S_p$ if and only if $b \in F^p_{\alpha/2}$. □

(As the family $\{H_b^\beta\}$ is independent of β, there is only one Hankel projection in $S_\infty(F_\alpha^2)$). Note the formula

$$H_b^\beta = H^\gamma_{b((\gamma/\beta)z)}, \qquad \beta, \gamma > 0, \tag{7.12}$$

which is proved as (7.10), or by checking $f = K_{\beta,z}$, $g = K_{\beta,w}$.

Remark 7.3. Strictly speaking, the argument above presumes $\beta < 4\alpha$, because otherwise e.g. the integral

$$\int \bar{b} K_z K_w \, d\mu_\beta$$

may diverge for $b \in F^\infty_{\beta 2/2\alpha}$. Theorem 7.5 is true for all β with a suitable interpretation of H_b^β (e.g. by (7.12)).

We may also study the Hankel form (7.9) when f and g are in two different Fock spaces $F^2_{\alpha_1}$ and $F^2_{\alpha_2}$. (Cf. Feldman and Rochberg (1986)).

Theorem 7.6. *Let* $1 \leqslant p \leqslant \infty$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$ *and assume that b is an entire function. Then* $H_b^\beta \in S_p(F^2_{\alpha_1} \times F^2_{\alpha_2})$ *if and only if* $b \in F^p_{\beta 2/(\alpha_1 + \alpha_2)}$.

PROOF. The case $\beta = \alpha_1 + \alpha_2$ is proved exactly as in §4, using the fact that $K_{\alpha_1+\alpha_2} = K_{\alpha_1} K_{\alpha_2}$. The general case follows by (7.12). □

Theorem 7.5 also generalizes to multi-linear forms as is shown in Section 5.

Theorem 7.7. *Let* $m > 2$, $\alpha > 0$, $\beta > 0$, $p = 1, 2$ *or* ∞, *and* $b \in \mathcal{H}(\mathbb{C}^n)$. *Then* $\int \bar{b} f_1 \cdot \ldots \cdot f_m \, d\mu_\beta$ *is an* S_p *multilinear form on* F^2_α *if and only if* $b \in F^p_{\beta 2/m\alpha}$.

PROOF. By Theorem 5.2 if $\beta = m\alpha$; the general case follows by a multilinear version of (7.12). □

We end this section with some remarks on the norms in Theorem 7.5 and their dependence on n. For simplicity we take $\beta = \alpha$; the general case is covered by (7.12). We obtain from the estimates in §4 (cf. Remark 4.1) by straight-forward computations.

$$A_p \|b\|_{F^p_{\alpha/2}} \leqslant \|H_b\|_{S_p} \leqslant B_p \|b\|_{F^p_{\alpha/2}} \tag{7.13}$$

with

$$A_p = 2^{n \min(1/p,\, 1-1/p)} p^{-n/p}$$

and

$$B_p = 2^{n \max(1/p,\, 1-1/p)} p^{-n/p}.$$

In particular, $\|H_b\|_{S_2} = \|b\|_{F^2_{\alpha/2}}$. However, if $p \neq 2$,

$$b_p / A_p = 2^{n|1-2/p|} \to \infty \quad \text{as} \quad n \to \infty$$

(a consequence of the fact that $\varkappa = 2^n \to \infty$ as $n \to \infty$). The constants given above are not best possible, as we soon will see for $p = 4$, but simple examples (take $b = 1, z, \dots$) show that $\|b\|$ and $\|H_b\|$ are not strictly proportional for any $p \neq 2, 4$, even when $n = 1$. By considering symbols of the type $b(z_1) \cdot b(z_2) \cdot \dots \cdot b(z_n)$ (i.e. tensor products), we then easily see that it is impossible to have A_p and B_p in (7.13) independent of n, except when $p = 2$ or 4. Surprisingly, however, there is an exact result for $p = 4$.

Theorem 7.8. *It b is entire, then*

$$\|H_b\|_{S_2} = \|b\|_{F^2_{\alpha/2}} = \|b\|_{L^2(\mu_{\alpha/2})}$$

and

$$\|H_b\|_{S_4} = \|b\|_{F^4_{\alpha/2}} = \|b\|_{L^4(\mu_\alpha)}.$$

PROOF. The S_2 result was given above. Let us for notational convenience assume $n = 1$ and $\alpha = 1$, and let

$$b(z) = \sum_0^N b_j z^j$$

be a polynomial. $\{z^j(j!)^{-1/2}\}_0^\infty$ is an ON-basis in F^2_1. In this basis, H_b has the matrix representation $\{h_{jk}\}_{j,k \geq 0}$ with

$$h_{jk} = H_b(z^j(j!)^{-1/2}, z^k(k!)^{-1/2}) = \bar{b}_{j+k}(j+k)!(j!)^{-1/2}(k!)^{-1/2}.$$

Let $\tilde{H}_b$ be the corresponding linear operator $F^2_1 \to (F^2_1)^*$. $\tilde{H}^*_b\tilde{H}_b$ corresponds to the matrix $\{\sum h_{ji}\bar{h}_{jk}\}_{ik}$. Thus

$$\begin{aligned} \|H_b\|^4_{S_4} &= \|\tilde{H}^*_b\tilde{H}_b\|^2_{S_2} = Tr\tilde{H}^*_b\tilde{H}_b\tilde{H}^*_b\tilde{H}_b \\ &= \sum_{ijkl} h_{ji}\bar{h}_{jk}h_{lk}\bar{h}_{li} \qquad (7.14) \\ &= \sum_{ijkl} \bar{b}_{i+j}b_{j+k}\bar{b}_{k+l}b_{l+i}\frac{(i+j)!(j+k)!(k+l)!(l+i)!}{i!j!k!l!}. \end{aligned}$$

Let $a_i = b_i \cdot i!$. Then (7.14) may be written

$$\|H_b\|^4_{S_4} = \sum_{m=0}^\infty \frac{1}{m!}\Sigma_m,$$

with

$$\Sigma_m = \sum_{i+j+k+l=m} \frac{m!}{i!j!k!l!}\bar{a}_{i+j}a_{j+k}\bar{a}_{k+1}a_{l+i}.$$

A combinatorial argument, which we omit, shows that

$$\Sigma_m = \left| \sum_{p=0}^{m} \binom{m}{p} a_p a_{m-p} \right|^2 = \left| \sum_{p=0}^{m} m! b_p b_{m-p} \right|^2.$$

Hence

$$\|H_b\|_{S_4}^4 = \sum_{m=0}^{\infty} m! \left| \sum_{p=0}^{\infty} b_p b_{m-p} \right|^2 = \left\| \sum_{m=0}^{\infty} \left(\sum_{p=0}^{\infty} b_p b_{m-p} \right) z^m \right\|_{L^2(\mu_1)}^2$$

$$= \|b^2\|_{L^2(\mu_1)}^2 = \|b\|_{L^4(\mu_1)}^4 \quad \square$$

If we study the Fock space in infinitely many dimensions (a well-known object in physics), we obtain (at least formally, ignoring all questions of definition etc.)

$$H_b \in S_2 \Leftrightarrow b \in L^2(\mu_{\alpha/2}) \quad \text{and} \quad H_b \in S_4 \Leftrightarrow b \in L^2(\mu_\alpha).$$

We repeat that Theorem 7.8 does not extend to any other p. (In particular, interpolation between $p = 2$ and $p = 4$ is not possible!)

Problem. What happens on the infinite-dimensional Fock space for $p \neq 2, 4$ (in particular for $p = \infty$)?

8. Decomposition, Approximation and (Pointwise) Interpolation

Theorem 6.1 yields, replacing α by $\alpha/2$, the following.

Theorem 8.1. *$f \in F_\alpha^1$ if and only if*

$$f(z) = \sum_1^\infty a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2},$$

for some sequences $\{z_j\} \subset \mathbb{C}^n$ and $\{a_j\} \in l^1$. $\square$

Let the Heisenberg group act on functions on $\mathbb{C}^n$ by $(w, t) \to e^{iat} C_\alpha(w)$, cf. (7.3)-(7.4). Then, by Section 6, we have the following.

Corollary 8.1. *F_α^1 is the smallest Heisenberg invariant Banach space that contains the constant functions. F_α^∞ is the largest Heisenberg invariant space such that $f \to f(0)$ is continuous.* $\square$

Theorem 8.1 says that the functions $k_z = K_z / \|K_z\|$ are atoms in F_α^1. We will show that suitable subsets of them can be employed as atoms in F_α^p also for $p > 1$.

We will call a set of points $\{z_j\} \subset \mathbb{C}^n$ ϵ-dense if every point of $\mathbb{C}^n$ is within distance ϵ of some z_j, i.e. if every ball with radius ϵ contains at least one z_j. We call the set separated if there exists a constant M such that any ball with radius 1 contains at most M points. (Any other fixed radius would do as well). In particular $\{z_j\}$ is separated if $\inf_{i \neq j} |z_i - z_j| > 0$. The lattice $\epsilon d^{-1/2}\mathbb{Z}^{2n}$ is ϵ-dense and separated.

Theorem 8.2. *There exists $\epsilon_0 > 0$ such that if $\{z_j\}$ is ϵ-dense with $\epsilon < \epsilon_0 \alpha^{-1/2}$ and separated, and $1 \leqslant p \leqslant \infty$, then $f \in F^p_\alpha$ if and only iff*

$$f(z) = \sum_1^\infty a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2} \tag{8.1}$$

with $\{a_j\} \in l^p$ (and similarly for f^∞_α and c_0). The norm $\|f\|_{F^p_\alpha}$ is equivalent to $\inf \|\{a_j\}\|_{l^p}$ *within constants depending on α, ϵ and the constant in the separation definition.*

Remark 8.1. The coefficients a_j are not unique, but the proof shows that they may be chosen as continuous linear functions of f.

Remark 8.2. A characterization of the lattices $\{z_i\}$ for which $\{e^{\alpha\langle z, z_j\rangle}\}$ span F^2_α is given by Bargmann et al. (1971).

PROOF. We assume, without loss of generality, that $\alpha = 1$. Let $G = \mathbb{C}^n \times \mathbb{T}$ be the quotient group of the Heisenberg group defined by

$$(z, u) \circ (w, v) = (z + w, uv \exp(-i\,Im\langle z, w\rangle))$$

cf. the discussion after (7.4). As Haar measure on G we choose $\mathrm{dm}(z)|du|/2\pi^{n+1}$.

Given a function f on $\mathbb{C}^n$ we define Tf on G by

$$Tf(z, u) = uf(z)e^{-|z|^2/2}, \qquad (z, u) \in G = \mathbb{C}^n \times \mathbb{T}.$$

T is a linear isometry of F^p_1 onto a subspace of $L^p(G)$ (with the norm in F^p_1 suitably renormalized).

Let $\phi = T1$. We write in this proof

$$g = (z, u) \quad \text{and} \quad h = (w, v).$$

Thus $\phi(g) = ue^{-|z|^2/2}$ and

$$\begin{aligned}\phi(gh^{-1}) &= \phi(z - w, uv^{-1}e^{i\,Im\langle z, w\rangle}) = uv^{-1}e^{i\,Im\langle z, w\rangle - |z - w|^2/2} \\ &= uv^{-1}e^{\langle z, w\rangle - |z|^2/2 - |w|^2/2}.\end{aligned} \tag{8.2}$$

Consequently, if $F = Tf$, the reproducing formula (Theorem 7.1) yields,

$$\begin{aligned}\phi * F(g) &= \int_G \phi(gh^{-1})F(h)\,dh \\ &= \int_G uv^{-1}e^{\langle z,w\rangle - |z|^2/2 - |w|^2/2} vf(w)e^{-|w|^2/2}\,dm\,(w)|dv|/2\pi^{n+1} \\ &= ue^{-|z|^2/2}\int f(w)e^{\langle z,w\rangle}\,d\mu_1(w) \\ &= ue^{-|z|^2/2}f(z) \\ &= F(g).\end{aligned} \tag{8.3}$$

Let $N = [2\pi/\epsilon] + 1$ and $h_{jk} = (z_j, e^{2\pi ik/N})$, $1 \leqslant j < \infty$, $1 \leqslant k \leqslant N$. Partition G into disjoint sets G_{jk} such that $|hh_{jk}^{-1} - (0, 1)| \leqslant 2\epsilon$ when $h \in G_{jk}$. ($(0, 1)$ is the unity in G).

Define

$$RF = \left\{\int_{G_{jk}} F(g)\,dg\right\}_{1 \leq j \leq \infty,\, 1 \leq k \leq N}$$

and

$$\begin{aligned}S(\{a_{jk}\})(z) &= \sum_{j,k} a_{jk} e^{-2\pi ik/N + \langle z, z_j\rangle - |z_j|^2/2} \\ &= \sum_{j=1}^{\infty}\left(\sum_{k=1}^{N} a_{jk}e^{-2\pi ik/N}\right)e^{\langle z, z_j\rangle - |z_j|^2/2}.\end{aligned} \tag{8.4}$$

It is easily seen that $R: L^p(G) \to l^p$. Since

$$\begin{aligned}|S(\{a_{jk}\})(z)e^{-|z|^2/2}| &\leqslant \sum_{j,k} |a_{jk}|e^{-|z-z_j|^2/2} \\ &\leqslant N\sum_j e^{-|z-z_j|^2/2}\sup_{j,k}|a_{jk}| \\ &\leqslant C\sup|a_{jk}|,\end{aligned}$$

because $\{z_j\}$ is separated, $S(\{a_{jk}\}) \in F_1^\infty$ when $\{a_{jk}\} \in l^\infty$. Also,

$$\begin{aligned}\|S(\{a_{jk}\})\|_{F_1^1} &\leqslant \sum_{j,k}|a_{jk}|\;\|e^{\langle z, z_j\rangle}\|_{F_1^1}e^{-|z_j|^2/2} \\ &= C\|\{a_{jk}\}\|_{l^1}.\end{aligned}$$

By interpolation, $S: l^p \to F_1^p$, $1 \leqslant p \leqslant \infty$. Next we observe that, by (8.2),

$$\begin{aligned}TS(\{a_{jk}\})(g) &= \sum_{j,k} a_{jk}ue^{-2\pi ik/N}e^{\langle z, z_j\rangle - |z_j|^2/2 - |z|^2/2} \\ &= \sum_{j,k} a_{jk}\phi(gh_{jk}^{-1}).\end{aligned}$$

Hence, if $f \in F_1^\infty$ and $F = Tf$,

$$TSRF(g) = \sum_{j,k} \int_{G_{jk}} F(h)\, dh \phi(gh_{jk}^{-1})$$

and, by (8.3),

$$F(g) - TSRF(g) = \sum_{j,k} \int_{G_{jk}} (\phi(gh^{-1}) - \phi(gh_{jk}^{-1}))F(h)\, dh. \tag{8.5}$$

Define

$$\delta(g, \epsilon) = \sup \{ |\phi(g) - \phi(h)| : |g^{-1}h - (0, 1)| \leqslant \epsilon \}$$

and

$$\delta(\epsilon) = \int_G \delta(g, \epsilon)\, dg.$$

Note that $\delta(g, \epsilon) \to 0$ as $\epsilon \to 0$ for every g, and thus, by dominated convergence, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. (8.5) now yields

$$|F(g) - TSRF(g)| \leqslant \sum_{j,k} \int_{G_{jk}} \delta(gh^{-1}, 2\epsilon)|F(h)|\, dh \leqslant \delta(2\epsilon) \|F\|_{L^\infty}.$$

Thus

$$\|Tf - TSRTf\|_{L^\infty} \leqslant \delta(2\epsilon) \|Tf\|_{L^\infty}.$$

Since T is an isometry of F_1^∞ into $L^\infty(G)$, this gives

$$\|I - SRT\|_{F_1^\infty} \leqslant \delta(2\epsilon).$$

Similarly, if $f \in F_1^1$,

$$\begin{aligned} \|F - TSRF\|_{L^1(G)} &\leqslant \iint \delta(gh^{-1}, 2\epsilon)|F(h)|\, dh\, dg \\ &= \int \delta(2\epsilon)|F(h)|\, dh \\ &= \delta(2\epsilon) \|F\|_{L^1(G)} \end{aligned}$$

and

$$\|I - SRT\|_{F_1^1} \leqslant \delta(2\epsilon).$$

It follows by interpolation (Theorem 7.3), that if ϵ is sufficiently small, $\|I - SRT\|_{F_1^p} \leqslant 1$, whence SRT is invertible and S maps l^p onto F_1^p, for every p, $1 \leqslant p < \infty$. The conclusion of the theorem follows easily. □

As a corollary we obtain an approximation definition of F_α^p. Let P_N be the set of entire functions of the type $\sum_1^N a_j e^{\langle z, z_j \rangle}$, with $a_j \in \mathbb{C}$ and $z_j \in \mathbb{C}^n$,

$j = 1, \ldots, N$. (P_N is not a linear space!). If $b \in P_N$, then H_b has rank $\leqslant N$; the converse is almost true, see Section 14.

Theorem 8.3. *Let $\alpha > 0$ and $1 \leqslant p < \infty$. Then $f \in F^p_\alpha$ if and only if $f \in F^\infty_\alpha$ and the sequence $\{d_N\}_0^\infty \in l^p$, where $d_N = \inf\{\|f - g\|_{F^\infty_\alpha} : g \in P_N\}$ is the distance from f to P_N in F^∞_α.*

PROOF. If $f \in F^p_\alpha$, then (8.1) holds for suitable sequences $\{z_j\}$ and $\{a_j\} \in l^p$. Reordering the sequences simultaneously, we may assume that $\{a_j\}$ is decreasing. Then

$$d_N \leqslant \left\| \sum_{N+1}^{\infty} a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2} \right\|_{F^\infty_\alpha} \leqslant C \|\{a_j\}_{N+1}^\infty\|_{l^\infty} = C|a_{N+1}|,$$

by another application of Theorem 8.2. Thus $\{d_N\} \in l^p$. Conversely, if $g \in P_N$ then H_g has rank $\leqslant N$ as a bilinear form on $F^2_{2\alpha}$. Thus, on that Hilbert space, $s_N(H_f) \leqslant \|H_f - H_g\|_{S_\infty(F^2_{2\alpha})} \leqslant C\|f - g\|_{F^\infty_\alpha}$, because of Theorem 7.5. Thus $s_N(H_f) \leqslant C d_N$. Hence $\{d_N\} \in l^p$ implies that $H_f \in S_p(F^2_{2\alpha})$ and, by Theorem 7.5 again, $f \in F^p_\alpha$. □

In the classical case $H^2(\mathbb{T})$, Adamjan, Arov and Kreĭn (1971) have proved that $s_N(H_f) = \inf\{\|H_f - H_g\| : H_g$ is a *Hankel* operator of rank $\leqslant N\}$. Theorems 8.3 and 7.5 suggest that something similar may be true on the Fock space too, possibly in a weaker version such as

$$S_N(H_f) \leqslant C_1 \inf\{\|H_f - H_g\| : g \in P_{C_2 N}\}.$$

Another consequence of Theorem 8.2 is the following weak factorization, cf. Theorem 6.2. We do not know whether strong factorization is possible (as for Bergman spaces by Horowitz (1977)).

Corollary 8.2. *Let $\alpha = \alpha_0 + \alpha_1$ and $1/p = 1/p_0 + 1/p_1$ with $\alpha_0, \alpha_1 > 0$, $p_0, p_1 \leqslant \infty$ and $1 \leqslant p < \infty$. Then $f \in F^p_\alpha$ if and only if $f = \sum_1^\infty a_i g_i h_i$, for some sequences $\{g_i\}$ and $\{h_i\}$ in the unit balls of $F^{p_0}_{\alpha_0}$ and $F^{p_1}_{\alpha_1}$, respectively. The norm of f is equivalent to $\inf \sum |a_i|$, extended over all such representations.*

PROOF. Let $\{z_j\}$ be an ϵ-dense lattice, with ϵ sufficiently small. Any $f \in F^p_\alpha$ has a representation of the type (8.1), and it is easily seen that it suffices to consider functions that has a finite representation

$$f = \sum_1^N a_j e^{\alpha\langle z, z_j\rangle - \alpha|z_j|^2/2}.$$

Let $b_j = |a_j|^{p/p_0} \operatorname{sign}(a_j)$ and $c_j = |a_j|^{p/p_1}$, and define, for each sequence $I = (\iota_1, \ldots, \iota_N)$ with each $\iota_j = \pm 1$,

$$g_I = \sum_j \iota_j b_j e^{\alpha_0 \langle z, z_j \rangle - \alpha_0 |z_j|^2/2}, \qquad h_I = \sum_j \iota_j c_j e^{\alpha_1 \langle z, z_j \rangle - \alpha_1 |z_j|^2/2}.$$

It is easily seen that

$$f = 2^{-N} \sum_I g_I h_I$$

and, by Theorem 8.2 again, for each I,

$$\| g_I \|_{F^{p_0}_{\alpha_0}} \leqslant C \| \{ b_j \} \|_{l^{p_0}} = C \| \{ a_j \} \|_{l^p}^{p/p_0} \quad \text{and} \quad \| h_I \|_{F^{p_1}_{\alpha_1}} \leqslant C \| \{ a_j \} \|_{l^p}^{p/p_1}.$$

The remaining, simple, details are left to the reader. □

Theorem 8.2 has also an interesting dual.

Theorem 8.4. *Let $\{z_j\}$ be ϵ-dense and separated with $\epsilon < \epsilon_0 \alpha^{-1/2}$. If $1 \leqslant p \leqslant \infty$ and $f \in F^p_\alpha$, then*

$$C_1 \| \{ f(z_j) e^{-\alpha |z_j|^2/2} \} \|_{l^p} \leqslant \| f \|_{F^p_\alpha} \leqslant C_2 \| \{ f(z_j) e^{-\alpha |z_j|^2/2} \} \|_{l^p}$$

with C_1 and C_2 depending only on α, ϵ and the constant in the separation definition.

PROOF. If $p > 1$, let p' be the conjugate exponent. The linear mapping

$$\{ a_j \} \to \sum a_j e^{-\alpha |z_j|^2/2} K_{z_j}$$

is by Theorem 8.2 a quotient mapping of $l^{p'}$ onto $F^{p'}_\alpha$, whence the adjoint map, which maps f to $\{ f(z_j) e^{-\alpha |z_j|^2/2} \}$ is an isomorphism of $F^p_\alpha \cong (F^{p'}_\alpha)^*$ (cf. Theorem 7.4) into l^p. If $p = 1$, we use c_0 and f^∞_α. □

In fact, the left inequality in (8.6) holds as soon as $\{z_j\}$ is separated and the right inequality holds as soon as $\{z_j\}$ is ϵ-dense, because then $\{z_j\}$ can be enlarged or reduced, respectively, to become both separated and ϵ_1-dense (for any $\epsilon_1 > \epsilon$). In particular, if $f \in F^p_\alpha$ vanishes on an ϵ-dense set (ϵ small enough), then it vanishes identically.

The right inequality in (8.6) is not valid without some a priori assumption on f. (E.g., if $n = 1$, Weierstrass' theorem shows that f may vanish at every z_j but not elsewhere). It is easy to show that the condition $f \in F^p_\alpha$ may be relaxed to $f \in F^\infty_\alpha$. We have proved that the mapping $f \to \{ f(z_j) e^{-\alpha |z_j|^2/2} \}$ maps F^p_α into l^p, if (and, as it is easily seen, only if) $\{z_j\}$ is separated. If the set is sufficiently well separated, this map is also onto.

Theorem 8.5. *There exists $D < \infty$ such that, if $\{z_j\}$ is a sequence in $\mathbb{C}^n$ with $\inf_{i \neq j} |z_i - z_j| > D\alpha^{-1/2}$, and $1 \leqslant p \leqslant \infty$, a sequence $\{a_j\}$ of complex numbers equals $\{f(z_j)\}$ for some $f \in F^p_\alpha$ if and only if $\{a_j e^{-\alpha|z_j|^2/2}\} \in l^p$.*

PROOF. We may assume that $\alpha = 2$. Define

$$Tf = \{f(z_j)e^{-|z_j|^2}\} \quad \text{and} \quad S\{a_j\} = \sum a_j e^{2\langle z, z_j\rangle - |z_j|^2}.$$

Let

$$\delta = \inf_{i \neq j} |z_i - z_j| > 0.$$

T maps F^p_α into l^p by Theorem 8.4, and it is easy to see, interpolating between $p = 1$ and $p = \infty$, that S maps l^p into F^p_α. Furthermore,

$$TS\{a_j\} = \left\{\sum_i a_i e^{2\langle z_j, z_i\rangle - |z_i|^2 - |z_j|^2}\right\}$$

and thus, again interpolating between $p = 1$ and $p = \infty$,

$$\|I - TS\|_{l^p} \leqslant \sup_i \sum_{j \neq i} |e^{2\langle z_j, z_i\rangle - |z_i|^2 - |z_j|^2}| = \sup_i \sum_{j \neq i} e^{-|z_i - z_j|^2} < 1,$$

provided δ in large enough. Hence TS is invertible and T is onto. $\square$

9. More on Projection, Duality and (Abstract) Interpolation

We have shown that the projection P_α is a bounded operator in $L^p_\alpha (1 \leqslant p \leqslant \infty)$, but it is also of interest to study the action of P_α on L^p_β when $\beta \neq \alpha$. In particular, this applies to the spaces $L^p(\mu_\alpha) = L^p_{2\alpha/p}$.

Theorem 9.1. *Let $\alpha > 0$, $\beta < 2\alpha$, $1 \leqslant p \leqslant \infty$. Then P_α maps L^p_β onto F^p_γ with $1/\gamma = 2/\alpha - \beta/\alpha^2$. P_α is not bounded on L^p_β unless $\beta = \alpha$.*

Remark 9.1. We may here allow $\beta \leqslant 0$. In particular, $P_\alpha(L^p) = F^p_{\alpha/2}$.

PROOF. We introduce explicitly the dilation and multiplication operators defined by

$$D_\delta f(z) = f(\delta z) \tag{9.1}$$
$$E_\epsilon f(z) = e^{\epsilon|z|^2} f(z), \tag{9.2}$$

the idea being that D_δ maps F^p_α isometrically onto $F^p_{\delta^2\alpha}$ $(\alpha, \delta > 0)$ and E_ϵ maps L^p_β isometrically onto $L^p_{\beta + 2\epsilon}$ $(\beta, \epsilon \in \mathbb{R})$. Furthermore, it follows by the same argument as in (7.10) that, for any $\alpha, \beta > 0$,

$$P_\alpha E_{\alpha-\beta} f(z) = \left(\frac{\alpha}{\beta}\right)^n P_\beta f\left(\frac{\alpha}{\beta} z\right),$$

(at least when $f \in L_\gamma^\infty$ for some $\gamma < 2\beta$), i.e.

$$P_\alpha E_{\alpha-\beta} = \left(\frac{\alpha}{\beta}\right)^n D_{\alpha/\beta} P_\beta. \tag{9.3}$$

Hence, substituting $2\alpha - \beta$ for β in (9.3),

$$\begin{aligned} P_\alpha(L_\beta^p) &= P_\alpha E_{\beta-\alpha}(L_{2\alpha-\beta}^p) = D_{\alpha/(2\alpha-\beta)} P_{2\alpha-\beta}(L_{2\alpha-\beta}^p) \\ &= D_{\alpha/(2\alpha-\beta)}(F_{2\alpha-\beta}^p) = F_{\alpha^2/(2\alpha-\beta)}^p. \end{aligned}$$

The last statement follows because $\gamma > \beta$ unless $\beta = \alpha$. □

Applying the theorem to $L^p(\mu_\alpha)$, we obtain (and refine) a result by Sjögren (1976).

Corollary 9.1. *Let $\alpha > 0$, $1 < p \leqslant \infty$ and $1/p + 1/p' = 1$. Then P_α maps $L^p(\mu_\alpha)$ onto $F_{p'\alpha/2}^p$. Hence P_α maps $L^p(\mu_\alpha)$ into $L^q(\mu_\alpha)$ $(0 < q \leqslant \infty)$ if and only if either $q < 4/p'$ or $p = q = 2$. P_α does not map $L^p(\mu_\alpha)$ into itself unless $p = 2$.*

PROOF. We may assume that $\alpha = 1$. Theorem 9.1 shows that P_1 maps $L^p(\mu_1) = L_{2/p}^p$ onto F_γ^p with $\gamma = (2 - 2/p)^{-1} = p'/2$, which is contained in $L^q(\mu_1) = L_{2/q}^q$ if and only if $2/q > \gamma = p'/2$ or $2/q = p'/2$ and $q \geqslant p$. Since $pp' > 4$ unless $p = 2$ (e.g. by the inequality between geometric and harmonic means), the latter case entails $p = q = 2$. □

Let $A^p(\mu_\alpha)$ be the space of analytic functions in $L^p(\mu_\alpha)$. Thus $A^p(\mu_\alpha) = F_{2\alpha/p}^p$. Sjögren (1976) used the above result to show that the dual of $A^p(\mu_\alpha)$ (for the pairing $\langle\ \rangle_\alpha$) is strictly larger than $A^{p'}(\mu_\alpha)$, unless $p = 2$. More generally, now we can prove the following.

Theorem 9.2. *Let $\alpha, \beta > 0$, $1 \leqslant p < \infty$ and $1/p + 1/p' = 1$. Then $(F_\beta^p)^* \cong F_{\alpha^2/\beta}^{p'}$ with the pairing $\langle\ \rangle_\alpha$. Similarly, $(f_\beta^\infty)^* \cong F_{\alpha^2/\beta}^1$.*

Remark 9.2. $\int f\bar{g}\, d\mu_\alpha$ does not necessarily converge when $f \in F_\beta^p$, $g \in F_{\alpha^2/\beta}^{p'}$, but $\langle\ \rangle_\alpha$ is easily extended. We omit the details.

PROOF. Since

$$\langle f, g\rangle_\beta = \left(\frac{\beta}{\alpha}\right)^n \langle f, E_{\alpha-\beta} g\rangle_\alpha, \qquad (L^p_\beta)^* \cong E_{\alpha-\beta} L^{p'}_\beta$$

with this pairing. The Hahn-Banach theorem and the fact that

$$\langle f, g\rangle_\alpha = \langle f, P_\alpha g\rangle_\alpha$$

shows, using (7.3), that

$$(F^p_\beta)^* \cong P_\alpha E_{\alpha-\beta}(L^{p'}_\beta) = D_{\alpha/\beta} P_\beta (L^{p'}_\beta) = D_{\alpha/\beta} F^{p'}_\beta = F^{p'}_{\alpha^2/\beta}.$$

$(f^\infty_\beta)^* \cong F^1_{\alpha^2/\beta}$ is proved similarly (using $C_0^* = M$), or by letting $p \to \infty$, noting that all constants stay bounded. □

Corollary 9.2. *If* $1 \leqslant p < \infty$, *then* $A^p(\mu_\alpha)^* \cong F^{p'}_{p\alpha/2}$, *with the pairing* $\langle\ \rangle_\alpha$. □

If $p \neq 2$, $p\alpha/2 > 2\alpha/p'$, whence $F^{p'}_{p\alpha/2} \supsetneq F^{p'}_{2\alpha/p'} = A^{p'}(\mu_\alpha)$, and we recover Sjögren's result.

We may now extend the interpolation theorem (Theorem 7.3) for the complex method. Since Fock spaces with different values of α are related by dilations, this is related to interpolation between spaces of functions defined in different discs, cf. Lions and Peetre (1964).

Theorem 9.3. *Let* $\alpha_0, \alpha_1 > 0$, $1 \leqslant p_0, p_1 \leqslant \infty$, *and* $0 < \theta < 1$. *Let* $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ *and* $\alpha_\theta = \alpha_0^{1-\theta}\alpha_1^\theta$. *Then*

$$[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]_\theta = F^{p_\theta}_{\alpha_\theta}, \qquad p_\theta < \infty, \tag{9.4}$$

$$[F^\infty_{\alpha_0}, F^\infty_{\alpha_1}]_\theta = f^\infty_{\alpha_\theta}, \qquad \alpha_0 \neq \alpha_1, \tag{9.5}$$

(*Note that* $p_\theta = \infty$ *if and only if* $p_0 = p_1 = \infty$). (9.4) *and* (9.5) *hold also with* $F^\infty_{\alpha_j}$ *replaced by* $f^\infty_{\alpha_j}$ ($j = 0$ *or* 1) *on the left hand side.*

PROOF. Define, for $\zeta \in \mathbb{C}$, $T_\zeta = D_{\alpha_0/\alpha_1}(\zeta - \theta)/2$, i.e.

$$T\zeta f(z) = f\left(\left(\frac{\alpha_0}{\alpha_1}\right)^{(\zeta-\theta)/2} z\right). \tag{9.6}$$

T_ζ is an isometry of $F^{p_0}_{\alpha_0}$ onto $F^{p_0}_{\alpha_\theta}$ when $\operatorname{Re}\zeta = 0$, and of $F^{p_1}_{\alpha_1}$ onto $F^{p_1}_{\alpha_\theta}$ when $\operatorname{Re}\zeta = 1$.

Furthermore, $T_\zeta f$ and $T_\zeta^{-1} f = T_{-\zeta} f$ are analytic in ζ (when f is analytic). Hence, by the abstract Stein interpolation therorem, see e.g. Cwikel and Janson (1984), Theorem 1, $T_\theta = I$ is an isometry of $[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]_\theta$ onto

$$[F^{p_0}_{\alpha_\theta}, F^{p_1}_{\alpha_\theta}]_\theta = F^{p_\theta}_{\alpha_\theta}$$

(using Theorem 7.3), provided p_0 and $p_1 < \infty$.

The same argument holds if $p_0 = \infty$ or $p_1 = \infty$, provided we use f^∞ instead of F^∞. However, an extra argument is needed for F^∞, because e.g. $t \to T_{it}f$ is, in general, not a continuous map of $\mathbb{R}$ into $F^\infty_{\alpha_0}$ when $f \in F^\infty_{\alpha_0}$, cf. Cwikel and Janson (1984). One possibility is to use duality (Theorem 9.2) and the result just proved (with p_0', α_0^{-1} etc.) to conclude that, see e.g. Bergh & Löfström (1976), Theorem 4.5.1,

$$[F^{p_0}_{\alpha_0}, F^{p_1}_{\alpha_1}]^\theta = F^{p_\theta}_{\alpha_\theta}, \qquad 1 \leqslant p_0, p_1 \leqslant \infty. \tag{9.7}$$

Berg (1979) proved that, for any Banach couple $[X_0, X_1]_\theta$ equals the closed hull of $X_0 \cap X_1$ in $[X_0, X_1]^\theta$, which gives (9.4) and (9.5) from (9.7). □

Remark 9.3. Also (9.7) remains valid if $F^\infty_{\alpha_j}$ is replaced by $f^\infty_{\alpha_j}$ on the left hand side (except in the trivial case $p_0 = p_1 = \infty$, $\alpha_0 = \alpha_1$). This follows from (9.4) and (9.7) when p_0 or p_1 is finite. That

$$[f^\infty_{\alpha_0}, f^\infty_{\alpha_1}] = F^\infty_{\alpha_\theta} \qquad (\alpha_0 \neq \alpha_1)$$

follows directly from the definition (Bergh & Löfström (1976), Chapter 4), taking

$$g(w) = \int_0^w T_{-\zeta} f \, d\zeta$$

with T as in (9.6) and $f \in F^\infty_{\alpha_\theta}$.

Note that α_θ is the (weighted) geometric mean of α_0 and α_1, while

$$[L^{p_0}_{\alpha_0}, L^{p_1}_{\alpha_1}]_\theta = L^{p_\theta}_{(1-\theta)\alpha_0 + \theta\alpha_1}$$

with the arithmetic mean of α_0 and α_1.

Exercise 9.1. Let us review Theorem 9.1 in the light of Theorem 9.3. First, Theorem 9.1 plainly may be restated as follows: Given α, the set of pairs (β, γ) such that $P_\alpha : L^p_\beta \to F^p_\gamma$ is precisely given by the inequality $\alpha^2/\gamma \leqslant 2\alpha - \beta$.

We leave it to the reader to show that this region cannot be enlarged by the interpolation theorem.

10. Addenda

10.1. Convolutions. The Hankel operator, which is a modified multiplication, is surprisingly also a convolution operator on the Fock space.

Theorem 10.1. *If $b \in F^{\infty}_{\alpha/2}$ and $f, g \in F^2_{\alpha}$, then*

$$\tilde{H}_b f(z) = \langle b(z + \bullet), f\rangle_{\alpha} = \int b(z + w)\overline{f(w)}\, d\mu_{\alpha}(w) \tag{10.1}$$

and

$$H_b(f, g) = \iint \overline{b(z + w)} f(z) g(w)\, d\mu_{\alpha}(z)\, d\mu_{\alpha}(w). \tag{10.2}$$

PROOF. Since both sides of (10.1) are continuous anti-linear functionals of f, it suffices to verify the formula when $f = K_{\zeta}$ for some ζ. Then

$$\begin{aligned}\tilde{H}_b K_{\zeta}(z) &= \langle \tilde{H}_b K_{\zeta}, K_z\rangle = \overline{H_b(K_z, K_{\zeta})} = \langle b, K_z K_{\zeta}\rangle_{\alpha} \\ &= \langle b, K_{z+\zeta}\rangle = b(z + \zeta) \\ &= \int b(z + w) K_{\zeta}(w)\, d\mu_{\alpha}(w).\end{aligned}$$

(10.2) follows. □

10.2. Finite rank. It follows from (10.1) that (for $b \in F^{\infty}_{\alpha/2}$) $\tilde{H}_b K_w = b(\bullet + w)$. Hence the linear span of the set of translates of b is a dense subspace of the range of $\tilde{H}_b$. In particular, invoking our general Kronecker's theorem (Corollary 14.1):

Theorem 10.2. *The following are equivalent* (*for b entire*).

(i) *H_b has finite rank.*
(ii) *Span $\{b(\bullet + w)\}$ has finite dimension.*
(iii) *$b(z) = \sum_{j=1}^{m} \sum_{|\nu| \le k_j} c_{j\nu} z^{\nu} e^{z\bar{w}_j}$ for some m, k_j, w_j.*
(iv) *$b \in \bar{P}_N$ for some N, with P_N as in §8.* (*Closure e.g. in F^{∞}_{α}*).

If $n = 1$, we can add:

(v) *$Db = 0$ for some constant coefficient linear differential operator D.*

10.3. An abstract characterization. Let H be any Hankel form on $F^2_{\alpha}(\mathbb{C})$. Then it is abstractly characterized by (cf. the introduction) the property

$$H(zf, g) = H(f, zg).$$

Notice that in terms of the associated *anti-linear* Hankel operator $\tilde{H}$ this can be written as

$$\tilde{H}A^* = A\tilde{H}$$

where, taking $\alpha = 1$, A^* and A are the creation and annihilation operator respectively, $Af = f'$, $A*f = zf$. This should be contrasted with the well-known abstract characterization of Hankel operators $\tilde{H}_b: H^2(\mathbb{T}) \to H^2(\mathbb{T})$ using the shift operator S, $Sf = zf$, and its compression to $H^2_-(\mathbb{T})$ (cf. e.g. Nikol'skii (1986), p. 180).

10.4. Other Gaussian measures. Let A be a positive definite matrix. Then we may define

$$\mu_A = \frac{\det A}{\pi^N} e^{-\langle Az, z\rangle} dm$$

and $F^2_A = A^2(\mu_A)$. The results of §7 extend immediately (because F^2_A is mapped onto F^2_α by a (linear) change of variables). The results of §9 extend too

$$(F^p_A)^* \cong F^{p'}_{A^{-1}} \quad \text{for} \quad 1 \leqslant p < \infty \qquad (\text{while } L^p_A \cong L^{p'}_{2-A})$$

in the duality $\langle\ \rangle_{\mu_1}$, and

$$[F^{p_0}_{A_0}, F^{p_1}_{A_1}]_\theta = F^{p_\theta}_{A_\theta} \qquad (p_0 < \infty)$$

for some A_θ (a «geometric mean» of A_0 and A_1).

11. Fock Space with a Different Gauge

In this section we for simplicity consider only the case $n = 1$. We write $z = x + iy$ and $w = u + iv$.

The gauge transformation $f(z) \to e^{-\alpha z^2/2} f(z)$ maps F^p_α onto the space of entire functions g such that

$$|g(z)| e^{\operatorname{Re} \alpha z^2/2 - \alpha |z|^2/2} = |g(z)| e^{-\alpha y^2} \in L^p(dx\, dy).$$

We denote this space by G^p_α. In particular, G^2_α is the Hilbert space

$$A^2\left(\frac{\alpha}{\pi} e^{-2\alpha y^2} dx\, dy\right).$$

The reproducing kernel for G^2_α is, by Proposition 1.2 and (7.2),

$$K(z, w) = e^{-\alpha z^2/2 - \alpha \bar{w}^2/2 + \alpha z \bar{w}} = e^{-\alpha(z - \bar{w})^2/2} \tag{11.1}$$

Thus $K(z, z) = e^{-\alpha(2iy)^2/2} = e^{2\alpha y^2}$ and $\omega(z) = e^{-2\alpha y^2}$. Consequently, $A^p_\omega = G^p_{2\alpha}$, $1 \leqslant p \leqslant \infty$, cf. §7. It follows from (7.3) that the Heisenberg group acts isometrically in every G^p_α by $(w, t) \to e^{i\alpha t} C'_\alpha(w)$, with

$$\begin{aligned} C'_\alpha(w) g(z) &= e^{-\alpha z^2/2} e^{\alpha(z - w)^2/2} g(z - w) e^{\alpha z \bar{w} - \alpha |w|^2/2} \\ &= g(z - w) e^{i\alpha v(w - 2z)} \end{aligned} \tag{11.2}$$

Remark 11.1 From the group theory point of view this is something which lives «half way» between the Bargmann-Segal representation of the Heisenberg group, which lives in Fock space $F^2_\alpha(\mathbb{C})$, and the Heisenberg representation of the same group, which lives in $L^2(\mathbb{R})$.

Let $g \in G^2_\alpha$, let $g_y(x) = g(x+iy)$ and let $\hat{g} \in S'(\mathbb{R})$ be the Fourier transform of $g(x)$, $x \in \mathbb{R}$. Then $\hat{g}_y(\xi) = e^{-y\xi}\hat{g}(\xi)$ and, by Plancherel's theorem,

$$\begin{aligned} \|g\|^2_{G^2_\alpha} &= \frac{\alpha}{\pi}\int |f(z)|^2 e^{-2\alpha y^2}\,dx\,dy \\ &= \frac{\alpha}{\pi}\int \|g_y\|^2 e^{-2\alpha y^2}\,dy \\ &= \frac{\alpha}{2\pi^2}\int e^{-2y\xi}|\hat{g}(\xi)|^2 e^{-2\alpha y^2}\,d\xi\,dy \\ &= \frac{\alpha^{1/2}}{(2\pi)^{3/2}}\int |\hat{g}(\xi)|^2 e^{\xi^2/2\alpha}\,d\xi. \end{aligned} \tag{11.3}$$

Conversely, it is easily seen that if

$$\int |h(\xi)|^2 e^{\xi^2/2\alpha}\,d\xi < \infty,$$

then $h = \hat{g}$ for some $g \in G^2_\alpha$. Thus, $g \to \hat{g}(\xi)e^{\xi^2/4\alpha}$ is an isomorphism of G^2_α onto $L^2(\mathbb{R})$; with a suitable constant factor we obtain an isometry.

Remark 11.2. The composition of the isometry $F^2_\alpha \to G^2_\alpha$ given above and this isometry is the Bargmann transform $F^2_\alpha \to L^2(\mathbb{R})$ which maps the function z^k to the k:th Hermite function h_k ($e^{-\xi^2/4\alpha}$ times the k:th Hermite polynomial; the normalization depends on α). Now let $g \in G^p_\alpha$ for some p, $1 \leqslant p \leqslant \infty$, let $\gamma(\xi) = \hat{g}(\xi)e^{\xi^2/4\alpha}$ and $\phi(\xi) = e^{-\xi^2/4\alpha}$. Then

$$\hat{g}_y(\xi) = \hat{g}(\xi)e^{-y\xi} = \gamma(\xi)\phi(\xi + 2\alpha y)e^{\alpha y^2}$$

and thus

$$\begin{aligned} \|g\|_{G^p_\alpha} &= \|\,\|g_y\|_{L^p(dx)}e^{-\alpha y^2}\|_{L^2(dy)} \\ &= \|\,\|\gamma(\xi)\phi(\xi+2\alpha y)\|_{L^p}\|_{L^p} \\ &= (2\alpha)^{-1/p}\|\,\|\gamma(\xi)\phi(\xi+y)\|_{L^p}\|_{L^p}. \end{aligned}$$

Again the converse holds, i.e. the mapping $g \to \gamma$ maps G^p_α onto the space of all distributions γ such that $\|\,\|\gamma \cdot \phi(\bullet + y)\|_{L^p}\|_{L^p} < \infty$. Furthermore, the latter space is a kind of generalized Besov space defined by translations instead of dilations of the kernel ϕ (cf. Peetre (1976), Chapter 10 with, formally, $Af = f(\bullet + i)$ on $L^2(\mathbb{R})$), and it is seen, just as for ordinary Besov spaces, that the space remains the same if ϕ is replaced by an arbitrary test function (except 0). Also, it suffices to restrict y to the integers, provided some nondegeneracy condition holds. We omit the details.

Theorem 11.1. *Let $\phi \in C_0^\infty(\mathbb{R})$ with $\phi \not\equiv 0$. Then the space*

$$E_p = \{\gamma \in \mathfrak{D}' : \| \, \|\phi(\cdot + y)\gamma\|_{\hat{L}^p} \|_{L^p(dy)} < \infty\} \tag{11.4}$$

does not depend on the choice of ϕ, and if $\phi \neq 0$ on [0, 1],

$$E_p = \{\gamma \in \mathfrak{D}' : \| \, \|\phi(\cdot + n)\gamma\|_{\hat{L}^p} \|_{l^p} < \infty\}.$$

The mapping $g \to \hat{g}(\xi)e^{\xi^2/4\alpha}$ is an isomorphism of G_α^p onto E_p, $1 \leqslant p \leqslant \infty$. □

Note that E_p does not depend on α. Define mappings M_a by

$$\widehat{M_a g(\xi)} = \hat{g}(\xi)e^{a\xi^2}.$$

Corollary 11.1. *Let $\alpha, \beta > 0$ and $a = 1/4\alpha - 1/4\beta$. Then M_a is a isomorphism of G_α^p onto G_β^p, $1 \leqslant p \leqslant \infty$.* □

We are now prepared to deal with the Hankel forms Γ_b and H_b on G_α^2. Cf. Theorem 7.5 and recall that the results for Γ_b are gauge invariant, but as we see here, not those for H_b.

Theorem 11.2. *Let b be entire and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $b \in G_{2\alpha}^p$, i.e. if and only if $b(z)e^{-2\alpha y^2} \in L^p(dx\,dy)$, and $H_b \in S_p$ if and only if $b \in G_{2\alpha/3}^p$, i.e. if and only if $b(z)e^{-2\alpha y^2/3} \in L^p(dx\,dy)$.*

PROOF. The result for Γ_b follows by Theorem 4.6. The result for H_b follows by Corollary 6.1 once we note that $Q(\omega^{-1}b) = cM_{1/4\alpha}b$ for some constant c and thus $Q(\omega^{-1}b) \in G_{2\alpha}^p$ if and only if $b \in G_{2\alpha/3}^p$ by Corollary 11.1. The latter formula follows by (6.8) and (11.3):

$$\begin{aligned}\langle Q(\omega^{-1}b), f\rangle_\nu = \langle b, f\rangle_\mu &= C_1 \int \hat{b}(\xi)\overline{\hat{f}(\xi)}e^{\xi^2/2\alpha}\,d\xi \\ &= C_1 \int (M_{1/4\alpha}b)^\wedge(\xi)\overline{\hat{f}(\xi)}e^{\xi^2/4\alpha}\,d\xi = C_2\langle M_{1/4\alpha}b, f\rangle_\nu. \quad \square\end{aligned}$$

One can similarly show, more generally, that if $0 < \beta < 4\alpha$, then the Hankel form H_b^ξ with $d\xi = e^{-2\beta y^2}\,dx\,dy$ belongs to $S_p(G_\alpha^2)$ if and only if $b \in G_\gamma^p$ with $1/\gamma = 2/\beta - 1/2\alpha$.

Remark 11.3 The spaces E_p are interesting in their own right . They are essentially special cases of more general function spaces studied intensively (also in the context of general locally compact Abelian groups) by Feichtinger. Here we briefly recapitulate some of their salient properties. They form an increasing scale of spaces of distributions, with $E_2 = L^2$. E_1 is the minimal strongly character invariant Segal algebra, see, e.g., Feichtinger (1981*a*), (1981*b*). The spaces are translation invariant (isometrically) and dilation invariant. They are also preserved by (Feichtinger) the Fourier and (new!) Mehler transforms. This follows because F^p_α is mapped onto E_p by the Bargmann transform (see Remark 11.2), which maps z^j to h_j with, for $\alpha = 1/2$, $\hat{h}_j = (2\pi)^{1/2}(-i)^j h_j$. Hence the Bargmann transform intertwines the rotation $f \mapsto f(iz)$ (which obviously preserves $F^p_{1/2}$) and the Fourier transform on E_p. (In particular, we may take a Fourier transform in (11.4) and obtain the definition of E_p given in the introduction). More generally, the Mehler transform $h_j \to \zeta^j h_j$ (ζ is fixed with $|\zeta| \leqslant 1$) corresponds to $f \mapsto f(\zeta z)$ in $F^p_{1/2}$. (Cf. Peetre (1980)).

Note also that Theorems 7.3 and 7.4 imply (already in Feichtinger (1981*b*))

$$[E_{p_0}, E_{p_1}]_\theta = (E_{p_0}, E_{p_1})_{\theta p_\theta} = E_{p_\theta}, \tag{11.5}$$

and $(E_p)^* \cong E_{p'}$ $(1 \leqslant p < \infty)$, with the usual pairing on $\mathbb{R}$.

Remark 11.4. An argument similar to the proof of Theorem 9.3, using the operators M_a with a complex, can be employed to show that (11.5) implies (for $p_\theta < \infty$)

$$[G^{p_0}_{\alpha_0}, G^{p_1}_{\alpha_1}]_\theta = G^{p_\theta}_{\alpha_\theta} \tag{11.6}$$

with p_θ as before and $1/\alpha_\theta = (1 - \theta)/\alpha_0 + \theta/\alpha_1$.

Thus α_θ is the harmonic mean of α_0 and α_1, while we obtain the geometric mean for F^p_α (Theorem 9.3) and the arithmetic mean for L^p_α. In fact, both these results can be understood from the point of view of the Shale-Weil representation of the so-called metaplectic groups. (Cf. again Peetre (1980)).

12. Bergman Spaces in a Ball

In this section we study the case Ω = the open unit ball in $\mathbb{C}^n$ and

$$d\mu = c(1 - |z|^2)^\gamma \, dm, \tag{12.1}$$

where $\gamma > -1$ is fixed, m is the Lebesgue measure, and

$$c = \pi^{-n}\Gamma(n + \gamma + 1)/\Gamma(\gamma + 1)$$

is a normalization constant making $\mu(\Omega) = 1$. Thus $A^2(\mu)$ is the (weighted) Bergman space in the unit ball.

It is easily seen that $\{z^\alpha\}$, where α ranges over the set of multiindices, is an orthogonal basis in $A^2(\mu)$. An integration shows that

$$\|z^\alpha\|^2 = \frac{\alpha!\Gamma(\gamma+n+1)}{\Gamma(|\alpha|+\gamma+n+1)} \tag{12.2}$$

and thus the reproducing kernel is

$$\begin{aligned} K(z, w) &= \sum_\alpha z^\alpha \bar{w}^\alpha / \|z^\alpha\|^2 \\ &= \sum_{m=0}^{\infty} \langle z, w\rangle^m \Gamma(m+\gamma+n+1)/\Gamma(\gamma+n+1) \qquad (12.3) \\ &= (1-\langle z, w\rangle)^{-\gamma-n-1} \end{aligned}$$

(Cf. Rudin (1980), Chapter 7.1.)

The group $\operatorname{Aut}(\Omega)$ is the Möbius group $PSU(n, 1)$ (Cf. Rudin (1980), Chapter 2). Every automorphism acts on μ as an analytic gauge transformation, i.e. $G(\mu) = \operatorname{Aut}(\Omega) = PSU(n, 1)$. Since the Möbius group is transitive and isotropic in the sense of (2.5) (choose $z = 0$ for convenience), the results Sections 2-6 apply. The invariant measure is by Corollary 2.1 and (12.3)

$$d\lambda(z) = K(z, z)\, d\mu(z) = c(1-|z|^2)^{-n-1}\, dm(z). \tag{12.4}$$

(Conversely, since this measure can be shown directly to be invariant, Proposition 1.5 yields an alternative proof of (12.3)). We observe that

$$\omega(z) = (1-\|z|^2)^{\gamma+n+1}. \tag{12.5}$$

An elementary computation yields

$$\varkappa = \prod_{j=1}^{n} \frac{n+2\gamma+1+j}{\gamma+j}$$

We will, after some preliminaries, apply the general theory to the Hankel forms H_b and Γ_b on $A^2(\mu)$. The same argument yields similar results for Hankel forms H_b^ξ with $\xi = c(1-|z|^2)^\beta$ for any $\beta > 0$, but that is left as an exercise for the reader. See also Burbea (1986), where (independently) this type of Hankel operator is treated by a different method.

It is convenient to relate the Bergman spaces to the (analytic) Besov spaces on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. These Besov spaces can be defined as follows, in complete analogy with the standard Besov spaces on $\mathbb{R}$, cf. e.g. Peetre (1976), Ahlmann (1984), Mitchell and Hahn (1976). Let $\phi \in L^1(\mathbb{R})$ with $\hat{\phi} \in C_0^\infty(0, \infty)$ and define, for any analytic function (or formal power series) $f(z) = \sum \hat{f}(\alpha) z^\alpha$,

$$\phi * f = \int \phi(e^{is}) f(e^{-is}z)\, ds = \sum \hat{\phi}(|\alpha|)\hat{f}(\alpha)z^{\alpha}. \tag{12.6}$$

Define $\phi_t(x) = t^{-1}\phi(x/t)$; thus $\hat{\phi}_t(\xi) = \hat{\phi}(t\xi)$, and

$$B_s^{pq} = \left\{ f: \int_0^{\infty} (t^{-s} \| \phi_t * f \|_{L^p(S^{2n-1})})^q \frac{dt}{t} < \infty \right\}. \tag{12.7}$$

Hence $-\infty < s < \infty$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$ (with the standard modification if $q = \infty$). These spaces are independent of the choice of ϕ; furthermore, it is equivalent to use only the discrete values $\{2^{-k}\}_0^{\infty}$ for t. In the sequel we let $B_s^p = B_s^{pp}$.

When $s < 0$, we may in the definition of B_p^{sq} allow $\hat{\phi}(\xi) = e^{-\xi}$, $\xi > 0$ (and $\hat{\phi}(0) = 0$; thus $\phi \notin L^1$, but that does not matter). This choice gives

$$\phi_t * f(z) = \sum_{\alpha \neq 0} e^{-t|\alpha|}\hat{f}(\alpha)z^{\alpha} = f(e^{-t}z) - f(0).$$

Restricting attention to $t \leqslant 1$, as we may do, and changing variables, we find that, when $s < 0$,

$$\begin{aligned} f \in B_s^{pq} &\Leftrightarrow t^{-s} \| f(e^{-t}z) \|_{L^p(S^{2n-1})} \in L^q((0,1), dt/t) \\ &\Leftrightarrow (1-r)^{-s} \| f(rz) \|_{L^p(S^{2n-1})} \in L^q((0,1), (1-r)^{-1}\, dr). \end{aligned} \tag{12.8}$$

Hence these Besov spaces coincide with the weighted Bergman spaces in the unit ball. In particular, in the important case $q = p$,

$$B_s^p = \{ f: (1-|z|^2)^{-s} f(z) \in L^p(\Omega, (1-|z|^2)^{-1}\, dm) \}, \tag{12.9}$$

provided $s < 0$.

We now see that $A^2(\mu) = B^2_{-(\gamma+1)/2}$ and, cf. the definitions in §3,

$$A_{\omega}^p = \{ f: (1-|z|^2)^{\gamma+n+1} f(z) \in L^p((1-|z|^2)^{-n-1}\, dm) \} = B^p_{n/p-n-1-\gamma}.$$

Define

$$Df = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}.$$

Thus $Dz^{\alpha} = |\alpha| z^{\alpha}$. The Taylor coefficient multiplier D^u gives an isomorphism of B_s^{pq} onto B_{s-u}^{pq} (modulo constants) for any real s, u and $1 \leqslant p,\ q \leqslant \infty$. Similarly, $f \in B_s^{pq}$ if and only if $\partial f/\partial z_j \in B_{s-1}^{pq}$, $j = 1, \ldots, n$. This yields a way to extend (12.8) and (12.9) to $s \geqslant 0$; e.g.

$$f \in B_0^{\infty} \Leftrightarrow (1-|z|^2)|Df(z)| \leqslant C \Leftrightarrow (1-|z|^2)|\nabla f(z)| \leqslant C$$

(B_0^{∞} is the n-dimensional Bloch space).

We will also consider a related multiplier. Taking $f = z^\alpha$ (and $\xi = \mu$) in (6.8), we see that

$$\langle Q(\omega^{-1}b), z^\alpha \rangle_\nu = \langle b, z^\alpha \rangle_\mu = \hat{b}(\alpha) \|z^\alpha\|_\mu^2,$$

and thus

$$Q(\omega^{-1}b) = \sum_\alpha \psi(\alpha)\hat{b}(\alpha)z^\alpha,$$

with

$$\psi(\alpha) = \|z^\alpha\|_\mu / \|z^\alpha\|_\nu^2 = \Gamma(|\alpha| + 2\gamma + 2n + 2) / \Gamma(|\alpha| + \gamma + n + 1)$$

(cf. (12.2)). It is easy to show that the multiplier $\psi(\alpha)|\alpha|^{-\gamma-n-1}$ maps any space B_s^{pq} onto itself. Hence,

$$Q(\omega^{-1}b) \in A_w^p = B_{n/p-n-1-\gamma}^p$$

if and only if

$$D^{n+1+\gamma} b \in B_{n/p-n-1-\gamma}^p$$

if and only if

$$b \in B_{n/p}^p.$$

The results of §§4 and 6 thus yields the following. (To see that (6.6) holds, let $g(z, w) = \langle f, K_z K_w \rangle$ and note that g is analytic and

$$\begin{aligned} D_z^\alpha D_w^\beta g(z, w)|_{w=z} &= \text{const} \langle f(\zeta), \zeta^\alpha (1 - \langle \zeta, z \rangle)^{-n-1-\gamma-|\alpha|} \\ &\quad \zeta^\beta (1 - \langle \zeta, z \rangle)^{-n-1-\gamma-|\beta|} \rangle \\ &= \text{const}\, D_z^{\alpha+\beta} g(z, z).) \end{aligned}$$

Theorem 12.1 *Let b be analytic and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $b \in B_{n/p-n-1-\gamma}^p$ if and only if $(1 - |z|^2)^{n+1+\gamma} b(z) \in L^p((1 - |z|^2)^{-n-1} dm)$ and $H_b \in S_p$ if and only if $b \in B_{n/p}^p$. In particular, H_b is bounded if and only if $|\nabla b(z)| = O((1 - |z|^2)^{-1})$, Γ_b is compact if and only if $|\nabla b(z)| = o((1 - |z|^2)^{-n-1-\gamma})$ and H_b is compact if and only if $|\nabla b(z)| = o((1 - |z|^2)^{-1})$ as $|z| \to 1$.* □

When $n = 1$, this result is due to Peller (1982). Note that $H^2(\mathbb{T})$ formally is the limit of $A^2(\mu)$ as $\gamma \to -1$, and recall that the result above for H_b is valid on $H^2(\mathbb{T})$ too, provided $p < \infty$, see Peller (1980), (1982).

For multilinear Hankel forms we obtain by Theorem 5.2 and arguments as above the following, with $H_b(f_1, \ldots, f_m) = \int \bar{b} f_1 \cdot \ldots \cdot f_m \, d\mu$.

Theorem 12.2. *Let $m > 2$ and $p = 1, 2$ or ∞. Then, if b is analytic, the m-linear form $\Gamma_b \in S_p$ if and only if $(1 - |z|^2)^{m(n+1+\gamma)/2} \in L^p((1 - |z|^2)^{-n-1} \, dm)$ if and only if $b \in B^p_{-m(n+1+\gamma)/2 + n/p}$ and $H_b \in S_p$ if and only if $b \in B^p_{(m/2-1)(\gamma+n+1)+n/p}$.* □

The same result is true for H_b on $H^2(\mathbb{T})$ too, cf. Peetre (1985), Lecture 5.

Theorems on decomposition, approximation and interpolation for Bergman spaces are given by Coifman and Rochberg (1980) and Rochberg (1985).

Theorem 3.1 shows that the Bergman projection Q is bounded on L^p_ω. The problem of telling exactly when such a projection is bounded on $L^p(\Omega, m)$ is solved by the Forelli-Rudin Theorem, see Rudin (1980), Chapter 7.1.

13. Bergman Spaces in a Half Plane

In this section we consider $\Omega =$ the upper half plane ($n = 1$) and $d\mu = y^\gamma \, dx \, dy$, $\gamma > -1$. Thus, by Plancherel's formula,

$$\begin{aligned} \|f\|^2_{A^2(\mu)} &= \iint |f(z)|^2 y^\gamma \, dx \, dy = \int_0^\infty \int_0^\infty |e^{-\xi y} \hat{f}(\xi)|^2 y^\gamma \, dy \, d\xi \\ &= c_\gamma \int_0^\infty |\hat{f}(\xi)|^2 \xi^{-\gamma-1} \, d\xi \qquad (z = x + iy). \end{aligned} \tag{13.1}$$

Hence the (weighted) Bergman space $A^2(\mu)$ equals the analytic Besov space $B^2_{-(\gamma+1)/2}$. (See Peetre (1976) for definition and properties of Besov spaces. We do not distinguish between functions in Ω and their boundary values on $\mathbb{R}$).

The change of variables $z \to (z - i)/(z + i)$ maps Ω onto the unit disc, and the measure $c(1 - |z|^2) \, dx \, dy$ $(c = (\gamma + 1)/\pi)$ on the disc corresponds to $c4^{\gamma+1}|z + i|^{-2\gamma-4} y^\gamma \, dx \, dy$ on Ω, which is mapped to $y^\gamma \, dx \, dy$ by the change of gauge $f \to 2^{\gamma+1}\sqrt{c}(1 - iz)^{\gamma+2} f(z)$.

It follows from (12.3) and Proposition 1.3 and 1.2 that the reproducing kernel in $A^2(\mu)$ is

$$K(z, w) = \frac{\gamma + 1}{4\pi} \left(\frac{z - \bar{w}}{2i} \right)^{-\gamma-2}. \tag{13.2}$$

Thus, the invariant measure

$$K(z, z) \, d\mu = \frac{\gamma + 1}{4\pi} y^{-2} \, dx \, dy.$$

We, again, for simplicity consider only the Hankel forms Γ_b and H_b. We obtain by §§4 and 6, if b is analytic in Ω,

$$\Gamma_b \in S_p \Leftrightarrow y^{\gamma+2} b(x+iy) \in L^p(y^{-2}\,dx\,dy) \Leftrightarrow b \in B^p_{-\gamma-2+1/p}.$$

If $g = Q(\omega^{-1}b)$, then by (13.1) and (6.8), for suitable f,

$$\langle g,f\rangle_\nu = C_1 \int_0^\infty \hat{g}(\xi)\overline{\hat{f}(\xi)}\xi^{-2\gamma-3}\,d\xi = \langle b,f\rangle_\mu = C_2 \int_0^\infty \hat{b}(\xi)\hat{f}(\xi)\xi^{-\gamma-1}\,d\xi.$$

Hence $\hat{g}(\xi) = \xi^{\gamma+2}\hat{b}(\xi)$, and $Q(\omega^{-1}b) \in B^p_s$ if and only if $b \in B^p_{s+\gamma+2}$, $-\infty < s < \infty$. We thus obtain

Theorem 13.1. *Let b be analytic in the upper half plane and $1 \leqslant p \leqslant \infty$. Then $\Gamma_b \in S_p$ if and only if $b \in B^p_{-\gamma-2+1/p}$ and $H_b \in S_p$ if and only if $b \in B^p_{1/p}$.* □

For this and similar results, see Peller (1982), Rochberg (1982), Semmes (1984), Janson and Peetre (1985).

Note that $H^2(\mathbb{R})$ formally is a limit of $B^2_{-(\gamma+1)/2}$ as $\gamma \to -1$.

Remark 13.1. By conformal mapping followed by a suitable change of gauge one can also get interesting formulations in other domains (not necessarily (generalized) discs). For instance in the case of the standard strip $0 < \operatorname{Im} z < 1$, the condition on the symbol takes the form

$$\int_{-\infty}^{\infty}\int_0^1 |b(z)|^p (\sin \pi y)^a\,dm(z) < \infty. \tag{13.3}$$

Such a condition can be made more explicit using Besov spaces on the boundary lines $\operatorname{Im} z = 0, 1$. Some limiting cases are likewise of interest. If one writes (13.3) for the strip $0 < \operatorname{Im} z < s$ then $s \to \infty$ gives back the upper halfplane. Similary taking the strip $-s/2 < \operatorname{Im} z < s/2$, so that we have the weight $(\cos \pi y/s)^a$, then if we let $a = \alpha\lambda$, $\pi^2/2s^2 = \lambda^{-1}$, $\lambda \to \infty$ then Fock space evolves once more (use $\cos \pi y/s \approx 1 - \pi^2 y^2/2s^2 = 1 - y^2/\lambda$).

14. A General Kroneckers's Theorem

The classical Kronecker's theorem asserts that an ordinary Hankel form (or operator), in the Hardy space $H^2(\mathbb{T})$, is of finite rank if and only if its symbol is a rational function. Here we wish to establish an analogous result in maximal generality. A closely related multivariable Kronecker theorem, containing the algebraic part of the proof below, is otherwise in Power (1982*b*).

We consider the following set-up, which differs considerably from the one used in the main part of the paper; in particular, we do not any longer require the Hankel forms to be defined on a Hilbert space.

Ω is a domain in $\mathbb{C}^n$. Let R be the ring of all polynomial functions in Ω. We say that a bilinear form H defined on a space X that contains R (or, more generally, or a pair X, Y of such spaces) is a Hankel form if

$$H(f, g) = H(fg, 1), \qquad f, g \in R. \tag{14.1}$$

We will study spaces X that satisfy the following assumptions.

W1: X is a topological vector space of analytic functions in Ω.
W2: X contains R as a dense subspace.
W3: The inclusion $X \subset H(\Omega)$ is continuous.
W4: If $z \in \mathbb{C}^n \setminus \Omega$, then the mapping $f \to f(z)$, $f \in R$, has no continuous extension to X.

Note that W3 implies that the mapping $f \to D^\nu f(z)$ is continuous for every multi-index ν and every $z \in \Omega$. Conversely, if e.g. X is a Banach space, it follows from the Banach-Steinhaus theorem that W3 is equivalent to

W3′: If $z \in \Omega$, then $f \to f(z)$ is continuous on X.

Hence, in that case, W3 and W4 may informally be summarized by «$f \to f(z)$ is continuous if and only if $z \in \Omega$».

The Fock spaces in Section 7-10 and the Bergman spaces in Section 12 are examples where these assumptions are satisfied (also for $p \neq 2$ as long as $p < \infty$), but W2 fails to hold for the related spaces in Sections 11 and 13. Another example where the assumptions hold is the classical Hardy space $H^2(\mathbb{T})$ (a limiting case of Bergman space hitherto not permitted).

Theorem 14.1. *Assume that X and Y are two vector spaces such that X satisfies* W1-W4 *and Y satisfies* W1-W3. *Then every (separately) continuous Hankel form H on $X \times Y$ of finite rank is given by*

$$H(f, g) = \sum_{j=1}^{N} \sum_{|\nu| \leq k_j} c_{j\nu} D^\nu (fg)(z_j), \tag{14.2}$$

for some finite sequence $\{z_j\}_1^N$ in Ω, integers k_j and constants $c_{j\nu}$. Conversely, (14.2) *defines a continuous Hankel form for any $\{z_j\}_1^N \subset \Omega$, k_j and $c_{j\nu}$.*

PROOF. The last statement is obvious by Leibniz' rule. In order to prove that a Hankel form H has the sought representation, it is by continuity sufficient to show that (14.2) holds for all $f, g \in R$. Hence we will study the restriction of H to R, and the remainder of the proof will be almost purely algebraic. Let

$$J = \{ f \in R : H(f, g) = 0 \quad \text{for all} \quad g \in R \}.$$

If $f \in J$ and $h, g \in R$, then

$$H(fh, g) = H(fhg, 1) = H(f, hg) = 0.$$

Thus $fh \in J$. This proves that J is an ideal in R. Furthermore, J has finite codimension because H has finite rank. (In fact, $\dim(R/J)$ equals the rank of H).

To fix the ideas, let us first study the case $n = 1$. The structure of the ideals in $R = \mathbb{C}[z]$ is well-known and, since $J \neq 0$, we conclude that here exist $z_1, \dots, z_N \in \mathbb{C}$ and integers k_j such that

$$J = \{f \in R: D^\nu f(z_j) = 0, 0 \leqslant \nu \leqslant k_j, j = 1, \dots, N\}. \tag{14.3}$$

Since J thus is described by finitely many linear functionals, and the linear functional $f \to H(f, 1), f \in R$, vanishes on J, there exist contants $c_{j\nu}$ such that

$$H(f, 1) = \sum_{j=1}^{N} \sum_{\nu=0}^{k_j} c_{j\nu} D^\nu f(z_j), \qquad f \in R. \tag{14.4}$$

The formula (14.2) follows by (14.1), but it remains to show that $z_j \in \Omega$. We may assume that $c_{jk_j} \neq 0$ for $j = 1, \dots, N$. Define, for $i \leqslant N$,

$$g_i(z) = \prod_{j=1}^{N} (z - z_j)^{k_j + 1 - \delta_{ij}}.$$

Then, by (14.1) and (14.4),

$$H(f, g_i) = H(fg_i, 1) = \sum_{j=1}^{N} \sum_{\nu=0}^{k_j} c_{j\nu} D^\nu (fg_i)(z_j) = c_{ik_i} k_i! f(z_i), \qquad f \in R.$$

Consequently the mapping $f \to f(z_i)$ is continuous, and $z_i \in \Omega$ follows by W4. This completes the proof when $n = 1$.

When $n > 1$, we will require the following result which is an exercise in commutative algebra, see Power (1982*b*). For completeness we will supply the details of the proof. Our reference will be van der Waerden (1959).

Remark 14.1 When one of the authors was a young student he bought a copy of that venerable treatise. Now after many years he has finally got use for it.

Lemma 14.1. *If J is an ideal of finite codimension in $R = \mathbb{C}[\zeta_1, \dots, \zeta_n]$, then there exist finitely many points $z_1, \dots, z_N \in \mathbb{C}^n$ and integers $k_1, \dots, k_N$ such that*

$$J \supset \{f \in R: D^\nu f(z_j) = 0, |\nu| \leqslant k_j, j = 1, \dots, N\}. \tag{14.5}$$

PROOF. Let $V = \{z \in \mathbb{C}^n: f(z) = 0 \text{ for all } f \in J\}$ be the algebraic variety corresponding to J. If V is an infinite set, let $\{z_j\}_1^\infty$ be distinct points in V and pick $f_1, f_2, \dots$ in R such that

$$f_i(z_1) = \cdots = f_i(z_{i-1}) = 0, \qquad f_i(z_i) = 1.$$

Then $f_1, f_2, \dots$ are linearly independent mod J, which contradicts the assumption that J has finite codimension. Hence V is finite, $V = \{z_1, \dots, z_N\}$.

Let us now invoke the primary decomposition: since R is Noetherian, J is a finite intersection of primary ideals (van der Waerden (1959), p. 73). Thereby it suffices to prove the lemma for primary ideals (of finite codimension).

Claim. If J is primary, then V is a point.

PROOF. Suppose that $N > 1$. Choose f in R such that $f(z_1) = 1$, $f(z_2) = f(z_3) = \cdots = f(z_N) = 0$. Then $f(1-f)$ vanishes on V. Thus, by Hilbert's Nullstellensatz (van der Waerden (1959), p. 102), $f^m(1-f)^m \in J$ for some $m \geqslant 1$. Since $(1-f)^m \notin J$ and J is primary, $f^{mk} \in J$ for some $k \geqslant 1$; a contradiction. The claim is proved.

Let now J be primary and $V = \{z\}$. Let M be the maximal ideal $\{f \in R: f(z) = 0\}$. If $f \in M$, then, by the Nullstellensatz again, some power of f lies in J. It follows that, for some k,

$$J \supset M^k = \{f \in R: D^\nu f(z) = 0, |\nu| < k\}$$

(van der Waerden (1959), p. 70). The lemma is proved for primary ideals, and thus in general. □

We may now complete the proof of Theorem 14.1 as in the case $n = 1$. It follows by (14.5) that $H(f, 1) = \sum\sum c_{j\nu} D^\nu f(z_j)$. Hence, using (14.1), (14.2) holds for some $\{z_j\}_1^N \subset \mathbb{C}^n$. Fix j. We may assume that $c_{j\nu} \neq 0$ for some ν with $|\nu| = k_j$. Let $g = (z - z_j)^\nu h^m$ where $h(z_j) = 1$, $h(z_i) = 0$ for $i \neq j$, and $m > \max k_i$. Then, by (14.2), $H(f, g) = c_{j\nu} \nu! f(z_j)$, and W4 implies that $z_j \in \Omega$. □

Let us now specialize to the case when X is a Hilbert space. Define the symbol of the Hankel form H as the function $b \in X$ which satisfies $H(f, 1) = \langle f, b \rangle$, $f \in X$. Equivalently, by (14.1) $H(f, g) = \langle fg, b \rangle$, $f, g \in R$.

Let K_z be the reproducing kernel defined in Section 1. Then K_z is the symbol of the Hankel form $(f, g) \to f(z)g(z)$. Recalling that K_z is an antianalytic X-valued function in Ω, we obtain the following.

Corollary 14.1. *Assume that X is a Hilbert space which satisfies* W1-W4. *Then a continuous Hankel form with finite rank on X has a symbol of the form*

$$b(w) = \sum_{j=1}^N \sum_{|\nu| \leq k_j} c_{j\nu} (\partial/\partial \bar{z})^\nu K_z(w), \tag{14.6}$$

with $z_1, \dots, z_N \in \Omega$, and every such symbol defines a continuous Hankel form with finite rank. □

There is no problem to extend the results above to multilinear Hankel forms of finite rank. We leave the details to the reader.

Remark 14.2. Theorem 14.1 implies that any finite rank continuous Hankel form H is a limit (pointwise, and uniformly on bounded subsets of $X \times Y$) of Hankel forms

$$H_m(f, g) = \sum_{j=1}^{N_m} c_{mj} fg(z_{mj}),$$

$z_m \in \Omega$, with $\{N_m\}$ bounded $\left(\text{e.g. } N_m \leqslant \sum_1^N (k_j + 1)^n\right)$. The converse is obvious. We conjecture that it is possible to take N_m as the rank of H, i.e. that the set of continuous Hankel forms of rank $\leqslant r$ coincides with the closure (in any reasonable topology) of the set $\left\{(f, g) \to \sum_1^r c_j fg(z_j): c_1, \ldots, c_r \in \mathbb{C},\ z_1, \ldots, z_r \in \Omega\right\}$. (For $n = 1$, this follows easily from (14.3), but we have been unable to find a proof in higher dimension). For the Fock space, this would imply, in the notation of Section 8, that H_b has rank $\leqslant r$ if and only if $b \in \overline{P_r}$ (e.g. in F_α^∞).

Remark 14.3. What can be said about the kernel $\{f: H(f, g) = 0 \text{ for all } g\}$ of a general Hankel form (not of finite rank)? Again restricting attention to R, we see that the kernel is an ideal J. Let V be the corresponding subvariety of Ω. Then the Hankel form is «concentrated» on V. Examples of such forms, given V, are those of the type $\int_V \bar{\beta} fg\, d\sigma$, where σ is (e.g) the are measure. What can be said about the boundedness or smoothness of such forms?

We end with another open question: How can the results of this section be extended to the case of a general complex manifold! Is there an analogue of the polynomial ring R?

References

[1] Adamjan, V. M., Arov, D. Z. & Kreĭn, M. G. (1971) Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem, *Mat. Sb.* **86**(1971), 34-75 (Russian). (English translation: *Math. USSR-Sb.* **15**(1971), 31-73).

[2] Ahlmann, M. (1984) The trace ideal criterion for Hankel operators on the weighted Bergman space $A^{\alpha 2}$ in the unit ball of $\mathbb{C}^n$. Univ. Lund. Technical report 1984-3.

[3] Arazy, J. & Fisher, S. (1984) Some aspects of the minimal Möbius invariant space of analytic functions on the unit disc. *Interpolation Spaces and Allied Topics in Analysis* (Proceedings, Lund 1983), Lecture Notes in Math. **1070**(1984), 24-44.

[4] Arazy, J., Fisher, S. & Peetre, J. Hankel operators in Bergman spaces. To appear in *Amer. J. Math.*

[5] Arazy, J. & Upmeier, H. (1985) Some remarks on Toeplitz operators on Bergman spaces. Manuscript, 1985.

[6] Aronszajn, N. (1950) Theory of reproducing kernels. *Trans. Amer. Math. Soc.* **68**(1950), 337-404.

[7] Axler, S. (1986) The Bergman space, the Bloch space, and commutators of multiplication operators. *Duke Math. J.*, **53**(1986), 315-332.

[8] Bargmann, V., Butera, P., Girardello, L. & Klauder, J. R. (1971) On the completeness of the coherent states. *Rep. Math. Phys.* **2**(1971), 221-228.

[9] Berezin, F. A. (1975) General concept of quantization. *Comm. Math. Phys.* **40**(1975), 153-174.

[10] Berger, C. A. & Coburn, L. A. (1987) Toeplitz operators on the Segal Bargmann space. *Trans. Amer. Math. Soc.* **301**(1987), 813-829.

[11] Berger, C. A. & Coburn, L. A. (1986) Toeplitz operators and quantum mechanics. *J. Funct. Anal.* **68**(1986), 273-299.

[12] Berger, C. A., Coburn, L. A. & Zhu, K. H. (1987) Toeplitz operators and fuction theory in n-dimensions. *Lecture Notes in Math.* **1256**(1987), 28-35.

[13] Bergh, J. (1979) On the relation between the two complex methods of interpolation. *Indiana J. Math.* **28**(1979), 775-778.

[14] Bergh, J. & Löfström, J. (1976) Interpolation Spaces. Grundlehren Math. Wiss. **223**(1976), Springer-Verlag.

[15] Bergman, S. (1950) The Kernel Function and Conformal Mapping. *Math. Surveys V, Amer. Math. Soc.* New York, 1950.

[16] Bochner, S. & Martin, W. T. (1948) Several Complex Variables. Princeton University Press, Princeton, 1948.

[17] Bonsall, F. F. (1984) Boundedness of Hankel matrices. *J. London Math. Soc.* 2, **29**(1984), 289-300.

[18] Burbea, J. (1986) Trace ideal criteria for Hankel operators on the ball of $\mathbb{C}^n$. In preparation, 1986.

[19] Coifman, R. R. & Rochberg, R. (1980) Representation theorems for holomorphic and harmonic functions in L^p. *Astérisque* **77**(1980), 11-66.

[20] Cwikel, M. & Janson, S.(1984) Interpolation of analytic families of operators. *Studia Math.* **79**(1984), 61-71.

[21] Van Dantzing, D. & Van der Waerden, B. L. (1928) Über metrisch homogene Räume. *Abh. Math. Sem. Hamburg.* **6**(1928), 367-376.

[22] Dzhrbashyan, A. E. (1983) Integral representation and continuous projections in certain spaces of harmonic functions. *Mat. Sb.* **121**(1983), 259-271 (Russian). (English translation: *Math. USSR-Sb.* **49**, 1 (1984), 255-267).

[23] Feichtinger, H. G. (1981) On a new Segal algebra. *Monatsh. f. Math.* **92**(1981), 269-289.

[24] Feichtinger, H. G. (1981) Banach spaces of distributions of Wiener's type. *Proc. Conf. Oberwolfach,* Aug. 1980. *Function Analysis and approximation.* Ed. P. L. Butzer, B. Sz.-Nagy and E. Görlich. Internat. Series Num. Math. Vol. 60, 153-165. Birkhäuser, Basel-Boston-Stuttgart, 1981.

[25] Feldman, M. & Rochberg, R. (1986) Hankel operators on the Hardy space of the Heisenberg group. In prepatation, 1986.

[26] Gindikin, S. G. (1964) Analysis in homogenous domains. *Uspekhi Mat. Nauk* **19** (1964), 3-92 (Russian). (English translation: *Russian Math. Surveys* **19**(1964), 1-89).

[27] Gohberg, I. C. & Krein, M. G. (1969) Introduction to the Theory of Linear Nonselfadjoint Operators. Nauka, Moscow 1965 (Russian). (English translation: *Trans. Math. Monographs* **18**, *Amer. Math. Soc.*, Providence, 1969).

[28] Mitchell, J. & Hahn, K. T. (1976) Representation of linear functionals in H^p spaces over bounded symmetric domains in $\mathbb{C}^n$. *J. Math. Anal. Appl.* **56**(1976), 379-396.

[29] Horowitz, C. (1977) Factorization theorems for functions in the Bergman spaces. *Duke Math. J.* **44**(1977), 201-213.

[30] Inoue, T. (1982) Orthogonal projections onto spaces of holomorphic functions on bounded homogeneous domains. *Japan. J. Math.* **8**(1982), 95-108.

[31] Janson, S. (1986) On interpolating a Banach space and its dual. To be prepared, 1986?

[32] Janson, S. & Peetre, J. (1985) Paracommutators-boundedness and Schatten-von-Neumann properties. To appear in *Trans. Amer. Math. Soc.*

[33] Kaup, W. (1967) Reelle Transformationsgruppen und invariante Metriken auf koplexen Raumen. *Invent. Math.* **3**(1967), 43-70.

[34] Kobayashi, S. (1959) Geometry of bounded domains. *Trans. Amer. Math. Soc.* **92**(1959), 267-290.

[35] Lions, J. L. & Peetre, J. (1964) Sur une classe d'espaces d'interpolation. *Inst. Hautes Études Sci. Publ. Math.* **19**(1964), 5-68.

[36] Lu Qi-Keng (1966) On Kaehler manifolds with constant curvature. *Acta Math. Sinica* **16**(1966), 269-281 (Chinese). (English translation: *Chinese Math. Acta* 8, 283-298).

[37] Luecking, D. (1987) Trace ideal criteria for Toeplitz operators. *J. Funct. Anal.* **73**(1987), 345-368.

[38] McCarthy, C. A. (1967) C_p. *Israel J. Math.* **5**(1967), 249-271.

[39] Narasimhan, R. (1971) Several Complex Variables. The University of Chicago Press, Chicago-London, 1971.

[40] Nikol'skiĭ, N. K. (1984) Ha-plitz operators: A survey of some recent results. *Operators and Function Theory* (ed. S. C. Power) (Proceedings, Lancaster 1984), 87-137, Reidel, Dordrecht.

[41] Nikol'skiĭ, N. K. (1986) Treatise on the Shift Operator. *Grundlehren Math. Wiss.* **273**, Springer-Verlag, 1986.

[42] Nikol'skiĭ, N. K., and Peller, V. V. (1988) Introduction to the Theory of Hankel and Toeplitz operators. To appear in *Math. Rep.*, 1988.

[43] Peetre, J. (1976) New Thoughts on Besov spaces. *Duke Univ. Math. Ser.* **1**(1976), Durham.

[44] Peetre, J. (1980) A class of kernels related to the inequalities of Beckner and Nelson. *A tribute to Åke Pleijel* (Proceedings, Uppsala 1979), 171-210, Uppsala University, 1980.

[45] Peetre, J. (1983) Hankel operators, rational approximation and allied questions of analysis. *Second Edmonton Conference on Approximation Theory*, 287-332, CMS Conference Proceedings 3, *Amer. Math. Soc.*, Providence, 1983.

[46] Peetre, J. (1984) Invariant function spaces connected with the holomorphic discrete series. *Anniversary Volume on Approximation Theory and Functional Analysis* (Proceedings, Oberwolfach 1983), 119-134, Intermat. Ser. Numer. Math. **65**, Birkhäuser, 1984.

[47] Peetre, J. (1985) Paracommutators and minimal spaces. *Operators and Function Theory* (ed. S. C. Power) (Proceedings, Lancaster, 1984), 163-224, Reidel, Dordrecht, 1985.

[48] Peller, V. V. (1980) Hankel operators of the class S_p and their applications (Rational approximation, Gaussian processes, the problem of majorizing operators). *Mat. Sb.* **113**(1980), 538-581 (Russian). (English translation: *Math. USSR-Sb.* **41**(1980), 443-479).

[49] Peller, V. V. (1982) Vectorial Hankel operators, commutators and related operators of the Schatten-von Neumann S_p. *Integral Equations Operator Theory* **5**(1982), 244-272.
[50] Power, S. C. (1980) Hankel operators on Hilbert space. *Bull. London Math. Soc.* **12**(1980), 422-442.
[51] Power, S. C. (1982) Hankel Operators on Hilbert Space. *Res. Notes in Math.* **64**, Pitman, 1982.
[52] Power, S. C. (1982) Finite rank multivariable Hankel forms. *Linear Algebra Appl.* **48**(1982), 237-244.
[53] Rochberg, R. (1982) Trace ideal criteria for Hankel operators and commutators. *Indiana Univ. Math. J.* **31**(1982), 913-925.
[54] Rochberg, R. (1985) Decomposition theorems for Bergman spaces and their applications. *Operators and Function Theory* (ed. S. C. Power) (Proceedings, Lancaster, 1984), 225-277, Reidel, Dordrecht, 1985.
[55] Rudin, W. (1980) Function Theory in the Unit Ball of $\mathbb{C}^n$. *Grundlehren Math. Wiss.* **241**, Springer-Verlag, 1980.
[56] Sarason, D. (1978) Function Theory on the Unit Circle. *Virginia Polytechnic Institute and State University*, Blacksburg, 1978.
[57] Selberg, A. (1957) Automorphic functions and integral operators. *Seminars on analytic functions* (Proceedings, Princeton (1957), Vol. II, 152-161.
[58] Semmes, S. (1984) Trace ideal criteria for Hankel operators, and applications to Besov spaces. *Integral Equations Operator Theory* **7**(1984), 241-281.
[59] Simon, B. (1979) Trace Ideals and their Applications. *London Math. Soc. Lecture Note Ser.* **35**(1979), Cambridge Univ. Press.
[60] Sjögren, P. (1975/76) Un contre-exemple pour le noyau reproduisant de la mesure gaussienne dans le plan complexe. *Seminaire Paul Krée (Equations aux dérivées partielles en dimension infinite)* 1975/76, Paris.
[61] Skwarczynski, M. (1969) The invariant distance in the theory of pseudo-conformal transformations and the Lu Qi-Keng conjecture. *Proc. Amer. Math. Soc.* **22**(1969), 305-310.
[62] Stoll, M. (1977) Mean value theorems for harmonic and holomorphic functions on bounded symmetric domains. *J. Reine Angew. Math.* **290**(1977), 191-198.
[63] Strichartz, R. (1986) L^p contractive projections and the heat semigroup for differential forms. *J. Funct. Anal.* **65**(1986), 348-357.
[64] Triebel, H. (1977) Interpolation Theory. Function Spaces. Differential Operators. VEB, Berlin, 1977.
[65] Van der Waerden, B. L. (1959) Algebra II, Vierte Aufl., Springer-Verlag, 1959.
[66] Zhu, K. H. (1987) VMO, ESV and Toeplitz operators on the Bergman space. *Trans. Amer. Math. Soc.* **302**(1987), 617-646.

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APPENDICES by Jaak Peetre

Appendix I. Hankel Forms in Weaker Assumptions

In the main body of the paper, referred to below as [JPR], a general theory of Hankel forms is developed but in rather severe restrictions (essentially a homogeneous situation). As the title indicates, the aim of this note is to establish all the essential general results of that paper in much weaker assumptions. We assume that the reader is somewhat familiar with the contents of [JPR] so we repeat only the most rudimentary notions.

Let Ω be a domain in $\mathbb{C}^d$ and μ a positive measure on Ω. Denote by $A^2(\Omega, \mu)$ the space of analytic functions over Ω which are square integrable with respect to μ and let $K(z, \bar{w})$ be the reproducing function in $A^2(\Omega, \mu)$, $L(z, \bar{w})$ the one in $A^2(\Omega, \nu)$, where ν is the «measure» associated to ν

$$d\nu(z) = \omega(z)\, d\mu(z) \quad \text{where} \quad \omega(z) = 1/K(z, \bar{z}).$$

We make the following hypothesis:

$$\text{(weak-V)} \qquad \begin{cases} \forall w \in \Omega \quad \text{we can write} \quad L_w = \Sigma_s u_s v_s \\ \text{with} \quad \Sigma_s \| u_s \|_{A^2(\Omega, \mu)} \| v_s \|_{A^2(\Omega, \mu)} \leqslant C/\omega(w), \end{cases}$$

(that is, L_w is in the image of $A^2(\Omega, \mu) \hat{\otimes} A^2(\Omega, \mu)$ with norm $\leqslant C/\omega(w)$).

Then we have in particular

$$(1) \qquad \int_\Omega |L(z, \bar{w})|\, d\mu(z) \leqslant C \cdot K(w, \bar{w}).$$

(PROOF. Just use Schwarz's inequality).

We consider Hankel forms Γ_b with (usually) analytic symbol b with respect to ν:

$$\Gamma_b(f, g) = \int_\Omega \overline{b(z)} f(z) g(z)\, d\nu(z).$$

We require the following spaces of symbols

$$\mathfrak{L}^p(\Omega, \nu) = \{ b : b \text{ locally integrable}, \ \omega b \in L^p(\Omega, \sigma) \},$$
$$\mathfrak{A}^p(\Omega, \nu) = \mathfrak{L}^p(\Omega, \nu) \cap \{ b : b \text{ analytic} \}.$$

Here σ is the «invariant» measure corresponding to μ, ν:

$$d\sigma(z) = d\mu(z)/\omega(z) = d\nu(z)/\omega(z)^2.$$

It is clear that

$$\mathcal{L}^2(\Omega, \nu) = L^2(\Omega, \nu),$$
$$\mathcal{L}^1(\Omega, \nu) = L^1(\Omega, \mu)$$

and, generally,

$$\mathcal{L}^p(\Omega, \nu) = L^p(\Omega, \omega^{p-2}\nu) = L^p(\Omega, \omega^{p-1}\mu).$$

Similarly with $\mathfrak{A}$ and A instead of $\mathcal{L}$ and L.

Proposition. *The projections Q: $\mathcal{L}^1(\Omega, \nu) \to \mathfrak{A}^1(\Omega, \nu)$ and Q: $\mathcal{L}^\infty(\Omega, \nu) \to \mathfrak{A}^\infty(\Omega, \nu)$ are continuous.*

PROOF. As the kernel of Q is $L(z, \bar{w})$,

$$Qf(z) = \int_\Omega L(z, \bar{w}) f(w)\, d\nu(w),$$

this follows from the estimate (1); the latter can also be rewritten as

$$\int_\Omega |L(z, \bar{w})| \,/\, \omega(z)\, d\nu(z) \leqslant C/\omega(w). \quad \square$$

By interpolation (real or complex) we obtain

Corollary. *The projections Q: $\mathcal{L}^p(\Omega, \nu) \to \mathfrak{A}^p(\Omega, \nu)$, $1 \leqslant p \leqslant \infty$, are continuous.* $\square$

Corollary. $(\mathfrak{A}^{p_0}, \mathfrak{A}^{p_1})_{\theta p} = [\mathfrak{A}^{p_0}, \mathfrak{A}^{p_1}]_\theta = \mathfrak{A}^p$ *if* $1/p = (1-\theta)/p_0 + \theta/p_1$ $(0 < \theta < 1)$. $\square$

We can now prove

Proposition. *Γ_b is bounded (on $A^2(\Omega, \mu) \times A^2(\Omega, \mu)$) if and only if $b \in \mathfrak{A}^\infty(\Omega, \nu)$.*

PROOF. $\Leftarrow$ If $b \in \mathfrak{A}^\infty(\Omega, \nu)$ then $|b(z)| \leqslant C\omega(z)^{-1}$. Therefore

$$\begin{aligned} |\Gamma_b(f, g)| &\leqslant \int_\Omega |b(z)|\,|f(z)|\,|g(z)|\, d\nu(z) \\ &\leqslant \int_\Omega |f(z)|\,|g(z)|\, d\mu(z) \\ &\leqslant C\|f\|_{A^2(\Omega,\mu)}\|g\|_{A^2(\Omega,\mu)}, \end{aligned}$$

where we in the last step used Schwarz's inequality.

$\Rightarrow$ Assume that Γ_b is bounded. We may write

$$|b(w)| \leqslant \|\Gamma_b\| \sum_s \|u_s\|_{A^2(\Omega,\mu)}\|v_s\|_{A^2(\Omega,\mu)} \leqslant C\|\Gamma_b\|/\omega(w),$$

completing the proof. $\square$

Without «any» assumptions we can prove

Proposition. *$b \in \mathcal{A}^1(\Omega, \nu)$ implies $\Gamma_b \in S_1$.*

PROOF. For each $w \in \Omega$ introduce the Hankel form

$$\Gamma_w(f, g) = f(w)g(w) = \langle f, K_w \rangle \langle g, K_w \rangle.$$

It is clear that

$$\|\Gamma_w\|_1 \leqslant \|K_w\|^2_{\mathcal{A}^2(\Omega, \mu)} = K(w, \bar{w}) = 1/\omega(w).$$

Now formally we may write

$$\Gamma_b = \int_\Omega \bar{b}(w) \Gamma_b \, d\nu(w).$$

Therefore, if is legitimate to use Minkowski's inequality in this situation, we get

$$\begin{aligned} \|\Gamma_b\|_1 &\leqslant \int_\Omega |b(w)| \, \|\Gamma_b\|_1 \, d\nu(w) \\ &\leqslant \int_\Omega |b(w)| \cdot (1/\omega(w)) \cdot \omega(w) \, d\mu(w) \\ &= \int_\Omega |b(w)| \, d\mu(w) \\ &= \|b\|_{\mathcal{A}^1(\Omega, \nu)}. \quad \square \end{aligned}$$

Remark. Notice that the constant in this imbedding is 1.

By interpolation we obtain at once

Corollary. $b \in \mathcal{A}^p(\Omega, \nu)$ implies $\Gamma_b \in S_p (1 < p < \infty)$. $\square$

Remark. In particular thus $b \in \mathcal{A}^2(\Omega, \nu) \Rightarrow \Gamma_b \in S_2$ (= Hilbert-Schmidt (H. S.) forms). Is it possible to prove this directly (without using interpolation)?

So far we have only proved «direct» results (except for $p = \infty$). We now come to the «converse».

We have the following formula

$$(*) \qquad \langle \Gamma_b, \Gamma_c \rangle_{H.S.} = \int_\Omega \overline{b(z)} \tilde{c}(z) \, d\nu(z),$$

where $\tilde{c}$ is determined from c via the formula

$$\tilde{c}(z) = \int_\Omega K(z, \bar{\zeta})^2 c(\zeta) \, d\nu(\zeta).$$

PROOF OF (*). The Hilbert space $A^2(\Omega, \mu)$ admits the «continuous» basis

$$\{K_w / \|K_w\|_{A^2(\Omega, \mu)}\}_{w \in \Omega}.$$

Therefore, for any bilinear forms B, C on $A^2(\Omega, \mu) \times A^2(\Omega, \mu)$ one has

$$\langle B, C\rangle_{H.S.} = \int_{\Omega\times\Omega} B(K_w, K_{w'})\overline{C(K_w, K_{w'})}\, d\mu(w)\, d\mu(w').$$

For a Hankel form $B = \Gamma_b$ the «matrix elements» in this basis are given by

$$\Gamma_b(K_w, K_{w'}) = \int_\Omega \overline{b(z)} K(z, \bar{w}) K(z, \bar{w}')\, d\nu(z).$$

Similarly for $C = \Gamma_c$. This gives

$$\begin{aligned}\langle \Gamma_b, \Gamma_c\rangle_{H.S.} &= \iint_{\Omega\times\Omega} \overline{b(z)} c(\zeta) \int_\Omega K(z, \bar{w})\overline{K(\zeta, \bar{w})}\, d\mu(w)\, \cdot \\ &\quad \cdot \int_\Omega K(z, \bar{w}')\overline{K(\zeta, \bar{w}')}\, d\mu(w') \cdot d\nu(z)\, d\nu(\zeta) \\ &= \iint_{\Omega\times\Omega} \overline{b(z)} c(\zeta) K(z, \bar{\zeta})^2\, d\nu(z)\, d\nu(\zeta),\end{aligned}$$

which is the «bilinear» form of formula (*). □

By a standard duality reasoning we now obtain

Proposition. $\Gamma_c \in S_p$ *implies* $\tilde{c} \in \mathfrak{A}^p(\Omega, \nu)$.

Proof. If $\Gamma_c \in S_p$ then by a previous proposition $\langle \Gamma_c, \Gamma_b\rangle_{H.S.}$ makes sense for any $b \in \mathfrak{A}^{p'}(\nu)$ (where $1/p + 1/p' = 1$). In other words we have a continuous linear functional $b \mapsto \langle \Gamma_b, \Gamma_c\rangle_{H.S.}$ on $\mathfrak{A}^{p'}(\Omega, \nu)$. By one of the corollaries then $\tilde{c} \in \mathfrak{A}^p(\Omega, \nu)$. □

Let us introduce an operator $\mathfrak{J}$ by the relation $\mathfrak{J}c = \tilde{c}$ and let us make the new assumption, supplementing the previous assumption (weak-V),

(I) $\mathfrak{J}$ is invertible in each of the spaces $\mathfrak{A}^p(\Omega, \nu)$ $(1 \leqslant p < \infty)$.

Then we can summarize our findings in an elegant

Theorem. $\Gamma_b \in S_p (1 \leqslant p \leqslant \infty)$ *if and only if* $b \in \mathfrak{A}^p(\Omega, \nu)$ (*or if and only if* $\tilde{b} \in \mathfrak{A}^p(\Omega, \nu)$). □

Remark. In [JPR], apparently, the case $\tilde{b} = \varkappa^{-1} b$ was considered, that is $\mathfrak{J} = \varkappa^{-1}$. (Identity operator) so hypothesis (I) is trivially fulfilled. (Also in this case the strong(er) hypothesis (V), implying our present (weak-V), is fulfilled). We know as yet no other cases when (I) is fulfilled.

Reference

[JPR] Janson, S., Peetre, J., Rochberg, R. Hankel forms and the Fock space. This issue.

Appendix II. Recent Progress in Hankel Forms

On these pages I would like to report very briefly on work done by me —in one instance, jointly with Svante Janson— since last summer ('86). One of my main objectives has been to push beyond the limitations on the entire theory put in [JPR]. (It is assumed that the reader us somewhat familiar with the main ideas of that paper).

Here is an appropriate quotation: «Then, English, French, and mere Spanish will disappear from this planet. The world will be Tlön.» ([B], p. 35).

1. Weak factorization and boundedness. Let Ω be a domain in $\mathbb{C}^d$. If μ is a positive measure on Ω we denote by $A^p(\Omega, \mu)$ the subspace of $L^p(\Omega, \mu)$ consisting of analytic functions. Let $K = K(z, \bar{w})$ denote the reproducing kernel in $A^2(\Omega, \mu)$ and $L = L(z, \bar{w})$ the one in $A^2(\Omega, \mu)$ where ν is the measure «associated» with μ (definition:

$$d\nu(z) = \omega(z)\, d\mu(z) \quad \text{where} \quad \omega(z) = 1/K(z, \bar{z}))$$

or, possibly, an equivalent measure. The basic hypothesis in [JPR] is («factorization» of the reproducing kernel):

$$\text{(V)} \qquad L = \varkappa K^2 \qquad (\varkappa \text{ a constant} > 1).$$

But already there the following weaker hypotehsis is mentioned («weak factorization»)

$$\text{(weak-V)} \qquad \begin{cases} \forall \in \Omega \quad \text{one can write} \quad L_w = \sum_s u_s v_s \\ \text{where} \quad \sum_s \|u_s\|_{A^2(\Omega,\mu)} \|v_s\|_{A^2(\Omega,\mu)} \leqslant c/\omega(w). \end{cases}$$

(Then sum may be finite or infinite). It is shown in Appendix 1 that, under the hypothesis of (weak-V), holds:

$$\Gamma_b \quad \text{is bounded on} \quad A^2(\Omega, \mu) \times A^2(\Omega, \mu) \quad \textit{if and only if} \quad b \in \mathfrak{A}^\infty(\Omega, \nu).$$

Here Γ_b is the Hankel form with (usually) analytic symbol b with respect to ν,

$$\Gamma_b(f, g) = \int_\Omega \overline{b(z)} f(z) g(z)\, d\nu(z),$$

and, generally speaking, the symbol class $\mathfrak{A}^p(\Omega, \nu)$, $0 < p \leqslant \infty$, is defined as

$$\{b\colon b \text{ analytic}, \ \omega b \in L^p(\Omega, \sigma)\},$$

where again σ is the «invariant» measure (definition:

$$d\sigma(z) = d\mu(z)/\omega(z) = d\nu(z)/\omega(z)^2).$$

The condition (weak-V) has been verified in several concrete cases.

In [P1] the case $d = 1$, Ω = an annulus $\{z: 1 < |z| < R\}$ is treated. In this case there is a natural family of measures $\mu = \mu_\alpha$ ($\alpha > -1$) to be considered: $d\mu_\alpha(z) = \lambda(z)^\alpha\, dE(z)$ where $ds = |dz|/\lambda(z)$ is the Poincaré metric on Ω and E is the Euclidean area measure $dE(z) = dx\,dy = i/2 \cdot dz\,d\bar{z}$). (This construction applies to any plane domain Ω bounded by finitely many smooth arcs (a «regular» domain); if Ω is the unit disk then $\lambda(a) = 1 - |z|^2$ so one gets back the usual weighted Bergman (or Dzhrbashyan) spaces). For α integer ($\alpha = 0, 1, 2, \ldots$) the weak factorization can be verified on the basis of the fact that the kernel L (and K) can be expressed in terms of *elliptic functions*. Here it is natural to take $\nu = \mu_\beta$, $\beta = 2\alpha + 2$, so it is not exactly the associated measure, only equivalent to it.

In [P2] I plan to extend the analysis in [P1] to the case of arbitrary regular planar domains. My idea is to invoke the Shottky double $\hat{\Omega}$ of Ω (= set theoretically the union $\Omega \cup \hat{\Omega} \cup \partial\Omega$, where $\hat{\Omega}$ is Ω with the «opposite» complex structure) and thus the theory of «symmetric» compact Riemann surfaces (= real algebraic curves). However, if the genus is >1, this cannot be done as explitly as in the above case of genus 1, because no such nice tool as the theory of elliptic functions is available.

Let me also remark that the case α not an integer is entirely open, also in genus 1.

In [JP] the case of «periodic» Fock space is considered, that is, entire periodic functions (with period, say, 2π) which are square integrable with respect to the measure $e^{-y^2}\,dx\,dy$ (if «Planck's constant» is taken to be 1/2). The basic fact about this case is now that the reproducing kernel can be written in terms of *theta functions* so the desired weak factorization can be obtained by just looking up in the literature the appropriate formulae for theta functions.

Again [P3] is addressed to the case of subspaces of Fock space singled out by symmetries. Example: $f(-z) = f(z)$ (even functions), $f(-z) = -f(z)$ (odd functions). In this case the reproducing kernel is expressed in terms of *hyperbolic functions* (cosh, sinh) and the weak factorization follows from the duplication formulae for the latter (viz. $\cosh 2x = \cosh^2 x + \sinh^2 x$, $\sinh 2x = 2 \sinh x \cosh x$). In a more general situation one similarly requires generalized hyperbolic functions.

I have assigned to a student the task of extending the analysis in [P3] to the case of weighted Bergman spaces. This seems to involve a sort of *generalization* of the generalized hyperbolic functions.

In [P4] I determine explicitly the reproducing kernels for certain Hilbert spaces of holomorphic tensor fields over the unit ball in $\mathbb{C}^d$ (the «Rudin ball»). Of course, these are then tensorial too. In this case «strong factorization» holds true (an appropriate tensor version of the previous condition (V)), so a corresponding boundedness result for Hankel like forms can be established.

Again the case of the ball is just the simplest case of a symmetric domain (essentially the rank one case). It is conceivable that one has similar results for other symmetric domains in É. Cartan's list.

Finally, in [P5] I have reformulated the relevant portions of [JPR], that is, as far as the issue of boundedness goes, in the language of holomorphic line bundles and, more generally, holomorphic vector bundles.

2. S_p Theory. In [P6] I address myself to the question of generalizing the S_p-theory in [JPR]-carried out in the hypothesis of condition (V)-to a more general setting. It turns out that besides condition (weak-V), which seems to be virtually indispensable, one requires basically only one more assumption. To formulate it let us introduce the «square operator» $\mathcal{J}$ on $\mathfrak{A}^2(\Omega, \nu)$ (Ω and μ are general now, as in the beginning of Sec. 1), defined by

$$\mathcal{J}f(z) = \int_\Omega K(z, \bar{w})^2 f(w)\, d\mu(w).$$

The relevant hypothesis is then

$$\text{(I)} \qquad \mathcal{J} \text{ is invertible in } \mathfrak{A}^p(\Omega, \nu).$$

In this hypotheses ((I) + (weak-V)) it is easy to establish that

$$\Gamma_b \in S_p \quad \text{if and only if} \quad b \in \mathfrak{A}^p(\Omega, \nu), \qquad 1 \leqslant p < \infty.$$

It is trivial that (V) $\Rightarrow$ (I). Indeed, if (V) is fulfilled then clearly $\mathcal{J} = \varkappa^{-1}$ (identity) where $\varkappa$ is a constant $\geqslant 1$. So far I have no non-trivial case when (I) is fulfilled but it should not be difficult to establish it in some of the simpler cases mentioned in Sec. 1. Work is in progress! (*Note* (added Jan. 88). See [JP1].)

3. Some related investigations. In [P7] I study the action of the metaplectic group on the spaces $F^p_\alpha(\mathbb{C})$, which are the natural L^p symbol classes corresponding to the scale of Fock spaces $F^p_\alpha(\mathbb{C})$; see [JPR] for details. (Similar results as those now described are expected in $\mathbb{C}^d$, $d > 1$). In particular I verify that this action is bounded continuous but not isometric, which is a result at least implicit in the work of Feichtinger (see e.g. [F1], [F2], [FG]). To get an isometric action one has to consider a new «caloric» representation of the Heisenberg group (and the *metaplectic* group as well), a «caloric Fock space». This leads also to the idea of a «caloric Bloch space» and a «caloric minimal space», which ought to be studied more. In this context, «caloric» means that the elements of the spaces are (analytic) solutions of the heat equation.

Turning to the situation of regular planar domains treated in [P1], [P2] (see Sec. 1), there is for any fixed halfinteger $l \in 1/2\mathbb{N}$ a natural duality between the following type of objects

$$\text{holomorphic } (1-l)\text{-forms } F(z)(dz)^{1-l} - \text{«integrals»}$$

and

$$\text{holomorphic } l\text{-forms } g(z)(dz)^{l} - \text{«differentials»},$$

embodied in the presence of the pairing

$$[F,f] = \int_{\partial\Omega} F(z)(dz)^{1-l} \cdot \overline{g(z)(dz)^{l}}.$$

In particular, it gives a possibility to lift invariant Hilbert metrics for differentials (for instance, the Dzrbashyan metric mentioned in Sec. 1) to a metric for integrals. One can then define corresponding «minimal» and «maximal» spaces, thus obtaining a new opportunity to extend Arazy's great program for Möbius invariant spaces (cf. [P8]). This connects also with lots of interesting notions such as (real) projective structures of Riemann surfaces, uniformization, Eichler cohomology, Schottky double, real algebraic curves etc. I have started a cooperation on these matters with Björn Gustafsson (Stockholm), who is a specialist on quadrature domains (see e.g. [G]). In particular, we have begun to study invariant differential operators on compact Riemann surfaces equipped with a projective structure (generalizing the classical Schwartz derivative).

It is not clear how much, if anything, of the above can be extended to several variables but on the whole I am about to believe that there must exist interesting illustrations to the theory of Hankel forms with higher dimensional algebraic varieties, especially algebraic surfaces (complex dimension 2).

4. A small selection of open problems.

4.1. To extend the AAK theorem beyond its classical $H^2(T)$-setting (see e.g. [N], App. 4).

4.2. The S_p-theory in [JPR] and its extension indicated in Sec. 3 below is confined to the case $1 \leqslant p < \infty$. It would be interesting to have any general results for $0 < p < 1$ too. In the case of Fock space rather complete results have been obtained by Svante Janson's student Robert Wallstén [W].

4.3. To extend the theory of higher weight Hankel forms in [JP2] to more general cases, for instance the unit ball in $\mathbb{C}^d$. This is basically a question of *Invariant Theory*.

4.4. Is there an analogue of the metaplectic group in the case of Bergman space (cf. [P6])?

References

[B] Borges, J. L. (1962) Ficciones. Grove, New York, 1962.

[F1] Feichtinger, H. (1981) A characterization of minimal homogeneous Banach spaces. *Proc. Amer. Math. Soc.* **81**(1981), 55-61.

[F2] Feichtinger, H. Modulation spaces on locally compact abelian groups II. Technical report, Vienna.

[FG] Feichtinger, H., Gröhening, K. H. Banach spaces related to integrable group representations and their atomic decompositions. Part I. General theory. In preparation.

[G] Gustafsson, B. (1983) Quadrature identities and the Shottky double. *Acta Applicandae Matematicae* **1**(1983), 209-240.

[JP1] Janson, S.-Peetre, J. (1987) Weak factorization in periodic Fock space. Technical report, Lund, 1987. Submitted to *Math. Nachr.*

[JP2] Janson, S.-Peetre, J. (1987) A new generalization of Hankel operators (the case of higher weights). *Math. Nachr.* **132**(1987), 313-328.

[JPR] Janson, S.-Peetre, J.-Rochberg, R. (1987) Hankel forms and the Fock space. This issue.

[N] Nikol'skiĭ, N. K. (1986) Treatise on the shift operator. *Grundlehren Math. Wiss.* **273**, Springer, 1986.

[P1] Peetre, J. (1988) Hankel forms over planar domains I. *Complex Variables* **10**(1988).

[P2] Peetre, J. Hankel forms over planar domains II. In preparation.

[P3] Peetre, J. (1987) Hankel forms on Fock space modulo C_N. Technical report, Lund, 1987. Submitted to *Resultate Math.*

[P4] Peetre, J. Reproducing formulae for holomorphic tensor fields. To appear in *Boll. Un. Mat. Ital.*

[P5] Peetre, J. (1987) Hankel forms over line bundles and vector bundles. Technical report, Lund, 1987. Submitted *Proc. London Math. Soc.*

[P6] Peetre, J. Hankel forms in weaker assumptions. This issue.

[P7] Peetre, J. Some calculations related to Fock space and the Shale-Weil representation. To appear in *Integral Equations Operator Theory.*

[P8] Peetre, J. (1986) Invariant function spaces and Hankel operators-a rapid survey. *Expositiones Mathematicae* **5**(1986), 3-16.

[W] Wallstén, R. The S^p-criterion for Hankel forms on the Fock space, $0 < p < 1$. To appear in *Math. Scand.*.

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A Harnack Inequality Approach to the Regularity of Free Boundaries. Part I: Lipschitz Free Boundaries are $C^{1,\alpha}$

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Introduction

1. This is the first in a series of papers where we intend to show, in several steps, the existence of «classical» (or as classical as possible) solutions to a general two-phase free-boundary problem.

2. We plan to do so by

(*a*) constructing rather weak generalized solutions of the free-boundary problems,

(*b*) showing that the free boundary of such solutions have nice measure theoretical properties (i.e., finite $(n-1)$-dimensional Hausdorff measure and the associated differentiability properties),

(*c*) showing that near a «flat» point free boundaries are Lipschitz graphs and

(*d*) showing that Lipschitz free boundaries are really $C^{1,\alpha}$.

From then on, the theory of regularity developed by Kinderlehrer-Nirenberg and Spruck applies.

We start here with the last part of the project, that is, to show that Lipschitz free boundaries are $C^{1,\alpha}$, mainly for two reasons: the first because many of the ideas in this part reappear in a much more entangled way than in the others, and the second, because this part is of immediate interest, since the existence of solutions to which these theorems will apply has been obtained already in many cases by different means.

An heuristic discussion of this paper can be found in [C]. The ideas presented here originated in a joint work with J. Athanasopoulus (see [At-C]).

Notion of Weak Solution

We denote a point in $\mathbb{R}^{n+1}$ as X or (x, y), with $x = (x_1, \ldots, x_n)$. To state the simplest version of our results, let us define what we mean by a weak solution of a free-boundary problem.

Definition 1. *In the unit cylinder $C_1 = B_1 \times [-1, 1]$ of $\mathbb{R}^{n+1}$, we are given a continuous function u satisfying*

(i) $\Delta u = 0$ *on* $\Omega^+ = \{u > 0\}$,

(ii) $\Delta u = 0$ *on* $\Omega^- = \{u \leqslant 0\}^0$,

(iii) *(The weak free-boundary condition). Along $F = \partial\{u > 0\}$ u satisfies the free-boundary condition*

$$u_{\nu+} = G(u_{\nu-})$$

in the following sense.

If $X_0 \in F$ and F has a one-sided tangent ball at X_0 (i.e. $\exists B_p(Y)$ such that $X_0 \in \partial B_p(Y)$ and $B_p(Y)$ is contained either in Ω^+ or Ω^-) then

$$u(X) = \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

and $\alpha = G(\beta)$.

The basic requirements on G will be strict monotonicity and continuity in u_ν.

Theorem 1. *Let u be a continuous function in the unit ball. Assume that u satisfies*

(i) $\Delta u = 0$ *in* $\Omega^+ = \{u > 0\}$ *and* $\Omega^- = \{u \leqslant 0\}^0$.

(ii) $\Omega^+ = \{(x, y): y > f(x)\}$, *with $f(x)$ a Lipschitz continuous function.*

(iii) $0 \in F = \partial\Omega^+$ *and along F, the free-boundary condition $u_\nu^+ = G(u_\nu^-)$ is satisfied in the sense described above.*

Assume further that $G(s)$ is strictly increasing and for some C large, $s^{-C}G(s)$ is decreasing. Then, on $B_{1/2}$, f is a $C^{1,\alpha}$ function.

1. Some Properties of Harmonic Functions in a Lipschitz Domain

In this section we recall some properties of nonnegative harmonic functions in a Lipschitz domain.

Lemma 1. (Dahlberg, see [D], see also [C-F-M-S]). *Let u_1, u_2 be two nonnegative harmonic functions in a (Lipschitz) domain D of $\mathbb{R}^{n+1}$ of the form*

$$D = \{|x| < 1, |y| < M, y > f(x)\}$$

with f a Lipschitz function with constant less than M and $f(0) = 0$. Assume further that u_1 and u_2 take continuously the value $u_1 = u_2 = 0$ along the graph of f. Then, on the domain

$$D_{1/2} = \left\{|x| < \frac{1}{2}, |y| < \frac{M}{2}, y < f(x)\right\},$$

we have

$$0 < C_1 \leqslant \frac{u_1(x,y)}{u_2(x,y)} \cdot \frac{u_2\left(0, \dfrac{M}{2}\right)}{u_1\left(0, \dfrac{M}{2}\right)} \leqslant C_2$$

with C_1, C_2 depending only on M. In particular, if

$$\frac{u_2(0, M/2)}{u_1(0, M/2)} = 1$$

we get

$$0 < C_1 \leqslant \frac{u_1(x,y)}{u_2(x,y)} \leqslant C_2.$$

Lemma 2 (Jerison and Kenig [J-K], see also [At-C]). *Let D, u_1 and u_2 be as in Lemma 1. Assume further that*

$$\frac{u_1(0, M/2)}{u_2(0, M/2)} = 1.$$

Then, $u_1(x,y)/u_2(x,y)$ is Hölder continuous in $\bar{D}_{1/2}$ (i.e. up to the graph of $f(x)$) for some coefficient α, both α and the C^α norm of u_1/u_2 depending only on M.

Lemma 3 (Dahlberg [D], see also [C-F-M-S]). *Let u be as u_1 (or u_2) above. Then, there exists a constant $\delta = \delta(M)$ such that for*

$$D_\delta = \{|x| < \delta, |y| < \delta M, y > f(x)\}$$

we have

$$u|_{D_\delta} \leqslant \frac{1}{2} u\left(0, \frac{M}{2}\right).$$

Lemma 4. *Let u be as in Lemma 3. Assume further that $D_y u \geqslant 0$ on D. Then,*

$$0 < C_1 \leqslant \frac{D_y u\left(0, \frac{M}{2}\right)}{u\left(0, \frac{M}{2}\right)} \leqslant C_2.$$

As usual $C_i = C_i(M)$.

PROOF. From Lemma 3,

$$\frac{1}{2} u\left(0, \frac{M}{2}\right) \leqslant \int_\delta^{M/2} D_y u(0, t)\, dt \leqslant u\left(0, \frac{M}{2}\right).$$

But D_y is positive and harmonic in Ω. Therefore, by Harnack's inequality, all the values along the segment of integration are comparable, and the formula with $d = M/2$ follows. For $0 < d < M/2$ we may use rescaling. □

Lemma 5. *Let u be as in Lemma 3. Then, in D_δ, for some $\delta(M)$, $D_y u \geqslant 0$.*

PROOF. Let $u_1 = u$ and u_2 be the (bounded) auxiliary function

$$\begin{cases} u_2 = C > 0, & \text{on } \partial D \backslash \operatorname{graph} f \\ u_2 = 0, & \text{on } \operatorname{graph} f \\ \Delta u_2 = 0, & \text{on } D. \end{cases}$$

If we compare u_2 with vertical translations in their common domain of definition, we obtain

$$D_y u_2 > 0 \quad \text{on} \quad D.$$

Let us adjust C so that

$$\frac{u_1\left(0, \frac{M}{2}\right)}{u_2\left(0, \frac{M}{2}\right)} = 1.$$

Then, from Lemma 2, on $D_{1/2}$

$$0 < C_1 \leqslant \frac{u_1(x,y)}{u_2(x,y)} \leqslant C_2$$

and further, from Lemma 3

$$\left| \frac{u_1(x,y)}{u_2(x,y)} - \frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})} \right| \leqslant C(|x-\bar{x}| + |y-\bar{y}|)^{\alpha}.$$

In particular, if we freeze $(\bar{x},\bar{y})$, at distance d from graph of f, and let (x,y) vary in a $d/2$-neighborhood of $(\bar{x},\bar{y})$, we get

$$\begin{aligned} \left| u_1(x,y) - u_2(x,y)\left[\frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})}\right] \right| &< Cu_2(x,y)(|x-\bar{x}| + |y-\bar{y}|)^{\alpha} \\ &\leqslant Cu_2(\bar{x},\bar{y})d^{\alpha} \\ &\leqslant CD_y u_2(\bar{x},\bar{y})d^{\alpha+1} \end{aligned}$$

(we may substitute $u_2(x,y)$ by $u_2(\bar{x},\bar{y})$ by Harnack's inequality, and $u_2(\bar{x},\bar{y})$ by $d(D_y u_2(\bar{x},\bar{y}))$, because of Lemma 4). Therefore, taking D_y derivative on the unfrozen variable y, and evaluating at $\bar{y}$, we get, from standard interior *a priori* estimates for $w = u_1 - u_2 k$, $k = u_1(\bar{x},\bar{y})/u_2(\bar{x},\bar{y})$

$$\left| D_y u_1(\bar{x},\bar{y}) - \left[\frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})}\right] D_y u_2(\bar{x},\bar{y}) \right| \leqslant CD_y u_2(\bar{x},\bar{y}) \cdot d^{\alpha}$$

that is

$$D_y u_1(\bar{x},\bar{y}) \geqslant \left\{ \left[\frac{u_1(\bar{x},\bar{y})}{u_2(\bar{x},\bar{y})}\right] - Cd^{\alpha} \right\} \cdot D_y u_2(\bar{x},\bar{y}).$$

And this last term is positive if d^{α} is small enough. □

2. Subsolutions to Our Free-Boundary Problems and Comparison Principles

In this section we define weak subsolutions to our free-boundary problem, and discuss a comparison principle.

We start by defining the notion of a weak subsolution.

Definition 2. *The continuous function $v(X)$ is a subsolution to our free-boundary problem in Ω if*

(i) $\Delta v \geqslant 0$ *both in* $\Omega^+ = \{v > 0\}$ *and* $\Omega^- = \{v \leqslant 0\}^0$

(ii) *let* $X_0 \in F = (\partial\Omega^+) \cap \Omega$, *assume that at X_0, F has a tangent ball B_ϵ from the Ω^+ side (i.e. $B_\epsilon \subset \Omega^+$, $X_0 \in \partial B_\epsilon \cap F$). Then, for some $\alpha \geqslant 0$, $\beta = G(\alpha)$, ν the unit inner radial direction of ∂B_ϵ at X_0,*

$$v(X) \geqslant \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|).$$

Definition 3. *Given a subsolution v to our F.B. Problem, a point $X_0 \in F$, at which F has a tangent ball from Ω^+ (as in Definition 2(ii)) will be called a regular point.*

We now state a strong comparison principle.

Lemma 6. *Let $v \leqslant u$ be two continuous functions in Ω, $v < u$ in $\Omega^+(v)$, v a subsolution and u a solution. Let $X_0 \in F(v) \cap F(u)$ (the free boundaries of v and u). Then X_0 cannot be a regular point for $F(v)$.*

PROOF. Since $\Omega^+(v) \subset \Omega^+(u)$, X_0 automatically will be a point for which u has the desired asymptotic development (Definition 1)

$$u(X) = \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with $\beta = G(\alpha)$

$$v(X) \geqslant \bar{\beta}\langle X - X_0, \nu\rangle^+ - \bar{\alpha}\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with $\bar{\beta} = G(\bar{\alpha})$. This implies that $\beta \geqslant \bar{\beta}$ and $\alpha \leqslant \bar{\alpha}$.

Since G is assumed to be monotone $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$. But $u - v$ is a positive superharmonic function in $\Omega^+(v)$. By Hopf principle, since X_0 is regular

$$(u - v)(X) \geqslant \epsilon|X - X_0|$$

radially into $\Omega^+(v)$, along ν from X_0. □

We refine the previous lemma to a continuous family of subsolutions.

Lemma 7. *Let v_t, for $0 \leqslant t \leqslant 1$, be a continuous family of subsolutions in Ω (continuous in $\bar{\Omega} \times [0, 1]$). Let u be a solution in Ω, continuous in $\bar{\Omega}$. Assume that*

(i) *$v_0 \leqslant u$ in Ω.*
(ii) *$v_t \leqslant u$ on $\partial\Omega$ and $v_t < u$ in $[\overline{\Omega^+(v_t)} \cap \partial\Omega]$ for $0 \leqslant t \leqslant 1$.*
(iii) *every point $X_0 \in F(v_t)$ is regular and*
(iv) *the family $\Omega^+(v_t)$ is continuous, that is $\Omega^+(v_{t_1}) \subset N_\epsilon(\Omega^+(v_{t_2}))$ whenever $|t_1 - t_2| < \delta(\epsilon)$ (N_ϵ denotes the ϵ-neighborhood of the set).*

Then $v_t \leqslant u$ in Ω for any t.

PROOF. The set of t's for which $v_t \leqslant u$ is obviously closed. Let us show that it is open: first, if $v_{t_0} \leqslant u$, it follows from (ii) and the strong maximum principle, that $v_{t_0} < u$ in $\Omega^+(v_{t_0})$. And since every point of $F(v_{t_0})$ is regular (assumption (iii)), it follows that $\overline{[\Omega^+(v_{t_0})]}$ is compactly contained in $\Omega^+(u)$ (up to $\partial\Omega$, from assumption (ii)). From assumption (iv), the openness follows. □

Remark. Since u may be the solution of a one-phase problem, that is $u|_{\Omega^-(u)} \equiv 0$, assumption (iv) is necessary (an easy counterexample where $\Omega^+(v_t) = \Omega$ for $t > 0$, can be constructed).

3. Continuous Families of Subsolutions

In this section we construct particular families of subsolutions, starting from a given solution. The simplest family is the following:

Lemma 8. *Let u, a continuous function in Ω, be a weak solution of our F. B. Problem. Let*

$$v_t(X) = \sup_{B_t(X)} u(Y), \qquad t > 0.$$

Then v_t is a subsolution of our F. B. Problem in its domain of definition. Furthermore, any point of $F(v_t)$ is regular.

PROOF. v_t is the supremum of a family of translations of u, and as such, v is subharmonic both in $\Omega^+(v_t)$ and $\Omega^-(v_t)$. Let now $X_0 \in F(v_t)$. That means that $B_t(X_0)$ is tangent from $\Omega^-(u)$ to $F(u)$ at a point Y_0. Therefore

(*a*) X_0 is regular since $B_t(Y_0) \subset \Omega^+(v)$ and is tangent to $F(v)$ at X_0.
(*b*) At Y_0, u has the asymptotic behavior

$$u = \beta\langle X - Y_0, \nu\rangle^+ - \alpha\langle X - Y_0, \nu\rangle^- + o(|X - Y_0|),$$

with $\beta = G(\alpha)$, and ν the outer normal to $\partial B_t(X_0)$ at Y_0, and hence

$$v \geqslant \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|). \quad \square$$

The family v_t on the previous lemma is an admissible family for the comparison lemma (Lemma 7) and as such it can be used for a comparison principle that says: «If u_1 and u_2 are two weak solutions, with $u_1 \leqslant u_2$ and near $\partial\Omega$, $\sup_{B_\epsilon(X)} u_1 \leqslant u_2(X)$, then also in the interior of Ω $\sup_{B_\epsilon(X)} u_1 \leqslant u_2(X)$», keeping, in particular $F(u_2)$, ϵ-away from $F(u_1)$.

This family has the problem of being too rigid. If u_2 is, for instance, much larger than u_1 in some section of $\partial\Omega$, one cannot exploit that fact. Therefore, we will now introduce a more delicate family of perturbations, where we make the radius of the ball $B_t(X_0)$ dependent on X_0 itself ($t = t(X_0)$).

The key lemma is the following.

Lemma 9. *Let $\varphi(x)$ be a C^2-positive function satisfying*

$$\Delta\varphi \geqslant \frac{C|\nabla\varphi|^2}{|\varphi|}$$

(for C large enough) in $B_1(0)$ (the unit ball of $\mathbb{R}^n$). Let u be continuous, defined in a domain Ω large enough so that the following function be defined in $B_1(0)$

$$w(X) = \sup_{|\nu|=1} u(X + \varphi(x)\nu).$$

Then, if u is harmonic in $\{u > 0\}$, w is subharmonic in $w > 0$.

PROOF. Assume $w(0)$ to be positive. We will show that

$$\varliminf_{r\to 0} \frac{1}{r^2}\left[\fint_{B_1(0)} (w(X) - w(0))\,dx\right] \geqslant 0.$$

For that purpose, we will estimate $w(x)$ by below near 0, choosing an appropriate value for $\nu = \nu(X)$: Choose the system of coordinates so

(1) $w(0) = u(\varphi(0)e_n)$
(2) $\nabla\varphi(0) = \alpha e_1 + \beta e_n$.

We evaluate w by below by choosing $\nu(X) = \nu^*/|\nu^*|$ with

$$(3)\quad \nu^*(X) = e_n + \frac{(\beta x_1 - \alpha x_n)}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)}\left(\sum_2^{n-1} x_i e_i\right).$$

Here γ is chosen so that

(4) $(1 + \gamma)^2 = (1 + \beta)^2 + \alpha^2$.

Let us examine the point Y obtained by such a choice.

$$Y = X + \left\{\varphi(0) + \nabla\varphi(0)X + \frac{1}{2}(D_{ij}\varphi)x_i x_j + o(|X|^2)\right\}$$
$$\left\{\left[e_n + \frac{(\beta x_1 - \alpha x_n)}{\varphi(0)} e_1 + \frac{\gamma}{\varphi(0)} \sum_2^{n-1} x_i e_i\right]\right.$$
$$\left.\left[1 - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi^2(0)} - \left(\frac{\gamma}{\varphi}\right)^2 \sum_2^{n-1} x_i^2 + o(|X|^4)\right]\right\}.$$

The above expression has a constant (translation) term $\varphi(0)e_n$. A first-order term

$$Y^* - \varphi(0)e_n = X + (\alpha x_1 + \beta x_n)e_n + (\beta x_1 - \alpha x_n)e_1 + \gamma \sum x_i e_i$$

than can be thought as a rotation followed by and expansion by $1 + \gamma$ since

$$\frac{1}{1+\gamma}[Y^* - \varphi(0)e_n] = \begin{bmatrix} \frac{1+\beta}{1+\gamma} & \cdots & \frac{-\alpha}{1+\gamma} \\ \vdots & \ddots & \vdots \\ \frac{\alpha}{1+\gamma} & \cdots & \frac{1+\beta}{1+\gamma} \end{bmatrix} X = MX.$$

where M is a rotation in the e_1, e_n plane (by the definition of γ) and a quadratic term

$$Y - Y^* = \left[\frac{1}{2}(D_{ij}\varphi)x_i x_j - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi(0)} + \frac{\gamma^2}{\varphi(0)} \sum_2^{n-1} x_i^2\right] e_n + O\left(\frac{|\nabla\varphi|^2}{\varphi}|X|^2\right)\mu$$

with $\mu \perp e_n$ and $|\mu| = 1$. Hence

$$\begin{aligned} ⨍ w(X) - w(0) &\geqslant ⨍ u(Y(X)) - u(Y(0)) \\ &= ⨍ u(Y(X)) - u(Y^*(X)) + ⨍ u(Y^*(X)) - u(Y(0)) \\ &= ⨍ u(Y(X)) - u(Y^*(X)). \end{aligned}$$

(Since the last term is zero, due to the fact that u is harmonic and Y^* is a rigid rotation plus a dilation of X). We now point out that, by the definition of w, ∇u must point in the direction of e_n at $Y(0)$. Hence

$$\begin{aligned} u(Y) - u(Y^*) &= \nabla u \circ (Y - Y^*) + O(|Y - Y^*|)^2 \\ &= |\nabla u| \left[\frac{1}{2} D_{ij}\varphi x_i x_j - \frac{(\beta x_1 - \alpha x_n)^2}{\varphi(0)} + \frac{\gamma^2}{\varphi(0)} \sum x_i^2\right] + O(|X|^4) \end{aligned}$$

and hence

$$\frac{1}{r^2} ⨍ u(Y) - u(Y^*) + O(|X|^2) =$$

$$= |\nabla u(Y(0))| \left\{\frac{1}{n}\left(\Delta\varphi - [\beta^2 + \alpha^2 + (n-2)\gamma^2]\frac{1}{\varphi(0)}\right)\right\} \geqslant 0$$

if

$$\Delta\varphi > C\frac{|\Delta\varphi|^2}{\varphi}.$$

Remark.

$$\Delta\varphi \geqslant C\frac{|\nabla\varphi|^2}{\varphi}$$

if φ^{1-c} is superharmonic.

We now study a more flexible family of perturbations, namely, given a solution u of our F. B. Problem and a function φ satisfying the properties of Lemma 9, we want to consider $v = v_\varphi$ defined by

$$v(x) = \sup_{B_{\varphi(X)}(X)} u(y).$$

We start with the asymptotic behavior of v at the free boundary.

Lemma 10. *Let u be a continuous function and*

$$v(X) = \sup_{B_{\varphi(X)}(X)} u(Y).$$

with φ a positive C^2 function, and $|\nabla\varphi| < 1$. Assume that

$$X_0 \in \partial\Omega^+(v), \qquad Y_0 \in \partial\Omega^+(u)$$

and that they are related by the fact that

$$Y_0 \in \partial B_{\varphi(X_0)}(X_0).$$

Then

(*a*) *X_0 is a regular point for $F(v)$.*
(*b*) *If near Y_0, u^+ (resp. u^-) has the asymptotic behavior*

$$u^+(\text{resp } u^-) = \alpha\langle Y - Y_0, \nu\rangle^+ + o(|Y - Y_0|)$$

then

$$v^+ \geqslant \alpha\langle X - X_0, \nu + \nabla\varphi\rangle^+ + o(|X - X_0|)$$

(*resp.* $v^- \leqslant \alpha\langle X - X_0, \nu + \nabla\varphi\rangle + o(|X - X_0|)$).
(*c*) *If $F(u)$ is a Lipschitz graph, and $|\nabla\varphi|$ is small enough (depending on the Lipschitz norm, λ, of $F(u)$),* then *$F(v)$ is a Lipschitz graph with Lipschitz norm*

$$\lambda' \leqslant \lambda + C\sup|\nabla\varphi|.$$

PROOF. To prove (*a*), we notice that $\Omega^+(v)$ contains the set

$$\Theta = \{|X - Y_0|^2 < \varphi^2(X)\}.$$

The boundary of this set is a smooth (C^2) surface, since

$$\nabla(|X - Y_0|^2 - \varphi^2(X)) = 2(X - Y_0 - \varphi(X)\nabla\varphi(X)) \neq 0$$

along the surface. Since this surface goes through X_0, (*a*) is proven.

To prove (*b*) we use the fact that near X_0,

$$\varphi(X) \geqslant \varphi(X_0) + \langle X - X_0 \cdot \nabla\varphi(X_0)\rangle + o(|X - X_0|^2).$$

Hence

$$v^+(X) \geqslant \alpha\langle X - X_0, \nu + \nabla\varphi(X_0)\rangle^+ + o(|X - X_0|)$$

and

$$v^-(X) \leqslant \alpha\langle X - X_0, \nu + \nabla\varphi(X_0)\rangle^- + o(|X - X_0|)$$

respectively.

To prove (*c*) it is enough to assume that $\Omega^+(u)$ is above the graph of a smooth convex cone $f(x)$, since the general case is a union of such sets. Then if X_0 and Y_0 are as before, $Y_0 - X_0$ is by definition parallel to the inner unit normal ν to a supporting plane to $F(u)$ at Y_0. About ν we can say that it must lie in a cone of apperture $\arctan\lambda$ around e_{n+1}. On the other hand at X_0, $F(v)$ has the upper and lower envelopes the implicit surfaces

$$S_1 = \{|X - Y_0|^2 - \varphi^2(X) = 0\}$$

and

$$S_2 = \{d(X, \pi)^2 - \varphi^2(X) = 0\}$$

where π is the support plane to $F(u)$ at Y_0. Both surfaces are smooth with unit normal vector, $\bar{\nu}$, parallel to

$$Y_0 - X_0 + \varphi(X_0)\nabla\varphi(X_0)$$

or to

$$\nu + \nabla\varphi(X_0).$$

Therefore, the angle between $\bar{\nu}$ and e_{n+1} is less than

$$\arctan\lambda + |\nabla\varphi|.$$

If $|\nabla\varphi|$ is small enough depending on λ, more precisely $|\nabla\varphi|$, a small multiple of $1/(1+\lambda)$, the angle between $\bar{\nu}$ and e_{n+1} is less than

$$\arctan(\lambda(1+(c+\lambda)|\nabla\varphi|))$$

i.e., $F(v)$ is Lipschitz, with Lipschitz constant

$$\lambda' = \lambda(1+(c+\lambda)|\nabla\varphi|).$$

An important corollary is our next lemma.

Lemma 11. *Let u be a solution of our F. B. Problem and both φ and $v = v_\varphi$ be the functions of Lemmas* 9 *and* 10 *(i.e. φ satisfies the hypothesis of both lemmas). Then*

(*a*) *v is subharmonic in $\Omega^+(v)$ and $\Omega^-(v)$.*
(*b*) *Every point of $F(v)$ is regular.*
(*c*) *At every point of $F(v)$, v satisfies the asymptotic inequality*

$$v(X) \geqslant \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with

$$\frac{\beta}{1-|\nabla\varphi|} \geqslant G\left(\frac{\alpha}{1+|\nabla\varphi|}\right).$$

4. Main Harnack

In this section we develop the basis of our iteration technique. First, two preliminary lemmas:

Lemma 12. *Let $0 \leqslant u_1 \leqslant u_2$ be harmonic functions in $B_\lambda(0)$. Let $\epsilon < \lambda/8$ and assume that on $B_{\lambda-\epsilon}(0)$*

$$v_\epsilon(X) = \sup_{B_\epsilon(X)} u_1(Y) \leqslant u_2(X)$$

and further

$$u_2(0) - v_\epsilon(0) \geqslant \sigma\epsilon u_2(0).$$

Then, for some $\bar{C} = \bar{C}(\lambda)$, $\mu = \mu(\lambda) > 0$, we have in $B_{(3/4)\lambda}$

$$u_2(X) - v_{(1+\mu\sigma)\epsilon}(X) \geqslant \bar{C}\sigma\epsilon u_2(0).$$

PROOF. For any $|\nu| < 1$

$$w(X) = u_2(X) - u_1(X + \epsilon\nu)$$

is harmonic and positive in $B_{\lambda-\epsilon}$. By Harnack's inequality in $B_{3\lambda/4}$

$$w(X) \geqslant Cw(0) \geqslant C\sigma\epsilon u_2(0).$$

Also, both

$$|\nabla u_i(X)| \leqslant \frac{C}{\lambda} u_i(0) \leqslant \frac{C}{\lambda} u_2(0)$$

on $B_{3\lambda/4}$. It follows that

$$\begin{aligned} u_2(X) - u_1(X + (1 + \sigma\mu)\epsilon\nu) &= w(X) + u_1(X + \epsilon\nu) - u_1(X + (1 + \sigma\mu)\epsilon\nu) \\ &\geqslant \sigma\epsilon u_2(0) - \frac{C\mu\sigma}{\lambda}\epsilon u_2(0) \\ &\geqslant \bar{C}\sigma\epsilon u_2(0) \end{aligned}$$

if μ is chosen small. □

Lemma 13. *Let $0 < \lambda < 1/8$, then there exists a θ and a $\mu > 0$, $(\mu(\lambda), \theta(\lambda))$ and a C^2 family of functions φ_t $(0 \leqslant t \leqslant 1)$ defined in $\bar{B}_1 \backslash B_{\lambda/2}(0, 3/4)$, such that*

(i) $1 \leqslant \varphi_t \leqslant 1 + t\mu$
(ii) $\varphi\Delta\varphi \geqslant C|\nabla\varphi|^2$
(iii) $\varphi \equiv 1$ *outside of* $B_{7/8}$
(iv) $\varphi|_{B_{1/2}} \geqslant 1 + \theta t\mu$
(v) $|\nabla\varphi| < Ct\mu$.

PROOF. It is not hard to construct a smooth function ψ_0 in $B_1 \backslash B_{\lambda/2}(0, 3/4)$ such that

$$\begin{cases} 0 \leqslant \psi_0 \leqslant 1 \\ \psi_0 \equiv 0 \quad \text{outside} \quad B_{7/8}(0) \\ |\nabla\psi_0| < C\Delta\psi_0, \quad \text{for some } C \text{ large} \\ \psi_0|_{B_{1/2}} \geqslant \gamma > 0. \end{cases}$$

Then $\varphi_t = 1 + t\mu\psi_0$ is our desired function, provided that μ is small enough. □

Now, a comparison theorem:

Lemma 14. *Let $u_1 \leqslant u_2$ be two solutions of our free-boundary problem in $B_1 \subset \mathbb{R}^{n+1}$ with $F_2 = F(u_2)$ a Lipschitz free boundary through the origin. Assume further that*

$$v_\epsilon(x) = \sup_{B_\epsilon(x)} u_1(y) \leqslant u_2(x)$$

in $B_{1-\epsilon}$, that

$$v_\epsilon\left(0, \frac{3}{4}\right) \leqslant (1 - \sigma\epsilon) u_2\left(0, \frac{3}{4}\right)$$

and that

$$B_\lambda\left(0, \frac{3}{4}\right) \subset \Omega^+(u_1).$$

Then, for ϵ small enough, there exists a δ, depending only on λ and the various constants C, such that on $B_{1/2}$

$$v_{(1+\delta\sigma)\epsilon}(x) = \sup_{B_{(1+\delta\sigma)\epsilon}(x)} u_1(y) \leqslant u_2(x).$$

PROOF. We construct a continuous family of subsolutions $\bar{v}_t$, such that $\bar{v}_0 \leqslant u_2$, $\bar{v}_1|_{B_{1/2}} \geqslant v_{(1+\delta)\epsilon}$, and for which the comparison lemma (Lemma 7), applies. More precisely

$$\bar{v}_t(x) = \sup_{B_{\epsilon\varphi_{\sigma t}}(x)} u_1(y) + C\sigma\epsilon w_t \equiv v_t(x) + C\sigma\epsilon w_t$$

for a small constant $C > 0$, with w_t a continuous function in

$$\Omega = B_{9/10} - B_{\lambda/2}(0, 3/4)$$

defined by

$$\begin{cases} \Delta w_t = 0 \quad \text{in} \quad \Omega^+(v_t) \cap \Omega = \Omega_1 \\ w_t|_{\partial(\Omega^+(v_t) \cap B_{9/10})} = 0 \\ w_t|_{\partial B_{\lambda/2}(0,3/4)} = u_2(0, 3/4). \end{cases}$$

Let us check that $\bar{v}_t$ satisfies the hypothesis of Lemma 7 in Ω with respect to $u = u_2$:

(i) comparison in $B_{9/10} - \Omega^+(v_0)$ is clear. In Ω_1 we compare the boundary values of $\bar{v}_0$ and u_2 thanks to Lemma 12

(ii) follows from our hypothesis and Lemma 12, provided that $\mu = \mu(\lambda)$ is kept small (we should really replace ϵ by any smaller ϵ^1, to ensure the validity of (ii) along ∂B_1, but that is a minor detail)

(iii) follows from part (a) of Lemma 10
(iv) is by constrution.

It only remains to check the fact that $\bar{v}_t$ are indeed subsolutions.

The subharmonicity in Ω^+ and Ω^- follows from Lemma 9. About the asymptotic behavior, we write

$$\bar{v}_t = v_t + C\sigma\epsilon w_t .$$

From Lemma 11, v_t satisfies the asymptotic inequality (c) with

$$\frac{\beta}{1 - \epsilon|\nabla\varphi_{\sigma t}|} \geqslant G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right).$$

Since outside $B_{7/8}$, $|\nabla\varphi| \equiv 0$ the right inequality is satisfied by v_t and hence by $\bar{v}_t$ since w_t is positive. Inside $B_{7/8} \cap \Omega^+(v_t)$, we notice that by Dahlberg's theorem (Lemma 1) $(w_t/v_t) \geqslant C$, provided that $\epsilon\mu$ and hence $\epsilon|\nabla\varphi|$, is kept small to make sure that the $F(v_t)$ are uniformly Lipschitz domains (see Lemma 10(c)). Therefore, from the asymptotic development of Lemma 11(c), we may say that

$$(v_t + C\sigma\epsilon w_t)^+ \geqslant \bar{\beta}\langle X - X_0, \nu\rangle^+ + o(|X - X_0|)$$

with $\bar{\beta} \geqslant (1 + C\sigma\epsilon)\beta$. Therefore, to complete the proof of the theorem, we must prove that, for μ in the definition of $\varphi_{\sigma t}$ small enough,

$$\bar{\beta} \geqslant G(\alpha).$$

From the properties of $G(s)$, $s^{-C}G(s)$ is decreasing. Hence

$$\alpha^{-C}G(\alpha) \leqslant \left[\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right]^{-C} G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right)$$

or

$$G(\alpha) \leqslant (1 + C\epsilon|\nabla\varphi_{\sigma t}|)G\left(\frac{\alpha}{1 + \epsilon|\nabla\varphi_{\sigma t}|}\right) \leqslant \frac{1 + C\epsilon|\nabla\varphi_{\sigma t}|}{1 - \epsilon|\nabla\varphi_{\sigma t}|}\beta$$

$$\leqslant \frac{1 + C\epsilon|\nabla\varphi_{\sigma t}|}{1 - \epsilon|\nabla\varphi_{\sigma t}|}\,\frac{\bar{\beta}}{1 + C\epsilon}.$$

Since $|\nabla\varphi_{\sigma t}| \leqslant C\mu t$, the proof is complete for μ small. □

5. Intermediate Cones

In this section we state an auxiliary lemma about cones in $\mathbb{R}^n$.

We denote by $\alpha(e,f)$ the angle between the vectors e and f, and by $\Gamma(\theta, e)$ the cone of axis e and apperture θ, i.e.

$$\Gamma(\theta, e) = \{\tau : \alpha(\tau, e) < \theta\}.$$

Lemma 16. *Let* $0 < \theta_0 < \theta < \pi/2$ *and let*

$$\Gamma(\theta, e) \subset \Gamma\left(\frac{\pi}{2}, \nu\right) = H(\nu).$$

For $\tau \in \Gamma(\theta/2, e)$, *let*

$$E(\tau) = \frac{\pi}{2} - \left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)$$

and for μ *small, define*

$$\rho(\tau) = |\tau| \sin\left(\frac{\theta}{2} + \mu E(\sigma)\right).$$

Finally, let

$$S_\mu = \bigcup_{\tau \in \Gamma(\theta/2, e)} B_{\rho(\tau)}(\tau).$$

Then, $\exists \bar{\theta}$, $\bar{e}$ *such that*

$$\Gamma(\theta, e) \subset \Gamma(\bar{\theta}, \bar{e}) \subset S_\mu$$

and

$$\frac{\bar{\theta} - \theta}{\pi/2 - \theta} \geqslant Q(\theta_0, \mu) > 0.$$

PROOF. We reduce it to a problem in the plane through stereographic projection. We first restrict ourselves to the sphere, and then project using ν as the north pole. By symmetry, the lemma reduces to the following question in the plane (changing slightly θ, θ_0, μ)

Let $D_\theta(e)$ be a disc in $\mathbb{R}$ of radius $\theta > \theta_0 > 0$. Assume that $D_\theta \subset D_1$, the unit disc. For $0 < \lambda_0 < \lambda < \lambda_1 < 1$, for any $\tau \in D_{\lambda\theta}(e)$, define

$$E(\tau) = (1 - [|\tau| + (1 - \lambda)\theta])$$

(note that $E(\tau) > 0$, since $D_\theta \subset D_1$) and $\rho(\tau) = (1 - \lambda)\theta + \mu E(\tau)$ $(0 < \mu < 1)$. Then

$$S_\mu = \bigcup_{\tau \in D_{\lambda\theta}(e)} B_{\rho(\tau)}(\tau) \supset D_{\bar{\theta}}(\bar{e}) \supset D_\theta(e)$$

with

$$\frac{\bar{\theta} - \theta}{1 - \theta} \geqslant Q(\mu, \theta_0, \lambda_0, \lambda_1) > 0.$$

The proof is an elementary computation. □

6. The Basic Iteration

We are now ready to prove our basic iterative lemmas.

Lemma 17. *Let u be a weak solution of our F. B. Problem on B_1. Assume that, for some $0 < \theta_0 < \theta \leqslant \pi/2$, u is monotonically increasing for any direction $\tau \in \Gamma(\theta, e_m)$. Then, $\exists \mu < 1$, $(\mu(\theta_0))$ and e a unit vector such that, for*

$$\bar{\theta} - \pi/2 = \mu(\theta - \pi/2),$$

the cone

$$\Gamma(\bar{\theta}, e) \supset \Gamma(\theta, e_n)$$

and, on $B_{1/2}$, u is monotonically increasing for any direction $\tau \in \Gamma(\bar{\theta}, e)$.

PROOF. We first point out that $B_{1/4 \sin\theta_0}\left(\frac{3}{4}e_n\right)$ is all contained in Ω^+ by the monotonicity of u. Let ν be the direction of ∇u at $\frac{3}{4}e_n$. Then for any $\tau \in \Gamma(\theta, e_n)$, we have that on $B_{1/4 \sin\theta_0}\left(\frac{3}{4}e_n\right)$, $D_\tau u$ is harmonic and nonnegative, and

$$D_\tau u\left(\frac{3}{4}e_n\right) = \langle \nabla u, \tau \rangle = |\nabla u| \langle \nu, \tau \rangle.$$

From Lemma 4 and Harnack's inequality applied to both $D_\tau u$ and u in $B_{1/4 \sin\theta_0}\left(\frac{3}{4}e_n\right)$, we get

$$D_\tau u|_{B_{1/4 \sin\theta_0}(3e_n/4)} \geqslant C(\sup |\nabla u|)\langle \tau, \nu \rangle \geqslant \sup D_{\epsilon_n} u \geqslant C\Big(\sup_{B_{1/8 \sin\theta_0}(3e_n/4)} u\Big)\langle \nu, \tau \rangle.$$

Let τ be a small vector in $\Gamma(\theta/2, e_n)$, and let $\bar{u}(x) = u(x - \tau)$. We now apply the main Harnack-type Lemma 14 with

$$u_1(x) = \bar{u}(x)$$

$$u_2(x) = u(x)$$

$$\epsilon = |\tau| \sin\frac{\theta}{2}$$

and σ defined as

$$\sigma = C\left(\frac{\pi}{2} - \left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)\right) \sim C\cos\left(\alpha(\tau, \nu) + \frac{\theta}{2}\right)$$

(C to be chosen). Then, the only nontrivial hypothesis is that

$$v_\epsilon\left(0, \frac{3}{4}\right) \leqslant (1 - \sigma\epsilon)u_2\left(0, \frac{3}{4}\right).$$

Let $Y \in B_\epsilon(X)$, $u_1(Y) = u(Y - \tau) = u(X - \tau - (X - Y)) = u(X - \bar{\tau})$ with

$$\alpha(\bar{\tau}, \tau) \leqslant \theta/2$$

(since $|\bar{\tau} - \tau| = |X - Y| \leqslant |\tau| \sin\theta/2$). Also

$$|\bar{\tau}| \geqslant |\tau| - |\tau| \sin\frac{\theta}{2} \geqslant \frac{1}{2}|\tau|.$$

since, $\bar{\tau} \in \theta/2 < \pi/4$. It follows that

$$\inf_{B_{1/8}(3/4\, e_n)} D_{\bar{\tau}}u \geqslant C\left[\sup_{B_{1/8}(3/4\, e_e)} u\right]\langle \nu, \bar{\tau}\rangle$$
$$= C(\sup u)|\bar{\tau}| \cos\alpha(\nu, \bar{\tau})$$
$$\overset{\text{def}}{\geqslant} \sigma\epsilon(\sup u).$$

(Here we chose C in the definition of σ). Hence

$$u(X - \bar{\tau}) \leqslant u(X) - D_{\bar{\tau}}u(\tilde{X}) \geqslant (1 - \sigma\epsilon)u(X)$$

and the hypotheses of the Harnack lemma (Lemma 14) are satisfied.

It follows that on $B_{1/2}$

$$\sup_{B_{(1+\delta\tau)\epsilon}(x)} u(y - \tau) \leqslant u(x).$$

Recalling that $\epsilon = |\tau| \sin\theta/2$, $\sigma = C(\pi/2 - (\alpha(\tau, \nu) + \theta/2))$, we get, for any τ in $\Gamma(\theta/2, e_n)$, that

$$(1 + \delta\sigma)\epsilon = |\tau|\left(\sin\frac{\theta}{2}\right)(1 + \delta CE(\tau))$$

(in the notation of Lemma 16) and for $\theta_0 < \theta < \pi/2$ we get

$$(1 + \delta\sigma)\epsilon \geqslant |\tau| \sin\left(\frac{\theta}{2} + \mu E(\tau)\right), \qquad \mu = \delta C\frac{\theta_0}{2}.$$

The statement above then translates into saying, for any Z of the form $Z = Y - \tau$ (for $Y \in B_{|\tau|\sin(\theta/2+\mu E(\tau))}(X)) = X - (Y - X) - \tau = X - \tau$, with τ in S_μ (of Lemma 16), we have

$$u(Z) \leqslant u(X).$$

That is, u is monotone for any direction τ in S_μ, and in particular in the intermediate cone $\Gamma(\bar{\theta}, \bar{e})$.

The proof of the lemma is now complete.

PROOF OF THEOREM 1. To prove Theorem 1, we repeat inductively Lemma 17, (notice that if u is a solution of our F. B. Problem; $u(\lambda X)/\lambda$ is also a solution in the corresponding domain). We get that if u is a weak solution as in Theorem 1, then on $B_{2^{-k}}$, u is monotone in a cone of directions

$$\Gamma(\theta_k, e_k)$$

with

$$\Gamma(\theta_{k+1}, e_{k+1}) \supset \Gamma(\theta_k, e_k)$$

and

$$\frac{\pi/2 - \theta_{k+1}}{\pi/2 - \theta_k} = \mu < 1.$$

It follows that $\pi/2 - \theta_k \leqslant b^k$ and hence the fact that the free boundary is $C^{1,\alpha}$ at the origin for some $\alpha(b) > 0$. (Note: the first step in the inductive process, i.e. the free boundary being Lipschitz implies u to be monotone in a cone of directions, follows from Lemma 5). □

7. A Generalization

In this last section, we show how to treat the case in which X and ν dependence is introduced in the free-boundary relation and how the restriction on G at infinity are unnecessary. That is, we now consider weak solutions to the free-boundary problem

$$u_\nu^+ = G(u_\nu^+, X, \nu)$$

in the same sense as before, i.e. whenever X_0 has a tangent ball from Ω^+ or Ω^-

$$u = \beta\langle X - X_0, \nu\rangle^+ - \alpha\langle X - X_0, \nu\rangle^-$$

with

$$\beta = G(\alpha, X_0, \nu)$$

(ν given by the radial direction of the tangent ball at X_0) and assume that

(a) $\log G$ is Lipschitz continuous on X and ν for bounded values of u_ν^-,
(b) for u_ν^- in a compact interval $[0, M]$, G is strictly monotone in u_ν^- and $s^{-C}G(s, X, \nu)$ is decreasing in s, $(C = C(M))$.

Then we have

Theorem 2. *Same geometric situation as in Theorem* 1, *u and G satisfying now the conditions above, the same conclusion as in Theorem* 1 *holds.*

In order to prove Theorem 2, we must do two things. First, to show that u is Lipschitz continuous, eliminating the need to impose conditions at infinity on G. Second, to verify that the dependence in X and ν introduce controllable perturbations in our argument. The first step is achieved by the following monotonicity formula, due to Alt, Friedman and myself.

Lemma 18. (See [A-C-F]). *Let u be a continuous function in* B_1, $u(0) = 0$. *Assume that on* $\{u > 0\}$, $\Delta u \geqslant 0$ *and on* $\{u < 0\}$, $\Delta u \leqslant 0$. *Then*, (ρ, σ *are radial and spherical coordinates in* $\mathbb{R}^n$)

$$g(r) = \frac{\int_{B_r} (\nabla u^+)^2 \rho \, d\rho \, d\sigma \int_{B_r} (\nabla u^-)^2 \rho \, d\rho \, d\sigma}{r^4}$$

is an increasing function of r.

Remark. g is shown to be finite from the continuity of u by an approximation of say, u^+, by a smooth function and the fact that

$$(\nabla u^+)^2 \leqslant (\nabla u^+)^2 + u^+ \Delta u^+ = \frac{1}{2} \Delta (u^+)^2$$

and

$$\rho \, d\rho \, d\sigma = \frac{1}{|X|^{n-2}} dx.$$

By integrating by parts, this allows us to control $g(r)$ for say, $r < 1/2$, by $(\sup_{B_1} |u|)^4$.

Lemma 19. (Corollary to Lemma 18). *Let u be a weak solution as in Theorem* 2. *Then u is Lipschitz continuous in* (*say*) $B_{1/4}$.

PROOF. It is enough to prove that $|u(X)| < Cd(X, F)$.

From Lemma 18 and from the remark following it,

$$g(r) \leqslant C\Big(\sup_{B_1} |u|\Big)^4$$

for any $r < 1/2$ and taking as origin any point X_0 on $F \cap B_{1/4}$. We consider two cases:

(*a*) $u|_{\Omega^-} \equiv 0$ or,
(*b*) $u|_{\Omega^-}$ is never zero.

In Case (*a*) let $X \in \Omega^+$, $u(X) = \sigma$, $d(X, F) = \rho$, and $X_0 \in \partial B_\rho(X) \cap F \cap B_{1/2}$. Then by Harnack's inequality, $u|_{B_{\rho/2}(X)} \geqslant C\sigma$ and hence

$$u|_{B_\rho(X)} \geqslant h$$

where h is the auxiliary radially symmetric harmonic function on $B_\rho(X) - B_{\rho/2}(X)$ with values $h|_{\partial B_\rho(X)} = 0$ and $h|_{\partial B_{\rho/2}(X)} = C\sigma$. Since h has linear behaviour

$$h = C\frac{\sigma}{\rho}\langle X - X_0, \nu\rangle$$

near X_0 and

$$\begin{aligned} u &= \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|) \\ &= G(0, X_0, \nu)\langle X - X_0, \nu\rangle^+ + o(|X - X_0|) \end{aligned}$$

we get

$$C\frac{\sigma}{\rho} \leqslant G(0, X_0, \nu) \leqslant C,$$

or

$$\sigma \leqslant C\rho$$

and Case (*a*) is complete.

Case (*b*) (we only prove it for u^+). We proceed as in Case (*a*) and we obtain at X_0 the estimate

$$u(X) = \alpha\langle X - X_0, \nu\rangle^+ + \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with

$$C\frac{\sigma}{\rho} \leqslant \alpha$$

and

$$\alpha = G(\beta, X_0, \nu).$$

We now bring into play the monotonicity formula by pointing out that

$$g(O^+) \geqslant C\alpha^2\beta^2.$$

(Indeed, in any non-tangential domain, $|\langle X - X_0, \nu\rangle| > \delta$, ∇u converges to $\alpha\nu$ (resp. $\beta\nu$)). Therefore,

$$\alpha^2\beta^2 \leqslant C\|u\|^4_{L^\infty(B_1)}$$

and

$$\alpha = G(\beta, X_0, \nu).$$

Since G is monotone in β, and

$$G(1, X_0, \nu) \geqslant \mu_0 > 0$$
$$\beta G(\beta, X_0, \nu) \geqslant \mu_0\beta.$$

Therefore,

$$\beta \leqslant C\|u\|^2_{L^\infty(B_1)} \leqslant C$$

and hence

$$\alpha \leqslant C.$$

It follows that $\sigma/\rho \leqslant C\alpha \leqslant C$ and Case (b) is also proven.

To complete the proof of the theorem, we only need to prove

Lemma 20. *Let $\bar{v}_t$ be the one parameter family of functions constructed in the proof of Lemma* 14. *There exists a $\theta > 0$, depending only on λ and the various constants C such that if*

$$|\log(\alpha, X, \nu) - \log(\alpha, Y, \nu)| \leqslant \theta\sigma|X - Y|$$

for any $\alpha \leqslant \|\nabla u\|_{L^\infty}$ for any $\nu \in S_1$, then $\bar{v}_t$ is still a subsolution of our generalized free boundary problem.

PROOF. We estimate once more the coefficients in the asymptotic inequality (c) of Lemma 11, satisfied by v_t at X_0 in $F(v)$. For that, we go back to Lemma 10, and with the notation there employed, we now have that v satisfies there the assymptotic inequality

$$v(X) \geqslant \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + \sigma(|X - X_0|)$$

with

$$\frac{\beta}{1 - C\epsilon|\nabla\varphi_{\sigma t}|} \geqslant G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_{\sigma t}|}, Y_0, \nu_0\right)$$

where

(a) $Y_0 \in \partial B_{\epsilon\varphi_t}(X_0)$

(b) $\nu_0 = \dfrac{Y_0 - X_0}{|Y_0 - X_0|}$ and

(c) ν is parallel to $\nu + \epsilon\nabla\varphi_{\sigma t}$.

It follows that

$$|Y_0 - X_0| \leqslant C\epsilon$$

and

$$|\nu - \nu_0| \leqslant \epsilon|\nabla\varphi_t|.$$

Therefore

$$\begin{aligned}\log\beta - \log 1 + C\epsilon|\nabla\varphi_{\sigma t}| &\geqslant \log G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_t|}, Y_0, \nu\right)\\ &\geqslant \log G\left(\frac{\alpha}{1 + C\epsilon|\nabla\varphi_t|}, X_0, \nu\right) - \theta\sigma\epsilon - C\epsilon|\nabla\varphi_{\sigma t}|\\ &\geqslant \log G(\alpha, X_0, \nu) - C\epsilon|\nabla\varphi_{\sigma t}| - C\theta\sigma\epsilon.\end{aligned}$$

But $\log\bar{\beta} \geqslant \log\beta + C\sigma\epsilon$ (β and $\bar{\beta}$ being bounded). The proof of the lemma is complete.

PROOF OF THEOREM 2. To prove Theorem 2, we now want to apply the equivalent of Lemma 17 inductively. We want, therefore, to make sure that the hypothesis of the Harnack type Lemma 14 (now Lemma 20) holds. This follows from the fact that after a first Lipschitz expansion,

$$\tilde{u}(X) = \frac{1}{\lambda}u(\lambda X),$$

the Lipschitz norm of $\log G$ in X becomes as small as we wish, ($= \theta$) and that after a k^{th} expansion, $\|\log\|_{\Lambda'(X)} \leqslant \theta 2^{-nk}$ and σ_k can be chosen $\geqslant 2^{-nk}$.

Remark. Only a Hölder condition in X and ν is necessary, but this requires a more careful argument.

References

[At-C] Athanasopoulos, I. and Caffarelli, L. A. A theorem of real analysis and its application to free-boundary problems, *Comm. Pure Appl. Math.* **XXXVIII** No. 5 (1985), 499-502.

[A-C-F] Alt, H. W., Caffarelli, L. A. and Friedman, A. Variational problems with two phases and their free boundaries, *Trans. Amer. Math. Soc.* **282** No. 2 (1984), 431-461.

[C] Caffarelli, L. A. A Harnack inequality approach to the regularity of free boundaries, *Comm. Pure Appl. Math.* **XXXIX** No 5 (Supplement 1986).

[C-F-M-S] Caffarelli, L. A., Fabes, E., Mortola, M. and Salsa, S. Boundary behavior of non-negative solutions of elliptic operators in divergence form, *Indiana J. Math.* **30** (1981), 621-640.

[D] Dahlberg, B. On estimates of harmonic measures, *Arch. Rational Mech. Anal.* **65** (1977), 272-288.

[J-K] Jerison, D. and Kenig, C. Boundary behaviour of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** No. 1 (1982), 80-147.

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Biorthogonalité et Théorie des Opérateurs

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On dispose maintenant de bases hilbertiennes explicites de $L^2(\mathbb{R}^n)$ infiniment plus performantes que la base de Haar. Les fonctions du système de Haar, qui s'écrivent, en dimension 1,

$$h_{j,k}(x) = 2^{j/2}h(2^jx - k),$$

où

$$h(x) = \chi_{[0,1/2[}(x) - \chi_{[1/2,1[}(x) \quad \text{et} \quad j, k \in \mathbb{Z},$$

si elles sont bien localisées en x sont en revanche discontinues, c'est-à-dire très mal localisées en fréquence, empêchant par là une analyse stable des fonctions. Pierre-Gilles Lemarié et Yves Meyer ont montré qu'on pouvait remplacer la fonction h par une fonction $\psi \in S_0(\mathbb{R})$ (l'espace des fonctions de la classe S de Schwartz dont tous les moments sont nuls) et obtenir encore une base hilbertienne de $L^2(\mathbb{R})$, et ils ont indiqué comment généraliser à $\mathbb{R}^n$ la construction ([9]). Leurs bases sont alors bien localisées à la fois en variable d'espace et en variable de fréquence, et permettent d'analyser avec précision n'importe quel signal: ce sont en fait des bases hilbertiennes de tous les Sobolev, et des bases inconditionnelles de la plupart des espaces fonctionnels classiques.

Il peut dès lors paraître surprenant de s'intéresser aux bases inconditionnelles de $L^2(\mathbb{R}^n)$. Cela oblige à étudier des couples de bases inconditionnelles biorthogonales, c'est-à-dire deux systèmes notés pour le moment (σ_α) et (τ_α), tels que

$$\text{(1)} \qquad \forall f \in L^2(\mathbb{R}^n) \qquad \int_{\mathbb{R}^n} |f|^2 \sim \sum_\alpha |\langle f, \sigma_\alpha \rangle|^2 \sim \sum_\alpha |\langle f, \tau_\alpha \rangle|^2$$

$$\text{(2)} \qquad \forall f \in L^2(\mathbb{R}^n) \qquad f = \sum_\alpha \langle f, \sigma_\alpha \rangle \tau_\alpha = \sum_\alpha \langle f, \tau_\alpha \rangle \sigma_\alpha$$

$$\text{(3)} \qquad \langle \sigma_\alpha, \tau_\beta \rangle = \delta_{\alpha,\beta}.$$

Mais justement, le fait essentiel que nous présentons dans cet article, et qui ne peut se produire avec une base hilbertienne, est que les deux systèmes (σ_α) et (τ_α) peuvent être très différents de nature. Par exemple, il peut arriver que les (τ_α) soient des fonctions bien régulières et à supports compacts, tandis que les (σ_α) présentent des singularités en certains points. C'est ce qui permet de prouver qu'il ne peut exister de calcul symbolique sur les opérateurs de Calderón-Zygmund (paragraphe VIII).

En nous plaçant au point de vue de l'analyse fonctionnelle, appelons l_{σ_α} la fonctionnelle qui, à chaque $f \in L^2$, associe sa composante suivant le vecteur de base σ_α. Alors, les fonctions σ_α peuvent être singulières et se comporter plus ou moins comme si elles étaient régulières: les propriétés de régularité sont en fait portées par les fonctionnelles l_{σ_α}. Nous appliquons cela en construisant de nouvelles bases inconditionnelles de $H^1(\mathbb{R}^n)$, constituées de fonctions en escalier (paragraphe VII).

Mais pour commencer, nous décrivons un algorithme construisant des couples de bases inconditionnelles biorthogonales de $L^2(\mathbb{R}^n)$. Les idées de départ, comme dans [9], sont celles de la transformation en ondelettes (voir [6] et [14]) que nous rappelons maintenant.

Cetains des résultats publiés ici ont déjà paru, sans démonstration, dans la note aux C.R.A.S., citée en référence [15].

1. Généralités

Pour le moment, nous restons en dimension 1, le cas de la dimension n faisant l'objet du paragraphe 5.

Si $g = ax + b$, $a > 0$ et $b \in \mathbb{R}$, est un élément du groupe affine G, son action sur une fonction $\sigma \in L^2$ donne la fonction $U(g)\sigma$ définie par

$$U(g)\sigma(x) = a^{1/2}\sigma(ax + b).$$

Suivant Grossmann, Morlet et Paul ([6]), nous appelons ondelette admissible toute fonction σ telle que

$$\text{(4)} \qquad C_\sigma^2 = \int_G |\langle U(g)\sigma, \sigma \rangle|^2 \, d\mu(g) < +\infty,$$

où $d\mu(g) = da\,db/a$ est la mesure de Haar invariante à droite du groupe G. Cette condition est équivalente à la suivante:

$$\forall \xi \neq 0 \qquad \int_0^{+\infty} |\hat{\sigma}(a\xi)|^2 \frac{da}{a} < +\infty.$$

Choisissons une autre ondelette admissible τ, et posons

$$C_{\sigma,\tau} = \frac{1}{\langle \sigma, \tau \rangle} \int_G \langle \sigma, U(g)\sigma \rangle \overline{\langle \tau, U(g)\tau \rangle}\, d\mu(g).$$

Dès que $0 < |C_{\sigma,\tau}| < +\infty$, on a la décomposition suivante de l'identité:

$$\forall f \in L^2, \qquad f = \frac{1}{C_{\sigma,\tau}} \int_G \langle f, U(g)\sigma \rangle U(g)\tau\, d\mu(g). \tag{5}$$

C'est la transformation continue en ondelettes sur L^2.

Pour obtenir des bases de L^2, nous commençons par discrétiser la formule précédente, en imposant $a = 2^j$, $j \in \mathbb{Z}$, et $b \in \mathbb{Z}$. Si I est l'intervalle dyadique $[k2^{-j}, (k+1)2^{-j}]$, où $j, k \in \mathbb{Z}$, nous posons $\sigma_I(x) = 2^{j/2}\sigma(2^j x - k)$, et de même pour τ. La notation rappelle que, en un sens à préciser, les fonctions σ_I et τ_I sont localisées autour de l'intervalle I en variable d'espace, et autour de la bande $1/(2|I|) \leqslant |\xi| \leqslant 2/|I|$ en variable de fréquence. Les relations (1) à (3) se réécrivent alors

$$\int_{\mathbb{R}} |f|^2 \sim \sum_I |\langle f, \sigma_I \rangle|^2 \sim \sum_I |\langle f, \tau_I \rangle|^2, \tag{1}$$

$$f = \sum_I \langle f, \sigma_I \rangle \tau_I = \sum_I \langle f, \tau_I \rangle \sigma_I, \tag{2}$$

$$\langle \sigma_I, \tau_J \rangle = \delta_{I,J} \tag{3}$$

oú il est entendu une fois pour toutes que I et J parcourent l'ensemble des intervalles dyadiques.

La relation (2) est la discrétisation de (5). Il faut remarquer que, parce que les $g_{j,k} = 2^j x - k$ ne forment pas un sous-groupe du groupe affine, il n'est pas du tout nécessaire que le système biorthogonal aux σ_I soit constitué de l'orbite d'une seule fonction τ sous l'action des $g_{j,k}$. C'est une hypothèse que nous imposons pour nous permettre de calculer.

Une deuxième remarque, beaucoup plus importante, est à faire. C'est la relation (1) qui fixe le cadre fonctionnel dans lequel on opère, ici L^2 et l^2. La relation (2) peut avoir un sens dans d'autres espaces fonctionnels que L^2, d'autant plus que la convergence des séries écrites peut être forte ou faible. Nous reviendrons sur ce point quand nous sortirons du cadre L^2. Quant à (3), elle est pratiquement vide de tout contenu fonctionnel, et a un sens dès que le crochet $\langle \sigma, \tau \rangle$ existe. Tout ceci a des répercussions cruciales sur l'invariance des relations (1) à (3).

Si L est un opérateur borné inversible sur L^2, en écrivant $f = L^{-1}Lf = LL^{-1}f$, on voit que les relations (1) à (3) restent vraies pour les familles $(L^*(\sigma_I))$ et $(L^{-1}(\tau_I))$, ou $(L(\sigma_I))$ et $(L^{*-1}(\tau_I))$.

En revanche, cela ne peut être le cas si L est non borné, à cause de (1). Supposons cependant que L ait un domaine dense, soit inversible sur son image, et surtout qu'il soit invariant par translation et jouisse d'une certaine propriété d'homogénéité:

$$\forall t > 0, \qquad L(\varphi(tx)) = t^\alpha L\varphi(tx)$$

pour un certain α (par exemple $L = D$, $L = \sqrt{-\Delta}$, $L = D^\alpha$). Dans ce cas, les relations (2) et (3) sont formellement vraies pour les familles $(L^*\sigma)_I$ et $(L^{-1}\tau)_I$ dès que $L^*\sigma$ et $L^{-1}\tau$ existent. Pour qu'elles aient un sens dans L^2, il reste à s'assurer que

$$\int_{\mathbb{R}} |f|^2 \sim \sum_I |\langle f, (L^*\sigma)_I \rangle|^2 \sim \sum_I |\langle f, (L^{-1}\tau)_I \rangle|^2.$$

En particulier, si les fonctions σ et τ engendrent deux bases biorthogonales de L^2, pour certaines valeurs de α, on aura de nouvelles bases à partir des fonctions $D^\alpha\sigma$ et $D^{-\alpha}\tau$: c'est pourquoi deux bases biorthogonales peuvent être de régularités très différentes.

Enfin, soulignons que les relations (1), (2) et (3) ne sont pas indépendantes. Pour le voir, il est nécessaire de faire un rappel sur la notion de «frame».

On appelle «frame» une famille de fonctions (ψ_α) telle que, pour toute $f \in L^2$, on ait

$$\int |f|^2 \sim \sum_\alpha |\langle f, \psi_\alpha \rangle|^2.$$

Dans [4], Ingrid Daubechies montre qu'un frame est nécessairement une partie génératrice, au sens qu'il existe un autre frame $(\tilde{\psi}_\alpha)$ tel que, si $f \in L^2$,

$$f = \sum_\alpha \langle f, \tilde{\psi}_\alpha \rangle \psi_\alpha = \sum_\alpha \langle f, \psi_\alpha \rangle \tilde{\psi}_\alpha.$$

Cela vient de ce que l'opérateur A qui, à $f \in L^2$, associe les coefficients $\langle f, \psi_\alpha \rangle$, est inversible sur son image. On en déduit que A^*A est inversible sur L^2, et on pose alors $\tilde{\psi}_\alpha = (A^*A)^{-1}\psi_\alpha$.

Ce résultat implique immédiatement que les relations (1) à (3) sont équivalentes aux deux relations

$$\int_{\mathbb{R}} |f|^2 \sim \sum_I |\langle f, \sigma_I \rangle|^2 \quad \text{et} \quad \langle \sigma_I, \tau_J \rangle = \delta_{I,J}.$$

Cette réduction des équations n'est cependant pas très utile dans la pratique, car pour prouver l'inégalité

$$\int_{\mathbb{R}} |f|^2 \leqslant C \sum_I |\langle f, \sigma_I \rangle|^2,$$

on est souvent obligé de passer par (1), et de prouver ensuite que

$$\sum_I |\langle f, \tau_I \rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2.$$

Au paragraphe suivant, nous expliquons pourquoi il est plus facile de démontrer une inégalité dans ce sens.

2. Un Algorithme de Construction de Bases Inconditionnelles de $L^2(\mathbb{R})$

Nous rentrons maintenant dans le vif du sujet, en prouvant le:

Théorème 1. *Soit g une fonction continue* $4\pi-$ *périodique telle que* $g(\xi) > 0$ *si* $\xi \in]0, 4\pi[$, *et* $g(\xi) = 0(\xi^{2p})$ *au voisinage de* 0. *On a normalisé g en imposant* $g(2\pi) = 1$.

Soient

$$p(\xi) = \frac{g(\xi + 2\pi)}{g(\xi) + g(\xi + 2\pi)}$$

et

$$G(\xi) = \frac{1}{g(\xi) + g(\xi + 2\pi)} \prod_{j=1}^{+\infty} g(2^{-j}\xi).$$

On suppose qu'il existe $q \in [1, p]$ *tel que* $(1 + \xi^2)^q G(\xi) \in L^\infty$.

Définissons les fonctions σ *et* τ *par*

$$\hat{\sigma}(\xi) = e^{-i\xi/2} \frac{g(\xi)}{|\xi|^q} \quad et \quad \hat{\tau}(\xi) = e^{-i\xi/2} |\xi|^q G(\xi). \tag{6}$$

Alors, les familles (σ_I) *et* (τ_I) *forment un couple de bases inconditionnelles biorthogonales de* $L^2(\mathbb{R})$.

On peut remplacer dans cet énoncé $|\xi|^q$ par $\mathrm{sgn}(\xi)|\xi|^q$: cela revient à utiliser l'invariance du problème par la transformation de Hilbert. Nous étudierons au paragraphe 6 la question plus délicate de savoir pour quelles valeurs de α

les familles $(D^\alpha \sigma)_I$ et $(D^{-\alpha}\tau)_I$ sont encore des couples de bases de $L^2(\mathbb{R})$. Nous donnerons également, après la démonstration du théorème, des exemples qui montrent que les hypothèses peuvent être satisfaites pour des valeurs de p et q arbitrairement grandes (paragraphe 3). Pour le moment, nous faisons simplement remarquer que ce théorème redonne le système de Haar, à partir de $g(\xi) = \sin^2 \xi/4$, et $p = q = 2$.

Démontrons le théorème, c'est-à-dire les relations (1), (2) et (3) pour σ et τ définies par (6).

Supposons pour commencer avoir montré (2) pour toute $f \in S_0(\mathbb{R})$ (espace des fonctions de la classe de Schwartz dont tous les moments sont nuls) et (3). Nous obtenons (2) pour toute $f \in L^2(\mathbb{R})$ et (1) grâce au lemme suivant, d'un usage constant dans la théorie:

Lemme 1. *Soit σ une ondelette et ν un réel tels que*

$$\sum_I |I|^\nu |\langle \sigma, \sigma_I \rangle| < +\infty \tag{7}$$

on a alors les deux inégalités

$$\forall (\lambda_I) \in l^2, \qquad \int_{\mathbb{R}} \Big| \sum_I \lambda_I \sigma_I \Big|^2 \leqslant C \sum_I |\lambda_I|^2, \tag{8}$$

$$\forall f \in L^2(\mathbb{R}), \qquad \sum_I |\langle f, \sigma_I \rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2 \tag{9}$$

En effert par changement de variable, la condition (7) implique l'existence d'une constante $C > 0$ telle que, pour tout intervalle J,

$$\sum_I \left(\frac{|I|}{|J|} \right)^\nu |\langle \sigma_J, \sigma_I \rangle| \leqslant C.$$

On en déduit (8) en écrivant:

$$\begin{aligned} \sum_I \sum_J \lambda_I \bar{\lambda}_J \langle \sigma_J, \sigma_I \rangle &\leqslant \left(\sum_I |\lambda_I|^2 \sum_J \left(\frac{|J|}{|I|} \right)^\nu |\langle \sigma_I, \sigma_J \rangle| \right)^{1/2} \\ &\quad \left(\sum_J |\lambda_J|^2 \sum_I \left(\frac{|I|}{|J|} \right)^\nu |\langle \sigma_I, \sigma_J \rangle| \right)^{1/2} \\ &\leqslant C \sum_I |\lambda_I|^2. \end{aligned}$$

Quant à l'inégalité (9), c'est la version adjointe de la précédente.

Un critère très utile dans la pratique pour vérifier (7) est le suivant

Lemme 2. *Soit σ une fonction telle que, pour un certain $\delta > 0$,*

$$|\sigma(x)| \leqslant C(1 + x^2)^{-(1+\delta/2)} = C\omega_\delta(x), \tag{10}$$

$$|\sigma(x) - \sigma(x')| \leqslant C|x - x'|^\delta[\omega_\delta(x) + \omega_\delta(x')], \tag{11}$$

$$\int_{\mathbb{R}} \sigma(x)\, dx = 0. \tag{12}$$

Alors, pour tout ν tel que $\left|\nu - \dfrac{1}{2}\right| < \delta$, *$\sigma$ vérifie* (7).

La preuve est laissée au lecteur. Le lemme est d'ailleurs classique en théorie des intégrales singulières: voir Lemarié [8] ou Meyer [12], par exemple.

Les lemmes 2 et 1 s'appliquent immédiatement aux ondelettes σ et τ définies par (6). On en déduit (1) et (2) pour toute $f \in L^2$, si on l'a prouvé pour $f \in S_0(\mathbb{R})$.

Soit donc $f \in S_0(\mathbb{R})$, et posons

$$f_j = \sum_{|I| = 2^{-j}} \langle f, \sigma_I \rangle \tau_I,$$

c'est-à-dire

$$f_j(t) = \sum_k 2^j \left(\int f(u)\sigma(2^j u - k)\, du \right) \tau(2^j t - k).$$

Traduisons cela en variable de Fourier:

$$\hat{f}_j(\xi) = \sum_k 2^{-j} \frac{1}{2\pi} \int \hat{f}(\eta) e^{ik2^{-j}\eta}\, \overline{\hat{\sigma}(2^{-j}\eta)}\, d\eta\, e^{-ik2^{-j}\xi}\, \hat{\tau}(2^{-j}\xi).$$

La formule de Poisson s'applique, et donne

$$\hat{f}_j(\xi) = \sum_{k \in \mathbb{Z}} \hat{f}(\xi + 2\pi k 2^j) \overline{\hat{\sigma}(2^{-j}\xi + 2\pi k)}\, \hat{\tau}(2^{-j}\xi). \tag{13}$$

En sommant sur j et en regroupant les termes, on obtient

$$\sum_{j \in \mathbb{Z}} \hat{f}_j(\xi) = \hat{f}(\xi) \sum_{j \in \mathbb{Z}} \overline{\hat{\sigma}(2^{-j}\xi)}\, \hat{\tau}(2^{-j}\xi)$$
$$+ \sum_{j \in \mathbb{Z}} \sum_{k_0 \in 2\mathbb{Z}+1} \hat{f}(\xi + 2\pi k_0 2^j) \sum_{n \geq 0} \overline{\hat{\sigma}(2^n[2^{-j}\xi + 2\pi k_0])}\, \hat{\tau}(2^{n-j}\xi)$$

(il faut remarquer que tout nombre dyadique $k2^j$ se décompose de manière unique en $k_0 2^{j_0}$, où k_0 est impair et $j_0 = j + n$ pour un certain $n \geqslant 0$).

Cette égalité est valable pour toute $f \in S_0$. On est donc ramené à prouver

$$\forall \xi \neq 0, \qquad \sum_{j \in \mathbb{Z}} \overline{\hat{\sigma}(2^{-j}\xi)}\, \hat{\tau}(2^{-j}\xi) = 1, \tag{14}$$

et

(15) $\forall \xi$ et $\forall k_0$ impair, si $\xi \neq -2\pi k_0$

$$\sum_{n \geq 0} \overline{\hat{\sigma}(2^n[\xi + 2\pi k_0])}\hat{\tau}(2^n \xi) = 0.$$

On remplace maintenant $\hat{\sigma}$ et $\hat{\tau}$ par leurs définitions. Grâce au choix des phases et à la 4π-périodicité de la fonction g, le système infini précédent se réduit au couple d'équations suivantes:

$$\forall \xi \neq 0, \quad \sum_{j \in \mathbb{Z}} g(2^{-j}\xi)G(2^{-j}\xi) = 1 \tag{16}$$

$$\forall \xi \neq 0, \quad \sum_{n \geq 1} g(2^n \xi)G(2^n \xi) = g(\xi + 2\pi)G(\xi). \tag{17}$$

En fait, il suffit de vérifier (17), car, en l'appliquant à $2^{-N}\xi$, on obtient

$$\sum_{j \geq -N+1} g(2^j \xi)G(2^j \xi) = g(2^{-N}\xi + 2\pi)G(2^{-N}\xi).$$

La limite de cette égalité quand N tend vers $+\infty$ donne exactement (16). Quant à (17), elle se déduit de l'équation fonctionnelle satisfaite par g et G:

$$G(\xi)g(\xi + 2\pi) = G(2\xi)[g(2\xi) + g(2\xi + 2\pi)]. \tag{18}$$

En itérant cette relation, et en tenant compte de ce que $\lim_{\xi \to \pm\infty} G(\xi) = 0$ on prouve (17), et par conséquent (2) pour toute $f \in S_0$.

Il reste enfin à prouver (3). Là encore, on utilise la formule de Poisson pour montrer que (3) est équivalent au système d'équations suivant, où $j \geqslant 1$:

$$(A_j) \qquad \sum_{k \in \mathbb{Z}} \overline{\hat{\sigma}(\xi + 2\pi k)}\hat{\tau}(2^j[\xi + 2\pi k]) = 0 \quad \text{pour tout } \xi,$$

$$(B_j) \qquad \sum_{k \in \mathbb{Z}} \overline{\hat{\sigma}(2^j[\xi + 2\pi k])}\hat{\tau}(\xi + 2\pi k) = 0 \quad \text{pour tout } \xi,$$

$$(C) \qquad \sum_{k \in \mathbb{Z}} \overline{\hat{\sigma}(\xi + 2\pi k)}\hat{\tau}(\xi + 2\pi k) = 1 \quad \text{pour tout } \xi.$$

En fonction de g et G, ces équations deviennent:

$$(A_j) \qquad \sum_{k \in \mathbb{Z}} (-1)^k g(\xi + 2\pi k)G(2^j[\xi + 2\pi k]) = 0,$$

$$(B_j) \qquad \sum_{k \in \mathbb{Z}} (-1)^k g(2^j[\xi + 2\pi k])G(\xi + 2\pi k) = 0,$$

$$(C) \qquad \sum_{k \in \mathbb{Z}} g(\xi + 2\pi k)G(\xi + 2\pi k) = 1.$$

La clé est le lemme suivant:

Lemme 3.

$$H(\xi) = \sum_{k\in\mathbb{Z}} G(\xi + 4\pi k) = \frac{1}{g(\xi) + g(\xi + 2\pi)} \quad \textit{pour tout } \xi.$$

Supposons-le acquis. On prouve (C) de la manière suivante:

$$\begin{aligned}\sum_{k\in\mathbb{Z}} g(\xi + 2\pi k)G(\xi + 2\pi k) &= \sum_{k \text{ pair}} + \sum_{k \text{ impair}} \\ &= g(\xi)H(\xi) + g(\xi + 2\pi)H(\xi + 2\pi) \\ &= 1.\end{aligned}$$

Les équations (B_j) deviennent

$$\begin{aligned}\sum_{k\in\mathbb{Z}} (-1)^k g(2^j[\xi + 2\pi k])G(\xi + 2\pi k) &= g(2^j\xi) \sum_{k\in\mathbb{Z}} (-1)^k G(\xi + 2\pi k) \\ &= g(2^j\xi)[H(\xi) - H(\xi + 2\pi)] \\ &= 0.\end{aligned}$$

La preuve des (A_j) est à peine moins simple. On commence par (A_1):

$$\sum_{k\in\mathbb{Z}} (-1)^k g(\xi + 2\pi k)G(2\xi + 4\pi k)$$
$$= g(\xi) \sum_{k \text{ pair}} G(2\xi + 4\pi k) - g(\xi + 2\pi k) \sum_{k \text{ impair}} G(2\xi + 4\pi k),$$

et on utilise la relation (18), qu'on écrit par commodité sous la forme

$$G(2\xi) = H(2\xi)g(\xi + 2\pi)G(\xi).$$

Cela donne

$$\sum_{k\in\mathbb{Z}} (-1)^k g(\xi + 2\pi k)G(2\xi + 4\pi k)$$
$$= H(2\xi)g(\xi)g(\xi + 2\pi)H(\xi) - H(2\xi)g(\xi + 2\pi)g(\xi)H(\xi + 2\pi) = 0.$$

Ensuite, si $j \geqslant 2$, on écrit

$$G(2^j[\xi + 2\pi k]) = H(2^j\xi)g(2^j\xi + 2\pi)G(2^{j-1}[\xi + 2\pi k]),$$

ce qui montre que $(A_{j-1}) \Rightarrow (A_j)$.

Pour achever la démonstration du théorème 1, prouvons le lemme 3. Il est équivalent à l'identité suivante:

$$S(\xi) = \sum_{k\in\mathbb{Z}} R(\xi + 4\pi k) = 1,$$

où l'on a posé

$$R(\xi) = \prod_{j=1}^{+\infty} p(2^{-j}\xi) \qquad \left(\text{et} \quad p(\xi) = \frac{g(\xi + 2\pi)}{g(\xi) + g(\xi + 2\pi)}\right).$$

On a $R(2\xi) = p(\xi)R(\xi)$, et $p(\xi) + p(\xi + 2\pi) = 1$, $p(\xi) > 0$ si $\xi \in]-2\pi, 2\pi[$. On en déduit

$$\begin{aligned} S(2\xi) &= \sum_{k\in\mathbb{Z}} p(\xi + 2\pi k)R(\xi + 2\pi k) \\ &= p(\xi) \sum_{k\,\text{pair}} R(\xi + 2\pi k) + p(\xi + 2\pi) \sum_{k\,\text{impair}} R(\xi + 2\pi k), \end{aligned}$$

ce qui donne l'équation fonctionnelle

$$S(2\xi) = p(\xi)S(\xi) + p(\xi + 2\pi)S(\xi + 2\pi).$$

S étant 4π-périodique, nous nous restreignons à $[-2\pi, 2\pi]$. Si S atteint son maximum en $\xi_0 \in [-2\pi, 2\pi]$, alors S doit l'atteindre aussi en $\xi_0/2$, et donc, par itération, en $\xi_0/2^n$ pour tout n, et en 0 à la limite: on a prouvé $S \leqslant 1$.

Le même raisonnement vaut pour les minima de S, ce qui conclut la démonstration du lemme 3 et du théorème 1.

3. Deux Familles D'exemples

PREMIER EXEMPLE: BASES D'ONDELETTES À SUPPORTS COMPACTS. Si, dans la construction du théorème 1, on veut que les fonctions σ et τ soient à supports compacts, alors il faut imposer à la fonction g une condition supplémentaire. Il faut que g soit un polynôme trigonométrique, et que $g(\xi) + g(\xi + 2\pi) = 1$. A ce moment-là, le produit infini G s'écrit $G(\xi) = \Pi\, g(2^{-j}\xi + 2\pi)$: c'est une fonction entière de type exponentiel, égal à celui de g.

Une classe générale d'exemples est décrite ci-dessous.

Lemme 4. *Soit l'ensemble*

$$A = \{P \in \mathbb{R}[X];\quad P(0) = 0,\quad P(u) > 0 \text{ si } 0 < u \leqslant 1,\quad P(u) + P(1-u) = 1\}.$$

Si $P \in A$, soient p la valuation de P et

$$B = \sup_{u\in[0,1]} \frac{P(u)}{u^p}.$$

On pose

$$g(\xi) = P(\sin^2 \xi/4) \quad \text{et} \quad G(\xi) = \prod_{j=1}^{+\infty} g(2^{-j}\xi + 2\pi).$$

Alors, si

$$q = p - \frac{\log B}{\log 4}, \qquad (1 + \xi^2)^q G(\xi) \in L^\infty.$$

Soit $\xi \geqslant 2$, et j_0 tel que $2^{-j_0-1}\xi \leqslant 1 \leqslant 2^{-j_0}\xi$. Alors,

$$\begin{aligned} G(\xi) &\leqslant \prod_{j=1}^{j_0} g(2^{-j}\xi + 2\pi) \\ &= \prod_{j=1}^{j_0} P(\cos^2 2^{-j}\xi/4) \\ &\leqslant B^{j_0} \prod_{j=1}^{j_0} (\cos^2 2^{-j}\xi/4)^p. \end{aligned}$$

Comme

$$\prod_{j=1}^{\infty} \cos 2^{-j}u = \frac{\sin u}{u},$$

on a

$$G(\xi) \leqslant B^{j_0}\left(\frac{\sin \xi/4}{\xi/4}\right)^{2p}\left(\inf_{|u| \leq 2} \frac{\sin u}{u}\right)^{-2p}.$$

Or, $j_0 \leqslant \log \xi/\log 2$, donc

$$G(\xi) \leqslant C \frac{(\sin^2 \xi/4)^p}{\xi^{2p - \log B/\log 2}}.$$

Construisons maintenant une suite de polynômes $P_n \in A$ telle que les constantes associées

$$q_n = p_n - \frac{\log B_n}{\log 4}$$

tendent vers l'infini.

L'ensemble A jouit de la propriété suivante: si $P, Q \in A$, alors $P \circ Q \in A$. D'autre part, $P_1(u) = 3u^2 - 2u^3 \in A$.

Posons alors $P_n = P_1 \circ \cdots \circ P_1$. La valuation de P_n est $p_n = 2^n$, et un calcul donne $B_n = 3^{2^n - 1}$. On a donc

$$q_n = 2^n\left(1 - \frac{\log 3}{\log 4}\right) + \frac{\log 3}{\log 4}.$$

Si, à n fixé, on prend $q = [q_n]$, les fonctions σ et τ définies par (6), avec g_n, G_n et q, sont à supports compacts et $(q-2)$ fois dérivables. La fonction σ est en fait une fonction spline, et $\sigma^{(q-2)}$ est lipschitzienne. La fonction τ est plus compliquée: on peut seulement affirmer que

$$|\tau^{(q-2)}(x+h) + \tau^{(q-2)}(x-h) - 2\tau^{(q-2)}(x)| \leqslant C|h|.$$

Au paragraphe 7, nous revenons plus en détail sur les fonctions obtenues avec le polynôme P_1.

DEUXIÈME EXEMPLE: LES BASES HILBERTIENNES D'ONDELETTES DE LEMARIÉ ET BATTLE. Chacun de leur côté et indépendamment de nous, P. G. Lemarié, puis G. Battle ont construit des bases orthonormales d'ondelettes, à localisation exponentielle et de régularité finie, mais arbitrairement grande.

C'est leur construction qui nous a suggéré de rechercher une version du théorème publié dans [15] un peu plus forte, qui puisse expliquer les similitudes troublantes qui apparaissent. C'est ainsi que nous avons abouti au théorème 1.

Pour obtenir les bases de Lemarié et Battle, il suffit de résoudre l'équation

$$G(\xi) = \frac{g(\xi)}{\xi^{2m}},$$

où $m \geqslant 1$ est fixé, et G est défini comme d'habitude à partir de g. En utilisant le lemme 3, le lecteur pourra se convaincre que les seules solutions sont données par

$$g(\xi) = \frac{A_m(\xi + 2\pi)}{A_m(\xi) + A_m(\xi + 2\pi)},$$

où

$$A_m(\xi) = \sum_k \frac{1}{(\xi + 4\pi k)^{2m}}.$$

Là encore, les ondelettes obtenues (avec $q = m$) sont $(m-2)$ fois dérivables, et $\sigma^{(m-2)} = \tau^{(m-2)}$ est une fonction lipschitzienne, σ est en fait une fonction-spline sur tout $\mathbb{R}$: sur chaque intervalle $[k/2, (k+1)/2]$, $k \in \mathbb{Z}$, σ est un polynôme de degré $m-1$.

Nous ne savons pas s'il existe des bases orthonormales d'ondelettes à supports compacts et de régularité arbitrairement grande. En tout cas, l'algorithme du théorème 1 n'en permet pas la construction.

Nous allons maintenant passer au cas de la dimension n. Comme dans [9], ce passage dépend du calcul des sommes partielles en dimension 1.

4. Cacul des Sommes Partielles en Dimension 1

Soient, si $f \in L^2(\mathbb{R})$,

$$S_0 f = \sum_{|I| \geq 2} \langle f, \sigma_I \rangle \tau_I \quad \text{et} \quad S_N f = \sum_{|I| \geq 2^{-N+1}} \langle f, \sigma_I \rangle \tau_I .$$

Si δ_N est l'opérateur de dilatation par 2^N: $\delta_N f(x) = f(2^N x)$ alors $S_N = \delta_N S_0 \delta_{-N}$. Pour calculer les S_N il suffit donc de calculer S_0.

Repartons de (13): en sommant sur les indices $j \leqslant -1$, on obtient

$$(S_0 f)^\wedge(\xi) = \sum_{j \leq -1} \sum_k \hat{f}(\xi + 2\pi k 2^j) \overline{\hat{\sigma}(2^{-j}\xi + 2\pi k)} \hat{\tau}(2^{-j}\xi).$$

On écrit alors les entiers k sous la forme $2^n k_0$, où $n \geqslant 0$ et k_0 impair

$$\begin{aligned}(S_0 f)^\wedge(\xi) &= \sum_{j \leq 1} \hat{f}(\xi) \overline{\hat{\sigma}(2^{-j}\xi)} \hat{\tau}(2^{-j}\xi) \\ &\quad + \sum_{j \leq 1} \sum_{n \geq 0} \sum_{\substack{k_0 \\ \text{impair}}} \hat{f}(\xi + 2\pi 2^{j+n} k_0) \overline{\hat{\sigma}(2^{-j}\xi + 2\pi 2^n k_0)} \hat{\tau}(2^{-j}\xi) \\ &= \sum_{j \leq 1} \hat{f}(\xi) \overline{\hat{\sigma}(2^{-j}\xi)} \hat{\tau}(2^{-j}\xi) \\ &\quad + \sum_{j \in \mathbb{Z}} \sum_{\substack{k_0 \\ \text{impair}}} \hat{f}(\xi + 2\pi 2^j k_0) \Bigg(\sum_{\substack{n \geq 0 \\ n \geq j+1}} \overline{\hat{\sigma}(2^n[2^{-j}\xi + 2\pi k_0])} \hat{\tau}(2^{n-j}\xi) \Bigg).\end{aligned}$$

D'après (15), le terme entre parenthèses est nul si $j + 1 \leqslant 0$. Il ne reste donc que les indices $j \in \mathbb{N}$ dans la dernière somme. Ecrivant $2^j k_0 = k$, on en déduit:

$$(S_0 f)^\wedge(\xi) = \sum_{k \in \mathbb{Z}} \hat{f}(\xi + 2\pi k) \Bigg[\sum_{j \geq 1} \overline{\hat{\sigma}(2^j[\xi + 2\pi k])} \hat{\tau}(2^j \xi) \Bigg]. \tag{19}$$

Grâce à (17), on calcule la somme entre crochets:

$$\begin{aligned}\sum_{j \geq 1} \overline{\hat{\sigma}(2^j[\xi + 2\pi k])} \hat{\tau}(2^j \xi) &= \frac{|\xi|^q}{|\xi + 2k\pi|^q} \sum_{j \geq 1} g(2^j[\xi + 2\pi k]) G(2^j \xi) \\ &= \frac{|\xi|^q}{|\xi + 2\pi k|^q} \sum_{j \geq 1} g(2^j \xi) G(2^j \xi) \\ &= \frac{|\xi|^q}{|\xi + 2\pi k|^q} g(\xi + 2\pi) G(\xi).\end{aligned}$$

On vérifie que ce calcul est vrai même si $\xi + 2\pi k = 0$ ou $\xi = 0$.

Soient les fonctions φ et γ définies par

$$\hat{\varphi}(\xi) = \left| \frac{\sin \xi/2}{\xi/2} \right|^q \quad \text{et} \quad \hat{\gamma}(\xi) = \left| \frac{\xi/2}{\sin \xi/2} \right|^q g(\xi + 2\pi) G(\xi).$$

Alors, pour tous ξ et k,

$$\hat{\varphi}(\xi + 2\pi k)\hat{\gamma}(\xi) = \frac{|\xi|}{|\xi + 2\pi k|^q} g(\xi + 2\pi)G(\xi).$$

Une nouvelle application de (13) montre que

$$S_0 f = \sum_{|I|=1} \langle f, \varphi_I \rangle \gamma_I. \tag{21}$$

Une dernière propriété, importante pour la suite, est que $\langle \varphi_I, \gamma_J \rangle = \delta_{I,J}$ si $|I| = |J|$. En effet, on a

$$\begin{aligned}\sum_k \overline{\hat{\varphi}(\xi + 2\pi k)}\hat{\gamma}(\xi + 2\pi k) &= \sum_k g(\xi + 2\pi k + 2\pi)G(\xi + 2\pi k)\\ &= g(\xi + 2\pi)H(\xi) + g(\xi)H(\xi + 2\pi)\\ &= 1.\end{aligned}$$

Les définitions de φ et γ peuvent être modifiées, en particulier pour mieux s'adapter aux ondelettes des exemples du paragraphe précédent (et sans rien changer aux propriétés qu'on vient de démontrer).

Si l'on veut que φ et γ soient à supports compacts, q étant alors un entier, on pose

$$\hat{\varphi}(\xi) = e^{-iq\xi/2}\left(\frac{\sin \xi/2}{\xi/2}\right)^q$$

et

$$\hat{\gamma}(\xi) = e^{-iq\xi/2}\left(\frac{\xi/2}{\sin \xi/2}\right)^q g(\xi + 2\pi)G(\xi)$$

Si l'on veut, dans le cas où $G(\xi) = g(\xi)/\xi^{2m}$, avoir $\varphi = \gamma$, il faut prendre

$$\hat{\varphi}(\xi) = \hat{\gamma}(\xi) = \frac{\sqrt{g(\xi)g(\xi + 2\pi)}}{\xi^m}.$$

Cela montre d'ailleurs que les fonctions φ et γ définissant les opérateurs de sommes partielles suivant (21) ne sont pas uniques.

Enfin, remarquons que la fonction φ, définie dans (20), est indépendante de g et de G. Dans le cas où q est entier, c'est un B-spline (B pour basic), égal à la convolution de la fonction caractéristique de $[-1/2, 1/2]$ q fois avec elle-même.

5. Bases d'Ondelettes en Dimension n

Nous imitons dans ce paragraphe la construction donnée dans [9].

La collection des intervalles dyadiques est maintenant remplacée par celle des cubes dyadiques. Q est un cube dyadique si on a $Q = I_1 \times I_2 \times \cdots \times I_n$, où les I_j sont des intervalles dyadiques de même longueur: Q est donc défini par un indice $j \in \mathbb{Z}$ et un multi-indice $k = (k_1, k_2, \ldots, k_n)$. Si $\psi \in L^2(\mathbb{R}^n)$, on pose $\psi_Q(x) = 2^{nj/2}\psi(2^j x - k)$.

Soit (en dimension 1) D_j la projection sur les τ_I pour $|I| = 2^{-j}$ et E_j la projection sur les γ_I pour les mêmes intervalles. Cela signifie que

$$D_j f = \sum_{|I| = 2^{-j}} \langle f, \sigma_i \rangle \tau_I \quad \text{et} \quad E_j f = \sum_{|I| = 2^{-j}} \langle f, \varphi_I \rangle \gamma_I.$$

Le calcul des sommes partielles se résume par l'égalité

$$D_j = E_{j+1} - E_j. \tag{22}$$

Passons en dimension 2. On a alors

$$D_j \otimes D_j + D_j \otimes E_j + E_j \otimes D_j = E_{j+1} \otimes E_{j+1} - E_j \otimes E_j. \tag{23}$$

Notons $\sigma^0 = \sigma$, $\sigma^1 = \varphi$, $\tau^0 = \tau$ et $\tau^1 = \gamma$. Soient les six ondelettes de $\mathbb{R}^2$ définies par $\sigma^{(\epsilon_1, \epsilon_2)} = \sigma^{\epsilon_1} \otimes \sigma^{\epsilon_2}$ et $\tau^{(\epsilon_1, \epsilon_2)} = \tau^{\epsilon_1} \otimes \tau^{\epsilon_2}$, où $(\epsilon_1, \epsilon_2) = (0,0)$, $(0,1)$ et $(1,0)$. La relation (23) implique

$$\forall f \in L^2(\mathbb{R}^2) \qquad f = \sum_Q \sum_\epsilon \langle f, \sigma_Q^\epsilon \rangle \tau_Q^\epsilon,$$

Q parcourant l'ensemble des cubes dyadiques, et $\epsilon = (\epsilon_1, \epsilon_2)$.

On vérifie ensuite que les familles (σ_Q^ϵ) et (τ_Q^ϵ) sont biorthogonales.

Pour cela, il faut remarquer que $S_0\gamma = \gamma$, ce qui s'écrit

$$\sum_{|I| \geq 2} \langle \gamma, \sigma_I \rangle \tau_I = \sum_I \langle \gamma, \sigma_I \rangle \tau_I.$$

Par conséquent, $\langle \gamma, \sigma_I \rangle = 0$ si $|I| \leqslant 1$. On prouve de la même façon les relations suivantes:

$$\langle \gamma_I, \sigma_J \rangle = 0 \quad \text{si} \quad |J| \leqslant |I| \quad \text{et} \quad \langle \varphi_I, \tau_J \rangle = 0 \quad \text{si} \quad |J| \leqslant |I|.$$

Jointes aux égalités déjà connues:

$$\langle \sigma_I, \tau_J \rangle = \delta_{I,J} \quad \text{et} \quad \langle \varphi_I, \gamma_J \rangle = \delta_{I,J} \quad \text{si} \quad |I| = |J|,$$

on en déduit $\langle \sigma_Q^\epsilon, \tau_{Q'}^{\epsilon'} \rangle = \delta_{Q,Q'}\delta_{\epsilon,\epsilon'}$.

Enfin, les lemmes 1 et 2 se généralisant sans problème, on prouve comme en dimension 1 que, si $f \in L^2$,

$$\int_{\mathbb{R}^2} |f|^2 \sim \sum_Q \sum_\epsilon |\langle f, \sigma_Q^\epsilon \rangle|^2 \sim \sum_Q \sum_\epsilon |\langle f, \tau_Q^\epsilon \rangle|^2.$$

Les choses se passent bien sûr de la même façon en dimension $n \geqslant 3$. On a donc le théorème suivant:

Théorème 2. *Avec les notations du théorème* 1 *et du paragraphe* 4, *soit* $\sigma^0 = \sigma$, $\sigma^1 = \varphi$, $\tau^0 = \tau$, $\tau^1 = \gamma$. *Soit E l'ensemble des indices* $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, *où* $\epsilon_i = 0$ *ou* 1 *et l'un au moins des* ϵ_i *est nul. Définissons* $2^{n+1} - 2$ *ondelettes de* $\mathbb{R}^n$ *par les formules* $\sigma^\epsilon = \sigma^{\epsilon_1} \otimes \cdots \otimes \sigma^{\epsilon_n}$, $\tau^\epsilon = \tau^{\epsilon_1} \otimes \cdots \otimes \tau^{\epsilon_n}$.

Alors, les (σ_Q^ϵ) *et les* (τ_Q^ϵ), *où Q parcourt l'ensemble des cubes dyadiques, forment un couple de bases inconditionnelles biorthogonales de* $L^2(\mathbb{R}^n)$.

Nous disposons maintenant d'une classe de bases de $L^2(\mathbb{R}^n)$, de régularités finies et arbitrairement grandes. Etudions ce qui se passe quand on fait varier la régularité, c'est-à-dire quand le couple (σ, τ) est remplacé par le couple $(D^\alpha \sigma, D^{-\alpha} \tau)$.

6. Nouvelles Bases de L^2 Obtenues en Intégrant par Parties

Nous nous cantonnons en dimension 1, laissant au lecteur le passage à la dimension n.

Ensuite, nous fixons l'exposant q qui intervient dans le théorème 1 et la définition des ondelettes σ et τ en imposant les conditions $(1 + \xi^2)^{q-\epsilon} G(\xi) \in L^\infty$ et $(1 + \xi^2)^{q+\epsilon} G(\xi) \notin L^\infty$ pour tout $\epsilon > 0$.

Notre problème est la détermination des valeurs de α pour lesquelles les familles $(D^\alpha \sigma)_I$ et $(D^{-\alpha} \tau)_I$ sont encore des bases de $L^2(\mathbb{R})$. Ce qui revient à demander: pour quelles valeurs de α a-t-on

$$\forall f \in L^2(\mathbb{R}) \qquad \int_{\mathbb{R}} |f|^2 \sim \sum_I |\langle f, (D^\alpha \sigma)_I \rangle|^2 \sim \sum_I |\langle f, (D^{-\alpha} \tau)_I \rangle|^2 ? \tag{24}$$

If faut naturellement que $D^\alpha \sigma$ et $D^{-\alpha} \tau$ soient dans $L^2(\mathbb{R})$, ce qui impose $|\alpha| < q - 1/2$. Il se trouve que cette condition est suffisante si l'on rajoute certaine hypothèse sur la fonction g.

Théorème 3. *Avec les notations du théorème* 1, *on suppose que*

$$\sum_{n \in \mathbb{Z}} n^2 |\hat{g}(n/2)| < +\infty. \tag{25}$$

Alors les familles $(D^{\alpha}\sigma)_I$ *et* $(D^{-\alpha}\tau)_I$ *forment un couple de bases biorthogonales de* $L^2(\mathbb{R})$ *dès que* $D^{\alpha}\sigma \in L^2(\mathbb{R})$ *et* $D^{-\alpha}\tau \in L^2(\mathbb{R})$.

Les relations (2) et (3) étant formellement stables par intégration par parties, le théorème se ramène à montrer que, si $|\alpha| < q - 1/2$, alors

$$\sum_I |\langle f, (D^{\alpha}\sigma)_I \rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2$$

et de même

$$\sum_I |\langle f, (D^{-\alpha}\tau)_I \rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2.$$

Dans le cas où $|\alpha| < q - 1$, les fonctions $D^{\alpha}\sigma$ et $D^{-\alpha}\tau$ vérifient toutes les deux les hypothèses du lemme 2, et il n'y a rien de plus à démontrer.

Plaçons-nous d'abord dans le cas $q - 1 < \alpha < q - 1/2$, c'est-à-dire celui où

$$(D^{\alpha}\sigma)^{\wedge}(\xi) = e^{-i\xi/2} \frac{g(\xi)}{|\xi|^{1/2+\epsilon}}$$

pour un certain $\epsilon \in]0, 1/2[$. Nous allons prouver que $D^{\alpha}\sigma$ vérifie l'inégalité (7) avec $\nu = 0$ (alors que $D^{\alpha}\sigma$ ne satisfait pour aucun $\delta > 0$ les hypothèses du lemme 2):

$$\sum_{|I| \leq 1} |\langle D^{\alpha}\sigma, (D^{\alpha}\sigma)_I \rangle| < +\infty. \tag{26}$$

Cela suffira, car $D^{-\alpha}\tau$ est bien régulière et décroissante à l'infini: on lui applique le lemme 2.

On a

$$g(\xi) = \sum_n a_n e^{in\xi/2},$$

et

$$\sum_n n^2 |a_n| < +\infty.$$

La somme écrite dans (26) devient

$$\sum_{j \geq 0} \sum_k 2^{j/2} \left| \int_{\mathbb{R}} e^{ik\xi} (D^{\alpha}\sigma)^{\wedge}(2^j\xi) \overline{(D^{\alpha}\sigma)^{\wedge}(\xi)}\, d\xi \right| =$$

$$= \sum_{j \geq 0} \sum_k 2^{-j\epsilon} \left| \int_{\mathbb{R}} e^{i(k + (1/2) - 2^{j-1})\xi} \frac{g(\xi) g(2^j\xi)}{|\xi|^{1+2}}\, d\xi \right|$$

$$\leqslant \sum_n |a_n| \sum_{j \geq 0} 2^{-j\epsilon} \sum_k \left| \int_{\mathbb{R}} e^{i(k + (1/2) + (n-1)2^{j-1})\xi} \frac{g(\xi)}{|\xi|^{1+2\epsilon}}\, d\xi \right| .$$

Tout se ramène donc à montrer

$$\sum_k \left| \int_{\mathbb{R}} e^{ik\xi} \frac{g(\xi)}{|\xi|^{1+2\epsilon}} d\xi \right| \leqslant C \qquad (\text{pour } j = 0)$$

et

$$\sum_k \left| \int_{\mathbb{R}} e^{i(k+(1/2))\xi} \frac{g(\xi)}{|\xi|^{1/2\epsilon}} d\xi \right| \leqslant C \qquad (\text{pour } j \geqslant 1).$$

Or,

$$\int_{\mathbb{R}} e^{ik\xi} \frac{g(\xi)}{|\xi|^{1+2\epsilon}} d\xi = C \sum_n a_n |k + n/2|^2.$$

On utilise alors la qualité du zéro de g en 0: on a au moins $g(\xi) = 0(\xi^2)$, donc $\sum_n a_n = \sum_n n a_n = 0$. Avec l'hypothèse (25) cela implique

$$\sum_n a_n |k + n/2|^{2\epsilon} = 0(|k|^{2\epsilon - 2})$$

à l'infini, et $2\epsilon - 2 < -1$ permet de conclure.

Le cas $\alpha = q - 1$ peut se traiter suivant les mêmes idées, ou bien aussi par interpolation. Comme nous développerons ce point dans le paragraphe suivant, nous laissons ce cas au lecteur.

Il reste enfin le cas $-q + 1/2 < \alpha \leqslant -q + 1$, qui se traite en utilisant la dualité entre les espaces de Beppo-Levi. Rappelons que l'espace B^α est défini par la condition $\int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^2 < +\infty$, et que B^α et $B^{-\alpha}$ sont duaux l'un de l'autre.

Soient les deux opérateurs

$$Sf = \sum_I \langle f, \sigma_I \rangle \sigma_I \quad \text{et} \quad Tf = \sum_I \langle f, \tau_I \rangle \tau_I.$$

Nous venons de prouver la continuité de S sur B^α, pour tout α dans l'intervalle $]-q + 1/2, -q + 1]$. En effet, cette continuité est équivalente à celle de l'opérateur $D^\alpha S D^{-\alpha}$ sur L^2, qui s'écrit

$$D^\alpha S D^{-\alpha} f = \sum_I \langle f, (D^{-\alpha}\sigma)_I \rangle (D^\alpha \sigma)_I.$$

La fonction $D^\alpha \sigma$ vérifiant les conditions du lemme 2, on a

$$\int_{\mathbb{R}} |D^\alpha S D^{-\alpha} f|^2 \leqslant C \sum_I |\langle f, (D^{-\alpha}\sigma)_I \rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2$$

d'après ce qui précède.

De même, nous avons prouvé que T est continu sur $B^{-\alpha}$. Comme S et T sont auto-adjoints, ils sont continus sur $B^{-\alpha}$ et B^{α} respectivement. Ainsi, $D^{-\alpha}SD^{\alpha}$ est continu sur L^2, et comme les $(D^{-\alpha}\sigma)_I$ forment une base de L^2, cela veut dire que

$$\sum_I |\langle f, (D^{\alpha}\sigma)_I\rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2.$$

De même,

$$\sum_I |\langle f, (D^{-\alpha}\tau)_I\rangle|^2 \leqslant C \int_{\mathbb{R}} |f|^2,$$

et le théorème est démontré.

Les valeurs limites de α montrent bien les différences qui peuvent apparaître entre deux bases biorthogonales associées.

Prenons par exemple $\alpha = q - 1/2 - \epsilon$, $0 < \epsilon < 1/2$. Alors,

$$(D^{\alpha}\sigma)(x) = 0\left(\left|x - \frac{n+1}{2}\right|^{-1/2+\epsilon}\right)$$

au voisinage de $(n+1)/2$, si $g(n/2) \neq 0$. $D^{\alpha}\sigma$ n'est donc dans $L^p_{\mathrm{loc}}(\mathbb{R})$ que si $p < 1/(1/2 - \epsilon)$. En revanche, $D^{-\alpha}\tau$ est une fonction suffisamment douce et régulière pour vérifier $|D^{-\alpha}\tau(x)| + |D^{-\alpha+1}\tau(x)| \leqslant C(1 + |x|)^{-3/2-\epsilon}$.

Dans l'autre cas limite $\alpha = -q + 1/2 - \epsilon$, $0 < \epsilon < 1/2$, $D^{-\alpha}\tau \in H^s$ seulement pour $s < \epsilon$, tandis que $D^{\alpha}\sigma \in H^s$ pour $s < 4q - 2\epsilon$.

Le cas $\alpha = q - 1$ est également intéressant, il permet d'obtenir une base inconditionnelle de H^1 (l'espace de Stein et Weiss) constituée de fonctions en escalier.

7. Bases Inconditionnelles sur les L^p, H^1 et BMO

Dans ce paragraphe, nous voulons décrire, dans l'échelle des espaces H^1, L^p et BMO, ceux pour lesquels les familles de fonctions que nous venons de construire sont des bases. Pour l'espace BMO, qui n'est pas séparable, cela doit être entendu au sens faible de la dualité avec H^1. Les autres espaces sont munis de leur topologie forte. Là encore, nous restons en dimension 1.

La théorie des opérateurs de Calderón-Zygmund intervient ici de façon cruciale.

Rappelons qu'un noyau de Calderón-Zygmund est une fonction $K(x, y)$ définie sur $\mathbb{R} \times \mathbb{R}$ privé de sa diagonale, et telle que

$$|K(x, y)| \leqslant \frac{C}{|x - y|}, \tag{27}$$

(28) $$|K(x,y) - K(x',y)| \leqslant C\frac{|x-x'|^{\delta}}{|x-y|^{1+\delta}} \quad \text{si} \quad |x-x'| \leqslant \frac{1}{2}|x-y|,$$

(29) $$|K(x,y) - K(x,y')| \leqslant C\frac{|y-y'|^{\delta}}{|x-y|^{1+\delta}} \quad \text{si} \quad |y-y'| \leqslant \frac{1}{2}|x-y|,$$

pour un certain $\delta \in]0, 1]$ et une certaine constante C.

Un opérateur de Calderón-Zygmund (CZO en abrégé) est un opérateur continu sur L^2 et dont le noyau est de Calderón-Zygmund.† Un CZO est automatiquement continu sur les L^p, $1 < p < +\infty$, de H^1 dans L^1, de BMO dans L^∞, et envoie L^1 dans L^1 faible. Il est continu sur H^1 si $T^*(1) = 0$ (T est le nom de l'opérateur), et continu sur BMO si $T(1) = 0$. Les CZO T tels que $T(1) = T^*(1) = 0$ forment une algèbre, découverte et étudiée par Lemarié ([2], [7] et [8]).

Pour nous, le point important est le suivant:

Lemme 5. *Soient deux ondelettes ψ et $\tilde{\psi}$ vérifiant les hypothèses du lemme* 2. *Alors l'opérateur A défini par*

$$Af = \sum_{i} \langle f, \psi_I \rangle \tilde{\psi}_I$$

est un CZO *appartenant à l'algèbre de Lemarié.*

Les estimations sur le noyau de A viennent de (10) et (11), la continuité sur L^2 de A vient du lemme 2, et l'appartenance à l'algèbre de Lemarié vient de (12).

Revenons à nos familles $(D^\alpha \sigma)_I$ et $(D^{-\alpha}\sigma)_I$, et supposons pour commencer que $|\alpha| < q - 1$.

Soit (ψ_I) une base d'Yves Meyer: $\psi \in S_0(\mathbb{R})$ et (ψ_I) est une base des L^p, $1 < p < +\infty$, de H^1 et de BMO, entre autres ([9]). Alors, les deux opérateurs suivants

$$S_\alpha f = \sum_I \langle f, \psi_I \rangle (D^\alpha \sigma)_I$$

et

$$T_\alpha f = \sum_I \langle f, (D^{-\alpha}\tau)_I \rangle \psi_I$$

sont des CZO inverses l'un de l'autre. Ils sont donc continus sur les L^p, $1 < p < +\infty$, sur H^1 et sur BMO, et inversibles. Cela implique que les $(D^\alpha \sigma)_I$ et les $(D^{-\alpha}\tau)_I$ sont des bases inconditionnelles biorthogonales de ces espaces.

Supposons maintenant que $\alpha = q - 1/2 - \epsilon$, $0 < \epsilon < 1/2$. Nous réservons $\alpha = q - 1$. Alors, T_α reste un CZO, mais pas S_α. On a vu en effet que $D^\alpha \sigma \in L^p_{\text{loc}}$ seulement si $p < 1/(1/2 - \epsilon)$.

Si $\alpha = -q + 1/2 + \epsilon$, $0 < \epsilon < 1/2$, c'est T_α qui cesse d'être un CZO.

Dans ces deux cas, le résultat est le suivant:

Théorème 4. *Avec les notations précédentes, les* $(D^{\alpha}\sigma)_I$ *et les* $(D^{-\alpha}\tau)_I$ *sont des bases inconditionnelles biorthogonales de* L^p *si et seulement si* $|1/p - 1/2| < \epsilon$*, c'est-à-dire si et seulement si* $D^{\alpha}\sigma$ *et* $D^{-\alpha}\tau$ *appartiennent à* $L^p \cap L^{p'}$*, p et p' étant conjugués.*

Démontrons ce théorème quand $\alpha = q - 1/2 - \epsilon$. La nécessité de la condition est évidente. Pour la réciproque, nous prouvons que, si $|1/p - 1/2| < \epsilon$, S_α est continu sur L^p. Comme T_α est un CZO, cela permet de conclure.

La continuité de S_α sur L^p se montre par interpolation complexe. Ecrivons $\alpha = (1 - \lambda)\alpha_1 + \lambda\alpha_2$, où $\alpha < \alpha_1 < q - 1/2$ et $\alpha_2 < q - 1$. Alors, l'ensemble des p pour lesquels il existe $p_1 \in]1, +\infty[$ et $\lambda \in]0, 2\epsilon[$ tels que

$$\frac{1}{p} = \frac{1 - \lambda}{2} + \frac{\lambda}{p_1}$$

est exactement l'intervalle $\left]\dfrac{1}{1/2 + \epsilon}, \dfrac{1}{1/2 - \epsilon}\right[$.

Soit

$$a = \frac{p_1 - 2}{p_1(\alpha - \alpha_2) + 2(\alpha_1 - \alpha)}.$$

Si $z \in \mathbb{C}$ et $\alpha_2 \leqslant Re(z) \leqslant \alpha_1$, posons

$$S_z f = e^{z^2 - \alpha^2} \sum_I \langle f_Z, \psi_I \rangle (\sigma_z)_I,$$

où tout à fait classiquement, si $f = |f| e^{i\theta}$, $f_z = |f|^{a(z-\alpha)+1} e^{i\theta}$, et où

$$\hat{\sigma}_z(\xi) = |\xi|^z \hat{\sigma}(\xi) = e^{-i\xi/2} |\xi|^{z-q} g(\xi).$$

On vérifie alors successivement que

1) Si $Re(z) = \alpha_1$, S_z est uniformément borné sur L_2.
2) Si $Re(z) = \alpha_2$, S_z est uniformément borné sur L_2, et S_z est un CZO, dont les constantes de Calderón-Zygmund (appelées C et δ dans (27), (28) et (29)) sont elles aussi uniformément bornées, grâce au terme $e^{z^2 - \alpha^2}$. On en déduit que S_z est uniformément borné sur L^{p_1}.
3) Si $f, g \in S_0$, alors $\langle S_z f, g \rangle$ est holomorphe dans un voisinage de la bande $\alpha_2 \leqslant Re(z) \leqslant \alpha_1$.

Cela implique que $S_\alpha = S$ est borné sur L^p.

On procède de la même façon quand $\alpha = -q + 1/2 + \epsilon$, en échangeant les rôles de σ et τ. Le théorème est démontré.

Le cas $\alpha = q - 1$ est spécialement intéressant. Les $(D^{\alpha}\sigma)_I$, qui sont des fonctions en escalier, sont, par ce qui précède, une base inconditionnelle de tous les L^p. En cela, elles ressemblent au système de Haar.

EXEMPLE. Soit $g(\xi) = P_1(\sin^2 \xi/4)$, où $P_1(u) = 3u^2 - 2u^3$, et s définie par $\hat{s}(\xi) = e^{-i\xi/2}(g(\xi)/\xi)$. La fonction s est la fonction en escalier à support compact *dont le graphe est représenté ci-dessous.*

Mais il y a pourtant une grande différence entre la famille des (σ_I), où $s = D^{q-1}\sigma$, et le système de Haar, qui réside dans la régularité du système biorthogonal associé. Cela a pour conséquence le

Théorème 5. *Soit g vérifiant les hypothèses du théorème* 3, *et s la fonction en escalier définie par $\hat{s}(\xi) = e^{-i\xi/2}(g(\xi)/\xi)$. Alors, les (σ_I) forment une base inconditionnelle de $H^1(\mathbb{R})$.*

Par dualité, ce théorème se ramène à l'énoncé suivant: $f \in \mathrm{BMO}(\mathbb{R})$ si et seulement si, pour tout intervalle I, on a

$$\sum_{J \subset I} |\langle f, s_J \rangle|^2 \leqslant C|I|, \tag{30}$$

la plus petite des constantes étant équivalente à $\|f\|^2_{\mathrm{BMO}}$. Car, si (t_I) est le système biorthogonal associé aux (s_I), cela veut dire que (t_I) est une base inconditionnelle (au sens faible) de BMO.

Supposons pour commencer $f \in \mathrm{BMO}$, et $m_I f = 0$. Par changement de variable, il suffit de montrer (30) quand $I = [0, 1]$. Posons, si $l \geqslant 1$, $\chi_l = \chi_{2^l \le |x| \le 2^{l+1}}$, $f_l = f\chi_l$, et $f_0 = f\chi_{\{|x| \le 2\}}$. Grâce au théorème 3, on a

$$\sum_{J\subset[0,1]} |\langle f_0, s_J\rangle|^2 \leqslant \int_{\mathbb{R}} |f_0|^2 \leqslant C\|f\|^2_{\mathrm{BMO}}.$$

Si $l \geqslant 1$, on écrit

$$\sum_{J\subset[0,1]} |\langle f_l, s_J\rangle|^2 = \langle Pf_l, f_l\rangle \leqslant \|\chi_l P \chi_l\|_{2,2} \|f_l\|_2^2,$$

où P est l'opérateur défini par

$$Pf = \sum_{J\subset[0,1]} \langle f, s_J\rangle s_J.$$

Puisque $f \in \mathrm{BMO}$, $\|f_l\|_2^2 \leqslant C(1+l^2)2^l\|f\|^2_{\mathrm{BMO}}$. D'autre part, le lemme de Schur montre que

$$\|\chi_l P \chi_l\|_{2,2} \leqslant \sup_{K\subset[0,1]} \sum_{J\subset[0,1]} |\langle \chi_l s_J, s_K\rangle|.$$

La majoration du membre de droite est un peu longue, mais sans difficulté spéciale. Nous nous contentons donc d'indiquer la marche à suivre. Il suffit d'écrire explicitement la fonction s.

Si

$$g(\xi) = \sum_n a_n e^{in\xi/2},$$

et si

$$r_p = \sum_{n\leq p} a_n,$$

on a

$$s(x) = \sum_p r_p \chi_{\{(p-1)/2 \leq x \leq p/2\}},$$

et

$$\sum_p p|r_p| < +\infty.$$

On calcule s_j et s_k, puis $\langle \chi_l s_j, s_k\rangle$. On majore brutalement $\sum_{J\subset[0,1]} |\langle \chi_l s_j, s_k\rangle|$, en tenant compte des localisations imposées par la présence de χ_l et les conditions $J \subset [0,1]$, $K \subset [0,1]$. Il vient alors $\|\chi_l P \chi_l\|_{2,2} \leqslant C2^{-2l}$.

Par conséquent,

$$\sum_{J\subset[0,1]} |\langle f_l, s_J\rangle|^2 \leqslant C(1+l^2)2^{-l}\|f\|^2_{\mathrm{BMO}},$$

d'où le résultat désiré

$$\sum_{J\subset[0,1]} |\langle f, s_J\rangle|^2 \leqslant C\|f\|^2_{\mathrm{BMO}}.$$

Supposons maintenant que, pout tout intervalle I, on ait

$$\sum_{J\subset I} |\langle f, s_J\rangle|^2 \leqslant C|I|.$$

On montre que $f\in \mathrm{BMO}$ en construisant un CZO, U, tel que $f = U(1)$. Pour cela, si $I = [k2^{-j}, (k+1)2^{-j}]$, et si φ est une fonction bosse ($\varphi \in D(\mathbb{R})$, $\varphi \geqslant 0$ et $\int\varphi = 1$), on pose $\varphi_I(x) = 2^j\varphi(2^jx - k)$ (noter la différence de normalisation). L'opérateur U est le paraproduit (voir [13]) donné par le noyau

$$U(x,y) = \sum_I \langle f, s_I\rangle t_I(x)\varphi_I(y),$$

où (t_I) est, rappelons-le, la famille biorthogonale associée aux (s_I). Il est connu que la condition (30) implique la continuité de U sur L^2 ([13]). U est un CZO grâce à la condition plus faible $|\langle f, s_I\rangle| \leqslant C|I|^{1/2}$, et surtout grâce à la régularité et à la décroissance des fonctions t_I. Donc, $U(1)\in\mathrm{BMO}$. On vérifie facilement que $U(1) = f$ au sens des distributions, ce qui achève de prouver le théorème.

Nous terminons cette étude banachique par l'espace L^1: il n'admet pas de bases inconditionnelles, cela est bien connu. Mais on a:

Théorème 6. *Soient σ et τ comme au théorème* 1, *et soient φ et γ les fonctions associées aux sommes partielles décrites au paragraphe* 1.

Alors, les deux systèmes $(\sigma_I)_{|I|\le 1}$, $(\varphi_I)_{|I|=2}$ et $(\tau_I)_{|I|\le 1}$, $(\gamma_I)_{|I|=2}$ forment un couple de bases conditionnelles biorthogonales de $L^1(\mathbb{R})$. De plus, si

$$f = \sum_{|I|\le 1}\lambda_I\sigma_I + \sum_{|I|=2}\lambda_I\varphi_I \in L^1 \quad et \quad \epsilon_I = \pm 1,$$

alors

$$f_\epsilon = \sum_{|I|\le 1}\epsilon_I\lambda_I\sigma_I + \sum_{|I|\le 1}\epsilon_I\lambda_I\varphi_I$$

est dans l'espace L^1_{faible}, et de même si σ et φ sont remplacées par τ et γ.

Rappelons que L^1_{faible} est l'espace des fonctions telles que, pour tout $\lambda > 0$, $|\{x\colon |f(x)| > \lambda\}| \leqslant C/\lambda$.

Montrons que les $(\sigma_I)_{|I|\le 1}$ et $(\varphi_I)_{|I|=2}$ sont une base de L^1. Il faut voir pourquoi, si $f\in L^1$,

$$\lim_{N\to\infty}\left\| f - \sum_{2^{-N-1}\le |I|\le 1}\lambda_I\sigma_I - \sum_{|I|=2}\lambda_I\varphi_I\right\|_1 = 0,$$

où $\lambda_I = \langle f, \tau_I\rangle$ si $|I| \leqslant 1$ et $\lambda_I = \langle f, \gamma_I\rangle$ si $|I| = 2$. Mais cela s'écrit

$$(31) \qquad \lim_{N\to+\infty}\left\| f - \sum_{|I|=2^{-N}}\langle f, \gamma_I\rangle\varphi_I\right\|_1 = 0.$$

Or, on vérifie que $\varphi(2\pi k) = \delta_{0,k}$ donc $\sum_k \varphi(x-k) = 1$. Par conséquent

$$f(x) - \sum_{|I|=2^{-N}} \langle f, \gamma_I \rangle \varphi_I(x) = \sum_k 2^N \varphi(2^N x - k) \int_{\mathbb{R}} [f(x) - f(t)] \gamma(2^N t - k)\, dt$$

d'où

$$\Big\| f - \sum_{|I|=2^{-N}} \langle f, \gamma_I \rangle \varphi_I \Big\|_1 \leqslant \iint |f(x) - f(t)| \sum_k 2^N |\gamma(2^N t - k)|\, |\varphi(2^N x - k)|\, dt\, dx.$$

$$\leqslant C \iint |f(x) - f(t)| \frac{2^N}{1 + (2^N |t - x|)^2}\, dt\, dx$$

Il est maintenant classique de conclure par (31).

Il reste à voir pourquoi $f_\epsilon \in L^1_{\text{faible}}$: cela vient de la théorie des CZO. En effet, $f_\epsilon = Y_\epsilon(f)$, où Y_ϵ est de noyau

$$\sum_{|I|\leq 1} \epsilon_I \sigma_I(x) \tau_I(y) + \sum_{|I|=2} \epsilon_I \varphi_I(x) \gamma_I(y).$$

Y_ϵ est continu sur L^2 d'après les paragraphes 2 et 4, et son noyau est de Calderón-Zygmund. Donc, Y_ϵ envoie L^1 dans L^1_{faible}.

Les preuves des résultats précédents sont, on l'a vu, fortement liées à la théorie des CZO. En fait, il y a presque toujours deux versants au moins d'un même résultat: un versant banachique et un versant théorie des opérateurs. Il est temps de passer de ce côte-ci.

8. De l'Impossibilité d'un Calcul Symbolique Classique sur les CZO

Nous allons reprendre, dans le cadre des opérateurs de Calderón-Zygmund, un programme introduit par R. Beals dans le cadre des opérateurs pseudo-différentiels.

Soit $\mathfrak{A} \subset \mathcal{L}(H, H)$ une algèbre d'opérateurs continus sur un espace de Hilbert H. Dans les exemples qui nous intéressent, cette algèbre n'est pas fermée (pour la norme d'opérateur). La question du calcul symbolique consiste à savoir si tout opérateur $T \in \mathfrak{A}$ inversible dans $\mathcal{L}(H, H)$ est inversible dans $\mathfrak{A}$. R. Beals a montré que c'est effectivement le cas pour les algèbres d'opérateurs pseudo-différentiels.

Nous allons voir que la réponse est non pour l'algèbre A de Lemarié où A est l'algèbre des $T \in \text{CZO}$ tels que $T(1) = T^*(1) = 0$.

Théorème 7. *Pour tout $\epsilon \in]0, 1/2[$, il existe $T \in A$ inversible sur L^2 et non inversible sur L^p pour $|1/p - 1/2| \geqslant \epsilon$. En particulier, $T^{-1} \notin A$.*

Il existe également un opérateur $T \in A$ inversible sur tous les L^p, $1 < p < +\infty$, et tel que T^{-1} ne soit pas un CZO.

Le théorème résulte immédiatement des résultats du paragraphe précédent. En gardant les mêmes notations, et si $\epsilon \in]0, 1/2[$ est fixé, soit T_ϵ l'opérateur défini par

$$T_\epsilon f = \sum_I \langle f, (D^{-\alpha}\tau)_I \rangle (D^{-\alpha}\tau)_I,$$

où $\alpha = q - 1/2 - \epsilon$.

Grâce au lemme 5, on voit que $T_\epsilon \in A$; T_ϵ est inversible sur L^2, et

$$T_\epsilon^{-1}((D^{-\alpha}\tau)_I) = (D^\alpha \sigma)_I.$$

D'après le théorème 5, T_ϵ est inversible sur L^p exactement quand $|1/p - 1/2| < \epsilon$.

Le choix $\alpha = q - 1$ donne de la même manière la deuxième partie de l'énoncé.

Remarques. 1) L'inversibilité d'un opérateur quelconque T sur L^2, supposé continu sur les L^p, entraine automatiquement l'existence d'un $\epsilon > 0$ tel que T soit inversible sur L^p pour $|1/p - 1/2| < \epsilon$ (communication personnelle de G. David et S. Semmes).

2) Nos contre-exemples ont des noyaux assez peu réguliers, qui vérifient (28) et (29) avec $\delta = 1/2$. P. G. Lemarié a construit des contre-exemples T_ϵ dont les noyaux vérifient, pour tous p, q, $|\partial_x^p \partial_y^q K(x - y)| \leqslant C_{p,q}|x - y|^{-p-q-1}$, et tels que $T_\epsilon(x^n) = T_\epsilon^*(x^n) = 0$ pour tout n.

9. Conclusion

Bien que la théorie des CZO joue un rôle central dans l'étude des bases hilbertiennes ou inconditionnelles de L^2 du type de celles d'Y. Meyer, de Battle et Lemarié, ou des théorèmes 1 et 2, la classe des opérateurs naturellement liés à la transformation en ondelettes, telle qu'elle est décrite dans le lemme 1, est strictement plus large que la classe des CZO de l'algèbre de Lemarié.

Du point de vue banachique, cela exprime que les bases de L^2 qu'on peut construire en discrétisant la transformation en ondelettes ne sont pas nécessairement des bases de L^p, H^1 ou BMO. La raison essentielle en est la différence de nature pouvant exister entre deux bases biorthogonales de L^2.

On aimerait comprendre d'où provient cette différence. Plus précisément, étant données des σ_I formant une base inconditionnelle de L^2, localisées autour de l'intervalle I en variable d'espace et autour des fréquences de module $1/|I|$, que peut-on dire du système biorthogonal associé?

En théorie des opérateurs, cette question peut par exemple devenir si T est un CZO de l'algèbre de Lemarié inversible sur H^1 et BMO, est-ce que T^{-1} est lui aussi un CZO?

Je tiens à remercier très vivement Yves Meyer.

Le théorème 5, en particulier, existe grâce aux discussions que nous avons eues. Et c'est l'ensemble de cet article qui doit beaucoup à sa chaleureuse attention.

Bibliographie

[1] Calderón, A. P. Commutators, singular integrals on Lipschitz curves and applications, *Actes du Congrès d'Helsinki*, 1978.

[2] Coifman, R. R. et Meyer, Y. Au-delà des opérateurs pseudo-différentiels. *Astérisque*, N.° 57.

[3] Coifman, R. R. et Meyer, Y. The discrete wavelet transform. Séminaire de Mathématique du Courant Institute.

[4] Daubechies, I. (Communication orale et texte en préparation).

[5] David, G., Journé, J. L., Semmes, S. Opérateur de Calderón-Zygmund et fonctions para-accrétives. *Revista Matemática Iberoamericana*, *1*(4), (1985), 1-56.

[6] Grossman, A., Morlet, J. et Paul, T. Transforms associated to square integrable group representations, I. General results, *J. Math. Phys.*, **26**(10), October 1985, 2473-2479.

[7] Journé, J. L. Calderón-Zygmund operators, pseudo-differential equations and the Cauchy integral of Calderón, L.N. 994, Springer Verlag.

[8] Lemarié, P. G. Algèbres d'opérateurs et semi-groupes de Poisson sur un espace de nature homogène, Thèse de 3e cycle, 1984, Univ. Paris XI, Centre d'Orsay.

[9] Lemarié, P. G. et Meyer, Y. Ondelettes et bases hilbertiennes.

[10] McIntoch, A. et Meyer, Y. Algèbres d'opérateurs définis par des intégrales singulières, *C.R. Acad. Sc.*, Paris.

[11] Meyer, Y. Remarques sur un théorème de J. M. Bony, *Suppl. Rend. Circ. Mat. Palermo,* **1**(1981), 1-20.

[12] Meyer, Y. Minimalité de certains espaces fonctionnels et applications à la théorie des opérateurs. Séminaire Bony-Sjöstrand-Meyer, 1984-1985, Centre de Mathématiques, Ecole Polytechnique.

[13] Meyer, Y. La transformation en ondelettes et les nouveaux paraproduits, A paraître aux Actes du Colloque d'Analyse non linéaire du CEREMADE, Univ. Paris-Dauphine.

[14] Meyer, Y. Principe d'incertitude, bases hilbertiennes et algèbrés d'opérateurs, Séminaire Bourbaki, 38ème année, 1985-86, N.° 662.

[15] Tchamitchian, P. Calcul symbolique sur les opérateurs de Calderón-Zygmund et bases inconditionnelles de $L^2(\mathbb{R})$, *C. R. Acad. Sci.*, Paris, t. 303, Série 1, N.°6, 1986.

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Hardy Spaces and the Dirichlet Problem on Lipschitz Domains

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Introduction

Our concern in this paper is to describe a class of Hardy spaces $H^p(D)$ for $1 \leqslant p < 2$ on a Lipschitz domain $D \subset \mathbb{R}^n$ when $n \geqslant 3$, and a certain smooth counterpart of $H^p(D)$ on $\mathbb{R}^{n-1}$, by providing an atomic decomposition and a description of their duals. For a Lipschitz domain D,

$$H^p(D, d\sigma) = \{u: \Delta u = 0 \text{ in } D \text{ and } Nu(Q) \in L^p(\partial D, d\sigma)\}$$

where $Nu(Q) = \sup_{\Gamma(Q)} |u(x)|$ is the nontangential maximal function. When $p \geqslant 2$ H^p and L^p are essentially the same. When the dimension $n = 2$, $H^p(D)$ can be understood in terms of conformal mappings onto the upper half plane (Kenig [20]).

In 1979, B. Dahlberg overcame one major obstacle in providing the atomic decomposition of $H^1(D, d\sigma)$ in higher dimensions by showing that appropriately defined atoms belong to H^1. However, the pairing between BMO and H^1 was not established since, as we show, the most natural class of measures arising from the harmonic extensions of BMO functions do not satisfy the right Carleson measure condition.

At this point we would like to mention the work of Jerison-Kenig [18] and Dahlberg-Kenig [10] where the analogous theory on Lipschitz (and even NTA) domains was carried out for $H^p(D, d\omega)$ for harmonic functions ([18]) and systems of conjugate harmonic functions ([10]).

The paper is organized as follows. At the begining of section 1 we describe the notation to be used throughout. We then explain why Dahlberg's lemma (which was stated in [8], without proof) on the harmonic extension of BMO functions fails. In addition, we give an example which shows that there is no Carleson measure condition on the *harmonic* extension of a BMO function.

In section 2 we give (for completeness) the proof of Dahlberg's lemma on atoms ([8]). In order to exhibit the duality between H^1 and BMO, one requires that some extension of a BMO function be a Carleson measure. At the end of this section, we discuss which properties such an extension must satisfy, and give the motivation for the work which follows.

There are two approaches to obtain such an extension result. One approach is by duality, giving an atomic decomposition via a grand maximal function; the other approach is constructive, as in Varopoulos [25]. In section 3 we consider a related space, $H^p(w\,dx)$, a space of distributions on $\mathbb{R}^{n-1}$, where the weight $w(x)$ appears in the kernel used to define the maximal functions, and not as a weight on Lebesgue measure dx. For this space of distributions we prove that definitions in terms of grand maximal functions or vertical maximal functions are equivalent, give an atomic decomposition, a description of the dual space, and a Varopoulos type extension theorem for the dual. At the end of this section, a constructive proof of the extension theorem is given.

In section 4 we use the extension result and a localization argument to obtain the atomic decomposition for $H^1(D, d\sigma)$ and duality with $\mathrm{BMO}_\sigma(\omega)$. This is carried out first for starlike domains, and a separate argument gives the duality and decomposition for a general Lipschitz domain. Section 5 is devoted to the analogous results for $H^p(D, d\sigma)$, $1 < p < 2$. In this case the dual of H^p is characterized by a weighted «sharp» function, which arises from the defining condition of $\mathrm{BMO}_\sigma(\omega)$.

We would like to thank J. O. Stromberg for helpful conversations on weighted $H^p(\mathbb{R}^n)$ spaces and for preprints of his joint work with A. Torchinsky on this topic.

1. We begin by reviewing some basic facts about harmonic functions on a Lipschitz domain $D \subseteq \mathbb{R}^n$ and by setting up the notation to be used throughout. For $P \in D$, $d\omega^P$ is harmonic measure evaluated at P and $k(P, \cdot) = d\omega^P/d\sigma$, where d$\sigma$ is surface measure on ∂D. Then if $G(P, \cdot)$ is the Green's function with pole at P, we have

$$k(P, \cdot) = \frac{\partial}{\partial n} G(P, \cdot).$$

In D we fix an arbitrary P_0 and set $G(X) = G(P_0, X)$, $k(Q) = k(P_0, Q)$ and $d\omega = d\omega^{P_0}$. If $f(Q)$ is defined on ∂D, $u(P) = \int_{\partial D} f(Q) k(P, Q)\, d\sigma(Q)$ is the harmonic extension of f to D, where $d\sigma(Q)$ is surface measure on ∂D.

To each point $Q \in \partial D$ is associated a cone

$$\Gamma(Q) = \{P \in D: |P - Q| \leqslant c \operatorname{dist}(P, \partial D)\}$$

contained in D. For u harmonic in D, $Nu(Q) = \sup_{x \in \Gamma(Q)} |u(x)|$ is the non-tangential maximal function of u and

$$Su(Q) = \left\{\int_{\Gamma(Q)} |\nabla u(X)|^2 d(X)^{-n+2} dX\right\}^{1/2}$$

is its square function.

Definition. $H^p(D, d\sigma) = \{u: \Delta u = 0$ *in* D, $u(P_0) = 0$ *and* $Nu \in L^p(\partial D, d\sigma)\}$, *and* $\|u\|_{H^p} = \|Nu\|_{L^p(d\sigma)}$.

By Dahlberg's theorem ([7]),

$$\|Nu\|_{L^p(d\sigma)} \approx \|Su\|_{L^p(d\sigma)} \quad \text{for all} \quad p > 0,$$

and so $H^p(D, d\sigma)$ could just as well be defined using square functions.

The normalizing condition $u(P_0) = 0$ says that if $u(P) = \int f(Q)\, d\omega^P(Q)$ for some function f then $\int_{\partial D} f\, d\omega = 0$. With this in mind we say that a harmonic function a on D is an *atom* if there exists a surface ball $\Delta \subset \partial D$ such that $a \equiv 0$ on $\partial D \setminus \Delta$, $\|a\|_\infty \leqslant \sigma(\Delta)^{-1}$ and $\int_\Delta a(Q)\, d\omega = a(P_0) = 0$. We will sometimes identify an atom a with its harmonic extension A.

Definitions.

(1) $H^1_{at}(\partial D, d\sigma) = \{f: f = \sum \lambda_k a_k$ *where the* a_k *are atoms and* $\sum |\lambda_k| < \infty\}$.
(2) $\mathrm{BMO}_\sigma(\omega) = \{g$: *there exists* $C < \infty$ *such that*

$$\sup_{\Delta \subseteq \partial D} \left\{\int_\Delta |g - g_\Delta|\, d\omega / s(\Delta) + \int_{\partial D} |g(Q)|\, d\omega \leqslant C\right\},$$

where

$$g_\Delta = \frac{1}{\omega(\Delta)} \int_\Delta g(Q)\, d\omega.$$

(3) $$VMO_\sigma(\omega) = \left\{g \in BMO_\sigma(\omega): \overline{\lim_{\sigma(\Delta) \to 0}} \int_\Delta |g - g_\Delta|\, d\omega / \sigma(\Delta) = 0\right\}.$$

The space $H^1(D, d\sigma)$ is in fact a space of extensions of distributions, as not every element of H^1 is the Poisson integral of its pointwise boundary values. We should like to identify these boundary distributions with $H^1_{at}(\partial D, d\sigma)$ and thereby show its dual to be $\mathrm{BMO}_\sigma(\omega)$. At this point we take a closer look at $\mathrm{BMO}_\sigma(\omega)$. In particular, an alternative Carleson characterization of $\mathrm{BMO}_\sigma(\omega)$ will be considered, and rejected.

Suppose that $u(x)$ and $v(x)$ are the Poisson extensions of functions f and g on ∂D satisfying $u(P_0) = v(P_0) = 0$. Assume $u \in H^1(D, d\sigma)$. Then by Green's formula, together with the relationship between $G(x)$ and harmonic measure,

$$\int_{\partial D} f(Q)g(Q)\,d\omega = \int_{\partial D} f(Q)g(Q)k(Q)\,d\sigma = \int_D G(x)\Delta(u \cdot v)(x)\,dx$$
$$= 2\int_D G(x)\nabla u \cdot \nabla v\,dx.$$

In this way, the pairing between BMO and H^1 is typically established by bounding the solid integral over the domain. In our situation we have (with $d(x)$ abbreviating $\operatorname{dist}(x, \partial D)$)

$$\int_D G(x)|\nabla u(x)|\,|\nabla v(x)|\,dx = \int_{\partial D}\int_{\Gamma(Q)} G(x)|\nabla u(x)|\,|\nabla v(x)|\,d(x)^{1-n}\,dx\,d\sigma(Q)$$
$$\leqslant \int_{\partial D}\left\{\int_{\Gamma(Q)} |\nabla u|^2 d(x)^{2-n}\,dx\right\}^{1/2} \cdot \left\{\int_{\Gamma(Q)} |\nabla v|^2 \frac{G^2}{d(x)} d(x)^{1-n}\,dx\right\}$$
$$= \int_{\partial D} Su(Q)Av(Q)\,d\sigma$$

with

$$A^2 v(Q) \equiv \int_{\Gamma(Q)} |\nabla v|^2 \frac{G^2(x)}{d(x)} d(x)^{1-n}\,dx.$$

Observe that

$$\int_{\partial D} A^2 v(Q)\,d\sigma = \int_D \frac{G^2(x)}{d(x)} \cdot |\nabla v|^2\,dx.$$

Then by mimicking the duality argument of Fefferman-Stein [13], it will follow that $\left|\int_{\partial D} fg\,d\omega\right|$ is finite if $|\nabla v|$ satisfies the Carleson condition

$$\iint_{T(\Delta)} |\nabla v|^2 \frac{G^2(X)}{d(X)}\,dX \leqslant c\sigma(\Delta) \tag{1.1}$$

where, for $\Delta = \Delta(Q_0, r_0)$, a surface ball centered at Q_0 with radius r_0, $T(\Delta) = \{X \in D: |X - Q_0| < r_0\}$.

This Carleson measure condition was introduced in Fabes-Kenig-Neri [11] for the analogous problem on C^1 domains and in the C^1 case was shown to be equivalent to the $\mathrm{BMO}_\sigma(\omega)$ boundary definition. However, for a Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 3$, the Poisson kernel does not satisfy the appropriate decay and a $\mathrm{BMO}_\sigma(\omega)$ function can fail to satisfy (1.1).

Example 1. Let $D \subseteq \mathbb{R}^3$ be that part of the complement of the cone

$$\Gamma(m) = \{|x| \leqslant my\}$$

with vertex at the origin which is contained in the unit sphere S. Then there exists a function $h(x)$, harmonic in D, given by $h(x) = |x|^\alpha \varphi(x/|x|)$ where φ vanishes on the spherical cap cut out by $\Gamma(m)$, and $\varphi(0,0,1) = 1$. Thus $h(x)$ vanishes on $\partial D \cap \partial\Gamma(m)$ and moreover $\alpha = \alpha_m$ becomes arbitrarily small as $m \to \infty$, i.e., as the Lipschitz constant of D increases. (See Dahlberg [6] for a similar construction). Let $\Delta_r \subseteq \partial\Gamma(m)$ be a surface ball with radius $r << 1$ centered at the origin. Let D' be a subdomain of D containing

$$D \cap \{|x|^2 + y^2 \leqslant 1/2\} = D \cap S_{1/2}$$

such that

$$\partial D' \cap \partial D = \partial\Gamma(m) \cap S_{1/2}$$

but with $\partial D'$ smooth except at the origin. Then $h(x)$ satifies the $\mathrm{BMO}_\sigma(\omega)$ condition on $\partial D'$ since it is a continuous function which vanishes where the boundary of D' fails to be smooth.

Let us assume that D' does not contain the pole of $G_D(x)$, the Green's function for D. Then both $G(x)$ and $h(x)$ are harmonic in D' and vanish on $\partial D \cap \partial\Gamma(m)$. Hence by the comparison theorem ([6]), we may fix some z_0 away from the origin and obtain the estimate

$$\frac{G(x)}{h(x)} \approx \frac{G(z_0)}{h(z_0)} = c_0, \tag{1.2}$$

for $x \in D \cap S_{1/4}$. Set $T(\Delta_r)^+ = \{x \in T(\Delta_r): d(x) > r/2\}$. The above estimate for $G(x)$ shows that

$$\int_{T(\Delta_r)^+} |\nabla h|^2 G^2(x)/d(x)\,dx \geqslant C_0 r^{2\alpha-2} \int_{T(\Delta_r)^+} r^{2\alpha}/r\,dx = Cr^{4\alpha}.$$

Here the constant C depends only on the comparability constant in (1.2) and not on the radius r. Observe now that if the Carleson measure condition holds, the estimate

$$\int_{T(\Delta_r)} |\nabla h|^2 G^2/d\,dX \lesssim \sigma(\Delta_r)$$

implies that $r^{4\alpha} \leqslant Cr^2$. Letting $r \to 0$ forces $\alpha \geqslant 1/2$, but in dimension $n \geqslant 3$, α tends to zero.

This argument fails, as it must, for dimension $n = 2$. In the plane the Green's function for the complement of a cone can be computed explicitly by a conformal mapping onto the upper half plane and one finds that the restriction $\alpha > 1/2$ is satisfied.

There is, however, a more fundamental reason for the failure of the Carleson measure condition of Example 1. Namely, if $u \in H^1(D, d\sigma)$ and v is the harmonic extension of a $\mathrm{BMO}_\sigma(\omega)$ function, the integral $\int_D G(X)|\nabla u|\,|\nabla v|\,dX$ need not be absolutely convergent.

Example 2. Let $D, D' \subseteq \mathbb{R}^n$ and $h(x)$ be as in Example 1. Again let Δ_r be the surface ball on $\partial D \cap \partial\Gamma(m)$ with radius r centered at the origin. A plane perpendicular to the axis of the cone divides Δ_r into two pieces Δ_1 and Δ_2 with $\sigma(\Delta_1) \approx \sigma(\Delta_2) \approx r^{n-1}$. We construct an H^1 function with support on Δ_r by setting $u(x) = \omega^x(\Delta_1) - \beta\omega^x(\Delta_2)$ where $\beta = \omega(\Delta_1)/\omega(\Delta_2)$. By the doubling property of harmonic measure ([17]), β is bounded above and below by universal constants depending only on D and not, in particular, on the radius r. Hence $u(x)/r^{n-1}$ is an atom and Dahlberg's lemma ([18], see 2.3 of this paper) yields $\|u\|_{H^1} \leqslant Cr^{n-1}$, where $C = C(D)$.

We claim that there is no constant C such that

$$\int_{D'} |\nabla u(x)| G^2(x)/d(x)\, dx \leqslant Cr^{n-1}.$$

Because $|\nabla h(x)| \lesssim G(x)/d(x)$, this would imply that $\int_D G(x)|\nabla h(x)|\,|\nabla u(x)|\,dx$ has no bound in terms of $\|u\|_{H^1}$. Assume, to the contrary, that such a constant C_0 exists. Define $v(x) = u(x) + \beta$ and choose three points z_1, z_2, z_3 in D as follows. Let Q_1 (resp. Q_2) be a point in Δ_1 (resp. Δ_2) at a distance $r/2$ from the origin, the vertex of $\Gamma(m)$. If $Q \in \partial D$, let n_Q denote the unit normal at Q and set $z_1 = Q_1 + rn_{Q_1}$, $z_2 = Q_2 + rn_{Q_2}$ and let z_3 be the point on the vertical axis of $\Gamma(m)$ at a distance r from the origin. From now on, C will denote a constant which depends only on the domain D, not necessarily the same at each occurence.

Because $\omega^{z_1}(\Delta_1) \geqslant C$ (see [7]) and $v(x) = (1+\beta)\omega^x(\Delta_1)$ we have $v(z_1) \geqslant C$. Moreover, by Harnack's principle, $v(z_1) \approx v(z_2) \approx v(z_3)$. Hence

$$\begin{aligned} C \leqslant v(z_2) &= v(Q_2 + rn_{Q_2}) \\ &= v(Q_2 + \epsilon rn_{Q_2}) + \int_{\rho=\epsilon r}^{r} \frac{\partial}{\partial\rho} v(Q_2 + \rho n_{Q_2})\, d\rho \\ &\leqslant C\epsilon^\gamma(1+\beta) + \int_{\epsilon r}^{r} |\nabla v(Q_1 + \rho n_{Q_2})|\, d\rho. \end{aligned}$$

This last estimate follows from Hölder continuity since v is non-negative, bounded by $(1+\beta)$ and vanishes on Δ_1. If $\tilde{\Delta}_1$ is a surface ball with $\sigma(\tilde{\Delta}_1) \approx \sigma(\Delta_1)$ whose double is contained in Δ_1, the above estimate is valid with $Q \in \tilde{\Delta}_1$ replacing Q_2. Thus if we set

$$T_{\epsilon r}(\tilde{\Delta}_1) = \{X \in D: X = Q + \rho n_Q, Q \in \tilde{\Delta}_1, \epsilon r < \rho\},$$

integrating the above inequality on $Q \in \tilde{\Delta}_1$ yields

$$C \leqslant C'\epsilon^\gamma(1+\beta) + \int_{T_{\epsilon r}(\tilde{\Delta}_1)} |\nabla v(X)|\, dX/r^{n-1}$$

and so

$$Cr^{n-1} \leqslant C'\epsilon^{\gamma}(1+\beta)r^{n-1} + \int_{T_{\epsilon r}(\tilde{\Delta}_1)} |\nabla v(X)| G^2(X)/d(X) \cdot d(X)/G^2(X)\, dX$$
$$\leqslant C'\epsilon^{\gamma}(1+\beta)r^{n-1} + C(\epsilon) r/G^2(z_1) \int_{D'} |\nabla u(X)| G^2(X)/d(X)$$

where $C(\epsilon) \geqslant [G(x)/G(z_1)]^2$ depends only on ϵ and on D. Since $G(z_1) \approx G(z_3) \approx r^{\alpha}$, and, by our assumption that C_0 exists, we have

$$C \leqslant C'\epsilon^{\gamma}(1+\beta) + C(\epsilon) r^{1-2\alpha}.$$

Now fix a small α and then fix ϵ so that $C - C'\epsilon^{\gamma}(1+\beta) > 0$. The above inequality leads to contradiction as $r \to 0$.

2. In this section we give Dahlberg's proof that atoms belong to $H^1(D, d\sigma)$ and then discuss the two methods by which the atomic decomposition of H^1 and its duality with $\mathrm{BMO}_{\sigma}(d\omega)$ will be obtained. Recall that a harmonic function A is an atom if there is a surface ball $\Delta \subseteq \partial D$ such that $A = 0$ on $\partial D \setminus \Delta$, $\|A\|_{\infty} \leqslant \sigma(\Delta)^{-1}$ and A has mean value zero with respect to harmonic measure $d\omega = d\omega^{P_0}$.

Lemma 2.1 (Dahlberg [8]). *There is a constant C such that for all atoms A*

$$\int_{\partial D} NA(Q)\, d\sigma(Q) \leqslant C.$$

PROOF. Suppose that A is supported on $\Delta_r = \{Q: |Q - Q_0| < r\}$, where r is small. (If $r > c\sigma(\partial D)$, then $\int_{\partial D} NA(Q)\, d\sigma(Q)$ is bounded by $\|A\|_{L^2(\partial D)} \sigma(\partial D)^{1/2} \leqslant C$).

Since $\|A\|_{L^2} \leqslant r^{1-n}$, by the L^2 theory for Lipschitz domains (Dahlberg [9]), we have

$$\int_{\Delta_{2r}} NA(Q)\, d\sigma(Q) \leqslant \sigma(\Delta_{2r})^{1/2} \|N(A)\|_{L^2} \leqslant C.$$

Thus it suffices to estimate $|A(X)|$ for $X \in D_{\rho} = \{P \in D: |P - Q_0| > \rho\}$ for $\rho > 2r$. Let $S_{\rho} = \{P \in D: |P - Q_0| = \rho\}$ and pick $Q_{\rho} \in S_{\rho}$ such that $\operatorname{dist}(Q_{\rho}, \partial D) = \max_{Q \in S_2} \operatorname{dist}(Q, \partial D)$. Extend A to all of $\mathbb{R}^n$ by putting $A = 0$ in $\mathbb{R}^n \setminus D$ and let $A^+ = \max(A, 0)$, $A^- = \max(-A, 0)$. Both A^+ and A^- are subharmonic in $\{X: |X - Q_0| > r\}$. If $d\tau$ denotes normalized surface measure on the unit sphere S, set

$$m_{\pm}(\rho) = \left(\int_Q |A^{\pm}(\rho\tau)|^2\, d\tau\right)^{1/2}.$$

By Huber [6] and Friedland and Hayman [14],

$$m_{\pm}(\rho) \leqslant \sqrt{2}\,(r/\rho)^{n-2} m_{\pm}(r) \exp\left(-\int_{2r}^{\rho} \alpha_{\pm}(t)\, dt/t\right) \tag{2.2}$$

where $\alpha_{\pm}(t)$ is the nonnegative root of the equation

$$\alpha(\alpha+n-2)=\lambda(U^{\pm}(t)),\ U^{\pm}(t)=\{Q\in S: A^{\pm}(tQ)>0\},$$

and

$$\lambda(U)=\inf\left\{\int|\nabla u|^2\,d\tau: u\in C_0^\infty(U), \int u^2\,d\tau=1\right\}\cdot$$

Set

$$h_\rho^{\pm}(X)=\int_{\partial D_\rho}A^{\pm}(Q)\,d\omega_{D_\rho}^X(Q)$$

where $d\omega_{D_\rho}^X$ is harmonic measure for the domain D_ρ and let $d\omega_{D_\rho}$ denote the harmonic measure for D_ρ evaluated at the point Q_ρ. Let $k_\rho=d\omega_{D_\rho}/d\sigma_\rho$ be the density of harmonic measure in surface measure on ∂D_ρ. Then

$$\begin{aligned}(2.3)\qquad h^{\pm}(Q_\rho)&\leqslant\left(\int_{\partial D_\rho}|A^{\pm}(Q)|^2\,d\sigma_\rho\right)^{1/2}\cdot\left(\int_{D_\rho}k_\rho^2\,d\sigma_\rho\right)^{1/2}\\&\leqslant\left(\int_{\partial D_\rho}|A^{\pm}(Q)|^2\,d\sigma_\rho\right)^{1/2}\cdot\sigma(\partial D_\rho)^{-1/2}\cdot\int_{\partial D_\rho}k_\rho\,d\sigma_\rho\\&\leqslant C\left(\int_{\partial D_\rho}|A^{\pm}(Q)|^2\,d\sigma_\rho\right)^{1/2}\rho^{1-n}=Cm_{\pm(\rho)},\end{aligned}$$

where the estimates above follow from the reverse Hölder condition on k_ρ (see Dahlberg [9]). By Harnack's inequality, $\max_{D_{2\rho}}|h_\rho^{\pm}|\leqslant Ch_\rho^{\pm}(Q_\rho)$, and by the comparison theorem ([16]),

$$\frac{h^+(Q_\rho)}{h^-(Q_\rho)}\approx\frac{h^+(P_0)}{h^-(P_0)}.$$

However, $A=h_\rho^+-h_\rho^-$ and $A(P_0)=0$, by assumption, so it follows that $h_\rho^+(Q_\rho)\approx h_\rho^-(Q_\rho)$. Hence, by (2.2) and (2.3)

$$\begin{aligned}(2.4)\ \operatorname*{Max}_{D_{2\rho}}|A|&\leqslant C\{h_\rho^+(Q_\rho)\cdot h_\rho^-(Q_\rho)\}^{1/2}\\&\leqslant Cr^{n-2}(m_+(r)m_-(r))^{1/2}\rho^{2-n}\exp\left(-\int_{2r}^{\rho}\frac{\alpha_+(t)+\alpha_-(t)}{2}\frac{dt}{t}\right).\end{aligned}$$

Again by the L^2 theory,

$$\begin{aligned}r^{n-1}m_{\pm}^2(r)&\leqslant\int_{\partial\Delta(Q_0,r)}|A^{\pm}(P)|^2\,d\sigma_r(P)\\&\leqslant\int_{\Delta_r}N^2(A)(Q)\,d\sigma(Q)\\&\leqslant C\int_{\Delta_r}|A(Q)|^2\,d\sigma(Q)\\&\leqslant Cr^{1-n}.\end{aligned}$$

This gives the estimate

$$\operatorname*{Max}_{D_{2\rho}} |A| \leqslant Cr^{-1}\rho^{2-n} \exp\left(-\int_{2r}^{\rho} \frac{\alpha_+(t) + \alpha_-(t)}{2} \frac{dt}{t}\right).$$

Let

$$a_{\pm}(t) = \int_{U^{\pm}(t)} d\tau$$

an set $\varphi(x) = 2(1-x)$ for $1/4 \leqslant x \leqslant 1$,

$$\varphi(x) = \frac{1}{2} \log (4x)^{-1} + \frac{3}{2}$$

otherwise. Because φ is decreasing and convex, it follows that $\alpha_{\pm}(t) \geqslant \varphi_{\pm}(t)$. (See Friedland-Hayman [14]). Since D is Lipschitz and $A^+A^- = 0$ there exists a positive β such that $a_+(t) + a_-(t) \leqslant 1 - \beta$. Hence

$$\frac{1}{2}(\alpha_+(t) + \alpha_-(t)) \geqslant \varphi\left(\frac{1}{2}(a_+(t) + a_-(t))\right) \geqslant 1 + \beta.$$

Substituting this estimate into (2.4) gives

$$\operatorname*{Max}_{D_{2\rho}} |A| \leqslant C\rho^{1-n-\beta} r^{\beta} \tag{2.5}$$

and therefore

$$NA(Q) \leqslant C \min \{r^{\beta} |Q - Q_0|^{1-n-\beta}, r^{1-n}\}. \quad \square$$

Observe that the argument yields a strong pointwise estimate (2.5) on the harmonic function $A(x)$ away from the support of its boundary values. We will need this fact later.

Let us now assume that D is Lipschitz and starlike with respect to the origin. Let $d\omega$ denote $d\omega^0$.

Definition.

$$H^1(\partial D, d\sigma) = \left\{f\colon f(Q) = \lim_{r \to 1} u(rQ), u \in H^1(D, \delta)\right\},$$

where the limit is taken in an appropriate sense to be made precise later on.

Definition. *For*

$$\mathrm{VMO}_\sigma(d\omega) = \left\{g \in \mathrm{BMO}_\sigma(\omega)\colon \lim_{\sigma(\Delta) \to 0} \frac{1}{\sigma(\Delta)} \int_\Delta |g - g_\Delta| \, d\omega = 0\right\},$$

let $VMO_\sigma^*(d\omega)$ be its dual, i.e., the linear functionals acting continuously on $\mathrm{VMO}_\sigma(d\omega)$.

Definition. $H^1_{at}(\partial D, d\sigma) = \{ f \in \mathrm{VMO}^*_\sigma(d\omega): f = \sum_j \lambda_j a_j$, *where the* a_j *are atoms,* $\sum_j |\lambda_j| < \infty$, *and the convergence takes place in* $\mathrm{VMO}^*_\sigma\ (d\omega) \}$.

Our goal in the next few sections is to establish the following.

Theorem 2.6.

$$H^1(\partial D, d\sigma) = \mathrm{VMO}^*_\sigma(d\omega) = H^1_{at}(\partial D, d\sigma)$$

and the dual of $H^1(\partial D, d\sigma)$ *is* $\mathrm{BMO}_\sigma(d\omega)$ *with pairing* $(f, g) = \int fg\, d\omega$, *on an appropiate dense subclass.*

There will be an analogous result for nonstarlike Lipschitz domains, which will be formulated at the end of section 4.

The main result we need is the following, establishing the pairing between $H^1(D, d\sigma)$ and $\mathrm{BMO}_\sigma(\omega)$.

Theorem 2.7. *If* $u \in \mathcal{L}(\bar{D})$, *the space of functions on D Lipschitz on* $\bar{D}$, $\Delta u = 0$, $u(0) = 0$, *and* $f \in \mathrm{BMO}_\sigma(d\omega)$ *then*

$$\int_{\partial D} u(Q) f(Q)\, d\omega \leqslant \|N(u)\|_{L^1(d\sigma)} \|f\|_{\mathrm{BMO}}.$$

The idea behind the proof of Theorem 2.7 is to find some (non-harmonic) extension v of f to the domain D which satisfies a Carleson measure condition. For suppose $v(X)$ is some smooth extension of f to all of D, for which the following formal argument is justified. Let $G(x)$ be the Green's function with pole at 0.

$$\begin{aligned}\int_{\partial D} u(Q) f(Q)\, d\omega &= \int_D G(x) \Delta(u \cdot v)\, dx \\ &= 2\int_D G(x) \nabla u(x) \cdot \nabla v(x)\, dx + \int_D G(x) u(x) \Delta v(x)\, dx \\ &= 2\int_D G(x) \nabla u(x) \cdot \nabla v(x)\, dx - \int_D G(x) \nabla u(x) \cdot \nabla v(x)\, dx \\ &\quad - \int_D u(x) \nabla G(x) \cdot \nabla v(x)\, dx.\end{aligned}$$

If $d(x) = \mathrm{dist}\,(x, \partial D)$, then $|\nabla G(x)| \lesssim G(x)/d(x)$, when $x \notin K$ for K some region around the pole, and we have

$$\begin{aligned}\left| \int_{\partial D} u(Q) f(Q)\, d\omega(Q) \right| &\leqslant \int_D G(x) |\nabla u(x)|\, |\nabla v(x)|\, dx + \int_K u |\nabla G|\, |\nabla v|\, dx \\ &\quad + \int_D |u(x)|\, |\nabla v(x)| G(x)/d(x)\, dx.\end{aligned}$$

The first integral on the right-hand side is bounded by $\int_{\partial D} S(u)(Q)\, d\sigma(Q)$ if $|\nabla v|^2 G^2/d(x)$ is a Carleson measure. (See the argument at the beginning of §1).

The third integral is bounded by $\int_{\partial D} Nu(Q)\, d\sigma(Q)$ as long as $|\nabla v| G(X)/d(X)$ is a Carleson measure. The second integral also has this bound if $|v(x)|$ is bounded in K. Thus we seek an extension v of f which satisfies, for some constant C and all surface balls $\Delta \subseteq \partial D$,

$$\text{(2.8)}\quad \begin{aligned} &\text{(i)} && \int_{T(\Delta)} |\nabla v(x)| G(x)/d(x)\, dx \leqslant C\sigma(\Delta) \\ &\text{(ii)} && |\nabla v(x)| \leqslant CG(x)^{-1}, \quad \text{for } x \text{ near } \partial D \\ &\text{(iii)} && |v(x)| \leqslant C_K \quad \text{on a compact subset } K \text{ of } D. \end{aligned}$$

We shall find such an extension by a localization procedure which allows one to translate the problem to an analog on the upper half space $\mathbb{R}^n_+$. Roughly speaking, a $\mathrm{BMO}_\sigma(\omega)$ function f on ∂D can be cut off and projected onto $\mathbb{R}^{n-1}$ so that the resulting function lies in $\mathrm{BMO}(w\,dx)$, a weighted space of bounded mean oscillation. The counterpart of (2.8) on $\mathbb{R}^n_+$ will be obtained in two ways. The first approach is by duality. We introduce a space of distributions on $\mathbb{R}^{n-1}$, a «weighted» space of homogeneous type, which could be regarded as a smooth version of $H^1(D, d\sigma)$. The dual of this H^1 space will be $\mathrm{BMO}(w\,dx)$ and a representation of a $\mathrm{BMO}(w\,dx)$ in terms of an appropriate kernel and a Carleson measure is obtained. Obtaining such an extension constructively is the approach to the classical duality taken by Carleson [2], Varopoulos [25], and Jones [19]. The theory here becomes somewhat elaborate, although we think it is of interest in itself, and so an alternative constructive approach is developed. The constructive argument parallels that of Varopoulos [25] and will be given at the end of §3. Basic to both methods of proof is the following observation about harmonic measure.

Lemma 2.9. *There exists a constant C, an $\alpha > 0$, and a radius r_0, all depending only on the domain D, such that for every surface ball*

$$\Delta(Q_0, r) = \{Q \in \partial D: |Q - Q_0| < r\}$$

with $2r < r_0$,

$$\omega(\Delta(Q_0, 2^{-j}r)) \leqslant C 2^{-j(n-2+\alpha)} \omega(\Delta(Q_0, r))$$

PROOF. By Lemma 5.8 of [17], there exists r_0 and M such that whenever $2r < r_0$,

$$M^{-1} < \omega(\Delta(Q, r))/r^{n-2} G(X_r) < M$$

where X_r is a point in D whose distance to ∂D is approximately $|X_r - Q_0|$. Thus

$$\omega(\Delta(Q_0, 2^{-j}r))/\omega(\Delta(Q_0, r)) \leqslant C_M 2^{-j(n-2)} G(X_{2^{-j}r})/G(X_r).$$

The Green's function is harmonic in $\Delta(Q_0, 2r)$ and vanishes on the boundary of D so by Hölder continuity ([17]) we have

$$G(X_{2^{-j}r}) \leqslant C(|X_{2^{-j}r} - Q_0|/r)^{\alpha} G(X_r) \leqslant C2^{-j\alpha} G(X_r), \quad \text{for some} \quad \alpha > 0.$$

In the next section we define an H^1 space relative to a kernel with a weight satisfying the condition of Lemma 2.9. This weight should therefore be thought of as the projection of harmonic measure onto $\mathbb{R}^{n-1}$. The localization arguments and the translation of the problem in D to the situation which follows are given later.

3. Suppose w is a weight on $\mathbb{R}^{n-1}$ with the following properties:

(i) For some constant C an some $0 < \alpha < 1$,

$$w(Q)/w(2^j Q) \leqslant 2^{-j^{(n-2+\alpha)}}$$

for all cubes Q and their 2^j-fold enlargements $2^j Q$.

(3.1) (ii) $w(Q) \approx |Q|$ when the length $l(Q)$ of Q is larger than 1.

(iii) w satifies a reverse Hölder inequality with exponent two; that is

$$\left(\frac{1}{|Q|}\int_Q w^2\,dx\right)^{1/2} \lesssim \frac{1}{|Q|}\int_Q w\,dx.$$

We remark that conditions (ii) and (iii) are stated in this seemingly strong way for convenience only. If we only ask that $w \in A_\infty$, i.e.,

$$w(E)/w(Q) \leqslant C(|E|/|Q|)^{\theta} \quad \text{for some } \theta,$$

then this would suffice instead of (ii) and (iii). (See Coifman-Fefferman [3] for the relevant properties of A_∞ weights).

Define BMO$(w\,dx)$ to be the space of functions $g \in L^1_{\text{loc}}(w\,dx)$ for which there exists a contant M such that

$$\sup_{Q:\,cube} \frac{1}{|Q|}\int_Q |g - g_Q| w\,dx < M$$

where $|Q|$ is the Lebesgue measure of Q and

$$g_Q = w(Q)^{-1}\int_Q g w\,dx.$$

Clearly if for each cube Q there exists some constant c_Q for which

$$\sup_Q \left\{|Q|^{-1}\int_Q |g - c_Q| w\,dx\right\}$$

is finite, then g belongs to BMO($w\,dx$). Let VMO denote the closure of Lip_0, the compactly supported Lipschitz functions, in the BMO norm. This definition makes sense for if ψ belongs to Lip_0, $|Q|$ is small, take Q_0 to be the cube of length 1 containing Q. Then

$$\begin{aligned}\int_Q |\psi(x) - \psi(x_0)|\,w\,dx &\leqslant \|\psi\|_{\mathrm{Lip}} \int_Q |x - x_0|\,w\,dx \\ &\leqslant \|\psi\|_{\mathrm{Lip}} l(Q) w(Q) \\ &\leqslant \|\psi\|_{\mathrm{Lip}} l(Q) w(Q)/W(Q_0) w(Q_0) \\ &\leqslant \|\psi\|_{\mathrm{Lip}} l(Q)^{n-1+\alpha},\end{aligned}$$

by conditions (3.1) (i) and (ii). And when Q is large,

$$\int_Q |\psi(x)|\,w\,dx \leqslant Cw(Q) \leqslant C|Q|.$$

Hence $\mathrm{Lip}_0 \subseteq$ BMO($w\,ds$). Fix, once and for all, a C^∞ bump function φ supported in the unit ball $B(0, 1)$ and $\varphi \equiv 1$ on $B(0, 1/2)$. The kernel function formed from φ, relative to the weight w, is

$$K(x, z, y) = \varphi(x - z/y)\left\{\int \varphi\left(\frac{x - z'}{y}\right) w(z')\,dz'\right\}^{-1}.$$

An f belonging to VMO*($w\,dx$), the dual of VMO($w\,dx$), will be called a distribution and the pairing $\langle f, \varphi\rangle$ will be denoted $\int f\varphi w\,dx$. For $f \in$ VMO*($w\,dx$), let

$$u(x, y) = \int_k (x, z, y) f(z) w(z)\,dz$$

be its «Poisson» extension to the upper half space.

Definition.

$$H^1(w\,dx) = \left\{f \in \mathrm{VMO}^*(w\,dx) : f^+(x) = \sup_{y>0} |u(x, y)| \in L^1(dx)\right\}.$$

The first goal is to give an atomic decomposition of $H^1(w\,dx)$ and for this purpose we need a definition of this space in terms of a grand maximal function. Let $\mathcal{S}$ be class of Schwartz functions and let

$$\mathfrak{A} = \left\{\psi \in \mathcal{S} : \int (1 + |x|)^N \Big(\sum_{|\alpha| \leq N} |D^\alpha \psi|^2\Big)\,dx \leqslant 1\right\},$$

for some large N. The pairing of an $f \in$ VMO*($w\,dx$) against $\psi \in \mathfrak{A}$ is well defined. To see this, fix a sequence of bump functions θ_j supported on $\{|x| \leqslant 2^{j+1}\}$, with $\theta_j \equiv 1$ for $|x| \leqslant 2^j$. Now if $\psi \in \mathfrak{A}$, $\theta_j\psi \in \mathrm{Lip}_0$ and we need $\|\theta_j\psi - \psi\|_{\mathrm{BMO}} \to 0$ as $j \to \infty$. Suppose $Q \subseteq \mathbb{R}^{n-1}$ is a cube with $l(Q) > 1$. We have

$$\begin{aligned}\int_Q |(1-\theta_j)\psi(x)|\,w\,dx &\leqslant \left(\int_{Q\cap\{|x|\geq 2^j\}} w^2(1+|x|)^{-N}dx\right)^{1/2}\\ &\cdot\left(\int_{\{|x|\geq 2^j\}} |\psi|^2(1+|x|)^N dx\right)^{1/2}\\ &\leqslant 2^{-jN/2}\left(\int_Q w^2\frac{dx}{|Q|}\right)^{1/2}\cdot|Q|^{1/2}\\ &\leqslant 2^{-jN/2}|Q|\end{aligned}$$

by (3.1) (ii) and (iii). Alternatively, if $l(Q)$ is small, and x_0 is the center of Q,

$$\begin{aligned}\int_Q [(1-\theta_j)\psi(x)-(1-\theta_j)\psi(x_0)]\,w\,dx &\leqslant l(Q)\cdot\int_Q |\nabla((1-\theta_j)\psi)|\,w\,dx\\ &\leqslant C_j l(Q)w(Q)\leqslant C_j|Q|\end{aligned}$$

where $C_j \to 0$ as $j\to\infty$. The last inequality, valid for small cubes, is derived from an argument we have used before. For Q_0 is the cube of length 1 containing Q, condition (3.1) (i) yields

$$w(Q)\leqslant l(Q)^{n-2+\alpha}w(Q_0)$$

and again by (3.1) (ii),

$$l(Q)w(Q)\leqslant l(Q)^{n-1+\alpha}\leqslant l(Q)^{n-1}.$$

One can now define, for $\eta\in\mathcal{Q}$ and $B(x,t)=\{x'\in\mathbb{R}^{n-1}: |x-x'|<t\}$, the extension

$$f\eta(x,t)=\int f(z)\eta\left(\frac{x-z}{t}\right)w(z)\,dz\Big/ w(B(x,t)).$$

The grand maximal function of f is

$$f^*(x)=\sup_{\eta\in\mathcal{Q}}\ \sup_{|x-x'|<t}|f\eta(x',t)|.$$

Lemma 3.2. *There is a constant C such that*

$$\frac{1}{C}\|f^+\|_{L^1(dx)}\leqslant\|f^*\|_{L^1(dx)}\leqslant C\|f^+\|_{L^1(dx)}$$

PROOF. The argument is an adaptation of that of Fefferman-Stein [13], so only those details which indicate how the properties of the weight come into play are provided. Recall that φ is our fixed «nice» bump function. The non-tangential maximal function is

$$Nf(x)=\sup_{|x-x'|<t}|f_\varphi(x',t)|,$$

where $f_\varphi(x, t)$ is the $u(x, t)$ defined before. We shall first see that

$$\|Nf\|_{L^1(dx)} \approx \|f^*\|_{L^1(dx)}.$$

Assume $Nf \in L^1(dx)$ and let $\eta \in \mathfrak{A}$. A purely geometric argument (p. 185 of [13]) shows that the tangential maximal function

$$f^{**}(x) = \sup_{t,x'} |f_\varphi(x', t)| \left(\frac{t}{|x - x'| + t}\right)^N$$

belongs to $L^1(dx)$ for sufficiently large N. Suppose that for some $\psi \in \mathfrak{A}$, and $s < 1$, $\eta = \psi * \varphi_s$ where $\varphi_s(\bullet) = s^{-n+1}\varphi(\bullet/s)$. Then

$$\left|\int f(y)\eta(x - y/t)w(y)\,dy\right| = \left|\int f(y)\left\{\int \psi_t(x - z)s^{-n+1}\varphi\left(\frac{x-y}{st}\right)dx\right\}w(y)\,dy\right|$$
$$= \left|\int \psi_t(x - z)\left(\int f(y)s^{-n+1}\varphi\left(\frac{z-y}{st}\right)w(y)\,dy\right)dx\right|.$$

This interchange of integration can be justified in the same way one justifies the interchange in the ordinary convolution case and we obtain

$$|f\eta(x, t)| \leqslant \int |\psi_t(x - z)|s^{-n+1}\frac{w(B(z, st))}{w(B(x, t))}|f_\varphi(z, st)|\,dz$$
$$\leqslant \int |\psi_t(x - z)|s^{-n+1}\frac{w(B(z, st))}{w(B(x, t))}\left(\frac{st}{|x - z| + st}\right)^{-N}dz \cdot f^{**}(x)$$
$$= f^{**}(x)s^{-n+1}\sum_j \int_{|x-z|\approx 2^j t} t^{-n+1}\left|\psi\left(\frac{x-z}{t}\right)\right|\left(\frac{st}{2^j t + st}\right)^{-N}$$
$$\cdot \frac{w(B(z, st))}{w(B(x, t))}\,dz.$$

By (3.1) and the fact that $w \in A^\infty$,

$$\frac{w(B(z, st))}{w(B(x, t))} = \frac{w(B(z, st))}{w(B(z, 2^j t))} \cdot \frac{w(B(z, 2^j t))}{w(B(x, t))} \leqslant (s/2^j)^{n-2+\alpha} \cdot 2^{Mj}.$$

Thus

$$|f\eta(x, t)| \lesssim f^{**}(x)s^{-n+1} \cdot s^{n-2+\alpha}\sum_j 2^{Mj}\int_{|x-z|\sim 2^j t} t^{-n+1}\left|\psi\left(\frac{x-z}{t}\right)\right|(s/2^j)^{-N}dz$$
$$\lesssim f^{**}(x)s^{-1+\alpha}\sum_j 2^{Mj}s^{-N}2^{-jN}\left(t^{-n+1}\int_{|x-z|\sim 2^j t}\left|\psi\left(\frac{x-z}{t}\right)\right|dz\right)$$
$$\lesssim f^{**}(x)s^{-N-1+\alpha}, \quad \text{since} \quad \psi \in \mathfrak{A}.$$

The remainder of the argument for $\|f^*\|_{L^1(dx)} \leqslant \|Nf\|_{L^1(dx)}$ follows Fefferman-Stein verbatim, cutting up the support of $\hat{\eta}$ to express it as a sum of convolutions $\psi * \varphi_{s_j}$, with $\psi \in \mathfrak{A}$.

To show that $\|Nf\|_{L^1(dx)} \leqslant \|f^+\|_{L^1(dx)}$ one uses a geometric argument of Burkholder-Gundy to prove

$$|Nf(x)| \leqslant \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q [f^+(z)]^r \, dz \right)^{1/r}, \qquad 0 < r < 1$$

and invoke the maximal theorem. Again, for details, we refer to [13].

The main step in the atomic decomposition for $H^1(w\,dx)$ is to provide an appropriate dense subclass of functions. Here we follow closely the method of Macías and Segovia [26].

Given a distribution $f \in H^1(w\,dx)$, let $\Omega_\lambda = \{x: f^*(x) > \lambda\}$. Associate to Ω_λ a partition of unity $\{\eta_j\}$ with $\sum_j \eta_j = \chi_{\Omega_\lambda}$, $\eta_j = 1$ on $B(x_j, r_j)$, supp $\eta_j \subseteq B(x_j, 2r_j)$, and $\|D^\alpha \eta_j\|_\infty \leqslant C^{r_j^{-|\alpha|}}$. Furthermore, $\bigcup B(x_j, r_j) = \Omega_\lambda$ and for some constant M, and some $c > 2$, no point of Ω_λ lies in more than M of the balls $B(x_j, cr_j)$. Define the mapping S_j on VMO$(w\,dx)$ by

$$S_j(\psi)(x) = \eta_j(x) \int [\psi(x) - \psi(z)] \eta_j(z) w(z) \, dz \cdot \left\{ \int \eta_j(z') w \, dz' \right\}^{-1}.$$

Lemma 3.3. *If* $\psi \in$ VMO, *then* $\sum_{j=1}^\infty S_j(\psi)$ *belongs to* VMO *and*

$$\left\| \sum_{j=1}^\infty S_j(\psi) \right\|_{\mathrm{BMO}} \leqslant C \|\psi\|_{\mathrm{BMO}}.$$

PROOF. Let $B_j = \operatorname{supp} \eta_j$ and $l(B_j) = \operatorname{radius}(B_j)$. We first show that $\sum_{j=1}^\infty \eta_j(x)(\psi - \psi_{B_j})$ belongs to VMO, where $\psi_{B_j} = 1/w(B_j) \int_{B_j} \psi \, dx$. The difference between this sum and $\sum_j S_j(\psi)$ will be controlled later. Fix a cube $J \subseteq \mathbb{R}^{n-1}$ with center x_J. We need only consider those balls B_j which intersect J. These balls B_j are of two types: $l(J) < l(B_j)$ or $l(J) > l(B_j)$. Let $j \in \mathfrak{J}_1$ if $l(J) > l(B_j)$ and $j \in \mathfrak{J}_2$ if $l(J) \leqslant l(B_j)$. Set

$$c_J = \sum_{j \in \mathfrak{J}_2} \eta_j(x_J)(\psi - \psi_{B_j})_J.$$

Then

$$\int_J \left| \sum_{j=1}^\infty \eta_j(x)(\psi - \psi_{B_j}) - c_J \right| w \, dx \leqslant \int_J \left| \sum_{j \in \mathfrak{J}_1} \eta_j(x)(\psi - \psi_{B_j}) \right| w \, dx$$

$$+ \int_J \left| \sum_{j \in \mathfrak{J}_2} \eta_j(x)(\psi - \psi_J) + \sum_{j \in \mathfrak{J}_2} (\eta_j(x) - \eta_j(x_J)(\;\;)(\psi_J - \psi_{B_j}) \right| w \, dx.$$

The first term above is bounded by

$$\sum_{j\in\mathcal{J}_1}\int_{B_j}|\psi-\psi_{B_j}|w\,dx \leqslant \|\psi\|_{\mathrm{BMO}}\sum_{j\in\mathcal{J}_1}|B_j| \leqslant C\int_{2J}\sum_j \chi_{B_j}(x)\,dx \leqslant C|J|.$$

We consider the two sums in the second term separately.

$$\int_J\left|\sum_{j\in\mathcal{J}_2}\eta_j(x)(\psi(x)-\psi_J)\right|w\,dx \leqslant M\int_J|\psi(x)-\psi_J|w(x)\,dx \leqslant M|J|$$

since no point lies in more than M of the B_j's. In the second sum we add and subtract $\psi(x)$ from $\psi_J-\psi_{B_j}$ and it remains to bound

$$\int_J\left|\sum_{j\in\mathcal{J}_1}(\eta_j(x)-\eta_j(x_J))(\psi-\psi_{B_j})\right|w\,dx.$$

The balls B_j which occur int this sum have $l(B_j)\geqslant l(J)$ and so at most M of them intersect J. Hence we have only to estimate a single term of the form

$$\int_J|\eta_j(x)-\eta_j(x_J)|\,|\psi(x)-\psi_{B_j}|w(x)\,dx \tag{3.4}$$

where $l(J)<l(B_j)$. Let J_n be the 2^n-fold enlargement of J and let n_0 be the smallest n for which $B_j\subseteq J_{n_0}$. Of course, for this n_0, $|J_{n_0}|$ is comparable to $|B_j|$. By the gradient estimate on η_j, (3.4) is less than

$$\frac{l(J)}{l(B_j)}\int_J\Bigg|(\psi-\psi_{B_j})-(\psi-\psi_{B_j})_J+\sum_{l=1}^{n_0}[(\psi-\psi_{B_j})_{J_l}-(\psi-\psi_{B_j})_{J_{l-1}}]$$
$$+(\psi-\psi_{B_j})_{J_{n_0}}\Bigg|w\,dx$$

Applying the BMO condition on ψ to the interval $J_l\cup J_{l-1}$ gives

$$|\psi_{J_l}-\psi_{J_{l-1}}|\leqslant|J_l|/w(J_l).$$

In particular,

$$\begin{aligned}\frac{l(J)}{l(B_j)}\int_J|\psi_{J_{n_0}}-\psi_{B_j}|w\,dx &\leqslant \|\psi\|_{\mathrm{BMO}}\frac{l(J)}{l(B_j)}\,\frac{|B_j|}{w(B_j)}\,w(J)\\ &\leqslant C|J|\frac{l(J)}{l(B_j)}\,\frac{|B_j|}{|J|}\left(\frac{l(J)}{l(B_j)}\right)^{n-2+\alpha}\\ &\leqslant C|J|\left(\frac{l(J)}{l(B_j)}\right)^{\alpha}.\end{aligned}$$

A similar estimate is achieved on the sum:

$$\begin{aligned}\frac{l(J)}{l(B_j)}\int_J\left|\sum_{l=1}^{n_0}|\psi_{J_l}-\psi_{J_{l-1}}|w\,dx\right| &\leqslant \frac{l(J)}{l(B_j)}\int_J\sum_{l=1}^{n_0}\frac{|J_l|}{w(J_l)}w\,dx\\ &\leqslant \frac{l(J)}{l(B_j)}\sum_{l=1}^{n_0}|J_l|\frac{w(J)}{w(J_l)}\\ &\leqslant |J|\sum_{l=1}^{n_0}\frac{(2^l l(J))^{n-1}}{l(J)^{n-1}}\frac{l(J)}{l(B_j)}\left(\frac{l(J)}{2^l l(J)}\right)^{n-2+\alpha}\\ &= |J|\sum_{l=1}^{n_0}(2^l)^{1-\alpha}\frac{l(J)}{l(B_j)}\\ &\leqslant |J|\sum_{l=1}^{n_0}(2^l)^{1-\alpha}2^{-n_0}\\ &\leqslant C|J|.\end{aligned}$$

To control the difference between the sum $\sum_j \eta_j(\psi-\psi_{B_j})$ and $\sum_j S_j(\psi)$, note that each term is bounded by

$$\eta_j(x)\frac{1}{w(B_j)}\int_B|\psi-\psi_{B_j}|w\,dx$$

and, as in the previous argument it is possible to obtain a bound of $|J|$ on the BMO($w\,dx$) norm of the difference.

We have shown that $\sum_j S_j(\psi)\in \mathrm{BMO}(w\,dx)$, and it remains to verify that it belongs to VMO($w\,dx$) when $\psi\in \mathrm{VMO}(w\,dx)$. Suppose that $\psi\in \mathrm{Lip}_0$. Then

$$\begin{aligned}\sum_j S_j(\psi)(x)-\sum_j S_j(\psi)(x_0) &\leqslant \sum_j[\eta_j(x)-\eta(x_0)]\int[\psi(x)-\psi(z)]\frac{\eta_j w\,dz}{\int\eta_j w\,dz'}\\ &\quad+\sum_j\eta_j(x_0)\int[\psi(x)-\psi(x_0)]\frac{\eta_j w\,dz}{\int\eta_j w\,dz'}\\ &\leqslant \sum_j[\eta_j(x)-\eta_j(x_0)]\int|\psi(x)-\psi(z)|\frac{\eta_j w\,dz}{\int\eta_j w\,dz'}\\ &\quad+C|x-x_0|.\end{aligned}$$

By considering, separately the cases $l(B_j)<|x-x_0|$ and $l(B_j)>|x-x_0|$ in the sum above, and using $|\psi(x)-\psi(z)|\leqslant l(B_j)$ we can see that $\sum_j S_j(\psi)$ is also Lip_0. If $\psi\in\mathrm{VMO}(w\,dx)$, let θ be a Lip_0 function such that $\|\psi-\theta\|_{\mathrm{BMO}}<\epsilon$. Then

$$\sum_j S_j(\psi) = \sum_j S_j(\psi - \theta) + \sum_j S_j(\theta).$$

The second term is Lip_0 and the first is BMO with norm bounded by $\|\psi - \theta\|_{\mathrm{BMO}}$. $\square$

Lemma 3.5. *If $\psi \in \mathrm{VMO}$, set*

$$\psi_t(x) = \int \psi(y) K(x, y, t) w(y)\, dy$$

where

$$K(x, y, t) = \varphi(x - y/t)\left\{\int \varphi(x - z'/t) w(z')\, dz'\right\}^{-1}.$$

Then ψ_t converges to ψ in $\mathrm{VMO}(w\,dx)$ as $t \to 0$.

PROOF. We first show that $\|\psi_t\|_{\mathrm{BMO}} \leqslant C\|\psi\|_{\mathrm{BMO}}$ for $t < 1$. Let Q be a cube in $\mathbb{R}^{n-1}$. If $t < l(Q)$,

$$\begin{aligned}\int_Q |\psi_t(x) - \psi_Q| w\, dx &\leqslant \int_Q \int_{y \in B(x,t)} |\psi(y) - \psi_Q| K(x, y, t) w(y)\, dy\, w(x)\, dx \\ &\leqslant \int_{y \in 2Q} |\psi(y) - \psi_Q| \int_{x \in B(y,t)} \frac{\varphi(x - y/t)}{w(B(x, t))} w(x)\, dx\, w(y)\, dy \\ &\leqslant C\|\psi\|_{\mathrm{BMO}} |Q|.\end{aligned}$$

If instead $t > l(Q)$, let Q_t be the cube of length $t \cdot l(Q)$ containing Q. Then

$$\begin{aligned}&\int_Q |\psi_t(x) - \psi_t(x_0)| w(x)\, dx \\ &\quad= \int_Q \left| \int (\psi(y) - \psi_{Q_t})((K(x, y, t) - K(x_0, y, t)) w(y)\, dy \right| w(x)\, dx \\ &\quad\leqslant \int_Q \int_{y \in Q_t} |\psi(y) - \psi_{Q_t}|\, |x - x_0| \Big/ t \frac{w\, dy}{w(Q_t)} w(x)\, dx \\ &\quad\lesssim \frac{l(Q)}{t} \|\psi\|_{\mathrm{BMO}} \frac{|Q_t|}{w(Q_t)} w(Q) \\ &\quad\lesssim \|\psi\|_{\mathrm{BMO}} (tl(Q))^{n-1} \left(\frac{l(Q)}{tl(Q))}\right)^{n-2+\alpha} \frac{l(Q)}{t} \\ &\quad\lesssim \|\psi\|_{\mathrm{BMO}} l(Q)^{n-\alpha} (l(Q)/t)^{\alpha} \\ &\quad\leqslant \|\psi\|_{\mathrm{BMO}} l(Q)^{n-1}\end{aligned}$$

since $l(Q)$ is also less than 1.

We now need to see that if $\psi \in \mathrm{Lip}_0$, then $\psi_t \in \mathrm{VMO}(w\,dx)$ and $\|\psi_t - \psi\|_{\mathrm{BMO}} \to 0$ as $t \to 0$. Clearly $\psi_t \in \mathrm{Lip}_0$ if ψ is Lip_0, so we now estimate $\|\psi_t - \psi\|_{\mathrm{BMO}}$. In fact, for $0 < \beta < 1$, we have $\|\psi_t - \psi\|_{\mathrm{Lip}(\beta)} \leqslant t^{1-\beta}$. Consider $|(\psi_t - \psi)(x) - (\psi_t - \psi)(x_0)|$. If $t < |x - x_0|$,

$$\begin{aligned}|\psi_t(x) - \psi(x)| &\leqslant \int |\psi(y) - \psi(x)| K(x, y, t) w(y)\, dy \\ &\leqslant t \leqslant t^{1-\beta}|x - x_0|^{\beta}.\end{aligned}$$

If $t > |x - x_0|$,

$$\begin{aligned}|(\psi_t - \psi)(x) - (\psi_t - \psi)(x_0)| &\leqslant \int |\psi(y) - \psi(x)|\, |K(x, y, t) - K(x_0, y, t)|\, w(y)\, dy \\ &\quad + |\psi(x) - \psi(x_0)| \int K(x_0, y, t) w(y)\, dy \\ &\leqslant |x - x_0| \\ &< t^{1-\beta}|x - x_0|^{\beta}.\end{aligned}$$

If we choose $\beta = 1 - \alpha$, and $l(Q)$ is small, then

$$\int_Q |(\psi_t - \psi)(x) - (\psi_t - \psi)(x_0)|\, w(x)\, dx \leqslant t^{\alpha}\|\psi\|_{\mathrm{Lip}} \int_Q |x - x_0|^{1-\alpha} w\, dx \leqslant t^{\alpha}|Q|,$$

which tends to zero with t. When $|Q|$ is large, $|\psi_t - \psi| \lesssim t$ and the same bound is achieved.

Finally, if $\psi \in \mathrm{VMO}$, write $\psi = \theta + \eta$ where $\theta \in \mathrm{Lip}_0$ and $\|\eta\|_{\mathrm{BMO}} < \epsilon$. Then $\|\psi_t - \psi\|_{\mathrm{BMO}} \leqslant \|\theta_t - \theta\|_{\mathrm{BMO}} + C\|\epsilon\| \leqslant C\epsilon$, if t is small.

Lemma 3.6. (Compare Macías-Segovia [22], Lemma 3.2). *Suppose f is a distribution in $H^1(w\,dx)$ and let $\Omega = \{f^* > \lambda\}$. Let $\{\eta_j\}$ be the partition of unity associated to Ω described above. Define the distribution b_j on* VMO$(w\,dx)$ *by setting*

$$\langle b_j, \psi\rangle = \langle f, S_j(\psi)\rangle$$

for all $\psi \in \mathrm{VMO}(w\,dx)$. Then

(3.7) $$Nb_j(x_0) \leqslant C\lambda(r_j/|x_0 - x_j|)^{n-1+\alpha}\chi_{{}^cB(x_j, cr_j)}(x_0) + Cf^*(x_0)\chi_{B(x_j, cr_j)}(x_0)$$

the series $\sum_j b_j$ converges weakly in VMO*$(w\,dx)$ *to a distribution b satisfying*

(3.8) $$\int b^*(x)\, dx \leqslant C\int_{\Omega} f^*(x)\, dx,$$

and if $g = f - b$,

(3.9) $$Ng(x_0) \leqslant C\lambda \sum_j \left(\frac{r_j}{|x_0 - x_j| + r_j}\right)^{n-1+2} + Cf^*(x_0)\chi_{c_\Omega}(x_0).$$

PROOF. We shall prove (3.7) and (3.8); the proof of (3.9) requires no new ideas and the argument is essentially provided in Macías-Segovia [22].

PROOF OF (3.7). To estimate $Nb_j(x_0)$, let us first assume that $x_0 \notin B(x_j, cr_j)$ for some constant c, which does not depend on j. Then

$$\left| \int b_j(y)\varphi(x_0 - y/t)w(y)\,dy \cdot w(B(x_0, t))^{-1} \right| = w(B(x_0, t))^{-1} \cdot \left| \int f(y) S_j(\varphi_{x_0, t})(y) w(y)\,dy \right|$$

where $\varphi_{x_0,t}(\bullet) = \varphi(x_0 - \bullet/t)$. For the above expression to be nonzero, we must have $t > c|x_0 - x_j|$. Therefore,

$$\begin{aligned} |S_j(\varphi_{x_0,t})(x)| &= \left| \eta_j(x) \left\{ \eta_j(z')w(z')\,dz' \right\}^{-1} \cdot \int (\varphi_{x_0,t}(x) - \varphi_{x_0,t}(z))\eta_j(z)w(z)\,dz \right| \\ &\leqslant \int_{B(x;2r_j)} (|x - x|/t) w(z)\,dz \cdot \left\{ \int \eta_j(z')w(z')\,dz' \right\}^{-1} \\ &\leqslant cr_j/t, \end{aligned}$$

and

$$|D^\alpha(S_j(\varphi_{x_0,t}))| \leqslant \sum_{\beta \leq \alpha} c_{\alpha,\beta}(1/r_j)^{|\alpha| - |\beta|} t^{-|\beta|} \leqslant c(r_j/t) r_j^{-|\alpha|}.$$

Now pick $y_j \in {}^c\Omega$ with $|x_j - y_j| \approx r_j$. The function $S_j(\varphi_{x_0,t})$ has support in $B(x_j, 2r_j) \subseteq B(y_j, cr_j)$ so we can write $S_j(\varphi_{x_0,t})(y) = \theta(y_j - y/r_j)$. The above computations show that $Ct/r_j \cdot \theta(z)$ belongs to $\mathfrak{A}$. This gives the estimate

$$\left| \left\langle f, \frac{t}{r_j} S_j(\varphi_{x_0,t}) \right\rangle \right| w(B(y_j, cr_j))^{-1} \lesssim f^*(y_j)$$

and so

$$|\langle b_j, \varphi_{x_0,t} \rangle| \cdot w(B(x_0, t))^{-1} \lesssim \frac{w(B(y_j, cr_j))}{w(B(x_0, t))} \frac{r_j}{t} f^*(y_j) \lesssim \lambda \left(\frac{r_j}{t} \right)^{n-2+\alpha} \frac{r_j}{t},$$

using (3.1) (i) and the fact that $y_j \in {}^c\Omega$. Because $|x_0 - x_j| \lesssim t$, the above is bounded by $c\lambda(r_j/|x_0 - x_j|)^{n-1+\alpha}$, which is the first summand in (3.7).

Let us assume now that $x_0 \in B(x_j, cr_j)$, and consider two cases. If $t > r_j$, then $\operatorname{supp} S_j(\varphi_{x_0,t}) \subseteq \operatorname{supp} \eta_j$. Moreover,

$$\|S_j(\varphi_j(\varphi_{x_0,t}))\|_\infty \leqslant r_j/t \leqslant 1 \quad \text{and} \quad \|D^\alpha S_j(\varphi_{x_0,t})\|_\infty \leqslant r_j^{-|\alpha|}.$$

Reasoning as before we conclude that

$$|\langle b_j, \varphi_{x_0,t} \rangle| / w(B(x_0, t)) \lesssim f^*(x_0).$$

On the other hand, if $t < r_j$, split $S_j(\varphi_{x_0,t})$ in two parts as

$$S_j(\varphi_{x_0,t})(y) = \eta_j(y)\varphi_{x_0,t}(y) - \eta_j(y)\left\{\int \eta_j(z')w(z')\,dz'\right\}^{-1}\int \eta_j(z)\varphi_{x_0,t}(z)w\,dz$$
$$= h_1(y) - h_2(y).$$

The function $h_1(y)$ has support in $B(x_0, 2t)$, and satisfies $\|h_1\|_\infty \leqslant 1$ and $\|D^\alpha h_1\|_\infty \leqslant t^{-|\alpha|}$. Consequently,

$$|\langle f, h_1\rangle|\, w(B(x_0, t))^{-1} \lesssim f^*(x_0).$$

The function h_2 has support in $B(x_j, 2r_j)$ and satisfies

$$\|h_2\|_\infty \leqslant w(B(x_0, t))/w(B(x_j, 2r_j))$$

and

$$\|D^\alpha h\|_\infty \leqslant w(B(x_0, t))/w(B(x_j, wr_j)) \cdot r_j^{-|\alpha|}.$$

Since $|x_0 - x_j| < r_j$,

$$|\langle f, h_2\rangle|\, w(B(x_0, t))^{-1} \leqslant \left|\left\langle f, \frac{w(B(x_0, r_j))}{w(B(x_0, t))} h_2\right\rangle\right| w(B(x_0, r_j)) \leqslant f^*(x_0).$$

Altogether, this gives (3.7).

PROOF OF (3.8). Let

$$b^k = \sum_{j \leq k} b_j(x).$$

The above estimates imply that $\{b^k\}$ is a Cauchy sequence in $H_1(w\,dx)$, since

$$\sum_{j=N+1}^{M} \int Nb_j(x)\,dx \leqslant \sum_{j=N+1}^{M} c\lambda \int_{{}^cB(x_j, cr_j)} (r_j/|x - x_j|)^{n-1+\alpha}\,dx$$
$$+ \sum_{j=N+1}^{M} \int_{B(x_j, cr_j)} f^*(x)\,dx$$
$$\leqslant c\lambda \sum_{j=N+1}^{M} \sum_{l>1} \int_{|x - x_j| \sim 2^l r_j} (r_j/2^l r_j)^{n-1+\alpha}\,dx$$
$$+ \sum_{j=N+1}^{M} \int_{B(x_j, cr_j)} f^*(x)\,dx$$
$$\leqslant c\lambda \sum_{j=N+1}^{M} \sum 2^{-l\alpha} r_j^{n-1} + \sum_{j=N+1}^{M} \int_{B(x_j, cr_j)} f^*(x)\,dx$$
$$\leqslant c \sum_{j=N+1}^{M} \int_{B(x_j, cr_j)} f^*(x)\,dx$$

as $f^* > \lambda$ on $B(x_j, r_j)$. By the finite overlap property of $\{B(x_j, cr_j)\}$ the above sum will be small once N and M are large.

We now want to see that $\{b^k\}$ converges (weakly) to a distribution in VMO*$(w\,dx)$. If $\psi \in \mathrm{VMO}(w\,dx)$,

$$|\langle b^k, \psi\rangle| = \left|\left\langle f, \sum_{j=1}^{k} S_j(\psi)\right\rangle\right| \leqslant \|f\|_{\mathrm{VMO}^*}\left\|\sum_{j=1}^{k} S_j(\psi)\right\|_{\mathrm{BMO}} \leqslant C\|f\|\ \|\psi\|_{\mathrm{BMO}},$$

by lemma 3.3. Since the $\{b^k\}$ are uniformly bounded in VMO*$(w\,dx)$, some subsequence $\{b^{k_j}\}$ has a weak limit, b, in VMO*$(w\,dx)$. Because the b^k are Cauchy in $H^1(w\,dx)$, the argument to follow shows that b is the H^1 limit of $\{b^{k_j}\}$, and hence of the entire sequence $\{b^k\}$. To compute the $H^1(w\,dx)$ norm of b, we test against φ and by weak convergence in VMO*$(w\,dx)$,

$$Nb(x) \leqslant \varlimsup_{k_j\to\infty} Nb^{k_j} \lesssim \lambda \sum_{j=1}^{\infty} (r_j/|x-x_j|)^{n-1+\alpha}\chi_{{}^cB(x_j,cr_j)}(x) + cf^*(x)\chi_{{}^cB(x_j,cr_j)}(x)$$

so that

$$\int b^*\,dx \leqslant C\int_{\Omega} f^*\,dx. \quad \square$$

When $f \in \mathrm{VMO}^*(w\,dx)$, set

$$\tilde{f}(y,t) = \langle f, K(\cdot, y, t)\rangle = \int f(x)\varphi(x-y)/t)\left\{\int \varphi(x - z'/t)w(z')\,dz'\right\}^{-1} w(x)\,dx.$$

Then one can show that $\langle \tilde{f}(\cdot, t), \psi\rangle = \langle f, \psi_t\rangle$ for all ψ and since $\psi_t \to \psi$ in VMO$(w\,dx)$ as $t \to 0$, by Lemma 3.4, $\tilde{f}(y,t)$ will converge to $f(y)$ in the ditribution sense as $t \to 0$.

Lemma 3.10. *If f is a distribution in* VMO*$(w\,dx)$ *which satisfies*

$$\int (f^*(x))^2\,dx < \infty,$$

then there exists a function $F \in L^2(dx)$ such that, for all $\psi \in \mathrm{Lip}_0$,

$$\langle f, \psi\rangle = \int F(x)\psi(x)w(x)\,dx.$$

PROOF. Notice that $\tilde{f}(y,t)$ is not quite the same as $f_\varphi(y,t)$, however

$$\int \varphi(x - z'/t)w(z')\,dz' \approx w(B(y,t)) \quad \text{for} \quad |x-y| < t.$$

This implies that $|\tilde{f}(y,t)| \lesssim f^*(y)$ for all t and so $\tilde{f}(y,t)$ is uniformly in $L^2(dy)$, by the same argument used in the proof of Lemma 3.2. By passing to a subsequence we get an $L^2(dx)$ function $F(x)$ such that $\tilde{f}(\cdot, t)$ converges weakly to $F(\bullet)$ as $t \to 0$. We also know that $\tilde{f}(\bullet, t) \to f$ when tested against Lip_0 functions. Let $\psi \in \mathrm{Lip}_0$. Then

$$\langle f, \psi \rangle = \lim_t \langle \tilde{f}(\bullet, t), \psi \rangle = \lim_{t \to 0} \int \tilde{f}(y, t)\psi(y)w(y)\, dy.$$

The reverse Hölder condition of exponent two tells us that $\psi_w \in L^2(dy)$, and we have shown that $\tilde{f}(\bullet, t) \in L^2(dy)$. Hence

$$\lim_t \langle \tilde{f}(\bullet, t), \psi \rangle = \int F(y)\psi(y)w(y)\, dy$$

which proves the lemma. $\square$

Lemma 3.11. *Suppose f is a distribution in* VMO*$(w\,dx)$ *with $\int f^*(x)\, dx < \infty$. Then, given $\epsilon > 0$, there exists a function $\tilde{g} \in L^2(dx)$ such that*

(i) *$\tilde{g}$ has a unique extension, g, to* VMO*$(w\,dx)$
(ii) $\int (f - g)^* w\, dx < \epsilon$.

PROOF. Choose $\lambda > \int f^*(x)\, dx$ such that

$$\int_\Omega f^*(x)\, dx < \epsilon, \quad \text{for} \quad \Omega = \{f^* > \lambda\}.$$

By Lemma (3.6), $f = g + b$, where

$$\int b^*(x)\, dx \leqslant c \int_\Omega f^*(x)\, dx$$

and

$$g^*(x) \leqslant c\lambda \sum_j (r_j/|x - x_j| + r_j)^{n-1+\alpha} + cf^*(x)\chi_{c\Omega}(x).$$

Therefore,

$$\begin{aligned}\int (g^*(x))^2\, dx &\leqslant c\lambda^2 \int \Big\{\sum_j (r_j/|x - x_j| + r_j)^{n-1+\alpha}\Big\}^2 dx + \int_{c\Omega} (f^*(x))^2\, dx \\ &\leqslant c\lambda^2|\Omega| + \lambda \int_\Omega f^*(x)\, dx \\ &\leqslant c\lambda \int f^*(x)\, dx\end{aligned}$$

By lemma (3.10), there exists an $L^2(dx)$ function $\tilde{g}$ such that $\langle g, \psi \rangle = \langle \tilde{g}, \psi \rangle$ for all $\psi \in \mathrm{Lip}_0$. And

$$\int (f - g)^*\, dx = \int b^*(x)\, dx < \epsilon. \quad \square$$

Definition. *A function $a(x)$ is an atom if the support of $a(x)$ is contained in some ball $B(x_0, r)$, $\|a\|_\infty \leqslant r^{-n+1}$ and $\int a(x)w(x)\,dx = 0$.*

Lemma 3.12. *There exists a constant C such that for all atoms $a(x)$,*

$$\int a^*(x)\,dx \leqslant C.$$

Proof. Let $a(x)$ be an atom supported in $B(x_0, r) = B_r$. Let

$$M_w(h)(x) = \sup_{Q \ni x}\left\{\int_Q |h(y)|w(y)\,dy/w(Q)\right\}$$

be the Hardy-Littlewood maximal function formed from the measure $w\,dx$. Consider

$$\begin{aligned}|a_\varphi(x,t)| &\leqslant \left|\int a(y)\varphi(x-y)/t)w(y)\,dy \cdot w(B(x,t))\right|\\ &\leqslant \frac{1}{w(B(x,t))}\int_{B(x,t)} |a(y)|w(y)\,dy,\end{aligned}$$

which shows that

$$\sup_t |a_\varphi(x,t)| \leqslant M_w a(x).$$

Now

$$\begin{aligned}\int_{B(x_0,2r)} Na(x)\,dx &\lesssim |B_r|^{1/2}\left(\int (M_w a)^2(x)\,dx\right)^{1/2}\\ &= |B_r|^{1/2}\left(\int (M_w a)^2(x)w^{-1}(x)w(x)\,dx\right)\\ &\lesssim |B_r|^{1/2}\left(\int a^2\,dx\right)^{1/2}\\ &\leqslant |B_r|\cdot\|a\|_\infty \leqslant C.\end{aligned}$$

The last few inequalities follow from the fact that $w^{-1} \in A_2(w\,dx)$, (Muckenhoupt [23]) and the properties of atoms. It remains to estimate $Na(x)$ when $x \in {}^cB(x_0, 2r)$. By the mean value property of atoms,

$$\begin{aligned}|a_\varphi(x,t)| &= \left|\left\{\int \varphi(x - z/t)w\,dz\right\}^{-1}\right.\\ &\quad\left.\cdot \int_{y\in B_r} a(y)[\varphi(x-y/t) - \varphi(x-x_0/t)]w(y)\,dy\right|\\ &\leqslant w(B(x,t))^{-1}\cdot\|a\|_\infty \int_{B_r} \frac{|y-x_0|}{t} w(y)\,dy\end{aligned}$$

Note that if $x \in {}^cB(x_0, 2r)$, then t must be larger than $c|x - x_0|$ in order that $\operatorname{supp}(x - \bullet/t) \cap \operatorname{supp} a$ be nonempty. So $|a_\varphi(x, t)|$ is bounded by

$$|B_r|^{-1} \cdot \frac{r}{|x - x_0|} \frac{w(B_r)}{w(B(x, t))} \leqslant |B_r|^{-1} \left(\frac{r}{|x - x_0|} \right)^{n-1+\alpha}$$

by (4.1) (i). This implies that

$$\begin{aligned} \int_{{}^cB(x_0, 2r)} Na(x)\, dx &\leqslant \sum_{l > 2} \int_{\{x: |x - x_0| \approx 2^l r\}} |B_r|^{-1} \cdot 2^{-l(n-1+\alpha)}\, dx \\ &\leqslant \sum_l |B_r|^{-1} \approx 2^{-l(n-1+\alpha)} \cdot (2^l r)^{n-1} \\ &\leqslant C. \quad \square \end{aligned}$$

Definition.

$$H^1_{at} = \left\{ f = \sum_k \lambda_k a_k \text{: the } a_k \text{ are atoms and } \|f\|_{H^1_{at}} = \sum_k |\lambda_k| \text{ is finite} \right\}$$

where $f \in \mathrm{VMO}^*(w\,dx)$ *and the convergence is in* $\mathrm{VMO}^*(w\,dx)$.

By Lemma 3.12, H^1_{at} is contained in $H^1(w\,dx)$. Standard arguments show that the dual of H^1_{at} is $\mathrm{BMO}(w\,dx)$.

Lemma 3.13. H^1_{at} *is the dual of* $\mathrm{VMO}(w\,dx)$.

Proof. We refer to Coifman-Weiss [5], p. 638, for a proof of

$$\mathrm{VMO}(\mathbb{R}^n, dx) = H^1_{at}(\mathbb{R}^{n-1})$$

which can be modified to yield our lemma. $\square$

Theorem 3.14. *Let* f *be a distribution in* $\mathrm{VMO}^*(w\,dx)$ *with* $\int f^*(x)\,dx < \infty$. *Then there exists a sequence of numbers* $\{\lambda_k\}$ *and atoms* $\{a_k\}$ *such that* $f = \sum \lambda_k a_k$ *in the sense that*

$$\langle f, \psi \rangle = \left\{ \sum_k \lambda_k a_k, \psi \right\}$$

for all $\psi \in \mathrm{Lip}_0$ *and the norms* $\|f\|_{H^1(w\,dx)}$ *and* $\sum_k |\lambda_k|$ *are equivalent.*

We shall confine ourselves to a few remarks about the proof of Theorem 4.14; the technical details of the construction are standard from the information we have at hand, and the reader is referred to Latter [21], Stromberg-Torchinsky [24], or Macías-Segovia [22]. We observe the following.

(1) The density Lemma (3.11) provides a class of $L^1_{\text{loc}}(w\,dx)$ functions on which the construction of the atoms can take place. This forms, essentially, the heart of the proof.

(2) We already know that elements of $H^1(w\,dx)$ have an atomic decomposition $H^1(w\,dx)$ as a subspace of $\text{VMO}^*(w\,dx) = H^1_{at}$. The point of course is the equivalence of the two norms. Moreover, by Lemma (4.13) and continuity, the relationship $\langle f, \psi\rangle = \langle \Sigma\lambda_k a_k, \psi\rangle$ is true for all $\psi \in \text{VMO}(w\,dx)$.

(3) Theorem (3.14) shows that $H^1(w\,dx)$ is complete in the norm $\int f^*(x)\,dx$. It is equivalent to this fact since lemma 4.12 guarantees that $\|f\|_{H^1(w\,dx)}$ is bounded by $\|f\|_{H^1_{at}}$ and so the comparability of these two norms is equivalent to the completeness in $\|\cdot\|_{H^1(w\,dx)}$ norm.

(4) Finite sums of atoms are dense in $H^1(w\,dx)$.

(5) If one uses the approach in Stromberg-Torchinsky [24], an atomic decomposition for $H^1(w\,dx)$ can be obtained with arbitrarily large vanishing moments on the atoms. That is, whenever $p(x)$ is a polynomial of degree less than N, the atoms $a(x)$ will satisfy

$$\int a(x)p(x)w(x)\,dx = 0.$$

Corollary 3.15. *The dual of* $H^1(w\,dx)$ *is* $\text{BMO}(w\,dx)$, *with pairing*

$$\langle f, \psi\rangle = \int f\psi w\,dx,$$

for f *a finite sum of atoms.*

Theorem 3.16. *Let* $\psi \in \text{BMO}(w\,dx)$ *with compact support. Set*

$$\varphi_s(\cdot) = s^{-n+1}\varphi(\cdot/s).$$

Then $\psi = b_0 + b$ *where* $b_0(x)w(x) \in L^\infty(dx)$ *and* b *has an extension* $h(x,s)$ *to the upper half plane (in the sense that* $h(x,s) \to b$ *weakly in* $L^1(w\,dx)$*) which satisfies*

(1) $|\nabla h(x,s)|\,w * \varphi_s(x)$ *is a Carleson measure, and*

(2) $|\nabla h(x,s)|\,w * \varphi_s(x) \cdot s \leqslant C$.

Remark. As in P. Jones [19], one can obtain such an extension for all $\text{BMO}(w\,dx)$ functions, once it is known for compactly supported functions.

PROOF. Recall that for $u \in H^1(w\,dx)$,

$$u(x,y) = \int \varphi(x - z/y)f(z)w(z)\,dz \cdot \left\{\int \varphi\left(\frac{x-z'}{y}\right)w(z')\,dz'\right\}^{-1} \tag{3.17}$$

We claim that if λ is a linear functional on $H^1(w\,dx)$ and $u \in H^1$

$$\lambda(u) = \int u(x) g_\infty(x)\,dx + \sum_{n=-\infty}^{n=\infty} \int u(x, y_n) g_n(x)\,dx$$

where $y_n \to -\infty$ as $n \to \infty$, $y_n \to +\infty$ as $n \to -\infty$, $|y_n - y_{n+1}| < \min\{\delta, y_n^2\}$ and

$$\|g\|_\infty + \left\| \sum_{n=-\infty}^{\infty} |g_n| \right\|_\infty \leqslant \|\lambda\|.$$

(We assume here, by the density lemma, that in (3.17), $f \in L^1 \cap L^2(dx)$). The argument for this is due to C. Fefferman; it depends on the fact that $u^*(x)$, the vertical maximal function defines an equivalent norm on $H^1(w\,dx)$. Our source for this is Garnett [15].

Fix a ball B, and let $\psi \in \mathrm{BMO}(w\,dx)$ with support in B. Then if f is an atom supported in B,

$$\begin{aligned}\int f(x)\psi(x)w(x)\,dx &= \int f(x) g_\infty(x)\,dx + \sum_{-\infty}^{\infty} \int u(x, y_n) g_n(x)\,dx \\ &= \int f g_\infty\,dx + \lim_{N\to\infty} \int f(z) w(z) \sum_{-N}^{N} \int \varphi\left(\frac{x-z}{y_n}\right) \\ &\quad \cdot \left\{ \int \varphi\left(\frac{x-z'}{y_n}\right) w\,dz' \right\}^{-1} g_n(x)\,dx\end{aligned}$$

Let $b_0 = g_\infty / w \in L^1_{\mathrm{loc}}(dw)$ and set $b = \psi - b_0$. Then b belongs to $\mathrm{BMO}(w\,dx)$. Set

$$h_N(z) = \sum_{-N}^{N} \int \varphi\left(\frac{x-z}{y_n}\right) \left\{ \int \varphi\left(\frac{x-z'}{y_n}\right) w(z')\,dz' \right\}^{-1} g_n(x)\,dx$$

so that

$$\int b f w\,dz = \lim_{N\to\infty} \int f(z) h_N(z) w(z)\,dz.$$

The function $h_N(z)$ is well defined, as

$$|h_N(z)| \leqslant \sum_{-N}^{N} |B(z, y_n)| / w(B(z, y_n)).$$

Now test h_N against an atom a:

$$\begin{aligned}\int a(z)h_N(z)w(z)\,dz &= \sum_{-N}^{N}\int a(x,y_n)g_n(x)\,dx\\ &\leqslant \int a^+(x)\sum_{-N}^{N}|g_n(x)|\,dx\\ &\leqslant C\|a\|_{H^1}\leqslant C.\end{aligned}$$

So $\|h_N\|_{\text{BMO}}$ is bounded by a constant independent of N.

We now claim that BMO $\subseteq L^{1+\epsilon}_{\text{loc}}(dw)$ for some $\epsilon > 0$. By the A_2 property of w^{-1}, $\psi^2w^2 \in L^1_{\text{loc}}(dx)$ for any $\psi \in \text{BMO}(w\,dx)$. But $w \in A_\infty(dx)$ so there exists a small $\delta > 0$ such that $w^{-\delta} \in L^1_{\text{loc}}(dx)$. (See Coifman-Fefferman [3]). In particular, $h_N \in L^{1+\epsilon}_{\text{loc}}(w\,dx)$ and so on B, there exists constant c_N and a constant A for which $\int |h_N - c_N|^{1+\epsilon}\,dw \leqslant A$, all N. Thus there is a weak limit for $\{h_N - c_N\}$; call this h. We have

$$\int fh\,w\,dx = \int fb\,w\,dx$$

for all $f \in L^\infty$ with support in B, and $\int f\,w\,dx = 0$. Hence $b = h + c$ for some constant C. As in Varopoulos [25], set

$$h_N(z,s) = \sum_{-N}^{N}\theta(s/y_n)\int \varphi\left(\frac{x-z}{y_n}\right)\left\{\int \varphi\left(\frac{x-z'}{y_n}\right)w(z')\,dz'\right\}^{-1}g_n(x)\,dx$$

where $\theta \in C^\infty$ satisfies $\theta(t) \equiv 1$ for $0 \leqslant t \leqslant 1/2$, $\theta(t) = 0$ for $t > 1$. Then

$$|\nabla h_N(z,s)| \leqslant \int_{y=s}^{\infty}\int_{\{|x-z|<y\}}\{yw(B(z,y))\}^{-1}\sum_{j=-N}^{N}g_j(x)\,dS_j(x,y) \tag{3.17}$$

where $dS_j(x,y) = dx$ on $y = y_j$. If we set

$$d\sigma(x,y) = \sum_{j=-N}^{N}g_j(x)\,dS_j(x,y)$$

then $d\sigma(x,y)$ is a Carleson measure. In fact

$$\int_{x\in Q}\int_0^\infty d\sigma(x,y) \leqslant C|Q|,$$

which we will refer to as the vertical Carleson measure estimate. Hence, using (3.1) (i)

$$\begin{aligned}|\nabla h_n(z,s)| &\leqslant \sum_{l>0}\int_{y\approx 2^ls}\int_{|x-z|<2^ls}(yw(B(z,s)))^{-1}\frac{w(B(z,s))}{w(B(z,y))}\,d\sigma(x,y)\\ &\leqslant \sum_{l>0}\int_{y\approx 2^ls}\int_{|x-z|<2^ls}(2^ls)^{-1}w(B(z,s))^{-1}(2^{-l})^{n-2+\alpha}\,d\sigma(x,y)\end{aligned}$$

$$|\nabla h_n(z,s)| \leqslant \sum_{l>0} (2^{-l})^{n-1+\alpha}\{sw(B(z,s))\}^{-1}\sigma(B(z,2^l s)\times[0,2^l s])$$
$$\leqslant \sum_{l>0} 2^{-l\alpha}\cdot s^{n-2}/w(B(z,s))$$

because $d\sigma$ is Carleson. This proves (2) for $h_N(x,s)$. Now consider the Carleson property of $|\nabla h_N|w*\varphi_s(\bullet)/s^{n-1}$. Fix a cube $Q\subseteq \mathbb{R}^{n-1}$ and let $S(Q) = Q\times[0,l(Q)]$. We split the integral in (3.17) in two parts, estimating each separately.

$$(3.18)\quad \int_{(z,s)\in S(Q)}\int_{y=s}^{2l(Q)}\int_{x\in B(z,y)} \frac{|w*\varphi_s(y)|}{yw(B(z,y))}\,d\sigma(x,y)\,dz\,ds$$
$$\leqslant C\int_{(x,y)\in S(4Q)}\int_{z\in B(x,y)}\int_{s=0}^{y} \frac{w(B(z,s))}{w(B(z,y))}y^{-1}s^{-n+1}\,dz\,ds\,d\sigma(x,y)$$
$$\leqslant C\int_{(x,y)\in S(4Q)}\int_{z\in B(x,y)}\int_{s=0}^{y} (s/y)^{n-2+\alpha}y^{-1}s^{-n+1}\,dz\,ds\,d\sigma(x,y)$$
$$\leqslant C\iint_{S(4Q)} d\sigma(x,y) \leqslant C|Q|.$$

$$(3.19)\quad \int_{(z,s)\in S(Q)}\sum_{k>0}\int_{y\approx s^k l(Q)}\int_{x\in 2^kQ} y^{-1}\frac{w(B(z,s)}{w(B(z,y)}s^{-n+1}\,d\sigma(x,y)\,dz\,ds$$
$$\leqslant \int_{(z,s)\in S(Q)}\sum_{k>0}\int_{y\approx 2^k l(Q)}\int_{x\in 2^kQ} 2^{-k}l(Q)^{-1}[s/2^k l(Q)]^{n-2+\alpha}s^{-n+1}\,d\sigma(x,y)\,dz\,ds$$
$$\leqslant \sum_{k>0}\int_{y\approx 2^k l(Q)}\int_{x\in 2^kQ} (2^{-k})^{n-1+\alpha}|Q|l(Q)^{-(n-1+\alpha)}\int_{s=0}^{l(Q)} s^{-1+\alpha}\,ds\,d\sigma(x,y)$$
$$\leqslant \sum_{k>0} 2^{-k\alpha}|Q| = C|Q|.$$

Combining (3.18) and (3.19) gives (3.16) (1) for $|\nabla h_N(x,s)|$ with a Carleson measure constant independent of N.

By (3.16) (2), when $s>0$, $|\nabla h_N(\bullet,s)|$ is uniformly bounded on compact sets. A similar result holds for higher order derivatives. Thus there exists a sequence of constans $\{a_N\}$ such that $h_N(x,s)+a_N\to h(x,s)$ uniformly on compact sets as $N\to\infty$, and also $\nabla h_N(x,s)\to\nabla h(x,s)$. This $h(x,s)$ satisfies conditions (1) and (2) of the theorem.

For fixed N, the expression

$$\sum_{-N}^{N}\theta(s/y_n)\varphi\left(\frac{x-y}{y_n}\right)g_n(x)\left\{\int\varphi\left(\frac{x-z'}{y_n}\right)w(z')\,dz'\right\}^{-1}$$

is bounded by some constant c_N and converges pointwise as $s \to 0$. So by dominated convergence, $h_N(x,s) \to h_N(x)$ as $s \to 0$ in $L^1_{\text{loc}}(w\,dx)$.

For fixed $s > 0$,

$$\int h(x,s)f(x)w(x)\,dx = \lim_{N\to\infty} \int h_N(x,s)f(x)w(x)\,dx$$

for all $f \in L^\infty(B)$ with $\int fw\,dx = 0$. Hence if f is an atom,

$$\int h(x)f(x)w(x)\,dx - \int h(x,s)f(x)w(x)\,dx$$
$$= \lim_{N\to\infty} \int [h_N(x) - h_N(x,s)]f(x)w(x)\,dx$$

and

$$|h_N(x) - h_N(x,s)| \leqslant \sum_{-\infty}^{\infty} |g_n(x)|u^+(x)[1 - \theta(s/y_n)]$$

which converges to zero as $s \to 0$. So for all atoms f,

$$\text{(3.20)}\quad \lim_{s\to 0} \int h(x,s)f(x)w(x)\,dx = \int h(x)f(x)w(x)\,dx = \int b(x)f(x)w(x)\,dx.$$

Now let f be an arbitrary L^∞ function supported in B. Set $g = (f - (f)_B)\chi_B$. By (3.20) with g in place of f, we get

$$\int fbw\,dx - (f)_B \int_B bw\,dx = \lim_{s\to 0} \int g(x)h(x,s)w(x)\,dx$$

which implies

$$b(x) = \lim_{s\to 0}\left[h(x,s) - \int_B h(x,s)w(x)\frac{dx}{w(B)}\right] + (b)_B,$$

where the limit is taken weakly in $L^1(w\,dx)$. Thus $(b - (b)_B)\chi_B$ has an extension $h(x,s) - C(s)$ to the upper half plane where

$$C(s) = \lim_{s\to 0} \int_B h(x,s)w(x)\,dx/w(B).$$

Using the vertical Carleson measure estimate, it is not hard to check that $C(s)$ verifies (1) and (2). □

We now describe an alternative, constructive mean of obtaining the extension theorem for BMO$(w\,dx)$ functions. We mimic the approach of Varopoulos [25]. Let us assume that $f \in \mathrm{BMO}(w\,dx)$ has support in Q_0 the unit cube) with $(f)_{Q_0} = 0$. Then we claim the following.

Lemma 3.21. *If $f \in \mathrm{BMO}(Q_0, w\,dx)$ with $\|f\|_{\mathrm{BMO}} \leqslant 1$, there exists a sequence of dyadic cubes $\{I_j\}$ and constants $\{\alpha_{I_j}\}$ such that*

(1) $\left|f(x) - \sum \alpha_{I_j}\chi_{I_j}(x)\right| \leqslant \frac{3}{2} w(x)^{-1}$.

(2) $|\alpha_{I_j}| \leqslant |I_j|/w(I_j)$.

(3) $\sum_{I_j \subseteq I_0} |I_j| \leqslant C|I_0|$, *for all dyadic I_0.*

PROOF. The proof is obtained in the same way as its counterpart in BMO(dx), the argument in that situation is due to Garnett ([15]. p.) and closely follows the stopping time procedure used to prove the John-Nirenberg theorem. We observe that one only needs to show that if f belongs to BMO($w\,dx$), then for any two consecutive dyadic intervals I, I', the difference $|(f)_I - (f)_{I'}|$ is less than $C|I|/w(I)$ where $I = I' \cup I''$. Moreover, this condition characterizes the difference between BMO($w\,dx$) and dyadic BMO($w\,dx$). □

The function $f_0(x) = \sum \alpha_I \chi_I(x)$ can be extended to the upper half space in a discrte way by setting

$$F(x, y) = \sum_I \alpha_I \chi_{\tilde{I}}(x, y),$$

where $\tilde{I} = I \times [0, L(I)]$.

Lemma 3.22. *In the distribution sense, $|\nabla F(x, y)|\, w * \varphi_y(x)/y^{n-1}$ is a Carleson measure on $\mathbb{R}^n_+$.* (Comparare Varopoulos [15], p. 226).

PROOF. We give the proof in $\mathbb{R}^2_+$, for the mesure $|\nabla F|\, w * \varphi_y(x)/y^{n-1}$. Note that $\partial\chi_{\tilde{I}}/\partial x$, with $I = [a, b]$, is Lebesgue measure on $\{x = a, 0 \leqslant y \leqslant l(I)\}$ and $\{x = b, 0 \leqslant y \leqslant l(I)\}$ and that $\partial\chi_{\tilde{I}}/\partial y$ is Lebesgue measure on $\{x \in I, y = l(I)\}$. Consider first $\sum \alpha_I(\partial\chi_{\tilde{I}}(x, y)/\partial y) w * \varphi_y(x)/y^{n-1}$. Then for $(x, y) \in \operatorname{supp} \partial\chi_{\tilde{I}}/\partial y$, $w * \varphi_y(x)/y^{n-1}$ is comparable to $w(I)/|I|$. Let $\beta_I = \alpha_I w(I)/|I|$. By (3.21) (2), $|\beta_I| \leqslant 1$ so we must show that $\sum_I \beta_I \partial\chi_{\tilde{I}}/\partial y$ is Carleson when the intervals $\{I\}$ have the packing property (3.2) (3). But this is precisely Varopoulos's result.

Now consider $\sum \alpha_I \partial\chi_{\tilde{I}}(x, y)/\partial x) w * \varphi_y(x)/y^{n-1}$. Let I_0 be a dyadic interval of the form $[p/2^m, p + 1/2^m]$ for $n, p \geqslant 0$. Let I_1 and I_2 be its adjacent dyadic intervals and set

$$\mu_i = \sum_{I \subseteq I_i} \alpha_I \partial\chi_{\tilde{I}}/\partial x, \qquad i = 0, 1, 2.$$

Then

$$\int_{x\in I_0}\int_{y=0}^{l(I_0)} w*\varphi_y(x)/y^{n-1}\,d\mu_i(x,y)=\int_{x\in I_0}\int_{y=0}^{l(I_0)}\sum_{I\subseteq I_0}\alpha_I w(B(a,y))/y^{n-1}\,dS_I(x,y)$$

where $dS_I(x,y)=dy$ for $\{x=a, 0<y<l(I)\}$ and $\{x=a+l(I), 0<y\leqslant l(I)\}$. By (3.2) (2) and our basic estimate for $w(B)/w(2^kB)$, the above is bounded by

$$\sum_{I\subseteq I_0}\int_{x\in I_0}\int_{y=0}^{l(I)}\frac{|I|}{w(I)}w(B(a,y))/y^{n-1}\,dS_I(x,y)$$

$$\leqslant \sum_{I\subseteq I_0}\int_{x\in I_0}\int_{y=0}^{l(I)}[l(I)/y]^{n-1}[y/l(I)]^{n-2+\alpha}\,dS_I(x,y)$$

$$\leqslant 2\sum_{I\subseteq I_0}\int_{y=0}^{l(I)}(l(I)/y)^{1-\alpha}\,dy$$

$$=2\sum_{I\subseteq I_0}l(I)\leqslant C|I_0|,$$

which gives the desired estimate for the measure μ_i. The rest of the argument of [25] goes through with the help of the following estimates:

$$|(f)_I-(f)_{2I}|\leqslant |I|/w(I)$$

and

$$\sum_{I\subseteq I_0}\alpha_I w(I)\leqslant |I_0|. \quad \square$$

The discrete version of our extension given by Lemma 3.22 can be smoothed to obtain a continuous one by setting $\bar{F}(x,y)=F*\varphi_y(x,y)$ for some smooth bump function φ. The estimate for $|\nabla_x\bar{F}(x,y)|$, both the pointwise bound and the Carleson measure condition, is not hard. The main difficulty lies in the estimate for $|\nabla_y\bar{F}(x,y)|$. This can be overcome however by writing $\partial\varphi_y(\cdot)/\partial y$ as $\nabla_x\eta_y(x,y)$ for some other smooth bump function η.

4. We shall prove in this section, the main results for $H^1(D,d\sigma)$, $D\subset\mathbb{R}^n$ and Lipschitz. The fundamental tool will be the extension Theorem 3.16. Let us begin by localizing to a part of the boundary of D which is the graph of a Lipschitz function.

There exists some $\delta>0$ and a finite convering of $\{x\colon \mathrm{dist}(x,\partial D)\leqslant\delta\}$ by balls $B_j=B(Q_j,r_j)$ such that

$$B(Q_j,4r_j)\cap D=B(Q_j,4r_j)\cap\{(x,y)\colon y>\Phi_j(x)\}$$

where each Φ_j is Lipschitz. Let $\{\psi_j\}$ be a finite partition of unity for $\{x: \operatorname{dist}(x, \partial D) \leqslant \delta\}$ subordinate to $\{B_j\}$ with $\psi_j \in C_0^\infty$. Let w_j be defined on $\mathbb{R}^{n-1}$ by $w_j(x) = k(x, \Phi_j(x))$ for $(x, \theta_j(x)) \in \partial D \cap B(Q_j, 4r_j)$, where $k = d\omega/d\sigma$.

Lemma 4.1. *The measure $w_j(x)\,dx$ on $\mathbb{R}^{n-1}$ obtained by extending w_j (the projection of harmonic measure) by reflection across the face of a cube*

$$I_j \subseteq \{x: (x, \Phi_j(x) \in \partial D \cap B(Q_j, r_j)\}$$

satisfies conditions (3.1) (i), (ii), *and* (iii).

PROOF. Let us assume that $w(x)$ is the projection of harmonic measure onto the unit cube Q_0 and then extended by repeated reflection across the faces of Q_0 to all of $\mathbb{R}^{n-1}$. Thus $\mathbb{R}^{n-1} = \bigcup Q_l$, each Q_l has unit size and $w(Q_l) = w(Q_k)$ all l, k; and when the cubes are large, conditions (i)-(iii) are obviously satisfied.

Let us consider the first condition (3.1) (i): $w(Q)/w(2^jQ) \leqslant C2^{-j(n-2+\alpha)}$ for all cubes Q. When both Q and 2^jQ are contained in some Q_l, this is clear. If the length of 2^jQ is large, let j_0 be the largest integer such that $l(2^{j_0}Q) < 1$. Then $w(Q)/w(2^jQ) = w(Q)/w(2^{j_0}Q) \cdot w(2^{j_0}(Q)/w(2^jQ)$ and by our previous remark it is enough to show (3.1) (i) under the assumption that $l(2^jQ)$ is small. We shall reduce this situation to the case where both cubes are contained in some Q_l.

We claim that there exists a Q' and Q'_j, both contained in some Q_l, with $w(Q')/w(Q'_j)$ comparable to $w(Q)/w(2^jQ)$ and $w(Q') \leqslant 2^{-j(n-2+\alpha)}w(Q'_j)$. To see this choose a Q_l such that $|Q \cap Q_l| > |Q|/2$. Then

$$w(Q) = \sum_k w(Q \cap Q_k) \leqslant 2^{n-1} w(Q \cap Q_l),$$

since w was extended by reflection. Inside $Q \cap Q_l$ there is a cube Q' with $|Q'| \approx |Q \cap Q_l|$ and since $w \in A_\infty(Q_l, dx)$ we also have $w(Q') \approx w(Q \cap Q_l)$. The same argument gives a cube Q'_j, $Q' \subset Q'_j \subset Q_l$ with $w(2^jQ) = w(Q'_j)$ and $l(Q'_j) \approx 2^j l(Q')$.

The argument for the reverse Hölder condition of exponent two consists of the same case by case analysis and will be omitted.

Lemma 4.2. *If $f \in \mathrm{BMO}_\sigma(d\omega)$ and $\theta \in C_0^\infty(\mathbb{R}^n)$ with $|\nabla\theta| \leqslant c$, then $f\theta \in \mathrm{BMO}_\sigma(d\omega)$ with $\|f\theta\|_{\mathrm{BMO}} \leqslant c\|f\|_{\mathrm{BMO}}$.*

PROOF. Let $\Delta = \Delta(Q_0, r)$ be a small surface ball contained in ∂D and let $c_\Delta = \theta(Q_0)(f)_\Delta$ where Q_0 is the center of Δ. Then

$$\begin{aligned}\int_\Delta |f\theta - c_\Delta|\, d\omega &\leqslant \int_\Delta |f(Q)[\theta(Q) - \theta(Q_0)]\, d\omega + \int_\Delta \theta(Q_0)[f - (f)_\Delta]\, d\omega \\ &\leqslant \sigma(\Delta)\|\theta\|_\infty \|f\|_{\mathrm{BMO}} + \int_\Delta |f(Q)|\, |Q - Q_0|\, d\omega \\ &\leqslant c\|f\|_{\mathrm{BMO}} \cdot \sigma(\Delta) + r \cdot \int_\Delta |f|\, dw.\end{aligned}$$

Let $2^j\Delta$ denote the surface ball contained in ∂D with $\sigma(2^j\Delta) = (2^j)^{n-1}\sigma(\Delta)$ and let $m = \inf\{j\colon \sigma(2^j\Delta) > 1/2\, \sigma(\partial D)\}$. Then

$$\int_\Delta |f|\, d\omega \leqslant \int_\Delta |f - f_\Delta|\, d\omega + \sum_{j=1}^m \int_\Delta |(f)_{2^{j-1}\Delta} - (f)_{2^j\Delta}|\, d\omega + \int_\Delta |(f)_{2^m\Delta}|\, d\omega \tag{4.3}$$

Applying the $\mathrm{BMO}_\sigma(d\omega)$ condition to the interval $2^{j-1}\Delta$ we see that

$$|(f)_{2^{j-1}\Delta} - (f)_{2^j\Delta}| \leqslant \sigma(2^j\Delta)/\omega(2^j\Delta)$$

Therefore

$$\begin{aligned}\sum_{j=1}^m \int_\Delta |(f)_{2^{j-1}\Delta} - (f)_{2^j\Delta}|\, d\omega &\leqslant \omega(\Delta) \cdot \sum_{j=1}^m \sigma(2^j\Delta)/\omega(2^j\Delta) \\ &\leqslant \sum_{j=1}^m (2^{-j})^{n-2+\alpha} \cdot \operatorname{diam}(\partial D) \cdot (2^j r)^{n-2} \\ &\leqslant Cr^{n-2}.\end{aligned}$$

Also

$$|(f)_{2^m\Delta}| \leqslant C \int_{\partial D} |f|\, dw \leqslant C\|f\|_{\mathrm{BMO}}.$$

Altogether then (4.3) is bounded by $\|f\|_{\mathrm{BMO}}\{\sigma(\Delta) + r^{n-2} + \omega(\Delta)\}$, and

$$\begin{aligned}r \cdot \int_\Delta |f|\, d\omega &\leqslant \|f\|_{\mathrm{BMO}}\{\operatorname{diam}(\partial D) \cdot \sigma(\Delta) + r^{n-1} + r \cdot r^{n-2+\alpha}\} \\ &\leqslant c\|f\|_{\mathrm{BMO}} r^{n-1}. \quad \square\end{aligned}$$

Lemma 4.4. *Assume $f \in \mathrm{BMO}_\sigma(d\omega)$. Then there exists an f_0 with $f_0 k \in L^\infty$ ($k = d\omega/d\sigma$) such that if $f_1 = f - f_0$, there is an extension F of f_1 in the sense that*

$$\int_{\partial D} u(Q) f_1(Q)\, d\omega = \int_D G(x)\nabla u(x) \cdot \nabla F(x)\, dx - \int_D u(x)\nabla G(x) \cdot \nabla F(x)\, dx$$

for all $u \in \mathcal{L}(\bar{D})$ and $|\nabla F(x)|G(x)/d(x)$ is a Carleson measure on D. Moreover, $|\nabla F(x)| \lesssim G^{-1}(x)$ near ∂D and $|\nabla F(x)| + |F(x)|$ is bounded in a compact subset of D. (Compare Lemma 2.3 of Fabes-Kenig [11]).

PROOF. Let $\{\psi_j\}$ be the partition of unity associated to D subordinate to the covering $\{B_j\}$ and let $f_{\psi_j}(x) = \psi_j(x, \Phi_j(x)) \cdot f(x, \Phi_j(x))$. Let $w_j(x)\,dx$ be the measure of Lemma 4.2. By Lemma 4.2, $f_{\psi_j}(x)$ belongs to $\mathrm{BMO}(\mathbb{R}^{n-1}, w_j\,dx)$. Let $\tilde{B}_j$ be the support of $f_{\psi_j}(x)$. Theorem 3.16 tells us that $f_{\psi_j} - g_j$ (where $g_j w_j \in L^\infty$) has an extension F_{ψ_j} to the upper half space such that $|\nabla F_{\psi_j}(x,s)|\, w_j * \varphi_s(x)/s^{n-1}$ is a Carleson measure and $|\nabla F_{\psi_j}(x,s)|(w_j * \varphi_s(x)/s^{n-2})^{-1}$ is bounded.

Pick $\theta_j \in C_0^\infty(B_j)$, identically 1 in a neighborhood of the support of ψ_j. Set $F_j(X) = \theta_j(X) \cdot F_{\psi_j}(x, y - \theta_j(x))$ for $X \in B_j \cap D$, $X = (x,y)$ and $y > \theta_j(x)$; put $F_j \equiv 0$ outside $B_j \cap D$. We will first show that

$$|\nabla(\theta_j(x, t + \Phi_j(x)) \cdot F_{\psi_j}(x,t)|\, w_j(B(x,t))/t^{n-1}$$

is a Carleson measure on $\mathbb{R}^{n-1}_+(dx\,dt)$ for all cubes Q with $l(Q) = r$, $r \leqslant r_0$, a number which depends only on the domain D. When the gradient falls on $F_{\psi_j}(x,t)$ our estimate is known. When the gradient falls on $\theta_j((x, t + \Phi_j(x))$, consider the integral

$$(4.5)\quad \int_0^r \int_{Q_r} |F_{\psi_j}(x,t)|\, w_j(B(x,t))/t^{n-1}\,dx\,dt$$
$$= \int_0^r \int_{Q_r} |F_{\psi_j}(x,t) - F_{\psi_j}(x,R)|\, w_j(B(x,t))/t^{n-1}\,dx\,dt$$
$$+ \int_0^r \int_{Q_r} |F_{\psi_j}(x,R)|\, w_j(B(x,t))/t^{n-1}\,dx\,dt,$$

where R is the radius of B_j, the support of f_{ψ_j}. The first integral is bounded by

$$\int_0^r \int_{Q_r} \int_{s=t}^R |\nabla F_{\psi_j}(x,s)|\,ds\, w_j(B(x,t))/t^{n-1}\,dx\,dt$$
$$\leqslant \int_0^r \int_{Q_r} \int_{s=t}^R |\nabla F_{\psi_j}(x,s)|\, w_j(B(x,s))/s^{n-1} \cdot (t/s)^{n-2+\alpha} \cdot (s/t)^{n-1}\,ds\,dx\,dt$$
$$\leqslant R \int_{X \in Q_r} \int_{s=0}^\infty |\nabla F_{\psi_j}(x,s)|\, w_j(B(x,s))/s^{n-1}\,dx\,dt$$
$$\leqslant CR|Q_r|$$

since the Carleson measure property is satisfied over all vertical lines for our extension. But R is just a constant which depends only on δ in the covering of $\{X: \operatorname{dist}(X, \partial D) < \delta\}$.

We claim that $|F_{\psi_j}(x,R)|$ is bounded by a constant which depends only on R and hence only on the domain D. Recall from the proof of (3.16) that $F_{\psi_j}(x,R)$ has the form $h(x,R) - (1/w(B)) \int_B h(y,R) w\,dy$. Hence

$$|F_{\psi_j}(x,R)| \leqslant \sup_{t \in B} |\nabla h(t,R)|\, |B|,$$

which is at most $(w(B)/R^{n-1}) \cdot |B| \leqslant w(B)$. Therefore, the second integral in (4.5) hast the bound

$$C_D \int_0^r \int_{Q_r} \frac{w_j(B(x,t))}{w(B(x,r))} \frac{w(B(x,r))}{t^{n-1}} \, dx \, dt$$

$$\begin{aligned} &\leqslant C_D \int_0^r \int_{Q_r} (t/r)^{n-2+\alpha} t^{-n+1} \, dx \, dt \cdot w(Q_r) \\ &\leqslant C_D w(Q_r) r^{-n+2} \cdot |W_r| \\ &\leqslant C_D r w(Q_r) \\ &\leqslant C_D r^{\alpha} |Q_r| \\ &\leqslant R^{\alpha} C_D |Q_r| . \end{aligned}$$

The pointwise gradient estimate holds for $\theta_j(x, t + \Phi_j(x)) \nabla F_{\psi_j}(x, t)$ and when the differentiation falls on θ_j, we have

$$\begin{aligned} |\nabla \theta_j(x, t + \Phi_j(x)) \cdot F_{\psi_j}(x, t)| &\leqslant |F_{\psi_j}(x, t) - F_{\psi_j}(x, R)| + |F_{\psi_j}(x, R)| \\ &\leqslant \int_{s=t}^{R} |\nabla F_{\psi_j}(x, s)| \, ds + C_R \\ &\leqslant \int_{s=t}^{R} \frac{s^{n-2}}{w(B(x,s)} \frac{w(B(x,t))}{t^{n-2}} \, ds \frac{t^{n-2}}{w(B(x,t))} + C_R \\ &\leqslant \frac{t^{n-2}}{w(B(x,t))} \cdot \left\{ \int_{s=t}^{R} \frac{s^{n-2}}{t^{n-2}} \left(\frac{t}{s} \right)^{n-2+\alpha} ds + C_R' \right\} \\ &\leqslant \frac{C t^{n-2}}{w(B(x,t))} \end{aligned}$$

This shows all the bounds for the extension. We now show that F is an extension in the required sense.

Consider

$$\begin{aligned} \int_{\partial D} u(Q) f(Q) \, d\omega &= \sum_j \int_{\partial D} u \psi_j \theta_j f \, d\omega \\ &= \sum_j \int u(x, \Phi_j(x)) \theta_j(x, \Phi_j(x)) [f \psi_j(x) - g_j(x) + g_j(x)] w_j(x) \, dx . \end{aligned}$$

Let $h_j(x) = f_{\psi_j}(x) - g_j(x)$. Then $f_0 = \sum_j g_j(x, \Phi_j(x))$ satisfies $f_0 k \in L^{\infty}(d\sigma)$. We look at a single term

$$\int u(x, \Phi_j(x)) \theta_j(x, \Phi_j(x)) h_j(x) w_j(x) \, dx .$$

Let F_{ψ_j} be the extension of h_j. Then the above is equal to

$$\lim_{s\to 0}\int u(x,\Phi_j(x)+s)\theta_j(x,\Phi_j(x)+s)F_{\psi_j}(x,s)w_j(x)\,dx$$

$$= \lim_{s\to 0}\int_{\partial D} u(x,\Phi_j(x)+s))F_j(x,\Phi_j(x)+s)\,d\omega.$$

Now for fixed $s>0$, $F_j(x,\Phi_j(x)+s)$ can be approximated by C^2 functions and since s is fixed, $F_j\in \mathcal{L}(\bar{D})$. Then Green's theorem can be applied and the formal reasoning of the argument following the statement of Theorem 2.7 is justified. Thus we have established

Theorem 2.7. *If $u\in\mathcal{L}(\bar{D})$, $\Delta u=0$ and $u(P_0)=0$, then*

$$\left|\int_{\partial D} u(Q)f(Q)\,dw\right| \leqslant C\|Nu\|_{L^1(d\sigma)}\|f\|_{\mathrm{BMO}}.$$

Lemma 4.6. *Let $\mathcal{L}(\partial D)$ denote the functions which are Lipschitz on ∂D. $\mathcal{L}(\partial D)$ is dense in* $\mathrm{VMO}_\sigma(w)$.

PROOF. Clearly $\mathcal{L}(\partial D)$ is contained in $\mathrm{VMO}_\sigma(w)$. By Lemma 4.2, if $\psi\in \mathrm{VMO}_\sigma(w)$ then $\theta\psi\in\mathrm{VMO}_\sigma(w)$, where $\theta\in C_0^\infty$ and supported on B where

$$B\cap\partial D=\{(x,\Phi(x)):\Phi \text{ Lipschitz on } \mathbb{R}^{n-1}\}.$$

Then if $\bar{\psi}(x)=(\theta\psi)(x,\Phi(x))$ we need only show that $\bar{\psi}\in\mathrm{VMO}(w\,dx)$ (see section 3) to establish the density. Write $\bar{\psi}=(\bar{\psi}-\bar{\psi}_t)+\bar{\psi}_t$. Clearly $\bar{\psi}_t$ is in Lip_0. Thus we only need to see that $\bar{\psi}-\bar{\psi}_t$ has small BMO norm when t is sufficiently small. We follow an argument in Garrett, p. 272. Let

$$M_\delta(\bar{\psi})=\sup_{l(J)<\delta}\frac{1}{|J|}\int_J|\bar{\psi}-\bar{\psi}_J|w\,dx.$$

We know that $M_\delta(\bar{\psi})\to 0$ as $\delta\to 0$. Fix a cube $Q\subseteq\mathbb{R}^{n-1}$ and a δ so that $M_\delta(\bar{\psi})<\epsilon$.

If $l(Q)>\delta$, express $Q=\bigcup Q_j$ with $l(Q_j)<\delta$. Let $h=\sum_j\bar{\psi}_{Q_j}\chi_{Q_j}(x)$. We estimate the BMO norms of $(\bar{\psi}-h)$, $(\bar{\psi}_t-h_t)$ and h_t-h. First,

$$\int_Q|\bar{\psi}-h|w\,dx\leqslant\sum_j\int_{Q_j}|\bar{\psi}-\bar{\psi}_{Q_j}|w\,dx\leqslant M_\delta(\bar{\psi})\sum_j|Q_j|<\epsilon|Q|.$$

For h_t-h we have

$$(h_t-h)(x)=\int K(x,y,t)[h(y)-h(x)]w(y)\,dy$$

and if $t < l(Q_j)$, $|h(y) - h(x)| \lesssim |\bar{\psi}_{Q_j} - \bar{\psi}_{Q_j'}|$ where Q_j' is adjacent to Q_j. By applying the BMO condition to $Q_j \cup Q_j'$ one sees that

$$|\bar{\psi}_{Q_j} - \bar{\psi}_{Q_j'}| \leqslant CM_\delta(\bar{\psi})|Q_j|/w(Q_j).$$

Hence

$$\sum_J \int_{Q_j} |h_t - h| w\, dx \leqslant CM_\delta(\bar{\psi}) \sum_J \frac{|Q_j|}{w(Q_j)} \cdot w(Q_j) \subset \epsilon |Q|.$$

Similarly, one can show that

$$\int_Q |\bar{\psi}_t - h_t| w\, dx \leqslant \epsilon |Q|. \quad \square$$

Lemma 4.7. $H'_{at}(\partial D, d\sigma) = \mathrm{VMO}_\sigma^*(w)$.

PROOF. The proof in Coifman-Weiss [5] can be modified to work in our situation, once we know Lemma 4.6.

Lemma 4.8. *$K(x, Q) = d\omega^x/d\omega$ belongs to* $\mathrm{VMO}_\sigma(\omega))$.

PROOF. Fix $x \in D$. If A is any atom,

$$|A(x)| = \left| \int A(Q) K(x, Q)\, dw(Q) \right| \leqslant \int_{\partial D} NA(Q)\, d\sigma(Q) \leqslant C,$$

which means $K(x, \bullet)$ belongs to $\mathrm{BMO}_\sigma(\omega)$. Then if $\Delta = \Delta(Q_0, r_0)$ is so small that dist $(x, Q_0) > 2r_0$, the pointwise estimate (2.5) for the harmonic extension of atoms, i.e.,

$$|A(x)| \leqslant r_0^\beta |x - Q_0|^{1-n-\beta}$$

implies that $K(x, \bullet)$ belongs to $\mathrm{VMO}_\sigma(\omega)$. $\square$

We now assume that D is starlike with respect to the origin. At this point we define

$$H^1(\partial D, d\sigma) = \left\{ f = \lim_{r \to 1} u(rQ), u \in H^1(D, d\sigma) \right\}$$

where the limit is taken in $\mathrm{VMO}_\sigma^*(\omega)$. The arguments to follow will show that this limit exists and there is uniqueness in the sense that $f = 0$ implies $u \equiv 0$.

Lemma 4.9. *Let $f \in H^1_{at}(\partial D, d\sigma)$. Set $u(x) = \int f(Q) K(x, Q)\, d\omega(Q)$. Then $u \in H^1(D, d\sigma)$ and $\lim_{r \to 1} u(rQ) = f(Q)$ in* $\mathrm{VMO}_\sigma^*(\omega)$.

PROOF. First, $u(x)$ is well-defined since $K(\bullet, Q) \in VMO_\sigma(\omega)$, by Lemma 4.8. Let

$$f = \sum_j \lambda_j a_j$$

where the a_j are atoms, and $\sum |\lambda_j| < \infty$. Then

$$Nu(Q) \leqslant \sum_j |\lambda_j| Na_j(Q),$$

so $u \in H^1(D, d\sigma)$.

Given $\epsilon > 0$, let N be large enough so that

$$f = \sum_{j=1}^{N} \lambda_j a_j + R_N$$

where $\|N(R_N)\|_{L^1(d\sigma)} < \epsilon$. Then

$$u(rQ) = \sum_{j=1}^{N} \lambda_j a_j(rQ) + R_N(rQ).$$

If $g \in \mathrm{VMO}_\sigma(\omega)$, then by Theorem 2.7,

$$\left| \int \left[u(rQ) - \sum_{j=1}^{N} \lambda_j a_j(rQ) \right] g(Q)\, d\omega \right| \leqslant \|N(R_N)\|_{L^1(d\sigma)} \|g\|_{\mathrm{BMO}} \leqslant \epsilon.$$

Moreover, by Theorem 2.7,

$$\lim_{r \to 1} \int \sum_j \lambda_j a_j(rQ) g(Q)\, d\omega = \int \sum_{j=1}^{N} \lambda_j a_j(q) g(Q)\, d\omega.$$

Thus $\lim_{r \to 1} u(rQ) = f(Q)$ in $\mathrm{VMO}_\sigma^*(w)$. □

At this point we can give the proof of

Theorem 2.6. *$H^1(\partial D, d\sigma) = H^1_{at}(\partial D, d\sigma)$, with comparable norms.*

PROOF. Lemma 4.9 implies that $H^1_{at}(\partial D, d\sigma)$ is continuously imbedded in $H^1(D, d\sigma)$. If $u \in H^1(D, d\sigma)$, then Theorem 2.7 shows that $\{u(rQ)\}$ is bounded in $\mathrm{VMO}_\sigma^*(\omega)$ and so there exists a subsequence $\{r_{j_k}\}$, $r_{j_k} \to 1$ as $k \to \infty$, such that $u(r_{j_k}) \to f \in \mathrm{VMO}_\sigma^*(\omega)$ as $k \to \infty$. This limiting distribution is unique, for by Theorem 2.7, if $r_j \to 1$ and $r_l \to 1$,

$$\left| \int [u(r_j Q) - u(r_l Q)] g(Q) \right| d\omega \leqslant C \|N(u(r_j, \bullet) - u(r_l, \bullet))\|_{L^1} \|g\|_{\mathrm{BMO}}$$

and the above tends to zero as $r_j, r_l \to 1$. By uniqueness and Lemma 4.9 this shows that every $u \in H^1(D, d\sigma)$ can be written as

$$u(x) = \int f(Q)K(x, Q)\, dw(Q)$$

where $f(Q) = \lim_{r \to 1} u(rQ)$ belongs to $H^1_{at}(\partial D, d\sigma)$. $\square$

We now consider the case of a general Lipschitz domain D, not assumed to be starlike. In this situation we have the following result.

Theorem 4.10. *If $\Delta u = 0$ and $Nu(Q) \in L^1(\partial D, d\sigma)$, there exists a sequence of constants $\{\lambda_j\}$ and atoms $\{a_j\}$ such that*

$$u(x) = \sum_j \lambda_j \int a_j(Q)K(x, Q)\, d\omega(Q)$$

for $x \in D$; with $\|N(u)\|_{L^1(d\sigma)} \approx \sum_j |\lambda_j|$. Moreover, the above sum of atoms determines $u(x)$ uniquely.

Let us fix our domain D and a u in $H^1(D, d\sigma)$, and set up some notation. There exists a finite collection of starlike Lipschitz domains $D_i \subseteq D$ such that $\bigcup D_i$ covers a neighborhood of ∂D in D, with $N_{D_i}(u) \in L^1(\partial D_i\, d\sigma_i)$ where $N_{D_i} u$ is the nontangential maximal of u relative to the domain D_i. Let $u_i = u|_{\partial D_i}$, so that u_i belongs to $H^1(D_i, d\sigma_i)$. The domains D_i will have the additonial property that there exists subdomains $\tilde{D}_i$ of D_i, with the same starcenter as D_i and with $\bigcup \tilde{D}_i$ covering a neighborhood of ∂D within D, such that $2\tilde{\Delta}_i = \Delta_i$, where $\Delta_i = \partial D_i \cap \partial D$ and $\tilde{\Delta}_i = \partial \tilde{D}_i \cap \partial D$. Fix the pole of the Green's function for D at $P \in D$ and let dw denote $d\omega_D^P$. Define $\mathrm{VMO}_\sigma(w)$ as the closure of $\mathcal{L}(\partial D)$ under the norm

$$\int_{\partial D} |\psi|\, d\omega + \sup_{\Delta \subseteq \partial D} \inf_{\psi_\Delta \text{contant}} \left\{ \int_\Delta |\psi - \psi_\Delta|\, d\omega/\sigma(\Delta) \right\}.$$

Its dual $\mathrm{VMO}_\sigma^*(d\omega)$ can be identified with $H^1_{at}(\partial D, d\sigma)$. Let $d\omega_i$ be harmonic measure for D_i evaluated at the starcenter.

Lemma 4.11. *There exists distributions f_i in $\mathrm{VMO}_\sigma^*(\partial D_i, d\omega_i)$ with the property that if $\psi \in \mathrm{VMO}_\sigma(\partial D, d\omega)$ and is supported in a compact subset of Δ_i, then (if we call the origin the starcenter of D_i),*

$$\langle f_i, \psi \rangle_{(\partial D_i, d\omega_i)} = \lim_{r \to 1} \int_{\partial D_i} u(rQ)\psi(Q)\, d\omega_i. \tag{4.12}$$

Moreover, if the above limit is zero for all such $\psi \in \mathrm{VMO}_\sigma(\partial D, d\omega)$, and all i, then $u \equiv 0$.

PROOF. If $K \subset\subset \Delta_i$, then $d\omega_i/d\omega$ and $d\omega/d\omega_i$ are bounded on K. Hence if ψ is supported on K, its $\mathrm{VMO}_{\sigma_i}(\partial D_i, d\omega_i)$ and $\mathrm{VMO}_\sigma(\partial D, d\omega)$ norms are comparable, with constants depending only on K. Since $u_i \in H^1(D_i, d\sigma_i)$, Theorem 2.6 for starlike domains gives distributions $f_i \in \mathrm{VMO}_\sigma(\partial D_i, d\omega_i)$ satisfying (4.12). To establish the uniqueness of the $\{f_i\}$, we use an argument of Dahlberg-Kenig [10]. If the expresiónes in (4.12) are zero, we will see that $N_{D_i}(u_i)$ belongs to $L^2(\partial\tilde{\Delta}_i, d\sigma)$ and that the nontangential limit of u_i is zero on $\tilde{\Delta}_i$. By the L^2 uniqueness in the Dirichlet problem (Dahlberg [9]) this would imply that u is identically zero.

With this in mind, let $\psi \in \mathrm{Lip}(\partial D_i)$ be supported in a compact subset of $\partial D_i \setminus \Delta_i$. Let 0 be the starcenter of D_i and assume that $\psi_i(0) = u(0)$, where $\psi_i(x) = \int_{\partial D_i} \psi(Q)\, d\omega_i^x(Q)$ is the harmonic extension of ψ to D_i. Set $v_i = u_i - \psi_i$. Then v_i belongs to $H^1(\partial D_i d\sigma_i)$, $v_i(0) = 0$ and its boundary distribution g_i is supported in $\partial D_i \setminus \Delta_i$. Then v_i has an atomic decomposition, $v_i = \sum_j \lambda_j a_j^i$, where we may assume that each a_j^i has support in $\partial D_i \setminus K_i$ where $\tilde{\Delta}_i \subset K_i \subset\subset \Delta_i$. By the pointwise estimate (2.5) on atoms, $N_{D_i}(v_i) \in L^2(\tilde{\Delta}_i)$. Let h_i be the nontangential limit of v_i on $\tilde{\Delta}_i$, an L^2 function on a neighborhood of $\tilde{\Delta}_i$. Then for all $\theta \in \mathrm{Lip}(\tilde{\Delta}_i)$, by dominated convergence we have

$$\int_{\tilde{\Delta}_i} h_i \theta \, d\omega_i = \lim_{r \to 1} \int_{\tilde{\Delta}_i} v_i(rQ)\theta(Q)\, d\omega_i = \lim_{r \to 1} \int_{\partial D_i} v_i(rQ)\theta(Q)\, d\omega_i = 0.$$

Hence h_i is zero almost everywhere $\tilde{\Delta}_i$, but therefore u_i has zero nontangential limit on Δ_i. □

Lemma 4.13. $L^2(\partial D, d\sigma) \subset H^1_{at}(\partial D, d\sigma)$.

PROOF. Let $\psi \in \mathrm{BMO}_\sigma(d\omega)$. Then $\int_{\partial D} f\psi\, d\omega = \int_{\partial D} f\psi k\, d\sigma$, but $\psi k \in L^2(d\sigma)$ since

$$\sup_\Delta \left\{ \int_\Delta |\psi - \psi_\Delta|^2 k^2 \, d\sigma/\sigma(\Delta) \right\}^{1/2} \quad \text{and} \quad \sup_\Delta \left\{ \int_\Delta |\psi - \psi_\Delta| \, d\omega/\sigma(\Delta) \right\}^{1/2}$$

define equivalent $\mathrm{BMO}_\sigma(\omega)$ norms.

Lemma 4.14. *Let $f \in H^1_{at}(\partial D, d\sigma)$ with $f = \sum_j \lambda_j a_j$. Suppose there exists a $g \in L^2(\partial D, d\sigma)$, with $g \equiv f$ on $\Delta \subseteq \partial D$ in the sense that*

$$\int_{\partial D} f\psi\, d\omega = \int_{\partial D} g\psi\, d\omega$$

for all $\psi \in \mathrm{VMO}_\sigma(\omega)$ with $\operatorname{supp}\psi \subseteq \Delta$. Then for any $\Delta' \subset\subset \Delta$, if

$$u = \sum_j \lambda_j \int a_j(Q) k(x, Q)\, d\omega,$$

$N(u)$ belongs to $L^2(\Delta', d\sigma)$.

PROOF. We can assume that there exists a starlike Ω with $\partial\Omega \cap \partial D = \Delta$ by taking Δ small, and that $u|_\Omega \in H^1(\Omega, d\sigma)$. Set $v(x) = \int_\Delta g(Q)k(x, Q)\,d\omega$. The proof of Lemma 4.11 on the domain Ω shows that $N(u - v) \in L^2(\Delta', d\sigma)$. □

Lemma 4.15. *Let a be an atom on ∂D_i with respect to $d\omega_i$ supported in $K \subset\subset \Delta_i$. Then there exists a constant $C = C(K)$ such that*

$$\|a\|_{H^1_{at}(\partial D, d\sigma)} \leqslant C.$$

PROOF. Clearly,

$$\|a\|_{H^1_{at}(\partial D, d\sigma)} \leqslant C(K) \sup\left\{\int_{\partial D} a\psi\,d\omega : \psi \in \mathrm{VMO}_\sigma(\omega), \|c\|_{\mathrm{BMO}} \leqslant 1, \text{ and } \operatorname{supp}\psi \subseteq K\right\}.$$

Fix such a ψ in $\mathrm{VMO}_\sigma(\omega)$. By the construction in Lemma 4.4, ψ has an extension F such that $|\nabla F|G(x)/d(x)$ is a Carleson measure and $F \equiv 0$ outside D_i. We can also assume that $F = 0$ at the poles of both G and G_i (the Green's function for D_i). Let $u_i(x) = \int a(Q)\,d\omega_i^x$ be the harmonic extension of a to D_i. Let θ be a C_1^∞ function with $\theta \equiv 1$ on $\operatorname{supp} F$ and $\operatorname{supp}\theta \subseteq \overline{D_i \cap D}$. Set $v_i = \theta u_i$. Then, by the argument used in the proof of Lemma 4.4 the following formal calculation can be justified.

$$\begin{aligned}\left|\int_{\partial D} a\psi\,d\omega\right| &= \left|\int_D \Delta(v_i F)G(x)\,dx\right| \\ &= \left|\int_D (\Delta v_i)FG\,dx + \int_D v_i\Delta(F)G\,dx + 2\int_D G\nabla v_i\cdot\nabla F\,dx\right| \\ &\left|\int_D G\nabla v_i\cdot\nabla F\,dx - \int_D v_i\nabla F\cdot\nabla G\,dx\right| \\ &\leqslant C\left(\int_{D_i} G_i|\nabla F|\,|\nabla u_i|\,dx + \int_{D_i}|\nabla F|(G_i/d_i)|u_i|\,dx\right)\end{aligned}$$

which is bounded by $\|Na\|_{L^1(\partial D_i, d\sigma)}$, as before. □

PROOF OF THEOREM 4.10. Let $\{f_i\}$ be the distributions obtained in Lemma 4.11 for $u \in H^1(D, d\sigma)$. Fix i and let $\eta_i \in C_0^\infty$ satisfy $\eta_i \equiv 1$ on a neighborhood of $\tilde\Delta_i$ and $\operatorname{supp}\eta_i \subset\subset \Delta_i$. Set $g_i = f_i\eta_i$. Let $\psi_i \in \operatorname{Lip}(\partial D_i)$ be compactly supported in Δ_i such that

$$g_i(0) = \int_{\partial\Omega_i} \psi_i(Q)\,d\omega_i^0.$$

By Lemma 4.15, g_i has an atomic decomposition on $(D, d\omega)$,

$$\|g_i\|_{H^1_{at}(\partial D, d\sigma)} \leqslant C \quad\text{and}\quad g_i = \psi_i + \sum_j \lambda_j^i a_j^i$$

where the a_j^i are atoms on D compactly supported in a neighborhood of Δ_i.

Now consider

$$v_i(x) = u(x) - \int_{\partial D} g_i(Q)K(x, Q)\, d\omega.$$

By Lemma 4.14, $N(v_i) \in L^2(K_i)$, $\tilde{\Delta}_i \subset K_i \subset \Delta_i$. On D_l, $l \neq i$, $w_{i,l} = v_{i,l} = v_i|_{D_l}$ belongs to $H^1(\partial D_l, d\sigma)$. Let $f_{i,l}$ denote the boundary value distribution of v_i on ∂D_l. As before, multiply $f_{i,l}$ by a cut off function η_l and set $g_{i,l} = \eta_l f_{i,l}$. Again, $\|g_{i,l}\|_{H^1(\partial D, dw)} \leqslant C$ and

$$g_{i,l} = \psi_l + \sum_k \lambda_k^l a_k^l$$

with a_k^l, and atom on $(\partial D, d\omega)$, compactly supported in a neighborhood of $\hat{\Delta}_l$. Set

$$u_{i,l}(x) = v_i(x) - \int_{\delta D} g_{i,l}(Q)K(x, Q)\, d\omega.$$

We have $N(v_{i,l}) \in L^2(\tilde{\Delta}_l)$, but we claim that $N(v_{i,l})$ is in L^2 on a neighborhood of $\tilde{\Delta}_i \cup \tilde{\Delta}_l$. By the pointwise estimate (2.5) on atoms, the harmonic extension of $g_{i,l}$ is in L^2 away from $\tilde{\Delta}_l$, and v_i is in $L^2(\tilde{\Delta}_i)$. It remains to consider the behavior of $N(v_{i,l})$ on the intersection of a neighborhood of $\tilde{\Delta}_i$ with a neighborhood of $\tilde{\Delta}_l$. At such a point, however, the boundary values of v_i and $g_{i,l}$ are in L^2 and so in this case, the claim follows by Lemma 4.14. Proceeding in this manner, we find $v_1(x), \ldots, v_N(x)$ such that $u(x) - \sum_i v_i(x)$ has non-tangential maximal function in $L^2(\partial D)$, $v_i(x) = \int_{\partial D} g_i(Q)K(x)Q)\, d\omega$ and with each $g_i \in H^1_{at}(\partial D, d\sigma)$. By Lemma 4.13 this proves the theorem. □

5. The results for the $H^p(D, d\sigma)$ spaces, $1 < p < 2$. In this section we discuss the results for the $H^p(D, d\sigma)$ spaces, $1 < p < 2$. As before, we introduce a related space on $\mathbb{R}^{n-1}(dx)$ and obtain our results in this setting first. In what follows we shall use the notation of section 3. Our weight $w(x)$ satisfies conditions 3.1 (i)-(iii). Note the difference in the normalization in our definition of atoms below.

Definitions.

(1) *An atom $a(x)$ on $\mathbb{R}^{n-1}$ has support in a cube Q and satisfies $\|a\|_\infty \leqslant 1$ and*

$$\int_p a(x)w(x)\, dx = 0.$$

(2) $L^{\#,q}_{dx}(w\, dx) = \{g \in L^1_{\mathrm{loc}}(w\, dx) : M^\#_w(g) \in L^q(dx)\}$ *where* $q > 1$ *and*

$$M^\#_w g(x) = \sup_{Q \ni x} \left\{ \frac{1}{|Q|} \int_Q |g - g_Q| w(x)\, dx \right\}.$$

(3) $\mathcal{L}_{dx}^{\#,q}(w\,dx)$ = *closure of the* Lip_0 *functions in* $L_{dx}^{\#,q}(w\,dx)$.

(*We observe that if* $\psi \in \mathrm{Lip}_0$, $M_w^{\#}\psi(x) \leqslant C$ *and if* x *is far from the support of* ψ, $M_w^{\#}\psi(x) \leqslant C/|x|^{n-1}$ *so that* Lip_0 *is contained in* $L_{dx}^{\#,q}(w\,dx)$).

(4) $H^p(\mathbb{R}^{n-1}, w\,dx) = \{f \in (\mathcal{L}_{dx}^{\#,p'}, w\,dx))^*: Nf \in L^p(dx)\}$ *where* ( )* *denotes the dual space and* $1/p + 1/p' = 1$.

(5) $H_{at}^p(w\,dx) = \{f: f = \sum \lambda_k a_k$ *where the* a_k *are atoms supported in balls* B_k, $\|\sum \lambda_k \chi_{B_k}\|_{L^p(dx)} < \infty$ *and the convergence takes place in* $(\mathcal{L}_{dx}^{\#,p'}(w\,dx))^*\}$.

Lemma 5.1. *A «distribution» f belongs to* $H^p(w\,dx)$ *if and only if either* f^+ *or* f^* *belongs to* $L^p(dx)$ *and all maximal functions have comparable norms.*

PROOF. To argue as before one needs only check that if $\psi \in \mathcal{A}$ then the pairing $\langle f, \psi\rangle$ is well-defined. To check this one must first see that $M_w^{\#}(\psi) \in L^{p'}(dx)$ (for $p' > 2$) and that $\theta_j\psi \to \psi$ in $L_{dx}^{\#,p'}(w\,dx)$ where θ_j is smooth bump function supported on $\{|x| \leqslant 2^j\}$. □

Lemma 5.2. *The functions in* $L^2(dx) \cap H^p(w\,dx)$ *are dense in* $H^p(w\,dx)$.

The proof of Lemma 5.2 proceeds exactly as the proof of Lemma 3.11 once we have the following facts. (See Lemmas 3.3-3.6 for the notation appearing below).

Proposition 5.2.1. *If* $\psi \in \mathcal{L}_{dx}^{\#,q}(w\,dx)$ *then so is* $\sum_{j=1}^{\infty} S_j(\psi)$ *and*

$$\left\| \sum_{j=1}^{\infty} S_j(\psi) \right\|_{L_{dx}^{\#,q}(w\,dx)} \lesssim \|\psi\|_{L_{dx}^{\#,q}(w\,dx)}.$$

PROOF. Our proof of Lemma 3.3 shows that in fact

$$M_w^{\#}\left(\sum_{j=1}^{\infty} S_j(\psi) \right) \leqslant CM(M_w^{\#}\psi)$$

where m is the Hardy-Littlewood maximal function. □

Proposition 5.2.2. *If* $\psi \in \mathcal{L}_{dx}^{\#,q}(w\,dx)$ *and* $\psi_t(x) = \int \psi(y)K(x,y,t)w(y)\,dy$, *then* $\psi_t \to \psi$ *in* $L_{dx}^{\#,q}(w\,dx)$ *as* $t \to 0$.

PROOF. If $\psi \in L_{dx}^{\#,q}(w\,dx)$, then we claim that

$$\|\psi_t\|_{L_{dx}^{\#,q'}(w\,dx)} \lesssim \|\psi\|_{L_{dx}^{\#,q}(w\,dx)}.$$

The proof of Lemma 3.5 shows that $M_w^{\#}(\psi_t)(x_0) \lesssim CM_w^{\#}\psi(x_0)$ so this is immediate. Then if $\psi \in \mathrm{Lip}_0$, we need $\psi_t \in \mathcal{L}_{dx}^{\#,q}(w\,dx)$ and $\|M_w^{\#}(\psi_t - \psi)\|_{L^q(dx)} \to 0$ as $t \to 0$. But again, in the proof of Lemma 3.5 we found that

$$\|\psi_t - \psi\|_{\mathrm{Lip}(\beta)} \lesssim t^{1-\beta}, \quad \text{for} \quad 1 > \beta > 0.$$

Hence, if $t < 1$, and x_Q is the center of Q,

$$\begin{aligned} M_w^{\#}(\psi_t - \psi)(x_0) &\leqslant \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q |(\psi_t - \psi)(x) - (\psi_t - \psi)(x_Q)|\,w\,dx \\ &\leqslant \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q t^{1-\beta}|x - x_Q|^{\beta} w\,dx \\ &\lesssim t^{1-\alpha} \quad \text{if} \quad l(Q) < t < 1. \end{aligned}$$

If $l(Q) > t$

$$\begin{aligned} \frac{1}{|Q|}\int_Q |(\psi_t - \psi)(x)|\,w\,dx &\leqslant \frac{1}{|Q|}\int_Q \int |\psi(y) - \psi(x)| K(x,y,t) w(y)\,dy\,w(x)\,dx \\ &\leqslant t \leqslant t^{1-\alpha} \end{aligned}$$

The estimate $M_w^{\#}(\psi_t - \psi)(x) \leqslant t^{1-\alpha}$ can be used for $x_0 \in \operatorname{supp}(\psi_t - \psi)$. If $(\psi_t - \psi)$ has support, say, in $B(0,1)$ and $|x_0| > 2$,

$$\begin{aligned} \sup_{Q \ni x_0} \frac{1}{|Q|}\int_Q |(\psi_t - \psi)(x)|\,w(x)\,dx &\leqslant \frac{1}{B(0,|x_0|)} \int_{B(0,1)} \|\psi\|_{\mathrm{Lip}_0} t \\ &\quad \cdot \int K(x,y,t) w(y)\,dy\,w(x)\,dx \\ &\leqslant \frac{ct}{|x_0|^{n-1}} \end{aligned}$$

and then $\|M_w^{\#}(\psi_t - \psi)\|_{L^q({}^cB(0,1),dx)} \lesssim t$. □

Theorem 5.3. *If $f \in H^p(w\,dx)$, then there exists a sequence of positive constants $\{\lambda_k\}$ and a sequence of atoms a_k, supported in balls B_k such that the sum $\sum_k \lambda_k a_k$ converges to f in $(\mathfrak{L}_{dx}^{\#,p'}(w\,dx))^*$ and in $H^p(w\,dx)$ norm, with*

$$\Big\|\sum_k \lambda_k \chi_{B_k}\Big\|_{L^p(dx)} \leqslant \|f\|_{H^p(w\,dx)}.$$

Moreover, if $\{a_k\}$ is a sequence of atoms such that $\Big\|\sum_k \lambda_k \chi_{B_k}\Big\|_{L^p(dx)} < \infty$, then

$$\sum_k \lambda_k a_k \in H^p(w\,dx)$$

and

$$\Big\|\sum_k \lambda_k a_k\Big\|_{H^p(w\,dx)} \leqslant C\Big\|\sum_k \lambda_k \chi_{B_k}\Big\|_{L^p(dx)}.$$

PROOF. Both the statement of the teorem and the ideas in its proof follow Stromberg-Torchinsky [24]. Again, the decomposition of an $f \in H^p(w\,dx)$ is fairly standard, using the ideas of Later [21], and we refer to Stromberg-Torchinsky [24] for the proof in this case. We turn to the proof of the second half of the theorem.

Let

$$f(x) = \sum_{k=1}^{N} \lambda_k a_k$$

be a finite linear combination of atoms. Consider

$$\begin{aligned} Na_k(x) &= \sup_t |(a_k)_\varphi(x,t)| \\ &= \sup_t \left| \int a_k(y)\varphi(x-y/t)w(y)\,dy \cdot \left\{ \int \varphi(x-y'/t)w(y')\,dy' \right\}^{-1} \right| \end{aligned}$$

when $x \in B_k = \operatorname{supp} a_k$,

$$\sup_t |(a_k)_\varphi(x,t)| \leqslant C$$

when $x \in 2^j B_k \setminus 2^{j-1} B_k$, $j > 1$, and r_k is the radius of B_k,

$$\begin{aligned} |(a_k)_\varphi(x,t)| &\leqslant \left| \int a_k(y) \left[\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-y}{t}\right) \right] \frac{w(y)\,dy}{w(B(x,t)} \right| \\ &\leqslant \frac{C}{w(B(x,t))} \int_{B_k} \frac{|y-y_k|}{t} w(y)\,dy \\ &\leqslant \frac{r_k}{t} \frac{w(B_k)}{w(B(x,t)}. \end{aligned}$$

But in order that $(a_k)_\varphi(x,t)$ be nonzero when $x \in 2^j B_k \setminus 2^{j-1} B_k$, we must have $t > c2^j r_k$, hence the above is bounded by

$$c\frac{r_k}{2^j r_k} \frac{w(B_k)}{w(2^j B_k)} \leqslant C\left(\frac{r_k}{2^j r_k}\right)^{n-1+\alpha} = c2^{-j(n-1+\alpha)}.$$

Let $a_{k,0} = \chi_{B_k}(x)$ and $a_{k,j}(x) = \chi_{2^j B_k}(x)$. Then we have shown that

$$Na_k(x) \leqslant ca_{k,0}(x) + \sum_{j=1}^{\infty} 2^{-j(n-1+\alpha)} a_{k,j}(x).$$

Therefore

$$\begin{aligned}\left\| N\left(\sum_{k=1}^{N}\lambda_k a_k\right)\right\|_{L^p(dx)} &\leqslant \left\|\sum_{k=1}^{N}\lambda_k a_{k,0}\right\|_{L^p(dx)} + \sum_{j=1}^{\infty}2^{-j(n-1+\alpha)}\left\|\sum_{k=1}^{N}\lambda_k a_{k,j}\right\|_{L^p(dx)}\\ &\leqslant \left\|\sum_{k=1}^{N}\lambda_k\chi_{B_k}\right\|_{L^p(dx)}\\ &\quad+\sum_{j=1}^{\infty}2^{-j(n-1+\alpha)}2^{j((n-1)/p)}\left\|\sum_{k=1}^{N}\lambda_k\chi_{B_k}\right\|_{L^p(dx)}\\ &\leqslant C\left\|\sum_{k=1}^{N}\lambda_k\chi_{B_k}\right\|_{L^p(dx)},\end{aligned}$$

where the next-to-last inequality follows by a change of variables in the integral. □

Corollary. *The dual of $H^p(w\,dx)$ is $L^{\#,p'}(w\,dx)$, with $1/p + 1/p' = 1$ and pairing $\langle f, g\rangle = \int f(x)g(x)w(x)\,dx$, for f a finite linear combination of atoms.*

PROOF. Suppose $f = \sum \lambda_k a_k$ is a sum of atoms with

$$\left\|\sum\lambda_k\chi_{B_k}\right\|_{L^p(dx)} \approx \|f\|_{H^p(w\,dx)}.$$

If $g \in L_{dx}^{\#,p'}(w\,dx)$, we have

$$\begin{aligned}\int f(x)g(x)w\,dx &= \int\sum_k\lambda_k a_k g(x)w\,dx\\ &\leqslant \sum_k\lambda_k\int_{B_k}|g - g_{B_k}|w\,dx\\ &\leqslant \sum_k\lambda_k|B_k|\inf_{x\in B_k}M_w^{\#}g(x)\\ &\leqslant \left\|\sum\lambda_k\chi_{B_k}\right\|_{L^p}\|M_w^{\#}g\|_{L^{p'}(dx)}.\end{aligned}$$

If Λ is a linear functional on $H^p(w\,dx)$, one can show that Λ is given by a $g \in L_{dx}^{\#,p'}(w\,dx)$, with pairing $\langle f, g\rangle = \int fgw\,dx$, as in Coifman-Weiss [5]. It can also be shown that H_{at}^p is the dual of $\mathcal{L}_{dx}^{\#,p'}(w\,dx)$.

Lemma 5.5. *If Λ is continuous linear functional on $H^p(w\,dx)$, there exists $\{y_n\}$ with $y_n \to 0$ as $n \to +\infty$ and $y_n \to \infty$ and $n \to -\infty$ and functions $g_\infty(x)$, $\{g_n(x)\}_{n=-\infty}^{n=\infty}$ such that for all $f \in L^2 \cap L^1(dx)$,*

$$\Delta(f) = \int f(x)g_\infty(x)\,dx + \sum_{n=-\infty}^{\infty}\int u(x, y_n)g_n(x)\,dx$$

where

$$u(x, y_n) = \int f(z)\varphi\left(\frac{x-z}{y_n}\right)w(z)\,dz \cdot \left\{\int \varphi\left(\frac{x-z'}{y_n}\right)w(z')\,dz'\right\}^{-1}$$

and

$$|g_\infty(x)| + \sum_{n=-\infty}^{\infty} |g_n(x)| \in L^{p'}(dx).$$

PROOF. Again, we refer to Garnett [15] for C. Fefferman's argument in the case $p = 1$, which may be easily modified to give the above characterization when $p > 1$.

Our strategy for proving the atomic decomposition for $H^p(D, d\sigma)$ and the duality result is the same as that for the $H^1(D, d\sigma)$ situation. To carry this out we need a Varopoulos-type extension theorem for $L_{dx}^{\#,p}(w\,dx)$ functions. We shall formulate our result within the framework of the theory of tent spaces. In what follows the functions and measures are defined on $\mathbb{R}^n_+$ and $\Gamma(x)$ denotes a cone with vertex at x.

Definitions.

(1) $T^p_\infty = \{f: A_\infty(f)(x) \in L^p(dx)\}$ *where* $A_\infty f(x) = \sup_{\Gamma(x)} |f(x', y)|$.

(2) $T^p_q = \{f: A_q f(x) \in L^p(dx)\}$, $p < \infty$, $q < \infty$, *where*

$$A_q f(x) = \left\{\int_{\Gamma(x)} |f(x', y)|^q\,dx'\,dy/y^{n-1}\right\}^{1/q}.$$

(3) $\tau^p_1 = \{\mu: A_1(\mu) \in L^p\}$ *for* $p < \infty$, *where* $d\mu$ *is a measure on* $\mathbb{R}^n_+$ *and*

$$A_1(\mu) = \int_{\Gamma(x)} y^{-n+1}\,d\mu(x', y).$$

(4) $\tau^\infty_1 = \{\mu: C_1(\mu) \in L^\infty\}$ *where*

$$C_1(\mu)(x) = \sup_{Q \ni x}\left\{\frac{1}{|Q|}\int_{Q \times [0, l(Q)]} d\mu(x', y)\right\}.$$

Theorem 5.6. (Coifman-Meyer-Stein [4] and Alvarez-Milman [1]).

(i) $(T^p_\infty)^* = \tau^{p'}_1$, $\quad \dfrac{1}{p} + \dfrac{1}{p'} = 1.$

(ii) $(T^p_2)^* = T^{p'}_2$, $\quad \dfrac{1}{p} + \dfrac{1}{p'} = 1.$

Theorem 5.7. *Suppose $g \in L^{\#,p}_{dx}(w\,dx)$. Then*

$$g(x) = g_0(x) + g_1(x),$$

where $g_0 \in L^p(w\,dx)$ and $g_1(x)$ has an extension $g(x,s)$ to the upper half space $\mathbb{R}^n_+$ in the sense that $g(x,s) \to g(x)$ weakly in $L^1(w\,dx)$ with

$$\text{(i)} \qquad |\nabla g(x,s)|\, w * \varphi_s(x) \in \tau_1^p$$

and

$$\text{(ii)} \qquad |\nabla g(x,s)|s \cdot w * \varphi_s(x) \in T_2^p.$$

PROOF. We argue as in Theorem 3.16, omitting those details which are merely repetitious.

By Lemma 5.5 since g determines a continuous linear functional on $H^{p'}(w\,dx)$ we have

$$\int g(x) f(x) w(x)\,dx = \int f(x) g_\infty(x)\,dx + \lim_{N\to\infty} \sum_{-N}^{N} \int u(x,y_j) g_j(x)\,dx$$

for all

$$f = \sum_{k=1}^{M} \lambda_k a_k,$$

a finite linear combination of atoms, and where

$$|g_\infty(x)| + \sum_{-\infty}^{\infty} |g_n(x)| \in L^p(dx).$$

Set $g_0(x) = g_\infty(x)/w(x)$ and get

$$h_N(z) = \sum_{j=-N}^{N} \int \varphi\left(\frac{x-z}{y_j}\right) \left\{ \int \varphi\left(\frac{x-z'}{y_j}\right) w\,dz' \right\}^{-1} g_j(x)\,dx.$$

For each N, h_N belongs to $L^{\#,p}_{dx}(w\,dx)$ with norm bounded by a constant which is independent of N. For $\theta \in C^\infty$ with $\theta(t) \equiv 1$ when $0 < t < 1/2$ and $\theta(t) \equiv 0$ when $t > 1$, define

$$h_N(z,s) = \sum_{-N}^{N} \theta(s/y_j) \int \varphi\left(\frac{x-z}{y_j}\right) \left\{ \int \varphi\left(\frac{y-z'}{y_j}\right) w\,dz' \right\}^{-1} g_j(x)\,dx.$$

If we assume that g has compact support then, as in our argument for (3.16), there are constants c_N such taht $\{h_N - c_N\}$ has a weak limit in $L^p(w\,dx)$, call it $h(x)$, and $g_1(x) = h_1(x) + c$ for some constant c. Then we need only prove the estimates (i) and (ii), uniformly in N, for $|\nabla h_N(z,s)|$.

Let us first check condition (i). We write

$$h_N(z,s) = \int_x \int_{y=s/2} K(x,y,z)\, d\sigma(x,y)$$

where

$$K(x,y,z) = \varphi\left(\frac{x-z}{y}\right)\left\{\int \varphi\left(\frac{x-z'}{y}\right) w(z')\, dz'\right\}^{-1},$$

$$d\sigma(x,y) = \sum_{j=-N}^{N} g_j(x)\, d\sigma_j(x,y) \quad \text{and} \quad d\sigma_j(x,y) = dx \quad \text{on} \quad y = y_j.$$

By the properties of $\{g_j\}$, a computation shows that $d\sigma \in \tau_1^p$, with

$$\|d\sigma\|_{\tau_1^p} \leqslant \|g\|_{L_{dx}^{\#,p(w\,dx)}}.$$

We must show then that $|\nabla h_N(z,s)|\, w * \varphi_s(z)$ belongs to τ_1^p whenever $d\sigma$ belongs to τ_1^p, with comparable norms. We have already (Theorem 3.16) argued for this in the case $p = \infty$. By interpolation (see Álvarez-Milman [1]) it suffices to check this in the case $p = 1$. Assume then that $d\sigma \in \tau_1^1$, i.e., that

$$\int_{\mathbb{R}^{n-1}} \int_{\Gamma(x)} y^{-n+1}\, d\sigma(x,y) = \int_0^\infty \int_{\mathbb{R}^{n-1}} d\sigma(x,y) < \infty.$$

Then, since

$$|\nabla h_N(z,s)| \leqslant \int_{y=s}^{\infty} \int_{\{|x-z|<y\}} [y w(B(x,y)]^{-1}\, d\sigma(x,y),$$

we have

$$\begin{aligned}
&\int_{s=0}^{\infty} \int_{\mathbb{R}^{n-1}} w * \varphi_s(x) |\nabla h_N(z,s)|\, dz\, ds \\
&\leqslant \int_{y=0}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{s=0}^{y} \int_{\{|x-z|<y\}} w(B(x,s))/w(B(x,y)) s^{-n+1} y^{-1}\, dz\, dx\, d\sigma(x,y) \\
&\leqslant \int_{y=0}^{\infty} \int_{\mathbb{R}^{n-1}} \int_{s=0}^{\infty} (s/y)^{n-2+\alpha} y^{n-2} s^{-n+1}\, ds\, d\sigma(x,y) \\
&\leqslant C \int_{y=0}^{\infty} \int_{\mathbb{R}^{n-1}} d\sigma(x,y) < \infty
\end{aligned}$$

where the second inequality used the basic estimate 3.1 (i) on the measure $w\,dx$.

We turn now to condition (ii). We want to show that

$$|\nabla h_N(z,s)| \cdot s(w * \varphi_s(x)) \in T_2^p$$

under the condition that $d\sigma \in \tau_1^p$. Recall that

$$T_2^\infty = \left\{ f\colon \sup_{Q \ni x,\, cube} \left\{ \frac{1}{|Q|} \int_{Q \times [0, l(Q)]} |f(x',y)|^2\, dx'\, dy/y \right\}^{1/2} \in L^\infty \right\}$$

We have shown (3.16) that the above condition on $|\nabla h_N(z,s)|$ holds in the case $p=\infty$. By interpolation (Coifman-Meyer-Stein [4]) it suffices to show that this statement holds for $p=1$. If $d\sigma \in \tau_1^1$, $|\nabla h_N(z,s)| w * \varphi_s(z) \in \tau_1^1$ and therefore

$$\int_{\Gamma(x)} |\nabla h_N(z,s)| w * \varphi_s(z) \cdot s\, dz\, ds/s^n$$

is in $L^1(dx)$. To show that

$$\left\{ \int_{\Gamma(x)} |\nabla h_N(z,s)|^2 |w * \varphi_s(x) \cdot s|^2 \, dz \frac{ds}{s^n} \right\}^{1/2}$$

belongs to $L^1(dx)$, we estimate

$$\sup_{\Gamma(x)} |\nabla h_N(z,s)| s \cdot w * \varphi_s(z),$$

which is less than

$$\begin{aligned} &\sup_{\Gamma(x)} s^{-n+1} w(B(z,s)) \int_{y=s}^{\infty} \int_{\{|x-z|<y\}} \{ y w(B(x,y)) \}^{-1} \, d\varphi(x,y) \\ &\qquad \leqslant \sup_{\Gamma(x)} s^{-n+2} \int_{y=s}^{\infty} \int_{\{|x-z|<y\}} y^{-1} (s/y)^{n-2+\alpha} \, d\sigma(x,y) \\ &\qquad \leqslant \sup_{\Gamma(x)} \int_0^{\infty} \int_{\{|x-z|<y\}} y^{-n+1} \, d\sigma(x,y) \end{aligned}$$

which belongs to $L^1(dx)$ since $d\sigma \in \tau_1^1$. □

Having obtained the main result for $H^p(w\,dx)$, we now give the description of $H^p(D, d\sigma)$ and duality with

$$\begin{aligned} &L_\sigma^{\#,p'}(\partial D, d\omega) \\ &\quad = \left\{ g \in L^1(d\omega) : \sup_{\Delta \ni Q} \left\{ 1/\sigma(\Delta) \int_\Delta |g - g_\Delta| \, d\omega \right\} + \int_{\partial D} |g| \, d\omega \in L^{p'}(\partial D, d\sigma) \right\}. \end{aligned}$$

We will use the same localization procedure and notation as in the beginning of section 4.

Lemma 5.8. *Let $\theta \in C_0^\infty(\mathbb{R}^n)$ with $|\nabla\theta| \leqslant c$. Then if $g \in L_\sigma^{\#,p'}(\partial D, d\omega)$, $g\theta \in L_\sigma^{\#,p'}(\partial D, d\omega)$ also.*

PROOF. The proof of Lemma 4.2 shows that $M_w^{\#}(\psi g)(x_0) \leqslant c g^{\#}(x_0) + c$. □

An $H^p(\partial D, d\sigma)$ atom is function A, harmonic in D, with $\|A\|_\infty \leqslant 1$ and with boundary values $A(Q)$ supported in a surface ball $\Delta \subseteq \partial D$ and satisfying $\int_\Delta A(Q)\, d\omega(Q) = 0$. We have the following analog of Lemma 4.6.

Lemma 5.2.4. *$\mathcal{L}(\partial D)$, the space of Lipschitz functions on ∂D, is dense in $L_\sigma^{\#,p'}(\partial D, d\omega)$.*

PROOF. Our argument for Lemma 4.6 (the density of $\mathcal{L}(\partial D)$ in $\mathrm{VMO}_\sigma(\omega)$) will work in this context if we know that a compactly supported ψ belonging to $L_{dx}^{\#,p'}(\mathbb{R}^{n-1}, w\,dx)$ satisfies a «small oscillation» condition. That is, we want to see that

$$\sup_{\substack{l(Q)<\delta \\ Q\ni x_0}} \frac{1}{|Q|}\int_Q |\psi - \psi_Q| w\,dx = M_\delta(x_0)$$

satisfies: $\|M_\delta(x_0)\|_{L^{p'}(dx)} \to 0$ as $\delta \to 0$. But this is just a consequence of the dominated convergence theorem together with the fact that $w(Q)/|Q| \to w(x_0)$ as $l \to 0$, which is finite almost everywhere.

One can then show

Lemma 5.2.5. $H_{at}^p = (L_\sigma^{\#,p'}(\partial D, dw))^*$ *(although this information is not necessary for the duality argument).*

Lemma 5.9. *If $\sum_{k=1}^\infty \lambda_k A_k$ is an infinite linear combination of atoms, the λ_k are positive and $\Delta_k = \operatorname{supp} A_k$, then*

$$\begin{aligned}\Big\|\sum \lambda_k A_k\Big\|_{H^p(D,d\sigma)} &\equiv \Big\{\int_{\partial D} N\Big(\sum \lambda_k A_k\Big)^p d\sigma\Big\}^{1/p} \\ &\leqslant \Big\|\sum \lambda_k \chi_{\Delta_k}\Big\|_{L^p(d\sigma)}.\end{aligned}$$

PROOF. Since $K(x, Q) \in \mathrm{VMO}_\sigma(\omega)$, the harmonic extension of this infinite linear combination of atoms makes sense by duality. Now let $B_j = B(x_j, r_j)$ be the finite covering of $\{\operatorname{dist}(x, \partial D) < \delta\}$ for $\delta = \delta(D)$ with $B(x_j, 4r_j) \cap \partial D = \{(x, y): y = \Phi_j(x)\}$. We can assume that for each k, $\sigma(\Delta_k) < \delta$ so that all atoms have support contained in one of these coordinate charts. Let m_k be the largest m such that $2^m\Delta_k$ $(= \Delta_k(x_k, 2^m r_k))$ is contained in a ball of radius no more than δ. By Dahlberg's pointwise estimate (2.5) on atoms, there is a $\beta > 0$ so that

$$NA_k(Q) \lesssim \chi_{\Delta_k}(Q) + \sum_{l=1}^{m_k} 2^{-l(n-1+\beta)}\chi_{2^l\Delta_k}(Q) + 2^{-(m_k+1)(n-1+\beta)}\chi_{\partial D}(Q).$$

Hence

$$\left\{\int_{\partial D} N^p\Big(\sum_k \lambda_k A_k\Big)\,d\sigma\right)^{1/p} \leqslant \left\{\int_{\partial D}\Big(\sum_k \lambda_k \chi_{\Delta_k}(Q)\Big)^p\right\}^{1/p}$$

$$+\left\{\int_{\partial D}\Big(\sum_k \lambda_k \sum_{l=1}^{m_k} 2^{-l(n-1+\beta)}\chi_{2^l\Delta_k}(Q)\Big)^p\,d\sigma(Q)\right\}^{1/p} + \sum_k \lambda_k 2^{-(m_k+1)(n-1+\beta)}.$$

Let $\epsilon_{k,l}=1$ if $l\leqslant m_k$ and $\epsilon_{m,l}=0$ otherwise. The second term in the sum is bounded by

$$\sum_{l=1}^{\infty} 2^{-l(n-1+\beta)}\left\{\int_{\partial D}\Big(\sum_k \epsilon_{k,l}\lambda_k\chi_{2^l\Delta_k}(Q)\Big)^p\,d\sigma\right\}^{1/p}$$

with by a change of variable, valid since each $2^l\Delta_k$ is contained in a coordinate chart, is less than

$$C_D\sum_{l=1}^{\infty} 2^{-l(n-1+\beta)}2^{l(n-1)/p}\Big\|\sum \lambda_k\chi_{\Delta_k}\Big\|_{L^p(d\sigma)}.$$

The third term in the sum is bounded by

$$C(\delta)\cdot\sum_k \lambda_k 2^{-(m_k+1)(n-1+\beta)}\int_{\partial D}\chi_{2^{m_k}\Delta_k}(Q)\,d\sigma$$

$$\leqslant C(\delta)\sum_k \lambda_k 2^{-(m_k+1)(n-1+\beta)}2^{m_k(n-1)/p}\int_{\partial D}\chi_{\Delta_k}(Q)\,d\sigma$$

$$\leqslant C(\delta)\sigma(\partial D)^{1/p'}\Big\|\sum_k \lambda_k\chi_{\Delta_k}\Big\|_{L^p(d\sigma)}. \quad \square$$

Lemma 5.10. *If $u\in\mathcal{L}(\bar{D})$ with $\Delta u=0$ and $u(p_0)=0$ then*

$$\left|\int_{\partial D} u(Q)f(Q)\,d\omega(Q)\right| \leqslant C\|N(u)\|_{L^p(d\sigma)}\|f\|_{L_\sigma^{\#,p}(\partial D,\,d\omega)}$$

for all $f\in L_\sigma^{\#,p'}(\partial D, d\omega)$.

PROOF. By Green's theorem, (see the argument following Theorem 2.7 in section 2),

$$\left|\int_{\partial D} u(Q)f(Q)\,d\omega(Q)\right| \leqslant \int_D G(x)|\nabla u|\,|\nabla v|\,dx + \int_D \nabla G\cdot\nabla v u\,dx$$

where $d(x)=\operatorname{dist}(x,\partial D)$ and v is some smooth extension of $f(Q)$ to D such that this formal argument is justified. Suppose that $|v|+|\nabla v|\leqslant C$ in $K\subset\subset D$ and that

$$(5.11)\quad \begin{array}{ll} \text{(i)} & |\nabla v(x)|G(x)/d(x) \in \tau_1^{p'} \\ \text{(ii)} & |\nabla v(x)|G(x) \in T_2^{p'} \end{array}$$

where τ_1^p, $T_2^{p'}$ have the obvious definitions on a Lipschitz domain D. Let $B(p_0)$ be a ball containing the pole of $G(x)$. Then

$$\begin{aligned} \int_{DB(p_0)} & G(x)|\nabla u|\,|\nabla v|\,|\nabla v|\,dx \\ & \leqslant \int_{\partial D}\int_{\Gamma(Q)} G(x)|\nabla u|\,|\nabla v|\frac{dx}{d(x)^{n-1}}\,d\sigma(Q) \\ & \leqslant \int_{\partial D} Su(Q)\left\{\int_{\Gamma(Q)} |\nabla v|^2 G^2(x) d(x)^{-n}\,dx\right\}^{1/2} d\sigma(Q) \\ & \leqslant \|Su\|_{L^p(d\sigma)}\,\||\nabla v|G\|_{T_2^{p'}(D)} \\ & \leqslant C\|Nu\|_{L^p(d\sigma)}\,\|f\|_{L^{\#,p}(\partial D,d\sigma)}. \end{aligned}$$

Using $|\nabla G(x)| \leqslant G(x)/d(x)$ away from p_0,

$$\begin{aligned} \int_{D\setminus B(p_0)} G/d\,|u|\,|\nabla v|\,dx & \leqslant \int_{\partial D}\int_{\Gamma(Q)} G/d\,|u|\,|\nabla v|\,d(x)^{-n+1}\,dx\,d\sigma(Q) \\ & \leqslant \int_{\partial D} Nu(Q)\int_{\Gamma(Q)} G/d\,|\nabla v|\,d(x)^{-n+1}\,dx \\ & \leqslant \|N(u)\|_{L^p(d\sigma)}\,\|G/d\,|\nabla v|\,\|_{\tau_1^{p'}} \\ & \lesssim \|Nu\|_{L^p(d\sigma)}\,\|f\|_{L^{\#,p'}}. \end{aligned}$$

Since G is integrable on $B(p_0)$ and $|v| + |\nabla v| \leqslant C$ here, the integral over $G(p_0)$ is handled as before. □

Thus it remains to find such an extension. Because we have the result on $\mathbb{R}^n_+$ (Theorem 5.7) and we can localize to a coordinate chart of ∂D (Lemma 5.8), the argument is just a variant of that given in Lemma 4.4 and we omit the details.

Assume now that D is starshaped with respect to the origin.

Definition.

(i) $H^p(\partial D, d\sigma) = \{f: f(Q) = \lim_{r\to 1} u(rQ),\ u \in H^p(D, d\sigma),$ *with convergence in* $(L_\sigma^{\#,p'}(\partial D, d\omega))^*\}$.

(ii) $H^p_{at}(\partial D, d\sigma) = \{f: f = \sum \lambda_k A_k$ *where the* A_k *are* H^p *atoms and* $\|\sum \lambda_k \chi_{\Delta_k}\|_{L^p(d\sigma)} \leqslant \infty\}$,

and the convergence takes place in $(L_\sigma^{\#,p'}(\partial D, dw))^*$.

Theorem 5.11. *$H^p_{at}(\partial D, d\sigma) = H^p(D, d\sigma)$ and $(H^p(D, d\sigma))^* = L^{\#, p'}_{\sigma}(\partial D, dw)$.*

PROOF. We refer the reader to the proof of Theorem 2.6 given in section 4. □

Finally, as before, we have the following theorem for domains, D, not assumed to be starlike.

Theorem 5.12. *If $\Delta u = 0$ and $Nu \in L^p(\partial D, d\sigma)$, $p \leqslant 2$ there exists a sequence of constants $\{\lambda_j\}$ and atoms $\{A_j\}$ such that*

$$u(x) = \sum_j \lambda_j \int A_j(Q) K(x, Q)\, dw(Q)$$

for $x \in S$, with $\|Nu\|_{L^p(d\sigma)} \approx \|\sum \lambda_j \chi_{\Delta_j}\|_{L^p(d\sigma)}$. Moreover, given $\{\lambda_j\}$ and $\{A_j\}$ sequences of constants and atoms satisfying $\|\sum \lambda_j \chi_{\Delta_j}\|_{L^p(d\sigma)} \leqslant \infty$, there exists a harmonic $u(x) = \sum \lambda_j A_j(x)$ with $Nu \in L^p(d\sigma)$. These boundary values determine $u(x)$ uniquely in the sense that the limiting distributions on the starlike subdomains of D are zero if $u \equiv 0$ on ∂D.

References

[1] Alvarez, J. and Milman, M. Spaces of Carleson measures; duality and interpolation, preprint.

[2] Carleson, L. Two remarks on H^1 and BMO, *Adv. in Math.* **22** (1976), 269-277.

[3] Coifman, R. R. and Fefferman, C. Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241-250.

[4] Coifman, R. R., Meyer, Y., Stein, E. M. Some new functions spaces and their applications to harmonic analysis, *J. Funct. Anal.* **62** (1985), 304-335.

[5] Coifman, R. R. and Weiss, G. Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569-645.

[6] Dahlberg, B. E. J. On estimates of harmonic measure, *Arch. Rat. Mech. Anal.* **65** (1977), 272-288.

[7] Dahlberg, B. E. J. Weighted norm inequalities for the Lusin area integral and the nontangential maximal function for functions harmonic in a Lipschitz domain, *Studia Math.* **67** (1980), 297-314.

[8] Dahlberg, B. E. J. A note in H^1 and BMO, Proceedings of the Plëyel Conference, 1979, 23-30.

[9] Dahlberg, B. E. J. On the Poisson integral for Lipschitz and C^1 domains, *Studia Math.*

[10] Dahlberg, B. E. J., and Kenig, C. E. Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains, to appear in *Annals of Math.*

[11] Fabes, E. B. and Kenig, C. E. On the Hardy spaces of a C^1 domain, *Ark. Mat.* **19** (1981), 1-22.

[12] Fabes, E. B., Kenig, C. E., Neri, U. Carleson measures, H^1 duality, and weighted BMO in non-smooth domains, *Indiana, J. Math.* **30** (4) (1981), 547-581.

[13] Fefferman, C. and Stein, E. M. H^p spaces of several variables, *Acta Math.* **129** (1972), 137-193.
[14] Friedland S. and Hayman, W. K. Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions, *Comm. Math. Helv.* **51** (1976), 133-161.
[15] Garnett, J. B. Bounded Analytic Functions, *Academic Press*, 1981.
[16] Huber, A. «Uber Wachsturmseigenschaften gewissen Klassen von subharmonischen Funcktionen», *Comm. Math. Helv.* **26** (1952), 81-116.
[17] Jerison, D. S. and Kenig, C. E. Boundary value problems on Lipschitz domains, *MAA Studies*, Vol. 23, 1-68.
[18] Jerison, D. S. and Kenig, C. E. Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1), (1982), 80-147.
[19] Jones P. W. Extension theorems for BMO, *Indiana J. Math.* **29** (1980), 41-66.
[20] Kenig, C. E. Weighted Hardy spaces on Lipschitz domains, *Amer. J. Math.* **102** (1980), 129-163.
[21] Latter, R. A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms, *Studia Math.* **62** (1978), 93-101.
[22] Macías, R. and Segovia, C. A decomposition into atoms of distributions on spaces of homogeneous type, *Adv. in Math.* **33** (1979), 171-209.
[23] Muckenhoupt, B. Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207-226.
[24] Stromberg, J. O. and Torchinsky, A. Book to appear. (See also *Bull. Amer. Math. Soc.* **3**, No. 3 (1980)).
[25] Varapoulos, N. Th. BMO functions and the $\bar{\partial}$-equation, *Pacific J. Math.* **71** (1977), 221-273.

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Homeomorphisms Preserving A_p

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Introduction

In a recent paper, Benedetto, Heinig and Johnson [1] showed that if w is a monotone A_2 weight, $w(1/x)$ is also an A_2 weight. Here A_2 denotes the set of weights satisfying the condition found by Muckenhoupt, defined in general by

$$w \in A_p \quad \text{iff} \quad \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1} < +\infty,$$

$1 < p < +\infty$, which was shown by Muckenhoupt [6] to characterize the weights w for which the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q f(y)\,dy, \qquad f \geqslant 0,$$

satisfies

$$\int Mf(x)^p w(x)\,dx \leqslant C^p \int f(x)^p w(x)\,dx.$$

Hunt, Muckenhoupt and Wheeden showed that A_p is the condition that characterizes weights for which the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\lim_{\epsilon \to 0}\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\,dy$$

satisfies

$$\int |Hf(x)|^p w(x)\,dx \leqslant C^p \int |f(x)|^p w(x)\,dx$$

(see [4] for the proof). The Hilbert transform commutes with the unitary operator on $L^2(\mathbb{R})$,

$$Uf(x) = \frac{1}{x} f(1/x),$$

and this gives the result that for any $w \in A_2$, $w(1/x) \in A_2$. We have considered the more general question: which homeomorphisms of $\mathbb{R}$ preserve the A_p class? We answer this question, (and its counterpart on $\mathbb{R}^n$) and apply it to determine the pointwise multipliers of A_p. In view of the close connection between A_p, reverse Hölder inequalities and $A_\infty = \bigcup_{p<\infty} A_p$, we also investigate the corresponding questions for these conditions, by means of a precise connection between reverse Hölder inequalities and A_∞. We also have results for double weights $(u, v) \in A_p$ though not so complete.

1. Notation and Preliminary Results

In addition to the A_p-classes we define for $1 < p < \infty$,

$$A_p(w) = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

We shall also need A_1, the class of all w for which $Mw \leqslant cw$ with

$$A_1(w) = \inf\{c : Mw \leqslant cw\}.$$

We let

$$A_\infty = \bigcup_{p<\infty} A_p,$$

and write

$$A_\infty(w) = \lim_{p\to\infty} A_p(w).$$

This limit exists since for $q \geqslant p$, $A_q(w) \leqslant A_p(w)$. We say that $w \in RH_{p_0}$ (reserve Hölder) if

$$\left(\frac{1}{|Q|} \int_Q w^{p_0} \right)^{1/p_0} \leqslant \frac{C}{|Q|} \int_Q w,$$

and we abbreviate by $RH_{p_0}(w)$ the infimum of all such C. It is easily seen by Hölder's inequality that $A_p(w) \geqslant 1$ and $RH_p(w) \geqslant 1$.

We will repeatedly use some common properties of A_p, namely,

(*a*) $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$,
(*b*) $w \in A_p$ implies $w \in A_q$, $q \geqslant p$, and $w^\alpha \in A_p$, $0 \leqslant \alpha \leqslant 1$,
(*c*) if $w_1, w_2 \in A_p$, then $w_1^\alpha w_2^{1-\alpha} \in A_p$, $0 \leqslant \alpha \leqslant 1$,
(*d*) $w \in A_p$, $1 < p < \infty$, if and only if there exists $u, v \in A_1$, so that $w = uv^{1-p}$,
(*e*) $w \in A_p$ for some p if and only if $w \in RH_q$ for some q,
(*f*) if $w \in A_p$, then $w^\tau \in A_p$ for some $\tau > 1$,
(*g*) if $w \in A_p$, $p > 1$, then $w \in A_{p-\epsilon}$ for some $\epsilon > 0$.

An excellent reference is [4].

We will also use the close connection between A_p and *BMO*, i.e., the space of functions satisfying

$$\sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| \, dx < \infty,$$

where

$$f_Q = \frac{1}{|Q|} \int_Q f.$$

The sup is a semi-norm on *BMO* which gives constants norm 0. The connection between the two classes is the following. For any $w \in A_p$, $\log w \in BMO$, and for any $u \in BMO$ and $1 < p < \infty$, $e^{\lambda u} \in A_p$ for some $\lambda > 0$. This last result is not true for $p = 1$.

The classes RH_{p_0} and A_∞ are closely related as the following theorem [12] shows.

Theorem 1.1. *$w \in RH_{p_0}$ if and only if $w^{p_0} \in A_\infty$.*

We postpone the simple proof till Section 3 where we need a quantitative version of this result.

As a first general composition type theorem we have

Theorem 1.2. *Let $h: \mathbb{R} \to \mathbb{R}$ be arbitrary. Then the following statements are equivalent for $1 < p_0 < \infty$.*

(1) *$w \in A_\infty$ implies $w \circ h \in A_\infty$,*
(2) *$w \in RH_{p_0}$ implies $w \circ h \in RH_{p_0}$,*
(3) *$w \in A_1$ implies $w \circ h \in \bigcap_{p>1} A_p$.*

Either one of these statements implies, but is not implied by

(4) *$w \in A_{p_0}$ implies $w \circ h \in A_{p_0}$.*

PROOF. Theorem 1.1 implies (1) $\Leftrightarrow$ (2). The implication (1) $\Rightarrow$ (4) follows from the fact that $w \in A_{p_0}$ if and only if w and $w^{1-p_0'}$ are in A_∞, [4, p.408]. This also shows (1) $\Rightarrow$ (3). For the proof of (3) $\Rightarrow$ (1), let $w \in A_\infty$. Then there is $1 < p < \infty$, and there are $u_1, u_2 \in A_1$ with $w = u_1 \cdot u_2^{1-p}$. Hence $w \circ h = u_1 \circ h \cdot (u_2 \circ h)^{1-p}$. Since $u_1 \circ h \in A_2$, there is $\tau > 1$ so that $(u_1 \circ h)^\tau \in A_2$ and $\tau'/(p'-1) + 1 \geqslant 2$. We let $q = \tau'/(p'-1) + 1$ and note that $(u_1 \circ h)^\tau \in A_q$ and $u_2 \circ h \in A_{q'}$. We claim that $w \circ h \in A_q$. To prove this, we note that

$$\frac{1}{|I|}\int_I u_1 \circ h \cdot (u_2 \circ h)^{1-p} \leqslant \left(\frac{1}{|I|}\int_I u_1 \circ h^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)}\right)^{1/\tau'}$$

and

$$\frac{1}{|I|}\int_I (u_1 \circ h)^{1-q'}(u_2 \circ h)^{(1-p)(1-q')} \leqslant \left(\frac{1}{|I|}\int_I (u_1 \circ h)^{\tau(1-q')}\right)^{1/\tau} \cdot \left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)(1-q')}\right)^{1/\tau'}.$$

Consequently,

$$\frac{1}{|I|}\int_I w \circ h\left(\frac{1}{|I|}\int_I (w \circ h)^{1-q'}\right)^{q-1} \leqslant \left(\frac{1}{|I|}\int_I (u_1 \circ h)^\tau\right) \cdot \left(\frac{1}{|I|}\int_I (u_1 \circ h)^{\tau(1-q')}\right)^{(q-1)/\tau} \cdot \left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)}\right)^{1/\tau'} \cdot \left(\frac{1}{|I|}\int_I (u_2 \circ h)^{\tau'(1-p)(1-q')}\right)^{(q-1)/\tau'}$$

Since $\tau'(1-p)(1-q') = 1$ or $\tau'(1-p) = 1-q$ we see that

$$A_q(w \circ h) \leqslant A_q(u_1 \circ h^\tau)^{1/\tau} A_{q'}(u_2 \circ h)^{(q-1)/\tau'}.$$

In order to complete the proof we need to give an example for (4) $\not\Rightarrow$ (1). We let $h(x) = 1/x$. Then, as observed in the introduction, (4) holds for $p_0 = 2$. However, h cannot preserve A_∞, since

$$w(x) = |x|^2 \in A_\infty, \quad \text{but} \quad w \circ h(x) = |x|^{-2} \notin A_\infty.$$

Remark. The main problem with which this paper is concerned is to find conditions on h so that (4) is equivalent with the other conditions of Theorem 1.2, and such a condition will have to be independent of p_0.

2. Homeomorphisms Preserving A_p

We begin with several preliminary lemmas some of which are known [3] and are included here for the sake of completeness.

Lemma 2.1. *Let $w \in A_\infty$ and assume that $w^\epsilon \in A_1$ for some $\epsilon > 0$. Then $w \in A_1$.*

PROOF. We may suppose that $\epsilon = p_0' - 1$, since we may decrease ϵ and increase p_0 by property (*b*) of Section 1. Then

$$\begin{aligned}\frac{1}{|I|}\int_I w\,dx &\leqslant A_{p_0}(w)\Big/\left(\frac{1}{|I|}\int_I w^{-\epsilon}\right)^{1/\epsilon}\\ &\leqslant A_{p_0}(w)\left(\frac{1}{|I|}\int_I w^{\epsilon}\right)^{1/\epsilon}\\ &\leqslant A_{p_0}(w)\left(A_1(w^\epsilon)\inf_I w^\epsilon\right)^{1/\epsilon}\\ &\leqslant A_{p_0}(w)A_1(w^\epsilon)^{1/\epsilon}\inf_I w,\end{aligned}$$

where in the second inequality we used Hölder's inequality in the form

$$1 \leqslant \left(\frac{1}{|I|}\int_I \varphi\right)\left(\frac{1}{|I|}\int_I \frac{1}{\varphi}\right).$$

Our necessary and sufficient condition will involve the space $\bigcap_{p>1} A_p$. We will see that this space contains A_1 properly, and for this we need the simple

Lemma 2.2. *If $w, w^{-1} \in A_1$, then $w \simeq 1$, i.e., w is bounded above and below.*

PROOF.

$$\frac{1}{A_1\left(\frac{1}{w}\right)\inf_I \frac{1}{w}} \leqslant \frac{1}{\frac{1}{|I|}\int_I \frac{1}{w}} \leqslant \frac{1}{|I|}\int_I w \leqslant A_1(w)\inf_I w.$$

Thus

$$\sup_I w \leqslant A_1\left(\frac{1}{w}\right)A_1(w)\inf_I w$$

and the result follows.

Lemma 2.3. $A_1 \subsetneqq \bigcap_{p>1} A_p$.

PROOF. If equality held, then

$$\left\{w\colon w, w^{-1} \in \bigcap_{p>1} A_p\right\} = \{w\colon w, w^{-1} \in A_1\} = \{w\colon w \simeq 1\}.$$

However, by [4, p. 474],

$$\left\{w\colon w, w^{-1} \in \bigcap_{p>1} A_p\right\} = \{e^{\varphi}\colon \varphi \in \text{BMO-Closure } L^{\infty}\}$$

$$\supseteq \{e^{Hf}\colon f \text{ continuous of compact support}\}$$

by [10]. Then $e^{Hf} \simeq 1$, for each continuous f of compact support, and this is impossible.

Remark. An explicit example of

$$w \in \bigcap_{p>1} A_p \setminus A_1$$

is

$$w(t) = \left(\log \frac{1}{|t|}\right)^{-1}, \quad \text{for} \quad |t| \quad \text{close to} \quad 0.$$

This example was communicated to us by Rubio de Francia.

As we shall see now, Lemma 2.1 remains valid for $\bigcap_{p>1} A_p$.

Lemma 2.4. *Let $w \in A_{p_0}$ for some $1 < p_0 < \infty$ and suppose that for some $\epsilon > 0$, $w^{\epsilon} \in \bigcap_{p>1} A_p$. Then $w \in \bigcap_{p>1} A_p$.*

PROOF. Since we may decrease ϵ and increase p_0, we may suppose $\epsilon = p_0' - 1$. Then

$$\frac{1}{|I|}\int_I w \leqslant A_{p_0}(w) \Big/ \left(\frac{1}{|I|}\int_I w^{-\epsilon}\right)^{1/\epsilon} \leqslant A_{p_0}(w)\left(\frac{1}{|I|}\int_I w^{\epsilon}\right)^{1/\epsilon}.$$

Fix $1 < p < \infty$, and let

$$r = \frac{\epsilon}{p'-1} + 1.$$

Since $w^{\epsilon} \in A_r$,

$$\frac{1}{|I|}\int_I w^{\epsilon}\left(\frac{1}{|I|}\int_I w^{\epsilon(1-r')}\right)^{r-1} \leqslant A_r(w^{\epsilon}),$$

and hence we have

$$\left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{\epsilon(1-r')}\right)^{(r-1)/\epsilon} \leqslant A_r(w^\epsilon)^{1/\epsilon}A_{p_0}(w).$$

Since $(r-1)/\epsilon = p-1$ and $\epsilon(1-r') = 1-p'$, $w \in A_p$ and $A_p(w) \leqslant A_r(w^\epsilon)^{1/\epsilon} \cdot A_{p_0}(w)$.

As we shall discuss further below, the case $n = 1$ is the most involved. We shall assume throughout unless otherwise noted, that $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism onto such that h, h^{-1} are locally absolutely continuous and, without loss of generality, that $h' \geqslant 0$.

Our next result implies a quantitative version of a result known qualitatively by the result of [4, p. 402] on comparability of measures.

Lemma 2.5. *$(h^{-1})' \in RH_q$ if and only if $h' \in A_{q'}$ and $RH_q((h^{-1})') = A_{q'}(h')^{1/q'}$.*

PROOF. If $(h^{-1})' \in RH_{q'}$ then for every interval J,

$$\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right)^{1/q} \leqslant RH_q((h^{-1})')\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt,$$

where $h(I) = J$. Let

$$L = \frac{1}{|I|}\int_I h'\left(\frac{1}{|I|}\int_I h'^{(1-q)}\right)^{q'-1}.$$

The first term of L is $|J|/|I|$ and the second is, by the change of variables $t = h(x)$,

$$\frac{1}{|I|}\int_I h'^{(1-q)}(x)\,dx = \frac{1}{|I|}\int_J \frac{1}{h'\circ h^{-1}(t)^q}\,dt$$

$$= \frac{|J|}{|I|}\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right) \leqslant \frac{|J|}{|I|}RH_q((h^{-1})')^q.$$

$$\left(\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt\right)^q = RH_q((h^{-1})')^q\left(\frac{|J|}{|I|}\right)^{1-q}.$$

Consequently,

$$L \leqslant \left(\frac{|J|}{|I|}\right)RH_q((h^{-1})')^{q(q'-1)}\left(\frac{|J|}{|I|}\right)^{(1-q)(q'-1)}$$

and this gives

$$A_{q'}(h') \leqslant RH_q((h^{-1})')^{q'}.$$

Conversely, if $h' \in A_{q'}$, we write with the change of variables $t = h(x)$

$$\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right)^{1/q}}{\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt} = \left(\frac{1}{|J|}\int_I h'(x)^{1-q}\,dx\right)^{1/q}\left(\frac{|I|}{|J|}\right)^{-1}.$$

Raise both sides to the q'^{th} power and then

$$\left\{\frac{\left(\frac{1}{|J|}\int_J (h^{-1})'(t)^q\,dt\right)^{1/q}}{\frac{1}{|J|}\int_J (h^{-1})'(t)\,dt}\right\}^{q'} \leqslant \frac{|J|}{|I|}\left(\frac{1}{|I|}\int_I h'(x)^{1-q}\,dx\right)^{q'-1},$$

and thus $RH_q((h^{-1})')^{q'} \leqslant A_{q'}(h')$.

An easy consequence is the following lemma.

Lemma 2.6. *$(h^{-1})' \in A_\infty$ if and only if $h' \in A_\infty$.*

PROOF. If $(h^{-1})' \in A_\infty$, then $(h^{-1})' \in RH_{p'}$, for some $1 < p' < \infty$, and hence $h' \in A_p \subset A_\infty$. The same argument gives the converse direction.

Below, and throughout the paper, we will use the notation $Q \prec P$ to mean that Q is majorized by and expresión depending on P only so that for $P \leqslant T$, $Q \prec T$.

Now we are ready for one of the main theorems of the paper.

Theorem 2.7. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a homeomorphism as above, and suppose $1 < p_0 < \infty$, $0 \leqslant \alpha \leqslant 1$. Then, for every $w \in A_{p_0}$, $w \circ h \cdot h'^{\alpha} \in A_{p_0}$ with $A_{p_0}(w \circ h \cdot h'^{\alpha}) \prec A_{p_0}(w)$ if and only if $h' \in \bigcap_{p>1} A_p$.*

PROOF. Suppose first that $h' \in \bigcap_{p>1} A_p$. If the result holds for $\alpha = 0$ and $\alpha = 1$, it holds for $0 < \alpha < 1$ by property (c) of A_p-weights. For $\alpha = 0$, we use Property (*f*) of A_p-weights. If $w \in A_{p_0}$, then there is $\tau > 1$ so that $w^\tau \in A_{p_0}$. Let

$$L = \frac{1}{|I|}\int_I w \circ h\left(\frac{1}{|I|}\int_I (w \circ h)^{1-p_0'}\right)^{p_0-1}$$

and make the change of variables $t = h(x)$, $h(I) = J$. The first term of the product can be estimated by

$$\frac{1}{|I|}\int_I w \circ h = \frac{1}{|I|}\int_J \frac{w(t)}{h' \circ h^{-1}(t)}\,dt$$

$$\leqslant \left(\frac{1}{|I|}\int_J w^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_J\left(\frac{1}{h' \circ h^{-1}}\right)^{\tau'}\right)^{1/\tau'}.$$

We change back to the x-variable in the last integral and obtain

$$\frac{1}{|I|}\int_I w \circ h \leqslant \left(\frac{|J|}{|I|}\right)^{1/\tau}\left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}\left(\frac{1}{|I|}\int_I h'^{(1-\tau')}\right)^{1/\tau'}$$

$$\leqslant \left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau} A_\tau(h')^{1/\tau}.$$

Similarly the second term in L is

$$\frac{1}{|I|}\int_I (w \circ h)^{1-p_0'} \leqslant \left(\frac{1}{|J|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau} A_\tau(h')^{1/\tau},$$

and hence

$$A_{p_0}(w \circ h) \leqslant A_{p_0}(w^\tau)^{1/\tau} A_\tau(h')^{p_0/\tau}.$$

It is well-known [4, p. 397-9] that τ and $A_{p_0}(w^\tau)$ depend only on $A_{p_0}(w)$; in fact, $A_{p_0}(w^\tau) \leqslant cA_{p_0}(w)$ once τ has been chosen to depend on $A_{p_0}(w)$.

The case $\alpha = 1$ is similar but requires more complicated indices. We call again

$$L = \left(\frac{1}{|I|}\int_I w \circ hh'\right)\left(\frac{1}{|I|}\int_I (w \circ hh')^{1-p_0'}\right)^{p_0-1},$$

and proceeding as above, estimate the first term by

$$\frac{1}{|I|}\int_I w \circ hh' = \frac{1}{|I|}\int_J w(t)\,dt$$

$$= \frac{1}{|I|}\int_I h'\left(\frac{1}{|J|}\int_J w\right)$$

$$\leqslant \frac{1}{|I|}\int_I h'\left(\frac{1}{|J|}\int_J w^\tau\right)^{1/\tau}.$$

The second term of the product is

$$\frac{1}{|I|}\int_I (w \circ hh')^{1-p_0'} \leqslant \left(\frac{1}{|I|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau}\left(\frac{1}{|I|}\int_J (h' \circ h^{-1})^{-\tau' p_0'}\right)^{1/\tau'}$$

$$= \left(\frac{|J|}{|I|}\right)^{1/\tau}\left(\frac{1}{|J|}\int_J w^{\tau(1-p_0')}\right)^{1/\tau}\left(\frac{1}{|I|}\int_I h'^{(1-\tau' p_0')}\right)^{1/\tau'}.$$

Hence

$$L \leqslant A_{p_0}(w^\tau)^{1/\tau}\left(\frac{1}{|I|}\int_I h'\right)^{1+(p_0-1)/\tau}\left(\frac{1}{|I|}\int_I h'^{(1-\tau' p_0')}\right)^{(p_0-1)/\tau'}.$$

Since $h' \in \bigcap_{p>1} A_p$, we have $h' \in A_{p'}$, $p' = \tau' p_0'$. We note that

$$p - 1 = \frac{1}{\tau' p_0' - 1} = \frac{(\tau - 1)(p_0 - 1)}{p_0 + \tau - 1}$$

and

$$\frac{p_0}{p} = \frac{p_0(p' - 1)}{p'} = \frac{\tau + p_0 - 1}{\tau},$$

and this allows us to write

$$\begin{aligned} L &\leqslant A_{p_0}(w^\tau)^{1/\tau} \left\{ \frac{1}{|I|} \int_I h' \left(\frac{1}{|I|} \int_I h'^{(1-p')} \right)^{\frac{p_0 - \tau}{\tau'(\tau + p_0 - 1)}} \right\}^{(\tau + p_0 - 1)/\tau} \\ &\leqslant A_{p_0}(w^\tau)^{1/\tau} \cdot A_{p_0}(h')^{p_0/p} \end{aligned}$$

which completes the proof for $\alpha = 1$.

Now we suppose that for a fixed α and p_0, and any $w \in A_{p_0}$, $w \circ h \cdot h'^\alpha \in A_{p_0}$ with $A_{p_0}(w \circ h \cdot h'^\alpha) \prec A_{p_0}(w)$. We will first show, using extrapolation, that there is $\eta > 0$ so that $h'^\eta \in \bigcap_{p>1} A_p$. Since $w \circ h \cdot h'^\alpha \in A_{p_0}$, by [6]

$$\int M f^{p_0} w \circ h h'^\alpha \leqslant C \int f^{p_0} w \circ h h'^\alpha$$

where $C = C_{p_0} A_{p_0}(w \circ h h'^\alpha)^{p(p_0'+1)}$ [2]. The substitution $t = h(x)$ gives

$$\int M f^{p_0}(h^{-1}(t)) \frac{w(t)}{h' \circ h^{-1}(t)^{1-\alpha}} \, dt \leqslant C \int f(h^{-1}(t))^{p_0} \frac{w(t)}{h' \circ h^{-1}(t)^{1-\alpha}} \, dt.$$

Let $g(t) = f \circ h^{-1}(t) / h' \circ h^{-1}(t)^{(1-\alpha)/p_0}$. Then $f(x) = h'(x)^{(1-\alpha)/p_0} g \circ h(x)$. The sublinear operator

$$Tg(t) = \frac{M(h'^{(1-\alpha)/p_0} g \circ h)(h^{-1}(t))}{h' \circ h^{-1}(t)^{(1-\alpha)/p_0}}$$

satisfies

$$\int Tg^{p_0} w \leqslant C \int g^{p_0} w, \qquad w \in A_{p_0} \quad \text{and} \quad C \prec A_{p_0}(w).$$

We can now apply the extrapolation theorem [4, p. 448] and obtain

$$\int Tg^p w \leqslant C \int g^p w,$$

$w \in A_p$, $1 < p < \infty$ with $C \prec A_p(w)$. We undo the change of variables and get

$$\int M(h'^{(1-\alpha)/p_0} g \circ h)^p w \circ h \cdot h'^{1-(1-\alpha)p/p_0} \leqslant C \int g \circ h^p w \circ hh'$$
$$= C \int (g \circ h \cdot h'^{(1-\alpha)/p_0})^p w \circ h \cdot h'^{1-(1-\alpha)p/p_0},$$

which in terms of $f = g \circ hh'^{(1-\alpha)/p_0}$ says

$$\int Mf^p w \circ h \cdot h'^{1-(1-\alpha)p/p_0} \leqslant C \int f^p w \circ hh'^{1-(1-\alpha)p/p_0}$$

guaranteeing that $w \circ h \cdot h'^{1-(1-\alpha)p/p_0} \in A_p$. We choose $w = 1$ and restrict $1 < p < p_0$, and then $h'^{\alpha} \in A_p$, $1 < p < p_0$, if $\alpha > 0$, while if $\alpha = 0$, we can restrict $1 < p < gp_0/2$ some $0 < g < 1$ and conclude $h'^{1-g/2} \in \bigcap_{p>1} A_p$, and this completes the proof of the claim that $h'^{\eta} \in \bigcap_{p>1} A_p$ for some $\eta > 0$.

To conclude that $h' \in \bigcap_{p>1} A_p$, we want to apply Lemma 2.4 which requires us to show that $h' \in A_\infty$. To do this, we first show that h preserves BMO. If $u \in \mathrm{BMO}$, then $e^{\lambda u} \in A_{p_0}$ for some $\lambda > 0$, and thus by hypothesis, $e^{\lambda u \circ h} h'^{\alpha} \in A_{p_0}$. Hence $\lambda u \circ h + \alpha \log h' \in \mathrm{BMO}$. Since $h'^{\eta} \in \bigcap_{p>1} A_p$, $\log h' \in \mathrm{BMO}$ and thus $u \circ h \in \mathrm{BMO}$. By [5], $(h^{-1})' \in A_\infty$ and hence by Lemma 2.6, $h' \in A_\infty$.

The situation for $p_0 = 1$ is different depending on whether $\alpha = 0$ or $0 < \alpha < 1$.

Theorem 2.8. *Let $h: \mathbb{R} \to \mathbb{R}$ be a homeomorphism as before and suppose that $0 < \alpha \leqslant 1$. Then for every $w \in A_1$, $w \circ h \cdot h'^{\alpha} \in A_1$, with $A_1(w \circ h \cdot h'^{\alpha}) \prec A_1(w)$ if and only if $h' \in A_1$.*

Theorem 2.9. *Let $h: \mathbb{R} \to \mathbb{R}$ be a homeomorphism as before. Then for every $w \in A_1$, $w \circ h \in A_1$ with $A_1(w \circ h) \prec A_1(w)$ if and only if $h' \in \bigcap_{p>1} A_p$.*

We begin with the proof of Theorem 2.9 since it requires the least change from the proof of Theorem 2.7. If $h' \in \bigcap_{p>1} A_p$ and $w \in A_1$, we first choose $\tau > 1$ so that $w^\tau \in A_1$. The argument gave

$$\frac{1}{|I|} \int_I w \circ h \leqslant \left(\frac{1}{|J|} \int_J w^\tau \right)^{1/\tau} A_\tau(h')^{1/\tau}.$$

Since $w^\tau \in A_1$, this can be estimated by

$$\frac{1}{|I|} \int_I w \circ h \leqslant \left(\inf_J w^\tau \right)^{1/\tau} A_\tau(h')^{1/\tau} = A_\tau(h')^{1/\tau} \inf_I w \circ h.$$

Conversely, if for every $w \in A_1$, $w \circ h \in A_1$, with $A_1(w \circ h) \prec A_1(w)$, we apply the factorization theorem [4, p. 434] to show that the conditions of Theorem 2.7 hold with, e.g., $p_0 = 2$. For $w \in A_2$, $w = u_1/u_2$, for $u_1, u_2 \in A_1$

with $A_1(u_j) \prec A_2(w)$. Since $w \circ h = u_1 \circ h/u_2 \circ h$, we see that $w \circ h \in A_2$ and $A_2(w \circ h) \prec A_2(w)$, and Theorem 2.7 gives the result.

The proof of Theorem 2.8 is easy with all the tools now available to us. We only need to show that, if $h' \in A_1$, then $w \circ h \cdot h' \in A_1$ (property (*c*) of A_p-weights and Theorem 2.9 show then $w \circ hh'^{\alpha} \in A_1$).

If $h' \in A_1$ and $h(I) = J$,

$$\frac{1}{|I|}\int_I w \circ hh' = \frac{1}{|I|}\int_J w = \frac{|J|}{|I|}\left(\frac{1}{|J|}\int_J w\right)$$

$$\leqslant A_1(h') \inf_I h' A_1(w) \inf_I w \circ h$$

$$\leqslant A_1(h') A_1(w) \inf_I w \circ hh'.$$

Conversely, suppose

$$A_1(w \circ h \cdot h'^{\alpha}) \prec A_1(w), \qquad 0 < \alpha \leqslant 1.$$

The choice $w = 1$ shows that $h'^{\alpha} \in A_1$. The factorization theorem again shows that $h' \in \bigcap_{p>1} A_p$ because

$$w \circ h = \frac{u_1 \circ h \cdot \mathrm{h}'^{\alpha}}{u_2 \circ h \cdot h'^{\alpha}}.$$

By Lemma 2.1, $h' \in A_1$.

Remarks. **1.** The extension to n-dimensions presents no real difficulties. If a homeomorphism preserves A_p, it must also preserve BMO and then h and h^{-1} are quasi-conformal by [9]. We will therefore assume that h is smooth when stating the next result. We denote by J_h the Jacobian of h.

Theorem 2.10. *Let h be a smooth quasi-conformal homeomorphism, h*: $\mathbb{R}^n \to \mathbb{R}^n$.

(*a*) *If* $1 < p_0 < \infty$, $0 \leqslant \alpha \leqslant 1$, *then* $A_{p_0}(w \circ h|J_h|^{\alpha}) \prec A_{p_0}(w)$ *if and only if* $|J_h| \in \bigcap_{p>1} A_p$.

(*b*) $A_1(w \circ h) \prec A_1(w)$ *if and only if* $|J_h| \in \bigcap_{p>1} A_p$.

(*c*) $A_1(w \circ h|J_h|^{\alpha}) \prec A_1(w)$, $0 < \alpha \leqslant 1$, *if and only if* $|J_h| \in A_1$.

2. The fundamental estimate of Theorem 2.7 contains a sufficient condition for a homeomorphism to preserve a single weight $w \in A_{p_0}$ since it says

$$A_{p_0}(w \circ h) \leqslant C A_{p_0}(w^{\tau})^{1/\tau} A_{\tau}(h')^{p_0/\tau}.$$

It can also be applied to the local A_p-classes $A_{p,\Omega}$ [4, p. 438].

3. Theorems 2.7, 2.8, 2.9 contain generalizations of themselves allowing negative powers of h', because if $h' \in \bigcap_{p>1} A_p$ and $w \in A_{p_0}$, then $w^{1-p_0'} \in A_{p_0'}$, and hence $(w \circ h)^{1-p_0'} h'^{\alpha} \in A_{p_0'}$, $0 \leqslant \alpha \leqslant 1$. We use duality again and see that $w \circ h \cdot h'^{\alpha(1-p_0)} \in A_{p_0}$. Thus $w \circ h \cdot h'^{\beta} \in A_{p_0}$ for $1 - p_0 \leqslant \beta \leqslant 1$.

The mapping $T_h w(x) = w \circ h(x)$, where $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism as before, is not an onto map from $A_{p_0} \to A_{p_0}$ in general, but it is possible to determine precisely when it is onto.

Theorem 2.11. *Let $1 \leqslant p_0 < \infty$. Then $T_h: A_{p_0} \to A_{p_0}$ is onto with*

$$A_{p_0}(T_h w) \prec A_{p_0}(w), \qquad A_{p_0}(T_h^{-1} w) \prec A_{p_2}(w)$$

if and only if $\log h' \in \text{BMO}$*-closure* L^{∞}.

PROOF. For the the necessity, if $T_h: A_{p_0} \to A_{p_0}$ is onto, T_h^{-1} is defined and $T_h^{-1} = T_{h^{-1}}$. It follows from Theorems 2.7 and 2.9 that h' and $(h^{-1})'$ are in $\bigcap_{p>1} A_p$. This implies that

$$\frac{1}{h'} \in \bigcap_{p>1} A_p,$$

for, if we fix $1 < p_1 < \infty$ and apply Theorem 2.7 to $h' \in \bigcap_{p>1} A_p$ and $(h^{-1})' \in A_{p_1}$ we obtain

$$(h^{-1})' \circ h(x) = \frac{1}{h'(x)} \in A_{p_1}.$$

By [4, p. 474] $\log h' \in$ BMO-closure L^{∞}.

For the sufficiency, if we suppose that $\log h' \in$ BMO-closure L^{∞}, then $h' \in \bigcap_{p>1} A_p$ and $h' \in \bigcap_{q<\infty} RH_q$. Then by Lemma 2.5, $(h^{-1})' \in \bigcap_{p>1} A_p$. By Theorem 2.7 $T_{h^{-1}}: A_{p_0} \to A_{p_0}$ and thus T_h is onto.

Remark. The situation for A_p contrasts with that for BMO, since Jones shows in [5] that whenever T_h is bounded on BMO, it is onto, because of the result we gave here as Lemma 2.6.

We can also characterize the pointwise multipliers of A_{p_0}.

Theorem 2.12. *Let $1 < p_0 < \infty$, and let $\varphi: \mathbb{R} \to \mathbb{R}_+$. Then for every $w \in A_{p_0}$, $w \cdot \varphi \in A_{p_0}$ with $A_{p_0}(w\varphi) \prec A_{p_0}(w)$ if and only if* $\log \varphi \in \text{BMO}$*-closure* L^{∞}.

PROOF. For the necessity, note that $\varphi^n \in A_{p_0}$, $n = 1, 2, \ldots$, and thus $\varphi \in A_{(p_0-1)/n+1}$ by [4, p. 394] and so $\varphi \in \bigcap_{p>1} A_p$.

Next, we claim

$$\frac{1}{\varphi} \in \bigcap_{p>1} A_p .$$

If $w \in A_{p_0'}$, then $w^{1-p_0} \in A_{p_0}$ and hence $w^{1-p_0}\phi^n \in A_{p_0}$, $n = 1, 2, \ldots$. Thus $w \cdot \varphi^{n(1-p_0')} \in A_{p_0'}$ from which we get by taking $w = 1$, $\varphi^{-1} \in A_{1+1/n}$ or $\varphi^{-1} \in \bigcap_{p>1} A_p$. Again by [4, p. 474], $\log \varphi \in$ BMO-closure L^∞.

For the sufficiency, suppose $\log \varphi \in$ BMO-closure L^∞. We define

$$h(x) = \int_0^x \varphi .$$

Then $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $h' = \varphi$. Our first claim is that h is onto $\mathbb{R}$, i.e., $\int_0^x \varphi \to \infty$ as $x \to \infty$. The condition on φ implies that $\varphi \in A_2$ and hence, if

$$\varphi(E) = \int_E \varphi ,$$

for any $E \subseteq I$,

$$\varphi(I) \leqslant c \left(\frac{|I|}{|E|} \right)^2 \varphi(E),$$

with c independent of E, I. Apply this to $I_n = [0, 2^n]$ and $J = [2^{n-1}, 2^n]$ and note that

$$\varphi(I_n) = \varphi(I_{n-1}) + \varphi(J).$$

Since

$$\varphi(I_{n-1}) \leqslant \varphi(I_n) \leqslant 4c\varphi(J),$$

we see that

$$\varphi(I_n) \geqslant \left(1 + \frac{1}{4c}\right) \varphi(I_{n-1}),$$

and iterating

$$\varphi(I_n) \geqslant \left(1 + \frac{1}{4c}\right)^n \varphi(I_0),$$

and our claim is proved.

Since our hypothesis is that $\log \varphi \in$ BMO-closure L^∞, by Theorem 2.12, T_h is onto A_{p_0} for any $1 \leqslant p_0 < \infty$, and thus by Theorem 2.7, $(h^{-1})' \in \bigcap_{p>1} A_p$. For a given $w \in A_{p_0}$, we write $w\varphi = [w \circ h^{-1} \circ h]h'$ which is in A_{p_0} because $w \circ h^{-1} \in A_{p_0}$. One checks that $A_{p_0}(w\varphi) \prec A_{p_0}(w)$.

Remark. The multiplier problem for BMO is known [11] but quite different.

The pointwise multipliers of A_1 require a slightly different approach. For $1 < r < \infty$, let $A_{1,r}$ be the class of all $f\colon \mathbb{R} \to \mathbb{R}_+$ for which $f^r \in A_1$.

Lemma 2.13. *$f \in \bigcap_{r<\infty} A_{1,r}$ if and only if $f \in A_1$ and $1/f \in \bigcap_{p>1} A_p$.*

PROOF. For the necessity simply observe that $f \in A_1$ and

$$\frac{1}{|I|}\int_I f^{-1}\left(\frac{1}{|I|}\int_I f^{p'-1}\right)^{p-1} \leqslant \frac{C}{|I|}\int_I \frac{1}{f}\cdot \inf_I f \leqslant C \sup_I \frac{1}{f}\cdot \inf_I f \leqslant C.$$

For the converse,

$$\frac{1}{|I|}\int_I f \leqslant C \inf_I f,$$

and since

$$1 \leqslant \frac{1}{|I|}\int_I f\left(\frac{1}{|I|}\int_I \frac{1}{f}\right),$$

we see that

$$C \sup_I \frac{1}{f} \leqslant \frac{1}{|I|}\int_I \frac{1}{f}.$$

Since for $1 < p < \infty$,

$$\frac{1}{|I|}\int_I \frac{1}{f}\left(\frac{1}{|I|}\int_I f^{p'-1}\right)^{p-1} \leqslant C_p < \infty,$$

we get

$$\left(\frac{1}{|I|}\int_I f^{p'-1}\right)^{p-1} \leqslant C \inf_I f \quad \text{or} \quad f^{p'-1} \in A_1.$$

Theorem 2.14. *Let $\varphi\colon \mathbb{R} \to \mathbb{R}_+$. Then $w \in A_1$ implies $w \cdot \varphi \in A_1$ with $A_1(w\varphi) \prec A_1(w)$ if and only if $\varphi \in \bigcap_{r<\infty} A_{1,r}$.*

PROOF. If $w\varphi \in A_1$ for $w \in A_1$, then $\varphi \in A_1$ by taking $w = 1$. Let $1 < p_0 < 2$ and $w \in A_{p_0}$. Then $w = u \cdot v^{1-p_0}$, $u, v \in A_1$, and $A_1(u)$, $A_1(v) \prec A_{p_0}(w)$ as well as

$$A_{p_0}(w) \leqslant A_1(u) A_1(v)^{p_0 - 1}$$

[4, p. 434]. Thus

$$w \cdot \varphi^{2-p_0} = u \cdot \varphi \cdot (v \cdot \varphi)^{1-p_0}$$

and $w \cdot \varphi^{2-p_0} \in A_{p_0}$ with $A_{p_0}(w \cdot \varphi^{2-p_0}) \prec A_{p_0}(w)$. Therefore, by Theorem 2.12, $\log \varphi$ is in the closure of L^∞ in BMO which implies that $1/\varphi \in \bigcap_{p>1} A_p$. Hence $\varphi \in \bigcap_{r>\infty} A_{1,r}$ by Lemma 2.13.

For the converse we use the technique of Theorem 2.12 and let

$$h(x) = \int_0^x \varphi.$$

Then, as shown there, $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism onto and $(h^{-1})' \in \bigcap_{p>1} A_p$ $h' \in A_1$. For $w \in A_1$, write $w \cdot \varphi = w \circ h^{-1} \circ h \cdot h'$, and, since $v = w \circ h^{-1} \in A_1$, by Theorem 2.9, we see that $v \circ h \cdot h' \in A_1$ by applying Theorem 2.8.

3. Homeomorphisms Preserving A_∞ and RH_{p_0}

We shall see that the homeomorphisms preserving A_∞ and RH_{p_0} will be the same as those preserving A_{p_0} under the condition $h' \in \bigcap_{p>1} A_p$, but the pointwise multiplier condition is different. We begin with some preliminary results, including a proof of Theorem 1.1 that gives the constants we need.

Lemma 3.1. *Let* $1 < p_0 < \infty$. *Then*

$$\frac{A_\infty(w^{p_0})^{1/p_0}}{A_\infty(w)} \leqslant RH_{p_0}(w) \leqslant A_\infty(w^{p_0})^{1/p_0}.$$

PROOF. For the first inequality simply note that

$$\frac{1}{|I|}\int_I w^{p_0} \leqslant RH_{p_0}(w)^{p_0}\left(\frac{1}{|I|}\int_I w\right)^{p_0},$$

and hence for $q < \infty$,

$$\frac{1}{|I|}\int_I w^{p_0}\left(\frac{1}{|I|}\int_I w^{p_0(1-q')}\right)^{q-1}$$
$$\leqslant RH_{p_0}(w)^{p_0}\left\{\frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{p_0(1-q')}\right)^{(q-1)/p_0}\right\}^{p_0}.$$

Thus

$$A_q(w^{p_0}) \leqslant RH_{p_0}(w)^{p_0} \cdot A_{\frac{p_0+q-1}{p_0}}(w)^{p_0}$$

Let $q \uparrow \infty$.

The second inequality can be obtained as follows

$$\frac{\left(\frac{1}{|I|}\int_I w^{p_0}\right)^{1/p_0}}{\frac{1}{|I|}\int_I w} = \frac{\left\{\frac{1}{|I|}\int_I w^{p_0}\left(\frac{1}{|I|}\int_I w^{p_0(1-q')}\right)^{q-1}\right\}^{1/p_0}}{\frac{1}{|I|}\int_I w\left\{\frac{1}{|I|}\int_I w^{p_0(1-q')}\right\}^{(q-1)/p_0}} \leqslant A_q(w^{p_0})^{1/p_0}$$

since the denominator is $\geqslant 1$. Hence

$$RH_{p_0}(w) \leqslant A_q(w^{p_0})^{1/p_0}.$$

Let $q \uparrow \infty$.

Remark. We shall see below in Theorem 3.4 that the quantity $RH_{p_0}(w)$ is not as convenient as $RH_{p_0}(w) \cdot A_\infty(w) \equiv \overline{RH}_{p_0}(w)$.

Lemma 3.2. *Let $1 < p_0 < \infty$. Then*

$$\max\{A_\infty(w), A_\infty(w^{1-p_0'})^{p_0-1}\} \leqslant A_{p_0}(w) \leqslant A_\infty(w)A_\infty(w^{1-p_0'})^{p_0-1}.$$

PROOF. The first inequality is easy since both terms on the left are $\leqslant A_{p_0}(w)$. For the second inequality simply note that

$$\frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{1-p_0'}\right)^{p_0-1} = \frac{1}{|I|}\int_I w\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1}$$
$$\cdot\left\{\frac{1}{|I|}\int_I w^{1-p_0'}\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{(1-q)/(p_0-1)}\right\}^{p_0-1}$$

Using

$$1 \leqslant \left(\frac{1}{|I|}\int_I \varphi\right)\left(\frac{1}{|I|}\int_I \frac{1}{\varphi}\right),$$

the expression in $\{\quad\}$ is

$$\leqslant \frac{1}{|I|}\int_I w^{1-p_0'}\left(\frac{1}{|I|}\int_I w^{q'-1}\right)^{(q-1)/(p_0-1)}$$
$$= \left(\frac{1}{|I|}\int_I w^{(1-p_0')(q'-1)/(1-p_0')}\right)^{(q-1)/(p_0-1)}\left(\frac{1}{|I|}\int_I w^{1-p_0'}\right).$$

If

$$r - 1 = \frac{q-1}{p_0-1},$$

one checks that

$$r' - 1 = \frac{p_0 - 1}{q - 1} = \frac{q' - 1}{p_0' - 1}.$$

Hence for $q < \infty$,

$$A_{p_0}(w) \leqslant A_q(w)\{A_{1+(q-1)/(p_0-1)}(w^{1-p_0'})\}^{p_0-1}.$$

Let $q \uparrow \infty$.

Lemma 3.3.

(i) $A_\infty(w^{1-q'})^{q-1} \searrow \sigma(w)$ *as* $q \nearrow \infty$ *and*
(ii) $\sigma(w) \leqslant A_\infty(w)$.

PROOF. (i) For $p < \infty$ and $q_1 > q_2$ we have

$$\left(\frac{1}{|I|}\int_I w^{1-q_1'}\right)^{q_1-1} \leqslant \left(\frac{1}{|I|}\int_I w^{1-q_2'}\right)^{q_2-1},$$

so that

$$A_p(w^{1-q_1'})^{q_1-1} \leqslant A_p(w^{1-q_2'})^{q_2-1}.$$

Let $p \to \infty$ and (i) follows. To prove (ii), let $\lambda < \sigma(w)$. We claim that for $q < \infty$, $\lambda \leqslant A_q(w)$. Fix q, and note that

$$\lambda < A_\infty(w^{1-q'})^{q-1} \leqslant A_p(w^{1-q'})^{q-1},$$

for any $p < \infty$. Since

$$\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1}\left(\frac{1}{|I|}\int_I w^{(1-q')(1-p')}\right)^{(p-1)(q-1)}$$
$$\leqslant \left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1}\left(\frac{1}{|I|}\int_I w\right),$$

if, $(p-1)(q-1) > 1$, the claim and hence (ii) follows.

Theorem 3.4. *Let* $1 < p_0 < \infty$. *Then the following statements are equivalent for a homeomorphism* h *as in section* 2.

(1) $h' \in \bigcap_{p>1} A_p$.

(2) $w \in A_\infty$ *implies* $w \circ h \in A_\infty$ *with* $A_\infty(w \circ h) \prec A_\infty(w)$.

(3) $w \in RH_{p_0}$ *implies* $w \circ h \in RH_{p_0}$ *with* $\overline{RH}_{p_0}(w \circ h) \prec \overline{RH}_{p_0}(w)$.

(4) $w \in A_{p_0}$ *implies* $w \circ h \in A_{p_0}$ *with* $A_{p_0}(w \circ h) \prec A_{p_0}(w)$.

PROOF. We will be brief. First (1) $\Rightarrow$ (4) is Theorem 2.7. The equivalence (2) $\Rightarrow$ (3) is Lemma 3.1. The implication (2) $\Rightarrow$ (4) follows from Lemma 3.2. There remains (4) $\Rightarrow$ (2). Since (4) $\Rightarrow$ (1) by Theorem 2.7. $A_p(w \circ h) \prec A_p(w), p < \infty$. Let $w \in A_\infty$. By Lemma 3.3 we can choose p_0 so that $A_\infty(w^{1-p_0'})^{p_0-1} \leqslant 2A_\infty(w)$. Then $A_\infty(w \circ h) \leqslant A_{p_0}(w \circ h) \prec A_{p_0}(w)$, and $A_{p_0}(w) \leqslant A_\infty(w)A_\infty(w^{1-p_0'})^{p_0-1}$ by Lemma 3.2. Hence $A_\infty(w \circ h) \prec A_\infty(w)$.

We come now to the pointwise multipliers of A_∞ and RH_{p_0}.

Theorem 3.5. *Let $1 < p_0 < \infty$. Then the following statements are equivalent for $\varphi: \mathbb{R} \to \mathbb{R}_+$.*

(1) $\varphi^n \in A_\infty$, $n = 1, 2, \ldots$.

(2) $\varphi \in \bigcap_{r<\infty} RH_r$.

(3) $w \in A_\infty$ *implies* $w \cdot \varphi \in A_\infty$.

(4) $w \in RH_{p_0}$ *implies* $w \cdot \varphi \in RH_{p_0}$.

PROOF. From Lemma 3.1 we get (1) $\Leftrightarrow$ (2). It is clear that (3) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (3) can be seen as follows. Let $w \in A_\infty$. Then for some $p_0 < \infty$, $w \in A_{p_0}$. Hence there is $\tau > 1$ so that $w^\tau \in A_{p_0} \subset A_p, p \geqslant p_0$. Since by (1) $\varphi^{\tau'} \in A_\infty$, we can choose $p \geqslant p_0$ so that $\varphi^{\tau'} \in A_p$. An easy application of Hölder's inequality shows that

$$A_p(w\varphi) \leqslant A_p(w^\tau)^{1/\tau} A_p(\varphi^{\tau'})^{1/\tau'}.$$

Hence $w \cdot \varphi \in A_\infty$. Since (4) implies $\varphi^n \in RH_{p_0} \subset A_\infty$, $n = 1, 2, \ldots$, we get (4) $\Rightarrow$ (1). We complete the proof by showing (3) $\Rightarrow$ (4). If $\alpha > 0$, then from (3), $w \in A_\infty$ implies $w \cdot \varphi^\alpha \in A_\infty$, since (3) $\Rightarrow$ (1). Let $w \in RH_{p_0}$. Then, by Lemma 3.1,

$$RH_{p_0}(w \cdot \varphi)^{p_0} \leqslant A_\infty(w^{p_0}\varphi^{p_0}) < \infty$$

sience $w^{p_0} \in A_\infty$.

Remark. If $\varphi = P(x)$ is a polynomial, then (1) is satisfied, and hence a polynomial is a pointwise multiplier of A_∞ and RH_{p_0}. However, it is clear that this is not the case for A_{p_0}.

4. Homeomorphisms Preserving Double Weights

The argument presented in Theorem 2.7 for single weights is not applicable to double weights $(u, v) \in A_p$ since it is no longer true that there is $\tau > 1$ with $(u^\tau, v^\tau) \in A_p$. In fact the existence of such a $\tau > 1$ takes us back to the single weight case [7]. For the study of the homeomorphisms which preserve $(u, v) \in A_p$ we need the following extrapolation theorem.

Theorem 4.1. *Let* $1 < p_0 < \infty$, *and let* $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ *be nondecreasing. Assume that T is a sublinear operator so that for every* $(u, v) \in A_{p_0}$,

$$u\{x: |Tf(x)| > y\} \leqslant \frac{\gamma(A_{p_0}(u, v))^{p_0}}{y^{p_0}} \|f\|_{p_0, v}^{p_0}.$$

Then, if $1 < p < p_0$, *and* $(u, v) \in A_p$,

$$u\{x: |Tf(x)| > y\} \leqslant \frac{C^p}{y^p} \|f\|_{p, v}^{p}.$$

where

$$C \leqslant c p_0 A_p(u, v)[\gamma(A_p(u, v)^{(p'-1)(p_0-1)})]^{1/p_0}.$$

We use the notation

$$A_p(u, v) = \max\left\{1, \sup_Q \left(\frac{1}{|Q|}\int_Q u\right)\left(\frac{1}{|Q|}\int_Q v^{1-p'}\right)^{p-1}\right\}.$$

We will not prove this theorem since the proof is the same as in [8] by keeping track of the constants involved. In the applications that follow, $\gamma(t) = ct^{1/p_0}$.

As before we shall assume that $h: \mathbb{R} \to \mathbb{R}$ is a homeomorphism onto so that h, h^{-1} are locally absolutely continuous and $h' \geqslant 0$.

Theorem 4.2. $1 < p < \infty$, $1 - p \leqslant \alpha \leqslant 1$. *If* $h' \in A_1$, *then for every* $(u, v) \in A_p$, $(u \circ h \cdot h'^{\alpha}, v \circ h \cdot h'^{\alpha}) \in A_p$ *with* $A_p(u \circ h \cdot h'^{\alpha}, v \circ h \cdot h'^{\alpha}) \leqslant CA_p(u, v)$, *C independent of* α.

PROOF. Let

$$L = \frac{1}{|I|}\int_I u \circ h \cdot h'^{\alpha}\left(\frac{1}{|I|}\int_I (v \circ h)^{1-p'} \cdot h'^{\alpha(1-p')}\right)^{p-1}.$$

Let $t = h(x)$, $h(I) = J$. Then

$$\frac{1}{|I|}\int_I u \circ h \cdot h'^{\alpha} = \frac{|J|}{|I|}\frac{1}{|J|}\int_J \frac{u(t)}{h'^{(1-\alpha)} \circ h^{-1}(t)},$$

and

$$\frac{1}{|I|}\int_I \frac{1}{v \circ h'^{p'-1} \cdot h'^{\alpha(p'-1)}} = \frac{|J|}{|I|}\frac{1}{|J|}\int_J \frac{1}{v(t)^{p'-1} \cdot (h' \circ h^{-1}(t))^{\alpha(p'-1)+1}}.$$

Hence

$$L \leqslant \left(\frac{|J|}{|I|}\right)^p \frac{1}{\inf h'^{(1-\alpha)} \cdot \inf h'^{(\alpha+p-1)}} A_p(u, v),$$

since $1 - \alpha \geqslant 0$, $\alpha + p - 1 \geqslant 0$. Since $h' \in A_1$, i.e.,

$$\frac{1}{|I|}\int_I h' = \frac{|J|}{|I|} \leqslant C \inf_I h',$$

the proof is complete.

Remark. We do not know whether or not the converse of Theorem 4.2 is true (except for $\alpha = 1$ as we will see). From Theorem 2.7, however, the A_p-norm inequality does imply that $h' \in \bigcap_{p>1} A_p$, if $0 \leqslant \alpha \leqslant 1$. If we change the problem somewhat, we obtain a necessary and sufficient condition.

Theorem 4.3. *Let* $1 < p_0 < \infty$, $1 - p_0 \leqslant \alpha$. *Then for every* $(u, v) \in A_{p_0}$, $(u \circ h \cdot h', v \circ h \cdot h'^{\alpha}) \in A_{p_0}$ *with* A_{p_0} $(u \circ h \cdot h', v \circ h \cdot h'^{\alpha}) \leqslant CA_{p_0}(u, v)$ *if and only if*

$$\frac{1}{|I|}\int_I h' \leqslant C \inf_I (h')^{\alpha/p_0 + 1/p_0'},$$

C independent of I.

PROOF. The sufficiency can be handled as in Theorem 5.

$$\frac{1}{|I|}\int_I u \circ h \cdot h' = \frac{|J|}{|I|} \frac{1}{|J|}\int_J u(t)\, dt$$

$$\frac{1}{|I|}\int_I \frac{1}{(v \circ h)^{p_0'-1} \cdot h'^{\alpha(p_0'-1)}} = \frac{|J|}{|I|} \frac{1}{|J|}\int_J \frac{1}{v(t)^{p_0'-1}(h' \circ h^{-1}(t))^{\alpha(p_0'-1)+1}}.$$

Hence

$$A_{p_0}(u \circ h \cdot h', v \circ h \cdot h'^{\alpha}) \leqslant A_p(u, v) \cdot \left(\frac{|J|}{|I|}\right)^{p_0} \frac{1}{\inf_I (h')^{\alpha + p_0 - 1}},$$

since $\alpha + p_0 - 1 \geqslant 0$.

For the necessity we note first that [6]

$$\int_{\{Mf > y\}} u \circ h \cdot h' \leqslant \frac{CA_{p_0}(u, v)}{y^{p_0}} \int f^{p_0}(x) v \circ h(x) \cdot h'^{\alpha}(x)\, dx.$$

We change variables $t = h(x)$ and get

$$\int_{\{t:\, Mf[h^{-1}(t)] > y\}} u(t)\,dt \leqslant \frac{CA_{p_0}(u, v)}{y^{p_0}} \int f[h^{-1}(t)]^{p_0} \frac{v(t)}{[h' \circ h^{-1}(t)]^{1-\alpha}}\,dt.$$

If we set

$$g(t) = \frac{f \circ h^{-1}(t)}{[h' \circ h^{-1}(t)]^{(1-\alpha)/p_0}},$$

then $f(x) = h'(x)^{(1-\alpha)/p_0} g \circ h(x)$, and

$$\int_{\{t:\, M(h'^{(1-\alpha)/p_0} g \,\circ\, h)[h^{-1}(t)] > y\}} u(t)\,dt \leqslant \frac{CA_{p_0}(u, v)}{y^{p_0}} \int g^{p_0} v.$$

The hypothesis of Theorem 4.1 are satisfied with

$$\gamma(t) = ct^{1/p_0}$$

and

$$Tg(t) = M(h'^{(1-\alpha)/p_0} g \circ h)[h^{-1}(t)].$$

Hence for $1 < p < p_0$ and $(u, v) \in A_p$,

$$u\{x: |Tg(x)| > y\} \leqslant \frac{C^p}{y^p} \|g\|_{p,v}^p$$

with

$$C = cp_0 A_p(u, v)[\gamma(A_p(u, v)^{(p'-1)(p_0-1)})]^{1/p_0}.$$

We let now $u = v = 1$ so that $A_p(u, v) = 1$. With $t = h(x)$ we get

$$\int_{\{x:\, M(h'^{(1-\alpha)/p_0} g \,\circ\, h)(x) > y\}} h'(x)\,dt \leqslant \frac{C^p}{y^p} \int (g \circ h)^p(x) \cdot h'(x)\,dx,$$

where C is an absolute constant independent of p. Since

$$g \circ h(x) = \frac{f(x)}{h'^{(1-\alpha)/p_0}(x)},$$

the above is

$$\int_{\substack{I \\ \{x:\, Mf(x) > y\}}} h'(x)\,dx \leqslant \frac{C^p}{y^p} \int f^p(x) h'(x)^{1-(1-\alpha)p/p_0}\,dx.$$

We test this inequality with $f = \chi_I h'^\gamma$, where γ will be chosen later. Since

$$I \subset \left\{ x: Mf(x) \geqslant \frac{1}{|I|} \int_I h'^\gamma \equiv y \right\}$$

we obtain

$$\int_I h'(x)\,dx \leqslant C^p \frac{|I|^p}{\left(\int_I h'^\gamma\right)^p} \int_I h'^{\gamma p + 1 - (1-\alpha)p/p_0}.$$

We choose γ so that

$$\gamma = \gamma p + 1 - \frac{1-\alpha}{p_0} p, \quad (p-1)\gamma = \frac{1-\alpha}{p_0} \cdot p - 1 \text{ or } \gamma = p' \left[\frac{1-\alpha}{p_0} - \frac{1}{p} \right].$$

Hence

$$\frac{1}{|I|} \int_I h' \leqslant C^p |I|^{p-1} \left(\int_I h'^\gamma \right)^{1-p}.$$

We let now $p \downarrow 1$. Then

$$\left(\int_I h'^{[(1-\alpha)/p_0 - 1/p]p'} \right)^{p/p'} \to \sup_I h'^{(1-\alpha)/p_0 - 1} = \frac{1}{\inf_I h'^{1-(1-\alpha)/p_0}}.$$

Hence

$$\frac{1}{|I|} \int_I h' \leqslant C \inf_I h'^{\alpha/p_0 + 1/p_0'}.$$

Corollary 4.4. *Let $1 < p_0 < \infty$. Then for every $(u, v) \in A_{p_0}$, $(u \circ h \cdot h', v \circ h \cdot h') \in A_{p_0}$ with $A_{p_0}(u \circ h \cdot h, v \circ h \cdot h') \leqslant C A_{p_0}(u, v)$ if and only if $h' \in A_1$.*

As in the single weight case we ask when $T_h(u, v) = (u \circ h \cdot h', v \circ h \cdot h'^\alpha)$ takes A_{p_0} *onto* A_{p_0}.

Theorem 4.5. *Let $1 < p_0 < \infty$ and $\alpha > 1 - p_0$. Then $T_h: A_{p_0} \to A_{p_0}$ is onto with $A_{p_0}(T_h(u, v))$, and $A_{p_0}(T_h^{-1}(u, v))$ both $\leqslant C A_{p_0}(u, v)$ if and only if $h' \simeq 1$.*

PROOF. First assume that T_h is onto. Then, if $(u, v) \in A_{p_0}$, there is $(\bar{u}, \bar{v}) \in A_{p_0}$ with $(\bar{u} \circ h \cdot h', \bar{v} \circ h \cdot h'^\alpha) = (u, v)$. Hence

$$\bar{u}(t) = u \circ h^{-1}(t) \cdot \frac{1}{h' \circ h^{-1}(t)} = u \circ h^{-1}(t) \cdot (h^{-1})'(t)$$

and

$$\bar{v}(t) = v \circ h^{-1}(t) \cdot (h^{-1})'^{\alpha}(t).$$

Hence $T_h^{-1} = T_{h^{-1}}$, and Theorem 4.3 gives the two inequalities

$$\frac{1}{|I|} \int_I h' \leqslant C \inf_I h'^{\alpha/p_0 + 1/p_0'},$$

$$\frac{1}{|J|} \int_J (h^{-1})' \leqslant C \inf_J (h^{-1})'^{\alpha/p_0 + 1/p_0'}.$$

If $h(I) = J$, the second inequality can be rewritten as

$$C \sup_I h'^{\alpha/p_0 + 1/p_0'} \leqslant \frac{1}{|I|} \int_I h'.$$

This together with the first inequality shows that

$$C \sup_I h' \leqslant \inf_I h'$$

(note $\alpha/p_0 + 1/p_0'$ since $\alpha > 1 - p_0$), and $h' \simeq 1$.

Conversely, if $h' \simeq 1$, then there are constants $0 < c_1 \leqslant h' \leqslant c_2 < \infty$, and so

$$\frac{1}{|I|} \int_I h' \leqslant \frac{C_2}{C_1^\gamma} C_1^\gamma \leqslant C \inf_I h'^\gamma, \qquad \gamma = \frac{\alpha}{p_0} + \frac{1}{p_0'}.$$

Similarly

$$\frac{1}{|I|} \int_I (h^{-1})' \leqslant C \inf_I (h^{-1})'^{\alpha/p_0 + 1/p_0'}.$$

Hence T_h is onto.

References

[1] Benedetto, J. J., Heinig H. P. and Johnson, R. Fourier inequalities with A_p-weights, to appear.

[2] Carbery, A., Chang, S. Y. A. and Garnett, J. Weights and $L \log L$, *Pac. J. Math.* **120** (1985), 33-45.

[3] Dyn'kin, E. M. and Osilenker, B. P. Weighted estimates of singular integrals and their applications, *Journal Soviet Math.* **30** (1985), 2094-2154.

[4] García-Cuerva, J. and Rubio de Francia, J. L. Weighted norm inequalities and related topics, *North Holland Math. Studies,* vol. 116, North Holland, Amsterdam, 1985.

[5] Jones, P. W. Homeomorphisms of the line which preserve BMO, *Arkiv Mat.* **21**(1983), 229-231.

[6] Muckenhoupt, B. Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207-226.
[7] Neugebauer, C. J. Inserting A_p-weights, *Proc. Amer. Math. Soc.* **87**(1983), 644-648.
[8] —, A double weight extrapolation theorem, *Proc. Amer. Math. Soc.* **93**(1985), 451-455.
[9] Reimann, M. Functions of bounded mean oscillation and quasiconformal mappings, *Comm. Math. Helv.* **49**(1974), 260-276.
[10] Sarason, D. Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207** (1975), 391-405.
[11] Stegenga, D. Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation, *Amer. J. Math.* **98**(1976), 543-589.
[12] Strömberg, J. O. and Wheeden, R. L. Fractional integrals on weighted H^p and L^p spaces, *Trans. Amer. Math. Soc.* **287**(1985), 293-321.

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The Relation Between the Porous Medium and the Eikonal Equations in Several Space Dimensions

P. L. Lions, P. E. Souganidis and J. L. Vázquez

Abstract

We study the relation between the porous medium equation $u_t = \Delta(u^m)$, $m > 1$, and the eikonal equation $v_t = |Dv|^2$. Under quite general assumptions, we prove that the pressure and the interface of the solution of the Cauchy problem for the porous medium equation converge as $m \downarrow 1$ to the viscosity solution and the interface of the Cauchy problem for the eikonal equation. We also address the same questions for the case of the Dirichlet boundary value problem.

Introduction

In this paper we investigate the relation between the porous medium equation

$$u_t = \Delta(u^m), \qquad m > 1, \tag{0.1}$$

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and the eikonal equation

$$v_t = |Dv|^2, \tag{0.2}$$

for appropriate initial and boundary data. Here $Dv = (v_{x_1}, \ldots, v_{x_N})$ denote the spatial gradient of v and Δ is the Laplace operator. The connection be tween these equations is made apparent when we perform the change o variables

$$v = \frac{m}{m-1} u^{m-1} \tag{0.3}$$

which transforms (0.1) into the «pressure» equation

$$v_t = (m-1) v \, \Delta v + |Dv|^2. \tag{0.4}$$

Letting now $m \downarrow 1$ we formally obtain (0.2).

Equation (0.1) arises naturally as a mathematical model in several areas o applications (e.g. percolation of gas through porous media [33], radiative hea transfer in ionized plasmas [34], thin liquid films spreading under gravity [12] crowd-avoiding population spreading [26], etc.). Equation (0.2), which is special case of a Hamilton-Jacobi equation, is of main interest in optimal cor trol theory [29], the theory of geometrical optics [29], where it describes th propagation of wave fronts [23], etc.

As far as mathematical properties are concerned, (0.1) exhibits bot parabolic and hyperbolic behavior. In particular, at all points where $u > 0$, is smooth. Moreover, it is known ([9], [10]), that the solution u of (0.1 depends continuously in appropriate norms on both the initial data and on m Thus, as $m \downarrow 1$, (0.1) can be regarded as a perturbation of the heat equatior The hyperbolic behavior of (0.1) is manifested by the existence of a finit speed of propagation and the development of interfaces. (For a detaile discussion of the above as well as a complete list of references, see [37]). O the other hand, (0.2) is a hyperbolic equation with only locally defined smoot solutions but with globally defined weak solutions, namely the viscosity solu tions [17]. A common method for approximating viscosity solutions is th method of artificial viscosity [19], [35]. This method, however, does not giv any information whatsoever about the interface of the hyperbolic problem. I order to control the interface one needs to use approximations which exhib interface. In this context a natural question is whether (0.4) can be regarde as a degenerate viscosity (or diffusion) approximation to (0.2).

In [5] D. G. Aronson and J. L. Vázquez explored the convergence, as $m \downarrow 1$ of the solutions of (0.4) to (0.2) in the case of the Cauchy problem in on space dimension. They proved that not only the solutions but also the inter faces of the solutions of (0.4) converge to the solution and the interfac respectively of (0.2), if the initial data are continuous, nonnegative and converge locally uniformly. The proofs rely on estimates that are very particular to the one-dimensional setting.

In this paper we consider the convergence in N space dimensions with general initial data both for the Cauchy problem in $\mathbb{R}^N$ and for the Dirichlet problem in a bounded domain $O \subset \mathbb{R}^N$. For the Cauchy problem we prove that solutions of (0.4) converge to the unique viscosity solutions of (0.2) (Section 1, Theorem 1). Moreover, we show that the positivity sets of solutions of (0.4) converge (in the sense of sets) to the positivity sets of solutions of (0.2) (Section 2, Theorem 2). The main point for the convergence of the solutions is a new type estimates for the gradient. Gradient estimates are easy to obtain in the case of the one space dimension but not obvious at all in higher dimensions (cf. [1], [14], etc.). For the interfaces we also need some new information. This follows from an important result of L. Caffarelli and A. Friedman [13]. In the case of the Dirichlet problem we investigate the convergence of the solutions. We show that the limit takes on natural boundary conditions, thus giving rise to a boundary layer (Section 3, Theorem 3). Finally, the Appendix is a short survey on (0.2). We examine the existence and uniqueness of viscosity solutions under optimal initial conditions as well as some of their properties (e.g. regularity, growth at infinity, interfaces etc.).

1. The Cauchy Problem

Let us consider the following two problems

$$\text{(1.1)} \qquad \begin{cases} v_{mt} = (m-1)v_m \Delta v_m + |Dv_m|^2 & \text{in } \mathbb{R}^N \times (0, T_m) \\ v_m = v_{m0} & \text{on } \mathbb{R}^N \times \{t = 0\} \end{cases}$$

and

$$\text{(1.2)} \qquad \begin{cases} v_t = |Dv|^2 & \text{in } \mathbb{R}^N \times (0, T) \\ v = v_0 & \text{on } \mathbb{R}^N \times \{t = 0\} \end{cases}$$

with nonnegative initial data v_{m0}, $v_0 \in C(\mathbb{R}^N)$. Here T_m and T denote the maximal time of existence for equations (1.1) and (1.2) respectively.

Problem (1.2) has a unique viscosity solution defined in a time interval $(0, T)$ if the initial data satisfy a quadratic growth condition of the form

$$\text{(1.3)} \qquad v_0(x) \leqslant a|x|^2 + b$$

with $a, b \geqslant 0$. Moreover if

$$\text{(1.4)} \qquad \alpha = \limsup_{|x| \to \infty} \frac{v_0(x)}{|x|^2},$$

we have

$$T = 1/4\alpha, \tag{1.5}$$

so that a global solution exists if and only if $\alpha = 0$. On the other han growth condition like (1.3) on v_{m0} ensures the existence of a unique c tinuous weak solution of problem (1.1) ([11], [21]) for a time

$$T_m \geqslant \frac{1}{2[N(m-1)+2]\alpha}. \tag{1.6}$$

Again v_m is global in time if and only if $\alpha = 0$. Observe that $\liminf_{m\to 1} T_m \geqslant$

Our first Theorem states the convergence of solutions of (1.1) to the o of (1.2) as $m \downarrow 1$.

Theorem 1. *Assume that for m close to* 1 *we are given nonnegative ini data $v_{m0} \in C(\mathbb{R}^N)$ satisfying* (1.3) *uniformly in m and such that as m $v_{m0} \to v_0$ locally uniformly in $\mathbb{R}^N$. Let v_m and v be the solutions to probl* (1.1) *and* (1.2). *Then $v_m \to v$ as $m \to 1$ locally uniformly in $\mathbb{R}^N \times [0, T)$.*

The proof of this result relies on obtaining gradient estimates in the c where the solutions are uniformly bounded from below away from zero a series of delicate approximations which use the uniqueness and continu dependence on the initial data of the solutions to problems (1.1) and (1.2). begin with the gradient estimate. We state the result in a generality that be also useful in Section 3, when dealing with problems in bounded domai The proof is based on a variation of Bernstein's trick ([22], [24], [29]).

Lemma 1.1. *For $m > 1$ let v_m be a smooth solution of the equation*

$$(v_m)_t = (m-1)v_m \Delta v_m + |Dv_m|^2$$

in O, where O is an open subset of $\mathbb{R}^N \times (0, T]$. Assume that

$$\beta \geqslant \sup_O v_m \geqslant \inf_O v_m \geqslant \gamma > 0 \tag{1.7}$$

with β and γ independent of m. Then for every compact subset K of O for $m-1$ sufficiently small depending on K, β and γ, there exists a const $C = C(K, \beta, \gamma)$ such that

$$|Dv_m| \leqslant C \quad in \quad K. \tag{1.8}$$

If Dv_{m0} is locally bounded, the above estimate holds down to $t = 0$, i.e.

PROOF. Let ζ be a cut-off function supported in O such that: $0 \leqslant \zeta \leqslant 1$ and $\zeta \equiv 1$ on K. We consider the function

$$Z = \zeta^2 |Dv|^2 + \lambda v \tag{1.9}$$

where λ is a constant to be chosen later. Here for notational simplicity we have dropped the subscript m from v_m. If Z has a maximum at some point (x_0, t_0) such that $\zeta(x_0, t_0) > 0$, then at (x_0, t_0) we have

$$\begin{aligned} Z_t &= 2\zeta\zeta_t |Dv|^2 + 2\zeta^2 v_{x_i} v_{x_i t} + \lambda v_t \geqslant 0, \\ Z_{x_k} &= 2\zeta\zeta_{x_k} |Dv|^2 + 2\zeta^2 v_{x_i} v_{x_i x_k} + \lambda v_{x_k} = 0, \qquad & k = 1, \ldots, N, \\ Z_{x_k x_k} &= (2\zeta_{x_k}^2 + 2\zeta\zeta_{x_k x_k})|Dv|^2 + 4\zeta\zeta_{x_k} v_{x_i} v_{x_i x_k} \\ &\quad + 2\zeta^2 (v_{x_i x_k})^2 + 2\zeta^2 v_{x_i} v_{x_i x_k x_k} + \lambda v_{x_k x_k} \leqslant 0, \qquad & k = 1, \ldots, N, \end{aligned}$$

and

$$0 \leqslant Z_t - (m-1) v \Delta Z - 2Dv \cdot DZ.$$

Substituting in the last inequality and using the equation we obtain

$$\begin{aligned} 0 \leqslant\ & 2(m-1)\zeta^2 |Dv|^2 \Delta v - \lambda |Dv|^2 - 2(m-1)v(|D\zeta|^2 + \zeta\Delta\zeta)|Dv|^2 \\ & + 2\zeta\zeta_t |Dv|^2 - 2(m-1)v\zeta^2 v_{x_i x_k}^2 - 4\zeta\zeta_{x_k} v_{x_k} |Dv|^2 \\ & - 8(m-1)\zeta\zeta_{x_k} v v_{x_i} v_{x_i x_k}. \end{aligned}$$

Applying the Cauchy-Schwartz inequality together with the elementary inequality

$$(\Delta v)^2 \leqslant N \sum_{i,j=1}^{N} \left[\frac{\partial^2 v}{\partial x_i \, \partial x_j} \right]^2$$

we get

$$\lambda |Dv|^2 \leqslant C\zeta |Dv|^3 + C|Dv|^2 + (m-1)\zeta^2 \left(2|Dv|^2 \Delta v - \frac{\gamma}{N} (\Delta v)^2 \right)$$

where γ is from (1.7) and C stands for a constant which depends only on $\|D\zeta\|_\infty$, $\|\Delta\zeta\|_\infty$, $\|\zeta_t\|_\infty$ and β from (1.7), and may change from line to line. The last inequality can be transformed into

$$\lambda |Dv|^2 \leqslant C\zeta |Dv|^3 + C|Dv|^2 + (m-1)\frac{N}{\gamma} \zeta^2 |Dv|^4$$

with all the functions evaluated at (x_0, t_0). Let

$$\lambda = \mu \left[\max_O (\zeta^2 |Dv|^2) + 1 \right] \tag{1.10}$$

where $\mu > 0$ is to be chosen. Substituing in the above inequality we obt

$$\mu\Big[\max_O \zeta^2|Dv|^2 + 1\Big]|Dv|^2 \leqslant C\Big[\frac{1}{\gamma}(m-1)\Big(\max_O \zeta^2|Dv|^2+1\Big)^{1/2}+1\Big]$$
$$\cdot\Big(\max_O \zeta^2|Dv|^2+1\Big)^{1/2}|Dv|^2.$$

Now, if $|Dv|^2(x_0, t_0) \neq 0$, then

$$\Big(\mu - \frac{(m-1)C}{\gamma}\Big)(\max \zeta^2|Dv|^2+1)^{1/2} \leqslant C,$$

so that, if $\mu > (m-1)C/\gamma$, we have

$$|Dv|^2 \leqslant \left(\frac{C}{\mu - \dfrac{(m-1)C}{\gamma}}\right)^2 \quad \text{for every} \quad (x,t) \in K. \tag{1.11}$$

On the other hand, if $|Dv|^2(x_0, t_0) = 0$, then by the definitions of Z and $(x_0$
we have

$$\zeta^2(\bar{x}, \bar{t})|Dv|^2(\bar{x}, \bar{t}) \leqslant \lambda\beta \tag{1.12}$$

where

$$\zeta^2(\bar{x}, \bar{t})|Dv|^2(\bar{x}, \bar{t}) = \max_O \zeta^2|Dv|^2.$$

Using (1.10) and (1.12) we get

$$|Dv|^2(x, t) \leqslant \frac{\beta\mu}{1-\mu\beta} \quad \text{for every} \quad (x, t) \in K, \tag{1.13}$$

provided that $1 - \mu\beta > 0$. Choosing μ such that $\mu\beta = 1/2$, then for $m - 1$
ficiently small we have $\mu > (m-1)C/\gamma$, therefore the result follows in the
where O is a subset of $\mathbb{R}^N \times (0, T)$. If O intersects the set $\mathbb{R}^N \times \{0\}$ the m
mum of Z may take place at $t_0 = 0$. In that case we obtain a local bound
$|Dv|$ depending only on β, λ and the sup of $|Dv_{m0}|$ on $K \cap (\mathbb{R}^N \times \{0\})$.

PROOF OF THEOREM.

Step 1. We assume that $0 < \gamma \leqslant v_{m0}(x)$ with γ independent of m. It t
follows from known properties of the porous medium equation [1], [11] t
for every m, $v_m(x, t) \geqslant \gamma$, $v_m \in C^\infty(\mathbb{R}^N \times (0, T_m))$ and the v_m's are loc
bounded in $\mathbb{R}^N \times [0, T_m)$ uniformly in m. Therefore we can apply Lemma
on any compact subset K of $\mathbb{R}^N \times (0, T)$ and obtain a bound for $|Dv_m|$ o
that is uniform in m for m sufficiently close to 1. By [25] it follows that

v_m's are also locally Hölder-continuous in t with exponent $1/2$ and coefficient independent of m if m is again sufficiently close to 1. The family $\{v_m\}_{m>1}$ is therefore relatively compact in $C(K)$. By a standard diagonal process we may extract a subsequence from every sequence $m_n \to 1$, which we again denote by m_n for simplicity, such that the v_{m_n}'s converge locally uniformly in $\mathbb{R}^N \times (0, T)$ to a function $v \in C(\mathbb{R}^N \times (0, T))$, which is locally Lipschitz continuous in x, Hölder continuous with exponent $1/2$ in t and a viscosity solution of (0.2) (cf. [15], [17]).

If, moreover, $Dv_{m0} \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ uniformly in m, then the gradient estimates hold in compact subsets of $\mathbb{R}^N \times [0, T_m)$ and the same argument implies that the convergence $v_{m_n} \to v$ holds locally uniformly in $\mathbb{R}^N \times [0, T)$. Since $v_{m0} \to v_0$ locally in $\mathbb{R}^N$ we conclude that $v \in C(\mathbb{R}^N \times [0, T))$ takes on the initial value v_0. Therefore, in view of Theorem A.1 of the Appendix, v is the unique viscosity solution of problem (1.2) and the whole family $\{v_m\}_{m>1}$ converges to v.

To prove that v is continuous down to $t = 0$ and $v(x, 0) = v_0(x)$ for $x \in \mathbb{R}^N$ in the case where we do not have a control on $|Dv_{m0}|$ we proceed by approximation. Indeed, we approximate v_{m0} by sequences $\{v^n_{m0}\}$, $\{v_{m0,n}\}$ such that:

(i) the functions v^n_{m0} and $v_{m0,n}$ are smooth in $\mathbb{R}^N$, and for fixed n the gradients are locally bounded in $\mathbb{R}^N$ uniformly in m.

(ii) for each fixed m we have the monotone convergence $v^n_{m0} \downarrow v_{m0}$ and $v_{m0,n} \uparrow v_{m0}$ uniformly in m and $x \in \mathbb{R}^N$.

(Such approximations can be easily obtained by partition of unity and convolution with a smooth kernel).

We conclude as follows: For each fixed n, v^n_{m0} and $v_{m0,n}$ converge along subsequences to some functions v^n_0 and $v_{0,n}$ respectively, which have gradients locally bounded in $\mathbb{R}^N$ and converge, as $n \to \infty$, locally uniformly to v_0. The ordering properties of the porous medium equation imply that $v^n_m \geqslant v_m \geqslant v_{m,n}$ in $\mathbb{R}^N \times [0, T_m)$ where v^n_m and $v_{m,n}$ are the solutions of problem (1.1) in $\mathbb{R}^N \times [0, T_m)$ with initial data v^n_{m0} and $v_{m0,n}$ respectively. The argument above then implies that for each fixed n, as $m \downarrow 1$, $v^n_m \to v^n$ and $v_{m,n} \to v_n$ locally uniformly in $\mathbb{R}^N \times [0, T)$ where v^n, v_n are the unique viscosity solutions of problem (1.2) in $\mathbb{R}^N \times [0, T]$ with initial data v^n_0 and $v_{0,n}$ respectively. Moreover,

$$v^n \geqslant \overline{\lim_{m\downarrow 1}}\, v_m \geqslant \underline{\lim_{m\downarrow 1}}\, v_m \geqslant v_n \quad \text{in} \quad \mathbb{R}^N \times [0, T).$$

Letting $n \to \infty$ and using the uniqueness result of Theorem A.1 we obtain

$$\lim_{n\to\infty} v^n = \lim_{n\to\infty} v_n = v$$

where $v \in C(\mathbb{R}^N \times [0, T])$ is the unique viscosity solution of (1.2) in $\mathbb{R}^N \times [0, T)$. The result follows.

Step 2. The general case. Let

$$v_{m0}^n = v_{m0} + 1/n.$$

If v_m^n is the solution of (0.1) in $\mathbb{R}^N \times [0, T_m)$, then the maximum princi yields $v_m \leqslant v_m^n$ in $\mathbb{R}^N \times [0, T)$. Using Step 1 and Theorem A.1 we get

$$\overline{\lim_{m \to 1}}\, v_m \leqslant v$$

uniformly on compact subsets of $\mathbb{R}^N \times [0, T)$. To conclude, we need to estab the inequality

$$\text{(1.14)} \qquad \underline{\lim_{m \to 1}}\, v_m \geqslant v \quad \text{locally uniformly in} \quad \mathbb{R}^N \times [0, T).$$

We first prove (1.14) in the case where the v_{m0}'s satisfy the inequalities

$$0 \leqslant v_{m0} \leqslant C \quad \text{and} \quad \Delta v_{m0} \geqslant -C \quad \text{in} \quad \mathbb{R}^N$$

where C is a constant independent of m.

Let v_m^n be the solution of (1.1) with initial data $v_{m0} + 1/n$. Tl $v_m^n \in C^\infty(\mathbb{R}^N \times [0, T_m))$, $0 < v_m^n \leqslant C + 1/n$ and $\Delta v_m^n \geqslant -C$ in $\mathbb{R}^N \times [0, 7$ Using (1.1) we see that the function

$$w_m^n = v_m^n + C\left(C + \frac{1}{n}\right)(m-1)t$$

is a smooth solution of

$$\begin{cases} w_{mt}^n \geqslant |Dw_m^n|^2 & \text{in} \quad \mathbb{R}^N \times [0, T_m) \\ w_m^n \geqslant v_{m0} & \text{on} \quad \mathbb{R}^N \times \{t = 0\}. \end{cases}$$

It then follows that $w_m^n \geqslant V_m$, the solution of (1.2) with initial data $v_{m($ $\mathbb{R}^N \times [0, T_m)$. Now we let $n \to \infty$. The continuous dependence of the soluti of the porous medium equation on the initial data ([11]) yields

$$v_m(x, t) + C^2(m-1)t \geqslant v(x, t).$$

Letting $m \downarrow 1$ and using Proposition A.10 we obtain (1.14).

Next we prove (1.14) under only the assumption that v_0 is bounded. $\delta > 0$. We can find functions $\tilde{v}_0, \tilde{v}_{m0}$ bounded in $W^{2,\infty}(\mathbb{R}^N)$ uniformly in m : such that $\tilde{v}_0, \tilde{v}_{m0} \geqslant 0$, $\tilde{v}_{m0} \leqslant v_{m0}$, $\tilde{v}_{m0} \geqslant \tilde{v}_0$ in $\mathbb{R}^N$ and $\tilde{v}_0(x) \geqslant v_0(x) - \delta$ in ! Then

$$\underline{\lim_{m \downarrow 1}}\, v_m \geqslant \underline{\lim_{m \downarrow 1}}\, \tilde{v}_m \geqslant \tilde{v} \quad \text{locally uniformly in} \quad \mathbb{R}^N \times [0, T).$$

Since $v - \delta$ is a solution of (1.2) with data $v_0 - \delta \leqslant \tilde{v}_0$ we have $v - \delta \leqslant \tilde{v}$. Letting $\delta \to 0$ we conclude (1.14).

For the general unbounded case we truncate the initial data at height n. If v_m^n and v^n are the solutions of (1.1) and (1.2) with the truncated initial data, the above and the maximum principle yield

$$\underline{\lim}_{m \to 1} v_m \geqslant \underline{\lim}_{m \to 1} v_m^n \geqslant v^n \quad \text{locally uniformly in } \mathbb{R}^N \times [0, T).$$

Letting $n \to \infty$ we obtain $v^n \to v$. The result follows. □

We continue with a remark concerning Lemma 1.1. In fact gradient estimates can be obtained in a similar way for general classes of equations like for instance

$$(1.15) \qquad u_t^\epsilon - \epsilon F(x, t, u^\epsilon, Du^\epsilon, D^2 u^\epsilon) + H(x, t, u^\epsilon, Du^\epsilon) = 0$$

under suitable assumptions on F and H and provided that the family of smooth solutions $\{u^\epsilon\}_{\epsilon > 0}$ is locally bounded from above and below away from zero uniformly in ϵ. Such bounds allow to pass to the limit $\epsilon \to 0$ and thus obtain viscosity solutions of the limit problem

$$(1.16) \qquad u_t + H(x, t, u, Du) = 0.$$

General equations of the form (1.15) have a certain usefulness. For instance, in some numerical codes the approximation of shocks is improved with the addition of some nonlinear artificial viscosity (so called numerical viscosity, cf. [32]). The assumptions that one has to make on F and H are rather cumbersome although quite general. We leave it to the reader to fill in the details in particular applications.

Our next remark deals with an alternative and simpler proof of the gradient estimate of Lemma 1.1. Though it needs stronger assumpions on the initial data, it can be of interest for some applications.

Lemma 1.2. *Assume that for every $m > 1$ the continuous functions v_{m0} satisfy $0 < \gamma \leqslant v_{m0} \leqslant \beta$ and $|Dv_{m0}| \leqslant M_0$ where β, γ and M_0 are positive constants. Then there exists a bound for $|Dv_m|$ of the form*

$$(1.17) \qquad |Dv_m(x, t)|^2 \leqslant \frac{M_0}{1 - (t/T_m)} \quad \textit{for} \quad 0 \leqslant t < T_m$$

where

$$T = \frac{2\gamma}{}.$$

PROOF. Let $w_m = |Dv_m|^2$. Using equation (1.1) we obtain

$$w_t - (m-1)v\,\Delta w - 2Dv\cdot Dw = 2(m-1)w\,\Delta v - 2(m-1)v\sum_{i,j=1}^{N}\left(\frac{\partial^2 v}{\partial x_i\,\partial x_j}\right.$$

$$\leqslant 2(m-1)w\,\Delta v - \frac{2(m-1)\gamma}{N}(\Delta v)^2$$
$$\leqslant \frac{(m-1)N}{2\gamma}w^2,$$

where we have dropped the m's for simplicity and have used the inequa

$$(\Delta v)^2 \leqslant N\sum_{i,j=1}^{N}\left(\frac{\partial^2 v}{\partial x_i\,\partial x_j}\right)^2.$$

We compare w_m to the explicit solution

$$W_m(t) = \frac{M_0 C}{C-t}, \qquad 0 < t < C = \frac{2\gamma}{M_0 N(m-1)}$$

of the problem

$$\begin{cases} W' = \dfrac{N(m-1)}{2\gamma}W^2 \\ W(0) = M_0 \end{cases}$$

The maximum principle implies that $w_m \leqslant W_m$ in $\mathbb{R}^N \times [0, C)$. If γ, β and do not depend on m, then $C\uparrow\infty$ as $m\downarrow 1$. Thus we obtain a bound for $|D$ on bounded time intervals which is uniform in m for m close to 1. □

We conclude with a further remark about a sharper gradient estimate solutions of (1.1) under the assumption that $0 < \gamma \leqslant v_{m0} \leqslant \beta$. Indeed a resul Ph. Bénilan [8] implies that if u_m is a solution of (0.1) and $m \leqslant 1 + (N-1)$ then there exists a gradient bound of the form

(1.18) $$|D(u_m)^{m-1/2}| \leqslant Ct^{-1/2}$$

where C depends on β but not on m. If $v_{m0} \geqslant \gamma > 0$, we conclude that

$$|Dv_m|^2 \leqslant \tilde{c}_1 t^{-1}$$

where $\tilde{c}_1$ depends on β and γ.

2. Convergence of Interfaces

In this section we prove that under the assumptions of Theorem 1 the interf of the solution of problem (1.1) converges to the interface of the solution of (1.2) as $m \downarrow 1$. The interfaces are described in terms of the functions

$$S_m(x) = \inf\{t \geqslant 0 : v_m(x,t) > 0\}, \tag{2.1}$$

and

$$S(x) = \inf\{t \geqslant 0 : v(x,t) > 0\}, \tag{2.2}$$

The so-called retention property implies that $v_m(x,t) > 0$ for every $t > S_m(x)$. On the other hand it is proved in [12] that, if v_{m0} has compact support, the function S_m is Hölder continuous in the open set

$$A_m = \mathbb{R}^N \setminus \bar{\Omega}_m(0), \tag{2.3}$$

where for $m > 1$ and $t \geqslant 0$, $\Omega_m(t) = \{x \in \mathbb{R}^N : v_m(x,t) > 0\}$. The above restriction is essential in view of the fact that S_m is discontinuous at points of the boundary of $\Omega_m(0)$ whenever a positive waiting time occurs. This phenomenon may appear even in one space dimension, (cf. [3], [4]). The restriction of compact support in [13] is inessential. The interface to problem (1.2) has similar properties as we show in the Appendix.

In view of these observations we prove the convergence of S_m to S away from the initial sets $\Omega_m(0)$. More precisely, let

$$A = \mathbb{R}^N \Big/ \bigcap_{\epsilon > 0} \text{closure}\Big(\bigcup_{1 < m < 1+\epsilon} \Omega_m(0) \Big) \tag{2.4}$$

The set A consists of points $x \in \mathbb{R}^N$ such that for some $\epsilon > 0$ and $r > 0$ and all $1 < m < 1 + \epsilon$, v_{m0} vanishes dentically on $B(x,r)$, the open ball centered at x with radius r.

Theorem 2. *S_m converges to S as $m \downarrow 1$ uniformly on compact subsets of A.*

PROOF. Let K be a compact subset of A and suppose that S_m does not converge uniformly to S on K. Then there exist $\epsilon > 0$, $x_n \in K$ and $m_n \downarrow 1$ such that

$$|S_{m_n}(x_n) - S(x_n)| \geqslant \epsilon. \tag{2.5}$$

Suppose first (upon passing to a subsequence if necessary) that we have

$$S_{m_n}(x_n) \geqslant S(x_n) + \epsilon. \tag{2.6.a}$$

A contradiction follows then easily from the uniform convergence of v_m to v on K and the continuity of S (see Appendix). In fact the definition of S_m (2.6.a) implies

$$v_{m_n}(x_n, S(x_n) + \epsilon) = 0,$$

so that for any limit point $\bar{x}$ of $\{x_n\}$ we have

$$v(\bar{x}, S(\bar{x}) + \epsilon) = 0,$$

against the definition of S.

Let us now discuss the case where, as $m_n \downarrow 1$,

$$S_{m_n}(x_n) \leqslant S(x_n) - \epsilon. \tag{2.6.b}$$

This case is significantly more difficult. To exclude it, we need to use a precis information about the growth of solutions and interfaces of (1.1) based on th results of [13]. This information is summarized in the following lemmata.

Lemma 2.1. *Let K be a compact set where $v_{m0} = 0$, $1 < m < 1 + \epsilon \leqslant 2$. The for every compact set $K' \subset K$ there exists a $\tau > 0$ depending only on K, K' bu not on m or ϵ, such that for every $m \in (1, 1 + \epsilon)$*

$$v_m \equiv 0 \quad on \quad K' \times [0, \tau].$$

Lemma 2.2. *Let $0 < \tau < t_0$, $x_0 \in \mathbb{R}^N$ and $R_0 > 0$. There exist $\delta = \delta(\tau, R_0) >$ and $R = R(\tau, t_0, R_0) > 0$ such that whenever $v_m(\bullet, \tau) \equiv 0$ on $\overline{B(x_0, R_0)}$ an $v_m \leqslant \delta$ on $\overline{B(x_0, R_0)} \times [\tau, t_0]$, then $v_m(\bullet, t_0) \equiv 0$ on $\overline{B(x_0, R)}$.*

We postpone the proof of the lemmata to the end of the section and cor tinue with the proof of Theorem 2. Let $\bar{x}$ be a limit point of $\{x_n\}$. The propei ties of S (cf. Appendix) imply that there exists $t_0 \geqslant S(\bar{x}) - \epsilon/2$ and $r > 0$ suc that $v(\bullet, t_0) \equiv 0$ on $B(\bar{x}, r)$ and, since $v_t \geqslant 0$ by (1.2),

$$v \equiv 0 \quad \text{on} \quad \overline{B(\bar{x}, r)} \times [0, t_0].$$

On the other hand, Lemma 2.1 yields the existence of a $\tau = \tau(\bar{x}, r) < t_0$ suc that, for m_n sufficiently close to 1,

$$v_{m_n} \equiv 0 \quad \text{on} \quad B\left(\bar{x}, \frac{r}{2}\right) \times \{\tau\}.$$

Finally, in view of Theorem 1 and the above there exists n_0 such that for $n \geqslant r$

$$v_{m_n} \leqslant \delta \quad \text{on} \quad \overline{B(\bar{x}, r)} \times [0, t_0],$$

where $\delta = \delta(\tau, r)$ is given by Lemma 2.2. Since all the assumptions of Lemm 2.2 are satisfied, we obtain

$$v_{m_n}(\bullet, t_0) \equiv 0 \quad \text{on} \quad \overline{B(x, r')},$$

where $r' = r'(\tau, t_0, r)$. This contradicts (2.6.b). $\square$

We continue with a discussion of the convergence in the complement of the set A. Firstly, for every $\bar{x} \in \Omega(0)$ we have $v_0(x) > 0$ and $v_{m0}(x) > 0$ for m near 1 and x near $\bar{x}$. Therefore $S(x) = S_m(x) = 0$; i.e., $S_m \to S$ locally uniformly in $\Omega(0)$. In the case where $\Omega_m(0) \subset \bar{\Omega}(0)$ for all m near 1, then $A \supset \mathbb{R}^N \setminus \overline{\Omega(0)}$. So the only place where the convergence may fail is the boundary of $\Omega(0)$. It is easy to construct examples with waiting times where this happens. Finally, we cannot expect convergence on the set

$$B = \limsup_{m \downarrow 1} \Omega_m(0) \setminus \overline{\Omega(0)}.$$

In fact for each $x \in B$ there exists a subsequence $m_n \downarrow 1$ such that $v_{m_n 0}(x) > 0$ and $S_{m_n}(x) = 0$. However, $S(x) > 0$. In particular, it may happen that $\Omega_m(0) = \mathbb{R}^N$ for every $m > 1$ so that $B = \mathbb{R}^N \setminus \overline{\Omega(0)}$ and the only convergence that we get is the trivial convergence on $\Omega(0)$.

We next formulate the convergence of the interfaces in terms of the positivity sets $\Omega_m(t)$ and $\Omega(t)$. Since the proof of this result is only a variation of the proof of Theorem 2 we leave it up to the reader to fill in the details.

Theorem 2′. *Under the assumptions of Theorem* 1 *we have*

(i) $\liminf_{m \downarrow 1} \Omega_m(t) \supset \Omega(t)$

(ii) $\limsup_{m \downarrow 1} \Omega_m(t) \subset \bar{\Omega}(t) \cup (\mathbb{R}^N \setminus A)$.

As explained above an inclusion of the type

$$\limsup_{m \downarrow 1} \Omega_m(t) \subset \bar{\Omega}(t)$$

cannot be true in general. It may happen e.g. that $\Omega_m(t) = \mathbb{R}^N$ for every $m > 1$, $t > 0$, while $\Omega(t)$ is bounded.

We conclude with the proof of the lemmata.

PROOF OF LEMMA 2.1. Without any loss of generality we may assume that $K = \overline{B(0, R_1)}$ and $K' = \overline{B(0, R)}$ with $R < R_1$.

We proceed by constructing a barrier function $V: B(0, R_1) \times [0, \tau] \to \mathbb{R}$ for an appropriate choice of τ. It is given by the formula

$$V(r, t) = \lambda[a^2 t + a(r - R - \theta)]^+ \tag{2.7}$$

where $r = |x|$, $s^+ = \max(s, 0)$, $R < R + \theta < R_1$ and $a, \lambda > 0$. We choose a, λ

(i) Whenever $V > 0$, V satisfies

$$V_t \geqslant (m-1)V\Delta V + |DV|^2 \tag{2.8}$$

(ii) $V(r, \tau) = 0$ for $r \leqslant R$.
(iii) $V(R_1, t) \geqslant A$ for $t \in [0, \tau]$, where A is the L^∞-bound of v_m o $K \times [0, T-\epsilon]$, which in view of the proof of Theorem 1, is indepe dent of m.

If all the above are satisfied, setting

$$U = \left(\frac{m-1}{m} V\right)^{1/(m-1)}, \qquad u = \left(\frac{m-1}{m} v_m\right)^{1/(m-1)}$$

we have $U_t \geqslant \Delta U^m$, $u_t = \Delta u^m$ in $B(0, R_1) \times (0, \tau]$ and $U \geqslant u$ on the parabol boundary of $B(0, R_1) \times (0, \tau]$. By the standard comparison principle for tl porous medium equation, it follows that $U \geqslant u$ throughout $B(0, R_1) \times [0, \tau$ hence, in particular, $v(x, \tau) \leqslant V(|x|, \tau) = 0$ if $|x| < R$ and thus the result.

We conclude by establishing (i), (ii) and (iii) above. We begin by observir that $V > 0$ if and only if

$$r \geqslant R + \theta - at. \tag{2.9}$$

To satisfy (ii) it suffices to have

$$a\tau \leqslant \theta \tag{2.10}$$

For (iii) we need

$$\lambda[a^2 t + a(R_1 - (R + \theta))] \geqslant A$$

which requires

$$\lambda a \geqslant \frac{A}{R_1 - (R + \theta)}. \tag{2.11}$$

Finally, V satisfies (2.8), whenever $V > 0$, if and only if

$$\lambda\left[(m-1)(N-1)\frac{at + r - R - \theta}{r} + 1\right] \leqslant 1.$$

For the latter to be satisfied, in view of (2.9) and (2.10), it suffices to ha

$$\lambda \leqslant \frac{1}{1 + N(1 - (R/R_1))}. \tag{2.12}$$

To conclude we choose λ so that the inequality holds in (2.12). Then for a su ficiently large and τ sufficiently small (2.10) and (2.11) can be achieved. |

For the proof of Lemma 2.2 we need the following result.

Lemma 2.3 [13]. *For any $\tau > 0$ and $m > 1$ there exist positive constants η, c depending only on m, N and τ such that the following is true: Let $t_0 > \tau$, $R > 0$, $0 < \sigma < \eta$. If*

$$v_m(\cdot, t_0) \equiv 0 \quad on \quad B(x_0; R) \qquad x_0 \in \mathbb{R}^N \tag{2.13}$$

and

$$\fint_{B(x_0, R)} v_m(x, t_0 + \sigma)\, dx \leqslant \frac{cR^2}{\sigma}, \tag{2.14}$$

then

$$v_m(\cdot, t_0 + \sigma) \equiv 0 \quad on \quad B(x_0, R/6), \tag{2.15}$$

where $\fint_{B(x_0, R)} v_m(x, s)\, dx$ denotes the average of $v_m(\cdot, s)$ over the ball $B(x_0, R)$.

A careful scrutiny of the proof shows that the constants *do not depend* on m in the range $1 < m < 2$.

PROOF OF LEMMA 2.2. Let η, c be the constants which correspond to τ via Lemma 2.3, let M be so large that $\sigma = (t_0 - \tau)/M < \eta$ and let $\delta > 0$ be such that $\delta < c\, 6^{-2(M-1)} R_0^2/\sigma$. For every $i = 1, \ldots, M$, we then have

$$\fint_{B(x_0, 6^{-(i-1)} R_0)} v_m(x_0, \tau + i\sigma)\, dx \leqslant \frac{c}{\sigma} 6^{-2(i-1)} R_0^2.$$

Using Lemma 2.3 and arguing inductively we obtain

$$v_m(\cdot, t_0) \equiv 0 \quad \text{on} \quad \overline{B(x_0, 6^{-M} R_0)}. \quad \square$$

3. The Initial-Boundary Value Problem

Here we focus our attention to the initial-boundary value problems

$$\begin{cases} v_{mt} = (m-1) v_m \Delta v_m + |Dv_m|^2 & \text{in} \quad O \times (0, T] \\ v_m = 0 & \text{on} \quad \partial O \times [0, T] \\ v_m = v_{m0} & \text{on} \quad O \times \{t = 0\} \end{cases} \tag{3.1}$$

and

$$\begin{cases} v_t = |Dv|^2 & \text{in} \quad O \times (0, T] \\ v = 0 & \text{on} \quad \partial O \times [0, T] \\ v = v_0 & \text{on} \quad O \times \{t = 0\}. \end{cases} \tag{3.2}$$

Problem (3.2) does not have a globally defined viscosity solution which is c
tinuous up to the boundary. There exists, however, a *minimal* viscosity sc
tion v (the value function of the underlying control problem) which assui
some natural boundary conditions, not necessarily zero ([27]). We begin
discussing this minimal viscosity solution. To make some of the formu
clearer, we will occasionally refer to their form when O is convex. To this e
for $x, y \in O$ and $t > 0$ we define

$$(3.3)\quad L(x,y,t) = \inf\left\{\int_0^t \frac{1}{4}|\xi_s|^2\,ds\colon \xi(0) = x,\ \xi(t) = y,\ \xi(s) \in \bar{\Omega} \text{ for } s \in [0, t\right.$$

If O is convex, then it is easy to see that

$$(3.4)\qquad L(x, y, t) = \frac{|x-y|^2}{4t}.$$

In order to have a viscosity solution of (3.2) which is continuous up to
boundary, one needs certain compatibility conditions which restrict the c
of allowed initial data and the time of existence [29]. In particular, to h
a viscosity solution $v \in C(\bar{O} \times [0, T])$ of the problem

$$(3.5)\qquad \begin{cases} v_t = |Dv|^2 & \text{in} \quad O \times (0, T] \\ v = \phi & \text{on} \quad \partial O \times (0, T] \\ v = v_0 & \text{on} \quad O \times \{t = 0\} \end{cases}$$

we need to assume

$$(3.6)\quad \begin{cases} \phi(x,t) \geqslant \phi(y,s) - L(x,y,t-s) & \text{for all } x, y \in \partial O,\ t \geqslant s > 0 \\ \phi(x,t) \geqslant v_0(y) - L(x,y,t) & \text{for all } x \in \partial O,\ t > 0 \text{ and } y \in (\end{cases}$$

Next we define

$$(3.7)\qquad v(x,t) = \sup_{y \in \bar{O}} \{v_0(y) - L(x,y,t)\}.$$

Arguments similar to the ones of [29, Chapter 11] yield that v is a visco
solution of $v_t = |Dv|^2$ in $O \times (0, \infty)$. More precisely, (cf. [29]) v is
minimum element of the set of Lipschitz-continuous solutions of

$$(3.8)\qquad \begin{cases} v_t - |Dv|^2 = 0 & \text{in} \quad O \times (0, \infty), \\ v \geqslant 0 & \text{on} \quad \partial O \times (0, \infty), \\ v = v_0 & \text{on} \quad O \times \{t = 0\}. \end{cases}$$

Moreover, on $\partial O \times (0, \infty)$ $v = \tilde{\phi}$, where $\tilde{\phi}$ is the minimum element of the
of functions $\psi \in C([\partial O \times (0, \infty)] \cup [O \times \{t = 0\}])$ satisfying (3.6) and ψ
on $\partial O \times [0, \infty)$.

In the case when O is convex and $\phi = 0$, (3.4) and (3.6) yield

$$v_0(x) \leqslant \frac{1}{4T} \operatorname{dist}(x, \partial O)^2. \tag{3.9}$$

Then v defined by (3.7) satisfies $v = 0$ on $\partial O \times [0, T]$ and it is the unique viscosity solution of (3.2) in $O \times (0, T)$. The maximal time T for which such a solution exists is given by

$$T^* = \inf_{x \in \bar{O}} \frac{\operatorname{dist}(x, \partial O)^2}{4v_0(x)}. \tag{3.10}$$

We remark that this is precisely the waiting time T for the interface of the Cauchy problem in $\mathbb{R}^N$ (cf. Proposition A.15. See also (1.5)). In general we say that v is the minimal viscosity solution to (3.2).

The relation between (3.1) and (3.2) in the interior of $\bar{O} \times [0, T]$ is the same as the one of (1.1) and (1.2). At the boundary, however, *boundary layers* appear. This is due to the fact that although we are forcing Dirichlet data on (3.1), the solution of (3.2) takes on natural boundary values as explained above.

Theorem. 3. *Assume that* $v_{m0} \to v_0$ *uniformly on* $\bar{O}$ *as* $m \downarrow 1$ *and let* v *be the viscosity solution of* (3.2) *in* $O \times [0, \infty)$ *given by* (3.7) *above. Then, as* $m \downarrow 1$, $v_m \to v$ *locally uniformly in* $O \times [0, \infty)$.

PROOF.

Step 1. We begin by assuming that $v_0 > 0$ in O. We may also assume that the v_{m0}'s are Lipschitz continuous with gradients bounded uniformly in m; the general case follows by approximating v_{m0} from above and below by Lipschitz-continuous functions much as in Theorem 1. Let $B(0, R)$ be a ball strictly included in O, and let $\overline{B(0, R_1)} \subset O$ for some $R_1 > R$. Since $v_{m0} \to v_0$ uniformly on $\overline{B(0, R_1)}$ as $m \downarrow 1$ and $v_0 > 0$ in O, there exist $m_0 = m_0(R_1)$ and $\gamma > 0$ such that

$$\min_{\overline{B(0, R_1)}} v_{m0} > \gamma > 0 \quad \text{for} \quad m < m_0.$$

We claim that for every $T > 0$ there exist $m_1 = m_1(T) > 1$ and $\beta = \beta(R, R_1) > 0$ such that for $m < m_1$

$$v_m \geqslant \beta \quad \text{on} \quad \overline{B(0, R)} \times [0, T]. \tag{3.11}$$

Indeed we consider the similarity solutions

$$V_m(x, t; a, \tau) = \frac{1}{2[N(m-1)+2]} \frac{1}{(t+\tau)} (a^2(t+\tau)^{2/(N(m-1)+2)} - |x|^2)^+$$

of $v_t = (m-1)v\Delta v + |Dv|^2$ in $\mathbb{R}^N \times (0,\infty)$, where $a, \tau > 0$. We choose a and τ so that for m near 1 we have

$$\begin{cases} \operatorname{supp} V_m(\cdot, t) \subset B(0, R_1) & \text{for} \quad t \in [0, T] \\ \overline{B(0, R)} \subset \operatorname{supp} V_m(\cdot, 0) \\ V_m \leqslant \gamma \quad \text{on} \quad \overline{B(0, R_1)} \times \{t = 0\}. \end{cases}$$

But then

$$v_m \geqslant V_m \quad \text{on} \quad (\{|x| = R_1\} \times [0, T]) \cup (\{|x| \leqslant R_1\} \times \{t = 0\}),$$

therefore

$$v_m \geqslant V_m \quad \text{on} \quad B(0, R_1) \times [0, T].$$

We conclude by observing that there exists a constant $\beta > 0$, independen m, such that

$$\min V_m \geqslant \frac{1}{2[N(m-1)+2]}\,\frac{1}{T+\tau}\,(a^2\tau^{2/(N(m-1)+2)} - R^2) \geqslant \beta.$$

Using Lemma 1.1 and the results fo [25], we obtain that, along subseque $m \downarrow 1$, $v_m \to \bar{v} \geqslant 0$ locally uniformly in $O \times [0,\infty)$, where $\bar{v}$ is a viscosity solu of (3.8). Since the function v given by (3.7) is the minimal viscosity solu of (3.8), we immediately have $\bar{v} \geqslant v$.

For the other inequality, $v \geqslant \bar{v}$, we use several approximations. To this let O^δ be the δ-neighborhood of O defined by

$$O^\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \bar{O}) \leqslant \delta\}. \tag{3.12}$$

We extend v_{m0}, v_0 to be zero in $O^\delta \setminus O$ and we denote by $\tilde{v}_{m0}$, $\tilde{v}_0$ their ex sions respectively. Let $\underline{v}_m^\delta$ be the minimal viscosity solution of

$$\begin{cases} \underline{v}_{mt}^\delta = |D\underline{v}_m^\delta|^2 & \text{in} \quad O^\delta \times (0,\infty) \\ \underline{v}_m^\delta = \delta & \text{on} \quad \partial O^\delta \times (0,\infty) \\ \underline{v}_m^\delta = \tilde{v}_{m0} + \delta & \text{on} \quad O^\delta \times \{t = 0\}, \end{cases} \tag{3.13}$$

and define the function $w\colon O^{\delta/2} \times [0,\infty) \to \mathbb{R}$

$$w = \underline{v}_m^\delta * \rho_\alpha = (\underline{v}_m^\delta)_\alpha \tag{3.14}$$

where $*$ denotes the standard convolution with a smooth kernel ρ_α. immediate that, for $\alpha < \alpha_0 = \alpha_0(\delta)$, w satisfies

$$\begin{cases} w_t \geqslant (m-1)w\Delta w - (m-1)C_\alpha w & \text{in} \quad O \times (0, T) \\ w \geqslant v_{m0} & \text{on} \quad O \times \{t = 0\} \\ w > 0 & \text{on} \quad \partial O \times (0,\infty), \end{cases}$$

where C_α is such that $\Delta w \leqslant C_\alpha$ on $O \times [0, T]$. Next for $\mu > 0$ we define

$$z = e^{\mu t} w\left(x, \frac{e^{\mu t} - 1}{\mu}\right).$$

A simple calculation shows that, for $\alpha < \alpha_0(\delta)$, z satisfies

$$z_t - (m-1) z \Delta z - |Dz|^2 \geqslant (\mu - (m-1) C_\alpha e^{\mu t}) z.$$

For any $T > 0$ choose m_1 so small that $(m-1) C_\alpha e^T < 1$ for $m < m_1$. Then there exists $\mu = \mu(\alpha, m_1)$ such that $\mu \geqslant (m-1) C_\alpha e^{\mu T}$. Using the standard comparison argument for the porous medium equation we obtain

$$v_m \leqslant e^{\mu t} (\underline{v}_m^\delta)_\alpha \left(x, \frac{e^{\mu t} - 1}{\mu}\right) \quad \text{in} \quad O \times [0, T].$$

Now we let $m \downarrow 1$ keeping μ, α, δ fixed and we get

$$\bar{v} \leqslant e^{\mu t} (\underline{v}^\delta)_\alpha \left(x, \frac{e^{\mu t} - 1}{\mu}\right) \quad \text{in} \quad O \times [0, T], \tag{3.15}$$

where $\underline{v}^\delta$ is the minimal viscosity solution of

$$\begin{cases} \underline{v}_t^\delta = |D\underline{v}^\delta|^2 & \text{in} \quad O^\delta \times [0, \infty) \\ \underline{v}^\delta = \delta & \text{on} \quad \partial O^\delta \times [0, \infty), \\ \underline{v}^\delta = \tilde{v}_0 + \delta & \text{on} \quad O^\delta \times \{t = 0\}. \end{cases}$$

Sending $\mu \to 0$, $\alpha \to 0$ and $\delta \to 0$ we obtain the following sequence of inequalities in $O \times [0, T]$

$$\bar{v} \leqslant (\underline{v}^\delta)_\alpha, \qquad \bar{v} \leqslant \underline{v}^\delta, \qquad \bar{v} \leqslant v.$$

The result follows.

Step 2. We next consider the general case where $v_0 \geqslant 0$ in $\bar{O}$. The main problem here is that, since we cannot bound the v_m's from below away from zero, we are unable to obtain local Lipschitz estimates. To circumvent this difficulty (i.e. the apparent lack of estimates), we will employ some of the recent ideas of H. Ishii [27] and G. Barles and B. Perthame [7]. To this end, we define the functions

$$v_*(x, t) = \liminf_{\substack{m \downarrow 1 \\ (y, s) \to (x, t)}} v_m(y, s)$$

and

$$v^*(x, t) = \limsup_{\substack{m \downarrow 1 \\ (y, s) \to (x, t)}} v_m(y, s).$$

It is known ([7]) that v_* is a lower-semicontinuous viscosity solution of

$$(3.16)\quad \begin{cases} v_{*t} - |Dv_*|^2 \geqslant 0 & \text{in} \quad O \times (0, \infty) \\ \max(v_{*t} - |Dv_*|^2, v_* - \psi) \geqslant 0 & \text{on} \quad [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\} \end{cases}$$

and v^* is an upper semicontinuous viscosity solution of

$$(3.17)\quad \begin{cases} v_t^* - |Dv^*|^2 \leqslant 0 & \text{in} \quad O \times (0, \infty) \\ \min(v_t^* - |Dv^*|^2, v^* - \psi) \leqslant 0 & \text{on} \quad [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\} \end{cases}$$

where $\psi: [\partial O \times (0, \infty)] \cup [\bar{O} \times \{t = 0\}] \to \mathbb{R}$ is given by

$$(3.18)\quad \psi = \begin{cases} 0 & \text{on} \quad \partial O \times (0, \infty) \\ v_0 & \text{on} \quad \bar{O} \times \{t = 0\}. \end{cases}$$

Our goal is to show that

$$v^* = v_* = v \quad \text{in} \quad O \times (0, \infty).$$

Since, by definition, $v_* \leqslant v^*$, we only have to show that

$$(3.19)\quad v \leqslant v_* \quad \text{and} \quad v^* \leqslant v \quad \text{in} \quad O \times (0, \infty).$$

We begin with the right-hand side of (3.19), which is more or less immedi Indeed, let $v_0^n > 0$ be such that v_{m0}, $v_0 \leqslant v_0^n$ and $v_0^n \to v_0$ as $n \to \infty$ unifor on $\bar{O}$. If v_m^n and v^n are the solutions of (3.1) and (3.2) with initial datum then the first part of this proof yields

$$\lim_{m \downarrow 1} v_m^n = v^n \quad \text{uniformly on compact subsets of} \quad O \times (0, \infty).$$

By the maximum principle we have that $v_m \leqslant v_m^n$ on $\bar{O} \times [0, \infty)$. Moreo it follows from the formulae that $v^n \to v$ uniformly on $O \times (0, \infty)$ as $n \to$ Combining all the above we obtain $v^* \leqslant v$ in $O \times (0, \infty)$.

To obtain the left-hand side of (3.19) we have to work a bit harder. begin by regularizing the v_*'s using the *inf-convolutions* introduced by J. Lasry and P.-L. Lions [28]. For $\alpha > 0$, let $O_\alpha = \{x \in O: \text{dist}(x, \partial O) \geqslant \alpha\}$ consider the functions

$$(3.20)\quad v_{*\alpha}(x, t) = \inf_{(y,s) \in \bar{O} \times (0,\infty)} \left\{ v_*(y, s) + \frac{|x - y|^2 + |t - s|^2}{2\alpha} \right\}.$$

It turns out (cf. P.-L. Lions and P. E. Souganidis [31]) that for each $\alpha >$ $v_{*\alpha}$ is a Lipschitz continuous viscosity solution of $w_t - |Dw|^2 \geqslant 0$ in $O_\alpha \times (\alpha,$ and $v_{*\alpha} \downarrow v_*$ as $\alpha \downarrow 0$. Next we consider the minimal viscosity solution of problem

$$\begin{cases} w_t - |Dw|^2 = 0 & \text{in} \quad O_\alpha \times (\alpha, \infty) \\ w = 0 & \text{on} \quad \partial O_\alpha \times (\alpha, \infty) \\ w = v^n_{*\alpha} & \text{on} \quad \bar{O}_\alpha \times \{t = \alpha\}, \end{cases}$$

where $v^n_{*\alpha} \in C(\bar{O}_\alpha)$, $v^n_{*\alpha|\partial O\alpha} = 0$ and $v^n_{*\alpha} \uparrow v_{*\alpha}(\cdot, t)$ as $n \to \infty$. The definition of w (given at the beginning of this section) yields

$$v_{*\alpha} \geqslant w \quad \text{in} \quad O_\alpha \times (\alpha, \infty). \tag{3.21}$$

Let $(x, t) \in O_\alpha \times (\alpha, \infty)$. Since

$$w(x, t) = \sup_{y \in O_\alpha} \{v^n_{*\alpha}(y, \alpha) - L(x, y, t - \alpha)\},$$

the properties of $v_{*\alpha}$ and (3.21) yield

$$v_*(x, t) \geqslant v_{*\alpha}(x, t) \geqslant v^n_{*\alpha}(x, \alpha) - L(x, y, t - \alpha) \quad \text{for all} \quad y \in \bar{O}_\alpha.$$

Upon letting $n \to \infty$ we obtain

$$v_*(x, t) \geqslant v_{*\alpha}(x, t) \geqslant v_{*\alpha}(x, \alpha) - L(x, y, t - \alpha) \quad \text{for all} \quad y \in \bar{O}_\alpha.$$

To conclude, we need to examine the behaviour of $v_{*\alpha}(x, \alpha)$ as $\alpha \downarrow 0$. Since $v_{*t} \geqslant 0$, (3.21) yields

$$v_{*\alpha}(x, \alpha) \geqslant \inf_{(y, s) \in \bar{O} \times [0, \infty)} \left\{ v_*(y, 0) + \frac{|x - y|^2 + |\alpha - s|^2}{\alpha} \right\}.$$

Using the lower semicontinuity of $v_*(\cdot, 0)$ we then see that

$$\varliminf_{\alpha \downarrow 0} v_{*\alpha}(s, \alpha) \geqslant v_*(x, 0).$$

Combining all the above we get

$$v_*(x, t) \geqslant v_*(x, 0) - L(x, y, t) \quad \text{for all} \quad y \in \bar{O}. \tag{3.22}$$

Finally, it follows from (3.16) and the definition of v_* (cf. [7]) tha

$$v_*(\cdot, 0) = v_0 \quad \text{on} \quad \bar{O}.$$

This together with (3.22) yields

$$\begin{aligned} v_*(x, t) &\geqslant \sup_{y \in \bar{O}} \{v_0(y) - L(x, y, t)\} \\ &= v(x, t); \end{aligned}$$

APPENDIX

We consider the questions of existence and uniqueness as well as o qualitative properties of viscosity solutions of $v_t = |Dv|^2$ defined in a s $Q_T = \mathbb{R}^N \times (0, T)$ for some $T > 0$. The uniqueness results we obtain genera the results of [16] and [18], in the sense that they allow more general in data. Some of the properties of the viscosity solutions we are intereste here are growth at infinity, reguralizing effects, domain of dependence, ir face, etc. Several of the results presented have also appeared in similar f in ([6], [16], [18], [29], [30], etc.); for this reason a lot of proofs are ra sketchy.

We recall here for the reader's convenience the definition of *viscosity solu* A continuous function u defined in a domain $\Omega \subset \mathbb{R}^{N+1}$ is called a visc solution of equation $v_t - |Dv|^2 = 0$ if for any function $\varphi \in C^1(\Omega)$ we l $\varphi_t - |D\varphi|^2 \leqslant 0$ at all points $P_0 = (x_0, t_0) \in \Omega$ at which $v - \varphi$ attains a l maximum and $\varphi_t - |D\varphi|^2 \geqslant 0$ where $v - \varphi$ attains a local minimum. We r the interested reader to the references at the end of this paper, especially and [17], for the theory of viscosity solutions.

1. Growth at Infinity and Initial Trace

Proposition A.1. *Let v be a viscosity solution of* (0.2) *defined in Q_T. every (x_1, t_1), $(x_2, t_2) \in Q_T$ with $0 < t_1 < t_2 < T$ we have*

$$v(x_1, t_1) \leqslant v(x_2, t_2) + \frac{|x_1 - x_2|^2}{4(t_2 - t_1)}. \tag{A.1}$$

Therefore, if $t \in (0, T)$, then

$$\limsup_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \leqslant \frac{1}{4(T - t)} \tag{A.2}$$

and

$$\liminf_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \geqslant -\frac{1}{4t}. \tag{A.3}$$

PROOF. We begin assuming that v is bounded below. Let $C, \delta > 0$ define the function $\phi \in C^\infty(\mathbb{R}^N \times [t_1, T))$ by

$$\phi(x, t) = v(x_1, t_1) - \frac{|x - x_1|^2}{4(t - t_1 + \delta)} - C(t + 1).$$

If we fix $C > 0$ and choose δ small enough, it is immediate that $\phi(\cdot, t_1) < v(x, t_1)$ on $\mathbb{R}^N$. We want to prove that $v \geqslant \phi$ in $\Omega = \mathbb{R}^N \times [t_1, t_2]$. In fact if $v - \phi$ attains a minimum in Ω at a point $(\bar{x}, \bar{t})$, then, by the definition of the viscosity solution, we must have $\phi_t - |D\phi|^2 \geqslant 0$ at $(\bar{x}, \bar{t})$. However, $\phi_t - |D\phi|^2 = -C < 0$ in Ω. Therefore the minimum of $v - \phi$ either is attained at $t = t_1$ and then $v > \phi$ in Ω or it is approached as $|x| \to \infty$. But v is bounded from below and $\phi \to -\infty$ as $|x| \to \infty$, therefore the latter cannot happen. Letting first $\delta \downarrow 0$ and then $C \downarrow 0$ we obtain (A.1), from which (A.2) and (A.3) follow easily.

If v is not bounded from below we have to suitably modify our test function ϕ. To simplify notation we assume that $t_1 = v(x_1, t_1)$ $x_1 = 0$ and $|x_2| \leqslant 1$. We consider the rectangle $R = \{(x, t) \colon |x| \leqslant 2, 0 \leqslant t \leqslant t_2\}$ and we define the function

$$\phi(x, t) = -\frac{1}{4}\psi\left(\frac{x^2}{t + \delta}\right) - C(t + 1)$$

where C and δ are positive constants and $\psi \in C^\infty(\mathbb{R}^+)$ satisfies $\psi(0) = 0$, $\psi'(s) \geqslant 1$ for $s > 0$, $\psi(s) = s$ for $0 < s \leqslant s_2 = 1/(t_2 + \delta)$ and $-(1/4)\psi(s) \leqslant v(x, t)$ for every $|x| = 2$, $0 \leqslant t \leqslant t_2$ and $s = 4/t + \delta$. With these assumptions ϕ satisfies $\phi_t - |D\phi|^2 \leqslant -C < 0$ in R. Repeating the argument of the first part of the proof, we see that the minimum of $v - \phi$ is attained either at $t = 0$ or at $|x| = 2$. It follows from the properties of ψ that $v - \phi \geqslant 0$ for $|x| = 2$, $0 \leqslant t \leqslant t_2$. Moreover, choosing δ very small for fixed $C > 0$ we have $v(x, 0) \geqslant \phi(x, 0)$. Therefore $v \geqslant \phi$ in R and, in particular,

$$v(x_2, t_2) \geqslant \phi(x_2, t_2) = -C(t_2 + 1) - \frac{1}{4}\psi\left(\frac{|x_2|^2}{t_2 + \delta}\right).$$

The properties of ψ imply, however, that

$$\psi\left(\frac{|x_2|^2}{t_2 + \delta}\right) = \frac{|x_2|^2}{t_2 + \delta},$$

hence letting first $\delta \downarrow 0$ and then $C \downarrow 0$ we conclude. □

Next we turn our attention to the question of the initial trace of viscosity solutions of (0.2). Since $v_t \geqslant 0$, the family $\{v(\cdot, t)\}_t$ is nondecreasing as $t \downarrow 0$ ([17]). Therefore the initial trace

$$v_0(\cdot) = \lim_{t \downarrow 0} v(\cdot, t)$$

exists. The following proposition is immediate.

Proposition A.2. *Every viscosity solution of* (0.2) *in* Q_T *has an initial tr* v_0, *which is an upper semicontinuous function* v_0: $\mathbb{R}^N \to \{-\infty\} \cup \mathbb{R}$ *satisfy*

$$\limsup_{|x| \to \infty} \frac{v_0(x)}{|x|^2} \leqslant \frac{1}{4T}. \tag{A.4}$$

2. Existence of Solutions

It follows from Proposition A.1 that for every viscosity solution v we h

$$v \geqslant \underline{v} \quad \text{on} \quad Q_T \tag{A.5}$$

where $\underline{v}$ is given by the Lax-Oleinik formula

$$\underline{v}(x, t) = \sup_{y \in \mathbb{R}^N} \left\{ v_0(y) - \frac{|x-y|^2}{4t} \right\}. \tag{A.6}$$

In fact, as we will see below, this last formula provides with the unique s tion of the Cauchy problem (1.2) in $\mathbb{R}^N \times [0, T)$, where T depends on

$$\limsup_{|x| \to \infty} v_0(x)|x|^{-2}.$$

The Lax-Oleinik formula has been studied rather extensively at least in case where v_0 is bounded ([6], [29], [30]). Next, in a series of propositions summarize the properties of (A.6) under assumption (A.4). The proofs lot of these propositions are slight modifications of the ones for bounde therefore we omit them.

Proposition A.3. *For every function* v_0: $\mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ *such that*

$$-\infty \not\equiv v_0(x) \leqslant A|x|^2 + B \quad in \quad \mathbb{R}^N \tag{A.7}$$

for some $A, B > 0$, *the Lax-Oleinik formula* (A.6) *provides with a continu viscosity solution of* (1.2) *in* Q_T, *where the maximal* T (*blow-up time*) *is g by*

$$T = 1/4\alpha \tag{A.8}$$

with

$$\alpha = \limsup_{|x| \to \infty} \{ v_0(x)|x|^{-2} \}. \tag{A.9}$$

In particular, v exists for all time if and only if $v_0(x) \leqslant o(|x|^2)$. *Moreover, every* $t \in (0, T)$, $v(\cdot, t) \geqslant v_0(\cdot)$ *and*

$$\text{(A.10)} \qquad \liminf_{|x|\to\infty} \frac{v(x,t)}{|x|^2} \geqslant -\frac{1}{4t} \quad \textit{and} \quad \limsup_{|x|\to\infty} \frac{v(x,t)}{|x|^2} \leqslant \frac{\alpha}{1-4\alpha t}.$$

Let $\mathfrak{F}_\alpha$ be the set of functions $v_0: \mathbb{R}^N \to \{-\infty\} \cup \mathbb{R}$ such that

$$-\infty \not\equiv v_0(x) \leqslant A|x|^2 + B \quad \text{for some} \quad A, B > 0$$

and

$$\limsup_{|x|\to\infty} v_0(x)|x|^{-2} \leqslant \alpha.$$

For $t \in (0, 1/4\alpha)$ and $\beta = \alpha(1-4\alpha t)^{-1}$, let $L_t: \mathfrak{F}_\alpha \to \mathfrak{F}_\beta$ be the nonlinear operator defined by the Lax-Oleinik formula.

Proposition A.4. *Let $\alpha > 0$, $t \in (0, 1/4\alpha)$, $\beta = \alpha/(1-4\alpha t)$ and $s \in (0, 1/4\beta)$. For any $v_0 \in \mathfrak{F}_\alpha$ we have*

$$L_s(L_t v_0) = L_{g+s}(v_0)$$

i.e. L_t has the semigroup property.

We have remarked in Proposition A.2 that a viscosity solution can only take on upper-semicontinuous initial data. On the other hand, we have not discussed yet about whether $L_t v_0$ assumes the initial datum v_0. The next proposition addresses this question and gives more precise information.

Proposition A.5. *Let v_0 be an upper-semicontinuous function in $\mathfrak{F}_\alpha$ for some α and let v given by* (A.6). *Then v takes on the initial value v_0. More precisely.*

$$\text{(A.11)} \qquad \limsup_{(x,t)\to(x_1,0)} v(x,t) \leqslant v_0(x_1)$$

If $v_0(x_1) > -\infty$, then

$$\text{(A.12)} \qquad \liminf_{\substack{(x,t)\to(x_1,0)\\ |x-x_1| = 0(t^{1/2})}} v(x,t) \geqslant v_0(x_1).$$

PROOF. We only prove (A.11). For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - x_1| \leqslant \delta$ then

$$v_0(x) \leqslant v_0(x_1) + \epsilon.$$

If $|x - x_1| < \delta/2$ and $|y - x| \leqslant \delta/2$, then for $t > 0$ we have

$$v_0(y) - \frac{|x-y|^2}{4t} \leqslant v_0(x_1) + \epsilon.$$

On the other hand, if $|x - x_1| < \delta/2$ and $|y - x| \geqslant \delta/2$, then

$$v_0(y) - \frac{|x-y|^2}{4t} \downarrow -\infty$$

uniformly in y as $t \downarrow 0$. □

Corollary. *If $v_0 \in \mathfrak{F}_\alpha$ the $L_t v_0$ converges as $t \downarrow 0$ to the upper semicontinu envelope of v_0 i.e. the minimal of the upper semicontinuous functi $w: \mathbb{R}^N \to \{-\infty\} \cup \mathbb{R}$ which are larger than v_0.*

3. Regularity properties of the Lax-Oleinik Formula

Proposition A.6. *Let $v_0 \in \mathfrak{F}_\alpha$ for some $\alpha > 0$. Then for every $\tau > 0$, ι Lipschitz continuous uniformly on compact subsets of $\mathbb{R}^N \times (\tau, T)$. If ($v_0(x) \leqslant (a|x| + b)^2$ in $\mathbb{R}^N$, then for almost every $x \in \mathbb{R}^N$ and $t \in (0, 1/4\alpha)$*

$$v_t = |Dv|^2 \leqslant \frac{(a|x| + b)^2}{t(1 - 2at^{1/2})^2}. \tag{A.13}$$

On the other hand, if v_0 is bounded from above by M, then

$$v_t = |Dv|^2 \leqslant \frac{M - v}{t} \leqslant \frac{M - v_0(x)}{t}. \tag{A.14}$$

The proofs are easy consequences of (A.6). Another regularity type qu tion is related to the optimality of the bounds (A.10).

Proposition A.7. *For every $t \in (0, T)$, we have ($\alpha = 1/4T$)*

$$\limsup_{|x| \to \infty} \frac{v(x,t)}{|x|^2} = \frac{\alpha}{1 - 4\alpha t} = \frac{1}{4(T-t)}. \tag{A.15}$$

PROOF. The inequality $\leqslant$ was proved in Proposition A.2. For the conve assume that for some $t_1 \in (0, T)$ we have

$$\limsup_{|x| \to \infty} \frac{v(x,t_1)}{|x|^2} = \alpha_1 < \frac{\alpha}{1 - 4\alpha t_1}.$$

Then the solution with initial value $v(\bullet, t_1)$ exists for a time

$$t_2 = \frac{1}{4\alpha_1} > \frac{1 - 4\alpha t_1}{4\alpha}.$$

By the semigroup property the original solution would then exist for a time

$$t_1 + t_2 > \frac{1}{4\alpha} = T,$$

which is a contradiction. □

As far as lower bounds are concerned we have the following result.

Proposition A.8. *Let $\beta \in [-\infty, \infty)$ be defined by*

$$\beta = \liminf_{|x| \to \infty} \frac{v_0(x)}{|x|^2}. \tag{A.16}$$

For every $t \in (0, T)$ we have

$$\liminf_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \geqslant \frac{\beta}{1 - 4t\beta}. \tag{A.17}$$

The equality is false in general.

PROOF. If $\beta = -\infty$, (A.17) reduces to (A.10). If $\beta \in (-\infty, 0)$, then for every $\epsilon > 0$ there exists a $B_\epsilon \in \mathbb{R}$ such that $v_0(x) \geqslant (\beta - \epsilon)|x|^2 - B_\epsilon$. We compare $v(x, t)$ to the explicit solution

$$\phi(x, t) = -B_\epsilon - \frac{|x|^2}{4t + \tau}$$

with $\tau = -1/(\beta - \epsilon)$ and $\beta - \epsilon \neq 0$. Using the Lax-Oleinik formula we conclude that $v \geqslant \phi$ in Q_T, hence as $\epsilon \to 0$ we obtain the inequality $\geqslant$ in (A.18). To show that equality does not hold in general we consider a v_0 defined as follows: Let $B(y_n, r_n)$ be a sequence of balls such that $r_n \to 0$ and $y_n \to \infty$ where v_0 is negative, continuous and $v(y_n)|y_n|^{-2} \to -\infty$. Outside these balls $v_0 \equiv 0$. Therefore $\beta = -\infty$. If $(x, t) \in \mathbb{R}^N \times (0, \infty)$ we have $v(x, t) = 0$ if $x \notin \bigcup_n B(y_n, r_n)$. If $x \in B(y_n, r_n)$, then there exists $y = y(x)$ such that $v_0(y) = 0$ and $|y - x| = r_n$. Hence

$$v(x, t) \geqslant v_0(y) - \frac{|x - y|^2}{4t} \geqslant -\frac{r_n^2}{4t}.$$

Therefore,

$$\lim_{|x| \to \infty} \frac{v(x, t)}{|x|^2} = 0. \quad \square$$

We conclude the presentation of regularity-related properties of the L Oleinik formula with a result concerning their semiconvexity. Since th properties are an immediate consequence of the formula, we again omit proof.

Proposition A.9. (1) *Let* $\chi \in \mathbb{R}^N$, $|\chi| = 1$. *Then*

(i)
$$\frac{\partial^2 v}{\partial \chi^2} \geqslant -\frac{1}{2t}.$$

(ii) *If for every* $\chi \in \mathbb{R}^N$, $|\chi| = 1$, $\partial^2 v_0/\partial \chi^2 \geqslant -\alpha$ *then for every* $t > 0$,

$$\frac{\partial^2 v}{\partial \chi^2} \geqslant -\frac{\alpha}{1 + 2\alpha t}.$$

(2) *If* $\Delta v_0 \geqslant -\alpha$ *then for* $t > 0$,

$$\Delta v(x, t) \geqslant -\frac{N\alpha}{N + 2\alpha t}.$$

All the above inequalities should be interpreted in the sense of distributio

4. Uniqueness and continuous dependence

We begin with a proposition concerning the domain of dependence of Lax-Oleinik formula. The proof of this result is based on the gradi estimates from Proposition A.6 and the proofs of M. G. Crandall and Newcomb [20] and P. E. Souganidis [36] concerning viscosity solutions on boundary. See [5] for $N = 1$. Since it is a long exercise, we omit it.

Proposition A.10. *Let* v_{01}, v_{02} *be two initial data in* $\mathbb{R}^N$ *such that*

$$v_{01}(x), v_{02}(x) \leqslant (a|x| + b)^2$$

for every $x \in \mathbb{R}^N$ *and some constants* $a, b \geqslant 0$. *Let* $t \in (0, 1/4a^2)$. *Then*

$$v_1(0, t) - v_2(0, t) \leqslant \sup_{y \in I_0} \{ v_{01}(y) - v_{02}(y) \}$$

where

$$I_0 = \left\{ y \in \mathbb{R}^N \colon |y| \leqslant \frac{b}{a} \left[\exp\left(\frac{\lambda}{1 - \lambda} \right) - 1 \right], \lambda = 2at^{1/2} \right\}$$

if $a, b > 0$ *and*

$$I_0 = \{ y \in \mathbb{R}^N : |y| \leqslant \sqrt{4bt} \}$$

if $a = 0$ *and* $b > 0$.

Corollary. *If* $v_0^n \to v_0$ *locally uniformly in* $\mathbb{R}^N$, *then* $v^n \to v$ *locally uniformly in* $\mathbb{R}^N \times [0, T)$.

Next we prove the uniqueness of viscosity solutions of (A.7) with upper-semicontinuous initial datum $v_0 : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$. This implies that the viscosity solution of (A.7) is given by the Lax-Oleinik formula, therefore it enjoys all the regularity presented above.

Theorem A.1. *The viscosity solution of* (1.2) *in* Q_T *with upper-semicontinuous initial datum* $v_0 : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ *is unique.*

PROOF. If $\underline{v}(x, t) = L_t(v_0)(x)$, in view of (A.5), we only have to show that $v \leqslant \underline{v}$. We argue as follows: Since v is defined in Q_T, for every $t \in [0, T)$ we have

$$\limsup_{|x| \to \infty} \frac{v(x, t)}{|x|^2} \leqslant \frac{1}{4(T - t)}.$$

Let $v_{0n} \in C(\mathbb{R}^N, \mathbb{R})$ be such that $v_{0n} > v_0$ and

$$\limsup_{|x| \to \infty} \frac{v_{0n}(x)}{|x|^2} = \frac{1}{4\left(T - \dfrac{1}{n}\right)}.$$

Then $v_n = L_t(v_{0n})$ exists for a time $T_n = T - 1/n$. Moreover, for $t \in (0, T_n)$,

$$\limsup_{|x| \to \infty} \frac{v_n(x, t)}{|x|^2} = \frac{1}{4\left(T - \dfrac{1}{n} - t\right)}.$$

If $v_{n\epsilon} = v_n * \rho_\epsilon$, where $*$ denotes the standard convolution, then for $t > \epsilon$ we have

$$(v_{n\epsilon})_t = |Dv_n|^2 * \rho_\epsilon \geqslant |Dv_{n\epsilon}|^2.$$

Let $w_{\epsilon n} : \mathbb{R}^N \times [0, T_n - \epsilon]$ be defined by

$$w_{n\epsilon}(x, t) = v_{n\epsilon}(x, t + \epsilon) + \epsilon t.$$

The sup of $v - w_{n\epsilon}$ in $\mathbb{R}^N \times [0, t_1)$ with $t_1 < t_n - \epsilon$ cannot be taken ir interior of $\mathbb{R}^N \times [0, t_1)$. Therefore, either it is approached as $|x| \to \infty$ or taken at $t = 0$. In either case, it is negative. It then follows that

$$v < w_{n\epsilon} \quad \text{in} \quad \mathbb{R}^N \times [0, T_n - \epsilon].$$

Letting $\epsilon \to 0$ yields

$$v \leqslant v_n \quad \text{in} \quad \mathbb{R}^N \times [0, T_n).$$

Sending $n \to \infty$ and using the continuous dependence of the Lax-Oleinik : tions on the initial data in local norms we conclude. □

5. Free Boundaries

In the case of solutions which are bounded from either above or belo makes sense to consider the boundary of the sets where the largest or sm: values are attained. Let us consider first the case of an upper-semicontin initial datum $v_0: \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ which is bounded from above by a con M. Let

$$\text{(A.18)} \qquad \begin{aligned} D_+ &= D_+(v_0) \\ &= \{x \in \mathbb{R}^N: v_0(x) = M\}. \end{aligned}$$

This is a closed, possibly empty, set. It is immediate from $v_t \geqslant 0$ that if $x \epsilon$ then for every $t > 0$, $v(x, t) = M$. Therefore the set D_+ is invariant in tim so is its boundary. On the contrary, if v_0 is bounded from below by a con which without any loss of generality we may assume to be zero, then th

$$\text{(A.19)} \qquad \Omega_0 = \{x \in \mathbb{R}^N: v_0(x) > 0\}$$

is not necessarily open or closed. We define:

$$\text{(A.20)} \qquad \begin{cases} \Omega = \{(x, t) \in \mathbb{R}^N \times [0, T): v(x, t) > 0\} \\ \Omega(t) = \{x \in \mathbb{R}^N: (x, t) \in \Omega\} \\ \Gamma = \text{boundary of } \Omega \text{ in } \mathbb{R}^N \times [0, T) \\ \Gamma(t) = \{x \in \mathbb{R}^N: (x, t) \in \Gamma\}. \end{cases}$$

Γ is called the *free boundary* of v. Since $v_t \geqslant 0$, the following result is imme

Proposition A.11. *For every $t_2 > t_1$ in $(0, T)$, $\Omega_0 \subset \Omega(t_1) \subset \Omega(t_2)$.*

Next we examine the behavior of v on Ω.

Proposition A.12. *Let $t \in (0, T)$. For almost every $x \in \Omega(t) \setminus \Omega_0$ there exists a point $y = y(x) \in \Omega_0$ such that*

$$Dv = -\frac{x - y}{2t}.$$

For all points $x \in \Omega(t) \setminus \Omega_0$, $-\dfrac{x - y}{2t}$ is a subdifferential of v at x.

PROOF. Since $v(x, t) > 0$ there exist $y_n \in \mathbb{R}^N$ such that

$$v_0(y_n) - \frac{|x - y_n|^2}{4t} \uparrow v(x, t).$$

It follows that $v_0(y_n) > 0$, i.e. $y_n \in \Omega_0$ and

$$|x - y_n|^2 \leqslant 4tv_0(y_n) < 4t(\alpha + \epsilon)(|y_n| + b_\epsilon)^2. \tag{A.21}$$

Since $4t(\alpha + \epsilon) < 1$ if ϵ is small enough, $|y_n| \leqslant C$ and, upon passing to a subsequence, we may assume that $y_n \to y$. The upper semicontinuity of v_0 yields $y \in \Omega_0$ and

$$v(x, t) = v_0(y) - \frac{|x - y|^2}{4t}, \tag{A.22}$$

Next let $h \in \mathbb{R}^N$ with $|h|$ small. Then

$$v(x + h, t) - v(x, t) \geqslant v_0(y) - \frac{|x + h - y|^2}{4t} - v_0(y) - \frac{|x - y|^2}{4t}$$

$$\geqslant -\frac{1}{2t} h \cdot (x - y) - \frac{|h|^2}{4t}. \quad \square$$

Since $-Dv$ is the local velocity of propagation of the solutions of (0.2), this result controls the speed with which the interface moves. In fact the interface consists of a stationary part Γ_0, a union of vertical segments $\{(x, t): 0 \leqslant t \leqslant t_1\}$ with $x \in \partial\Omega_0$ fixed, and the moving interface

$$\Gamma_1 = \{(x,t) \in \Gamma: x \notin \Omega_0\}.$$

Proposition A.13. *The moving interface Γ_1 can be described by a Lipschitz continuous function $t = S(x)$ for $x \in \mathbb{R}^N \setminus \bar{\Omega}_0$. More precisely, for every $(\bar{x}, \bar{t}) \in \Gamma_1$ there is a conical region $K = \{(x, t): |x - \bar{x}| < h, |x - \bar{x}| < c|t - \bar{t}|\}$ with $0 < c < \operatorname{dist}(\bar{x}, \Omega_0)(2\bar{t})^{-1}$ and h small depending on c, $\bar{x}$, such that*

$$K_+ = \{(x, t) \in K: t > \bar{t}\} \subset \Omega \quad and \quad K_- = \{(x, t) \in K: t < \bar{t}\}$$

is disjoint with Ω.

PROOF. We only prove the result concerning K_+. The result about K follows in a similar way. To this end, let $(x,t) \in K_+$ and set $x - \bar{x} = z$ a $t - \bar{t} = \tau$. If there exists a $y \in \Omega_0$ with the properties described in Propositi A.12 (since $v(\bar{x},\bar{t}) = 0$ this is not necessarily the case) we have

$$\begin{aligned} v(x,t) &\geqslant v_0(y) - \frac{|x-y|^2}{4t} \\ &= \frac{|\bar{x}-y|^2}{4\bar{t}} - \frac{|x-y|^2}{4t} \\ &= \frac{|\bar{x}-y|^2}{4\bar{t}} - \frac{|\bar{x}-y|^2}{4t} - \frac{(\bar{x}-y)\cdot z}{2t} - \frac{|z|^2}{4t} \\ &= \frac{|\bar{x}-y|}{4t}\left(\frac{|\bar{x}-y|}{\bar{t}}\tau - 2z\cdot\theta\right) - 0(|z|^2) \end{aligned}$$

where $\theta = \bar{x} - y/|\bar{x}-y|$. Therefore if $z/\tau \leqslant d(\bar{x}, D_0)/2\bar{t} \leqslant |\bar{x}-y|/2\bar{t}$ a $0 < |z| < h$ with h small we have $v(x,t) > 0$.

If such a y does not exist we select a sequence of points $x_n \in \Omega_t$, $x_n \to x$, find $y_n \in \Omega_0$, construct a cone K_n with vertex $(x_n, \bar{t})$ and let $n \to \infty$ to obta K_+. We define

$$S(x) = \sup\{t \geqslant 0 : v(x,t) = 0\}. \tag{A.23}$$

The Lipschitz continuity of S at $(\bar{x},\bar{t})$ follows from the fact for every x su that $|x-\bar{x}| < h$ then $(x,t) \in \Gamma_1$ implies $t - \bar{t} \leqslant C|x-\bar{x}|$. □

Corollary. *Let* $x \in \Omega_t \setminus \Omega_0$. *If* $d(x) = \operatorname{dist}(x, \Omega_0)$, *then*

$$\frac{d(x)}{2t} < |Dv| < \frac{c(|x|)}{2t} \tag{A.24}$$

where $c(r)$ *is a continuous function of* r.

PROOF. Take $y \in \Omega_0$ as in (A.22). We have $|x-y| \geqslant d(x)$ and, from (A.2

$$|y| \leqslant \frac{|x| + b_\epsilon k}{1-k}, \qquad k = (4t(\alpha+\epsilon))^{1/2}.$$

We conclude. □

The above proof also shows that at every point x where S is differentiab we have $|DS| < 1/c$ for any c as in Proposition A.13. Therefore

$$|DS| \cdot \frac{d(x)}{2t} < 1. \tag{A.25}$$

In other words, $(2t)^{-1}d(x)$ is a lower bound for the velocity with which Ω_t grows. Thus if $\operatorname{dist}(\Omega_t, \Omega_0) = d > 0$ then for every $\tau > 0$ small enough $\Omega_{t+\tau}$ contains an ϵ-neighborhood of Ω_t, if $\epsilon < d\tau/2t$. In fact Ω_t moves with speed bounded from above. More precisely, we have:

Proposition A.14. *If $\tau > 0$ is small enough, then for every $\bar{x} \in \partial Q_{\bar{t}+\tau}$ there exists $C = C(|\bar{x}|)$ such that*

$$\operatorname{dist}(\bar{x}, \Omega_{\bar{t}}) \leqslant \frac{C\tau}{2\bar{t}} \cdot \tag{A.26}$$

Moreover, if v_0 is bounded from above, the bound on (A.26) *is independent of $\bar{x}$ and*

$$\operatorname{dist}(\Gamma_{t+\tau}, \Omega_t) \leqslant \frac{C\tau}{2t} \cdot \tag{A.27}$$

PROOF. Let $r = \operatorname{dist}(\bar{x}, \Omega_{\bar{t}})$ and c be an upper bound on v in $B(x, 2r)$ (which should be separated from Ω_0). The function

$$V(x,t) = c(c(t - \bar{t}) + |x - \bar{x}| - r)^+$$

is a supersolution of (1.2). The result follows. □

We conclude by characterizing the existence of a stationary interface Γ_0. A careful look at the Lax-Oleinik formula yields the following proposition.

Proposition A.15. *Let $x \in \Omega_0$. Then $v(x,t) = 0$ for $t \in [0, t^*]$ if and only if the quantity*

$$\gamma(x) = \sup_{y \in \mathbb{R}^N} \frac{v_0(y)}{|x-y|^2} \tag{A.28}$$

is finite. The starting time t^* is given by $1/4\gamma$.

6. Generalizations

All the above can be easily generalized to the Cauchy problems

$$\begin{cases} v_t = |Dv|^p & \text{in } \mathbb{R}^N \times (0, T) \\ v = v_0(x) & \text{on } \mathbb{R}^N \times \{t = 0\}, \end{cases} \tag{A.29}$$

with $p > 1$. The formula for viscosity solutions of (A.29) is

$$\underline{v}(x) = \sup\left\{v_0(y) - C_p \frac{|x-y|^{p/(p-1)}}{t^{1/(p-1)}}\right\} \tag{A.30}$$

for $C_p = (p-1)p^{-p/(p-1)}$.

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References

[1] Aronson, D. G. The porous medium equation, in Nonlinear Diffusion blems, A. Fasano and M. Primicerio eds., *Lecture Notes in Math.* 1224, Spri 1986.

[2] Aronson, D. G. and Bénilan, Ph. Régularité des solutions de l'équation milieux poreux dans $\mathbb{R}^N$, *C.R. Acad. Sci. Paris* **288**(1979), 103-105.

[3] Aronson, D. G., Caffarelli, L. A. and Kamin, S. How an initially statio interface begins to move in porous medium flow, *SIAM J. Math. Anal.* **14**(1! 639-658.

[4] Aronson, D. G., Caffarelli, L. A. and Vázquez, J. L. Interfaces with a cc point in one-dimensional porous medium flow, *Comm. Pure Appl. M* **38**(1985), 375-404.

[5] Aronson, D. G. and Vázquez, J. L. The porous medium equation as a fi speed approximation to a Hamilton-Jacobi equation, *J. Anal. Nonlinéaire, Inst. H. Poincaré,* **4**(1987), 203-330.

[6] Bardi, M. and Evans, L. C. On Hopf's formulas for solutions of Hami Jacobi equations, *Nonlinear Anal. TMA.* **8**(1984), 1373-1381.

[7] Barles, G. and Perthame, B. Discontinuous Solutions of deterministic opt stopping time problems, *Meth. Math. Anal. Num.*, to appear.

[8] Bénilan, Ph. Evolution equations and accretive operators, *Lecture N* University of Kentucky, Spring 1981.

[9] Bénilan, Ph. and Crandall, M. G. The continuous dependence on ϕ of the tions of $u_t - \Delta\phi(u) = 0$, *Indiana U. Math. J.* **30**(1981), 161-177.

[10] Bénilan, Ph. and Crandall, M. G. Regularizing effects of homogeneous e tion equations, in *Contributions to Analysis and Geometry*, Edited by I Clark, G. Pecelli, R. Sacksteder, The Johns Hopkins University Press, (1! 23-39.

[11] Bénilan, Ph., Crandall, M. G. and Pierre, M. Solutions of the porous me equation in $\mathbb{R}^N$ under optimal conditions on initial values, *Indiana U. Mat* **33**(1984), 51-87.

[12] Buckmaster, J. Viscous sheets advancing over dry bed, *J. Fluid Mech.* **81** (1977), 735-756.
[13] Caffarelli, L. A. and Friedman, A. Regularity of the free boundary of a gas flow in an n-dimensional porous medium, *Indiana U. Math. J.* **29** (1980), 361-391.
[14] Caffarelli, L. A., Vázquez, J. L. and Wolansky, N. Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation, *Indiana U. Math. J.* **36** (1987), 373-401.
[15] Crandall, M. G., Evans, L. C. and Lions, P. -L. Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282**(1984), 487-502.
[16] Crandall, M. G., Ishii, H. and Lions, P. -L. Uniqueness of viscosity solutions revisited, *Math. Soc. of Japan*, to appear.
[17] Crandall, M. G. and Lions, P. -L. Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277**(1983), 1-42.
[18] Crandall, M. G. and Lions, P. -L. On existence and uniqueness of solutions of Hamilton-Jacobi equations, *Nonlinear Anal.* **10**(1986), 353-370.
[19] Crandall, M. G. and Lions, P.-L. Two approximations of solutions of Hamilton-Jacobi equations, *Math. Comp.* **43**(1984), 1-19.
[20] Crandall, M. G. and Newcomb, R. Viscosity solutions of Hamilton-Jacobi equations at the boundary, *Proc. Amer. Math. Soc.* **94**(1985), 283-290.
[21] Dahlberg, B. E. and Kenig, C. E. Nonnegative solutions of the porous medium equation, *Comm. PDE.* **9**(1984), 409-437.
[22] Evans, L. C. and Ishii, H. Personal communication.
[23] Evans, L. C. and Souganidis, P. E. A PDE approach to geometric optics for certain reaction-diffusion equations, *Indiana U. Math. J.*, to appear.
[24] Fleming. W. H. and Souganidis, P. E. PDE-viscosity solution approach to some problems of large deviations, *Ann. Scuola Norm. Sup.* **13**(1986), 171-192.
[25] Gilding, B. H. Hölder continuity of solutions of parabolic equations, *J. London Math. Soc.* **13** (1976), 103-106.
[26] Gurtin, M. and MacCamy, R. C. On the diffusion of biological populations, *Math. Biosc.* **33**(1977), 35-49.
[27] Ishii, H. Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open subsets.
[28] Lasry, J. M. and Lions, P.-L. A remark on regularization on Hilbert spaces, *Israel J. Math.*, to appear.
[29] Lions, P.-L. *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, Boston, 1982, to appear
[30] Lions, P.-L. and Rochet, J.-C. Hopf formula and multi-time Hamilton-Jacobi equations, *Proc. Amer. Math. Soc.* **96**(1986), 79-84.
[31] Lions, P.-L. and Souganidis, P. E. Viscosity solutions of second-order equations, stochastic control and stochastic differential games, Proceedings of Workshop on Stochastic Control and PDE's, IMA, June 1986.
[32] MacHyman, personal communication with P.-L. Lions.
[33] Muskat, M. *The Flow of Homogeneous Fluids Through Porous Media*, McGrawHill, New York, 1937.
[34] Raizer, Yu. P., and Zeldovich, Ya. B. *Physics of Shock-Waves and High-Temperature hydrodynamic phenomena*, Vol. II, Academic Press, New York, 1966.
[35] Souganidis, P. E. Existence of viscosity solutions of Hamilton-Jacobi equations, *J. Diff. Equations.* **56**(1985), 345-390.

[36] Souganidis, P. E. A remark about viscosity solutions of Hamilton-Jacobi eq tions at the boundary, *Proc. Amer. Math. Soc.* **96**(1986), 323-330.
[37] Vázquez, J. L. Hyperbolic aspects in the theory of the porous medium eq tion. *Metastability and Incompletely Posed Problems.* IMA Volumes in Ma and Applications No. 3, S. Antman et al. eds., Springer 1987, 325-342.

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Norm Inequalities for Potential-Type Operators

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Introduction

The purpose of this paper is to derive norm inequalities for potentials of the form

$$Tf(x) = \int_{\mathbb{R}^n} f(y) K(x,y)\, dy, \qquad x \in \mathbb{R}^n,$$

when K is a kernel which satisfies estimates like those that hold for the Green function associated with the degenerate elliptic equations studied in [3] and [4]. Thus, for $0 \leqslant r < \infty$, $x \in \mathbb{R}^n$, and a nonnegative function $a(r,x)$ to be specified, we assume that

(i) $$|K(x,y)| \leqslant C \frac{a(|x-y|, x)}{|x-y|^n},$$

and, in some cases, that there exists ϵ_0, $0 < \epsilon_0 \leqslant 1$, such that

(ii) $$|K(x,y) - K(x,z)| \leqslant C \left(\frac{|y-z|}{|x-y|} \right)^{\epsilon_0} \frac{a(|x-y|, x)}{|x-y|^n}$$

if

$$|y-z| < \frac{1}{2} |x-y|.$$

It will be convenient to think of $a(r, x)$ as a function of the ball $B = B(r, x)$ with radius r and center x, and to write

$$a(B) = a(B(r, x)) = a(r, x).$$

We then require that there is a constant $C > 0$ such that

(iii) $$a(B_1) \leqslant Ca(B_2) \quad \text{if} \quad B_1 \subset B_2, \quad \text{and}$$

(iv) there exist $\mu, d > 0$ so that if tB denotes the ball concentric with B whose radius is t times that of B, then

$$C^{-1} t^{\mu} a(B) \leqslant a(tB) \leqslant Ct^{nd} a(B), \qquad t > 1.$$

Conditions (iii) and (iv) can be weakened in some of our results: see the comments later in this section. We also remark that the results which require condition (ii), which is a first-order smoothness condition, have analogues which reflect any higher order smoothness that K may have.

The simplest examples of such kernels are the classical fractional integral kernels $K(x, y) = 1/|x - y|^{n - \alpha}$, $\alpha > 0$. In this case, $a(r, x) = r^{\alpha}$ and (ii) holds with $\epsilon_0 = 1$. A more typical example is

(*) $$a(r, x) = \frac{r^{n + \alpha}}{w(B(r, x))},$$

where w is a weight function, *i.e.*, a nonnegative locally integrable function on $\mathbb{R}^n$, and $w(B) = \int_B w(x)\, dx$. In this case, estimate (i) becomes

$$|K(x, y)| \leqslant C \frac{|x - y|^{\alpha}}{w(B(|x - y|, x))}.$$

Such kernels with $\alpha = 2$ arise naturally if we consider a bounded domain Ω and a divergence form differential operator

$$L = -\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

whose coefficient matrix $A(x) = (a_{ij}(x))$ satisfies

$$c_1 w(x) |\xi|^2 \leqslant \langle A(x)\xi, \xi \rangle \leqslant c_2 w(x) |\xi|^2, \qquad \xi \in \mathbb{R}^n.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$ and $0 < c_1 < c_2 < \infty$. In [4], estimates are derived for the Green function $G(x, y)$ associated with such an operator. If w is suitably restricted (see the comments later in this section), these estimates imply that in the interior of Ω, $G(x, y)$ satisfies (i) and (ii) for some $\epsilon_0 > 0$, with $a(B)$ given by (*) and $\alpha = 2$.

If $0 < p < \infty$ and v is a weight function, let

$$L_v^p = \left\{ f \colon \|f\|_{L_v^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx \right)^{1/p} < \infty \right\}.$$

We also consider the Hardy space H_v^p defined as follows. Let $\mathcal{S}$ denote the Schwartz class of rapidly decreasing functions on $\mathbb{R}^n$, and let $\mathcal{S}'$ denote the class of tempered distributions. For $0 < p < \infty$,

$$H_v^p = \{ f \in \mathcal{S}' \colon \|f\|_{H_v^p} = \|N(f)\|_{L_v^p} < \infty \},$$

where

$$N(f)(x) = N_\phi(f)(x) = \sup_{t > 0} |(f * \phi_t)(x)|$$

for $\phi \in \mathcal{S}$, $\int_{\mathbb{R}^n} \phi\, dx \neq 0$, $\phi_t(x) = t^{-n}\phi(x/t)$. By [10], the finiteness of $\|N(f)\|_{L_v^p}$ is independent of ϕ if v satisfies the doubling condition $v(2B) \leqslant cv(B)$. We write $v \in D_\infty$ for such v.

In case $1 < p < \infty$ and $v \in A_p$, *i.e.*, if

$$\left(\frac{1}{|B|} \int_B v\, dx \right) \left(\frac{1}{|B|} \int_B v^{-1/(p-1)}\, dx \right)^{p-1} \leqslant c$$

for all balls B, it is well-known that $H_v^p \equiv L_v^p$ with equivalence of norms. This identification is also valid for certain other v's (see [1] and [11]) and is important for some of our results. We mention here that the class $\mathcal{S}_{0,0}$ of Schwartz functions whose Fourier transforms have compact support not containing the origin is dense in all the spaces L_v^p and H_v^p which we will consider. Note that all the moments of an $f \in \mathcal{S}_{0,0}$; in particular, $\int_{\mathbb{R}^n} f\, dx = 0$ for such f.

In order to state our main results, we now introduce several conditions on a pair of weight functions u and v. Let χ_E denote the characteristic function of a set $E \subset \mathbb{R}^n$. For $0 < p, q < \infty$, we consider the following kinds of conditions:

(1) $$a(B)u(B)^{1/q} \leqslant cv(B)^{1/p}$$

for all balls B;

(1)′ $$a(B)^p \left(\frac{1}{|B|} \int_B u^s\, dx \right)^{1/s} \leqslant c \frac{v(B)}{|B|}$$

for some $s > 1$ and all balls B;

(2) $$\|\Sigma a(B_k)\lambda_k \chi_{B_k}\|_{L_u^q} \leqslant c \|\Sigma \lambda_k \chi_{B_k}\|_{L_v^p}$$

for all sequences $\{\lambda_k\}$, $\lambda_k > 0$, and all balls B_k.

Note that (2) is stronger than (1) since it reduces to (1) in the case of a single ball. Also note that (1)′ with $s = 1$ is the same as (1) with $q = p$; thus, (1)′ represents a strengthening of (1) in case $q = p$. Some other relations between these conditions are given in Theorems 5 and 6, and an alternate form of (1)′ is given in Lemma 6.5.

We shall also use the condition

$$\|\Sigma\lambda_k\chi_{tB_k}\|_{L^p_v} \leqslant ct^\delta\|\Sigma\lambda_k\chi_{B_k}\|_{L^p_v}, \qquad t > 1. \tag{3}$$

This is related to the doubling condition: in fact, if $v \in D_\sigma$, by which we mean

$$v(tB) \leqslant ct^{n\sigma}v(B), \qquad t > 1,$$

then (3) holds with $\delta = n\sigma$ and $1 \leqslant p < \infty$ (see [12]). Moreover, as shown in [10], if $v \in A_r \cap D_\sigma$ then (3) holds with

$$\delta = \frac{n}{p}\left(\frac{r-p}{r-1}\right)(\sigma - 1) + n, \qquad r \geqslant p.$$

It is clear by considering a single term that (3) implies that $v \in D_{\delta p/n}$.

The principal results to be proved are as follows.

Theorem 1. *Let* (i)-(iv) *hold for a kernel* $K(x, y)$. *Let* $0 < p$, $q < \infty$, *u and v satisfy* (2), *and v satisfy* (3) *for some* $\delta < n + \epsilon_0$. *Then if* $f \in \mathcal{S}_{0,0}$, *the operator defined by*

$$Tf(x) = \int_{\mathbb{R}^n} f(y)K(x, y)\, dy$$

satisfies $\|Tf\|_{L^q_u} \leqslant c\|f\|_{H^p_v}$ *with c independent of f.*

Without assuming the smoothness condition (ii), we have

Theorem 2. *Let* (i), (iii) *and* (iv) *hold for a kernel* $K(x, y)$. *Let* $1 < p < \infty$, *u and v satisfy* (2), *and* $v \in A_p$. *Then if Tf is defined as above,*

$$\|Tf\|_{L^q_u} \leqslant c\|f\|_{L^p_v} \quad \textit{for all} \quad f \in L^p_v.$$

Theorem 2 turns out to be a corollary of Theorem 1 and the fact that $H^p_v \equiv L^p_v$ if $v \in A_p$. We can use Theorem 1 to derive results of the L^q_u, L^p_v type when $v \notin A_p$ provided $H^p_v \equiv L^p_v$, although we need the smoothness condition (ii) in this case. The next theorem is an example of such a result. It includes the power weights $v(x) = |x|^\beta$ for $n(p-1) < \beta < n(p-1) + \epsilon_0 p$; the range of β for which $|x|^\beta \in A_p$ is $-n < \beta < n(p-1)$. More general theorems of this type can also be derived.

Theorem 3. *Let* (i)-(iv) *hold for a kernel* $K(x, y)$. *Let* $1 < p < \infty$, $0 < q < \infty$, *u and v satisfy* (2), $|x|^{-\epsilon p} v \in A_p$ *for some* ϵ *with* $0 < \epsilon \leqslant \epsilon_0$, *and* $|x|^{-np} v \in A_p$. *If S is defined by*

$$Sf(x) = \int_{\mathbb{R}^n} f(y)[K(x, y) - K(x, 0)]\, dy$$

then $\| Sf \|_{L^q_u} \leqslant c \| f \|_{L^p_v}$ *for all* $f \in L^p_v$.

Moreover, the integral defining Sf converges absolutely a.e. Of course, $Sf = Tf$ *if* $\int f\, dy = 0$.

The next three theorems are intended to help understand condition (2) and show its relation to (1) and (1)′. For $1 < p < \infty$, p' is defined by $1/p + 1/p' = 1$.

Theorem 4. *Let* (iii) *hold,* $v \in D_\infty$ *and*

$$Mf(x) = \sup_{B \ni x} \frac{a(B)}{v(B)} \int_B |f(y)| u(y)\, dy.$$

If $1 < p$, $q < \infty$, *then* (2) *holds if and only if* $\|Mf\|_{L^{p'}_v} \leqslant c \| f \|_{L^{q'}_u}$.

Theorem 5. *Let* (iii) *and* (iv) *hold,* $1 < p < q < \infty$, $u \in D_\infty$ *and* $v \in A_p$. *Then* (1) *implies* (2).

Theorem 6. *Let* (iii) *and* (iv) *hold,* $1 < p < \infty$ *and* $v \in A_p$. *Then* (1)′ *implies* (2) *with* $q = p$.

By combining Theorems 2, 5 and 6, we see that if (i), (iii) and (iv) hold, $u \in D_\infty$, $1 < p < \infty$ and $v \in A_p$, then $\| Tf \|_{L^q_u} \leqslant \| f \|_{L^p_v}$ provided either

(*a*) $p < q < \infty$ and (1) holds, or
(*b*) $p = q$ and (1)′ holds.

Theorems 1-6 are proved in §1-6 respectively. Some of the methods of the proofs are related to those in [2], [6] and [12]. We also use results about H^p_v from [1] and [10], such as theorems stating when H^p_v and L^p_v can be identified, the atomic decomposition, and the fact that if $v \in D_\infty$ $(= \bigcup_{\sigma \geq 1} D_\sigma)$, then $\| f \|_{H^p_v}$ is equivalent to the L^p_v norm of the «grand» maximal function f^* of f defined in [5].

In addition to the notation already introduced, we write $v \in A_\infty$ if $v \in A_p$ for some p, and $v \in RD_\nu$ (reserve doubling), $\nu > 0$, if

$$v(tB) \geqslant ct^\nu v(B), \qquad t > 1,$$

for some $c > 0$ independent of t and B. We will use the same letter c to denote different constants, and we often write $\int f$ for $\int_{\mathbb{R}^n} f(x)\, dx$.

Finally, we make a few comments on two points raised earlier. First, it is not hard to see from the estimates derived in [4] that the Green function mentioned above satisfies (i) and (ii), for some $\epsilon_0 > 0$ and $a(r, x) = r^{n+2}/w(B(r, x))$, provided $w \in A_2 \cap RD_\nu$ for some $\nu > 2$. Furthermore, in this case, (iii) and (iv) hold if $w \in A_{1+2/n}$, or more generally, if $w \in D_\sigma$ for some $\sigma < 1 + 2/n$. Second, some of our results can be proved under weaker conditions than those listed in (i)-(iv). Generally speaking, we only use the full force of condition (iii) as well as of the first inequality in condition (iv) when we prove Theorems 5 and 6. Elsewhere we can weaken these by requiring only that $a(r, x) \leqslant ca(s, y)$ when $r \approx s$ and $|x - y| \leqslant cr$, provided we also require some local integrability of K. For example, Theorem 1 remains true if (i)-(iv) are replaced by assuming that $K(x, y)(1 + |y|)^{-L}$ is an integrable function of y for some L, and

$$\int_{B(r,z)} |K(x, y)|\, dy \leqslant ca(r, z) \quad \text{if} \quad x \in B(4r, z),$$

and

$$\int_{B(r,z)} |K(x, y) - K(x, z)|\, dy \leqslant c\left(\frac{r}{|x - z|}\right)^{n+\epsilon_0} a(|x - z|, z)$$

if $x \notin B(4r, z)$, where $a(\bullet, \bullet)$ is a function which satisfies $a(r, x) \leqslant ca(s, x)$ when $r \approx s$.

1. Proof of Theorem 1

Let f be an atom associated with $B(r, y_0)$, *i.e.*, let $|f| \leqslant 1$, supp $(f) \subset B(r, y_0)$, $\int f = 0$. Write

$$\begin{aligned} Tf(x) &= \int f(y)K(x, y)dy \\ &= \int f(y)[K(x, y) - K(x, y_0)]\, dy. \end{aligned}$$

Note that Tf converges absolutely since $K(x, y)$ is locally integrable as a function of y: by (i) and (iv),

$$\begin{aligned} |K(x, y)| &\leqslant c\frac{a(|x - y|, x)}{|x - y|^n} \\ &\leqslant c\frac{a(1, x)}{|x - y|^{n-\mu}} \end{aligned}$$

if $|x - y| < 1$. Also, if $|x - y| > 1$, $|K(x, y)| \leqslant ca(1, x)|x - y|^{n(d-1)}$ by (iv).

We want to estimate the size of $|Tf(x)|$.

Case 1. $x \in B(4r, y_0)$. By (i) and the first representation of Tf above,

$$\begin{aligned} |Tf(x)| &\leqslant c \int_{B(r,y_0)} |f(y)| \frac{a(|x-y|,x)}{|x-y|^n} \, dy \\ &\leqslant c \int_{|x-y|<5r} \frac{(|x-y|/r)^\mu a(r,x)}{|x-y|^n} \, dy \quad \text{by (iv)} \\ &= ca(r,x) \\ &\leqslant ca(r,y_0). \end{aligned}$$

Case 2. $x \notin B(4r, y_0)$. By (ii) and the second representation of Tf above,

$$\begin{aligned} |Tf(x)| &\leqslant c \int_{B(r,y_0)} |f(y)| \left(\frac{|y-y_0|}{|x-y_0|} \right)^{\epsilon_0} \frac{a(|x-y_0|,y_0)}{|x-y_0|^n} \, dy \\ &\leqslant \frac{cr^{n+\epsilon_0}}{|x-y_0|^{n+\epsilon_0}} a(|x-y_0|,y_0) \end{aligned}$$

since $|f| \leqslant 1$.

Thus, in any case,

$$|Tf(x)| \leqslant c \sum_{j=1}^{\infty} 2^{-j(n+\epsilon_0)} a(2^j r, y_0) \chi_{B(2^j r, y_0)}(x).$$

Now let

$$f = \Sigma \lambda_k g_k$$

be a finite sum, $\lambda_k > 0$, and g_k be an atom associated with $B(r_k, y_k)$. Then

$$|Tf(x)| \leqslant c \sum_j 2^{-j(n+\epsilon_0)} \sum_k \lambda_k a(2^j r_k, y_k) \chi_{B(2^j r_k, y_k)}(x).$$

For $q \geqslant 1$, by Minkowski's inequality,

$$\begin{aligned} \|Tf\|_{L_u^q} &\leqslant c \sum_j 2^{-j(n+\epsilon_0)} \left\| \sum_k \lambda_k a(2^j r_k, y_k) \chi_{B(2^j r_k, y_k)} \right\|_{L_u^q} \\ &\leqslant c \sum_j 2^{-j(n+\epsilon_0)} \left\| \sum_k \lambda_k \chi_{B(2^j r_k, y_k)} \right\|_{L_v^p} \quad \text{by (2)} \\ &\leqslant c \sum_j 2^{-j(n+\epsilon_0)} 2^{j\delta} \left\| \sum_k \lambda_k \chi_{B(r_k, y_k)} \right\|_{L_v^p} \quad \text{by (3)} \\ &= c \left\| \sum_k \lambda_k \chi_{B(r_k, y_k)} \right\|_{L_v^p} \end{aligned}$$

since $\delta < n + \epsilon_0$. If $0 < q < 1$,

$$\begin{aligned}\|Tf\|_{L^q_u}^q &\leqslant c\sum_j 2^{-j(n+\epsilon_0)q}\Big\|\sum_k \lambda_k a(2^j r_k, y_k)\chi_{B(2^j r_k, y_k)}\Big\|_{L^q_u}^q \\ &\leqslant c\sum_j 2^{-j(n+\epsilon_0)q}\Big\|\sum_k \lambda_k \chi_{B(2^j r_k, y_k)}\Big\|_{L^p_v}^q \\ &\leqslant c\sum_j 2^{-j(n+\epsilon_0)q}2^{j\delta q}\Big\|\sum_k \lambda_k \chi_{B(r_k, y_k)}\Big\|_{L^p_v}^q \\ &= c\Big\|\sum_k \lambda_k \chi_{B(r_k, y_k)}\Big\|_{L^p_v}^q\end{aligned}$$

since $\delta < n + \epsilon_0$. This shows that if

$$f_N = \sum_1^N \lambda_k g_k, \qquad \lambda_k > 0,$$

and g_k is an atom associated with B_k, then

$$\|Tf_N\|_{L^q_u} \leqslant c\Big\|\sum_1^N \lambda_k \chi_{B_k}\Big\|_{L^p_v}.$$

Let $f \in \mathcal{S}_{0,0}$. Write $f = \sum_1^\infty \lambda_k g_k$ with $\lambda_k > 0$, g_k an atom with support B_k, and (see [10])

$$\Big\|\sum \lambda_k \chi_{B_k}\Big\|_{L^p_v} \leqslant c\|f\|_{H^p_v}.$$

If

$$f_N = \sum_1^N \lambda_k g_k, \qquad \|Tf_N\|_{L^q_u} \leqslant c\Big\|\sum_1^N \lambda_k \chi_{B_k}\Big\|_{L^p_v} \leqslant c\|f\|_{H^p_v}$$

and

$$\|Tf_N - Tf_M\|_{L^q_u} \leqslant c\Big\|\sum_M^N \lambda_k \chi_{B_k}\Big\|_{L^p_v} \to 0 \qquad (N > M \to \infty).$$

Thus, Tf_N converges in L^q_u to a function h with $\|h\|_{L^q_u} \leqslant c\|f\|_{H^p_v}$. Of course, the integral defining Tf converges since $f \in \mathcal{S}_{0,0}$, and we wish to show that $Tf = h$ a.e. It is enough to show that $Tf_N \to Tf$ pointwise. We know $f_N \to f$ pointwise and $f_N \leqslant cf^*$ (f^* = grand maximal function of f). Thus, by the Lebesgue dominated convergence theorem, it suffices to prove that

$$\int f^*(y)|K(x,y)|\,dy < \infty.$$

This is clear from the earlier estimates on K since $f^*(y) \leqslant c_L(1 + |y|)^{-L}$ for all L. This proves Theorem 1.

2. Proof of Theorem 2

It is enough by considering $|f|$ and $|K|$ to prove the result for

$$T_1 f(x) = \int f(y) K_1(x, y)\, dy, \qquad K_1(x, y) = \frac{a(|x-y|, x)}{|x-y|^n}.$$

We claim it is also possible to assume that K_1 satisfies (ii) with $\epsilon_0 = 1$. To see this, let $\phi(z)$ be a nonnegative smooth function supported in $|z| < 1$ with $\phi(0) = 1$. Let $\phi_t(z) = t^{-n}\phi(z/t)$, $t > 0$, and define

$$\tilde{K}_1(x, y) = \int K_1(x, z) \phi_{|x-y|/2}(y - z)\, dz.$$

It is easy to check that there exist constants c' and c with $0 < c' < c < \infty$ such that

$$c' K_1(x, y) \leqslant \tilde{K}_1(x, y) \leqslant c K_1(x, y) \quad \text{and} \quad |\nabla_y \tilde{K}_1(x, y)| \leqslant \frac{c}{|x-y|} K_1(x, y)$$

so that $\tilde{K}_1$ satisfies (ii) with $\epsilon_0 = 1$.

Since $v \in A_p$, (3) holds with $\delta = n$. Thus, by Theorem 1 and the fact that $H_v^p \equiv L_v^p$ for $v \in A_p$, we have $\|T_1 f\|_{L_u^q} \leqslant c \|f\|_{L_v^p}$ if $f \in \mathcal{S}_{0,0}$. For general $f \in L_v^p$, since $\mathcal{S}_{0,0}$ is dense in L_v^p, there exist $f_j \in \mathcal{S}_{0,0}$ with $f_j \to f$ in L_v^p. Thus $\{T_1 f_j\}$ converges in L_u^q to a function h with $\|h\|_{L_u^q} \leqslant c \|f\|_{L_v^p}$. It is enough to show that the integral defining $T_1 f$ converges a.e. and that $h = T_1 f$ a.e. We claim that if $N < \infty$ and $v \in A_p$, then

$$\int_{|x| < N} |T_1 f(x)|\, dx \leqslant c_{N, v} \|f\|_{L_v^p}. \tag{2.1}$$

This will imply (by replacing f by $|f|$) that $T_1 f$ converges absolutely a.e.; moreover, it implies that $T_1 f_{j_k} \to T_1 f$ a.e. in $|x| < N$ for some subsequence. Since $T_1 f_{j_k} \to h$ in L_u^q, a further subsequence converges pointwise a.e. to h, and so $h = T_1 f$ a.e.

We now prove (2.1). For $f \geqslant 0$

$$T_1 f(x) \leqslant \left(\int_{|y|<1} + \int_{|y|<2|x|} + \int_{\substack{|y|>1 \\ |y|>2|x|}} \right) f(y) K_1(x, y)\, dy = F_1 + F_2 + F_3.$$

Then

$$\int_{|x|<N} (F_1(x) + F_2(x))\, dx \leqslant \int_{|y|<2N+1} f(y) \left(\int_{|x-y|<3N+1} K_1(x, y)\, dx \right) dy.$$

Also, by (iv),

$$\int_{|x-y|<3N+1} K_1(x, y)\, dx \leqslant c_N \int_{|x-y|<3N+1} \frac{dx}{|x-y|^{n-\mu}} = c_{N,\mu},$$

and

$$\int_{|y|<2N+1} f(y)\,dy \leqslant \left(\int_{|y|<2N+1} v^{-1/(p-1)}\,dy\right)^{1/p'} \|f\|_{L^p_v}$$
$$= c_{N,v}\|f\|_{L^p_v}.$$

Finally, if $|y| > 2|x|$, then $K_1(x,y) \leqslant ca(|y|,0)/|y|^n$ and

$$(2.2)\quad \int_{|x|<N} F_3(x)\,dx \leqslant c_N \int_{|y|>1} f(y)\frac{a(|y|,0)}{|y|^n}\,dy$$
$$\leqslant c_N\|f\|_{L^p_v}\left(\int_{|y|>1} v(y)^{-1/(p-1)}\frac{a(|y|,0)^{p'}}{|y|^{np'}}\,dy\right)^{1/p'}.$$

Thus, (2.1) will follow if

$$A = \int_{|y|>1} v(y)^{-1/(p-1)}\frac{a(|y|,0)^{p'}}{|y|^{np'}}\,dy < \infty.$$

Let $B_0 = \{y: |y| < 1\}$. Cover $\{y: |y| > 1\}$ by balls $\{B_k\}_{k=1}^\infty$ with $|B_k|^{1/n} \approx \operatorname{dist}(B_k, 0)$ and $\sum_1^\infty \chi_{B_k} \leqslant c$. Let $\bar{B}_k = 10B_k$ and note that $\bar{B}_k \supset B_0$. Clearly,

$$A \approx \sum_1^\infty \frac{a(B_k)^{p'}}{|B_k|^{p'}}\int_{B_k} v^{-1/(p-1)}\,dy$$
$$\approx \sum_1^\infty a(B_k)^{p'}v(B_k)^{-1/(p-1)}$$

since $v \in A_p$. If $\lambda_k \geqslant 0$,

$$\sum_1^M \lambda_k a(\bar{B}_k)u(B_0)^{1/q} \leqslant \left\|\sum_1^M \lambda_k a(\bar{B}_k)\chi_{\bar{B}_k}\right\|_{L^q_u} \quad \text{since} \quad \bar{B}_k \supset B_0$$
$$\leqslant c\left\|\sum_1^M \lambda_k \chi_{\bar{B}_k}\right\|_{L^p_v} \quad \text{by (2)}$$
$$\leqslant c\left\|\sum_1^M \lambda_k \chi_{B_k}\right\|_{L^p_v} \quad \text{by (3)}$$
$$\leqslant c\sum_1^M \lambda_k^p v(B_k)$$

since the B_k's have bounded overlaps. Pick λ_k so that

$$\lambda_k a(\bar{B}_k) = \lambda_k^p v(B_k), \quad \textit{i.e.}, \quad \lambda_k = [a(\bar{B}_k)/v(B_k)]^{1/(p-1)}.$$

Then from above,

$$\left[\sum_1^M \lambda_k a(\bar{B}_k)\right]^{1-1/p} \leqslant cu(B_0)^{-1/q}.$$

Since the constant c is independent of M, we obtain

$$\sum_1^\infty \lambda_k a(\bar{B}_k) \leqslant cu(B_0)^{-p'/q}.$$

But the sum on the left equals

$$\sum_1^\infty a(\bar{B}_k)^{1+1/(p-1)} v(B_k)^{-1/(p-1)},$$

and since $1 + 1/(p-1) = p'$, it follows that $A \leqslant cu(B_0)^{-p'/q}$. This proves that A is finite and completes the proof of Theorem 2.

3. Proof of Theorem 3

The proof is similar to that of Theorem 2. We first note from [1] that $H_v^p \equiv L_v^p$ with equivalence of norms if $v(x)/|x|^p \in A_p$ and $v(x)/|x|^{np} \in A_p$: in fact, we can then write $v(x) = |x|^p w(x)$ with $w \in A_p$ and $w(x)/|x|^{(n-1)p} \in A_p$, which fulfills the requirements in [1]. We also note by Lemma 6.3 of [12] that if $w \in D_\infty$ and $\gamma \geqslant 0$, then $|x|^\gamma w \in D_\infty$.

Now suppose that $|x|^{-\epsilon p} v \in A_p$ for some ϵ with $0 < \epsilon \leqslant \epsilon_0$, $\epsilon_0 \leqslant 1$, and $|x|^{-np} v \in A_p$. We claim that $|x|^{-p} v \in A_p$. Consider first the case of a ball B which is small compared to its distance to 0. Then $|x|$ is essentially constant on B and, consequently,

$$(3.1)\qquad \left(\frac{1}{|B|}\int_B |x|^{-p} v\,dx\right)\left(\frac{1}{|B|}\int_B [|x|^{-p} v]^{-1/(p-1)}\,dx\right)^{p-1}$$

'is bounded by a fixed multiple of the A_p constant of $|x|^{-\epsilon p} v$ (or $|x|^{-np} v$). If, n the other hand, B is not small compared to its distance to 0, we may assume ' enlarging B by a fixed factor that $B = \{x: |x| < R\}$ for some R. Next, note ıt both $|x|^{-p} v$ and $(|x|^{-p} v)^{-1/(p-1)}$ are in D_∞ since they may be written, ›ectively, as

$$|x|^{(n-1)p}(|x|^{-np} v) = |x|^\gamma w, \qquad \gamma \geqslant 0, \qquad w \in A_p,$$

and

$$|x|^{(1-\epsilon)p'}(|x|^{-\epsilon p} v)^{-1/(p-1)} = |x|^\delta w', \qquad \delta \geqslant 0, \qquad w' \in A_{p'}.$$

Hence, the product in (3.1) is equivalent to a similar product with the domains of integration replaced by $R/2 < |x| < R$. This essentialy reduces the situation to the first case and proves the claim.

It is also not difficult to see that if w satisfies $w \in A_p \cap D_d$ and $\alpha > 0$, then $|x|^\alpha w \in A_r$ for any $r > \max\{p, d + \alpha/n\}$. Of course, if $w \in A_p$ then $w \in D_p$. It follows that if v is a weight which satisfies $|x|^{-\epsilon p} v \in A_p$, then $v \in A_r$ (and D_r) if $r > p(1 + \epsilon/n)$. Thus, if $|x|^{-\epsilon p} v \in A_p$, (3) holds for any $\delta > \epsilon + n$. The requirement in Theorem 1 that $\delta < n + \epsilon_0$ is then satisfied if $\epsilon < \epsilon_0$. In case $\epsilon = \epsilon_0$, this requirement is also satisfied since if $|x|^{-\epsilon_0 p} v \in A_p$, then $|x|^{-\epsilon p} v \in A_p$ for some $\epsilon < \epsilon_0$: this is a corollary of the fact that if a weight $w \in A_p$, then $|x|^\eta w \in A_p$ for some $\eta > 0$.

Combining facts and applying Theorem 1, we see that under the hypothesis of Theorem 3, $\|Tf\|_{L^q_u} \leqslant c\|f\|_{H^p_v} \approx c\|f\|_{L^p_v}$ if $f \in \mathcal{S}_{0,0}$. Moreover, $Tf = Sf$ for such f since $\int f = 0$. Hence, $\|Sf\|_{L^q_u} \leqslant c\|f\|_{L^p_v}$ if $f \in \mathcal{S}_{0,0}$. To show the same inequality holds for any $f \in L^p_v$, it is enough as in the proof of Theorem 2 (since $\mathcal{S}_{0,0}$ is dense in L^p_v for v satisfying the hypothesis of Theorem 3) to show that Sf converges a.e. if $f \in L^p_v$ and

$$\int_{\eta < |x| < N} |Sf(x)|\, dx \leqslant c_{\eta, N, v} \|f\|_{L^p_v}, \qquad 0 < \eta < N < \infty. \tag{3.2}$$

Inequality (3.2) is similar to (2.1) and serves the same purpose. We will actually prove a stronger version of (3.2), namely, its analogue for the operator

$$S_1 f(x) = \int |f(y)|\, |K(x,y) - K(x,0)|\, dy.$$

This will prove (3.2) and show that Sf converges absolutely a.e. in $\eta < |x| < N$, and so a.e. in $\mathbb{R}^n$.

Write

$$F_1(x) = \int_{|y| < |x|/2} + \int_{|x|/2 < |y| < 2|x|} + \int_{|y| > 2|x|} = F_1 + F_2 + F_3.$$

By (ii) and (iv),

$$\begin{aligned} F_1(x) &\leqslant c \int_{|y| < |x|/2} |f(y)| \left(\frac{|y|}{|x|}\right)^{\epsilon_0} \frac{a(|x|, 0)}{|x|^n}\, dy \\ &\leqslant c \frac{a(|x|, 0)}{|x|^{n+\epsilon_0}} \int_{|y| < N} |f(y)|\, |y|^{\epsilon_0}\, dy \quad \text{if} \quad |x| < N. \end{aligned}$$

We have

$$\begin{aligned} \int_{|y| < N} |f(y)|\, |y|^{\epsilon_0} &\leqslant \|f\|_{L^p_v} \left(\int_{|y| < N} |y|^{\epsilon_0 p'} v(y)^{-p'/p}\, dy\right)^{1/p'} \\ &= c_{N, v} \|f\|_{L^p_v}, \end{aligned}$$

since the fact that $v = |y|^{\epsilon p} w$, $w \in A_p$, $\epsilon \leqslant \epsilon_0$, implies that

$$\int_{|y|<N} |y|^{\epsilon_0 p'} v(y)^{-p'/p}\, dy = \int_{|y|<N} |y|^{(\epsilon_0 - \epsilon)p'} w(y)^{-1/(p-1)}\, dy \leqslant N^{(\epsilon_0-\epsilon)p'} \int_{|y|<N} w(y)^{-1/(p-1)}\, dy.$$

It follows easily that

$$\int_{\eta<|x|<N} |F_1|\, dx \leqslant c_{\eta,N,v} \|f\|_{L_v^p}.$$

Now consider F_2. First note that f is integrable away from 0: for $\eta > 0$,

$$\int_{|y|>\eta} |f(y)|\, dy \leqslant \|f\|_{L_v^p} \left(\int_{|y|>\eta} v(y)^{-1/(p-1)}\, dy \right)^{1/p'} = c_{\eta,v} \|f\|_{L_v^p}$$

since the fact that $v = |y|^{np} w_1$, $w_1 \in A_p$, gives

$$\int_{|y|>\eta} v(y)^{-1/(p-1)}\, dy = \int_{|y|>\eta} \frac{w_1(y)^{-1/(p-1)}}{|y|^{np'}}\, dy,$$

which is finite since $w_1^{-1/(p-1)} \in A_{p'}$ (see, *e.g.*, (2.3) of [7] for the case $n = 1$). We have

$$F_2(x) \leqslant \int_{|x|/2<|y|<2|x|} |f(y)| \{ |K(x,y)| + |K(x,0)| \}\, dy.$$

Hence, by (i) and (iv),

$$\begin{aligned}
\int_{\eta<|x|<N} &|F_2(x)|\, dx \\
&\leqslant c_N \int_{|y|>\eta/2} |f(y)| \left[\int_{|x-y|<3N} \frac{dx}{|x-y|^{n-\mu}} + \int_{\eta<|x|<N} \frac{dx}{|x|^n} \right] dy \\
&= c_{N,\eta} \int_{|y|>\eta/2} |f(y)|\, dy \\
&\leqslant c_{N,\eta,v} \|f\|_{L_v^p}.
\end{aligned}$$

Finally, since $|K(x,y)| \leqslant ca(|y|,0)|y|^{-n}$ for $|y| > 2|x|$,

$$F_3(x) \leqslant c \int_{|y|>2|x|} |f(y)| \left[\frac{a(|y|,0)}{|y|^n} + \frac{a(|x|,0)}{|x|^n} \right] dy,$$

and

$$\int_{\eta<|x|<N} F_3(x)\,dx \leqslant c_N \int_{|y|>2\eta} |f(y)| \frac{a(|y|,0)}{|y|^n}\,dy + \int_{|y|>2\eta} |f(y)|\,dy \int_{\eta<|x|<N} \frac{a(|x|,0)}{|x|^n}\,dx.$$

The second term on the right is at most $c_{\eta,N,v}\|f\|_{L^p_v}$ as above. The first term is like the integral on the right in (2.2) and can be treated by the argument given there. This completes the proof of Theorem 3.

4. Proof of Theorem 4

We first show that (2) is necessary for $\|Mf\|_{L^{p'}_v} \leqslant c\|f\|_{L^{q'}_u}$ if $1<p,\ q<\infty$ and

$$Mf(x) = \sup_{B\ni x} \frac{a(B)}{v(B)} \int_B |f|u.$$

The proof is by duality. For $\lambda_k > 0$,

$$\left\|\sum \lambda_k a(B_k)\chi_{B_k}\right\|_{L^q_u} = \sup_{\|g\|_{L^{q'}_u}=1} \int \left(\sum \lambda_k a(B_k)\chi_{B_k}\right) gu.$$

The integral on the right equals

$$\begin{aligned}\sum \lambda_k a(B_k) \int_{B_k} gu &= \sum \lambda_k v(B_k) \left\{\frac{a(B_k)}{v(B_k)} \int_{B_k} gu\right\}\\ &\leqslant \sum \lambda_k v(B_k) \inf_{B_k} M(g)\\ &\leqslant \sum \lambda_k \int_{B_k} M(g)v\\ &= \int \left(\sum \lambda_k \chi_{B_k}\right) M(g)v.\end{aligned}$$

By Hölder's inequality and hypothesis, this is at most

$$\left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v} \|Mg\|_{L^{p'}_v} \leqslant \left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v} c\|g\|_{L^{q'}_u} = c\left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v}.$$

Taking the supremum over g, we obtain (2). Note that the proof works without any hypothesis on a, u, and v except $a \geqslant 0$ and $v(B) > 0$.

For the converse, we also assume (iii) and $v \in D_\infty$. If I is a cube, let $a(I) = a(B)$ where B is the smallest ball containing I. It is then easy to see that (2) for balls is equivalent to (2) for cubes. For example, if I_k is a cube and B_k is the smallest ball containing I_k, and if (2) holds for balls, then

$$\begin{aligned}\left\|\sum \lambda_k a(I_k)\chi_{I_k}\right\|_{L^q_u} &\leqslant \left\|\sum \lambda_k a(B_k)\chi_{B_k}\right\|_{L^q_u} \qquad (\lambda_k \geqslant 0)\\ &\leqslant c\left\|\sum \lambda_k \chi_{B_k}\right\|_{L^p_v}\\ &\leqslant c\left\|\sum \lambda_k \chi_{I_k}\right\|_{L^p_v}\end{aligned}$$

by (3), since $B_k \subset \alpha I_k$, $\alpha = \alpha_n$. A similar argument shows that (2) for cubes implies (2) for balls. Similarly, since $v \in D_\infty$ and (iii) holds, Mf is equivalent to its analogue defined by using cubes rather than balls.

Let G be a fixed dyadic grid of cubes. For $t \in \mathbb{R}^n$, let ${}^tG$ be the grid obtained by shifting G by t. Define

$${}^tMf(x) = \sup_{\substack{I \ni x\\ I \in {}^tG}} \frac{a(I)}{v(I)} \int_I |f|u.$$

We need the following analogue of Lemma 2 of [9].

Lemma 4.1. *Let a satisfy* (iii) *and $v \in D_\infty$. Then*

$$\sup_{\substack{I \ni x\\ |I| \leq r^n}} \frac{a(I)}{v(I)} \int_I |f|u \leqslant c\frac{1}{r^n}\int_{|t|<r} {}^tMf(x)\,dt, \qquad 0 < r < \infty,$$

'th c independent of r and x.

'. Fix $x = x_0$ and a cube I_0 containing x_0 with edgelength $h_0 \leqslant r$. Con-
ose dyadic I in $B_{2r}(x_0)$ with edgelength $2h_0$. The number of such I is
'. If I is such a cube and x_I is the center of I, then shifting I by
$+ \eta$ for any η with $|\eta| < h_0/2$ gives a cube ${}^tI$ containing I_0 with
'ince different I's lead to essentially disjoint sets of t's, the
set E of all t is $\approx (r^n/|I_0|)(h_0/2)^n \approx r^n$. If $t \in E$ and ${}^tI$ is the
then by (iii) and the doubling property of v,

$$\begin{aligned}\frac{a(I_0)}{v(I_0)}\int_{I_0}|f|u &\leqslant c\frac{a({}^tI)}{v({}^tI)}\int_{{}^tI}|f|u\\ &\leqslant c\,{}^tMf(x_0).\end{aligned}$$

Hence,

$$\frac{a(I_0)}{v(I_0)}\int_{I_0}|f|u \leqslant c\frac{1}{r^n}\int_{|t|<r} {}^tMf(x_0)\,dt,$$

and the lemma follows.

Denote the expression on the left in Lemma 4.1 by $M_r f(x)$ and apply Minkowski's integral inequality to obtain

$$\|M_r f\|_{L_v^{p'}} \leqslant c \frac{1}{r^n} \int_{|t|<r} \|{}^t M f\|_{L_v^{p'}} dt$$

with c independent of r. Hence, if we prove that

$$\|{}^t M f\|_{L_v^{p'}} \leqslant c \|f\|_{L_u^{q'}} \tag{4.2}$$

with c independent of t, then we will obtain $\|M_r f\|_{L_v^{p'}} \leqslant c\|f\|_{L_u^{q'}}$ with c independent of r, and Theorem 4 will follow by letting $r \to \infty$.

To prove (4.2), fix t and f and let

$$E_k = \{x: 2^k < {}^t M f(x) \leqslant 2^{k+1}\},$$

$k = 0, \pm 1, \pm 2, \ldots$ The E_k are disjoint. By considering maximal cubes, it follows from the dyadic nature of ${}^t G$ that we may write $E_k = \bigcup_j I_{j,k}$, $I_{j,k} \in {}^t G$, $I_{j,k}$ disjoint in j (and so in j and k), and

$$2^k < \frac{a(I_{j,k})}{v(I_{j,k})} \int_{I_{j,k}} |f| u \leqslant 2^{k+1}.$$

Therefore,

$$\begin{aligned} 2^{kp'} v(E_k) &= \sum_j 2^{kp'} v(I_{j,k}) \\ &\approx \sum_j \left(\frac{a(I_{j,k})}{v(I_{j,k})} \int_{I_{j,k}} |f| u \right)^{p'} v(I_{j,k}) \\ &= \sum_j \left(\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u \right)^{p'}. \end{aligned}$$

Since

$$\|{}^t M f\|_{L_v^{p'}} \approx \left(\sum_j 2^{kp'} v(E_k) \right)^{1/p'},$$

we obtain

$$\begin{aligned} \|{}^t M f\|_{L_v^{p'}} &\approx \left[\sum_{j,k} \left(\frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u \right)^{p'} \right]^{1/p'} \\ &\approx \sup \sum_{j,k} b_{j,k} \frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u, \end{aligned}$$

where the supremum is taken over all sequences $\{b_{j,k}\}$ with $b_{j,k} \geqslant 0$ and

$$\|\{b_{j,k}\}\|_{l^p} = \Big(\sum_{j,k} b_{j,k}^p\Big)^{1/p} = 1.$$

Now

$$\begin{aligned}
\sum_{j,k} b_{j,k} \frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} |f| u &= \int |f| \Big\{ \sum_{j,k} b_{j,k} \frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \varkappa_{I_{j,k}} \Big\} u \\
&\leqslant \|f\|_{L_u^{q'}} \Big\| \sum_{j,k} b_{j,k} \frac{a(I_{j,k})}{v(I_{j,k})^{1/p}} \varkappa_{I_{j,k}} \Big\|_{L_u^q} \\
&\leqslant c \|f\|_{L_u^{q'}} \Big\| \sum_{j,k} b_{j,k} \frac{1}{v(I_{j,k})^{1/p}} \varkappa_{I_{j,k}} \Big\|_{L_v^p}
\end{aligned}$$

by hypothesis, with c independent of t and f. We shall use duality to show that

$$\Big\| \sum_{j,k} b_{j,k} \frac{1}{v(I_{j,k})^{1/p}} \varkappa_{I_{j,k}} \Big\|_{L_v^p} \leqslant c_{p,v}. \tag{4.3}$$

For $g \in L_v^{p'}$, $g \geqslant 0$,

$$\begin{aligned}
\int \sum_{j,k} b_{j,k} \frac{1}{v(I_{j,k})^{1/p}} \varkappa_{I_{j,k}} g v &= \sum_{j,k} b_{j,k} \frac{1}{v(I_{j,k})^{1/p}} \int_{I_{j,k}} g v \\
&\leqslant \|\{b_{j,k}\}\|_{l^p} \Big(\sum_{j,k} \Big[\frac{1}{v(I_{j,k})} \int_{I_{j,k}} g v \Big]^{p'} v(I_{j,k}) \Big)^{1/p'} \\
&\leqslant \Big(\sum_{j,k} \int_{I_{j,k}} H_v(g)^{p'} v \Big)^{1/p'} \\
&\leqslant \|H_v(g)\|_{L_v^{p'}},
\end{aligned}$$

where $H_v(g)$ denotes the maximal function of Hardy-Littlewood type defined by

$$H_v(g)(x) = \sup_{I \ni x} \frac{1}{v(I)} \int_I |g| v \, dy.$$

Since $v \in D_\infty$, $\|H_v(g)\|_{L_v^{p'}} \leqslant c \|g\|_{L_v^{p'}}$, and it follows that (4.3) holds. Collecting estimates, we obtain (4.2). This completes the proof of Theorem 4.

5. Proof of Theorem 5

It is enough to prove the analogous result for cubes instead of balls, *i.e.*, to show that under the hypothesis of Theorem 5,

$$\left\|\sum \lambda_k a(I_k)\chi_{I_k}\right\|_{L^q_u} \leqslant c\left\|\sum \lambda_k \chi_{I_k}\right\|_{L^p_v} \tag{5.1}$$

if $\lambda_k \geqslant 0$ and the I_k are cubes. In fact, it is enough to prove this for dyadic cubes as we now show. If I is any cube, there are 2^n dyadic cubes J_i with $|J_i| \approx |I|$ and $I \subset UJ_i$. By (iii) and (iv), $a(I) \leqslant ca(J_i)$ with c independent of I and J_i. Find such a covering for each I_k and denote the dyadic cubes by $J_{k,i}$. Then

$$\begin{aligned}\sum \lambda_k a(I_k)\chi_{I_k} &\leqslant \sum_k \lambda_k a(I_k) \sum_i \chi_{J_{k,i}} \\ &\leqslant c\sum_i \left[\sum_k \lambda_k a(J_{k,i})\chi_{J_{k,i}}\right].\end{aligned}$$

Thus

$$\begin{aligned}\left\|\sum \lambda_k a(I_k)\chi_{I_k}\right\|_{L^q_u} &\leqslant c\sum_i \left\|\sum_k \lambda_k a(J_{k,i})\chi_{J_{k,i}}\right\|_{L^q_u} \\ &\leqslant c\sum_i \left\|\sum_k \lambda_k \chi_{J_{k,i}}\right\|_{L^p_v},\end{aligned}$$

assuming that (5.1) holds for dyadic cubes. However, it is easy to see from (3) that

$$\left\|\sum_k \lambda_k \chi_{J_{k,i}}\right\|_{L^p_v} \leqslant c\left\|\sum_k \lambda_k \chi_{I_k}\right\|_{L^p_v}$$

with c independent of i. Thus, since the number of i's is finite, we obtain by combining inequalities that

$$\left\|\sum \lambda_k a(I_k)\chi_{I_k}\right\|_{L^q_u} \leqslant c\left\|\sum \lambda_k \chi_{I_k}\right\|_{L^p_v},$$

as desired.

For the rest of the proof, we will assume that the cubes $\{I_k\}$ are dyadic. Fix $\{\lambda_k\}$, $\lambda_k \geqslant 0$, and let

$$M_\epsilon(x) = \sup_k \lambda_k a(I_k) u(I_k)^\epsilon \chi_{I_k}(x). \tag{5.2}$$

We claim that

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant c[M_\epsilon(x) M_{-\epsilon}(x)]^{1/2}, \qquad \epsilon > 0, \tag{5.3}$$

with c depending only on ϵ, n, and u. Fix x. Note that there is at most one I_k of a given size which contains x since the cubes are dyadic. We may then assume that those I_k containing x are ordered in size, *i.e.*, that $I_k \subset I_j$ if $k < j$. For $k_0 = k_0(x)$ to be chosen and $\epsilon > 0$, write

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) = \sum_{k \le k_0} \{\lambda_k a(I_k)u(I_k)^{-\epsilon}\chi_{I_k}(x)\}u(I_k)^{\epsilon}$$
$$+ \sum_{k > k_0} \{\lambda_k a(I_k)u(I_k)^{\epsilon}\chi_{I_k}(x)\}u(I_k)^{-\epsilon}$$
$$\leqslant M_{-\epsilon}(x) \sum_{k \le k_0} u(I_k)^{\epsilon} + M_{\epsilon}(x) \sum_{k > k_0} u(I_k)^{-\epsilon}.$$

The I_k's in these sums are dyadic cubes containing x, and they are ordered in size. There may not be cubes containing x of every (dyadic) size, but if there are missing sizes we just add cubes of those sizes, thereby increasing the sums on the right side above. We do not add any cubes to the collection used to define M_ϵ and $M_{-\epsilon}$. It follows easily from the doubling condition on u and the dyadic nature of the cubes that for any k_0

$$\sum_{k \le k_0} u(I_k)^{\epsilon} \leqslant cu(I_{k_0})^{\epsilon}$$

and

$$\sum_{k \le k_0} u(I_k)^{-\epsilon} \leqslant cu(I_{k_0})^{-\epsilon},$$

with c depending only on n, ϵ and u. Thus,

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant c[M_{-\epsilon}(x)u(I_{k_0})^{\epsilon} + M_{\epsilon}(x)u(I_{k_0})^{-\epsilon}].$$

Pick k_0 so that

$$u(I_{k_0})^{\epsilon} \approx \{M_{\epsilon}(x)/M_{-\epsilon}(x)\}^{1/2},$$

and (5.3) follows immediately.

By (5.3)

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L^q_u} \leqslant c\left(\int M_{\epsilon}^{q/2}M_{-\epsilon}^{q/2}u\,dx\right)^{1/q}.$$

For small $\epsilon > 0$ to be chosen, let q_1 and q_2 be defined by

$$\frac{1}{q_1} = \frac{1}{q} - \epsilon, \qquad \frac{1}{q_2} = \frac{1}{q} + \epsilon.$$

Then $1/q_1 + 1/q_2 = 2/q$, so that $2q_1/q$ and $2q_2/q$ are conjugate indices. By Hölder's inequality

$$\left(\int M_{\epsilon}^{q/2}M_{-\epsilon}^{q/2}u\,dx\right)^{1/q} \leqslant \|M_{\epsilon}\|_{L^{q_1}_u}^{1/2}\|M_{-\epsilon}\|_{L^{q_2}_u}^{1/2}.$$

The hypothesis (1) that $a(I)u(I)^{1/q} \leqslant cv(I)^{1/p}$ may be rewritten as both

$$a(I)u(I)^{1/q_1+\epsilon} \leqslant cv(I)^{1/p} \quad \text{and} \quad a(I)u(I)^{1/q_2-\epsilon} \leqslant cv(I)^{1/p}.$$

Also, since $q > p$, we have $q_1, q_2 \geqslant p$ for small $\epsilon > 0$. To complete the proof, we need the following lemma.

Lemma 5.4. *Let $u \in D_\infty$ and $a(I)$ satisfy (iii) and (iv). Let M_ϵ be defined by (5.2) for a collection of dyadic cubes $\{I_k\}$, and assume that*

$$a(I)u(I)^{1/q+\epsilon} \leqslant cv(I)^{1/p}$$

for all cubes, $1 < p \leqslant q < \infty$. There is a number $\eta > 0$ depending on u and a so that if $\epsilon > -\eta$ and $v \in A_p$,

$$\|M_\epsilon\|_{L_u^q} \leqslant c\Big\|\sum \lambda_k \chi_{I_k}\Big\|_{L_v^p}.$$

As we shall see, the value of η can be taken to be $\mu/n\sigma$ where μ is the parameter in (iv) and $u \in D_\sigma$.

Before proving the lemma, we note that Theorem 5 follows by combining it with the facts above, since then for small $\epsilon > 0$

$$\|M_\epsilon\|_{L_u^{q_1}}^{1/2}\|M_{-\epsilon}\|_{L_u^{q_2}}^{1/2} \leqslant c\Big\|\sum \lambda_k \chi_{I_k}\Big\|_{L_v^p}^{1/2+1/2} = c\Big\|\sum \lambda_k \chi_{I_k}\Big\|_{L_v^p}.$$

PROOF OF LEMMA 5.4. Write

$$\begin{aligned} \|M_\epsilon\|_{L_u^q}^q &= \int \sup_k \{\lambda_k a(I_k) u(I_k)^\epsilon \chi_{I_k}\}^q u\,dx \\ &\leqslant \sum \lambda_k^q a(I_k)^q u(I_k)^{\epsilon q+1}. \end{aligned}$$

Let

$$g(x) = \sum \lambda_k \chi_{I_k}(x).$$

For $j = 0, \pm 1, \pm 2, \ldots$, let $\mathfrak{I}_j = \{I_k : |I_k| = 2^{jn}\}$ and define

$$g_j(x) = \left(\frac{1}{|I_k|}\int_{I_k} g\right)\chi_{I_k}(x), \qquad I_k \in \mathfrak{I}_j.$$

Then

$$\begin{aligned} \sum \lambda_k^q a(I_k)^q u(I_k)^{\epsilon q+1} &= \sum_j \sum_{I_k \in \mathfrak{I}_j} \lambda_k^q a(I_k)^q u(I_k)^{\epsilon q+1} \\ &\leqslant \sum_j \sum_{I_k \in \mathfrak{I}_j} \left(\frac{1}{|I_k|}\int_{I_k} g_j^q\right) a(I_k)^q u(I_k)^{\epsilon q+1} \end{aligned}$$

$$= \sum_j \int g_j(x)^q \Big\{2^{-jn} \sum_{I_k \in \mathfrak{I}_j} a(I_k)^q u(I_k)^{\epsilon q+1}\chi_{I_k}(x)\Big\}\,dx. \tag{5.5}$$

Think of $g_j(x)$ as a function on $\mathbb{R}^n \times \{2^j\}$, and think of

$$\mathrm{dm}\,(x,j) = 2^{-jn} \sum_{I_k \in \mathfrak{I}_j} a(I_k)^q u(I_k)^{\epsilon q+1} \chi_{I_k}(x)\,dx$$

as a measure. We claim that dm is a (q,p)-Carleson measure with respect to $v(x)\,dx$ *i.e.*, that if J is a cube in $\mathbb{R}^n$ with edgelength h, then

$$\sum_{2^j \leq h} \int_J \left\{ 2^{-jn} \sum_{I_k \in \mathfrak{I}_j} a(I_k)^q u(I_k)^{\epsilon q+1} \chi_{I_k}(x) \right\} dx \leqslant cv(J)^{q/p} \tag{5.6}$$

with c independent of J. Since $v \in D_\infty$, we may assume that J is dyadic. The I_k's in a given $\mathfrak{I}_j$ are disjoint, and those above which intersect J have edgelength $2^j \leqslant h$, *i.e.*, are smaller than J, and so are contained in J. Hence, after performing the integration with respect to x, we see that the left side of (5.6) equals

$$\sum_{2^j \leq h} \sum_{\substack{|I_k| = 2^{jn} \\ I_k \subset J}} a(I_k)^q u(I_k)^{\epsilon q+1}. \tag{5.7}$$

From (iv)

$$a(I_k) \leqslant c\left(\frac{|I_k|}{|J|}\right)^{\mu/n} a(J) = c\left(\frac{2^j}{h}\right)^{\mu} a(J).$$

Thus, (5.7) is at most

$$c\frac{a(J)^q}{h^{\mu q}} \sum_{2^j \leq h} 2^{j\mu q} \sum_{\substack{I_k \in \mathfrak{I}_j \\ I_k \subset J}} u(I_k)^{\epsilon q+1}.$$

In case $\epsilon \geqslant 0$, we have

$$u(I_k)^{\epsilon q+1} \leqslant u(J)^{\epsilon q} u(I_k) \quad \text{if} \quad I_k \subset J,$$

and therefore the last estimate is bounded by

$$c\frac{a(J)^q}{h^{\mu q}} \sum_{2^j \leq h} 2^{j\mu q} u(J)^{\epsilon q} \sum_{\substack{I_k \in \mathfrak{I}_j \\ I_k \subset J}} u(I_k) \leqslant c\frac{a(J)^q}{h^{\mu q}} h^{\mu q} u(J)^{\epsilon q} u(J)$$
$$= ca(J)^q u(J)^{\epsilon q+1}$$

since the cubes in each $\mathfrak{I}_j$ are disjoint. By hypothesis,

$$a(J)^q u(J)^{\epsilon q+1} \leqslant cv(J)^{q/p},$$

and (5.6) follows in this case. If instead $\epsilon < 0$, then since $u \in D_\sigma$

$$u(J) \leqslant c(|J|/|I_k|)^{\sigma} u(I_k) \quad \text{if} \quad I_k \subset J,$$

and we see the estimate above is at most

$$c\frac{a(J)^q}{h^{\mu q}}\sum_{2^j\le h} 2^{j\mu q}\sum_{\substack{I_k\in\mathfrak{I}_j\\ I_k\subset J}}[(|I_k|/|J|)^\sigma u(J)]^{\epsilon q}u(I_k)$$

$$= c\frac{a(J)^q}{h^{\mu q+n\sigma\epsilon q}}\sum_{2^j\le h} 2^{j(\mu q+n\sigma\epsilon q)}u(J)^{\epsilon q}\sum_{\substack{I_k\in\mathfrak{I}_j\\ I_k\subset J}}u(I_k)$$

$$= ca(J)^q u(J)^{\epsilon q+1},$$

provided $\mu q + n\sigma\epsilon q > 0$, *i.e.*, $\epsilon > -\mu/n\sigma$. Thus, (5.6) again follows.

Using (5.6) and Carleson's theorem, we see that (5.5) is at most $c\|g^*\|_{L^p_v}$, where

$$g^*(x) = \sup\{g_j(y): (y,j) \text{ satisfies } |x-y| < 2^j\}.$$

Fix x and suppose $|x-y| < 2^j$. By definition of g_j, if $y\in I_k\in\mathfrak{I}_j$,

$$g_j(y) = \frac{1}{|I_k|}\int_{I_k} g \leqslant \frac{c}{|I|}\int_I g$$

where I is any interval containing x of edgelength $\approx 2^j$. If there is no such I_k for y, then $g_j(y) = 0$. Thus, g^* is majorized by a multiple of the Hardy-Littlewood maximal function of g. Therefore, $\|g^*\|_{L^p_v} \leqslant c\|g\|_{L^p_v}$ by [8] since $v\in A_p$. Combining estimates, we obtain Lemma 5.4; this completes the proof of Theorem 5.

6. Proof of Theorem 6

The proof is similar to that of Theorem 5. As there, it is enough to prove the result for dyadic cubes instead of balls. Fix x and order those I_k containing x according to size as before. Let $g(x) = \sum\lambda_k\chi_{I_k}(x)$ and

$$A_\epsilon(x) = \sup_k\{\lambda_k a(I_k)^{1+\epsilon}\chi_{I_k}(x)\}, \qquad \epsilon > 0.$$

For k_0 and ϵ to be chosen, write

$$\sum\lambda_k a(I_k)\chi_{I_k}(x) = \sum_{k\le k_0} + \sum_{k>k_0}$$

$$\leqslant \Big[\sup_{k\le k_0} a(I_k)\chi_{I_k}(x)\Big]g(x) + A_\epsilon(x)\sum_{k>k_0} a(I_k)^{-\epsilon}\chi_{I_k}(x).$$

By adding cubes if necessary, we may assume that $\sup_{k\le k_0} a(I_k)\chi_{I_k}(x)$ and $\sum_{k>k_0} a(I_k)^{-\epsilon}\chi_{I_k}(x)$ are taken over cubes of every dyadic size containing x.

We do not add any cubes to the collection used to define g or A_ϵ. Thus, by (iii) and (iv),

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant ca(I_{k_0})g(x) + ca(I_{k_0})^{-\epsilon}A_\epsilon(x).$$

Pick k_0 so that $a(I_{k_0}) \approx (A_\epsilon(x)/g(x))^{1/(1+\epsilon)}$. Then

$$\sum \lambda_k a(I_k)\chi_{I_k}(x) \leqslant cg(x)^{1/r'}A_\epsilon(x)^{1/r}, \qquad r = 1+\epsilon.$$

By Hölder's inequality,

$$\left\| \sum \lambda_k a(I_k)\chi_{I_k} \right\|_{L^p_u} \leqslant c\left(\int g^p v \right)^{1/pr'} \left(\int A^p_\epsilon \left(\frac{u}{v} \right)^r v \right)^{1/pr}. \tag{6.1}$$

If we show that

$$\int A^p_\epsilon \left(\frac{u}{v} \right)^r v \leqslant c \int g^p v, \qquad r = 1+\epsilon, \tag{6.2}$$

then the right side of (6.1) is at most $c\|g\|_{L^p_v}$, and Theorem 6 follows.

To prove (6.2), note that since

$$A_\epsilon(x)^p \leqslant \sum \lambda^p_k a(I_k)^{pr}\chi_{I_k}(x), \qquad r = 1+\epsilon,$$

the integral on the left is bounded by

$$\sum \lambda^p_k a(I_k)^{pr} \int_{I_k} \left(\frac{u}{v} \right)^r v.$$

Using the same notation as in the proof of Theorem 5, we see this equals

$$\sum_j \sum_{I_k \in \mathfrak{Z}_j} \left(\frac{1}{|I_k|} \int_{I_k} g^p_j \right) a(I_k)^{pr} \int_{I_k} \left(\frac{u}{v} \right)^r v$$

$$= \sum_j \int g_j(x)^p \left\{ \sum_{I_k \in \mathfrak{Z}_j} a(I_k)^{pr} \left(\frac{1}{|I_k|} \int_{I_k} \left(\frac{u}{v} \right)^r v \right) \chi_{I_k}(x) \right\}.$$

Thus, (6.2) will be as before if we show that the expression in curly brackets is a (p,p) Carleson measure with respect to $v(x)\,dx$. If J is a cube in $\mathbb{R}^n$ and $|J| = h^n$, we must show that

$$\sum_{2^j \leq h} \int_J \left\{ \sum_{I_k \in \mathfrak{Z}_j} a(I_k)^{pr} \left(\frac{1}{|I_k|} \int_{I_k} \left(\frac{u}{v} \right)^r v \right) \chi_{I_k}(x) \right\} dx \leqslant cv(J). \tag{6.3}$$

Arguing as before with J dyadic, we see this amounts to proving that

$$\sum_{2^j \leq h} \sum_{\substack{I_k \in \mathfrak{Z}_j \\ I_k \subset J}} a(I_k)^{pr} \int_{I_k} \left(\frac{u}{v} \right)^r v \leqslant cv(J).$$

Since $a(I_k) \leqslant c(|I_k|/|J|)^{\mu/n} a(J)$ if $I_k \subset J$, the sum on the left is majorized by

$$ca(J)^{pr} \sum_{2^j \le h} \left(\frac{2^j}{h}\right)^{\mu pr} \sum_{\substack{I_k \in \mathfrak{I}_j \\ I_k \subset J}} \int_{I_k} \left(\frac{u}{v}\right)^r v \leqslant ca(J)^{pr} \sum_{2^j \le h} \left(\frac{2^j}{h}\right)^{\mu pr} \int_J \left(\frac{u}{v}\right)^r v,$$

since the cubes in $\mathfrak{I}_j$ are disjoint and have edgelength 2^j. The last expression equals

$$ca(J)^{pr} \int_{I_k} \left(\frac{u}{v}\right)^r v. \tag{6.4}$$

Since (1)′ holds, the following lemma shows that if r is chosen near 1 (*i.e.*, ϵ is chosen small), then (6.4) is at most $cv(J)$, and the proof of (6.3) is complete.

Lemma 6.4. *If $v \in A_\infty$, the following two conditions are equivalent:*

(*a*) *condition* (1)′, *i.e., there exists $s > 1$ such that*

$$a(I)^p \left(\frac{1}{|I|} \int_I u^s\right)^{1/s} \leqslant c \frac{v(I)}{|I|}$$

for all cubes I;

(*b*) *there exists $r > 1$ such that*

$$a(I)^p \left(\frac{1}{v(I)} \int_I \left(\frac{u}{v}\right)^r v\right)^{1/r} \leqslant c$$

for all cubes I.

The proof is essentially the same as that given in [2] for the case $a(I) = |I|^{1/n}$ and we shall not repeat the details.

References

[1] Adams, E. On the identification of weighted Hardy spaces, *Indiana Univ. Math. J.* **32**(1983), 477-489.

[2] Chanillo, S. and Wheeden, R. L. L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators, *Comm. P.D.E.* **10**(1985), 1077-1116.

[3] —. Existence and estimates of Green's function for degenerate elliptic equations, *Am. Scuola Norm. Sup. Pisa.*, to appear.

[4] Fabes, E. B., Jerison, D. S. and Kenig, C. E. The Wiener test for degenerate elliptic equations, *Ann. Inst. Fourier.* **32**(1982), 151-182.

[5] Fefferman, C. L. and Stein, E. M. H^p spaces of several variables, *Acta Math.* **129**(1972), 137-193.

[6] Gatto, A. E., Gutiérrez, C. E. and Wheeden, R. L. Fractional integrals on weighted H^p spaces, *Trans. Amer. Math. Soc.* **289**(1985), 575-589.
[7] Hunt, R., Muckenhoupt, B. and Wheeden, R. L. Weighted norm inequalities for the conjugate function and Hilbert transform, *Trans. Amer. Math. Soc.* **176**(1973), 227-251.
[8] Muckenhoupt, B. Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165**(1972), 207-226.
[9] Sawyer, E. A characterization of a two-weight norm inequality for maximal operators, *Studia Math.* **75**(1982), 1-11.
[10] Strömberg, J. O. and Torchinsky, A. Weighted Hardy Spaces, to appear.
[11] Strömberg , J. O. and Wheeden, R. L. Relations between H^p_u and L^p_u with polynomial weights, *Trans. Amer. Math. Soc.* **270**(1982), 439-467.
[12] —. Fractional integrals on weighted H^p and L^p spaces, *Trans. Amer. Math. Soc.* **287**(1985), 293-321.

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Structural Stability and Generic Properties of Planar Polynomial Vector Fields

Douglas S. Shafer

Dedicated to my father, Philip S. Shafer,
on his seventieth birthday
May 5, 1988

In 1962 Peixoto [14] gave a complete characterization of the structurally stable C^1 vector fields on any compact, two-dimensional manifold without boundary, and showed that they form a dense, open set in the space of all C^1 vector fields with the uniform C^1 topology (see also [6], [7]). There later followed examples showing that, on any non-compact two-manifold, there exists an open set of vector fields, none of which is structurally stable ([10]; see also [16], [20]). Nevertheless, Kotus, Krych, and Nitecki [10] showed how to control behavior «at infinity» so as to guarantee stability of a vector field on any two-manifold under strong C^r peturbation, and gave a complete chracterization of the structurally stable vector fields on $\mathbb{R}^2$ (see also [3]). In this paper we consider the set $\mathfrak{P}_n$ of polynomial vector fields of degree $\leqslant n$ on $\mathbb{R}^2$, and give sufficient conditions for structural stability of $X \in \mathfrak{P}_n$ with respect to perturbation in $\mathfrak{P}_n$ (Theorem 3.2). Up to an added condition that asymptotically stable (or unstable) limit cycles be hyperbolic, these same conditions are also necessary (Theorem 3.3). We use the coefficient topology on $\mathfrak{P}_n$, and require that the equivalence homeomorphism lie in a pre-assigned compact-open neighborhood of $id_{\mathbb{R}^2}$. Briefly, in addition to the usual conditions that X be Morse-Smale, we have regularity conditions on the associated Poincaré vector field $\pi(X)$ along the equator S^1 of the Poincaré sphere S^2, and conditions on certain distinguished orbits, so-called separatrices of saddles-at-infinity, in their relation to the saddle points of X and to the critical points of $\pi(X)$ at infinity. The polynomial nature of the problem simplifies the dynamics to the point that we are able to prove existence of a dense, open set of structurally stable vector fields (Theorem 4.1). On the other hand, the analysis is complicated by the fact that the set of allowable perturbations is small, and every one of them is global and large at infinity. In particular, we have been unable to resolve the question, raised explicitly in [19] and implicitly in [1, §6.3], of whether a limit cycle of odd multiplicity must be hyperbolic in order to be locally structurally stable. (This question seems to have been overlooked in [21] and [23].)

We note that the results of this paper do not depend on the validity of Dulac's Theorem asserting that a polynomial vector field has at most finitely many limit cycles, a correct proof of which has not yet appeared in print. If a valid proof were given, then the statements and proofs of a number of results here would simplify greatly, and the similarities to the case of stability of smooth vector fields on compact manifolds increase.

In comparing this work with previous work on stability of polynomial vector fields ([19], [21], [23], for example), it should be noted that, while we exploit the Poincaré compactification of $\mathbb{R}^2$ (see §1), we do not require that the Poincaré vector fields $\pi(X)$ and $\pi(Y)$ be equivalent on S^2 in order for X and Y to be equivalent. This approach seems more natural, more closely mimics the compact case, and allows closer comparison with results of the general theory of stability of smooth vector fields on $\mathbb{R}^2$. Of course, more complicated behavior is now consistent with structural stability.

Moreover, in our setting it is natural to allow smooth as well as polynomial perturbations of $X \in \mathfrak{P}_n$. Building on work of Kotus, Krych, and Nitecki [10], we fully characterize elements of $\mathfrak{P}_n$ that are stable under small perturbation (with respect to the Whitney C^r topology) by C^r vector fields, $r \geqslant 1$ (Theorem 3.1). Again the equivalence homeomorphism must be close to the identity. Strong control at infinity leads to a particularly simple result: X is structurally stable if and only if it is «Morse-Smale», in the sense that saddles-at-infinity are included in the «no saddle connections» condition (and only hyperbolicity, not finitude, of critical elements is explicitly required). Structural stability is generic in this setting as well (Theorem 4.1).

Peixoto [13] also showed that on compact two-manifolds, any requirement that the equivalence homeomorphism be near the identity is redundant. This is not always the case on open two-manifolds (see pages 20-22 of [10]), but we are able to show that it holds for polynomials under general (C^r) perturbation (Theorem 5.1). For polynomial perturbation of polynomial vector fields, the effect of restriction of the equivalence homeomorphism depends on the degree of the vector fields involved; this question is discussed briefly in section 5, and will be treated in detail in a forthcoming paper with F. Dumortier [4].

This paper is organized as follows. Section 1 is devoted to background material, and Section 2 to the statements and proofs of a few results needed later. In Section 3 the structural stability theorems are stated and proved, and are used in Section 4 to show that stability is generic. Section 5 treats redundancy of restrictions on the equivalence homeomorphism.

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1. Background

Definitions and constructions in this section are generally confined to $\mathbb{R}^2$. For general background and meaning of terms not defined here, consult Hartman [8] or Palis-deMelo [12]. For extensive discussion of various aspects of structural stability on open manifolds, see §2 of the memoir of Kotus, Krych, and Nitecki [10].

$\mathfrak{P}_n$ will denote the set of vector fields on $\mathbb{R}^2$ of the form

$$X(x, y) = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

where P and Q are polynomial functions in x and y of degree $\leqslant n$. Any such X is uniquely specified by the $(n + 1)(n + 2)$ coefficients of P and Q, hence may be identified with a unique point of $\mathbb{R}^{(n+1)(n+2)}$. The topology induced on $\mathfrak{P}_n$ from the usual topology on $\mathbb{R}^{(n+1)(n+2)}$ by this identification is the *coefficient topology* on $\mathfrak{P}_n$. Further notation:

$\mathfrak{X}^r$: the C^r vector fields on $\mathbb{R}^2$, Whitney C^r topology (see [9])

$\mathfrak{Y}^r$: the C^r vector fields on S^2, uniform C^r topology

H: the homeomorphisms of $\mathbb{R}^2$, compact-open topology

J: the homeomorphisms of S^2, uniform C^0 topology

For $X \in \mathfrak{P}_n$ as written above, we let $X^{\perp}$ denote the vector field

$$-Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y},$$

also in $\mathfrak{P}_n$.

Let $\mathfrak{D}$ be $\mathfrak{P}_n$ or $\mathfrak{X}^r$ $(r \geqslant 1)$, and $X \in \mathfrak{D}$. Then X generates a local flow on $\mathbb{R}^2$, which will be denoted $\eta_X(t, p)$. X and another element X' of $\mathfrak{D}$ are *topologically equivalent* if there exists $h \in H$ carrying orbits of the flow induced by X

onto orbits of the flow induced by X', preserving sense but not necessarily parametrization; h is termed an *equivalence homeomorphism* between X and X'. X is *structurally stable* (*with respect to perturbation in* $\mathfrak{D}$) if for any neighborhood M of $id_{\mathbb{R}^2}$ in H, there exists a neighborhood $\mathfrak{N}$ of X in $\mathfrak{D}$, such that every X' in $\mathfrak{N}$ is topologically equivalent to X by an equivalence homeomorphism h lying in M.

A critical point of X is *hyperbolic* if no eigenvalue of the linear part $dX(p)$ of X at p has real part zero. A closed orbit γ is *hyperbolic* if $\int_\gamma \operatorname{div}(X)$ is non-zero. Choosing a sufficiently short line segment Σ through a point of γ, and a coordinate s on Σ with $s = 0$ corresponding to γ, the Poincaré first return map $f(s)$ is defined near zero, and both it and the difference map $d(s) = f(s) - s$, whose roots correspond to closed orbits of X near γ, are analytic. Thus closed orbits accumulate on γ only if a neighborhood of γ is composed of an *annular band* of closed orbits. The *multiplicity* of γ is the multiplicity of the zero of d at $s = 0$; γ is hyperbolic if and only if it has multiplicity 1.

A positive [negative] semi-orbit $o^+(p)$ $[o^-(p)]$ of X is *bounded* if it is contained in a compact set, *escapes to infinity* if for every compact set K there exists $p' \in o^+(p)[p' \in o^-(p)]$ such that $o^+(p')$ $[o^-(p')]$ is disjoint from K, and *oscillates* if it is neither bounded nor escapes to infinity.

A *saddle-at-infinity* (SAI) is a pair $(o^+(p), o^-(q))$ of semi-orbits, each escaping to infinity , such that there exist sequences $p_n \to p$ in $\mathbb{R}^2$ and $t_n \to \infty$ in $\mathbb{R}$ such that $\eta(t_n, p_n) \to q$ in $\mathbb{R}^2$. We identify this SAI with any formed using some $p' \in o^+(p)$ in the place of p or some $q' \in o^-(q)$ in the place of q, and further require that if for some sequence $\tilde{t}_i \to \infty$, $\eta(\tilde{t}_i, p_i) \to r$, then $r \in o^+(p) \cup o^-(q)$. Then $o^+(p)[o^-(q)]$ is termed the *stable* [*unstable*] *separatrix* of the SAI. The reader is cautioned that a SAI is a different object from a saddle (point) (of $\pi(X)$) on the line at infinity (see below).

A *saddle connection* is an orbit $o(p)$ such that $o^+(p)$ is a stable separatrix of a saddle, or of a SAI, while $o^-(p)$ is a separatrix of a saddle, or of a SAI. Note that, as this definition makes allowance for existence of SAIs, it is more general than the definition on a compact manifold. A *separatrix cycle* (elsewhere sometimes referred to as a *graph*) is a sequence $p_1, \sigma_1, p_2, \sigma_2, \ldots,$ p_k, σ_k, p_{k+1} of orbits, each p_j a critical point, each σ_j a separatrix at p_j or at p_{j+1} and tending from p_j to p_{j+1}, and $p_{k+1} = p_1$. The definition is identical for vector fields on S^2.

We let $W^+(X)$ $[W^-(X)]$ denote the union of all orbits containing a stable [unstable] separatrix (of a saddle or SAI) of X. $\Omega(X)$ will denote the set of non-wandering points of X and $\operatorname{Per}(X)$ the set of critical points and points on closed orbits. For $p \in \mathbb{R}^2$, $\alpha_X(p)$ and $\omega_X(p)$ denote the α- and ω-limit sets of p under X respectively.

For $X \in \mathfrak{P}_n$, an analytic vector field $\pi(X)$, the *Poincaré vector field corresponding to X*, is induced on S^2 as follows. Let

$$H^{\pm}(S^2) = \{(x, y, z) \in S^2 \subset \mathbb{R}^3 \mid \pm z > 0\},$$

identify $\mathbb{R}^2$ with $T_{(0,0,1)}S^2$, and let $f^{\pm}: T_{(0,0,1)}S^2 \to H^{\pm}(S^2)$ be central projection. Then $\pi(X)$ is the unique analytic extension of $z^{n-1}(f^{\pm})_* X$ to all of S^2, which is then termed the Poincaré sphere. We call $\mathbb{R}^2$ the *finite part of the plane* (*f.p.p.*), and $S^1 \subset S^2$, to which $\pi(X)$ is tangent, (*the line at*) *infinity*. Letting U_1 and U_2 be the hemispheres corresponding to $x > 0$ and $y > 0$ respectively, and choosing $\phi_i: U_i \to \mathbb{R}^2$, $i = 1, 2$, to be the inverses of the central projection from the vertical planes tangent to S^2 at $(1, 0, 0)$ and $(0, 1, 0)$ respectively, we have the following particularly simple coordinate representations of $\pi(X)$:

$$\text{(in } U_1) \qquad D(s,t)t^n\left[Q\left(\frac{1}{t}, \frac{s}{t}\right) - sP\left(\frac{1}{t}, \frac{s}{t}\right), -tP\left(\frac{1}{t}, \frac{s}{t}\right)\right] \tag{1.1}$$

$$\text{(in } U_2) \qquad D(s,t)t^n\left[P\left(\frac{s}{t}, \frac{1}{t}\right) - sQ\left(\frac{s}{t}, \frac{1}{t}\right), -tQ\left(\frac{s}{t}, \frac{1}{t}\right)\right] \tag{1.2}$$

where $D(s, t)$ is a positive function which is independent of X, where $\{(s, t) \mid t = 0\}$ corresponds to the equator $S^1 \subset S^2$, i.e., to infinity, with positive s-direction agreeing with the positive y-direction (in U_1) or the positive x-direction (in U_2) and where $\{(s, t) \mid t > 0\}$ corresponds to the f.p.p. Choosing V_i to be the open hemisphere opposite to U_i, $i = 1, 2$, and analogous coordinate mappings, we obtain the same expressions for $\pi(X)$ as (1.1) and (1.2), valid in V_1 and V_2 respectively, except for a multiplicative factor of $(-1)^{n-1}$. (The positive directions of the s and t axes disagree with the positive directions of the axes in $\mathbb{R}^3$ in these cases.) Complete details may be found in González [5]. We note in particular that, except for a positive scale factor, $\pi(X)$ appears in these coordinates as a polynomial vector field of degree $n + 1$, and that $S^1 \subset S^2$ is always invariant under $\pi(X)$.

Suppose that for some $r \in \mathbb{Z}^+ \cup \{0\}$, $\Pi \equiv \{\pi(X) \mid X \in \mathfrak{P}_n\}$ is given the subspace topology induced as a subset of $\mathfrak{Y}^r$. Then the coordinate expressions for $\pi(X)$ given above quickly lead to the fact that if $\mathfrak{N}$ is a neighborhood of $\pi(X)$ in Π, then, $\pi^{-1}(\mathfrak{N})$ is a neighborhood of X in $\mathfrak{P}_n$ (coefficient topology).

We note finally that because $\mathbb{R}^2$ is not compact, the flow $\eta_X(t, p)$ generated by $X \in \mathfrak{P}_n$ need not be complete. But the vector field $\tilde{X}$ formed from X by rescaling by $(1 + \|X\|)^{-1/2}$, say, is uniformly bounded, hence generates a complete flow with orbits identical as point-sets to those of X. Since equivalence homeomorphisms ignore parametrizations of orbits, we may safely ignore possible incompleteness of the flows involved, and treat all flows under discussion as if complete (as was already done in the definitions of this section).

2. Preliminary Propositions and Genericity Theorem

This section is devoted to the statement and proof of a few propositions that will be needed later. Several of them combine to show that a dense, open set of $\mathfrak{P}_n$ consists of vector fields whose structure is particularly simple, both on $\mathbb{R}^2$ and when extended to S^2 (Theorem 2.9). This latter result is practically the same as a theorem of G. Tavares dos Santos [21]. See also the book by J. Sotomayor [18].

Proposition 2.1. *Suppose $X \in \mathfrak{P}_n$ is such that, for some $r \in \mathbb{Z}^+ \cup \{0\}$, for every neighborhood M of id_{S^2} in J there is a neighborhood $\mathfrak{R}$ of $\pi(X)$ in $\mathfrak{Y}^r$ such that if $Y \in \mathfrak{R}$ is tangent to $S^1 \subset S^2$, then Y is topologically equivalent to X by some $h \in M$ satisfying $h(S^1) \subset S^1$. Then for any compact set $K \subset \mathbb{R}^2$ and any $\delta > 0$, there is a neighborhood $\mathfrak{U}$ of X in $\mathfrak{P}_n$ such that if $X_1 \in \mathfrak{U}$, then X_1 is topologically equivalent to X by some h that is C^0-δ-close to $id_{\mathbb{R}^2}$ on K.*

PROOF. Given K and δ, choose a compact neighborhood $L \subset H^+(S^2)$ of an ϵ-neighborhood of $f^+(K)$, some $\epsilon > 0$. By uniform continuity of $(f^+)^{-1}|L$, for some $\xi > 0$, $x, y \in L$ and $\text{dist}_{S^2}(x,y) < \xi$ imply $\text{dist}_{\mathbb{R}^2}((f^+)^{-1}(x), (f^+)^{-1}(y))$ $< \delta$. Choosing $\mu = \min(\epsilon, \xi)$, let M correspond to C^0-μ-closeness to id_{S^2}, and let $\mathfrak{R}$ be the neighborhood of $\pi(X)$ in $\mathfrak{Y}^r$ hypothesized to exist. We claim that $\mathfrak{U} \equiv \pi^{-1}(\mathfrak{R})$ is as required. It is a neighborhood of X, and for $X_1 \in \mathfrak{U}$, by hypothesis these exists $h \in M$ which satisfies $h(S^1) \subset S^1$ and is an equivalence between $\pi(X)$ and $\pi(X_1)$, which by choice of ϵ, ξ, and μ is easily seen to be uniformly δ-close to $id_{\mathbb{R}^2}$ on K. □

Proposition 2.2. *For $X \in \mathfrak{P}_n$, any one of the following conditions implies the failure of the inclusion $Cl(W^+(X)) \cap Cl(W^-(X)) \subset \text{Per}(X)$:*

(i) *$\pi(X)$ has a separatrix cycle containing a point not in $S^1 \subset S^2$;*
(ii) *X has an oscillating orbit;*
(iii) $\Omega(X) \not\subset \text{Per}(X)$;
(iv) *there is a point $p \in \mathbb{R}^2$ for which $\alpha_X(p)$ or $\omega_X(p)$ is not empty, nor precisely one critical point, nor precisely one closed orbit.*

PROOF. Call the inclusion I. If (i) holds, I obviously fails. If (ii) holds, then for $\pi(X)$ on S^2 there is an arc in the f.p.p. that is composed of orbits of X and which joins critical points A and B of $\pi(X)$ on $S^1 \subset S^2$ ($A = B$ possible), such that every point in the arc is in $\alpha_X(p)$ [or in $\omega_X(p)$] (where $o(p)$ oscillates). Since $\alpha_{\pi(X)}(p)$ [$\omega_{\pi(X)}(p)$] is connected, a saddle connection of X (among separatrices of SAIs) is formed, so I fails. If (iv) holds, then either $\alpha_X(p)$ is compact, hence X has a separatrix cycle, so I fails, or $\alpha_X(p)$ is not compact, so $o^-(p)$ oscillates, so I fails by (ii).

Finally, we show that (iii) implies failure of I by showing the contrapositive. Hence assume I holds, and suppose $p \in \mathbb{R}^2 \setminus \operatorname{Per}(X)$. Passing to $\pi(X)$ on S^2, by the Poincaré-Bendixson Theorem the non-empty, connected set $\omega_{\pi(X)}(p)$ must be either a single equilibrium or a single closed orbit, else (i) holds, which would imply failure of I. But if $\omega_{\pi(X)}(p)$ is such, then either it is a sink for $\pi(X)$ in S^2, which implies $p \notin \Omega(X)$, or it is a saddle for $\pi(X)$ containing p in a stable separatrix, which by I implies that $\alpha_{\pi(X)}(p)$ is a source for $\pi(X)$ in S^2, which again implies $p \notin \Omega(X)$. □

Proposition 2.3. *If $X \in \mathfrak{P}_n$ has only finitely many critical points, all critical points and closed orbits are hyperbolic, and there are no saddle connections (even when* SAI*s are taken into account), then* $\Omega(X) = \operatorname{Per}(X)$.

PROOF. If X is as hypothesized and $p \notin \operatorname{Per}(X)$, then $\omega_{\pi(X)}(p)$ $[\alpha_{\pi(X)}(p)]$ is a nonempty, compact, connected subset of S^2. By the Poincaré-Bendixson Theorem and the hypotheses, if it contains a separatrix, that separatrix is wholly contained in $S^1 \subset S^2$. In any event, it is composed of a single sink [source] (which could be a separatrix cycle) of $\pi(X)$, whose basin of attaction [region of repulsion] contains p. Thus $p \notin \Omega(X)$ □

Proposition 2.4. *The set $\mathfrak{G}_1$ of all vector fields X in $\mathfrak{P}_n$ such that $\pi(X)$ has only finitely many critical points, all hyperbolic, is open and dense in $\mathfrak{P}_n$.*

PROOF. G. Tavares [21] has shown that the set of X in $\mathfrak{P}_n$ such that X has finitely many critical points, all hyperbolic, is open and dense. Examining the form of $\pi(X)$ in the charts on S^2 mentioned in section one, formulas (1.1) and (1.2), we see that the same arguments carry over to give a relatively open, relatively dense set with only finitely many singularities of $\pi(X)$ on $S^1 \subset S^2$, all hyperbolic. □

This proposition is also an immediate consequence of a more general result, Proposition 4.1 of Chapter Two of [18].

Remark 2.5. The set of all vector fields in $\mathfrak{P}_n$ which have only finitely many critical points, all hyperbolic, is dense, but not open. For example, let

$$H_a(x, y) = 2ax^3 - a^2x^2y - x^2 - 2axy + a^2y^2 + y,$$

$a \in \mathbb{R}$, and let $X_a \in \mathfrak{P}_2$ be the corresponding gradient vector field. Then X_0 is non-singular, but X_a has a non-hyperbolic critical point at

$$(x, y) = \left(\frac{1}{a}, \frac{1}{a^2}\right).$$

Proposition 2.6. *The set* $\mathfrak{G}_2$ *of all vector fields* X *in* $\mathfrak{G}_1 \subset \mathfrak{P}_n$ *such that* $\pi(X)$ *has no saddle connections, except in* $S^1 \subset S^2$, *is open and dense in* $\mathfrak{P}_n$.

PROOF. If $X_\mu(x,y) = X(x,y) + \mu X^\perp(x,y)$, then for non-zero μ near zero, X_μ is a rotation and scaling of X, hence breaks all saddle connections between saddle points of X. Tavares [21, Lemma 3.4] used Sotomayor's criterion to show that passage to X_μ breaks connections between saddles of $\pi(X)$ as well. Since X_μ is near X in $\mathfrak{P}_n$, the set in question is dense. It is open because the finite portion of a saddle separatrix varies continuously with $\pi(X)$ (see for example [1], Lemma 3 of §9.2). □

Proposition 2.7. *The set* $\mathfrak{G}_3$ *of all vector fields* X *in* $\mathfrak{G}_2 \subset \mathfrak{G}_1 \subset \mathfrak{P}_n$ *such that* $\pi(X)$ *has finitely many closed orbits, all hyperbolic, is open and dense in* $\mathfrak{P}_n$.

PROOF. Suppose $X \in \mathfrak{G}_2$ has infinitely many closed orbits. They cannot accumulate on a critical point or a separatrix cycle, since X is in $\mathfrak{G}_2$, hence accumulate on a closed orbit γ of $\pi(X)$. Analyticity of $\pi(X)$, hence of a Poincaré first return map on a section through γ, implies that a neighborhood of γ is made up enterely of closed orbits of $\pi(X)$, hence that there is an annular band of closed orbits of X in $\mathbb{R}^2$. Analyticity of X similarly implies that each boundary of the band is either a single critical point, a single closed orbit, or a separatrix cycle. (See [17] for a detailed discussion and proof.) While the outer boundary could be $S^1 \subset S^2$, the inner boundary must be a center or a separatrix cycle, both of which are impossible for X in $\mathfrak{G}_2$. Thus every X in $\mathfrak{G}_2$ has at most finitely many closed paths.

Suppose now that $X \in \mathfrak{G}_2$, and that X has some non-hyperbolic closed paths. Passing from X to X_a as in the proof of Proposition 2.6, for a non-zero a close enough to zero, any particular closed orbit of X of even multiplicity either disappears entirely or decomposes into two hyperbolic closed orbits, while any particular closed orbit of odd multiplicity persists and becomes hyperbolic (see for example [1], Theorems 72 and 73 of §32.4). If $S^1 \subset S^2$ was a closed orbit of $\pi(X)$ originally, it is easy to make a small adjustment in X to make it hyperbolic (see González [5] for example). Thus we have density of $\mathfrak{G}_3$. Openness is clear. □

Remark 2.8. The set of vector fields in $\mathfrak{G}_2 \subset \mathfrak{G}_1 \subset \mathfrak{P}_n$ having only hyperbolic closed orbits is not itself open. For example, let

$$X_a(x,y) = [a^4x^5 - x^4y - 2x^2y^3 - y^5 - 2a^2x^3 - 2x^2y - 2y^3 + x - y]\frac{\partial}{\partial x}$$
$$+ [x^5 + 3a^4x^4y + 2x^3y^2 + 3a^4x^2y^3 + xy^4 + a^4y^5$$
$$+ 2x^3 - 4a^2x^2y + 2xy^2 - 2a^2y^3 + x + y]\frac{\partial}{\partial y}.$$

In polar coordinates this becomes

$$X_a(r, \theta) = (ar - 1)^2(ar + 1)^2 r \frac{\partial}{\partial r}$$
$$+ [(r^2 + 1)^2 + a^2 r^2 \cos\theta \sin\theta (a^2 r^2 (1 + \cos\theta) - 2)] \frac{\partial}{\partial \theta},$$

so that for $|a| < 2/\sqrt{5}$, the coefficient of $\partial/\partial\theta$ is positive. Thus X_a has a hyperbolic unstable focus at $(0, 0)$ as its sole critical point, and a semi-stable cycle lying in the circle $x^2 + y^2 = 1/a^2$ as its sole closed orbit, if $a \neq 0$. The field X_0 is Morse-Smale, and $X_a \to X_0$ in the coefficient topology on $\mathfrak{P}_5$ as $a \to 0$.

Theorem 2.9. *There is a dense, open subset $\mathfrak{G} \subset \mathfrak{P}_n$, each of whose elements X has the following properties*:

(i) *$\pi(X)$ (hence X) has only finitely many critical points and closed orbits, all of them hyperbolic*;
(ii) *$\pi(X)$ (hence X) has no saddle connections, except in $S^1 \subset S^2$; and*
(iii) $\Omega(X) = \mathrm{Per}(X)$.

PROOF. Let $\mathfrak{G}$ be the set $\mathfrak{G}_3$ of Proposition 2.7. Then $\mathfrak{G}$ is a dense, open subset of $\mathfrak{P}_n$, all of whose elements satisfy conditions (i) and (ii). But then they satisfy (iii) as well, by Proposition 2.3. □

3. Structural Stability Theorems

The first theorem of this section is an application of the characterization theorem of Kotus, Krych, and Nitecki [10] to the situation of smooth perturbation of polynomial vector fields. Recall Section 1 for definitions of terms.

Theorem 3.1. *$X \in \mathfrak{P}_n$ is structurally stable with respect to perturbation in $\mathfrak{X}^r$, $r \geqslant 1$ (Whitney C^r topology) if and only if*

(1) *X has only hyperbolic singularities and closed orbits (and there are only finitely many of the former)*;
(2) *X has no saddle connections (where separatrices of* SAIs *are taken into account); and*
(3) $\Omega(X) = \mathrm{Per}(X)$.

PROOF. While the theorem can easily proved directly, for brevity we will simply show that the three conditions stated are equivalent to the conditions characterizing structural stability with respect to perturbation in $\mathfrak{X}^r$ given in Theorems A and B of [10]:

(i) there are no non-trivial minimal sets or oscillatory orbits;
(ii) every critical point and closed orbit is hyperbolic; and
(iii) $Cl(W^-(X)) \cap Cl(W^+(X)) \subset \mathrm{Per}(X)$.

Hence suppose X satisfies (1), (2), and (3). Since no flow on $\mathbb{R}^2$ has a non-trivial minimal set, the first half of (i) holds automatically. Proposition 2.2(ii), finitude of the set of separatrices, and (2) combine to exclude oscillatory semi-orbits, so (i) holds. Condition (ii) follows from (1), and (iii) from (2) and finitude of the number of separatrices.

Conversely, suppose conditions (i), (ii), and (iii) hold. First, since by (ii) all critical points of X are hyperbolic, they are isolated, hence by Bézout's Theorem [22] there are at most n^2 of them. This and (ii) imply that (1) holds. Certainly (iii) implies (2). Truth of (3) follows from Proposition 2.2(iii). □

Turning to polynomial perturbations of polynomial vector fields, we have the following sufficient conditions for structural stability. These same conditions will prove necessary for stability, with the possible exception noted in Theorem 3.3.

Theorem 3.2. *$X \in \mathfrak{P}_n$ is structurally stable with respect to perturbation in $\mathfrak{P}_n$ (coefficient topology) if*

(1) *X has finitely many critical points and closed orbits, all of them hyperbolic;*
(2) *X has no saddle connections (where separatrices of* SAIs *are taken into account);*
(3) *if $S^1 \subset S^2$ is a closed orbit of $\pi(X)$, it is hyperbolic, or*
(3′) *if p is a critical point of $\pi(X)$ on $S^1 \subset S^2$, then it is hyperbolic, or $d\pi(X)(p)$ has a non-zero eigenvalue with corresponding eigenvector not in $T_pS^1 \subset T_pS^2$; and*
(4a) *no separatrix $o^+(p)$ $[o^-(p)]$ of a* SAI *tends under $\pi(X)$ to a saddle-node on $S^1 \subset S^2$ in forward [reverse] time, and*
(4b) *no separatrix $o^+(p)$ $[o^-(p)]$ common to two distinct* SAIs *tends under $\pi(X)$ to a non-hyperbolic saddle on $S^1 \subset S^2$ in forward [reverse] time.*

PROOF. Suppose X satisfies the hypotheses of the theorem and $\pi(X)$ has $S^1 \subset S^2$ a closed orbit. Then $\pi(X)$ is Morse-Smale, hence structurally stable in $\mathfrak{Y}^r$, $r \geqslant 1$. Moreover, the equivalence homeomorphism can be required to preserve $S^1 \subset S^2$ and to be close to id_{S^2}. Hence by Proposition 2.1, X is structurally stable.

If X satisfies the hypotheses and $S^1 \subset S^2$ is composed entirely of critical points, then condition (3′) implies that on a neighborhood of S^1, $\pi(X)(x, y, z) = zY(x, y, z)$, where $Y(x, y, 0)$ is non-zero and not tangent to S^1. Thus for all

$R \in \mathbb{R}^+$ sufficiently large, X points everywhere outward from (or inward to) $B_R(0,0)$ all along $\partial B_R(0,0)$. Moreover X has no SAIs. If a neighborhood N of $id_{\mathbb{R}^2}$ in the compact-open topology is specified, then there is a compact set K and a number $\epsilon > 0$ such that if h is ϵ-close to $id_{\mathbb{R}^2}$ on K, then $h \in N$. By hypothesis (2) and Proposition 2.2(iii), $\Omega(X) \subset \mathrm{Per}(X)$, hence we may choose $R \in \mathbb{R}^+$ so large that $\Omega(X) \cup K \subset Int(B_R(0,0))$, and non-tangency of X to $\partial B_R(0,0)$ is true. Since $X|Cl(B_R(0,0))$ is Morse-Smale, there is a neighborhood $\mathfrak{M}$ of X in $\mathfrak{X}^r(Cl(B_R(0,0)))$ (uniform C^r topology) so that $Y \in \mathfrak{M}$ implies Y equivalent to X by h uniformly C^0-ϵ-close to $id_{\mathbb{R}^2}$ on $Cl(B_R(0,0))$, and Y is nowhere tangent to $\partial B_R(0,0)$. By compactness of $B_R(0,0)$, clearly there is a neighborhood $\mathfrak{O}$ of X in $\mathfrak{P}_n$ such that $Y \in \mathfrak{O}$ implies $Y \in \mathfrak{M}$. Given $Y \in \mathfrak{M}$, the corresponding h extends to a homeomorphism of $\mathbb{R}^2$ (possibly far from $id_{\mathbb{R}^2}$ off $B_R(0,0)$) as usual: for $p \in \mathbb{R}^2 \setminus Cl(B_R(0,0))$, there exist unique $\tau(p) \in \mathbb{R}$, $\tilde{p} \in \partial B_R(0,0)$, such that $\eta_X(\tau(p), \tilde{p}) = p$; define $h(p) \equiv \eta_Y(-\tau(p), h(\tilde{p}))$. Since X has no SAIs, there are no separatrices other than those of the saddles of X, and these all lie in $B_R(0,0)$, hence h preserves all distinguished orbits, and $\mathfrak{O}$ is the required neighborhood of X in $\mathfrak{P}_n$.

Since in the coordinate charts discussed in Section 1 $\pi(X)$ is a scaled polynomial vector field, the only case remaining is X satisfying conditions (1) through (4) and $\pi(X)$ having a finite non-zero number of critical points on $S^1 \subset S^2$. Let a neighborhood N of $id_{\mathbb{R}^2}$ be specified, and let compact set K and $\epsilon > 0$ be such that if h is any homeomorphism that is C^0-ϵ-close to id on K, then $h \in N$.

Let C_X denote $\mathrm{Per}(X)$ together with all points lying in separatrices that limit on elements of $\mathrm{Per}(X)$ in both directions. Choose $R \in \mathbb{R}^+$ so large that the ϵ-neighborhood of the compact set $C_X \cup K$ lies in $Int(B_R(0,0))$. About each saddle point q of X choose a closed neighborhood N_q, bounded by a quadrilateral whose sides are line segments transverse to X, which is contained in an $\epsilon/4$-neighborhood of q, and which contains no critical point besides q, and no entire closed orbit. About each critical point q [closed orbit γ] which is a source or a sink choose a closed neighborhood $N_q[N_\gamma]$, bounded by a circle [pair of simple closed curves] transverse to X, similarly.

Each separatrix of a SAI, and each separatrix of a saddle point that escapes to infinity, tends under $\pi(X)$ to a unique critical point of $\pi(X)$ on $S^1 \subset S^2$. By assumption (3′), that critical point is a node, topological saddle, or saddle-node, in a neighborhood of which $\pi(X)$, expressed in local coordinates (1.1) or (1.2), is nowhere horizontal besides along the s-axis, corresponding to S^1. Thus for R sufficiently large, the separatrix in question crosses $\partial B_R(0,0)$ precisely once, without tangency, or, in the case of a separatrix of a SAI that escapes to infinity in both directions, exactly twice. Fix such a number R. On such a separatrix, choose a point p_1 outside $Cl(B_R(0,0))$ and a point p_2 in the interior of the isolating neighborhood of the critical element on which it limits, or, in the case of a separatrix of a SAI tending to infinity in both directions, outside $Cl(B_R(0,0))$ and near the second critical point on $S^1 \subset S^2$ to which the separatrix tends. Choose points p_1, p_2 similarly for each separatrix joining critical elements of X in $\mathbb{R}^2$. For each of the finitely many compact orbit segments $[p_1, p_2]$ so obtained, there exists a number $\delta > 0$ such that if at every point p on the orbit segment a perpendicular segment Σ_p of length 2δ (centered at p) is erected, then $\Sigma_p \cap \Sigma_q = \emptyset$ for $p \neq q$. Choose the minimum of all such numbers (one for each separatrix) and $\epsilon/4$, and for each orbit segment $[p_1, p_2]$ form the set $N[p_1, p_2]$ composed of all the transverse segments Σ_p for $p \in [p_1, p_2]$. Shrinking δ if necessary we can insure that the neighborhoods of orbit segments are pairwise disjoint. Let $\mathfrak{J}$ denote the closed set which is the union of all the neighborhoods $N[p_1, p_2]$ of orbit segments, the isolating neighborhoods of critical elements of X, and the complement of $Int(B_R(0,0))$.

Now consider a perturbation of X to a sufficiently close element Y of $\mathfrak{P}_n$. By well-known theorems, hypotheses (1) and (2), and the fact that choosing Y close enough to X in $\mathfrak{P}_n$ makes Y arbitrarily C^r-close to X on pre-assigned compact sets, the critical elements and separatrices of saddles limiting on them of Y properly persist and lie in $\mathfrak{J}$. We now show that if σ is a separatrix of a saddle point of X escaping to infinity, then the corresponding separatrix σ' of Y also escapes to infinity, never leaving $\mathfrak{J}$; that if σ is a separatrix of a SAI of X, there is a unique corresponding separatrix of a SAI of Y, lying in $\mathfrak{J}$ (and that Y has no additional separatrices of SAIs); and that if the semi-orbit opposite to that in σ forming the SAI escapes to infinity, then so does the corresponding opposite semi-orbit of σ'.

First let σ be an unstable separatrix of a saddle q of X which escapes to infinity, with orbit segment (p_1, p_2), $p_2 \in N_q$ and $p_1 \notin B_R(0,0)$, and tending to critical point r_0 of $\pi(X)$ in $S^1 \subset S^2$. If r_0 is a node, then for Y close enough to X, $\pi(Y)$ is so close to $\pi(X)$ that σ' tends to a critical point r_0' of $\pi(Y)$ arbitrarily close to r_0, $\sigma' \subset \mathfrak{J}$, and for all $p \in [p_1, p_2]$, σ' crosses Σ_p precisely once. If r_0 is a saddle, then by hypothesis (2), moving along S^1 in either direction from r_0, we encounter finitely many (possibly none) saddle-nodes, $r_1, r_2, \ldots, r_l$, all contracting onto S^1, followed by a stable node r_{l+1}. Under sufficiently small perturbation, σ' clearly remains in $\mathfrak{J}$ and tends to a critical point arbitrarily near one of $r_0, r_1, \ldots, r_{l+1}$, and crosses Σ_p precisely once, for each $p \in [p_1, p_2]$. If r_0 is a saddle-node, then by hypothesis (4a), proceeding in the direction induced on S^1 by the flow of $\pi(X)$ near r_0, we must again have a sequence of saddle-nodes, each contracting onto S^1, followed by a stable node, and reach the same conclusion. Note that in all these cases there is a subinterval $(a, b) \subset \Sigma_{p_1}$ containing p_1 in its interior, every point of which escapes to infinity under both X and Y. The same arguments handle the case of a stable separatrix of X that escapes to infinity in backward time.

Now let $\sigma = o^+(p)$ be a stable separatrix of a SAI of X with orbit segment $[p_1, p_2]$ and tending to $r_0 \in S^1 \subset S^2$. By hypothesis (4*a*), r_0 is a saddle of $\pi(X)$, and of the two critical points adjacent to r_0 on S^1, at least one is a saddle point of $\pi(X)$. If both adjacent critical points, call them r_1 and r_2, are saddles, then by hypothesis (4*b*) r_0 is hyperbolic, so for Y close enough to X, $\pi(Y)$ has a unique hyperbolic saddle point r_0' near r_0 with stable separatrix σ' near σ and meeting each transverse segment in $N[p_1, p_2]$ precisely once. There are also unique closest critical points r_1', r_2' to r_0', near r_1 and r_2, and each with a unique unstable separatrix, so each SAI of which σ' is now a stable separatrix persists. If precisely one of the critical points of $\pi(X)$ adjacent to r_0 on S^1, call it r_1, is a saddle point, then proceeding in the opposite direction along S^1 away from r_0, by hypothesis (4*a*) we encounter finitely many saddle-nodes (possibly none), all contracting onto S^1, followed by a stable node. Thus there is a subinterval $(a, b) \subset \Sigma_{p_1}$, containing p in its interior, such that every point of $(a, p_1]$ escapes to infinity, but no point of (p_1, b) does. Under small enough perturbation, Y has a unique pair of critical points r_0' near r_0 and r_1' near r_1 having no critical points between them along the arc of S^1 under consideration; r_0' has a unique stable separatrix σ', and r_1' has a unique unstable separatrix, so the SAI persists, and its stable separatrix σ' is appropriately near σ for Y close enough to X. If σ' meets Σ_{p_1} at p_1', then every point on $(a, p_1']$ escapes to infinity, but no point of (p_1', b) does. The same arguments handle the case of an unstable separatrix of a SAI.

The persistence of escape to infinity of the opposite semi-orbit from a semi-orbit forming a separatrix of a SAI has exactly the same proof as persistence of escape to infinity of separatrices of saddes.

By hypotheses (3′) and (4*a*), SAIs of X are in one-to-one correspondence with saddle connections of $\pi(X)$ that are subarcs of $S^1 \subset S^2$, hence there are finitely many of them, and clearly none are created under sufficiently small perturbation of X. Since therefore X has finitely many critical elements, saddle separatrices, and SAI separatrices, by the discussion of the previous paragraphs, there is a neighborhood $\mathfrak{M}$ of X in $\mathfrak{P}_n$ such that if $Y \in \mathfrak{M}$, then separatrices and critical elements of Y lie in $\mathfrak{J}$, and by a long but straightforward procedure, we can construct a homeomorphism $\hat{h}_Y: \mathfrak{J} \to \mathfrak{J}$ which (*a*) is $id_{\mathbb{R}^2}$ on $\partial\mathfrak{J}$, (*b*) is C^r-$\epsilon/2$-close to id on $B_R(0, 0)$, and (*c*) carries critical elements and separatrices of Y back onto the corresponding objects of X. We can now finish the proof in several ways. On the one hand, we can apply the classical techniques of M. C. Peixoto and M. M. Peixoto [15] to create a homeomorphism of each canonical region of $\pi(X)|H^+(S^2)$ to the corresponding canonical region of $\pi(Y)|H^+(S^2)$. There are several new types of canonical regions, but their techniques carry over, inducing an equivalence homeomorphism of X and Y in $\mathbb{R}^2$, which as in [15] can be made ϵ-close to $id_{\mathbb{R}^2}$ on the compact set K, simply by choosing Y close enough to X. On the other hand, imitating the procedure in Kotus, Krych, and Nitecki [10], we can increase $\mathfrak{J}$ in a natural way so as to include one orbit from each canonical region of X, hence of Y, in $\mathbb{R}^2$, and can extend $\hat{h}_Y$ to a homeomorphism $h_Y: \mathbb{R}^2 \to \mathbb{R}^2$ by the identity off $\mathfrak{J}$, and define a C^0-flow on $\mathbb{R}^2$ by $\mu(t,p) = h_Y(\eta_Y(t, h^{-1}(p)))$. Obviously h_Y^{-1} carries μ-orbits back onto η_Y-orbits, and is C^0-$\epsilon/2$-close to $id_{\mathbb{R}^2}$ on $B_R(0,0)$. But μ has exactly the same critical elements and separatrices as η_X, and exactly the same orbits as η_Y off $\mathfrak{J}$. Hence for Y close enough to X, as outlined in [10] we can use techniques of Neumann [11] to construct a homeomorphism h of $\mathbb{R}^2$ that carries η_X-orbits to μ-orbits, and is C^0-ϵ-close to id on K, and carries η_X-orbits onto η_Y-orbits, as required. □

We now turn to the necessity of the conditions of Theorem 3.2 for structural stability. We have the following partial result.

Theorem 3.3. *If $X \in \mathfrak{P}_n$ is structurally stable with respect to perturbation in $\mathfrak{P}_n$, and has no non-hyperbolic limit cycles of odd multiplicity, then X satisfies conditions* (1) *through* (4) *of Theorem* 3.2.

PROOF. Let $X \in \mathfrak{P}_n$ be structurally stable with respect to perturbation in $\mathfrak{P}_n$. Since X has any topological property satisfied by a dense subset of $\mathfrak{P}_n$, it follows from Theorem 2.9 that X has only finitely many critical points, each one a node, focus, or topological saddle, finitely many closed orbits, none semi-stable, and no saddle connections (including when SAIs are taken into account), and that $\Omega(X) = \operatorname{Per}(X)$. Condition (2) of Theorem 3.2 thus follows immediately. The additional hypothesis on limit cycles of odd multiplicity implies that all cycles are hyperbolic.

To establish hyperbolicity of critical points, choose disjoint compact neighborhoods $N_1, N_2, \ldots, N_k$ and $M_1, M_2, \ldots, M_l$ of the closed orbits $\gamma_1, \gamma_2, \ldots, \gamma_k$ and critical points $p_1, p_2, \ldots, p_l$, isolating them from one another. Suppose X has at least one non-hyperbolic critical point, say p_1. If $\det dX(p_1) > 0$ and $\operatorname{Tr} dX(p_1) = 0$, then p_1 is a focus, since it is not a center. But then for any neighborhood $\mathfrak{N}$ of X there is a rotation and scaling $X_\mu = X + \mu X^\perp$ of X that is in $\mathfrak{N}$, but has a closed orbit wholly contained in M_1 (*cf.* [1, Remark 3 of §10.3]). Then X_μ has at least $k+1$ closed orbits, hence is not equivalent to X, a contradiction. Thus all non-hyperbolic critical points of X are nodes and saddles, and the M_i can be chosen so that the ∂M_i are circles and quadrilaterals to which X is transverse, except at the corners of the latter. It follows that there is a neighborhood $\mathfrak{N}'' \subset \mathfrak{N}$ each of whose elements has a least one critical point in each of the M_i. For we simply choose $\mathfrak{N}' \subset \mathfrak{N}$ so that $Y \in \mathfrak{N}'$ is also transverse to ∂M_i in the case of each non-hyperbolic node, and apply the Poincaré-Bendixson Theorem. Then choose $\mathfrak{N}'' \subset \mathfrak{N}'$ so that $Y \in \mathfrak{N}''$ is nowhere opposed to X on ∂M_i in the case of any non-hyperbolic saddle, so that the index of Y with respect to ∂M_i is the same as that of X, namely (by Bendixson's formula) -1, hence ∂M_i surrounds a critical point of Y.

But now if $\det dX(p_1) = 0$, we use the well-known fact that for any neighborhood $\mathfrak{N}$ of X, there is a $Y \in \mathfrak{N}$ having at least two critical points in M_1, so Y has at least $l + 1$ critical points, a contradition. Thus X satisfies condition (1) of Theorem 3.2.

Condition (2) was mentioned already. To establish (3) for X suppose first that $S^1 \subset S^2$ is a closed orbit of $\pi(X)$, but non-hyperbolic, say (by condition (1)) asymptotically stable in $H^+(S^2)$. Then n is odd, and letting

$$m = \frac{n+1}{2} - 1 \in \mathbb{Z},$$

the vector field

$$Y(x, y) = [P(x, y) - \alpha(n + 1)x(x^2 + y^2)^m] \frac{\partial}{\partial x}$$

$$+ [Q(x, y) - \alpha(n + 1)y(x^2 + y^2)^m] \frac{\partial}{\partial y}$$

is in $\mathfrak{P}_n$. It is readily verified that if $\alpha > 0$ is sufficiently small, $S^1 \subset S^2$ is a hyperbolic, asymptotically unstable limit cycle for $\pi(X)$, so that by Poincaré's Theorem there is a closed orbit of Y outside any pre-assigned compact subset of $\mathbb{R}^2$. Since the closed orbits of X are all hyperbolic, they all persist under perturbation to Y, so Y is not equivalent to X, having one more closed orbit than X does, a contradition. If condition (3′) fails for X at $p \in S^1 \subset S^2$, it is not difficult to find an arbitrarily small change in the coefficients of X to produce a vector field Y for which $\pi(Y)$ has a critical point in the f.p.p. arbitrarily close to p, so that Y has at least one more critical point than X, contradicting stability of X. Since condition (3′) holds for X, it follows from Theorem 65, §21 of [2] that every critical point of $\pi(X)$ on $S^1 \subset S^2$ is topologically a node, saddle, or saddle-node.

Finally, we must establish (4a) and (4b). First suppose that X is structurally stable but that (4a) fails, say $(o^+(r_0), o^-(s_0))$ a SAI such that under $\pi(X)$, $r_0 \to r_1 \in S^1$ and $s_0 \to s_1 \in S^1$ (reverse time), and r_1 is a saddle-node of $\pi(X)$. Let r_2 be the first critical point of $\pi(X)$ encountered in moving away from r_1 along S^1 in the direction opposite to that of s_1, and let $\rho = \text{dist}_{S^1}(r_2, s_1)$. Rotating the coordinate system in $\mathbb{R}^2$ so that r_1 is at the end of the x-axis, it is not difficult to check, using chart (1.1), that there is an arbitrarily small change in the coefficients of X that removes the critical point of $\pi(X)$ at r_0, so that either the SAI is destroyed completely, or has separatrices tending to points of $S^1 \subset S^2$ that are at least $\rho/2 > 0$ apart. Thus choosing $K = Cl(B_R(0, 0))$

of sufficiently large radius R, and $\epsilon = 1$, the new vector field Y obtained from X by the perturbation is either not equivalent to X at all, or else not by a homeomorphism that is C^0-ϵ-close to $id_{\mathbb{R}^2}$ on K, contradicting stability of X.

It (4b) fails, we can always split the offending critical point in $S^1 \subset S^2$ into three or more by a small perturbation, similarly violating structural stability of X. □

Let us say that a closed orbit γ of $X \in \mathfrak{D}$ is *structurally unstable in* $\mathfrak{D}$ if for any neighborhoods N of γ in $\mathbb{R}^2$ and $\mathfrak{N}$ of X in $\mathfrak{D}$, there exists $X_1 \in \mathfrak{N}$ which has other than exactly one closed orbit entirely contained in N. It is well known that if $\mathfrak{D} = \mathfrak{X}^r$, then γ is structurally unstable if and only if it is non-hyperbolic. Thus when $\mathfrak{D} = \mathfrak{P}_n$, hyperbolicity implies stability. The converse, however, has so far eluded proof:

Question 3.4. (*Cf.* [1 §6.3], [19]). *Is a non-hyperbolic limit cycle γ of $X \in \mathfrak{P}_n$ necessarily structurally unstable in $\mathfrak{P}_n$?*

If γ has even multiplicity, a simple geometric argument shows that there is an arbitrarily small rotation of X producing a vector field Y having no closed orbit in N, so the answer to the question is «yes» in this case. An affirmative answer in general would mean that the conditions listed in Theorem 3.2 fully characterize structural stability in $\mathfrak{P}_n$.

4. Genericity of stability

Results of the previous sections yield the following genericity result.

Theorem 4.1. *There is a dense, open subset $\mathfrak{G}$ of $\mathfrak{P}_n$, every element of which is structurally stable with respect to perturbation in either $\mathfrak{X}^r$, $r \geqslant 1$ (Whitney C^r topology), or in $\mathfrak{P}_n$ (coefficient topology).*

PROOF. Let $\mathfrak{G}$ be the set described in Theorem 2.9, which is dense and open in $\mathfrak{P}_n$. Structural stability in $\mathfrak{X}^r$ and $\mathfrak{P}_n$ follow from Theorems 3.1 and 3.2 respectively. □

5. Equivalence Homeomorphisms

This section examines the role of restrictions on the equivalence homeomorphism in the definition of structural stability. In the case of arbitrary smooth perturbation, we are able to duplicate Peixoto's theorem from the compact case.

Theorem 5.1. *In the definition of structural stability of $X \in \mathfrak{P}_n$ with respect to perturbation in $\mathfrak{X}^r$, $r \geqslant 1$, the requirement that the equivalence homeomorphism lie in a pre-assigned neighborhood of $id_{\mathbb{R}^2}$ is superfluous. That is, Theorem* 3.1 *is also valid in the case that no restriction is put on h.*

PROOF. It is obvious that if $X \in \mathfrak{P}_n$ is structurally stable when h is restricted to being close to $id_{\mathbb{R}^2}$, then X is structurally stable when h is unrestricted. To prove the converse, we suppose that $X \in \mathfrak{P}_n$ is structurally stable in the setting of arbitrary equivalence homeomorphisms, and demonstrate that X satisfies conditions (1) through (3) of Theorem 3.1, hence by the sufficiency statement of that theorem is structurally stable in the original sense.

To begin with, we claim that X has no semi-stable limit cycles. For if it did, then given any neighborhood $\mathfrak{N}$ of X in $\mathfrak{X}^r$, we could find a positive C^∞ function $\epsilon(x, y)$: $\mathbb{R}^2 \to \mathbb{R}$ so small that $Y(x, y) = X(x, y) + \epsilon(x, y)X^{\perp}(x, y)$ would lie in $\mathfrak{N}$. Yet such Y has only hyperbolic limit cycles ([1], Theorem 71, §32), contradicting stability of X. But then all the limit cycles of X are in fact hyperbolic, since the fact that we have smooth perturbations at our disposal means that if X were to have a non-hyperbolic limit cycle γ, then by an arbitrarily small C^r perturbation of X supported in a neighborhood of γ, a semi-stable limit cycle can be made to bifurcate off from γ (see for example [10], Corollary 8.7(ii)).

If X had more than n^2 critical points, then there would be an algebraic curve of critical points of X [22]. In such a case there exists a $Y \in \mathfrak{P}_n$ which is arbitrarily close to X in $\mathfrak{P}_n$ and which has finitely many critical points. Then letting $f(x, y)$ be a C^∞ bump function which is identically 1 on $B_R(0, 0)$ and vanishes off $B_{2R}(0, 0)$, for R large enough and Y close enough to X, $f(x, y)Y(x, y)$ is close to X in $\mathfrak{X}^r$, but is not equivalent to X, a contradiction. It is well known that an arbitrarily C^r-small perturbation of X supported on a neighborhood of each of its critical points will yield a vector field with only hyperbolic critical points, hence each of the finitely many critical points of X is a node, focus, or topological saddle. At none of them can $\det dX$ vanish, else by a C^r perturbation the critical point in question can be split into several, contradicting stability of X. Thus any non-hyperbolic critical point p of X is a weak focus. Regardless of its multiplicity there is a C^r perturbation of X which causes an odd number of limit cycles to bifurcate from p, so that either the stability of p changes, or the new vector field near X has a semi-stable limit cycle near p, in either case contradicting the stability of X. In sum, X has finitely many critical points, all hyperbolic. Thus (1) of Theorem 3.1 holds.

The requirement of condition (2) now makes sense, and it is clear that it must actually hold for X, since saddle connections are easily destroyed by local smooth perturbation.

Finally, (3) follows from Proposition 2.3. □

In the case of polynomial perturbation of $X \in \mathfrak{P}_n$, stability of X depends on whether or not h is restricted to be close to $id_{\mathbb{R}^2}$, at least for n large, the reason being that separatrices of SAIs can make a sudden jump as X is changed. On the other hand, if closeness to id is overly restricted, say uniformly C^0-close rather than close in the compact-open topology, then as in the case of general smooth X ([10], Proposition 2.10), any X with an orbit escaping to infinity in both time-directions will be structurally unstable. A detailed analysis of the effects of restriction of h in the setting of polynomial perturbations will be given in [4].

References

[1] Andronov, A. A., Leontovich, E. A., Gordon, I. I. and Maier, A. G. Theory of Bifurcations of Dynamic Systems on a Plane, Israel Program for Scientific Translations, John Wiley & Sons, 1973.

[2] — Qualitative Theory of Second-Order Dynamic Systems, Israel Program for Scientific Translations, John Wiley & Sons, 1973.

[3] Camacho, C., Krych, M., Mañe, R., Nitecki, Z. An extension of Peixoto's structural stability theorem to open surfaces with finite genus, in «*Geometric Dynamics*». Lecture Notes in Math. **1007**, Springer-Verlag, 1983.

[4] Dumortier, F. and Shafer, D. S. Restrictions on the equivalence homeomorphism in stability of polynomial vector fields.

[5] González Velasco, E. A. Generic properties of polynomial vector fields at infinity, *Trans. Amer. Math. Soc.* **143**(1969), 201-222.

[6] Gutiérrez, C. Structural stability for flows on the torus with crosscap, *Trans. Amer. Math. Soc.* **241**(1978), 311-320.

[7] — Smooth non-orientable non-trivial recurrence on two-manifolds, *J. Diff. Equations.* **29**(1978), 388-395.

[8] Hartman, P. Ordinary Differential Equations, Birkhauser-Boston, 1982.

[9] Hirsch, M. W. Differential Topology, Springer-Verlag, 1976.

[10] Kotus, J., Krych, M., Nitecki, Z. Global structural stability of flows on open surfaces, *Memoirs Amer. Math. Soc.* **261**, 1982.

[11] Neumann, D. Classification of continuous flows on 2-manifolds, *Proc. Amer. Math. Soc.* **48**(1975), 73-81.

[12] Palis, J. and deMelo W. Geometric Theory of dynamical system, Springer-Verlag, 1982.

[13] Peixoto, M. M. On structural stability. *Annals of Math.* **69**(1959), 199-222.

[14] — Structural stability on two-dimensional manifolds, *Topology* **1**(1962), 101-120.

[15] Peixoto, M. C. and Peixoto, M. M. Structural stability in the plane with enlarged boundary conditions. *An. Acad. Bras. Cien.* **31**(1959), 135-160.

[16] Peixoto, M. M. and Pugh, C. C. Structurally stable systems on open manifolds are never dense, *Annals of Math.* **87**(1968), 423-430.

[17] Perko, L. M. On the accumulation of limit cycles, *Proc. Amer. Math. Soc.* **99**(1987), 515-526.

[18] Sotomayor, J. Curvas definidas por equações diferenciais no plano, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1981.

[19] — Stable planar polynomial vector fields, *Revista Mat. Iberoamericana.* **1**(1985), 15-23.
[20] Takens, F. and White, W. Vector fields with no nonwandering points, *Amer. J. Math.* **98**(1976), 415-425.
[21] Tavares dos Santos, G. Classification of generic quadratic vector fields with no limit cycles, in «Geometry and Topology. Proceedings, 1976». *Lecture Notes in Mathematics* **597**, Springer-Verlag, 1977.
[22] Walker, R. J. Algebraic Curves, Dover, 1962.
[23] Ye, Y. Q. Theory of Limit Cycles, Translations of Mathematical Monographs Vol. 66, Amer. Math. Soc., Providence, R. I., 1986.

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On the Spaces of Eisenstein Series of Hilbert Modular Groups

Toshitsune Miyake

Introduction

In the theory of automorphic forms, it is a basic result due to Hecke [1] that the space of elliptic modular forms of integral weight is the direct sum of the space of cusp forms and the space of Eisenstein series. This has been expected to be true in general and in fact it has been proved for Hilbert modular forms and for modular forms of half integral weight by several people, such as Kloosterman [2], Petersson [4], Pei [3], Shimizu [5], [6] and Shimura [9]. Especially Shimura [9] investigated automorphic eigen forms, which includes holomorphic automorphic forms, in great detail and proved that in most cases the orthogonal complements of (eigen) cusp forms in the spaces of the automorphic eigen forms of Hilbert modular groups are generated by Eisenstein series (and some other forms derived from Eisenstein series in certain special cases). There he omitted the case when the set of eigen values of differential operators is multiple in his sense. The purpose of the present paper is to prove that his result, which is the generalization of the classical fact mentioned in the beginning, is true without any restriction on eigenvalues.

Here we explain our result breifly restricting ourselves to only the case of integral weight, though our result includes also the case of half integral weight. Let F be a totally real algebraic number field, and $\mathbf{a}$ the set of all archimedean primes of F. Let H be the upper half plane. By $\mathbb{Z}^{\mathbf{a}}$, $\mathbb{C}^{\mathbf{a}}$ and $H^{\mathbf{a}}$, we understand copies the product of $\mathbf{a}$ of $\mathbb{Z}$, $\mathbb{C}$ and H, respectively. Then

$SL_2(F)$ acts on $H^{\mathbf{a}}$ in the usual way. For $\sigma \in \mathbb{Z}^{\mathbf{a}}$ and $v \in \mathbf{a}$, we define the differential operator L_v^σ on $H^{\mathbf{a}}$ by

$$L_v^\sigma = -4y_v^{2-\sigma_v}(\partial/\partial z_v)y_v^{\sigma_v}(\partial/\partial \bar{z}_v),$$

where $z = (z_v)$ is the variable on $H^{\mathbf{a}}$ and $y_v = \mathrm{Im}(z_v)$. Let $\lambda = (\lambda_v) \in \mathbb{C}^{\mathbf{a}}$. For each congruence subgroup Γ of $SL_2(F)$, we denote by $\mathcal{A}(\sigma, \lambda, \Gamma)$ the set of all C^∞-functions f on $H^{\mathbf{a}}$ satisfying the following conditions:

(1) $f(\gamma z) = \prod_{\gamma \in \mathbf{a}} (c_v z_v + d_v)^{\sigma_v} f(z)$ for every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$;

(2) $L_v^\sigma f = \lambda_v f$ for every $v \in \mathbf{a}$;

(3) f is slowly increasing at every cusp.

We also denote by $\mathcal{S}(\sigma, \lambda, \Gamma)$ the space of cusp forms in $\mathcal{A}(\sigma, \lambda, \Gamma)$ which is defined by the condition at cusps as usual. Then our main result is that the orthogonal complement of $\mathcal{S}(\sigma, \lambda, \Gamma)$ in $\mathcal{A}(\sigma, \lambda, \Gamma)$ with respect to the Petersson inner product is generated by the special values of Eisenstein series with parameters (and some other functions derived from Eisenstein series in some special cases).

Recently Shimizu proved that it is also valid for automorphic eigen forms on GL_2 over any algebraic number field in the case of integral weight using representation theory ([6]).

1. Automorphic Eigen Forms

Let F be a totally real number field and $\mathbf{a}$ the set of all archimedian primes of F. For each set X, we denote by $X^{\mathbf{a}}$ the product of $\mathbf{a}$ copies of X or the set $\{(x_v)_v \mid v \in \mathbf{a}\}$. For each element x of $X^{\mathbf{a}}$, we denote by x_v the v-component of x. For two elements c and x of $\mathbb{C}^{\mathbf{a}}$, we put

$$c^x = \prod_v c_v^{x_v}$$

whenever each factor is well defined.

Let H be the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. For each $g \in SL_2(\mathbb{R})$ and $z \in H$, we put

$$g(z) = \frac{az+b}{cz+d}, \qquad j(g, z) = cz + d \quad \text{if} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By $z \mapsto g(z)$, $SL_2(\mathbb{R})$ acts on H and therefore $SL_2(\mathbb{R})^{\mathbf{a}}$ acts on $H^{\mathbf{a}}$. For $g \in SL_2(\mathbb{R})^{\mathbf{a}}$, $z = (z_v) \in H^{\mathbf{a}}$ and $\sigma = (\sigma_v) \in \mathbb{Z}^{\mathbf{a}}$, we put

$$j_g(z)^\sigma = j(g, z)^\sigma = \prod_v j(g_v, z_v)^{\sigma_v}.$$

We shall define automorphic forms of integral weight and half integral weight. Put $u = (1, 1, \ldots, 1) \in \mathbb{R}^{\mathbf{a}}$. A weight will be either an element of $\mathbb{Z}^{\mathbf{a}}$ (Case I, integral weight) or an element of $(1/2)u + \mathbb{Z}^{\mathbf{a}}$ (Case II, half integral weight). For each weight σ, we denote by $\mathcal{G}_\sigma$ the set of all pairs $(g, l(z))$ with an element $g \in SL_2(F)$ and a holomorphic function $l(z)$ on $H^{\mathbf{a}}$ such that

$$l(z)^2 = t \cdot j(g, z)^\sigma, \qquad t \in \mathbb{C}, \qquad |t| = 1.$$

The set $\mathcal{G}_\sigma$ is a group by the group law defined by

$$(g, l)(g', l') = (gg', l(g'(z))l'(z)).$$

We denote the projection of $\mathcal{G}_\sigma$ to $SL_2(F)$ by pr, or $pr((g, l)) = g$. For $\alpha = (g, l(z)) \in \mathcal{G}_\sigma$, we denote $l(z)$ also by $l_\alpha(z)$ and put $\alpha(z) = g(z)$ for $z \in H^{\mathbf{a}}$ and $j_\alpha^\sigma = j_g^\sigma$. For a function f on $H^{\mathbf{a}}$ and $\alpha \in \mathcal{G}_\sigma$, we define the function $f \| \alpha$ by

$$(f \| \alpha)(z) = l_\alpha(z)^{-1} f(\alpha(z)), \qquad z \in H^{\mathbf{a}}.$$

For $z = (z_v) \in H^{\mathbf{a}}$, we put $y_v = \operatorname{Im}(z_v)$ and $y = (y_v)$, and consider y as an $\mathbb{R}^{\mathbf{a}}$-valued function on $H^{\mathbf{a}}$. Then

$$(1.1) \qquad y^p \| \alpha = l_\alpha^{-1} |j_\alpha|^{-2p} y^p \qquad (p \in \mathbb{R}^{\mathbf{a}}, \alpha \in \mathcal{G}_\sigma).$$

For $v \in \mathbf{a}$ and $\sigma \in \mathbb{R}^{\mathbf{a}}$, we define differential operators ϵ_v, δ_v^σ and L_v^σ operating on C^∞-functions f on $H^{\mathbf{a}}$ by

$$(1.2) \qquad \epsilon_v f = -y_v^2 \, \partial f / \partial \bar{z}_v,$$

$$(1.3) \qquad \delta_v^\sigma = -y_v^{\sigma v} \, \partial(y_v^{\sigma v} f) / \partial z_v,$$

$$(1.4) \qquad L_v^\sigma = 4 \, \delta_v^{\sigma'} \epsilon_v, \qquad \sigma'_v = \sigma_v - 2.$$

Let W be the subset of $SL_2(F)$ defined by [9, (1.10)] and h_g the holomorphic function on $H^{\mathbf{a}}$ for each $g \in W$ given in [9, Prop. 3.2]. For each weight σ, let Λ_σ be the injection

$$\begin{aligned} \Lambda_\sigma:\ SL_2(F) &\to \mathcal{G}_\sigma \qquad \text{(Case I)}, \\ \Lambda_\sigma:\ W &\to \mathcal{G}_\sigma \qquad \text{(Case II)} \end{aligned}$$

given by

$$\Lambda_\sigma(g) = \begin{cases} (g, j_g^\sigma), & g \in SL_2(F) \qquad \text{(Case I)}, \\ (g, h_g j_g^{\sigma - u/2}), & g \in W \qquad \text{(Case II)}. \end{cases}$$

Let $\mathfrak{g}$ be the maximal order of F, $\mathfrak{g}^\times$ the unit group of $\mathfrak{g}$ and $\mathfrak{d}$ the different of F. For each integral ideal $\mathfrak{c}$ of F, we put

$$\Gamma(\mathfrak{c}) = \{\alpha \in SL_2(F) \cap M_2(\mathfrak{g}) \mid \alpha - 1 \in \mathfrak{c} M_2(\mathfrak{g})\}.$$

We call a subgroup Δ of $\mathcal{G}_\sigma$ a congruence subgroup if it satisfies the following two conditions:

(1.5) Δ is isomorphic to a subgroup of $SL_2(F)$ by pr,

(1.6) Δ contains $\Lambda_\sigma(\Gamma(\mathfrak{c}))$ as a subgroup of finite index for some $\mathfrak{c}$ ($\mathfrak{c} \subset 8\mathfrak{d}^{-1}$ in Case II).

A real analytic function f on $H^{\mathbf{a}}$ is called an *automorphic eigen form* with respect to a congruence subgroup Δ of $\mathcal{G}_\sigma$, if it safisfies the following three conditions:

(1.7) $f \| \alpha = f$ for every $\alpha \in \Delta$;

(1.8) $L_v^\sigma f = \lambda_v f$ with $\lambda_v \in \mathbb{C}$ for every $v \in \mathbf{a}$;

(1.9) for every $\alpha \in \mathcal{G}_\sigma$, there exist positive numbers A, B and C (depending on f and α) such that

$$y^{\sigma/2}|(f \| \alpha)(x+iy)| \leqslant A y^{cu} \quad \text{if} \quad y^u > B.$$

For $\lambda = (\lambda_v) \in \mathbb{C}^{\mathbf{a}}$, we denote by $\mathcal{A}(\sigma, \lambda, \Delta)$ the set of all such f and by $\mathcal{A}(\sigma, \lambda)$ the union of $\mathcal{A}(\sigma, \lambda, \Delta)$ for all congruence subgroups Δ of $\mathcal{G}_\sigma$. We know $\mathcal{A}(\sigma, \lambda, \Delta)$ is finite dimensional and $\mathcal{A}(\sigma, \lambda)$ is stable under the action of $\alpha \in \mathcal{G}_\sigma$. If $f \in \mathcal{A}(\sigma, \lambda, \Delta)$, then it has a Fourier expansion of the form

$$f(x+iy) = b(y) + \sum_{0 \neq h \in \mathfrak{m}} b_h W(hy; \sigma, \lambda) e(hx)$$

with $\mathfrak{m}$ a lattice of F, $b_h \in \mathbb{C}$, $b(y)$ a function on $\mathbb{R}^{\mathbf{a}}$, $e(z) = \exp(2\pi i \Sigma z_v)$ for $z \in \mathbb{C}^{\mathbf{a}}$, and W the Whittaker function defined by [9, (2.19) and (2.20)]. We call $b(y)$ the constant term of f and call f a *cusp form* if the constant term of $f \| \alpha$ vanishes for every $\alpha \in \mathcal{G}_\sigma$. We denote the set of all cusp forms in $\mathcal{A}(\sigma, \lambda)$ by $\mathcal{S}(\sigma, \lambda)$ and put $\mathcal{S}(\sigma, \lambda, \Delta) = \mathcal{A}(\sigma, \lambda, \Delta) \cap \mathcal{S}(\sigma, \lambda)$.

For two continuous functions f and g satisfying (1.7), we put

$$\langle f, g \rangle = \mu(\Delta \backslash H^{\mathbf{a}})^{-1} \int_{\Delta \backslash H^{\mathbf{a}}} \bar{f} g y^\sigma \, d\mu(z)$$

where

$$d\mu(z) = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \, dy_v.$$

This does not depend on the choice of Δ. We define subspaces $\mathcal{N}(\sigma, \lambda)$ and $\mathcal{N}(\sigma, \lambda, \Delta)$ of $\mathcal{A}(\sigma, \lambda)$ by

$$\mathcal{N}(\sigma, \lambda, \Delta) = \{ g \in \mathcal{A}(\sigma, \lambda, \Delta) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda, \Delta) \},$$
$$\mathcal{N}(\sigma, \lambda) = \{ g \in \mathcal{A}(\sigma, \lambda) \mid \langle f, g \rangle = 0 \quad \text{for all} \quad f \in \mathcal{S}(\sigma, \lambda) \}.$$

Then $\mathcal{N}(\sigma, \lambda, \Delta) = \mathcal{N}(\sigma, \lambda) \cap \mathcal{A}(\sigma, \lambda, \Delta)$.

Let U be a subgroup of $\mathfrak{g}^\times$ of finite index. We call $\tau = (\tau_v) \in \mathbb{R}^{\mathbf{a}}$ *U-admissible* if

$$\Sigma\tau_v = 0 \quad \text{and} \quad |a|^{i\tau} = 1 \quad \text{for all} \quad a \in U. \tag{1.10}$$

We call τ ($\in \mathbb{R}^{\mathbf{a}}$) *admissible* if it is U-admissible for some U and denote by T_U the set of all U-admissible τ.

Hereafter we fix a weight σ and write $\mathcal{G} = \mathcal{G}_\sigma$ and $\Lambda = \Lambda_\sigma$. We call $\lambda \in \mathbb{C}^{\mathbf{a}}$ *critical* if $4\lambda_v = (1-\sigma_v)^2$ for all $v \in \mathbf{a}$, and call λ *non-critical* if it is not critical.

Proposition 1.1. ([9, Prop. 3.1]). *The constant term $b(y)$ of an element $f \in \mathcal{Q}(\sigma, \lambda)$ has one of the following forms.*

(1) *If λ is critical then*

$$b(y) = a_1 y^q + a_2 y^q \log y^u,$$

where $q = (q_v)$ and q_v is the multiple root of $X^2 - (1-\sigma_v)X + \lambda_v = 0$.

(2) *If λ is non-critical then $b(y)$ is a linear combination of y^p with $p = (p_v) \in \mathbb{C}^{\mathbf{a}}$ satisfying*

(1.11) *p_v is a root of $\psi_v(X) = X^2 - (1-\sigma_v)X + \lambda_v$,*

(1.12) *$p = su - (\sigma - i\tau)/2$ with $s \in \mathbb{C}$ and an admissible $\tau \in \mathbb{R}^{\mathbf{a}}$.*

When λ is non-critical, an element $p \in \mathbb{C}^{\mathbf{a}}$ is called an *exponent* attached to λ if it satisfies (1.11) and (1.12). We denote by $C(\sigma, \lambda)$ the set of all exponents attached to λ. For $p = (p_v) \in \mathbb{C}^{\mathbf{a}}$, we put $\bar{p} = (\bar{p}_v) \in \mathbb{C}^{\mathbf{a}}$. Then $C(\sigma, \bar{\lambda}) = \{\bar{p} \mid p \in C(\sigma, \lambda)\}$. We note if $C(\sigma, \lambda) = \phi$, then $\mathcal{Q}(\sigma, \lambda) = \mathcal{S}(\sigma, \lambda)$. We call λ *simple* either if λ is critical or if λ is non-critical and $C(\sigma, \lambda)$ consists of exactly two elements. We also call λ *multiple* if $C(\sigma, \lambda)$ has more than two elements.

Lemma 1.2. *Assume λ is non-critical. For $p \in C(\sigma, \lambda)$, put $p' = u - \sigma - p$. Then $p' \in C(\sigma, \lambda)$ and $p' \neq p$. Furthermore $\bar{\lambda}$ is non-critical and $\overline{p'} = \bar{p}' \in C(\sigma, \bar{\lambda})$.*

Proof. Since p_v is a root of $\psi_v(X) = 0$, $1 - \sigma_v - p_v$ is also a root of $\psi_v(X) = 0$. As $\psi_v(X) = 0$ has simple roots for at least one v, we have $p' \neq p$. Further since $p' = (1-s)u - (\sigma + i\tau)/2$, p' satisfies also (1.12). The last statement is obvious. □

2. A Bilinear Relation of Coefficients of Constant Terms

The purpose of this section is to generalize [9, Theorem 6.1] to multiple λ.

Hereafter we assume λ is non-critical. Let Δ be a congruence subgroup of $\mathcal{G}$ and put $\Gamma = pr(\Delta)$. Put

$$P = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(F) \,\middle|\, c = 0 \right\}, \qquad \mathcal{P} = \{\alpha \in \mathcal{G} \mid pr(\alpha) \in P\}.$$

Then $\mathcal{P} \backslash \mathcal{G}/\Delta$ is a finite set. We call classes of $\mathcal{P} \backslash \mathcal{G}/\Delta$ cusp classes. Take a complete set of representatives X for $\mathcal{P} \backslash \mathcal{G}/\Delta$. For each $\xi \in X$, we put $Q_\xi = P \cap pr(\xi\Delta\xi^{-1})$. Let Θ be a subgroup of P of the form

$$\Theta = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \,\middle|\, a \in U_1, b \in \mathfrak{m} \right\},$$

with a fractional ideal $\mathfrak{m}$ of F and a subgroup U of $\mathfrak{g}^\times$ of finite index. Take U_1 and $\mathfrak{m}$ so that $\Lambda(\Theta) \subset \xi\Delta\xi^{-1}$ for all $\xi \in X$ and every a ($\in U_1$) is totally positive. For $0 < r \in \mathbb{R}$, put

$$T_r = \{z \in H^{\mathbf{a}} \mid y^u > r\}, \qquad M_r = \{z \in H^{\mathbf{a}} \mid y^u = r\}.$$

Put $U = \{a^2 \mid a \in U_1\}$. Then $\Theta \backslash M_r$ is isomorphic to the product of $\mathbb{R}^{\mathbf{a}}/\mathfrak{m}$ and $\{y \in \mathbb{R}^{\mathbf{a}} \mid y^u = r, y >> 0\}/U$ up to the difference of orientations of $(-1)^{n(n-1)/2}$. For a fixed $v \in \mathbf{a}$, we put

$$\omega = y^{-2u} \prod_{v \in \mathbf{a}} dx_v \wedge dy_v,$$

$$\zeta_v = (i/2) y_v^{-2u} d\bar{z}_v \prod_{w \neq v} dx_w \wedge dy_w.$$

Since [9, Lemma 6.2] holds only when $\tau = 0$, its proper statement should be the following

Lemma 2.1. *For $s \in \mathbb{C}$ and $\tau \in T_U$, we have*

$$\int_{\Theta \backslash M_r} y^{su+\tau+v} \zeta_v = \begin{cases} (-i/2)\mu(\mathbb{R}^{\mathbf{a}}/\mathfrak{m}) R_U r^{s-1} & (\tau = 0) \\ 0 & (\tau \neq 0), \end{cases}$$

where $R_U = R_F[\mathfrak{g}^\times : U \cdot \{\pm 1\}]$ with the regulator R_F of F and $\mu(\mathbb{R}^{\mathbf{a}}/\mathfrak{m})$ is the volume of $\mathbb{R}^{\mathbf{a}}/\mathfrak{m}$.

Let $f \in \mathcal{A}(\sigma, \lambda, \Delta)$ and $g \in \mathcal{A}(\sigma, \bar{\lambda}, \Delta)$. For each $\xi \in X$, we write

$$f \,\|\, \xi^{-1} = \Sigma a_{p,\xi} y^p + \text{(non-constant terms)}$$

and

$$g \,\|\, \xi^{-1} = \Sigma b_{\bar{p},\xi} y^{\bar{p}} + \text{(non-constant terms)},$$

where p is taken over the elements of $C(\sigma, \lambda)$. Let $C_1 = C_1(\sigma, \lambda)$ be a subset of $C(\sigma, \lambda)$ such that

$$C(\sigma, \lambda) = C_1(\sigma, \lambda) \cup \{p' \mid p \in C_1(\sigma, \lambda)\} \qquad \text{(disjoint)}.$$

Then we can generalize [9, Theorem 6.1] to the following

Theorem 2.2. *We have*

$$\sum_{p \in C_1} (p_v - p'_v) \sum_{\xi \in X} \nu_\xi (a_{p,\xi} \bar{b}_{\bar{p}',\xi} - a_{p',\xi} \bar{b}_{\bar{p},\xi}) = 0,$$

where $\nu_\xi = [Q_\xi \cdot \{\pm 1\} : \Theta \cdot \{\pm 1\}]^{-1}$.

PROOF. Take a positive number r so that any two sets $\xi^{-1} Q_\xi \xi \setminus \xi^{-1}(T_r)$ $(\subset \Gamma \setminus H^{\mathbf{a}}, \xi \in X)$ have no intersection points. Let J be a union of small neighbourhoods of elliptic points on $\Gamma \setminus H^{\mathbf{a}}$, which are compact manifolds with boundary. Inducing a natural orientation into each set, we see that

$$\partial K = \sum_{\xi \in X} \xi^{-1} Q_\xi \xi \setminus \xi^{-1}(T_r) - \partial J.$$

Therefore for a Γ-invariant C^∞-form ϕ on $H^{\mathbf{a}}$ of codegree 1, we have

$$\int_K d\phi = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} \phi \circ \xi^{-1} - \int_{\partial J} \phi,$$

where $B_\xi = \xi^{-1} \Theta \xi \setminus \xi^{-1}(M_r)$. We put $\phi = \bar{f}(\epsilon_v g) y^\sigma \zeta_v$ and $\phi' = \bar{g}(\epsilon_v f) y^\sigma \zeta_v$. Then

$$d\phi = \frac{1}{4} \bar{f} L_v^\sigma g y^\sigma \omega - \overline{(\epsilon_v f)}(\epsilon_v g) y^{\sigma'} \omega \qquad (\sigma' = \sigma - 2v)$$

and

$$d(\bar{\phi} - \phi') = \frac{1}{4}(f \overline{L_v^\sigma g} - L_v^\sigma f \bar{g}) y^\sigma \omega$$

$$= \frac{1}{4}(\lambda_v - \lambda_v) f \bar{g} y^\sigma \omega = 0.$$

This implies that

$$\int_{\partial J} (\bar{\phi} - \phi') = \sum_{\xi \in X} \nu_\xi \int_{B_\xi} (\bar{\phi} - \phi') \circ \xi^{-1}.$$

Now we have the expansions

$$\bar{\phi} \circ \xi^{-1} = \frac{i}{2} \sum_{p,q} q_v a_{p,\xi} \bar{b}_{\bar{q},\xi} y^{\bar{p} + \bar{q} + v + \sigma} \zeta_v + \cdots,$$

and

$$\phi' \circ \xi^{-1} = -\frac{i}{2}\sum_{p,q} p_v a_{p,\xi}\bar{b}_{\bar{q},\xi} y^{\bar{p}+\bar{q}+v+\sigma}\zeta_v + \cdots,$$

where p and q are taken over $C(\sigma, \lambda)$. Though the unwritten terms also contain terms which do not contain $e(hx)$, they tend to 0 in our later process of letting $r \to \infty$ since they decrease rapidly by [9, Prop. 2.1(2)]. Applying Lemma 2.1, we obtain

$$\int_{\partial J}(\bar{\phi} - \phi') = \mu(\mathbb{R}^{\mathfrak{a}}/\mathfrak{m})R_U \sum_{p \in C_1} (p_v - p'_v) \sum_{\xi \in X} \nu_\xi(a_{p,\xi}\bar{b}_{\bar{p}',\xi} - a_{p',\xi}\bar{b}_{\bar{p},\xi}) + \cdots,$$

where $C_1 = C_1(\sigma, \lambda)$. Since the unwritten terms tend to 0 when $r \to \infty$ as we mentioned above and $\int_{\partial J}(\bar{\phi} - \phi') \to 0$ when $J \to \phi$, we have the assertion. □

3. Eisenstein Series

Let ρ be an element of $\mathbb{C}^{\mathfrak{a}}$ such that

$$\rho = (\sigma - \tau)/2$$

with an admissible τ. A cusp class $\mathcal{P}\xi\Delta$ ($\xi \in X$) is called ρ-regular if

$$y^{-\rho} \parallel \alpha = y^{-\rho} \quad \text{for every} \quad \alpha \in \mathcal{P} \cap \xi\Delta\xi^{-1}.$$

We denote by $Y(\rho)$ the subset of X that represents all ρ-regular cusps. We also denote by $\varkappa(\rho)$ the number of elements of $Y(\rho)$. For each congruence subgroup Δ, we define its Eisenstein series by

$$E(z, s; \rho, \Delta) = \begin{cases} \sum_{\alpha \in \mathcal{P} \cap \Delta \backslash \Delta} y^{su-\rho} \parallel \alpha & \text{if } \mathcal{P}\Delta \text{ is } \rho\text{-regular}, \\ 0 & \text{otherwise.} \end{cases}$$

The series is convergent for $\mathrm{Re}(s) > 1$ and can be continued as a meromorphic function in s to the whole s-plane. If $\Delta \supset \Delta'$, we see that

$$\text{(3.1)} \qquad [\mathcal{P} \cap \Delta : \mathcal{P} \cap \Delta']E(z, s; \rho, \Delta) = \sum_{\gamma \in \Delta \backslash \Delta'} E(z, s; \rho, \Delta') \parallel \gamma.$$

For each cusp class $\mathcal{P}\xi\Delta$, we put

$$E_\xi(z, s; \rho, \Delta) = E(z, s; \rho, \xi\Delta\xi^{-1}) \parallel \xi.$$

Then we see

$$\text{(3.2)} \qquad E_\xi(z, s; \rho, \Delta) \parallel \alpha = E_\xi(z, s; \rho, \Delta) \quad \text{for every} \quad \alpha \in \Delta.$$

We denote by $\mathcal{E}[\rho, \Delta]$ the complex vector space generated by the functions $E_\xi(z, s; \rho, \Delta)$ for all $\xi \in X$. Using (3.1), we can prove that if $\Delta \supset \Delta'$, then $\mathcal{E}[\rho, \Delta] \subset \mathcal{E}[\rho, \Delta']$ and

$$\mathcal{E}[\rho, \Delta] = \{g = g(z, s) \in \mathcal{E}[\rho, \Delta'] \mid g \parallel \alpha = g \text{ for every } \alpha \in \Delta\}. \tag{3.3}$$

Now by [9, Prop. 5.2], we have for $\xi, \eta \in Y(\rho)$,

$$E_\xi \parallel \eta^{-1} = \delta_{\xi\eta} y^{su-\rho} + f_{\xi\eta}(s) y^{u-su-\bar{\rho}} + \sum_{0 \neq h \in \mathfrak{n}} g_{\xi\eta}(h, s, y) e(hx) \tag{3.4}$$

where $f_{\xi\eta}$ and $g_{\xi\eta}$ are meromorphic functions in s, $\delta_{\xi\eta}$ is the Kronecker's delta and $\mathfrak{n}$ is a lattice in F.

For $s_0 \in \mathbb{C}$, we denote by $\mathcal{E}[s_0, \rho, \Delta]$ the subspace of $\mathcal{E}[\rho, \Delta]$ consisting of all functions $g(z, s)$ that are holomorphic at s_0. We put

$$\mathcal{E}(s_0, \rho, \Delta) = \{g(z, s_0) \mid g \in \mathcal{E}[s_0, \rho, \Delta]\}.$$

Then by [9, Prop. 7.1],

$$\mathcal{E}(s_0, \rho, \Delta) \subset \mathcal{Q}(\sigma, \lambda, \Delta)$$

with $\lambda = (\lambda_v)$, $\lambda_v = (s_0 - \rho_v)(1 - s_0 - \bar{\rho}_v)$.

The following lemma is stated in [9, Prop. 7.2] under the assumption that λ is simple, but the assertion holds also for multiple λ without any changes of the proof.

Lemma 3.1. (1) $\dim \mathcal{E}[\rho, \Delta] = \varkappa(\rho)$.

(2) *The map* $g(z, s) \to g(z, s_0)$ *gives an isomorphism of* $\mathcal{E}[s_0, \rho, \Delta]$ *onto* $\mathcal{E}(s_0, \rho, \Delta)$.

Conversely for a fixed non-critical λ, we express $p \in C(\sigma, \lambda)$ as

$$p = s_p u - (\sigma - i\tau_p)/2$$

with $s_p \in \mathbb{C}$ and an admissible τ_p. We put $\rho_p = (\sigma - i\tau_p)/2$ and also set $Y(p) = Y(\rho_p)$ and $\varkappa(p) = \varkappa(\rho_p)$. We note

$$p' = (1 - s_p)u - \rho_p \quad \text{and} \quad \varkappa(p') = \varkappa(p)$$

by [9, Prop. 7.5].

Theorem 3.2. *Suppose* λ *is non-critical,* $\mathcal{E}[\rho_p, \Delta] = \mathcal{E}[s_p, \rho_p, \Delta]$ *and* $\mathcal{E}[\bar{\rho}_p, \Delta] = \mathcal{E}[\bar{s}_p, \bar{\rho}_p, \Delta]$ *for any* $p \in C_1(\sigma, \lambda)$. *Then*

$$\mathfrak{N}(\sigma, \lambda, \Delta) = \bigoplus_{p \in C_1} \mathcal{E}(s_p, \rho_p, \Delta) \qquad (C_1 = C_1(\sigma, \lambda)).$$

PROOF. It is easy to see that the right-hand side is a direct sum and is contained in $\mathfrak{N}(\sigma, \lambda, \Delta)$ by [9, Prop. 7.1]. Therefore we have

$$\dim(\mathfrak{N}(\sigma, \lambda, \Delta)) \geqslant \sum_{p \in C_1} \varkappa(p).$$

For each $p \in C_1$, let $Y'(p)$ be the set of all $\xi \in X$ such that $\mathcal{P}\xi\Delta$ is $\bar{\rho}_p$-regular. Then the number of elements of $Y'(p)$ is $\varkappa(p)$ by [3, Prop. 7.5]. For $f \in \mathfrak{A}(\sigma, \lambda, \Delta)$ and $\xi \in X$, write

$$f \parallel \xi^{-1} = \sum_{p \in C_1} (a_{p,\xi} y^p + a_{p',\xi} y^{p'}) + (\text{non-constant terms}).$$

Then the map $\Psi: f \to ((a_{p,\xi})_{p \in C_1, \xi \in Y(p)}, (a_{p',\xi})_{p \in C_1, \xi \in Y'(p)})$ gives an injection of $\mathfrak{A}(\sigma, \lambda, \Delta)/\mathcal{S}(\sigma, \lambda, \Delta)$ into $\mathbb{C}^{2\mu}(\mu = \sum_{p \in C_1} \varkappa(p))$. For each $p \in C_1$, take $v \in \mathbf{a}$ so that $p_v \neq p'_v$. Let $g \in \mathcal{E}(\bar{s}_p, \bar{\rho}_p, \Delta)$ and $\xi \in Y'(p)$. Denote the Fourier expansion of $g \parallel \xi^{-1}$ by

$$g \parallel \xi^{-1} = b_{\bar{p},\xi} y^{\bar{p}} + b_{\bar{p}',\xi} y^{\bar{p}'} + (\text{non-constant terms}).$$

The using Theorem 2.2, we have a linear relation

$$\sum_{\xi \in Y'(p)} \nu_\xi (a_{p,\xi} \bar{b}_{\bar{p}',\xi} - a_{p',\xi} \bar{b}_{\bar{p},\xi}) = 0$$

among $(a_{p,\xi}, a_{p',\xi})$ for each g. Since these linear relations are independent if $p's$ are different, we have at least μ independent linear relations. This implies the dimension of the image of Ψ is at most μ and therefore is equal to μ. □

If λ is non-critical, by [9, Remark 7.4 (1), (2)], we can take the set $C_1(\sigma, \lambda)$ and s_p for each $p \in C_1(\sigma, \lambda)$ so that they satisfy the conditions of Theorem 3.2, except for the case when $\lambda = 0$, $\sigma = 0$ $(p = u, p' = 0, s_p = 1)$ in Case I and the case when $\sigma_v - 1/2$ is either an even non-negative integer or an odd negative integer for every $v \in \mathbf{a}$ $(p = 3/4 - (1/2)\sigma, p' = 1/4 - (1/2)\sigma, s_p = 3/4)$ in Case II. To discuss these cases, we denote by $\mathcal{E}^*[s_0, \rho, \Delta]$ the set of elements of $\mathcal{E}[\rho, \Delta]$ that have at most a simple pole at s, and by $\mathcal{E}^*(s_0, \rho, \Delta)$ the set of residues of elements in $\mathcal{E}^*[s_0, \rho, \Delta]$. The following theorem is a generalization of [9, Theorem 7.9], which can be proved similarly as [9, Theorem 7.9] together with the modification used in Theorem 3.2.

Theorem 3.3. *Suppose λ is real and non-critical. Suppose also for all $p \in C_1(\sigma, \lambda)$, $\mathcal{E}[\rho_p, \Delta] = \mathcal{E}^*[s_p, \rho_p, \Delta]$ and a cusp class of Δ is ρ_p-regular if and only if it is $\bar{\rho}_p$-regular. Then $\mathfrak{N}(\sigma, \lambda, \Delta)$ has dimension $\sum_{p \in C_1} \varkappa(p)$ and is the direct sum*

$$\bigoplus_{p \in C_1} (\mathcal{E}(s_p, \rho_p, \Delta) \oplus \mathcal{E}^*(s_p, \rho_p, \Delta)), \qquad (C_1 = C_1(\sigma, \lambda)).$$

Using Theorem 3.2 and Theorem 3.3, we obtain the following

Theorem 3.4. *If* λ *is non-critical, then* $\mathfrak{N}(\sigma, \lambda, \Delta)$ *is generated by special values and the residues of Eisenstein series and has dimension* $\sum_{p \in C_1} \varkappa(p)$ $(C_1 = C_1(\sigma, \lambda))$.

PROOF. This is a direct result of Theorem 3.2 and Theorem 3.3 together with [9, Remark 7.10]. The only thing we would like to mention here is that in Theorem 3.3, we have assumed a cusp class of Δ is ρ_p-regular if and only if it is $\bar{\rho}_p$-regular for all $p \in C_1$. To avoid this restriction, take a subgroup Δ' of Δ of finite index so small that any cusp class of Δ' is ρ_p-regular and also $\bar{\rho}_p$-regular for all $p \in C_1$. Then by (3.3), we have the result. □

4. Remarks on Multiple λ

In [9, Remark 5.5], Shimura gave an example of multiple λ, when the field F is a quadratic field. We explain it in a slightly more general situation, because it seems to be the only case when multiple λ appears. Let F be a totally real number field of degree $2n$ containing a quadratic field L as a subfield. Denote by $\{v, w\}$ the set of archimedian primes of L and by $\mathbf{a} = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ the set of archimedian primes of F of which $v_1, \ldots, v_n$ are lying over v and $w_1, \ldots, w_n$ are over w. Let U_1 be a subgroup of the unit group of L of finite index and take θ so that $T_{U_1} = \mathbb{Z}\theta$. Let U be a subgroup of the unit group of F of finite index such that $\{N_{F/L}(\epsilon) \mid \epsilon \in U\} \subset U_1$. Let τ be an element of $\mathbb{R}^{\mathbf{a}}$ such that

$$\tau_{v_i} = \theta_v, \qquad \tau_{w_i} = \theta_w \quad \text{for any} \quad i \ (1 \leqslant i \leqslant n).$$

Then $|\epsilon|^{i\tau} = |N_{F/L}(\epsilon)|^{i\theta} = 1$ for any $\epsilon \in U$. Put for integers m, n $(m \neq n)$,

$$p = \{(1 + 2ni\theta_v)u - (\sigma - 2mi\tau)\}/2,$$
$$q = \{(1 + 2mi\theta_v)u - (\sigma - 2ni\tau)\}/2.$$

Then p, $\bar{p}$, q, $\bar{q}$ are all distinct and are exponents belonging to $C(\sigma, \lambda)$ with $\lambda = (\lambda_{v_1}, \ldots, \lambda_{v_n}, \lambda_{w_1}, \ldots, \lambda_{w_n})$ given by

$$4\lambda_{v_i} = (1 - \sigma_{v_i})^2 + (m + n)^2\theta_v^2, \qquad 4\lambda_{w_i} = (1 - \sigma_{w_i})^2 + (m - n)^2\theta_w^2.$$

Therefore λ is multiple.

Now we return to the general situation and assume λ is multiple. Then by [9, Prop. 3.2]

(4.1) λ is real and $X^2 - (1 - \sigma_v)X + \lambda_v = 0$ has either a multiple root or two simple roots which are complex conjugate.

Therefore if p is an exponent attached to λ, then $p' = \bar{p}$ and for any other exponent $q \in C(\sigma, \lambda)$, we see that q_v is either p_v or $\bar{p}_v$ for all $v \in \mathbf{a}$. The following proposition suggests that even for multiple λ, $C(\sigma, \lambda)$ cannot contain so many exponents.

Proposition 4.1. *Let F be a totally real number field of degree $(n \geqslant 3)$. Assume λ is multiple. Let $p = (p_1, \ldots, p_n)$ $(p_i = p_{v_i}$ for $v_i \in \mathbf{a})$ be an exponent attached to λ. Assume $\bar{p}_i \neq p_i$ and put $q = (p_1, \ldots, \bar{p}_i, \ldots, p_n)$. Then q is not an exponent attached to λ.*

PROOF. We may assume $i = 1$ by changing the indices. Assume q is also an exponent attached to λ and put

$$p = su - (\sigma - i\tau)/2, \qquad q = s'u - (\sigma - i\tau')/2 \qquad (s, s' \in \mathbb{C})$$

with admissible τ and τ'. Since $\operatorname{Re}(s) = \operatorname{Re}(s') = 1/2$ by (1.11) and (4.1), we can write

$$s = 1/2 + it, \qquad s' = 1/2 + it'$$

with $t, t' \in \mathbb{R}$. Then we see

$$t' + \tau_1' = -(t + \tau_1), \qquad t' + \tau_j' = t + \tau_j \qquad (2 \leqslant j \leqslant n).$$

Therefore $\tau_j - \tau_j' = t' - t$ for all j $(2 \leqslant j \leqslant n)$. Putting $a = t - t'$, we obtain

$$\tau_1 - \tau_1' = -\left(\sum_{j=2}^{n} \tau - \sum_{j=2}^{n} \tau'\right) = (n-1)a.$$

This implies

$$\tau - \tau' = -au + (na, 0, \ldots, 0).$$

Take a subgroup of the unit group of F of finite index such that τ, τ' are U-admissible. Then we see for any $\epsilon \in U$,

$$1 = |\epsilon|^{i(\tau - \tau')} = |\epsilon_i|^{ina}.$$

Since the rank of U is $n - 1$, we see $a = 0$ if $n \geqslant 3$. This implies $t = t'$, $\tau = \tau'$ and therefore $p_1 = \bar{p}_1$, which is a contradiction. □

It is an interesting problem to determine whether all multiple λ can be obtained as Shimura's example mentioned in the beginning of this section or not, though it is a problem solely on the structure of the unit group of number fields.

Corollary 4.2. *If* $[F:\mathbb{Q}]=3$, *then there exists no multiple* λ *for any weight* σ.

PROOF. Assume λ is a multiple eigenvalue and let p be an exponent attached to λ. Then $\bar{p}$ is also an exponent. If q is an exponent attached to λ, then q is obtained by changing p or $\bar{p}$ at only one prime v. But this is not allowed by Proposition 4.1. □

References

[1] Hecke, E. Theorie der Eisensteinschen Reihen höher Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, *Abh. Math. Sem. Hamburg*, **5**(1927), 199-224.

[2] Kloosterman, H. D. Theorie der Eisensteinschen Reihen von mehreren Veränderlichen, *Abh. Math. Sem. Hamburg*, **6**(1928), 163-188.

[3] Pei, T.-y. Eisenstein series of weight 3/2: I, II, *Trans. Amer. Math. Soc.*, **274** (1982), 573-606, **283**(1984), 589-603.

[4] Petersson, H. Über die Entwicklungskoeffizienten der ganzen Modulformen unt ihre Bedeutung für die Zahlentheorie, *Abh. Math. Sem. Hamburg*, **8**(1931), 215-242.

[5] Shimizu, H. A remark on the Hilbert modular forms of weight 1, *Math. Ann.*, **265**(1983), 457-472.

[6] Shimizu, H. The Space of Eisenstein Series in the Case of GL_2, *Advanced Studies in Pure Mathematics*, **13**(1987), 585-621.

[7] Shimura, G. On Eisenstein series, *Duke Math. J.* **50**(1983), 417-476.

[8] Shimura, G. On Eisenstein series of half integral weight, *Duke Math. J.* **52**(1985), 281-314.

[9] Shimura, G. On the Eisenstein series of Hilbert modular groups. *Revista Mat. Iberoamer.* **1**(1985), 1-42.

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Eigenvalue Problems of Quasilinear Elliptic Systems on $\mathbb{R}^n$

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Abstract

In this paper, we get the existence results of the nontrivial weak solution (λ, u) of the following eigenvalue problem of quasilinear elliptic systems

$$-D_\alpha(a_{\alpha\beta}(x,u)D_\beta u^i) + \frac{1}{2}D_{u^i}a_{\alpha\beta}(x,u)D_\alpha u^j D_\beta u^j + h(x)u^i = \lambda|u|^{p-2}u^i,$$

for $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant N$ and

$$u = (u^1, u^2, \ldots, u^N) \in E = \{v = (v^1, v^2, \ldots, v^N) \mid v^i \in H^1(\mathbb{R}^n), 1 \leqslant i \leqslant N\},$$

where $a_{\alpha\beta}(x,u)$ satisfy the natural growth conditions. It seems that this kind of problem has never been dealt with before.

1. Introduction

We consider eigenvalue problems of the following quasilinear elliptic systems on $\mathbb{R}^n$

$$(1.1) \quad -D_\alpha(a_{\alpha\beta}(x,u)D_\beta u^i) + \frac{1}{2}D_{u^i}a_{\alpha\beta}(x,u)D_\alpha u^j D_\beta u^j + h(x)u^i = \lambda|u|^{p-2}u^i,$$

for $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant N$ and

$$u = (u^1, u^2, \ldots, u^N) \in E = \{v = (v^1, v^2, \ldots, v^N) \mid v^i \in H^1(\mathbb{R}^N), 1 \leqslant i \leqslant N\}$$

where $R < p < 2\hat{n}/(\hat{n} - 2)$, $\hat{n} = n$ if $n > 2$, $2\hat{n}/(\hat{n} - 2)$ is any positive number lager than 2 if $n \leqslant 2$,

$$D_\alpha = \frac{\partial}{\partial x_\alpha}, \quad D_{u^i} = \frac{\partial}{\partial u^i} \qquad (1 \leqslant \alpha \leqslant n, \quad 1 \leqslant i \leqslant N)$$

and the summation conventions have been used and will be used in the following, *i.e.* the repeated Greek letters and Latin letters denote the sum from 1 to n and 1 to N respectively.

Problem (1.1) comes from the theory of harmonic mappings. There have been some results of (1.1) in bounded domains ([1], [2]). In [1], the existence of solutions for (1.1) is discussed under the conditions

$$\mu_1 |\xi|^2 \leqslant a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leqslant \mu_2 |\xi|^2 \qquad \mu_1, \mu_2 > 0$$
$$\lim_{u \to +\infty} u D_u a_{\alpha\beta}(x, u) = 0$$

for every $(u, \xi) \in \mathbb{R}^1 \times \mathbb{R}^n$, $x \in \Omega \subset \mathbb{R}^n$, where $N = 1, p = 2n/(n - 2)$, $n > 2$ if $n > 2$. In [2] the existence theorem is obtained when $N \geqslant 1$, $h \equiv 0$, $2 < p < 2n/(n - 2)$, $n > 2$ under the conditions

$$\begin{cases} a_1 |\xi|^2 \leqslant \sigma(|u|)|\xi|^2 \leqslant a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leqslant a_2 \sigma(|u|)|\xi|^2 \\ |u^i D_{u^i} a_{\alpha\beta}(x, u)| \leqslant C\sigma(|u|) \\ |D_{u^i} a_{\alpha\beta}(x, u)| \leqslant C\sigma(|u|), \qquad |D_{u^i} a_{\alpha\beta}(x, u)| \leqslant \eta(|u|) \\ -\dfrac{u^i}{2} D_{u^i} a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leqslant a_3 a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \qquad (0 < a_3 < 1), \end{cases}$$

for every $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^n$, where $\sigma(t)$, $\eta(t)$ are nonnegative continuous functions on $[0, +\infty)$ satisfying that for any $c_1 > 1$, there exists c_2, such that $\sigma(c_1 t) \leqslant c_2 \sigma(t)$ for all $t \geqslant 0$.

However, there have not been any results for (1.1) in the unbounded domain $\mathbb{R}^n$. Formally, if the minimum of the functional

$$I(u) = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx \tag{1.2}$$

over the set $\left\{ u \in E \,\middle|\, \int_{\mathbb{R}^n} |u|^p dx = \mu \right\}$ $(\mu > 0)$ were achieved by some u, there should be a $\lambda \in \mathbb{R}^1$ such that (λ, u) solves (1.1) in a weak sense. But there are some difficulties in dealing with the functional $I(u)$. Firstly, because of the unboundedness of $\mathbb{R}^n$, the Sobolev embedding is not compact and the standard convex-compactness techniques can not be used, at least in a straightforward way as in the case of bounded domains, and this makes the problem of the existence of a minimizer more difficult. Secondly, the space where I is differentiable is $L_\infty \cap E$ (see [3]), so even if we had found a minimizer $u \in E$ of I, we could not conclude the existence of (λ, u) $\mathbb{R}^1 \times E$ solving (1.1), unless we had known that $u \in L_\infty$. But, usually, the fact that $\|u\|_\infty$ is finite is obtained because u satisfies the related Euler equation which in turn is a consequence of the differentiability of I at u. This makes the problem complicated.

To overcome the first difficulty, we use the concentration compactness principle, recently developed by P. L. Lions ([4], [5]), when treating the constrained variational problems in unbounded domains. To overcome the second difficulty, we first show that, for any minimizer u of I and some $\varphi \in E$,

$$\frac{d}{dt} I(u + t\varphi)\Big|_{t=0} = 0$$

i.e. the Euler equation related to the functional I holds in a weak sense for u over special test functions in E. We then use the Nash-Moser methods to show that $\|u\|_\infty$ is finite and finally we get the existence of a nontrivial solution (λ, u) of (1.1).

2. Main Results

In this section, we present the main results of this paper. First of all, we give some notations and conditions.

Let $H^1(\mathbb{R}^n)$ be the usual Sobolev space, $N \geqslant 1$ be a natural number and $E = \{u = (u^1, u^2, \ldots, u^N) \mid u^i \in H^1(\mathbb{R}^n), 1 \leqslant i \leqslant N\}$. The scalar product of $u, v \in E$ is defined by

$$\langle u, v \rangle = \int_{\mathbb{R}^n} [D_\alpha u^i D_\alpha v^i + u^i v^i]\, dx$$

and $(E, <, >)$ is a Hilbert space, the norm of $u \in E$ is $\|u\|_E = (\| |Du| \|_2^2 + \|u\|_2^2)^{1/2}$ where hereafter $\|f\|_q$ denotes the $L^q(\mathbb{R}^n)$ norm of the function f and $|f|$ denotes the Euclidean norm of the function f (possibly vector valued). For simplicity, we denote $\|u\|_E$ by $\|u\|$ for $u \in E$.

The main conditions imposed on (1.1) will be the following

(i) $2 < p < 2\hat{n}/(\hat{n} - 2)$ where $\hat{n} = n$ if $n > 2$; and $2\hat{n}/(\hat{n} - 2)$ is any positive number larger than 2 if $n \leqslant 2$.

(ii) $a_{\alpha\beta}(x, u) \in C^1(\mathbb{R}^n \times \mathbb{R}^N)$, $a_{\alpha\beta} = a_{\beta\alpha}$ for any α, β and $a_1 > 0$, $a_2 > 1$ such that for any $(x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$

$$(2.1) \qquad a_1 |\xi|^2 \leqslant \sigma(|u|)|\xi|^2 \leqslant a_{\alpha\beta}(x, u)\xi_\alpha \xi_\beta \leqslant a_2 \sigma(|u|)|\xi|^2$$

holds, where $\sigma(t)$ is a nonnegative nondecreasing continuous function on $[0, +\infty)$ satisfying: for any $l > 1$, there exists $C_l > 0$, such that

$$\sigma(lt) \leqslant C_l\sigma(t), \qquad \text{for all} \quad t \geqslant 0 \tag{2.2}$$

and C_l are bounded whenever l are bounded. Moreover, there is a constant $C > 0$ with

$$\sigma(t) \leqslant C(1 + |t|^q) \tag{2.3}$$

where $0 \leqslant q \leqslant 4/(n-2)$ if $n > 2$ and $0 \leqslant q$ if $n \leqslant 2$.

(iii) $a_{\alpha\beta}(x, u) \to \bar{a}_{\alpha\beta}(u)$ as $|x| \to +\infty$ uniformly for u bounded.

(iv) There exists, $s \geqslant 0$, $s < p - 2$ such that

$$a_{\alpha\beta}(x, \lambda u)\xi_\alpha\xi_\beta \leqslant \lambda^s a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \tag{2.4}$$

$$a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \leqslant \bar{a}_{\alpha\beta}(u)\xi_\alpha\xi_\beta \tag{2.5}$$

for any $(x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$, where p is given in (i) and $\bar{a}_{\alpha\beta}$ are defined in (iii), and $\lambda > 1$ is arbitrary.

(v) $h \in C(\mathbb{R}^n)$ and there are $\bar{h}, c > 0$ such that $h(x) \geqslant c, h(x) \leqslant \bar{h}$ for any $x \in \mathbb{R}^n$ and $\lim_{|x| \to \infty} h(x) = \bar{h}$.

(vi) There is a constant $c > 0$ such that

$$|u^i D_{u^i} a_{\alpha\beta}(x, u)| \leqslant c\sigma(|u|) \tag{2.6}$$

$$|D_{u^i} a_{\alpha\beta}(x, u)| \leqslant c\eta(|u|) \tag{2.7}$$

for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^N$, where $\eta(t)$ is a nonnegative nondecreasing continuous function on $[0, +\infty)$ and $\sigma(t)$ is given in (ii).

(vii) There is a constant a_3 with $0 < a_3 < 1$ such that

$$-\tfrac{1}{2}u^i D_{u^i} a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \leqslant a_3 a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta \tag{2.8}$$

for any $(x, u, \xi) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$.

Remark 2.1. If $a_{\alpha\beta}(x, u), h(x)$ satisfy (i)-(vii), then $\bar{a}_{\alpha\beta}(u)$, $\bar{h}$ satisfy (i)-(vii).

If $a_{\alpha\beta}(x, u), h(x)$ satisfy (i)-(v), we set, for any $u \in E$

$$I(u) = \int_{\mathbb{R}^n} (a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + h(x)|u|^2)\, dx \tag{2.9}$$

$$I^\infty(u) = \int_{\mathbb{R}^n} (\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i + \bar{h}|u|^2)\, dx \tag{2.10}$$

For any $\lambda > 0$, we set

$$I_\lambda = \inf\left\{I(u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\, dx = \lambda\right\} \tag{2.11}$$

$$I_\lambda^\infty = \inf\left\{I^\infty(u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\, dx = \lambda\right\} \tag{2.12}$$

It is clear that

$$I_\lambda = \inf\left\{I(\lambda^{1/p}u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\,dx = 1\right\} \tag{2.13}$$

$$I_\lambda^\infty = \inf\left\{I^\infty(\lambda^{1/p}u) \mid u \in E, \int_{\mathbb{R}^n} |u|^p\,dx = 1\right\} \tag{2.14}$$

The pair $(\lambda, u) \in \mathbb{R}^1 \times E$ will be called a weak solution of (1.1) if

$$\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta \varphi^i + \varphi^i D_{uj} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x)u^i\varphi^i]\,dx = \lambda \int_{\mathbb{R}^n} |u|^{p-2}u^i\varphi^i\,dx$$

for any $\varphi \in L_\infty \cap E$.

It is evident that $u = 0$ is a trivial solution of (1.1) for any λ.

The main results of this paper are the following

Theorem 2.1. *Suppose that* (i)-(vi) *hold, then for any* $\lambda > 0$, I_λ^∞ *is achieved by some* $u \in E$.

Theorem 2.2. *Suppose that* (i)-(vi) *hold, then there is a* $\lambda_0 > 0$ *such that* I_{λ_0} *is achieved by some* $u \in E$. *Moreover, if* $I_\lambda < I_\lambda^\infty$ *for any* $\lambda > 0$, *then* I_λ *is achieved by some* $u \in E$ *for any* $\lambda > 0$.

Theorem 2.3. *Suppose that* (i)-(vii) *hold, then* (1.1) *posesses at least a nontrivial weak solution* $(\lambda, u) \in \mathbb{R}^1 \times E$ *and* $\|u\|_\infty < \infty$.

Remark 2.2. By (iv)-(v), it is trivial that $I_\lambda \leqslant I_\lambda^\infty$, and by Theorem 2.1, $I_\lambda < I_\lambda^\infty$ (for all $\lambda > 0$) if

$$\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + h(x)|u|^2]\,dx < \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\,dx \tag{2.15}$$

holds for $u \in E$, $\int_{\mathbb{R}^n} |u|^p\,dx = \lambda$ with $I^\infty(u) = I_\lambda{}^\infty < \infty$. (2.15) is valid, for instance, when $h(x) < \bar{h}$ for any $x \in \mathbb{R}^n$, or $a_{\alpha\beta}(x, u)\xi_\alpha\xi_\beta < \bar{a}_{\alpha\beta}(u)\xi_\alpha\xi_\beta$ for any $(x, u, \xi) \in \mathbb{R}^n \times (\mathbb{R}^N - \{0\}) \times (\mathbb{R}^n - \{0\})$.

EXAMPLE 2.1. In (1.1), if $n = 3$, $p = 5$, $h(x)$ satisfies (v), and

$$a_{\alpha\beta}(x, u) = (1 + |u|^2)b_{\alpha\beta}(x) \qquad (\text{or, } a_{\alpha\beta}(x, u) = b_{\alpha\beta}(x)/(1 + |u|^2))$$

where $b_{\alpha\beta}(x) \in C^1(\mathbb{R}^n)$ and $b_{\alpha\beta} = b_{\beta\alpha}$ $(1 \leqslant \alpha, \beta \leqslant n)$ satisfy

$$0 < \lambda|\xi|^2 \leqslant b_{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqslant M|\xi|^2$$

for any $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ where $\lambda, M > 0$ are constants, and $\lim_{|x| \to \infty} b_{\alpha\beta}(x) = \bar{b}_{\alpha\beta}$, then, it is easy to see that $a_{\alpha\beta}(x, u)$, $h(x)$ satisfy conditions (i)-(vii), and thus we conclude that (1.1) possesses at least a nontrivial weak solution by using Theorem 2.3.

The above is only a simple example, the theorems in this section are applicable to many other cases.

3. Proof of Theorems 2.1 and 2.2

In this section, we prove Theorem 2.1 and Theorem 2.2. We need some lemmata and we always suppose that conditions (i)-(v) hold in this section.

Lemma 3.1. *I_λ, I_λ^∞ are continuous functions of λ on $[0, +\infty)$.*

PROOF. It is evident that I_λ, I_λ^∞ are all finite for each $\lambda \geqslant 0$. Let $\lambda_m \to \lambda_0 \in (0, +\infty)$. We may assume that $\lambda_m > 0$ for any $m > 0$. Given $\epsilon > 0$ we have by (2.13), that there are $(u_m) \subset E$ such that $\int_{\mathbb{R}^n} |u_m|^p\,dx = 1$ and

$$I(\lambda_m^{1/p} u_m) \leqslant I_{\lambda_m} + \epsilon.$$

We claim that $|I_{\lambda_m}| \leqslant C$ (hereafter C denotes a constant independent of m). In fact, for fixed $u_0 \in C_0^\infty \subset E$ with $\int_{\mathbb{R}^n} |u_0|^p\,dx = 1$, we have by (2.1), the fact that $|\lambda_m| \leqslant C$ and the continuity of $\sigma(t)$, that

$$\begin{aligned} I_{\lambda_m} &\leqslant I(\lambda_m^{1/p} u_0) = \lambda_m^{2/p} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \lambda_m^{1/p} u_0) D_\alpha u_0^i D_\beta u_0^i + h(x)|u_0|^2]\,dx \\ &\leqslant \lambda_m^{2/p} \int_{\mathbb{R}^n} \sigma(|\lambda_m^{1/p} u_0|)|Du_0|^2 + \lambda_m^{2/p} \int_{\mathbb{R}^n} h(x)|u_0|^2\,dx \leqslant C < +\infty. \end{aligned}$$

Hence, by (ii) we get

$$\int_{\mathbb{R}^n} [\sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2]\,dx \leqslant I_{\lambda_m} + \epsilon \leqslant C. \tag{3.1}$$

Since $\sigma(t)$ is nondecreasing in t, it is trivial that

$$\int_{\mathbb{R}^n} [\sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2]\,dx \leqslant C$$

when $\lambda_m \geqslant \lambda_0$, while if $\lambda_m < \lambda_0$, we have by (2.2) and the boundedness of $(\lambda_0/\lambda_m)^{1/p}$, that

$$\begin{aligned} &\int_{\mathbb{R}^n} [\sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2]\,dx \\ &\qquad = \int_{\mathbb{R}^n} \left[\sigma\left(\left(\frac{\lambda_0}{\lambda_m}\right)^{1/p} \lambda_m^{1/p}|u_m|\right)|Du_m|^2 + h(x)|u_m|^2\right] dx \\ &\qquad \leqslant C_m \int_{\mathbb{R}^n} [\sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2]\,dx \leqslant C < +\infty. \end{aligned}$$

Thus, we always have

$$\int_{\mathbb{R}^n} [\sigma(\lambda_0^{1/p}|u_m|)|Du_m|^2 + h(x)|u_m|^2]\, dx \leqslant C. \tag{3.2}$$

It is clear that

$$\begin{aligned} I_{\lambda_m} + \epsilon &\geqslant I(\lambda_m^{1/p} u_m) \\ &= I(\lambda_m^{1/p} u_m) - I(\lambda_0^{1/p} u_m) + I(\lambda_0^{1/p} u_m) \\ &\geqslant I(\lambda_m^{1/p} u_m) - I(\lambda_0^{1/p} u_m) + I_{\lambda_0}, \end{aligned}$$

but

$$\begin{aligned} I(\lambda_m^{1/p} u_m) - I(\lambda_0^{1/p} u_m) &= \lambda_m^{2/p} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \lambda_m^{1/p} u_m) - a_{\alpha\beta}(x, \lambda_0^{1/p} u_m)] D_\alpha u_m^i D_\beta u_m^i\, dx \\ &\quad + (\lambda_m^{2/p} - \lambda_0^{2/p}) \int_{\mathbb{R}^n} a_{\alpha\beta}(x, \lambda_0^{1/p} u_m) D_\alpha u_m^i D_\beta u_m^i\, dx \\ &\quad + (\lambda_m^{2/p} - \lambda_0^{2/p}) \int_{\mathbb{R}^n} h(x)|u_m|^2\, dx \\ &\equiv I_m^1 + I_m^2 + I_m^3. \end{aligned}$$

It is trivial that $\lim_{m\to\infty} I_m^3 = 0$ and by (2.1) and (3.2) we have that $\lim_{m\to\infty} I_m^2 = 0$. On the other hand, by the mean value theorem, we have

$$\begin{aligned} &|[a_{\alpha\beta}(x, \lambda_m^{1/p} u_m) - a_{\alpha\beta}(x, \lambda_0^{1/p} u_m)] D_\alpha u_m^i D_\beta u_m^i| \\ &\qquad = |(\lambda_m^{1/p} - \lambda_0^{1/p}) u_m^j D_{u^j} a_{\alpha\beta}(x, \xi_m(x) u_m) D_\alpha u_m^i D_\beta u_m^i|, \end{aligned}$$

where $\xi_m(x)$ is between $\lambda_m^{1/p}$ and $\lambda_0^{1/p}$, hence $|\xi_m(x)| \geqslant C > 0$. So, by (2.6), (3.1) and (3.2) we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} u_m^j D_{u^j} a_{\alpha\beta}(x, \xi_m(x) u_m) D_\alpha u_m^i D_\beta u_m^i\, dx \right| &\leqslant C \int_{\mathbb{R}^n} \sigma(\xi_n(x)|u_m|)|Du_m|^2\, dx \\ &\leqslant \max_{0\leq m} C \int_{\mathbb{R}^n} \sigma(\lambda_m^{1/p}|u_m|)|Du_m|^2\, dx \\ &\leqslant C \end{aligned}$$

from which $\lim_{m\to\infty} I_m^1 = 0$ and hence $\liminf_{m\to\infty} I_{\lambda_m} + \epsilon \geqslant I_{\lambda_0}$. Thus we have $\liminf_{m\to\infty} I_{\lambda_m} \geqslant I_{\lambda_0}$ which shows that I_λ is lower-semi continuous on $(0, +\infty)$. On the other hand, it is trivial to see that $\limsup_{m\to\infty} I_{\lambda_m} \leqslant I_{\lambda_0}$, which gives that I_λ is upper-semi continuous on $(0, +\infty)$. So we see that I_λ is continuous on $(0, +\infty)$. It is trivial that I_λ is continuous at $\lambda = 0$ and the lemma is proved. □

Lemma 3.2. *For any* $\lambda > 0$, *we have*

(3.3) $$I_\lambda \leqslant I_\lambda^\infty$$

(3.4) $$I_\lambda^\infty < I_\alpha^\infty + I_{\lambda-\alpha}^\infty \quad \text{for every} \quad \alpha \in (0, \lambda)$$

(3.5) $$I_\lambda < I_\alpha + I_{\lambda-\alpha} \quad \text{for every} \quad \alpha \in (0, \lambda)$$

If $I_\beta < I_\beta^\infty$ *for any* $\beta > 0$, *then*

(3.6) $$I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty \quad \text{for every} \quad \alpha \in [0, \lambda).$$

PROOF. By (iv) and (v), it is trivial that (3.3) holds. To prove (3.5), we only need to show that

(3.7) $$I_{\theta\gamma} < \theta I \quad \text{for every} \quad \gamma \in (0, \lambda), \theta \in \left(1, \frac{\lambda}{\gamma}\right)$$

(see Lemma II.1 of [4]). Given $\gamma \in (0, \lambda), \theta \in \left(1, \frac{\lambda}{\gamma}\right)$, we have by (2.13) and (2.4), that

$$\begin{aligned} I_{\theta\gamma} &= (\theta\gamma)^{2/p} \inf\left\{\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, (\theta\gamma)^{1/p}u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx : u \in E, \right. \\ &\qquad \left. \int_{\mathbb{R}^n} |u|^p\, dx = 1\right\} \\ &\leqslant \theta^{2/p}\gamma^{2/p}\theta^{s/p} \inf\left\{\int_{\mathbb{R}^n} [a_{\alpha\beta}(x, \gamma^{1/p}u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx : u \in E, \right. \\ &\qquad \left. \int_{\mathbb{R}^n} |u|^p\, dx = 1\right\} \\ &= \theta^{(2+s)/p} I_\gamma < \theta I_\gamma \end{aligned}$$

here we have made use of $I_\gamma > 0$ (for all $\gamma > 0$) which can easily be derived from the definition. Thus (3.7) holds and hence (3.5) holds. Similarly, by Remark 2.1 we see that (3.4) holds. By (3.3), (3.5) and $I_\beta < I_\beta^\infty$ (for all $\beta > 0$), we see that (3.6) holds. □

PROOF OF THEOREM 2.1 AND THEOREM 2.2. Let $(u_m) \subset E$ be a minimizing sequence of I_λ (or I_λ^∞) with

$$\int_{\mathbb{R}^n} |u_m|^p\, dx = \lambda > 0$$

and

$$I(u_m) < I_\lambda + 1/m \qquad (\text{or } I_\lambda^\infty(u_m) < I_\lambda^\infty + 1/m).$$

Since I_λ is finite, by (ii) we have

(3.8) $$\int_{\mathbb{R}^n} [\sigma(|u_m|)|Du_m|^2 + h(x)|u_m|^2]\, dx \leqslant C$$

(or

$$\int_{\mathbb{R}^n} [\sigma(|u_m|)|Du_m|^2 + \bar{h}|u_m|^2]dx \leqslant C$$

in the case of I_λ^∞) and $\|u_m\| \leqslant C$.

By the Sobolev embedding theorem, we may assume the existence of a $u_0 = (u_0^1, u_0^2, \dots, u_0^N) \in E$ such that

$$\begin{aligned} &u_m \rightharpoonup u_0 \quad \text{in} \quad E \\ &u_m^i \rightharpoonup u_0^i \quad \text{in} \quad H^1(\mathbb{R}^n), \qquad 1 \leqslant i \leqslant N \\ &u_m \to u_0 \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^n \\ &u_m^i \to u_0^i \quad \text{in} \quad L_{\text{loc}}^t(\mathbb{R}^n), \qquad 2 \leqslant t < \frac{2\hat{n}}{\hat{n}-2} \end{aligned} \tag{3.9}$$

where « $\rightharpoonup$ » designates weak convergence, while « $\to$ » means strong convergence.

Let

$$\rho_m = a_{\alpha\beta}(x, u_m)D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2$$

(respectively

$$\rho_m = \bar{a}_{\alpha\beta}(u_m)Du_m^i Du_m^i + \bar{h}|u_m|^2$$

in the case of I_λ^∞), and

$$\lambda_m = \int_{\mathbb{R}^n} \rho_m \, dx,$$

we easily see that $\lambda_m \geqslant C > 0$. We need the following concentration compactness lemma:

Lemma 3.3. *Let u_m, ρ_m, λ_m be as above, then there exists a subsequence of (ρ_m), still denoted by (ρ_m), satisfying one of the three following possibilities:*

(i) (*Compactness*) *There exists $y_m \in \mathbb{R}^n$ such that $\rho_m(x + y_m)$ is tight, i.e. for every $\epsilon > 0$, there exists R such that*

$$\int_{y_m + B_R} \frac{\rho_m(x)}{\lambda_m} dx \geqslant 1 - \epsilon,$$

where

$$y_m + B_R = \{x \in \mathbb{R}^n : |x - y_m| \leqslant R\}.$$

(ii) (*Vanishing*) $\lim_{m\to\infty} \sup_{y \in \mathbb{R}^n} \int_{y + B_R} \rho_m(x)\, dx = 0$ *for all* $R < +\infty$.

(iii) (*Dichotomy*) *There exist* $\alpha \in (0, 1)$ *and a positive function* $\mu(\epsilon)$, *with* $\lim_{\alpha \to 0} \mu(\epsilon) = 0$, *such that for every* $\epsilon > 0$ *there exist* $m_0 \geqslant 1$ *and* $u_m^1, u_m^2 \in E$ *with* $\|u_m^1\|, \|u_m^2\| \leqslant C$, *so that*

$$\lim_{m \to \infty} \operatorname{dist}(\operatorname{supp} u_m^1, \operatorname{supp} u_m^2) = +\infty \tag{3.10}$$

$$\|u_m - (u_m^1 + u_m^2)\|_2 \leqslant \mu(\epsilon) \tag{3.11}$$

$$\|u_m - (u_m^1 + u_m^2)\|_p < \mu(\epsilon) \tag{3.12}$$

$$\left| \frac{I(u_m^1)}{\lambda_m} - \alpha \right| < \mu(\epsilon) \tag{3.13}$$

$$\left| \frac{I(u_m^2)}{\lambda_m} - (1 - \alpha) \right| < \mu(\epsilon) \tag{3.14}$$

$$I(u_m) \geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon) \tag{3.15}$$

or, respectively, in the case of I_λ^∞,

$$\left| \frac{I^\infty(u_m^1)}{\lambda_m} - \alpha \right| < \mu(\epsilon) \tag{3.16}$$

$$\left| \frac{I^\infty(u_m^2)}{\lambda_m} - (1 - \alpha) \right| < \mu(\epsilon) \tag{3.17}$$

$$I^\infty(u_m) \geqslant I^\infty(u_m^1) + I^\infty(u_m^2) - \mu(\epsilon). \tag{3.18}$$

PROOF. For any $t \geqslant 0$, let

$$Q_m(t) = \sup_{y \in \mathbb{R}^n} \int_{y + B_t} \frac{\rho_m}{\lambda_m} dx.$$

Then $Q_m(t)$ is nondecreasing in t and $|Q_m(t)| \leqslant 1$, so by Helly's principle there is a subsequence of $Q_m(t)$, still denoted by $Q_m(t)$ with $\lim_{m \to \infty} Q_m(t) = Q(t)$ for any $t \geqslant 0$, where $Q(t)$ is a nondecreasing function on $[0, +\infty)$ and $|Q(t)| \leqslant 1$.

Let $\lim_{t \to \infty} Q(t) = \alpha \in [0, 1]$. If $\alpha = 0$, then $Q(t) \equiv 0$, hence $\lim_{m \to \infty} Q_m(t) = 0$ and (ii) (vaninshing) occurs.

If $\alpha = 1$, we can easily show that (i) (compactness) occurs by using the same method as in the proof of Lemma I.1 of [4].

Now, letting $\alpha \in (0, 1)$, we want to show that (iii) (dichotomy) occurs.

Given $\epsilon > 0$, there exists $R_0 = R_0(\epsilon) > 0$ such that

$$\alpha - \epsilon < Q(R_0) < \alpha + \epsilon$$
$$\alpha - 2\epsilon < Q(2R_0) < \alpha + 2\epsilon$$

hence there exists $m_0(\epsilon) > 0$ with

$$\alpha - \epsilon < Q_m(R_0) < \alpha + \epsilon \tag{3.19}$$

$$\alpha - 2\epsilon < Q_m(2R_0) < \alpha + 2\epsilon \tag{3.20}$$

whenever $m \geqslant m_0$.

We may choose $R_m \to +\infty$ such that

$$Q_m(2R_m) < \alpha + 1/m. \tag{3.21}$$

By the absolute continuity of Lebesgue integrals, there are $(z_m) \subset \mathbb{R}^n$ such that

$$Q_m(R_0) = \int_{z_m + B_{R_0}} \frac{\rho_m}{\lambda_m}\, dx. \tag{3.22}$$

Let ξ, $\varphi \in C_b^\infty(R^n)$, $0 \leqslant \xi$, $\varphi \leqslant 1$, $\xi \equiv 1$ and $\varphi \equiv 0$ if $|x| \leqslant 1$; $\xi \equiv 0$ and $\varphi \equiv 1$ if $|x| \geqslant 2$ and set $\xi_m = \xi[(x - z_m)/\tilde{R}]/\tilde{R}(\tilde{R} \geqslant R_0$ is to be determined) $\varphi_m = \varphi[(x - 3m)/R_m]$ and $u_m^1 = \xi_m u_m$, $u_m^2 = \varphi_m u_m$. It is trivial that (3.10) holds and that $\|u_m^1\|$, $\|u_m^2\| \leqslant C$.

By (3.22) we have

$$\begin{aligned} Q_m(R_0) &= \frac{1}{\lambda_m}\int_{z_m + B_{R_0}} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\ &= \frac{1}{\lambda_m}\int_{z_m + B_{R_0}} [a_{\alpha\beta}(x, \xi_m u_m) D_\alpha(\xi_m u_m^i) D_\beta(\xi_m u_m^i) \\ &\qquad + h(x)|\xi_m u_m|^2]\, dx \\ &= \frac{1}{\lambda_m} I(u_m^1) \\ &\quad - \frac{1}{\lambda_m}\int_{|x - z_m| \geq R_0} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i \\ &\qquad + h(x)|u_m^1|^2]\, dx \end{aligned} \tag{3.23}$$

We want to show that

$$\frac{1}{\lambda_m}\int_{|x - z_m| \geq R_0} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i + h(x)|u_m^1|^2]\, dx < \mu(\epsilon). \tag{3.24}$$

Since

$$\begin{aligned} &\frac{1}{\lambda_m}\int_{|x - z_m| \geq R_0} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i + h(x)|u_m^1|^2]\, dx \\ &\leqslant \frac{1}{\lambda_m}\int_{R_0 \leq |x - z_m| \leq 2\tilde{R}} [a_{\alpha\beta}(x, u_m^1)(u_m^i D_\alpha \xi_m + \xi_m D_\alpha u_m^i)(u_m^i D_\beta \xi_m + \xi_m D_\beta u_m^i) \\ &\qquad + h(x)|u_m|^2]\, dx \end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \xi_m^2 a_{\alpha\beta}(x, u_m^1) D_\alpha u_m^i D_\beta u_m^i \, dx \\
&+ \frac{2}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \xi_m u_m^i a_{\alpha\beta}(x, u_m^1) D_\alpha \xi_m D_\beta u_m^i \, dx \\
&+ \frac{1}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} a_{\alpha\beta}(x, u_m^1) D_\alpha \xi_m D_\beta \xi_m \cdot u_m^i u_m^i \, dx \\
&+ \frac{1}{\lambda_m} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} h(x) |u_m|^2 \, dx \\
&\equiv J_m^1 + J_m^2 + J_m^3 + J_m^4 .
\end{aligned}
$$

By (3.19), (3.21) and the fact that $Q_m(t)$ is nondecreasing, it is evident that

$$
|J_m^4| \leqslant Q_m(2\tilde{R}) - Q_m(R_0) < \alpha + 1/m - (\alpha - \epsilon) = 1/m + \epsilon < \mu(\epsilon)
$$

(for m large enough).

By (2.1), (2.2) and (2.3) and since $\|u_m\| \leqslant C$, we have that

$$
\begin{aligned}
|J_m^3| &\leqslant 2a_2 \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \sigma(|\xi_m u_m|) |D\xi_m|^2 |u_m|^2 \, dx \\
&\leqslant \frac{C}{\tilde{R}^2} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \sigma(|u_m|) |u_m|^2 \, dx \\
&\leqslant \frac{C}{\tilde{R}^2} \int_{\mathbb{R}^n} (|u_m|^2 + |u_m|^{q+2}) \, dx \\
&\leqslant \frac{C}{\tilde{R}^2} < \mu(\epsilon),
\end{aligned}
$$

for $\tilde{R}(\epsilon)$ large enough. In the same way, using (2.3) and (3.8) we have that

$$
\begin{aligned}
|J_m^2| &\leqslant \frac{C}{\tilde{R}} \int_{R_0 \le |x - z_m| \le 2\tilde{R}} |a_{\alpha\beta}(x, \xi_m u_m) D_\alpha u_m^i D_\beta u_m^i| \, dx \\
&\leqslant \frac{C}{\tilde{R}} \int_{\mathbb{R}^n} \sigma(|u_m|) |Du_m| \, |u_m| \\
&\leqslant \frac{C}{\tilde{R}} \int_{\mathbb{R}^n} \sigma(|u_m|)(|Du_m|^2 + |u_m|^2) \, dx \\
&< \frac{C}{\tilde{R}} < \mu(\epsilon)
\end{aligned}
$$

for $\tilde{R}(\epsilon)$ large enough. By (2.1), (3.19), (3.21) and (3.22) we have that

$$\begin{aligned}0 \leqslant J_m^1 &\leqslant C \int_{R_0 \le |x - z_m| \le 2\tilde{R}} \sigma(|u_m|)|Du_m|^2 \\ &\leqslant C \int_{R_0 \le |x - z_m| \le 2\tilde{R}} a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i \\ &\leqslant Q_m(2R_m) - Q_m(R_0) < \alpha + 1/m - (\alpha - \epsilon) \\ &= 1/m + \epsilon < \mu(\epsilon)\end{aligned}$$

(for m large enough).

Combining the above estimates, we see that (3.24) holds and (3.13) holds by (3.23). Similarly, (3.16) holds.

It is easy to show (see *e.g.* Lemma I.1 of [4]) that

$$\left| \int_{|x - z_m| \ge 2R_m} \frac{1}{\lambda_m} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx - (1 - \alpha) \right| < \mu(\epsilon) \tag{3.26}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{\lambda_m} I(u_m^2) &= \frac{1}{\lambda_m} \int_{|x - z_m| \ge R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m^2|^2]\, dx \\ &= \frac{1}{\lambda_m} \int_{R_m \le |x - z_m| \le 2R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m^2|^2]\, dx \\ &\quad + \frac{1}{\lambda_m} \int_{|x - z_m| \ge 2R_m} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx\end{aligned} \tag{3.27}$$

Similarly to (3.24), we can prove that

$$\frac{1}{\lambda_m} \int_{R_m \le |x - z_m| \le 2R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m|^2]\, dx \leqslant \mu(\epsilon) \tag{3.28}$$

Thus (3.26) and (3.27) imply that (3.14) holds. Similarly, (3.17) holds.

By (3.19) and (3.21) we have that

$$\begin{aligned}\| u_m - (u_m^1 + u_m^2) \|_2^2 &= \int_{\mathbb{R}^n} |1 - \xi_m - \varphi_m|^2 |u_m|^2\, dx \\ &\leqslant C \int_{\tilde{R} \le |x - z_m| \le 2R_m} |u_m|^2 \\ &\leqslant C[Q_m(2R_m) - Q_m(R_0)] < \mu(\epsilon).\end{aligned}$$

So we have (3.11). Similarly, by $\|u_m\| \leqslant C$ and $\|u_m^1\| \leqslant C$, $\|u_m^2\| \leqslant C$, we see that (3.12) holds.

Finally we prove (3.15). Since

$$\begin{aligned} I(u_m) \geqslant & \int_{|x-z_m| \le \tilde{R}} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\ & + \int_{|x-z_m| \ge 2R_m} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\ = & I(u_m^1) + I(u_m^2) \\ & - \int_{\tilde{R} \le |x-z_m| \le 2\tilde{R}} [a_{\alpha\beta}(x, u_m^1) D_\alpha (u_m^1)^i D_\beta (u_m^1)^i + h(x)|u_m^1|^2]\, dx \\ & - \int_{R_m \le |x-z_m| \le 2R_m} [a_{\alpha\beta}(x, u_m^2) D_\alpha (u_m^2)^i D_\beta (u_m^2)^i + h(x)|u_m^2|^2]\, dx \end{aligned}$$

and because of (3.24) and (3.28), we deduce that

$$I(u_m) \geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon).$$

Thus (3.15) holds. Similarly (3.18) holds. □

Lemma 3.4. (*cf. Lemma* 1.1 *of* [5].) *Let* $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, *with* $q \neq Np/(N-p)$ *if* $p < N$. *Assume that* (u_m) *is bounded in* $L^q(\mathbb{R}^N)$, $|Du_m|$ *is bounded in* $L^p(\mathbb{R}^N)$ *and*

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_m|^q\, dx \to 0 \quad as\ m \to \infty, \quad for\ some \quad R > 0.$$

Then $u_m \to 0$ *in* $L^t(\mathbb{R}^N)$ *for any t between q and* $Np/(N-p)$.

We now turn to prove Theorem 2.1 and Theorem 2.2. We already know that there is a minimizing sequence (u_m) of I_λ (or I_λ^∞) such that Lemma 3.3 holds.

If «vanishing» occurs, then

$$\lim_{m\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx = 0 \tag{3.29}$$

for all R. We know also that (Du_m) is bounded in $L^2(\mathbb{R}^n)$ and by (3.29) we know that

$$\lim_{m\to\infty} \sup_{y\in\mathbb{R}^n} \int_{y+B_R} |u_m|^2\, dx = 0 \qquad \text{(for any } R > 0).$$

So Lemma 3.4 gives that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^n}|u_m|^p\,dx=0$$

and this contradicts

$$\int_{\mathbb{R}^n}|u_m|^p\,dx=\lambda.$$

Thus we have ruled out «vanishing».

If «dichotomy» occurs, then Lemma 3.3 shows that for any $\epsilon>0$, there are $u_m^1, u_m^2\in E$ such that (3.10)-(3.15) hold (or (3.10), (3.12), (3.5) and (3.18) hold in the case of I_λ^∞). Therefore we would have that

$$\begin{aligned}I_\lambda+\epsilon &\geqslant I(u_m)\\ &\geqslant I(u_m^1)+I(u_m^2)-\mu(\epsilon)\\ &\geqslant I_{\int_{\mathbb{R}^n}|u_m^1|^p dx}+I_{\int_{\mathbb{R}^n}|u_m^2|^p dx}-\mu(\epsilon).\end{aligned}\tag{3.30}$$

We may assume that

$$\lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m^1|^p\,dx=\lambda_1(\epsilon),\qquad \lim_{m\to\infty}\int_{\mathbb{R}^n}|u_m^2|^p\,dx=\lambda_2(\epsilon).$$

Now

$$\lambda=\int_{\mathbb{R}^n}|u_m|^p\,dx$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^n}|u_m|^p\,dx-\int_{\mathbb{R}^n}|u_m^1|^p\,dx-\int_{\mathbb{R}^n}|u_m^2|^p\,dx\right| &\leqslant \int_{\mathbb{R}^n}|1-\varphi_m^p-\xi_m^p|\,|u_m|^p\,dx\\ &\leqslant C\int_{R_0\le|x-z_m|\le 2R_m}|u_m|^p\,dx\\ &\leqslant C\left(\int_{R_0\le|x-z_m|\le 2R_m}|u_m|^2\,dx\right)^{p/2}\\ &<\mu(\epsilon),\end{aligned}$$

(where we have made use of notations in the proof of Lemma 3.3.)

We conclude that

$$|\lambda-(\lambda_1(\epsilon)+\lambda_2(\epsilon))|\leqslant\mu(\epsilon)\tag{3.31}$$

Letting $m\to\infty$ in (3.30) and using Lemma 3.1 we obtain that

$$I_\lambda+\epsilon\geqslant I_{\lambda_1(\epsilon)}+I_{\lambda_2(\epsilon)}-\mu(\epsilon).$$

We assume now that $\lambda_1(\epsilon) \to \lambda_1$, $\lambda_2(\epsilon) \to \lambda_2$ as $\epsilon \to 0$. Then we have by Lemma 3.1, that

$$I_\lambda \geqslant I_{\lambda_1} + I_{\lambda_2}. \tag{3.32}$$

By Lemma 3.3 and the fact that $\lambda_m \geqslant c > 0$ we have that

$$|I(u_m^1) - \tilde{\alpha}| < \mu(\epsilon), \quad \text{where} \quad \tilde{\alpha} > 0$$
$$|I(u_m^2) - \tilde{\beta}| < \mu(\epsilon), \quad \text{where} \quad \beta > 0.$$

Thus, if $\lambda_1 = 0$ then by (3.31) $\lambda_2 = \lambda$. Since

$$I_\lambda + \epsilon \geqslant I(u_m) \geqslant I(u_m^1) + I(u_m^2) - \mu(\epsilon)$$

we obtain that

$$I_\lambda \geqslant \tilde{\alpha} + I_{\lambda_2(\epsilon)} - \mu(\epsilon).$$

Hence

$$I_\lambda \geqslant \tilde{\alpha} + I_\lambda.$$

This is a contradiction and so $\lambda_1 > 0$; similarly $\lambda_2 > 0$. And now $\lambda_1 + \lambda_2 = \lambda$ and (3.32) contradict (3.5). Thus we have ruled out the «dichotomy» for I_λ. Similarly we can rule out the «dichotomy» for I_λ^∞ using (3.4).

So we only have «compactness» *i.e.* there exists $(y_m) \subset \mathbb{R}^n$ such that for any $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ with

$$\int_{|x-y_m| \leq R} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \geqslant \lambda_m(1 - \epsilon).$$

Hence

$$\int_{|x-y_m| \geq R} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \leqslant \lambda_m \epsilon \tag{3.33}$$
$$\int_{|x-y_m| \geq R} [|Du_m|^2 + |u_m|^2]\, dx \leqslant \mu(\epsilon)$$

or, in the case of I_λ^∞, we have

$$\int_{|x-y_m| \geq R} [\bar{a}_{\alpha\beta}(u_m) D_\alpha u_m^i D_\beta u_m^i + \bar{h}|u_m|^2]\, dx \leqslant \lambda_m \epsilon \tag{3.34}$$
$$\int_{|x-y_m| \geq R} [|Du_m|^2 + |u_m|^2]\, dx \leqslant \mu(\epsilon)$$

We first prove Theorem 2.1. Let $\bar{u}_m(x) = u_m(x + y_m)$, then $\|\bar{u}_m\| \leqslant C < +\infty$ and by (3.34) and the Sobolev embedding theorem we may assume the existence of a $u = (u^1, u^2, \ldots, u^N) \in E$ such that

$$\text{(3.35)}\qquad \begin{cases} \bar{u}_m \rightharpoonup u & \text{in } \; E \\ \bar{u}^i_m \rightharpoonup u^i & \text{in } \; H^1(R^n) \\ \bar{u}^i_m \to u^i & \text{in } \; L^t(R^n) \qquad 2 \leqslant t < 2\hat{n}/(\hat{n}-2) \\ \bar{u}_m \to u & \text{a.e. in } \; R^n, \end{cases}$$

for $1 \leqslant i \leqslant N$, and

$$\lambda = \int_{\mathbb{R}^n} |u_m|^p \, dx = \int_{\mathbb{R}^n} |\bar{u}_m|^p \, dx \to \int_{\mathbb{R}^n} |u|^p \, dx \qquad (\text{as} \quad m \to \infty).$$

Also

$$I^\infty_\lambda = \lim_{m \to \infty} \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(\bar{u}_m) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m + \bar{h}|u_m|^2] \, dx.$$

By (3.35) and (ii), (iii) of Section 2 we see that

$$\bar{a}_{\alpha\beta}(\bar{u}_m) \to \bar{a}_{\alpha\beta}(u) \quad \text{a.e. in} \quad \mathbb{R}^n.$$

So for any bounded domain $\Omega \subset \mathbb{R}^n$ and $\delta > 0$, there is a $\Omega_\delta \subset \Omega$ with

$$|\Omega - \Omega_\delta| < \delta$$

and

$$\bar{a}_{\alpha\beta}(\bar{u}_m) \to \bar{a}_{\alpha\beta}(u)$$

uniformly for $x \in \Omega_\delta$ where $|A|$ denotes the Lebesgue measure of A for any $A \subset \mathbb{R}^n$. So that for any $\epsilon > 0$ and m large enough we have, by (2.2), that

$$\begin{aligned} \int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx &\geqslant \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(\bar{u}_m) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx \\ &\geqslant \int_{\Omega_\delta} [\bar{a}_{\alpha\beta}(\bar{u}^i_m) - \bar{a}_{\alpha\beta}(u)] D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx \\ &\quad + \int_{\Omega_\delta} a_{\alpha\beta}(u) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx \\ &\geqslant -\epsilon \int_{\mathbb{R}^n} |D\bar{u}_m|^2 \, dx + \int_{\Omega_\delta} a_{\alpha\beta}(u) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx \\ &\geqslant -\epsilon C + \int_{\Omega_\delta} a_{\alpha\beta}(u) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx. \end{aligned}$$

By (3.35), Mazur's theorem (see [6]) and Fatou's lemma, we see that

$$\liminf_{m \to \infty} \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u) D_\alpha \bar{u}^i_m D_\beta \bar{u}^i_m \, dx \geqslant \int_{\Omega_\delta} \bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i \, dx,$$

and hence we get, for any N, that

$$\liminf_{m\to\infty}\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha\bar{u}^i_m D_\beta\bar{u}^i_m\,dx \geqslant \int_{\Omega_\delta}\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i\,dx$$

$$\geqslant \int_{\Omega_\delta}[\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i]_N\,dx$$

where the function $[f]_N$ for any $f\geqslant 0$ is given by

$$[f]_N = \begin{cases} f & \text{if } f\leqslant N \\ N & \text{if } f\geqslant N\end{cases}$$

Since $[\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i]_N \in L^1(\Omega)$, and since $|\Omega_\delta|\to|\Omega|$ we have that

$$\liminf_{m\to\infty}\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha\bar{u}^i_m D_\beta\bar{u}^i_m\,dx \geqslant \int_\Omega [\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i]_N\,dx.$$

Letting $N\to\infty$, we have that

$$\liminf_{m\to\infty}\int_\Omega \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha u^i_m D_\beta u^i_m \geqslant \int_\Omega \bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i\,dx \tag{3.36}$$

Thus, since the supremum of any sequence of lower-semicontinuous functions is still lower-semicontinuous, we have that

$$\liminf_{m\to\infty}\int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha u^i_m D_\beta u^i_m\,dx \geqslant \int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i\,dx \tag{3.37}$$

On the other hand, by (3.35) we have that

$$\lim_{m\to\infty}\int_{\mathbb{R}^n}\bar{h}|u_m|^2\,dx = \int_{\mathbb{R}^n}\bar{h}|u|^2\,dx$$

and so we get that

$$I^\infty_\lambda \geqslant \liminf_{m\to\infty}\int_{\mathbb{R}^n}\bar{a}_{\alpha\beta}(\bar{u}_m)D_\alpha u^i_m D_\beta u^i_m\,dx + \lim_{m\to\infty}\int_{\mathbb{R}^n}\bar{h}|u_m|^2\,dx$$

$$\geqslant \int_{\mathbb{R}^n}[\bar{a}_{\alpha\beta}(u)D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\,dx.$$

But

$$\int_{\mathbb{R}^n}|u|^p\,dx = \lambda$$

and so

$$I_\lambda^\infty = \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\, dx$$

so I_λ^∞ is achieved and Theorem 2.1 is proved.

In the case of I_λ, by (3.33) we still have (3.35) with $\bar{u}_m(x) = u_m(x + y_m)$. If there is $\lambda_0 \in (0, \lambda]$ such that $I_{\lambda_0} = I_{\lambda_0}^\infty$, then by Theorem 2.1 there exists $u_0 \in E$ with $\int_{\mathbb{R}^n} |u_0|^p\, dx = \lambda_0$ and such that $I_{\lambda_0}^\infty = I^\infty(u_0)$, and hence $I_{\lambda_0} \leqslant I(u_0) \leqslant I^\infty(u_0) = I_{\lambda_0}^\infty = I_{\lambda_0}$ implies that $I(u_0) = I_{\lambda_0}$ and therefore I_{λ_0} is achieved by u_0 Theorem 2.2 is proved.

Now we assume that for any $0 < \mu \leqslant \lambda$ we have $I_\mu < I_\mu^\infty$. If (y_m) is unbounded, say $|y_m| \to \infty$, we have, by (ii) of Section 2 and (3.35), that

$$a_{\alpha\beta}(x + y_m, \bar{u}_m) \to \bar{a}_{\alpha\beta}(u) \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^n.$$

So we have, as in (3.37), that

$$\liminf_{i \to \infty} \int_{\mathbb{R}^n} a_{\alpha\beta}(x + y_m, \bar{u}_m) D_\alpha u_m^i D_\beta u_m^i\, dx \geqslant \int_{\mathbb{R}^n} \bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i\, dx. \tag{3.38}$$

By (v) of Section 2, (3.35) and the Lebesgue's theorem we have that

$$\lim_{m \to \infty} \int_{\mathbb{R}^n} h(x + y_m)|\bar{u}_m|^2\, dx = \int_{\mathbb{R}^n} \bar{h}|u|^2\, dx. \tag{3.39}$$

Combining (3.38), (3.39) and

$$\int_{\mathbb{R}^n} |u|^p\, dx = \lambda$$

we have that

$$\begin{aligned} I_\lambda &= \lim_{m \to \infty} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2]\, dx \\ &= \lim_{m \to \infty} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x + y_m, \bar{u}_m) D_\alpha \bar{u}_m^i D_\beta \bar{u}_m^i + h(x + y_m)|\bar{u}_m|^2]\, dx \\ &\geqslant \int_{\mathbb{R}^n} [\bar{a}_{\alpha\beta}(u) D_\alpha u^i D_\beta u^i + \bar{h}|u|^2]\, dx \\ &\geqslant I_\lambda^\infty \end{aligned}$$

which contradicts that $I_\mu < I_\mu^\infty$ for any $0 < \mu \leqslant \lambda$. Thus we have $|y_m| \leqslant C$ and by (3.34) we see that for any $\epsilon > 0$, there is a $R(\epsilon) > 0$ such that

$$\int_{|x| \geq R} [|Du_m|^2 + |u_m|^2]\, dx \leqslant \epsilon \tag{3.40}$$

and hence we may assume the existence of a $u_0 \in E$ such that

$$
(3.41) \qquad \begin{cases} u_m \rightharpoonup u_0 & \text{in } E \\ u_m^i \rightharpoonup u_0^i & \text{in } H^1(\mathbb{R}^n), \quad (1 \leqslant i \leqslant N), \\ u_m^i \to u_0^i & \text{in } L^t(\mathbb{R}^n) \quad 2 \leqslant t < \dfrac{2\hat{n}}{\hat{n}-2} \quad (1 \leqslant i \leqslant N), \\ u_m \to u_0 & \text{a.e. in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} |u_0|^p \, dx = \lambda \end{cases}
$$

Thus, similarly to (3.38) and (3.39) we can prove that

$$
\liminf_{m\to\infty} \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i \, dx \geqslant \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u_0) D_\alpha u_0^i D_\beta u_0^i \, dx
$$

$$
\lim_{m\to\infty} \int_{\mathbb{R}^n} h(x)|u_m|^2 \, dx = \int_{\mathbb{R}^n} h(x)|u_0|^2 \, dx.
$$

Since $\int_{\mathbb{R}^n} |u_0|^p \, dx = \lambda$ we have

$$
\begin{aligned}
I_\lambda &\geqslant \liminf_{m\to\infty} \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u_m) D_\alpha u_m^i D_\beta u_m^i + h(x)|u_m|^2] \, dx \\
&\geqslant \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u_0) D_\alpha u_0^i D_\beta u_0^i + h(x)|u_0|^2] \, dx \\
&\geqslant I_\lambda
\end{aligned}
$$

and hence I_λ is achieved by $u_0 \in E$. Theorem 2.2 is proved. □

4. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. The main difficulty is that I_λ is in general not in $C^1(E, \mathbb{R})$. To overcome this difficulty, we first prove that

$$
\frac{d}{dt} I\left(\frac{\lambda(u + t\varphi)}{\|u + t\varphi\|_p} \right) \bigg|_{t=0}
$$

exists for special $\varphi \in E$ and then show that $\|u\|_\infty$ is finite where u is a minimizer of I_λ for some $\lambda > 0$. Finally we prove the theorem.

Proof of Theorem 2.3. By Theorem 2.2 we may assume without loss of generality the existence of $u \in E$, with $\int_{\mathbb{R}^n} |u|^p \, dx = 1$ and such that

$$
I_1 = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x)|u|^2] \, dx.
$$

We first prove that for any $\tau \geqslant 0$,

$$\frac{d}{dt} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right)\Bigg|_{t=0} = 0 \tag{4.1}$$

where

$$|u|_L = \begin{cases} |u| & \text{if} \quad |u| \leqslant L \\ L & \text{if} \quad |u| \geqslant L \end{cases}$$

It is easy to see that $u + t|u|_L^\tau u = (1 + t|u|_L^\tau)u \in E$ for any $t \geqslant 0$ and since u achieves I_1, (4.1) will hold if

$$\frac{d}{dt} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right)\Bigg|_{t=0}$$

exists.

Because $0 \leqslant |u|_L^\tau \leqslant L^\tau$, there is a $M > 0$, depending on β and L, such that

$$\tfrac{1}{2} \leqslant \|u + t|u|_L^\tau u\|_p \leqslant M \tag{4.2}$$

for t small enough.

It is easy to prove that

$$\frac{d}{dt}(\|u + t|u|_L^\tau u\|_p)\Bigg|_{t=0} = \int_{\mathbb{R}^n} |u|^p |u|_L^\tau \, dx \tag{4.3}$$

and hence

$$\frac{d}{dt}\left[\int_{\mathbb{R}^n} \frac{h(x)|u + t|u|_L^\tau u|^2}{\|u + t|u|_L^\tau u\|_p} dx\right]\Bigg|_{t=0} \tag{4.4}$$

$$= 2\int_{\mathbb{R}^n} h(x)|u|^2 |u|_L^\tau \, dx - 2\int_{\mathbb{R}^n} h(x)|u|^2 \, dx \int_{\mathbb{R}^n} |u|^p |u|_L^\tau \, dx.$$

On the other hand

$$\begin{aligned} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right) &= \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right) \frac{D_\alpha u^i D_\beta u^i}{\|u + t|u|_L^\tau u\|_p^2} dx \\ &+ 2t \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t|u|_L^2 u}{\|u + t|u|_L^\tau u\|_p}\right) \frac{D_\alpha u^i D_\beta |u|_L^\tau u^i}{\|u + t|u|_L^\tau u\|_p^2} dx \\ &+ t^2 \int_{\mathbb{R}^n} a_{\alpha\beta}\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right) \frac{D_\alpha(|u|_L^\tau u^i) D_\beta(|u|_L^\tau u^i)}{\|u + t|u|_L^\tau u\|_p^2} dx \\ &+ \int_{\mathbb{R}^n} \frac{h(x)|u + t|u|_L^\tau u|^2}{\|u + t|u|_L^\tau u\|_p^2} dx \end{aligned}$$

$$(4.5)\quad I\left(\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau u\|_p}\right)=I^1(t)+I^2(t)+I^3(t)+\int_{\mathbb{R}^n}\frac{h(x)|u+t|u|_L^\tau u|^2}{\|u+t|u|_L^\tau u\|_p^2}\,dx$$

Using (ii), (iii) of Section 2, (4.2) and (2.1), the inequality

$$\left|a_{\alpha\beta}\left(x,\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau\|_p}\right)\frac{u^it|u|^{\tau-1}D_\alpha u^2 D_\beta|u|_L}{\|u+t|u|_L^\tau u\|_p^2}\right|\leqslant C|D_\alpha u^i D_\beta|u|_L|u|^\tau|,L^1(\mathbb{R}^n)$$

(which holds if $|u|\leqslant L$) and the Dominated Convergence Theorem, we have that

$$(4.6)\qquad \frac{d}{dt}[I^2(t)]\bigg|_{t=0}=\lim_{t\to0}\frac{I^2(t)}{t}=2\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^iD_\beta(|u|_L^\tau u^i)\,dx.$$

Similarly, we have that

$$(4.7)\qquad \frac{d}{dt}[I^3(t)]\bigg|_{t=0}=0$$

On the other hand

$$\begin{aligned}\frac{d}{dt}[I^1(t)]\bigg|_{t=0}&=\lim_{t\to0}\int_{\mathbb{R}^n}\frac{1}{t}\left[a_{\alpha\beta}\left(x,\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau\|_p}\right)\|u+t|u|_L^\tau u\|_p^{-2}\right.\\&\qquad\left.-a_{\alpha\beta}(x,u)\right]D_\alpha u^iD_\beta u^i\,dx\\&=\lim_{t\to0}\int_{\mathbb{R}^n}\frac{1}{t}\left[a_{\alpha\beta}\left(x,\frac{u+t|u|_L^\tau u}{\|u+t|u|_L^\tau u\|_p}\right)-a_{\alpha\beta}(x,u)\right]\\&\qquad\|u+t|u|_L^\tau u\|_p^{-2}D_\alpha u^iD_\beta u^i\,dx\\&\qquad+\lim_{t\to0}\frac{1}{t}(\|u+t|u|_L^\tau u\|_p^{-2}-1)\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^iD_\beta u^i\,dx\\&\equiv\lim_{t\to0}I^4(t)+\lim_{t\to0}I^5(t).\end{aligned}$$

By (4.3), we have

$$(4.8)\qquad \lim_{t\to0}I^5(t)=-2\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^iD_\beta u^i\,dx.$$

Using the mean value theorem we get that

$$\begin{aligned}\lim_{t\to0}I^4(t)=\lim_{t\to0}\int_{\mathbb{R}^n}&D_{u^j}a_{\alpha\beta}\left(x,\frac{u+t'|u|_L^\tau u}{\|u+t'|u|_L^\tau u\|_p}\right)\\&\left[\frac{|u|_L^\tau u^j}{\|u+t'|u|_L^\tau u\|_p}-\frac{u^j+t'|u|_L^\tau u^j}{\|u+t'|u|_L^\tau u\|_p^2}\frac{d}{dt}\|u+t|u|_L^\tau u\|_{p|t=t'}\right]\\&\|u+t|u|_L^\tau u\|_p^{-2}D_\alpha u^iD_\beta u^i\,dx\end{aligned}$$

$$= \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)\frac{|u|_L^\tau u^j}{\|u + t'|u|_L^\tau u\|_p}$$

$$\|u + t|u|_L^\tau u\|_p^{-2} D_\alpha u^i D_\beta u^i\, dx$$

$$- \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)$$

$$\frac{u^j + t'|u|_L^\tau u^j}{\|u + t'|u|_L^\tau u\|_p^2}\frac{d}{dt}\|u + t|u|_L^\tau u\|_{p|t=t'}\|u + t|u|_L^\tau u\|_p^{-2} D_\alpha u^i D_\beta u^i\, dx$$

$$(4.9)\qquad \equiv \lim_{t\to 0} I^6(t) - \lim_{t\to 0} I^7(t) \qquad (0 < t' = t'(x) < t)$$

By (vi) of Section 2 and (3.2) we have that

$$\left| D_{u^j}a_{\alpha\beta}\left(x, \frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)\frac{|u|_L^\tau u^j}{\|u + t'|u|_L^\tau u\|_p} D_\alpha u^i D_\beta u^i \|u + t|u|_L^\tau u\|_p^{-2}\right|$$

$$\leqslant C\sigma\left(\frac{u + t'|u|_L^\tau u}{\|u + t'|u|_L^\tau u\|_p}\right)\frac{|u|_L^\tau}{1 + t'|u|_L^\tau}|Du|^2$$

$$\leqslant C\sigma(|u|)|Du|^2 \in L^1(\mathbb{R}^n),$$

hence by the Dominated Convergence Theorem

$$(4.10)\qquad \lim_{t\to 0} I^6(t) = \int_{\mathbb{R}^n} |u|_L^\tau u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx.$$

Similarly, by (vi) of Section 2, (4.2) and (4.3) we get

$$(4.11)\qquad \lim_{t\to 0} I^7(t) = \int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx \int_{\mathbb{R}^n} u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^j\, dx.$$

Combining (4.4)-(4.11) we see that (4.1) holds and that

$$0 = \frac{d}{dt} I\left(\frac{u + t|u|_L^\tau u}{\|u + t|u|_L^\tau u\|_p}\right)\Bigg|_{t=0}$$

$$= 2\int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta(|u|_L^\tau u^i)\, dx$$

$$+ \int_{\mathbb{R}^n} |u|_L^\tau u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx$$

$$- \int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx \int_{\mathbb{R}^n} u^j D_{u^j}a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx$$

$$- 2\int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i\, dx + 2\int_{\mathbb{R}^n} h(x)|u|^2 |u|_L^\tau\, dx$$

$$- 2\int_{\mathbb{R}^n} h(x)|u|^2\, dx \int_{\mathbb{R}^n} |u|^p |u|_L^\tau\, dx$$

which implies that

$$\text{(4.12)}\quad \int_{\mathbb{R}^n} a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta(|u|_L^\tau u^i)\,dx + \tfrac{1}{2}\int_{\mathbb{R}^n} |u|_L^\tau u^j D_{u^j} a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx$$
$$+ \int_{\mathbb{R}^n} h(x)|u|^2|u|_L^\tau\,dx = \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx \quad (\text{for every } \tau \geqslant 0 \text{ and } L \geqslant 0),$$

where

$$\lambda = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i + \tfrac{1}{2}u^j D_{u^j} a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i + h(x)|u|^2]\,dx.$$

Now we are ready to prove that $\|u\|_\infty < +\infty$. By (4.12), we have for any $\tau \geqslant 0$, that

$$\text{(4.13)}\quad \int_{\mathbb{R}^n}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j\,dx + \tau\int_{\mathbb{R}^n}|u|_L^{\tau-1}u^i a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta|u|_L\,dx$$
$$+ \tfrac{1}{2}\int_{\mathbb{R}^n}|u|_L^\tau u^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx + \int_{\mathbb{R}^n}h(x)|u|^2|u|_L^\tau\,dx$$
$$= \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx.$$

It is easy to see that

$$\int_{\mathbb{R}^n}|u|_L^{\tau-1}u^i a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta|u|_L\,dx = \int_{\mathbb{R}^n}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha|u|D_\beta|u|_L\,dx$$
$$= \int_{\{|u|\leq L\}}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha|u|D_\beta|u|\,dx$$
$$\geqslant 0.$$

So by (2.8) we have

$$\text{(4.14)}\qquad (1-a_3)\int_{\mathbb{R}^n}|u|_L^\tau a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\,dx \leqslant \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx$$

hence

$$\mu(1-a_3)\int_{\mathbb{R}^n}|Du|^2|u|_L^\tau\,dx \leqslant \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx.$$

It is easy to see that

$$|D|u||^2 \leqslant |Du|^2$$

and from this and (4.14) we get that

$$\mu(1-a_3)\int_{\mathbb{R}^n}|D|u||^2|u|_L^\tau\,dx \leqslant \lambda\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx.$$

Thus, there is a $C > 0$ such that for any $\tau \geqslant 0$

$$\text{(4.15)}\qquad \int_{\mathbb{R}^n}|D|u|\,|u|_L^{\tau/2}|^2\,dx \leqslant C\int_{\mathbb{R}^n}|u|^p|u|_L^\tau\,dx$$

holds.

By (4.15) we have, for any $\tau \geqslant 1$, that

$$\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx \leqslant C\int_{\mathbb{R}^n} |u|^p|u|_L^{2(\tau-1)}\,dx. \tag{4.16}$$

Let $w_L = |u|\,|u|_L^{\tau-1}$, then we have

$$Dw_L = D|u|\,|u|_L^{\tau-1} + (\tau-1)|u|_L^{\tau-2}D|u|_L^{\tau-2}D|u|_L|u|.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^n} |Dw_L|^2\,dx &\leqslant C\Big[\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx + (\tau-1)^2\int_{\mathbb{R}^n} |\,|u|_L^{\tau-2}|u|D|u|_L|^2\,dx\Big]\\ &\leqslant C\Big[\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx + (\tau-1)^2\int_{\{|u|\le L\}} |D|u|\,|u|^{\tau-1}|^2\,dx\Big]\\ &\leqslant C(1+(\tau-1)^{2)}\int_{\mathbb{R}^n} |D|u|\,|u|_L^{\tau-1}|^2\,dx\\ &\leqslant C\tau^2\int_{\mathbb{R}^n} |u|^p|u|_L^{2\tau-2}\,dx.\end{aligned}$$

So we get

$$\int_{\mathbb{R}^n} |Dw_L|^2\,dx \leqslant C\tau^2\int_{\mathbb{R}^n} |u|^{p-2}|w_L|^2\,dx \tag{4.17}$$

By (4.17), the Sobolev embedding theorems and Hölder's inequality we have that

$$\begin{aligned}\|w_L\|_{2^*}^2 &\leqslant C\|\,|Dw_L|\,\|_2^2\\ &\leqslant C\tau^2\Big(\int_{\mathbb{R}^n} |u|^{(p-2)\frac{2^*}{p-2}}\,dx\Big)^{\frac{p-2}{2^*}}\Big(\int_{\mathbb{R}^n} |w_L|^{2\frac{2^*}{2^*-(p-2)}}\,dx\Big)^{\frac{2^*-(p-2)}{2}}\\ &= C\tau^2\|u\|_{2^*}^{p-2}\|w_L\|_{\frac{2q}{2-2}}^2\end{aligned} \tag{4.18}$$

where $2q/(q-2) = 2\cdot 2^*/(2^*-(p-2))$, *i.e.* $q = 2\cdot 2^*/(p-2)$. It is easy to see that $q > n$ when $n > 2$ or $n \leqslant 2$ by choosing 2^* large enough, hence $2^* > 2^* > 2q/(q-2)$. If $|u|^2 \in L^{2q/(q-2)}(\mathbb{R}^n)$, letting $L \to +\infty$ in (4.18) and using the Dominated Convergence Theorem and Fatou's lemma together with the fact that $|w_L| \leqslant |u|^\tau$ we get that

$$\|\,|u|^\tau\|_{2^*}^2 \leqslant C\tau^2\|\,|u|^\tau\|_{\frac{2q}{q-2}}^2.$$

Thus $u \in L^{2\tau q/(q-2)}(\mathbb{R}^n)$ implies that $u \in L^{\tau 2^*}(\mathbb{R}^n)$. If we set $q^* = 2q/(q-2)$, $\chi = 2^*/q^*$ then $\tau\chi q^* = \tau 2^*$ and we have that

$$\|\,|u|\,\|_{\tau\lambda q^*}^\tau \leqslant C\tau\|\,|u|\,\|_{\tau q^*}^\tau$$

that is

$$\|\,|u|\,\|_{\tau\chi q^*} \leqslant C^{1/\tau}\tau^{1/\tau}\|\,|u|\,\|_{\tau q^*}.$$

Let $\tau = \chi^m$, $m = 0, 1, \ldots$, then we have

$$\|u\|_{\chi^N q^*} \leqslant \prod_{m=0}^{N-1} (C\chi^m)^{-\chi^m} \|u\|_{q^*} \leqslant C^\sigma \chi^\tau \| |u| \|_{q^*} \leqslant C\|u\|_{q^*} \tag{4.19}$$

where

$$\sigma = \sum_{m=0}^{N-1} \chi^{-m}, \qquad \tau = \sum_{m=0}^{N-1} m\chi^{-m}$$

and C is independent of N for $\sum_{m=0}^{\infty} \chi^{-m}$, $\sum_{m=0}^{\infty} m\chi^{-m}$ are all convergent. Letting $N \to \infty$ in (4.19) we get

$$\|u\|_\infty \leqslant C\|u\|_{q^*} < +\infty. \tag{4.20}$$

Thus $u \in L_\infty \cap E$.

Finally, we show that for any $\varphi \in L_\infty \cap E$, we have

$$\frac{d}{dt} I\left(\frac{u + t\varphi}{\|u + t\varphi\|_p}\right)\bigg|_{t=0} = 0. \tag{4.21}$$

Note that we only need to show that

$$\frac{d}{dt} I\left(\frac{u + t\varphi}{\|u + t\varphi\|_p}\right)\bigg|_{t=0}$$

exists for any $\varphi \in L_\infty \cap E$. (4.21) can be proved by using the same method for proving (4.1). In fact, similarly to (4.2), (4.3), (4.4) and (4.5) we may obtain

$$\frac{1}{2} \leqslant \|u + t\varphi\|_p \leqslant M \qquad \text{(for } t \text{ small enough)} \tag{4.22}$$

$$\frac{d}{dt} \|u + t\varphi\|_{p|t=0} = \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx \tag{4.23}$$

$$\frac{1}{dt} \int_{\mathbb{R}^n} \frac{h(x)|u + t\varphi|^2}{\|u + t\varphi\|_p^2} dx \bigg|_{t=0} = 2\int_{\mathbb{R}^n} h(x) u^i \varphi^i \, dx - 2\int_{\mathbb{R}^n} h(x)|u|^2 \, dx \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx \tag{4.24}$$

$$I\left(\frac{u + t\varphi}{\|u + t\varphi\|_p}\right) = \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) \frac{D_\alpha u^i D_\beta u^i}{\|u + t\varphi\|_p^2} dx + 2t\int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u + t\varphi}{\|u + t\varphi\|_p}\right) \frac{D_\alpha u^i D_\beta \varphi^i}{\|u + t\varphi\|_p^2} dx \tag{4.25}$$

$$+ \frac{t^2}{\|u+t\varphi\|_p^2} \int_{\mathbb{R}^n} a_{\alpha\beta}\left(x, \frac{u+t\varphi}{\|u+t\varphi\|_p}\right) D_\alpha \varphi^i D_\beta \varphi^i \, dx$$

$$+ \int_{\mathbb{R}^n} \frac{h(x)|u+t\varphi|^2}{\|u+t\varphi\|_p^2} \, dx$$

$$\equiv J^1(t) + J^2(t) + J^3(t) + \int_{\mathbb{R}^n} \frac{h(x)|u+t\varphi|^2}{\|u+t\varphi\|_p^2} \, dx.$$

Using that $\|u\|_\infty \leqslant C$, $\|\varphi\|_\infty \leqslant C$, (4.22) and (ii) of Section 2, and similarly to (4.6) and (4.7) we obtain that

$$\frac{d}{dt} J^2(t)\bigg|_{t=0} = 2 \int_{\mathbb{R}^n} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta \varphi^i \, dx, \qquad \frac{d}{dt} J^3(t)\bigg|_{t=0} = 0.$$

On the other hand, we have that

$$\frac{d}{dt} J^1(t)\bigg|_{t=0} = \lim_{t \to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[a_{\alpha\beta}\left(x, \frac{u+t\varphi}{\|u+t\varphi\|_p}\right) \|u+t\varphi\|_p^{-2} - a_{\alpha\beta}(x,u) \right] D_\alpha u^i D_\beta u^i \, dx$$

$$= \lim_{t \to 0} \int_{\mathbb{R}^n} \frac{1}{t} \left[a_{\alpha\beta}\left(x, \frac{u+t\varphi}{\|u+t\varphi\|_p}\right) - a_{\alpha\beta}(x,u) \right] \|u+t\varphi\|_p^{-2} D_\alpha u^i D_\beta u^i \, dx$$

$$+ \lim_{t \to 0} \frac{1}{t} (\|u+t\varphi\|_p^{-2} - 1) \int_{\mathbb{R}^n} a_{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^i \, dx$$

$$= \lim_{t \to 0} J^4(t) + \lim_{t \to 0} J^5(t).$$

By (4.23) and similarly to (4.8) we obtain that

$$\lim_{t \to 0} J^5(t) = -2 \int_{\mathbb{R}^n} |u|^{p-2} u^i \varphi^i \, dx \int_{\mathbb{R}^n} a_{\alpha\beta}(x,u) D_\alpha u^i D_\beta u^i \, dx.$$

Using the mean value theorem we have that

$$\lim_{t \to 0} J^4(t) = \lim_{t \to 0} \int_{\mathbb{R}^n} D_{u^j} a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right) \left[\frac{\varphi^j}{\|u+t'\varphi\|_p} - \frac{u^j + t'\varphi^j}{\|u+t'\varphi\|_p^2} \frac{d}{dt} \|u+t\varphi\|_p \bigg|_{t=t'} \right] \|u+t\varphi\|_p^{-2} D_\alpha u^i D_\beta u^i \, dx$$

$$= \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right)\frac{\varphi^j}{\|u+t'\varphi\|_p}$$

$$\|u+t\varphi\|_p^{-2}D_\alpha u^i D_\beta u^i\, dx$$

$$- \lim_{t\to 0}\int_{\mathbb{R}^n} D_{u^j}a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right)\frac{u^j+t'\varphi^j}{\|u+t'\varphi\|_p^2}$$

$$\|u+t\varphi\|_p^{-2}D_\alpha u^i D_\beta u^i \frac{d}{dt}\|u+t\varphi\|_p\Big|_{t=t'} dx$$

$$= \lim_{t\to 0} J^6(t) - \lim_{t\to 0} J^7(t),$$

where $0 < t'(x) < t$. By (2.7) and (4.22) we see that

$$\left|D_{u^j}a_{\alpha\beta}\left(x, \frac{u+t'\varphi}{\|u+t'\varphi\|_p}\right)\frac{\varphi^j}{\|u+t'\varphi\|_p}\|u+t\varphi\|_p^{-2}D_\alpha u^i D_\beta u^i\right|$$

$$\leqslant C\eta\left(\frac{|u|+t'|\varphi|}{\|u+t'\varphi\|_p}\right)\|u+t\varphi\|_p^{-2}|Du|^2$$

$$\leqslant C|Du|^2 \in L^1(\mathbb{R}^n).$$

So, by the Dominated Convergence Theorem we have that

$$\lim_{t\to 0} J^6(t) = \int_{\mathbb{R}^n}\varphi^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\, dx. \tag{4.26}$$

Similarly to (4.11), we have that

$$\lim_{t\to 0} J^7(t) = \int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\, dx\int_{\mathbb{R}^n}u^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\, dx. \tag{4.27}$$

Combining (4.24)-(4.27) we have that

$$\begin{aligned}0 = {} & 2\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta\varphi^i\, dx + \int_{\mathbb{R}^n}\varphi^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\, dx\\ & - \int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\, dx\int_{\mathbb{R}^n}u^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\, dx\\ & - 2\int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\, dx\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\, dx\\ & + 2\int_{\mathbb{R}^n}h(x)u^i\varphi^i\, dx - 2\int_{\mathbb{R}^n}h(x)|u|^2\, dx\int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\, dx\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\mathbb{R}^n}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta\varphi^i\, dx + \tfrac{1}{2}\int_{\mathbb{R}^n}\varphi^j D_{u^j}a_{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^i\, dx\\ &\qquad + \int_{\mathbb{R}^n}h(x)u^i\varphi^i\, dx = \lambda\int_{\mathbb{R}^n}|u|^{p-2}u^i\varphi^i\, dx\end{aligned} \tag{4.28}$$

for every $\varphi \in L_\infty \cap E$ where

$$\lambda = \int_{\mathbb{R}^n} [a_{\alpha\beta}(x, u)D_\alpha u^i D_\beta u^i + \tfrac{1}{2} u^j D_{u^j} a_{\alpha\beta}(x, u) D_\alpha u^i D_\beta u^i + h(x)|u|^2]\, dx$$

i.e. u is a weak solution of (1.1) with $\|u\|_\infty < \infty$ and Theorem 2.3 is completely proved. □

References

[1] Ma Li. On the positive solutions of quasilinear elliptic eigenvalue problem with limiting exponent (preprint).

[2] Shen Yiao-tian Eigenvalue problems of quasilinear elliptic systems (preprint).

[3] Giaquinta, M. *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* Princeton University Press, 1983.

[4] P. L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case, Part 1 *Ann. I.H.P. Anal. non linéaire*, **1** (1984), 109-145.

[5] P. L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case, Part. 2 *Ann. I.H.P. Anal. non linéaire*, **1** (1984), 223-283.

[6] Yosida, K. *Funcional Analysis.* Springer-Verlag, 1978.

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An Atomic Decomposition of the Predual of BMO(ρ)

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Abstract

We study the Orlicz type spaces H_ω, defined as a generalization of the Hardy spaces H^p for $p \leqslant 1$. We obtain an atomic decomposition of H_ω, which is used to provide another proof of the known fact that BMO(ρ) is the dual space of H_ω (see S. Janson, 1980, [J]).

Introduction

The purpose of this work is to study the spaces H_ω, obtained as a generalization of the Hardy spaces H^p taking $\omega(t) = t^p$. For more general ω, the space H_ω was considered before by Janson in [J]. There, the author proves that BMO(ρ) is the dual space of H_ω, with ρ and ω related by $t^n \rho(t) \omega^{-1}(1/t^n) = 1$. The main result of this paper is an atomic decomposition of H_ω.

The atomic decomposition of H^p spaces starts with the work of Hertz in the martingale setting ([H]). Since then many authors have been studying the problem in different situations: R. R. Coifman [CO], Latter [L], Latter and Uchiyama [LU], Calderón [C], Macías and Segovia [MS], etc. Since most of the work in the atomic decomposition relies on Calderón-Zygmund type lemmas, we accomplish the problem in the setting of spaces of homogeneous type for which the Calderón-Zygmund method has been worked out by Macías [M] and Macías and Segovia [MS].

As corollary of the atomic decomposition we obtain another proof of the fact that BMO(ρ) is the dual space of H_ω. As a by-product we get the equivalence of BMO(ρ) and BMO(ρ, q) ($1 < q < \infty$) without using John-Niremberg type Lemmas (see [A] for a proof of John-Niremberg lemma in this context). We would like to point out that the atomic space is given by an Orlicz type norm which in the case of $\rho(t) = t^{1/p-1}$ provides the usual atomic H^p spaces.

In the first section we give the notation and definitions that we shall use in the sequel. We introduce the atomic spaces $H^{\rho,q}$, $1 < q \leqslant \infty$, the maximal spaces H_ω and the spaces BMO(ρ, q).

In Section 2 we state the main results: atomic decomposition, Theorem 2.1, and duality, Theorem 2.2.

In Section 3 we prove the basic properties of the growth functions ω and ρ, in particular Lemma 3.1 provides the tool for further work with this type of functions.

In Section 4 we prove Theorem 2.1. The key for this proof is the Calderón-Zygmund type Lemma 4.9. Other important tools in the proof of Theorem 2.1 are interesting by themselves: the maximal space H_ϕ is continuouly included in the Orlicz space L_ϕ for ϕ a Young function (Theorem 4.15). Theorem 2.1 also provides an important consequence namely, the spaces $H^{\rho,\infty}$ and $H^{\rho,q}$ ($1 < q < \infty$) are equivalent.

Finally, Theorem 2.2 is proved in Section 5.

1. Notation and Definitions

Let X be a set. A function $d: X \times X \to \mathbb{R}^+ \cup \{0\}$ shall be called a quasi-distance on X if there exists a finite constant K such that

(1.1) $$d(x,y) = 0 \quad \text{if and only if} \quad x = y,$$

(1.2) $$d(x,y) = d(y,x),$$

and

(1.3) $$d(x,y) \leqslant K[d(x,z) + d(z,y)]$$

for every x, y, and z in X.

In a set X, endowed with a quasi-distance $d(x,y)$, the balls

$$B(x,r) = \{y : d(x,y) < r\}, \qquad r > 0,$$

form a basis of neighbourhoods of x for the topology induced by the uniform structure on X.

We shall say that a set X, with a quasi-distance $d(x,y)$ and a non-negative measure μ defined on a σ-algebra of subsets of X which contains the balls $B(x,r)$, is a normal space of homogeneous type if there exist four positive and finite constans A_1, A_2, K_1 and $K_2 \leqslant 1 \leqslant K_1$, such that

$$A_1 r \leqslant \mu(B(x,r)) \quad \text{if} \quad r \leqslant K_1 \mu(X) \tag{1.4}$$

$$B(x,r) = X \quad \text{if} \quad r > K_1 \mu(X) \tag{1.5}$$

$$A_2 r \geqslant \mu(B(x,r)) \quad \text{if} \quad r \geqslant K_2 \mu(\{x\}) \tag{1.6}$$

$$B(x,r) = \{x\} \quad \text{if} \quad r < K_2 \mu(\{x\}). \tag{1.7}$$

We note that, under these conditions, there exist two finite constants, $a > 1$ and A, such that

$$0 < \mu(B(x,ar)) \leqslant A\mu(B(x,r)) \tag{1.8}$$

holds for every x in X and $r > 0$.

We shall say that a normal space of homogeneous type (X,d,μ), is of order α, $0 < \alpha < \infty$, if there exists a finite constant K_3 satisfying

$$|d(x,z) - d(y,z)| \leqslant K_3 r^{1-\alpha} d(x,y)^\alpha, \tag{1.9}$$

for every x, y and z in X, whenever $d(x,z) < r$ and $d(y,z) < r$ (see [MS]).

In this paper $X = (X,d,\mu)$ shall mean a normal space of homogeneous type of order $\alpha, 0 < \alpha \leqslant 1$, and we shall often refer to the constants that appear in (1.3) to (1.9) as the constants of the space.

Let ρ be a positive function defined on $\mathbb{R}^+$. We shall say that ρ is of upper type m (respectively, lower type m), if there exists a positive constants c such that

$$\rho(st) \leqslant ct^m \rho(s) \tag{1.10}$$

for every $t \geqslant 1$ (respectively, $0 < t \leqslant 1$). A non-decreasing function ρ of finite upper type such that $\lim_{t \to 0^+} \rho(t) = 0$ is called a growth function.

We shall say that a positive function ρ is quasi-increasing if there exists a constant c such that

$$\rho(s) \leqslant c\rho(t) \quad \text{for} \quad s \leqslant t.$$

Let ρ be a quasi-increasing function. We consider the function $\rho^{-1}(s)$ defined by

$$\rho^{-1}(s) = \sup\{t : \rho(t) \leqslant s\}$$

for those values of s at which the supremum is a real number. Clearly $\rho^{-1}(\rho(s)) \geqslant s$. It is easy to see that if the function $\rho(t)$ is continuous and strictly increasing then the function $\rho^{-1}(t)$ is the ordinary inverse function of $\rho(t)$.

We shall understand that two positive functions are equivalents if their quotient is bounded above and below by two positive constants.

Let $\psi(x)$ be an integrable function on bounded subsets of X. For any ball B, we denote

$$m_B(\psi) = \mu(B)^{-1} \int_B \psi(x)\, d\mu(x)$$

and, as is usual, the Hardy-Littlewood maximal function by

$$M(\psi)(x) = \sup m_B(|\psi|),$$

where the supremum is taken over all balls B containing x.

Definition 1.11. *Let $1 \leqslant q < \infty$ and ρ a growth function plus a non-negative constant or $\rho \equiv 1$. A function $f(x)$, integrable on bounded subsets, belongs to* BMO(ρ, q) *if there exists a constant c such that the inequality*

$$\left[\mu(B)^{-1} \int_B |f(x) - m_B(f)|^q \, d\mu(x)\right]^{1/q} \leqslant c\rho(\mu(B))$$

holds for every ball B. The least constant c satisfying the inequality above shall be denoted by $\|f\|_{\mathrm{BMO}(\rho, q)}$. When ρ is the constant function $\rho \equiv 1$ and $1 \leqslant q < \infty$, the space BMO$(1, q)$ *coincides with the space of functions of bounded mean oscillation* BMO. *The space* BMO$(\rho, 1)$ *shall be denoted by* BMO(ρ).

Let ρ be a growth function. We shall say that a function $\psi(x)$ belongs to Lip(ρ), *if there exists a finite constant c such that*

$$|\psi(x) - \psi(y)| \leqslant c\rho(d(x, y)) \tag{1.12}$$

for every x and y in X. The least constant c satisfying this condition shall be denoted by $\|\psi\|_{\mathrm{Lip}(\rho)}$. When $\rho(t)$ is the function t^β, $0 < \beta < \infty$, we shall say that $\psi(t)$ is in Lip(β) *and, in this case, $\|\psi\|_\beta$ indicates its norm.*

In [MS], Macías and Segovia introduce the space of distributions $(E^\alpha)'$ as the dual of the space E^α consisting of all functions with bounded support, belonging to Lipschitz β, $0 < \beta < \alpha$.

For x in X and $0 < \gamma < \alpha$ we consider the class $T_\gamma(x)$, of functions ψ belonging to E^α satisfying the following condition: there exists r such that $r \geqslant K_2\mu(\{x\})$, supp $\psi \subset B(x, r)$ and

$$r\|\psi\|_\infty \leqslant 1 \quad \text{and} \quad r^{1+\gamma}\|\psi\|_\gamma \leqslant 1. \tag{1.13}$$

Given γ, $0 < \gamma < \alpha$, we define the γ-maximal function $f_\gamma^*(x)$ of a distribution f on E^α by

$$f_\gamma^*(x) = \sup\{|f(\psi)| : \psi \in T_\gamma(x)\}. \tag{1.14}$$

Definition 1.15. *Let ρ be a growth function plus a non-negative constant or $\rho \equiv 1$. A (ρ, q) atom, $1 < q \leqslant \infty$, is a function $a(x)$ on X satisfying:*

(1.16) $$\int_X a(x)\, d\mu(x) = 0,$$

(1.17) *the support of $a(x)$ is contained in a ball B and*

(1.18) $$\left[\mu(B)^{-1} \int_B |a(x)|^q\, d\mu(x)\right]^{1/q} \leqslant [\mu(B)\rho(\mu(B))]^{-1}$$

if $q < \infty$ or

$$\|a\|_\infty \leqslant [\mu(B)\rho(\mu(B))]^{-1}, \quad \text{if} \quad q = \infty.$$

If $\mu(X)$ is finite, we may assume that $\mu(X) = 1$. In this case, we also suppose that $\rho(1) = 1$ and we consider the characteristic function of X as a (ρ, q) atom. Clearly, when $\rho(t) = t^{1/p-1}$, $p \leqslant 1$, a (ρ, q) atom is a (p, q) atom in the sense of [M].

Definition 1.19. *Let $0 < \gamma < \alpha$. Assume that ω is a growth function of lower type l such that $l(1 + \gamma) > 1$. We define*

$$H_\omega = H_\omega(X) = \left\{f \in (E^\alpha)' \colon \int \omega[f_\gamma^*(x)]\, d\mu(x) < \infty\right\}$$

and we denote

$$\|f\|_{H_\omega} = \inf\left\{\lambda > 0 \colon \int \omega\left(\frac{f_\gamma^*(x)}{\lambda^{1/l}}\right) d\mu(x) \leqslant 1\right\}.$$

It is easy to see that H_ω is a complete metrizable topological vector space with respect to the quasi-distance induced by $\|\ \ \|_{H_\omega}$. Moreover, H_ω is continuously included in $(E^\alpha)'$.

Definition 1.20. *Let ω be a growth function of positive lower type l. If $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$, we define $H^{\rho,q}(X) = H^{\rho,q}$, $1 < q \leqslant \infty$, as the linear space of all distributions f on E^α which can be represented by*

(1.21) $$f(\psi) = \sum_i b_i(\psi),$$

for every ψ in E^α, where $\{b_i\}_i$ is a sequence of multiples of (ρ, q) atoms such that if supp$(b_i) \subset B_i$ *then*

(1.22) $$\sum_i \mu(B_i)\omega(\|b_i\|_q \mu(B_i)^{-1/q}) < \infty.$$

We introduce a quasi-distance in $H^{\rho,q}$. Given a sequence of multiples of (ρ, q)-atoms, $\{b_i\}_i$, we denote

$$\Lambda_q(\{b_i\}) = \inf\left\{\lambda\colon \sum_i \mu(B_i)\omega\left(\frac{\|b_i\|_q \mu(B_i)^{-1/q}}{\lambda^{1/l}}\right) \leqslant 1\right\} \tag{1.23}$$

and we define

$$\|f\|_{H^{\rho,q}} = \inf \Lambda_q(\{b_i\}), \tag{1.24}$$

where the infimum is taken over all possible representations of f of the form (1.21).

2. Statement of the results

The main theorems in this work are the following.

Theorem 2.1. *Let ω be a function of lower type l such that $l(1+\gamma) > 1$, $0 < \gamma < \alpha$. Assume that $\omega(s)/s$ is non-increasing. Let $\rho(t)$ be the function defined by $t\rho(t) = 1/\omega^{-1}(1/t)$. Then $H_\omega \equiv H^{\rho,q}$ for every $1 < q \leqslant \infty$.*

Theorem 2.2. *Let $\rho(t)$ and $\omega(t)$ as in Theorem* 2.1. *Then, $(H_\omega)' \equiv$* BMO (ρ).

We observe that from Proposition 3.10, it turns out that the function $\rho(t)$ defined in Theorem 2.1 satisfies the conditions of Definitions 1.11 and 1.15.

3. Basic Lemmas on Growth Functions

Proposition 3.1. *Let ω be a function of positive lower type l such that $\omega(s)/s$ is non increasing. Then the following properties hold*

$$l \leqslant 1, \tag{3.2}$$

$$\frac{\omega(s)}{s^p} \tag{3.3}$$

is quasi-increasing for $p \leqslant l$,

$$\tilde{\omega}(t) = \int_0^t \frac{\omega(s)}{s}\, ds \tag{3.4}$$

is a continuous function of positive lower type $l \leqslant 1$ equivalent to ω, (3.5) $$\lim_{t\to 0^+} \tilde{\omega}(t) = 0,$$

(3.6) $\tilde{\omega}$ *is strictly increasing.*
(3.7) $\tilde{\omega}$ *is subaditive,*

(3.8) $$\frac{\tilde{\omega}(s)}{s}$$

is non-increasing and

(3.9) $$\frac{\tilde{\omega}(s)}{s^p}$$

is quasi-increasing for $p \leqslant l$.

PROOF. Since $\omega(s)/s$ is non-increasing and $\omega(s)$ is of lower type l, we have

$$\omega(1) \leqslant \frac{\omega(s)}{s} \leqslant cs^{l-1}\omega(1)$$

for $s < 1$, and (3.2) holds. To prove (3.3) take $s \leqslant t$ and $p \leqslant l$. From the lower type property of ω, we obtain

$$\frac{\omega(s)}{s^p} \leqslant c\left(\frac{s}{t}\right)^l \frac{\omega(t)}{s^p} \leqslant c\frac{\omega(t)}{t^p}.$$

Let us prove (3.4). Clearly $\omega(t) \leqslant \tilde{\omega}(t)$. On the other hand using the lower type l of ω, we get

$$\tilde{\omega}(t) = \int_0^1 \frac{\omega(st)}{s}\,ds \leqslant c\omega(t)\int_0^1 s^{l-1}\,ds = cl^{-1}\omega(t).$$

Properties (3.5) and (3.6) follow immediately from the definition fo $\tilde{\omega}$. In order to prove (3.7), let us observe that

$$\begin{aligned}\tilde{\omega}(a+b) &= \int_0^a \frac{\omega(s)}{s}\,ds + \int_a^{a+b} \frac{\omega(s)}{s}\,ds \\ &\leqslant \tilde{\omega}(a) + \int_0^b \frac{\omega(s)}{s}\,ds \\ &= \tilde{\omega}(a) + \tilde{\omega}(b),\end{aligned}$$

where the inequality follows from the fact that $\omega(s)/s$ is non-increasing. Let $t_1 \geqslant t_2$, then

$$\frac{\tilde{\omega}(t_1)}{t_1} = \frac{1}{t_1}\int_0^{t_1} \frac{\omega(s)}{s}\,ds = \frac{1}{t_2}\int_0^{t_2} \frac{\omega(st_1/t_2)}{st_1/t_2}\,ds \leqslant \frac{1}{t_2}\int_0^{t_2} \frac{\omega(s)}{s}\,ds = \frac{\tilde{\omega}(t_2)}{t_2},$$

which proves (3.4). Finally, (3.9) can be proved as (3.3) using (3.8). □

Let us observe that the results stated in Section 2 are invariant under change of equivalent growth functions. So that, there is no loss of generality in assuming that ω satisfies all properties of $\tilde{\omega}$ in Proposition 3.1, that is, we shall suppose that ω verifies (3.4) to (3.9).

Proposition 3.10. *Given $\omega(t)$, let $\rho(t)$ be the function defined on $\mathbb{R}^+$ by*

$$\rho(t) = \frac{t^{-1}}{\omega^{-1}(t^{-1})}.$$

Then $\rho(t)$ is a positive non decreasing function of upper type $l^{-1} - 1 \geqslant 0$.

PROOF. In order to prove that ρ is non-decreasing, let us take $t_1 \geqslant t_2 > 0$. Since $\omega^{-1}(t_1^{-1}) \leqslant \omega^{-1}(t_2^{-1})$ and $\omega(s)/s$ is non-increasing, we have

$$\rho(t_1) = \frac{\omega(\omega^{-1}(t_1^{-1}))}{\omega^{-1}(t_1^{-1})} \geqslant \frac{\omega(\omega^{-1}(t_2^{-1}))}{\omega^{-1}(t_2^{-1})} = \rho(t_2).$$

On the other hand, it is easy to check that ω is of lower type l if and only if ω^{-1} is of upper type l^{-1}. So that, ω^{-1} satisfies

$$\omega^{-1}(s\tau) \geqslant c\tau^{1/l}\omega^{-1}(s)$$

for every $s \in \mathbb{R}^+$ and $\tau \leqslant 1$. Therefore, for $s \in \mathbb{R}^+$ and $t \geqslant 1$, we obtain

$$\rho(st) = \frac{s^{-1}t^{-1}}{\omega^{-1}(s^{-1}t^{-1})} \leqslant \frac{s^{-1}t^{-1}}{ct^{-1/l}\omega^{-1}(s^{-1})} = ct^{1/l-1}\rho(s),$$

which proves that $\rho(t)$ is of upper type $l^{-1} - 1$. □

4. Atomic Decomposition

In [MS], Macías and Segovia obtain the atomic decomposition of H_ω, with $\omega(t) = t^p$, on spaces of homogeneous type. In this section we shall adapt to our situation their scheme of proof. Therefore we shall prove only the technical lemmas which require suitable modifications, the results that we state without proof can be found in that paper.

In this and the following section we shall assume that ω satisfies (3.4) to (3.9) with $l(1 + \gamma) > 1$ for some γ, $0 < \gamma < \alpha$. The first two lemmas deal with geometric properties of the space.

Lemma 4.1. *Let $r > 0$, $x_0 \in X$ and $p > 1$. Then*

$$\int_{B(x_0, r)^c} [r/d(x, x_0)]^p \, d\mu(x) \leqslant c\mu(B(x_0, r)),$$

where c depends only on p and the constants of the space.

Lemma 4.2. *Let $0 < \beta$, $1 < q(1 + \beta)$ and M a positive integer. There exists a finite constant $c_{q,\beta,M}$ such that given any sequence of points $\{x_n\}$, and any sequence of positive numbers $\{r_n\}$, satisfying the condition that no point in X belongs to more than M balls $B(x_n, r_n)$, then*

$$\int \left\{ \sum_n \left[\frac{\mu(B(x_n, r_n))}{\mu(B(x_n, r_n)) + d(x, x_n)} \right]^{1+\beta} \right\}^q d\mu(x) \leqslant c_{q,\beta,M}\,\mu\Big(\bigcup_n B(x_n, r_n)\Big).$$

The following lemma and its corollary show that (ρ, q) atoms are in H_ω. Lemma 4.7 is a technical result to be used in Theorem 2.2.

Lemma 4.3. *Let $b(x)$ be a function in $L^q(X, d\mu)$, $1 < q \leqslant \infty$, with support contained in $B = B(x_0, R)$ and $\int b(x)\, d\mu(x) = 0$. Then, there exists a constant c, independent of $b(x)$, such that*

$$\int \omega[b_\gamma^*(x)] \, d\mu(x) \leqslant c\mu(B)\omega(\|b\|_q \mu(B)^{-1/q}).$$

PROOF. Let $\psi(x) \in T_\gamma(y)$ such that $\operatorname{supp}(\psi) \subset B(y, r)$. Then, from (1.6) and (1.13) it follows that

$$\left| \int b(x)\psi(x)\, d\mu(x) \right| \leqslant A_2 M(b)(y).$$

Therefore, $b_\gamma^*(y) \leqslant A_2 M(b)(y)$. Let $m = \|b\|_q \mu(B)^{-1/q}$. Thus, using (3.6) and (3.8), we have

$$\begin{aligned}\omega[b_\gamma^*(y)] &\leqslant c\omega[M(b)(y)] \leqslant c\omega[M(b)(y) + m] \\ &\leqslant c\left(\frac{M(b)(y)}{m} + 1\right)\omega(m).\end{aligned}$$

Integrating on $B(x_0, 2KR)$, by (1.8) and the L^q-boundedness of M, we get

$$\begin{aligned}(4.4) \qquad \int_{B(x_0, 2KR)} \omega[b_\gamma^*(y)]\, d\mu(y) &\leqslant c\omega(m)\left[\frac{1}{m}\|M(b)\|_q \mu(B)^{1-1/q} + \mu(B)\right] \\ &\leqslant c\omega(m)\mu(B).\end{aligned}$$

On the other hand, let $y \notin B(x_0, 2KR)$ and $\psi \in T_\gamma(y)$ as before. We can assume that $B(x_0, R) \cap B(y, r) = \varnothing$. Consequently, $R < r$ and $d(y, x_0) < 2Kr$. Then,

$$\left|\int b(x)\psi(x)\,d\mu(x)\right| = \left|\int_B b(x)[\psi(x) - \psi(x_0)]\,d\mu(x)\right|$$
$$\leqslant \|b\|_q\Big(\int_B |\psi(x) - \psi(x_0)|^{q'}\,d\mu(x)\Big)^{1/q'}$$
$$\leqslant \|b\|_q \|\psi\|_\gamma R^\gamma \mu(B)^{1/q'}$$
$$\leqslant \|b\|_q \left[\frac{2KR}{d(y, x_0)}\right]^{1+\gamma} R^{-1}\mu(B)^{1/q'}$$
$$= m\left[\frac{2KR}{d(y, x_0)}\right]^{1+\gamma} R^{-1}\mu(B).$$

We can suppose that $R \geqslant K_2\mu(\{x_0\})$, since otherwise $b \equiv 0$. Thus, by (1.6), we have

$$b_\gamma^*(y) \leqslant A_2 m\left[\frac{2KR}{d(y, x_0)}\right]^{1+\gamma}.$$

Applying ω to both sides, since ω is of lower type l and $\omega(s)/s$ is decreasing, we obtain

$$\omega[b_\gamma^*(y)] \leqslant c\omega(m)\left[\frac{2KR}{d(y, x_0)}\right]^{(1+\gamma)l} \tag{4.5}$$

for $y \notin B(x_0, 2KR)$. Integrating (4.5) on the complementary set of $B(x_0, 2KR)$, since $l(1 + \gamma) > 1$, by Lemma 4.1, we get

$$\int_{B(x_0, 2KR)^c} \omega[b_\gamma^*(y)]\,d\mu\,(y) \leqslant c\mu(B)\omega(m),$$

which together with (4.4) completes the proof of the lemma. □

Corollary 4.6. *Let $\rho(t)$ be the function defined in* (3.10). *If $a(x)$ is a (ρ, q) atom, $1 < q \leqslant \infty$, then there exists a constant c, independent of $a(x)$, such that*

$$\|a\|_{H_\omega} \leqslant c.$$

Lemma 4.7. *Set $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$. Let $\{b_i\}_i$ be a sequence of multiples of (ρ, q) atoms, $1 < q \leqslant \infty$, such that $\Lambda_q(\{b_i\}) < \infty$ and $\alpha_i = \|b_i\|_q\mu(B_i)^{-1/q}/\omega^{-1}(\mu(B_i)^{-1})$, where $B_i \supset \operatorname{supp}(b_i)$. Then there exists a contant c independent of the sequence $\{b_i\}$ such that $\sum_i \alpha_i \leqslant c(\Lambda_q(\{b_i\}) + 1)^{1/l^2}$.*

PROOF. By definition of α_i, we get

$$\mu(B_i)^{-1} = \omega(\|b_i\|_q\mu(B_i)^{-1/q}\alpha_i^{-1}). \tag{4.8}$$

We first note that $\{\alpha_i^l\}$ is bounded. In fact, for those i such that $\alpha_i > 1$, since ω is of lower type l with constant c_0, by (4.8), we have

$$\begin{aligned}\alpha_i^l &= \mu(B_i)\alpha_i^l\omega(\|b_i\|_q\mu(B_i)^{-1/q}\alpha_i^{-1})\\ &\leqslant c_0\mu(B_i)\omega(\|b_i\|_q\mu(B_i)^{-1/q})\\ &\leqslant c_0\sum_j\mu(B_j)\omega\left(\|b_j\|_q\mu(B_j)^{-1/q}\frac{(\Lambda_q(\{b_i\})+1)^{1/l}}{(\Lambda_q(\{b_i\})+1)^{1/l}}\right)\\ &\leqslant c_0(\Lambda_q(\{b_i\})+1)^{1/l} = a.\end{aligned}$$

Applying again (4.8) we obtain

$$\sum_i\alpha_i = a^{1/l}\sum_i\mu(B_i)\frac{a_i}{a^{1/l}}\omega\left(\frac{\|b_i\|_q\mu(B_i)^{-1/q}}{\alpha_i}\right).$$

Using that $\omega(s)/s$ is non-increasing, this is bounded by

$$a^{1/l}\sum_i\mu(B_i)\omega\left(\frac{\|b_i\|_q\mu(B_i)^{-1/q}}{a^{1/l}}\right)\leqslant a^{1/l}$$

since ω is increasing and $a^l \geqslant \Lambda_q(\{b_j\})$. □

One of the main tools in the proof of atomic decomposition of Hardy spaces is provided by a Calderón-Zigmund type lemma which allows us to split a given function into «good» and «bad» parts. In order to do this, let us take f belonging to H_ω. Consider $\omega(t) > \int\omega[f_\gamma^*(x)]\,d\mu(x)/\mu(X)$ and $\Omega = \{x: f_\gamma^*(x) > t\}$. By a Whitney's type lemma applied to the open set Ω, following [MS], we get a sequence of balls $B_n = B(x_n, r_n)$ and a partition of the unity $\{\phi_n\}$ associated to it. For each n, the expression

$$S_n(\psi)(x) = \phi_n(x)\left[\int\phi_n(z)\,d\mu(z)\right]^{-1}\int[\psi(x)-\psi(x)]\phi_n(z)\,d\mu(z)$$

defines a continuous operator from E^α into itself.

Lemma 4.9. Calderón-Zygmund type. *Let f in H_ω and $b_n(\psi) = f(S_n(\psi))$ for $\psi \subset E^\alpha$. Then*

$$(4.10)\quad (b_n)_\gamma^*(x) \leqslant ct\left[\frac{r_n}{d(x,x_n)+r_n}\right]^{1+\gamma}\chi_{B(x_n,4Kr_n)^c}(x) + cf_\gamma^*(x)\chi_{B(x_n,4Kr_n)}(x)$$

and

$$(4.11)\qquad \int\omega[(b_n)_\gamma^*(x)]\,d\mu(x) \leqslant c\int_{B(x_n,4Kr_n)}\omega[f_\gamma^*(x)]\,d\mu(x).$$

Moreover, the series $\sum_n b_n$ *converges in* $(E^\alpha)'$ *to a distribution b satisfying*

$$(4.12)\qquad b_\gamma^*(x) \leqslant ct \sum_n \left[\frac{r_n}{d(x, x_n) + r_n} \right]^{1+\gamma} + cf_\gamma^*(x)\chi_\Omega(x)$$

and

$$(4.13)\qquad \int \omega[b_\gamma^*(x)]\, d\mu(x) \leqslant c \int_\Omega \omega[f_\gamma^*(x)]\, d\mu(x).$$

The distribution $g = f - b$ *satisfies*

$$(4.14)\qquad g_\gamma^*(x) \leqslant ct \sum_n \left[\frac{r_n}{d(x, x_n) + r_n} \right]^{1+\gamma} + cf_\gamma^*(x)\chi_{\Omega^c}(x).$$

PROOF. We shall only prove (4.11) and (4.13). To obtain (4.11), we first apply ω to inequality (4.10) and then we integrate on X. Thus, since ω is of lower type l, $l(1 + \gamma) > 1$, by Lemma 4.1 we get

$$\begin{aligned} \int_X \omega[(b_n)_\gamma^*(x)]\, d\mu(x) &\leqslant c\omega(t) \int_{B(x_n, 4Kr_n)^c} \left[\frac{r_n}{d(x, x_n) + r_n} \right]^{(1+\gamma)l} d\mu(x) \\ &\quad + c \int_{B(x_n, 4Kr_n)} \omega[f_\gamma^*(x)]\, d\mu(x) \\ &\leqslant c \int_{B(x_n, 4Kr_n)} \omega[f_\gamma^*(x)]\, d\mu(x). \end{aligned}$$

Applying ω to (4.12), using the sub-additivity of ω and proceeding as above, (4.13) follows. □

The next result which shall be often used in the sequel, is also an statement of the inclusion $H_\phi \subset L_\phi$, where ϕ is a Young function, *i.e.* a convex, positive and increasing function on $\mathbb{R}^+$ such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. Its proof is similar to that of Theorem 3.25 in [MS].

Theorem 4.15. *Let f be a distribution on* E^α *and assume that* $f_\gamma^*(x)$ *belongs to* $L_\phi(X, d\mu)$. *Then, there exists a function* $f(x)$ *such that* $|f(x)| \leqslant cf_\gamma^*(x)$ *and*

$$f(\psi) = \int f(x)\psi(x)\, d\mu(x),$$

for every ψ *in* E^α.

Applying the preceding theorem with $\phi(t) = t^q$, $1 \leqslant q < \infty$, we obtain the density of L^q in H_ω.

Theorem 4.16. *Let f be a distribution on E^α belonging to H_ω. Then for $\epsilon > 0$ and $1 \leqslant q < \infty$ given, there exists a function $h(x)$ in $L^q(X, d\mu)$ such that*

$$\int \omega[(f-h)^*_\gamma(x)]\, d\mu(x) < \epsilon.$$

PROOF. There exists $\omega(t) > \mu(X)^{-1} \int \omega[f^*_\gamma(x)]\, d\mu(x)$ such that the inequality

$$\int_\Omega \omega[f^*_\gamma(x)]\, d\mu(x) < \epsilon \tag{4.17}$$

holds for $\Omega = \{x: f^*_\gamma(x) > t\}$. For this value of t, by lemma (4.9), we get the decomposition

$$f = g + b.$$

Moreover, from (4.14), (4.2) and the fact that $\omega(s)/s$ is non-increasing, we have

$$\begin{aligned}\int g^*_\gamma(x)^q\, d\mu(x) &\leqslant ct^q\mu(\Omega) + c\int_{\Omega^c} f^*_\gamma(x)^q\, d\mu(x)\\ &\leqslant c\frac{t^q}{\omega(t)}\int \omega[f^*_\gamma(x)]\, d\mu(x) + ct^{q-1}\int_{\Omega^c} f^*_\gamma(x)\, d\mu(x)\\ &\leqslant c\frac{t^q}{\omega(t)}\int \omega[f^*_\gamma(x)]\, d\mu(x).\end{aligned}$$

Consequently, $g^*_\gamma(x)$ belongs to $L^q(X, d\mu)$. Hence, Theorem 4.15 with $\phi(t) = t^q$ implies that there exists a function $h(x)$ such that $|h(x)| < cg^*_\gamma(x)$ and the distribution on E^α induced by $h(x)$ coincides with g. Therefore $h(x) \in L^q(X, d\mu)$. On the other hand, from (4.13) and (4.17) it follows that

$$\begin{aligned}\int \omega[(f-h)^*_\gamma(x)]\, d\mu(x) &= \int \omega[(f-g)^*_\gamma(x)]\, d\mu(x) = \int \omega[b^*_\gamma(x)]\, d\mu(x)\\ &\leqslant c\int_\Omega \omega[f^*_\gamma(x)]\, d\mu(x) \leqslant c\epsilon. \quad \square\end{aligned}$$

The analogue to Lemma 4.9 in the case that f is a function is contained in the following Lemma.

Lemma 4.18. *Let $f(x) \in L^q(X, d\mu)$, $1 \leqslant q < \infty$. Assume that the distribution f on E^α induced by $f(x)$ belongs to H_ω and $|f(x)| \leqslant cf^*_\gamma(x)$ almost everywhere on X. With the same notation used in Lemma 4.9, let*

$$m_n = \left(\int \phi_n(z)\, d\mu(z)\right)^{-1} \int f(y)\phi_n(y)\, d\mu(y).$$

We have

$$|m_n| \leqslant ct. \tag{4.19}$$

(4.20) *The distribution on E^α induced by the function*

$$b_n(x) = (f(x) - m_n)\phi_n(x),$$

coincides with b_n.

(4.21) *The series $\sum b_n(x)$ converges for $x \in X$ and in $L^q(X, d\mu)$. Its sum induces a distribution on E^α which coincides with b and shall be denoted by $b(x)$.*

(4.22) *The function $g(x) = f(x) - b(x)$ satisfies*

$$g(x) = f(x)\chi_{\Omega^c}(x) + \sum m_n \phi_n(x)$$

and

$$|g(x)| \leqslant ct.$$

Moreover, $g(x)$ induces a distribution on E^α which coincides with g.

We shall need the following lemma, which is a consequence of Lemma 4.7.

Lemma 4.23. *Let $\rho(t)$ and $\{b_i\}_i$ as in Lemma 4.7. Then the series $\sum_i b_i$ converges in $(E^\alpha)'$.*

PROOF. Let us first assume that $l < 1$. Let D be a bounded subset of E^α, therefore there exists a ball $B(x_0, R)$ and a constant c such that $R > K_2\mu(\{x_0\})$ and for every $\psi \in D$ we have $\operatorname{supp}\psi \subset B(x_0, R)$, $\|\psi\|_\infty \leqslant c$ and $\|\psi\|_{l^{-1}-1} \leqslant c$. Observe that $\operatorname{Lip}(l^{-1} - 1) \cap D \subset \operatorname{Lip}(\rho)$ and

$$\|\psi\|_{\operatorname{Lip}(\rho)} \leqslant c(D)\|\psi\|_{l^{-1}-1} \leqslant \tilde{c}(D), \tag{4.24}$$

because $\rho(t)$ is of upper type $l^{-1} - 1$. From (4.24) and the definition of $\rho(t)$ we get

$$\begin{aligned}
\sup_{\psi\in D}\left|\sum_m^n b_i, (\psi)\right| &\leqslant \sup_{\psi \in D}\sum_m^n \int_{B_i} |b_i(x)|\,|\psi(x) - \psi(x_i)|\,d\mu(x) \\
&\leqslant c \sup_{\psi\in D}\|\psi\|_{\operatorname{Lip}(\rho)} \sum_m^n \|b_i\|_q \rho(\mu(B_i))\mu(B_i)^{1/q'} \\
&\leqslant c\frac{\sum_m^n \|b_i\|_q \mu(B_i)^{-1/q}}{\omega^{-1}(\mu(B_i)^{-1})} \\
&= c\sum_m^n \alpha_i .
\end{aligned}$$

Applying Lemma 4.7 we obtain the desired result. If $l = 1$, the series $\sum_i b_i$ actually converges in L^1, since $\rho \simeq$ constant. □

In order to prove Theorem 2.1 we shall need the following lemma, which gives a (ρ, ∞) decomposition for a suitable function.

Lemma 4.25. *Let $\rho(t)$ be the function defined by $1/\rho(t) = t\omega^{-1}(1/t)$. Let $h(x)$ be a function in $(L^2 \cap L^\infty)(X, d\mu)$. Suppose that for some p, such that $(1+\gamma)^{-1} < p < l$, $0 < \gamma < \alpha$, the γ-maximal function $h_\gamma^*(x)$ belongs to $L^p(X, d\mu)$. Then there exists a sequence $\{b_n(x)\}$ of multiples of (ρ, ∞) atoms such that*

$$h = \sum_n b_n \quad \text{in} \quad E^{\alpha'}$$

and

$$\sum_n \mu(B_n)\omega(\|b_n\|_\infty) \leqslant c\omega(\|h\|_\infty)\|h\|_\infty^{-p} \int h_\gamma^*(x)^p \, d\mu(x), \tag{4.26}$$

where c is a constant independent of $h(x)$.

PROOF. Let ϵ be any number, $0 < \epsilon < 1$. We shall construct, by recurrency a sequence of functions, $\{H_i(x)\}$ in the following way: $H_0(x) = H(x)$. Suppose that $H_{i-1}(x)$ is defined. Then, if

$$\omega(\|h\|_\infty \epsilon^i) \leqslant \int \omega[(H_{i-1})_\gamma^*(x)] \, d\mu(x)/\mu(X),$$

we stop the construction obtaining a finite sequence. If on the contrary

$$\omega(\|h\|_\infty \epsilon^i) > \int \omega[(H_{i-1})_\gamma^*(x)] \, d\mu(x)/\mu(X),$$

we choose $H_i(x)$ to be the function $g(x)$ associated in Lemma 4.18 to $f(x) = H_{i-1}(x)$ and $t = \|h\|_\infty \epsilon^i$. Thus, for those values of $i \geqslant 1$ for which $H_i(x)$ is defined, we have

$$H_i(x) = H_{i-1}(x) - B_i(x) = H_{i-1}(x) - \sum_n b_{i,n}(x), \tag{4.27}$$

and, by (4.22),

$$|H_i(x)| \leqslant c\|h\|_\infty \epsilon^i. \tag{4.28}$$

Moreover, using (4.12), (4.27), (4.28) and proceeding by induction, it can be proved that

$$(H_i)_\gamma^*(x) \leqslant h_\gamma^*(x) + c \sum_{j=1}^{i} \|h\|_\infty \epsilon^j \sum_n \left[\frac{r_{j,n}}{d(x, x_{j,n}) + r_{j,n}} \right]^{1+\gamma}. \tag{4.29}$$

First, we shall study the case when the sequence $\{H_i(x)\}$ is infinite, this is the case if $\mu(X) = \infty$. From (4.27) it follows that

$$h(x) = H_i(x) + \sum_{j=1}^{i} \sum_n b_{j,n}(x).$$

Using (4.28) we obtain that

$$h = \sum_{j=1}^{\infty} \sum_n b_{j,n} \quad \text{in} \quad E^{\alpha'}.$$

Now by (4.19), (4.20) and (4.28) with $i = j - 1$ we have that $b_{j,n}$ are multiples of (ρ, ∞) atoms with $\operatorname{supp}(b_{j,n}) \subset B(x_{j,n}, r_{j,n}) = B_{j,n}$ and

$$\|b_{j,n}\|_\infty \leqslant c\|h\|_\infty \epsilon^{j-1}. \tag{4.30}$$

Let us prove (4.26). Denoting

$$\Omega_j = \{x \in X: (H_{j-1})^*_\gamma(x) > \|h\|_\infty \epsilon^j\},$$

from (4.30) and the fact that $\omega(s)$ is of lower type l, we get

$$\begin{aligned}\sum_{j=1}^{\infty} \sum_n \mu(B_{j,n})\omega(\|b_{j,n}\|_\infty) &\leqslant \sum_{j=1}^{\infty} \omega(c\|h\|_\infty \epsilon^{j-1}) \sum_n \mu(B_{j,n}) \\ &\leqslant \frac{c}{\epsilon} \sum_{j=1}^{\infty} \epsilon^{j(l-p)} \epsilon^{jp} \omega(\|h\|_\infty)\mu(\Omega_j).\end{aligned} \tag{4.31}$$

On the other hand, applying Lemma 4.1 and (4.29), we have

$$\mu(\Omega_j)\epsilon^{jp}\|h\|^p_\infty \leqslant \int (H_{j-1})^*_\gamma(x)^p \, d\mu(x) \leqslant (c+2)^j \int h^*_\gamma(x)^p \, d\mu(x). \tag{4.32}$$

So that (4.31) is bounded by

$$\frac{c}{\epsilon} \sum_{j=1}^{\infty} [\epsilon^{l-p}(c+2)]^j \omega(\|h\|_\infty)\|h\|_\infty^{-p} \int h^*_\gamma(x)^p \, d\mu(x).$$

Choosing ϵ small enough, (4.26) holds and the proof of the lemma finishes for the case when $\{H_i(x)\}$ is an infinite sequence.

Assume now that the sequence $\{H_j(x)\}$ is finite, in this case $\mu(X) < \infty$, and we can suppose without loss of generality that $\mu(X) = 1$. Let $H_m(x)$ be the last function of the sequence, thus

$$\omega(\|h\|_\infty \epsilon^{m+1}) \leqslant \int \omega[(H_m)^*_\gamma(x)] \, d\mu(x). \tag{4.33}$$

Moreover, for $j \leqslant m$, the function $H_j(x)$ satisfies (4.27) through (4.29).

Therefore, as before, we get that

$$h(x) = H_m(x) + \sum_{j=1}^{m} \sum_n b_{j,n}(x), \tag{4.34}$$

where $b_{j,n}$ are multiples of (ρ, ∞) atoms for every $j \leqslant m$. Let

$$b_{m+1,1}(x) = \int H_m(y)\, d\mu(y) \chi_X(x)$$

and

$$b_{m+1,2}(x) = H_m(x) - \int H_m(y)\, d\mu(y).$$

These functions have their supports contained in

$$X = B(x_0, 2K_1) \quad \text{and} \quad \int b_{m+1,2}\, d\mu(x) = 0.$$

Therefore, both functions are multiples of (ρ, ∞) atoms. Hence, by (4.34)

$$h(x) = \sum_{j=1}^{m+1} \sum_n b_{j,n}(x).$$

In order to prove (4.26), we first observe that by (4.28) and (4.33)

$$\begin{aligned} \omega(\|b_{m+1,1}\|_\infty) + \omega(\|b_{m+1,2}\|_\infty) &\leqslant \omega(c\|h\|_\infty \epsilon^m) \\ &\leqslant c\epsilon^{-1} \int \omega[(H_m)^*_\gamma(x)]\, d\mu(x). \end{aligned} \tag{4.35}$$

In view of (3.3) and (4.28), it follows that (4.35) is bounded by

$$\begin{aligned} &\frac{c}{\epsilon} \frac{\omega(\|h\|_\infty \epsilon^m)}{(\|h\|_\infty \epsilon^m)^p} \int (H_m)^*_\gamma(x)^p\, d\mu(x) \\ &\qquad \leqslant c\epsilon^{-1} \epsilon^{(l-p)m} \omega(\|h\|_\infty) \|h\|_\infty^{-p} \int (H_m)^*_\gamma(x)^p\, d\mu(x). \end{aligned}$$

On the other hand, as in (4.32) we get

$$\int (H_m)^*_\gamma(x)^p\, d\mu(x) \leqslant (c+2)^m \int h^*_\gamma(x)^p\, d\mu(x).$$

Then, (4.35) is less than or equal to

$$\epsilon^{-1} c[\epsilon^{l-p}(c+2)]^m \omega(\|h\|_\infty) \|h\|_\infty^{-p} \int h^*_\gamma(x)^p\, d\mu(x).$$

Thus, arguing as in the case of an infinite sequence, we have

$$\sum_{j=1}^{m+1} \sum_n \mu(B_{j,n}) \omega(\|b_{j,n}\|_\infty) \leqslant c\omega(\|h\|_\infty) \|h\|_\infty^{-p} \int h^*_\gamma(x)^p\, d\mu(x). \quad \square$$

We are in position to prove Theorem 2.1. The inclusion of $H^{\rho,q}$ in H_ω is, as we shall see, an immediate consequence of Lemma 4.3. The proof of the converse is based on the Calderón-Zygmund type Lemmas 4.9 and 4.18, which are applied to obtain the atomic decomposition of a distribution belonging to H_ω.

PROOF OF THEOREM 2.1. *First inclusion*: $H^{\rho,q} \subset H_\omega$. Let f be a distribution in $H^{\rho,q}$. So, by (1.24), for every $\epsilon > 0$ there exists a sequence $\{b_i(x)\}_i$ of multiples of (ρ, q) atoms such that $f = \sum_i b_i$ in $(E^\alpha)'$ and

$$(1 + \epsilon)\|f\|_{H^{\rho,q}} > \Lambda_q(\{b_i\}). \tag{4.36}$$

On the other hand, let η be a positive real constant. Then, by Lemma 4.3, we obtain

$$\int \omega\left[\frac{f_\gamma^*(x)}{(\eta\Lambda_q(\{b_i\}))^{1/l}}\right] d\mu(x) \leqslant \sum_j \int \omega\left[\frac{(b_j)_\gamma^*(x)}{(\eta\Lambda_q(\{b_i\}))^{1/l}}\right] d\mu(x)$$

$$\leqslant \sum_j \mu(B_j)\omega\left[\frac{c^{1/l}\|b_j\|_q \mu(B_j)^{-1/q}}{(\eta\Lambda_q(\{b_i\}))^{1/l}}\right].$$

Taking $\eta = c$, by (1.23) and (1.19) we get

$$\|f\|_{H_\omega} \leqslant c\Lambda_q(\{b_i\}),$$

which by (4.36), is bounded by $c\|f\|_{H^{\rho,q}}$, as we wanted to prove.

Second inclusion: $H_\omega \subset H^{\rho,q}$. Since $H^{\rho,\infty}$ is continuously included in $H^{\rho,q}$, $1 < q < \infty$, it is enough to show $H_\omega \subset H^{\rho,\infty}$. Assume that $\mu(X) = \infty$, the case $\mu(X) < \infty$ follows the same lines. Given $f \in H_\omega$, we shall prove that there exists a sequence $\{b_n(x)\}$, of multiples of (ρ, ∞) atoms satisfying

$$f = \sum_n b_n \tag{4.37}$$

in the sense of $(E^\alpha)'$, and

$$\|f\|_{H^{\rho,\infty}} \leqslant c\|f\|_{H_\omega}, \tag{4.38}$$

where c is a constant independent of f.

We first assume that f is a distribution in H_ω such that $f_\gamma^*(x)$ belongs to $L^2(X, d\mu)$. Thus, by Theorem 4.15, f can be represented in $(E^\alpha)'$ by a function $f(x)$ belonging to $L^2(X, d\mu)$ satisfying $|f(x)| \leqslant cf_\gamma^*(x)$. For $k \in \mathbb{Z}$, let us consider $\Omega_k = \{x: f_\gamma^*(x) > 2^k\}$. Applying Lemma 4.18 with $t = 2^k$ and $t = 2^{k+1}$, we obtain

$$f(x) = B_k(x) + G_k(x) = B_{k+1}(x) + G_{k+1}(x).$$

So, we can write

$$G_{k+1}(x) - G_k(x) = B_k(x) - B_{k+1}(x) = H_k(x). \tag{4.39}$$

Then, from (4.22) we have

$$|H_k(x)| \leqslant c2^k. \tag{4.40}$$

Therefore, the inequality

$$(H_k)^*_\gamma(x) \leqslant c2^k \sum_{j=k}^{k+1} \sum_i [r_{j,i}/(d(x, x_{j,i}) + r_{j,i})]^{1+\gamma}, \tag{4.41}$$

follows from (4.12) if $x \notin \Omega_k$ and from (4.40) if $x \in \Omega_k$. Consequently, by Lemma 4.1, for any p satisfying $(1+\gamma)^{-1} < p \leqslant 1$, we obtain

$$\int (H_k)^*_\gamma(x)^p \, d\mu(x) \leqslant c2^{kp}\mu(\Omega_k). \tag{4.42}$$

Let us see that $\sum_{k\in\mathbb{Z}} H_k$ converges to f in $(E^\alpha)'$. In fact, by (4.39) we have

$$f - \sum_{k=-n}^{n} H_k = f - G_{n+1} + G_{-n} = B_{n+1} + G_{-n}.$$

From (4.13) it follows that

$$\int \omega[(B_{n+1})^*_\gamma(x)] \, d\mu(x) \leqslant c \int_{\Omega_{n+1}} \omega[f^*_\gamma(x)] \, d\mu(x).$$

Thus, B_n converges to zero in H_ω when n tends to infinity and consequently, B_n converges to zero in $(E^\alpha)'$. On the other hand, since by (4.22) $|G_{-n}(x)| \leqslant c2^{-n}$, G_{-n} converges to zero in $(E^\alpha)'$ as n tends to infinity. Then,

$$f = \sum_k H_k, \quad \text{in} \quad (E^\alpha)'. \tag{4.43}$$

Let us now observe that H_k satisfies the hypothesis of Lemma 4.25. Since $f(x)$ belongs to $L^2(X, d\mu)$, $H_k(x)$ is in $L^2(X, d\mu)$ and by (4.40), $H_k(x)$ also belongs to $L^\infty(X, d\mu)$. Furthermore, from (4.42), $(H_k)^*_\gamma(x)$ is in $L^p(X, d\mu)$, for any p satisfying $(1+\gamma)^{-1} < p < l$. Then, Lemma 4.25 implies that there exists a sequence $\{b_i^k\}_i$ of multiples of (ρ, ∞) atoms such that

$$H_k = \sum_i b_i^k \quad \text{in} \quad (E^\alpha)'$$

and

$$\sum_i \mu(B_i^k)\omega(\|b_i^k\|_\infty) \leqslant c\omega(\|H_k\|_\infty)\|H_k\|_\infty^{-p} \int (H_k)^*_\gamma(x)^p \, d\mu(x).$$

Therefore, using (3.3), (4.40) and (4.42), we get

$$\sum_i \mu(B_i^k)\omega(\|b_i^k\|_\infty) \leqslant c\omega(c2^k)(c2^k)^{-p}\int (H_k)_\gamma^*(x)^p\,d\mu(x) \tag{4.44}$$

$$\leqslant \omega(c2^k)\mu(\Omega_k).$$

On the other hand, from (4.43), we have

$$f = \sum_k\Big(\sum_i b_i^k\Big). \tag{4.45}$$

Let $\eta \geqslant 1$ be a constant to be determined later, and denote $\lambda = \eta\|f\|_{H_\omega}$. We now estimate the sum

$$\sum_{k\in\mathbb{Z}}\sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{\lambda^{1/l}}\right).$$

By (4.44) applied to $\lambda^{-1/l}H_k$ this sum is bounded by

$$\sum_{k\in\mathbb{Z}}\mu(\Omega_k)\omega\left(\frac{c2^k}{\lambda^{1/l}}\right) = \sum_{k\in\mathbb{Z}}\omega\left(\frac{c2^k}{\lambda^{1/l}}\right)\int_{\{x:f_\gamma^*(x)>2^k\}} d\mu(x)$$

$$\leqslant \int_X \sum_{k<\log_2(f_\gamma^*(x))}\omega\left(\frac{c2^k}{\lambda^{1/l}}\right),$$

applying that $\omega(s)/s$ is non increasing this is bounded by

$$\leqslant 2\int_X\left[\sum_{k<\log_2(f_\gamma^*(x))}\int_{c\lambda^{-1/l}2^{k-1}}^{c\lambda^{-1/l}2^k}\frac{\omega(s)}{s}\,ds\right]d\mu(x)$$

$$\leqslant 2\int_X\left[\int_0^{c\lambda^{-1/l}f_\gamma^*(x)}\frac{\omega(s)}{s}\,ds\right]d\mu(x),$$

which by (3.4) is less than or equal to

$$\int_X \omega\left[\frac{cf_\gamma^*(x)}{\lambda^{1/l}}\right]d\mu(x).$$

Choosing $\eta = c^l$, we get

$$\sum_{k\in\mathbb{Z}}\sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{c\|f\|_{H_\omega}^{1/l}}\right) \leqslant \int_X \omega\left(\frac{f_\gamma^*(x)}{\|f\|_{H_\omega}^{1/l}}\right)d\mu(x) \leqslant 1.$$

This proves that $\Lambda_\infty(\{b_i^k\}_{i,k}) \leqslant c\|f\|_{H_\omega}$. Applying Lemma 4.23 and using (4.45), the Theorem follows under the assumption that $f_\gamma^*(x)$ belongs to $L^2(X, d\mu)$. Next, we shall remove that assumption. Let f be a distribution in

H_ω. In view of Theorem 4.16, we have that, for any positive integer k, there exists a function $f_k(x)$ in $L^2(X, d\mu)$ such that

$$\|f - f_k\|_{H_\omega} \leqslant 2^{-k}\|f\|_{H_\omega}. \tag{4.46}$$

Defining $f_0(x) = 0$, since H_ω is immerse in $(E^\alpha)'$, we get that

$$f = \sum_{k=1} f_k - f_{k-1},$$

in $(E^\alpha)'$. On the other hand, since $f_k - f_{k-1}$ is a distribution in H_ω satisfying that $(f_k - f_{k-1})^*_\gamma(x)$ belongs to $L^2(X, d\mu)$, we have an atomic decomposition for $f_k - f_{k-1}$, *i.e.* $f_k - f_{k-1} = \sum_i b_i^k$, where $\{b_i^k\}_i$ is a sequence of multiples of (ρ, ∞) atoms such that

$$\Lambda_\infty(\{b_i^k\}_i) \leqslant (1+\epsilon)\|f_k - f_{k-1}\|_{H^{\rho,\infty}} \leqslant c\|f_k - f_{k-1}\|_{H_\omega}$$

for every $\epsilon > 0$. Therefore, by (4.73),

$$\Lambda_\infty(\{b_i^k\}_i) \leqslant c2^{-k}\|f\|_{H_\omega}.$$

Thus

$$\begin{aligned}\sum_{k=1}^\infty \sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{(c\|f\|_{H_\omega})^{1/l}}\right) &\leqslant \sum_{k=1}^\infty \sum_i \mu(B_i^k)2^{-k}\omega\left(\frac{\|b_i^k\|_\infty}{(2^{-k}c\|f\|_{H_\omega})^{1/l}}\right)\\ &\leqslant \sum_{k=1}^\infty 2^{-k}\sum_i \mu(B_i^k)\omega\left(\frac{\|b_i^k\|_\infty}{\Lambda_\infty(\{b_i^k\}_i)^{1/l}}\right)\\ &\leqslant \sum_{k=1}^\infty 2^{-k} = 1.\end{aligned}$$

This implies that

$$f = \sum_{k=1}^\infty \sum_i b_i^k,$$

in $(E^\alpha)'$ and $\|f\|_{H^{\rho,\infty}} \leqslant c\|f\|_{H_\omega}$, as we wanted to prove. □

Remark. Observe that the statement of the theorem implies in particular that all the spaces $H^{\rho,q}$ are equivalent, for $1 < q \leqslant \infty$. In fact the original proof of the first inclusion was obtained by proving directly this equivalence and using the inclusion $H^{\rho,\infty} \subset H_\omega$ proved by applying Lemma 4.3 restricted to $q = \infty$. This path is longer than the approach presented here. I want to thank the referee for suggesting this shortcut based on the extension of Lemma 4.3 to the general case $1 < q \leqslant \infty$.

5. Dual Spaces

In this section we use the atomic decomposition obtained in Theorem 2.1 to show that BMO (ρ) is the dual space of H_ω. Let us point out that as by product of the proof of this characterization we get that BMO (ρ) coincides with BMO (ρ, q) for $1 < q < \infty$.

We shall work, as before, on a normal space $(X, d\mu)$ of order α with the additional assumption that μ is a regular measure. It is well known the density of Lip (β), $0 < \beta < \alpha$ in $L^p(X, d\mu)$, $1 \leqslant p < \infty$ (see [MS]). Consequently, if g belongs to BMO (ρ), for every ball B and $\epsilon > 0$ there exists a bounded continuous function h satisfying

(5.1)
$$\int_B |g(x) - h(x)|\, d\mu(x) < \epsilon.$$

We shall denote by $g_t(x)$ the function defined by

(5.3)
$$g_t(x) = \int \phi(x, y, t) g(y)\, d\mu(y),$$

where $\phi(x, y, t)$ is the function constructed in Lemma 3.15 of [MS]. In the proof of Theorem 2.2 we shall need the following two lemmas.

Lemma 5.2. *Let g belongs to* BMO (ρ). *Then*

(5.3)
$$\|g_t\|_{\mathrm{BMO}(\rho)} \leqslant c\|g\|_{\mathrm{BMO}(\rho)}$$

and for every ball B

(5.4)
$$\lim_{t\to 0} \int_B |g_t(x) - g(x)|\, d\mu(x) = 0,$$

PROOF. The proof is similar to Lemma 5.3 in [MS] and it makes use of remark 5.1. □

Lemma 5.5. *Let* $\{b_i\}_i$ *be as in Lemma* 4.7 *with* $q = \infty$ *and such that* $\sum b_i$ *converges to zero in* $(E^\alpha)'$. *Then* $\sum b_i$ *converges to zero in* BMO (ρ).

PROOF. Following the same argument given in Theorem 5.9 of [MS], it is easy to see that we only need to prove the convergence for functions g in BMO (ρ) with bounded support and non-negative. In fact, by Lemma 4.7, for any $\epsilon > 0$, there exists N such that $\sum_{i>N} \alpha_i < \epsilon$. Let $B = B(x_0, r)$ be a ball containing the support of $b_i(x)$ for every $1 \leqslant i \leqslant N$. Now, if g is as above, g_t is in E^α for $0 < t \leqslant 1$, so that $\sum b_i(g_t) = 0$. Thus,

$$\sum_i b_i(g) = \sum_i b_i(g - g_t) = \sum_{i\leq N} b_i(g - g_t) + \sum_{i>N} b_i(g - g_t).$$

By (1.16) and (1.11), we get

$$\left|\sum_{i>N} b_i(g-g_t)\right| = \left|\sum_{i>N}\int b_i(x)[g(x)-g_t(x)-m_{B_i}(g-g_t)]\,d\mu(x)\right|$$

$$\leqslant \|g-g_t\|_{\mathrm{BMO}(\rho)}\sum_{i>N}\mu(B_i)\rho(\mu(B_i))\|b_i\|_\infty .$$

Using (5.3) and the definition of $\rho(t)$, we have

$$\left|\sum_{i>N} b_i(g-g_t)\right| \leqslant c\|g\|_{\mathrm{BMO}(\rho)}\sum_{i>N}\alpha_i < c\|g\|_{\mathrm{BMO}(\rho)}\epsilon.$$

On the other hand, we obtain

$$\left|\sum_{i\leq N} b_i(g-g_t)\right| \leqslant \sum_{i\leq N}\|b_i\|_\infty\int_B |g(x)-g_t(x)|\,d\mu(x),$$

which by (5.4), tends to zero with t. This proves the lemma. □

PROOF OF THEOREM 2.2. *First inclusion*: $\mathrm{BMO}(\rho)\subset(H_\omega)'$. Let f be a distribution in H_ω with $\|f\|_{H_\omega}\leqslant 1$, represented by $f=\sum b_i$, where b_i are multiples of (ρ,∞) atoms. Given g in $\mathrm{BMO}(\rho)$, we first prove that the series $\sum b_i(g)$ is absolutely convergent. By (1.16), (1.11) and Lemma 4.7, we get

$$\sum_i |b_i(g)| \leqslant \|g\|_{\mathrm{BMO}(\rho)}\sum_i \alpha_i \leqslant c\|g\|_{\mathrm{BMO}(\rho)}. \tag{5.6}$$

On the other hand, by Lemma 5.5, $\sum b_i(g)$ is independent of the representation of f. Therefore, we obtain that the linear functional

$$L_g(f)=\sum_i b_i(g)$$

is well defined and from (5.6) it satisfies

$$|L_g(f)|\leqslant c\|g\|_{\mathrm{BMO}(\rho)}\|f\|^{1/l}.$$

Second inclusion: $(H_\omega)'\subset\mathrm{BMO}(\rho)$. By Hölder inequality, we have that $\mathrm{BMO}(\rho,q)\subset\mathrm{BMO}(\rho)$, for every $1<q<\infty$. Thus, it is enough to prove that $(H_\omega)'\subset\mathrm{BMO}(\rho,q)$. Let us only consider the case $\mu(X)=\infty$. Let $1<q<\infty$ and $q'=q/(q-1)$. Given a ball B, we define

$$L_0^q(B)=\left\{f\in L^q(B)\colon \int_B f=0\right\}.$$

Therefore, if f is in $L_0^q(B)$, then

$$b(x)=\frac{f(x)\mu(B)^{1/q}}{\|f\|_q\mu(B)\rho(\mu(B))}=\frac{f(x)}{\|f\|_q\mu(B)^{1/q'}\rho(\mu(B))}$$

is a (ρ, q) atom. Let L be an element of $(H_\omega)'$. Hence, for f in $L_0^q(B)$, we have that $L(f)$ is defined and

$$|L(f)| \leqslant \|L\|_{(H_\omega)'}\mu(B)^{1/q'}\rho(\mu(B))\|f\|_q. \tag{5.3}$$

Consequently, L is a bounded linear functional on $L_0^q(B)$. Applying the Hahn-Banach Theorem L can be extended to $L^q(B)$ with the same norm. By the Riesz's Representation Theorem there exists a $h \in L^{q'}(B)$ such that

$$L(f) = \int_B f(x)h(x)\,d\mu(x), \tag{5.4}$$

for every f in $L^q(B)$. It is easy to check that h is determined in $L^{q'}(B)$ up to constants. Let $C(B)$ be the space of constant functions on B. Then, there exists g in $L^{q'}(B)/C(B)$ such that (5.4) holds for every $h \in g$ and every f in $L_0^q(B)$. Consider now an increasing sequence of balls, $\{B_k\}_{k=1}^\infty$, such that $\bigcup_k B_k = X$ and denote by T_k the operator which takes a function on X and restrict it to B_k. Thus, if $g_{k+1} \in L^{q'}(B_{k+1})/C(B_{k+1})$ and for every $h \in g_{k+1}$, (5.4) holds with $B = B_{k+1}$, then

$$L(f) = \int_{B_k} hf\,d\mu,$$

for every f in $L_0^q(B_k)$. Consequently $T_k(g_{k+1}) = g_k$. Therefore, there exists g in $L_{\text{loc}}^q(X)/C(X)$ such that $T_k(g) = g_k$, for every k. Let L_g be the operator defined on atoms by

$$L_g(a) = \int_X h(x)a(x)\,d\mu(x),$$

for every $h \in g$. From Lemma 4.7 we see that L_g is continuous on the space spanned by the set of atoms. So that there is only one continuous extension of L_g to the space $H^{\rho,q}$. Clearly $L_g \equiv L$.

It remains to show that every $h \in g$ belongs to BMO (ρ, q'). Let B be a ball, then, using (5.3) we have

$$\begin{aligned}
\|(g - m_B(g))\chi_B\|_{q'} &= \sup_{\|f\chi_B\|_q = 1} \left| \int_B (g - m_B(g)) f\,d\mu \right| \\
&= \sup_{\|f\chi_B\|_q = 1} \left| \int_B (g - m_B(g))(f - m_B(f))\,d\mu \right| \\
&= \sup_{\|f\chi_B\|_q = 1} |L[(f - m_B(f))\chi_B]| \\
&\leqslant \sup_{\|f\chi_B\|_q = 1} \|L\|_{(H_\omega)'}\mu(B)^{1/q'}\rho(\mu(B))\|(f - m_B(f))\chi_B\|_q \\
&\leqslant 2\|L\|_{(H_\omega)'}\mu(B)^{1/q'}\rho(\mu(B)).
\end{aligned}$$

Consequently,

$$\|g\|_{\mathrm{BMO}(\rho,q')} \leqslant 2\|L\|_{(H_\omega)'}. \quad \square$$

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References

[A] Aimar, H. A. Rearrangement of BMO (ρ) Functions on Spaces of Homogeneous Type, preprint.

[C] Calderón, A. P. An Atomic Decomposition of Distributions in Parabolic H^p Spaces, *Advances in Math.* **25**(1977), 216-225.

[CO] Coifman, R. R. A Real Variable Characterization of H^p, *Studia Math.* **51**(1974), 267-272.

[H] Herz, C. H^p Spaces of Martingales, $0 < p \leqslant 1$, *Zeit. für Wahrscheinlichkeitstheorie*, **28**(1974), 189-205.

[J] Janson, S. Lipschitz Spaces and Bounded Mean Oscillation, *Duke Math. J.* **47**(1980), 959-982.

[L] Latter, R. H. A Characterization of $H^p(\mathbb{R}^n)$ in Terms of Atoms, *Studia Math.* **62**(1978), 92-101.

[LU] Latter, R. H. and Uchiyama, A. The Atomic Decomposition for Parabolic H^p Spaces, *Trans. Amer. Math. Soc.* **256**(1979), 391-398.

[M] Macías, R. A. Interpolation Theorems on Generalized Hardy Spaces, Doctoral Dissertation, Washington University, 1974.

[MS] Macías, R. A. and Segovia, C. A Decomposition into Atoms of Distributions on Spaces of Homogeneous Type, *Advances in Math.* **33**(1979), 271-309.

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Global Models of Riemannian Metrics

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Introduction

In this paper we give certain Riemannian metrics on the manifolds $S^{n-1} \times S^1$ and S^n ($n \geqslant 2$), which have the property to determine these manifolds, up to diffeomorphisms.

The global expressions used for Riemannian metrics are based on the global expression for exterior forms studied in [4]. In [3] one finds certain metrics using global expressions that differ from the type we propose.

To some extent, Theorem 3 is a «generalization for metrics» in an arbitrary dimension, of a theorem proved in [2] for certain volume forms on surfaces.

1. Examples and Theorems on Surfaces

The following example illustrates the context in which our statements are made.

Let us consider in $\mathbb{R}^3 - 0$ the quadratic form:

$$m = (x_1\, dx_2 - x_2\, dx_1)^2 + (x_1\, dx_3 - x_3\, dx_1)^2 + (x_2\, dx_3 - x_3\, dx_2)^2.$$

A simple calculation proves that a vector v is isotropic if and only if

$$v = \lambda\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right), \qquad \lambda \in \mathbb{R}. \qquad (*)$$

Hence, m is a Riemannian metric over all surfaces in $\mathbb{R}^3$ whose tangent plane is transverse to the position vector field. In particular, if $i: S^2 \to \mathbb{R}^3 - 0$

is the ordinary inclusion, the metric $i^*(m)$ on S^2 admits the global expression

$$i^*(m) = (f_1\, df_2 - f_2\, df_1)^2 + (f_1\, df_3 - f_3\, df_1)^2 + (f_2\, df_3 - f_3\, df_2)^2$$

where $f_j\colon S^2 \to \mathbb{R}$, $j = 1,2,3$ are global functions given by $f_j = x_j \cdot i$.

As a consequence of (*) the following theorem is easily proved

Theorem 1. *Let M be a compact connected surface having a Riemannian metric m that admit the global expression:*

$$m = (f_1\, df_2 - f_2\, df_1)^2 + (f_1\, df_3 - f_3\, df_1)^2 + (f_2\, df_3 - f_3\, df_2)^2$$

where $f_i\colon M_2 \to \mathbb{R}$, $i = 1,2,3$ are C^∞-global functions. Then M_2 is diffeomorphic to the sphere S^2.

PROOF OF THEOREM 1. Given the metric m on M_2, let us consider the map $\varphi\colon M_2 \to \mathbb{R}^3 - 0$ expressed by

$$\varphi(p) = (f_1(p), f_2(p), f_3(p)).$$

Lemma 1. *The following statements are equivalent*

(*a*) *m is a Riemannian metric on M_2.*

(*b*) *φ is an immersion transverse to the vector field*

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \quad on \quad \mathbb{R}^3.$$

The proof of this lemma is basically the remark made in (*).

Let $\Pi\colon \mathbb{R}^3 - \{0\} \to S^2$ be given by $\Pi(x) = x/|x|$. Lemma 1 proves that if m is a Riemannian metric, $\pi \cdot \varphi\colon M_2 \to S^2$ is a covering map, hence $\pi \cdot \varphi$ is a diffeomorphism. □

Let us now consider the metric

$$m_1 = (1 + 4\sin^2\theta_2 \cos^2\theta_2)\, d\theta_1^2 + d\theta_2^2$$

on the torus $T^2 = S^1 \times S^1$ and

$$m_2 = (x_1\, dx_2 - x_2\, dx_1)^2 + (x_1\, dx_3 - 2x_3\, dx_1)^2 + (x_2\, dx_3 - 2x_3\, dx_2)^2$$

on the sphere S^2.

An easy calculation proves that both metrics admit the global expression

$$m = (f_1\, df_2 - f_2\, df_1)^2 + [f_1(f\, dg - g\, df) - 2fg\, df_1]^2 + [f_2(f\, dg - g\, df) - 2fg\, df_2]^2$$

where f_1, f_2, f and g are global C^∞-functions.

Theorem 2. *Let M_2 be a compact connected surface having a Riemannian metric that admits the global expression*:

$$m = (f_1\,df_2 - f_2\,df_1)^2 + [f_1(f\,dg - g\,df) - 2fg\,df_1]^2 + [f_2(f\,dg - g\,df) - 2fg\,df_2]^2$$

and let

$$H = \{p \in M_2 \mid f_1(p) = f_2(p) = 0\}.$$

(*a*) *If $H \neq \varnothing$ then M_2 is diffeomorphic to the sphere S^2.*
(*b*) *If $H = \varnothing$ then M_2 is diffeomorphic to the torus T^2.*

PROOF OF THEOREM 2.

Lemma 2. $(f_1\,df_2 - f_2\,df_1)(p) = 0$ *if and only if $p \in H$. Moreover, H is finite.*

PROOF. There is an obvious implication. If $f_1(p)\,df_2(p) - f_2(p)\,df_1(p) = 0$ with $f_1(p) \neq 0$ then

$$df_2(p) = \frac{f_2(p)}{f_1(p)}\,df_1(p),$$

and substituting in the expression for the metric m at that point, we obtain

$$m(p) = \left(1 + \frac{f_2^2}{f_1^2}\right)[f_1(f\,dg - g\,df) - 2fg\,df_1]^2(p)$$

which admits isotropic vectors. Hence $f_1(p) = 0$ and likewise $f_2(p) = 0$.

Finally, if $p \in H$, then

$$m(p) = 4f^2g^2(df_1^2 + df_2^2)(p)$$

and therefore $(df_1 \wedge df_2)(p) \neq 0$, hence H is finite. □

We define

$$\omega_1 = f_1(f\,dg - g\,df) - 2fg\,df_1$$

and

$$\omega_2 = f_2(f\,dg - g\,df) - 2fg\,df_2.$$

Lemma 3. $\omega_1 \wedge \omega_2 = 0$ *if and only if* $fg = 0$.

PROOF. From the definition of ω_1 and ω_2, if $fg = 0$ it is obvious that $\omega_1 \wedge \omega_2 = 0$. Conversely, if $\omega_1 \wedge \omega_2 = 0$, from the relation $2fg(f_1\,df_2 - f_2\,df_1) = f_2\omega_1 - f_1\omega_2$ we obtain

$$2fg(f_1\,df_2 - f_2\,df_1)\wedge\omega_1 = 0$$
$$2fg(f_1\,df_2 - f_2\,df_1)\wedge\omega_2 = 0.$$

The fact that m is a metric implies that two of the three forms $f_1\,df_2 - f_2\,df_1$, ω_1, ω_2 must be independent at each point, then either $(f_1\,df_2 - f_2\,df_1)\wedge\omega_1 \neq 0$ or $(f_1\,df_2 - f_2\,df_1)\wedge\omega_2 \neq 0$, hence $fg = 0$. □

Remark 1. From the expression for m, it is deduced that f and df (g and dg, respectively) cannot have common zeros, and either the set $f = 0$ ($g = 0$ respectively) is empty or it is made up of a finite number of disjoint circles. Hence ω_1 and ω_2 are independent in a dense open set.

Let us now define $\omega = f_1\omega_1 + f_2\omega_2$. We have

Lemma 4. *$\omega(p) = 0$ if and only if $p \in H$.*

PROOF. If $p \in H$ obviously $\omega(p) = 0$. If $\omega(p) = 0$ and $f_1^2(p) + f_2^2(p) \neq 0$, Lemma 3 implies that $fg = 0$. Moreover,

$$\omega = f_1\omega_1 + f_2\omega_2 = (f_1^2 + f_2^2)(f\,dg - g\,df) - 2fg(f_1\,df_1 + f_2\,df_2)$$

which implies $0 = (f_1^2 + f_2^2)(f\,dg - g\,df)(p)$ and then $(f\,dg - g\,df)(p) = 0$ and the expression for the metric at this point would be $(f_1\,df_2 - f_2\,df_1)^2$, which is a contradiction. □

Remark 2. Since H is finite (Lemma 2), ω has a finite number of singularities.

Let us consider on M_2 the vector fields X, Y which are dual, with respect to the metric m, of the 1-forms $f_1\,df_2 - f_2\,df_1$ and ω. Lemma 2 and Remark 2 imply that X and Y have a finite number of singularities.

Lemma 5. *X and Y are orthogonal with respect to the metric m.*

PROOF. The vector fields are defined by the relations

$$m(X, \bullet) = f_1\,df_2 - f_2\,df_1$$
$$m(Y, \bullet) = \omega.$$

Lemma 5 is equivalent to proving that $\omega(X) = 0$.
From the expression for m, it is deduced that

$$m(X, \bullet) = (f_1\,df_2 - f_2\,df_1)(X)(f_1\,df_2 - f_2\,df_1) + \omega_1(X)\omega_1 + \omega_2(X)\omega_2$$

which implies

$$[1 - (f_1\,df_2 - f_2\,df_1)(X)](f_1\,df_2 - f_2\,df_1) = \omega_1(X)\omega_1 + \omega_2(X)\omega_2$$

and from the relation used in Lemma 3 we have

$$[1 - (f_1\,df_2 - f_2\,df_1)(X)]\frac{f_2\omega_1 - f_1\omega_2}{2fg} = \omega_1(X)\omega_1 + \omega_2(X)\omega_2 .$$

From Lemma 3 it is deduced that in the dense open set $fg \neq 0$ the following relations are satisfied:

$$\lambda f_2 = \omega_1(X)$$
$$-\lambda f_1 = \omega_2(X)$$

where

$$\lambda = \frac{1}{2fg}[1 - (f_1\,df_2 - f_2\,df_1)(X)].$$

By multiplying the relations above by f_1 and f_2 respectively, and adding we obtain $(f_1\omega_1 + f_2\omega_2)(X) = 0$ and therefore $\omega(X) = 0$ if $fg \neq 0$.

Since the function $\omega(X)$ is defined on all of M_2 and it is zero in a dense open subset, we obtain $\omega(X) \equiv 0$ on M_2. □

Corollary 1. *M_2 is orientable.*

PROOF. From Lemma 5 it is concluded that $X \wedge Y$ is a 2-vector field on M_2 with a finite number of singularities, hence M_2 is orientable. □

2. Conclusion

(*a*) Let $H \neq \emptyset$ and $p \in H$. From the global expression for m it can be deduced that $df_1 \wedge df_2(p) \neq 0$ and hence f_1 and f_2 can be taken as coordinates in a neighbourhood of p. Therefore the vector field $f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}$ is well defined in a neighbourhood of each singularity of the vector field Y.

As the equality

$$(f_1\,df_2 - f_2\,df_1)\left(f_1\frac{\partial}{\partial f_1} + f_2\frac{\partial}{\partial f_2}\right) = f_1 f_2 - f_2 f_1 = 0$$

is satisfied in each of these neighbourhoods, the vector field $f_1 \frac{\partial}{\partial f_1} + f_2 \frac{\partial}{\partial f_2}$ is orthogonal to X and it follows from Lemma 5 that

$$Y = \mu\left(f_1 \frac{\partial}{\partial f_1} + f_2 \frac{\partial}{\partial f_2} \right)$$

in a neighbourhood of each singularity of Y.

Consequently, Y has a finite number of singularities (Remark 2), every singularity has an index of $+1$ (following from before), M_2 is orientable (Corollary 1) and hence M_2 is diffeomorphic to the sphere S^2.

(*b*) If $H \neq \emptyset$, according to Remark 2 Y is a vector field without singularities, hence the Euler characteristic is $\chi(M_2) = 0$. As M_2 is orientable, it is deduced that $M_2 = T^2$.

The proof of the Theorem is complete. □

3. Examples of Metrics in Arbitrary Dimension

The generalization of Theorem 2 to an arbitrary dimension is motivated by the following examples

(*a*) Let us consider the mapping

$$h_r \colon S^{n-1} \times S^1 \to \mathbb{R}^n \times \mathbb{R}^2,$$

given by

$$h_r(p, \theta) = (p, \cos r\theta, \sin r\theta); \qquad r = 1, 2, 3, \ldots$$

and the quadratic form in $\mathbb{R}^n \times \mathbb{R}^2$:

$$m = \sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} \omega_{ij}^2 + \sum_{k=1}^{n} [x_k(y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2\, dx_k]^2$$

where $(x_i)_{i=1,\ldots,n}$, $(y_i)_{i=1,2}$ are the coordinates in $\mathbb{R}^n$ and $\mathbb{R}^2$ respectively, and $\omega_{ij} = (x_i\, dx_j - x_j\, dx_i)$ for $i < j$.

A simple calculation proves that $h_r^*(m)$ is a Riemannian metric in $S^{n-1} \times S^1$.

For $r = 1$ and $n = 2$, the metric $h_1^*(m)$ in $S^1 \times S^1$ is the metric m_1 that appears in Section 1.

(*b*) Let us consider in $\mathbb{R}^{n+1}$ the quadratic form

$$m = \sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} \omega_{ij}^2 + \sum_{k=1}^{n} [x_k(y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2\, dx_k]^2,$$

with $y_1 = 1$, $y_2 = x_{n+1}$. If $i: S^n \to \mathbb{R}^{n+1}$ is the ordinary inclusion, it can easily be proved that $i^*(m)$ is a metric in S^n.

In the case where $n = 2$, we have the metric m_2 of S^2 from Section 1.

(c) Let us consider in $\mathbb{R}^{n+1}$ and for each $r = 0, 1, 2, \ldots$, the quadratic form

$$m_r = \sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} \omega_{ij}^2 + \sum_{k=1}^{n} \tau_k^2$$

where

$$\omega_{ij} = x_i\, dx_j - x_j\, dx_i \quad \text{for} \quad i < j$$

$$\tau_k = [x_k(\cos f(x_{n+1}) d \sin f(x_{n+1}) - \sin f(x_{n+1}) d \cos f(x_{n+1})) - 2 \sin f(x_{n+1}) \cos f(x_{n+1})\, dx_k].$$

$$f(x_{n+1}) = \frac{\pi}{4} + \left(\frac{\pi}{2} + r\pi\right)\left(\frac{x_{n+1} + 1}{2}\right) + \frac{\pi}{2}.$$

In the following remark we show that for every $r \in \mathbb{N}$, $i^*(m_r)$ is a Riemannian metric on S^n.

Remark 3. If we define

$$\alpha = \frac{df}{dx_{n+1}} = \frac{1}{2}\left(\frac{\pi}{2} + r\pi\right), \qquad \beta = \sin 2f(x_{n+1}),$$

then

$$\tau_k = x_k \alpha\, dx_{n+1} - \beta\, dx_k, \qquad k = 1, \ldots, n.$$

To see that $i^*(m_r)$ is a metric on S^n it is enough to check that at every point of S^n one can choose n independent 1-forms among the ω_{ij}'s and τ_k's. To do this we shall calculate the external product of certain n 1-forms by the form

$$\Omega = x_1\, dx_1 + x_2\, dx_2 + \cdots + x_n\, dx_n + x_{n+1}\, dx_{n+1}.$$

(1) If $\beta \neq 0$.

$$\Omega \wedge \tau_1 \wedge \cdots \wedge \tau_n = \beta^{n-1}[\alpha(1 - x_{n+1}^2) + \beta x_{n+1}]\, dx_1 \cdots dx_{n+1}$$

and the expressions for α and β imply that

$$\alpha(1 - x_{n+1}^2) + \beta x_{n+1} = 0$$

has no solution for $r = 0, 1, 2, \ldots$ with $-1 \leqslant x_{n+1} \leqslant 1$ [1].

(2) It $\beta = 0$, then $\tau_k = x_k \alpha\, dx_{n+1}$ and we calculate the following exterior products

$$\Omega \wedge \omega_{12} \wedge \omega_{13} \wedge \cdots \wedge \omega_{1n} \wedge \tau_1 = x_1^{n-1}\alpha(1 - x_{n+1}^2)\, dx_1 \wedge \cdots \wedge dx_{n+1}$$
$$\Omega \wedge \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{2n} \wedge \tau_2 = x_2^{n-1}\alpha(1 - x_{n+1}^2)\, dx_1 \wedge \cdots \wedge dx_{n+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Omega \wedge \omega_{1n} \wedge \omega_{2n} \wedge \cdots \wedge \omega_{n-1n} \wedge \tau_n = x_n^{n-1}\alpha(1 - x_{n+1}^2)\, dx_1 \wedge \cdots \wedge dx_{n+1}$$

Consequently, since $\alpha \neq 0$ and $x_{n+1}^2 \neq 1$ (because $x_{n+1} = \pm 1$ implies that $\beta \neq 0$), if $x_j \neq 0$ then $\omega_{1,j}$; $\omega_{2,j}$; $\ldots$; $\omega_{j-1,j}$; $\omega_{j,j+1}$; $\ldots$; $\omega_{j,n}$ are independent. On the other hand, if all the x_j's are zero then $x_{n+1} = \pm 1$ and therefore $\beta \neq 0$.

4. The Theorem in Arbitrary Dimension

Theorem 3. *Let M_n be a compact connected n-dimensional Hausdorff manifold, having a Riemannian metric m that admits the global expression*

$$m = \sum_{\substack{i=1,j=2\\ i<j}}^{n-1,n} (f_i\, df_j - f_j\, df_i)^2 + \sum_{k=1}^{n} [f_k(f\, dg - g\, df) - 2fg\, df_k]^2$$

where $f_1, f_2, \ldots, f_n, f, g$ are global C^∞-functions.

Let $H = \{p \in M_n \mid f_i(p) = 0 \quad i = 1, \ldots, n\}$.

Then

(*a*) *$H = \emptyset$ implies that M_n is diffeomorphic to $S^{n-1} \times S^1$.*

(*b*) *$H \neq \emptyset$ implies that M_n is diffeomorphic to the sphere S^n.*

Remark 4. Examples (*a*), (*b*) and (*c*) of Section 3 prove that on $S^{n-1} \times S^1$ and S^n there are metrics that admit the expression above.

Before beginning the proof of the Theorem, we include some comments on the quadratic form of example (*a*) which are essential for the proof.

Let us now consider in $\mathbb{R}^n \times \mathbb{R}^2$, with coordinates $(x_1, \ldots, x_n, y_1, y_2)$ the quadratic form

$$m_0 = \sum_{\substack{i=1,j=2\\ i<j}}^{n-1,n} (x_i\, dx_j - x_j\, dx_i)^2 + \sum_{k=1}^{n} [x_k(y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2\, dx_k]^2$$

and the vector fields

$$X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$$

$$Y = -2y_1 \frac{\partial}{\partial y_1} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

By a simple calculation it can be proved that X and Y are isotropic and independent in the open set U of $\mathbb{R}^n \times \mathbb{R}^2$ where the following inequality is satisfied

$$y_1^2 y_2^2 + (y_1^2 + y_2^2)\left(\sum_{i=1}^{n} x_i^2\right) \neq 0.$$

If $p \notin U$ the quadratic form m_0 reduces to

$$m_0 = \sum_{i=1, j=2}^{n-1, n} (x_i\,dx_j - x_j\,dx_i)^2,$$

its rank being less than n because

$$X \rfloor m_0 = 0 \quad \text{if} \quad X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\cdot$$

Lemma 6. *Let $p \in U$ and let v_p be a vector where $v_p \rfloor m_0 = 0$ (i.e. $m_0(v_p v_p) = 0$ as m_0 is semidefined positive). Then $v_p = \lambda Y + \mu X$.*

PROOF. Let $v_p = v_1 + v_2$ be a vector where $p \in U$ and

$$v_1 = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i},$$

$$v_2 = \mu_1 \frac{\partial}{\partial y_1} + \mu_2 \frac{\partial}{\partial y_2}$$

$$m_0(v_p, v_p) = 0 \Leftrightarrow \begin{cases} (1)\ v_1 \rfloor \left(\displaystyle\sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} (x_i\,dx_j - x_j\,dx_i)^2 \right) = 0 \\ \quad \Leftrightarrow v_1 = \lambda \displaystyle\sum_{i=1}^{n} x_i \frac{\delta}{\partial x_i} \Leftrightarrow \lambda_k = \lambda x_k \\ (2)\ x_k [v_2 \rfloor (y_1\,dy_2 - y_2\,dy_1) - 2y_1 y_2] = 0;\ k = 1, 2, \ldots, n. \end{cases}$$

If $x_k(p) = 0$ for all values of k, then $y_1 y_2(p) \neq 0$ and (1) implies that $v_1 = 0$. Hence

$$v_p = \mu_1 \frac{\partial}{\partial y_1} + \mu_2 \frac{\partial}{\partial y_2} \quad \text{with} \quad X = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \qquad Y = -2y_1 \frac{\partial}{\partial y_1},$$

and therefore v_p satisfies Lemma 6.

If

$$\sum_{k=1}^{n} x_k^2(p) \neq 0,$$

we would have from (2) that

$$v_2 \rfloor (y_1\, dy_2 - y_2\, dy_1) - 2y_1 y_2 \lambda = 0,$$

whence

$$v_2 = -2\lambda y_1 \frac{\partial}{\partial y_1} + \mu\left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right).$$

Since

$$v_1 = \lambda \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i},$$

we have

$$v_p = \lambda\left(-2y_1 \frac{\partial}{\partial y_1} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right) + \mu\left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right). \quad \square$$

Corollary 2. *The quadratic form m_0 has constant rank n in U. If $p \notin U$ the rank of m_0 is less than n.*

The vector fields X, Y define (Lemma 6) a completely integrable 2-dimension ditribution $\mathfrak{F}$ in the open set U. Given also that $[X, Y] = 0$, the leaves of $\mathfrak{F}$ are the orbits of the Abelian group action $\mathbb{R}^2$ on the open set U, defined $\mathbb{R}^2 \times U \to U$ by $(s, t, p) \to \psi_s \cdot \varphi_t(p)$ where ψ_s and φ_t are the one-parameter group generated by Y and X respectively.

The relation between the manifold M_n, the metric m and the space $\mathfrak{F}$ is expressed by the Lemma below, which follows directly from Lemma 6.

Lemma 7. *Let M_n be a compact connected Hausdorff manifold, having a quadratic form that admits the global expression*:

$$m = \sum_{\substack{i=1, j=2 \\ i<j}}^{n-1, n} (f_i\, df_j - f_j\, df_i)^2 + \sum_{k=1}^{n} [f_k(f\, dg - g\, df) - 2fg\, df_k]^2$$

and let $\varphi: M_n \to \mathbb{R}^n \times \mathbb{R}^2$ be given by $x_i = f_i$, $y_1 = f$, $y_2 = g$, $i = 1, \ldots, n$.
The following properties are equivalent

(*a*) *m is a Riemannian metric.*
(*b*) *φ is a transverse, immersion with respect to the distribution $\mathfrak{F}$.*

Remark 5. As a result, if m is a Riemannian metric on M_n expressed as above, and $\Pi: U \to \bar{U}$ is the quotient mapping to the leaf space of $\mathfrak{F}$, Lemma 7 proves that if $\bar{U}$ admits a quotient manifold structure, $\pi \cdot \varphi: M_n \to \bar{U}$ is then a local diffeormorphism.

5. The Leaf Space $\bar{U}$

The orbit passing through a point $p = (x_i, y_1, y_2)$ $i = 1, \ldots, n$, is expressed by $\psi_s\varphi_t(x_i, y_1, y_2) = (Ax_i, A^{-2}By_1, By_2)$ where $A = e^s$ and $B = e^t$. In order to calculate a model of the quotient of U by the action $\psi_s\varphi_t$, we consider the open sets

$$U_1 = \left\{ p \in U \,\middle|\, \sum_{i=1}^{n} x_i^2 \neq 0, y_1^2 + y_2^2 \neq 0 \right\}$$

and

$$U_1' = \{ p \in U \mid y_1 y_2 \neq 0 \}.$$

It is obvious that $U_1 \cup U_1' = U$, $U_1 \cap U_1' \neq \varnothing$ and both U_1 and U_1' are stable due to the action of $\mathbb{R}^2$.

In U_1, let us consider the submanifold

$$S^{n-1} \times S^1 = \left\{ p \in U_1 \,\middle|\, \sum x_i^2 = 1, y_1^2 + y_2^2 = 1 \right\}.$$

Lemma 8. *The leaf passing through $p \in U_1$ cuts $S^{n-1} \times S^1$ transversely at a single point, hence the mapping α: $U_1 \to S^{n-1} \times S^1$,*

$$\alpha(p) = \psi_s \cdot \psi_t(p) \cap (S^{n-1} \times S^1)$$

is a submersion.

Direct calculation proves that for

$$p = (x_i, y_1, y_2), \qquad \alpha(p) = (Ax_i, A^{-2}By_1, By_2)$$

where

$$A = \frac{1}{\sqrt{\sum_{i=1}^{n} x_i^2}} \quad \text{and} \quad B = \frac{1}{\sqrt{A^{-4}y_1^2 + y_2^2}}$$

The fact that the intersection is transversal follows immediately since these relations

$$\left(\sum_{i=1}^{n} x_i\, dx_i\right)(\lambda x + \mu Y) = 0 \qquad (y_1\, dy_1 + y_2\, dy_2)(\lambda x + \mu Y) = 0$$

imply that $\lambda = \mu = 0$. □

Remark 6. The mapping $p \to (\alpha(p), A, B)$ of $U_1 \to (S^{n-1} \times S^1) \times \mathbb{R}_+^2$ is a diffeomorphism.

The open set U_1' has four connected components:

$$V_1 = \{p \in U_1' \mid y_1 > 0, y_2 > 0\}$$
$$V_2 = \{p \in U_1' \mid y_1 > 0, y_2 < 0\}$$
$$V_3 = \{p \in U_1' \mid y_1 < 0, y_2 > 0\}$$
$$V_4 = \{p \in U_1' \mid y_1 < 0, y_2 < 0\};$$

in each of which the following manifolds are considered

$$\Pi(1,1) = \{p \in V_1 \mid y_1 = 1, y_2 = 1\}$$
$$\Pi(1,-1) = \{p \in V_2 \mid y_1 = 1, y_2 = -1\}$$
$$\Pi(-1,1) = \{p \in V_3 \mid y_1 = -1, y_2 = 1\}$$
$$\Pi(-1,-1) = \{p \in V_4 \mid y_1 = -1, y_2 = -1\}.$$

Lemma 9. *The leaf passing through* $p \in V_i$, $(i = 1,2,3,4)$ *cuts* $\Pi_{(k,l)} \subset V_i$ $(k = 1, -1; l = 1, -1)$ *transversely at a single point. Consequently, the mapping* $\beta \colon U_1' \to \bigcup_{k,l=1,-1} \Pi_{(k,l)}$, defined by

$$\beta(p) = \psi_s \varphi_t(p) \cap \Pi_{(k,l)}, \quad \text{for} \quad p \in V_i \supset \Pi_{(k,l)}$$

is a submersion.

PROOF. The same calculation mentioned in Lemma 8 proves that

$$\psi_s \varphi_t(p) \cap \Pi_{(k,l)} = (Ax_i, A^{-2}By_1, By_2)$$

where

$$A = \left|\frac{y_1}{y_2}\right|, \qquad B = \frac{1}{|y_2|},$$

and $p \in V_i \supset \Pi_{(k,l)}$ $(k, l = 1, -1)$.

The transversality is now a consequence of the fact that the relations

$$dy_1(\lambda x + \mu Y) = 0$$
$$dy_2(\lambda x + \mu Y) = 0$$

imply that $\lambda = \mu = 0$ when $|y_1| = 1$ and $|y_2| = 1$. □

Due to the fact that $U_1 \cap U_1' \neq \varnothing$ the quotient $\bar{U}$ is obtained by identifying

$$(U_1 \cap U_1') \cap \Big(\bigcup_{k,l=1,-1,-1} \Pi_{(k,l)}\Big) \quad \text{with} \quad (U_1 \cap U_1') \cap (S^{n-1} \times S^1)$$

by means of the diffeomorphism which associates to each point of $U_1 \cap \Pi_{(k,l)}$, the intersection of the leaf passing through the point and $V_j \cap (S^{n-1} \times S^1)$, $\Pi_{(k,l)} \subset V_j$, *i.e.*

$$(**) \qquad U_1 \cap \Pi_{(k,l)} \to V_j \cap (S^{n-1} \times S^1)$$

given by

$$(x_i, k, l) \to \left(\frac{x_i}{\sqrt{\sum_{i=1}^{n} x_1^2}}, \frac{k \sum x_i^2}{\sqrt{\left(\sum_{i=1}^{n} x_i^2\right)^2 + 1}}, \frac{l}{\sqrt{\left(\sum_{i=1}^{n} x_i^2\right)^2 + 1}} \right)$$

with $\Pi_{(k,l)} \subset V_j$ for $j = 1, 2, 3, 4$ and $k, l = 1, -1$.

The fact that $\bar{U}$ is obtained by identifying open sets of the manifolds $S^{n-1} \times S^1$ and $\bigcup_{k,l=1,-1} \Pi_{(k,l)}$ by means of a diffeomorphism proves that $\bar{U}$ is a manifold. Moreover, the canonic applications

$$\alpha^1 : S^{n-1} \times S^1 \to \bar{U}, \qquad \beta^1 : \bigcup_{k,l=1,-1} \Pi_{(k,l)} \to \bar{U}$$

are diffeomorphic to their image.

Remark 7.

(*a*) $\bar{U}$ comprises $S^{n-1} \times S^1$ and four points

$$p_1 = \beta^1((0, \ldots, 0, 1, 1)),$$
$$p_2 = \beta^1((0, \ldots, 0, 1, -1)),$$
$$p_3 = \beta^1((0, \ldots, 0, -1, 1)),$$
$$p_4 = \beta^1((0, \ldots, 0, -1, -1)),$$

because

$$\Pi_{(k,l)} - (U_1 \cap \Pi_{(k,l)}) = (0, \ldots, 0, k, l), \quad \text{for} \quad k, l = 1, -1,$$

where $S^{n-1} \times S^1$ has the usual differentiable structure.

(*b*) A base of open neighbourhoods of

p_1 is $p_1 \cup (S^{n-1} \times W^1)$ where W^1 is an open interval of S^1 with extremes (0, 1) contained in $y_1 > 0$, $y_2 > 0$.

p_2 is $p_2 \cup (S^{n-1} \times W^2)$ where W^2 is an open interval of S^1 with extremes (0, −1) contained in $y_1 > 0$ and $y_2 < 0$.

p_3 is $p_3 \cup (S^{n-1} \times W^3)$ where W^3 is an open interval of S^1 with extremes (0, 1) contained in $y_1 < 0$ and $y_2 > 0$.

p_4 is $p_4 \cup (S^{n-1} \times W^4)$ where W^4 is an open interval of S^1 with extremes (0, −1) contained in $y_1 < 0$, $y_2 < 0$.

See Fig. 1.

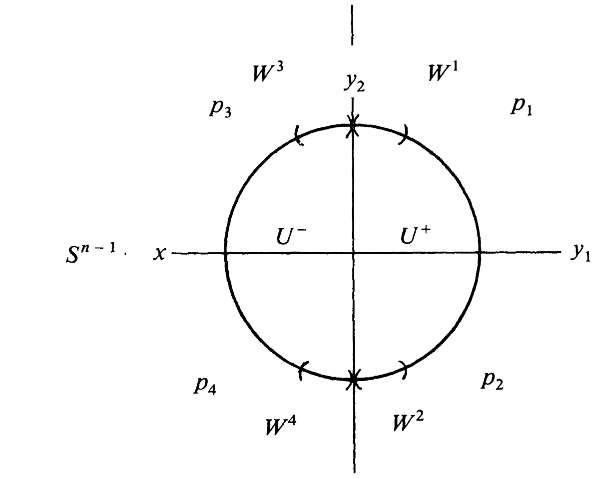

Fig. 1.

In fact, since β^1 is an open map it is sufficient to calculate

$$\beta^1[(B_\epsilon(0) - 0) \times (k, l)]$$

where $B_\epsilon(0)$ is the open ball of $\mathbb{R}^n$, centred at the origin with radius ϵ. This is obtained directly from the diffeomorphism (**) which transforms $(B(0) - 0) \times (k, l)$ into

$$S^{n-1} \times \left\{\left(\frac{k\sum x_i^2}{\sqrt{(\sum x_i^2)^2 + 1}}, \frac{l}{\sqrt{(\sum x_i^2)^2 + 1}}\right) \middle| \sum x_i^2 < \epsilon\right\} = S^{n-1} \times W^i$$

where W^i is an interval in S^1 contained in the quadrant corresponding to the point $\beta^1(0, \ldots, 0, k, l)$, of extremes $(0, l)$.

(*c*) $\alpha'\alpha\colon U_1 \to \bar{U}$ and $\beta'\beta\colon U_1' \to \bar{U}$ constitute the composition of the submersions α, β (Lemmas 8 and 9) with the diffeomorphisms α', β' respectively, $\alpha'\alpha$ and $\beta'\beta$ coincide on $U_1 \cap U_1'$ and define the quotient application $\Pi\colon U \to \bar{U}$, hence Π is a submersion.

(*d*) It should be noted that $\bar{U}$ is a compact, connected, non-Hausdorff manifold because p_1 and the points of $S^{n-1} \times (0, 1)$ do not have disjoint neighbourhoods. The same occurs with p_2 and $S^{n-1} \times (0, -1)$, p_3 and $S^{n-1} \times (0, 1)$ and p_4 and $S^{n-1} \times (0, -1)$.

(*e*) Note that $p \in M_n$ and $f_i(p) = 0$ for all $i = 1, \ldots, n$ is equivalent to $\pi \cdot \varphi(p)$ being equal to some p_i, $(i = 1, 2, 3, 4)$.

(*f*) Finally, $p \in M_n$ and $f(p) = 0$ is equivalent to $\pi \cdot \varphi(p) \in S^{n-1} \times (0, l)$, for $l = 1, -1$.

PROOF OF (*a*), THEOREM 3. From (*e*) of Remark 7, it is deduced that if $H = \emptyset$, $\pi\varphi$ is a local diffeomorphism from M_n onto $S^{n-1} \times S^1$ and consequently it is a covering map. From the classification of covering maps and since M_n is connected and compact, it is deduced that M_n is diffeomorphic to $S^{n-1} \times S^1$.

PROOF OF (*b*), THEOREM 3. For this proof it is necessary to assume that $n > 2$. Let

$$U^+ = S^{n-1} \times \left(\frac{3\pi}{2}, \frac{\pi}{2}\right) \quad \text{and} \quad U^- = S^{n-1} \times \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

(see Fig. 1) be open subsets of $\bar{U}$ and let $\pi\varphi: M_n \to \bar{U}$ be the local diffeomorphism obtained in Remark 5 of Lemma 7.

For the following we shall be using the next lemma, which is easy to prove.

Lemma 10. *Let $\tau: M_n \to M'_n$ be a local diffeomorphism between two compact connected manifolds. Let U be an open set of M'_n satisfying the following conditions*

(*a*) $\tau^{-1}(U) \neq \emptyset$.
(*b*) *For all values of $x \in U$* and *$y \in M'_n$ there are disjoint open neighbourhoods of x and y, U^x, U^y.*

Then if C is a connected component of $\tau^{-1}(U)$, the mapping $\tau_{|C}: C \to U$ is a covering map with a finite number of folds.

Corollary 3. *If C^+ (C^- respectively) is a connected component of $(\pi\varphi)^{-1}(U^+)$ ($(\pi\varphi)^{-1}(U^-)$ respectively), then $\pi \cdot \varphi_{|C^+}: C^+ \to U^+$ ($\pi \cdot \varphi_{|C^-}: C^- \to U^-$ respectively) is a diffeomorphism.*

This follows directly from $n > 2$ and the fact that the open set U^+ (U^- respectively) of the manifold $\bar{U}$ fulfills the conditions of Lemma 10. □

Lemma 11. *Let $p \in M_n$ such that $\pi \cdot \varphi(p) = p_1$ or p_2. There exists a unique connected component C^+ of $(\pi\varphi)^{-1}(U^+)$ such that*

(1) *$p \cup C^+$ is open.*
(2) *$\pi \cdot \varphi_{|p \cup C^+}: p \cup C^+ \to p_1 \cup U^+$ ($p_2 \cup U^+$ respectively) is a diffeomorphism.*

(If $\pi \cdot \varphi(p) = p_3$ or p_4, an analogous statement is obtained by substituting C^+ by C^- and U^+ by U^-.)

PROOF. Let us assume that $\pi\varphi(P) = p_1$ and that W^p is an open connected neighbourhood of p such that $\pi\varphi_{|W^p}: W^p \to V^{p_1}$ is a local diffeomorphism, where

$$V^{p_1} = p_1 \cup \left(S^{n-1} \times \left(\frac{\pi}{2}, \frac{\pi}{2} - \epsilon \right) \right)$$

and $\epsilon > 0$ (see Fig. 1). Since $\pi \cdot \varphi(W^p - p) \subset U^+$ and $W^p - p$ is connected, there is a unique connected component C^+ of $(\pi \cdot \varphi)^{-1}(U^+)$ such that $W^p - p \subset C^+$. Hence, $p \cup C^+$ is open.

Part 2 of Lemma 11 results from Corollary 3. □

Let us suppose that the function f appearing in the global expression for the metric m, satisfies $f(p) \neq 0$ for every $p \in M_n$ (this is the case of the metric on S^n in (*b*) of Section 3 where $f \equiv 1$). If $f(p) > 0$ for every $p \in M_n$ as $\pi\varphi(M_n)$ is compact, it follows from Lemma 11 that there will be points $p, q \in M_n$ such that

(1) $p \cup C^+ \cup q$ is open in M_n.
(2) $\pi \cdot \varphi_{|p \cup C^+ \cup q}: p \cup C^+ \cup q \to p_1 \cup U^+ \cup p_2$ is a diffeomorphism.

Since $p_1 \cup U^+ \cup p_2$ is diffeomorphic to the sphere S^n (see Fig. 1) and M_n is compact and connected, we conclude that M_n and S^n are diffeomorphic. Likewise, if $f(p) < 0$ for every $p \in M_n$. This proves Theorem 3(*b*) in the case where $f(p) \neq 0$ for every $p \in M_n$.

The metrics constructed on S^n in (*c*) of Section 3 have the property $\{p \in S^n \,|\, f(p) = 0\} \neq \varnothing$.

From the global expression for the metric in Theorem 3, we conclude that $df(p) \neq 0$ in the case $f(p) = 0$, hence $f^{-1}(0)$ is the union of a finite number of $(n-1)$-dimension compact manifolds. Moeover,

$$f^{-1}(0) = (\pi \cdot \varphi)^{-1}\left(S^{n-1} \times \frac{\pi}{2} \cup S^{n-1} \times \frac{3\pi}{2} \right)$$

(see Fig. 1).

We will denote by $M^{\pi/2}_{n-1}$ a connected component of $(\pi\varphi)^{-1}\left(S^{n-1} \times \left\{\frac{\pi}{2}\right\}\right)$ and by $M^{3\pi/2}_{n-1}$ a connected component of $(\Pi\varphi)^{-1}\left(S^{n-1} \times \left\{\frac{3\pi}{2}\right\}\right)$.

Lemma 12.

$$\pi\varphi_{|M^{\pi/2}_{n-1}}: M^{\pi/2}_{n-1} \to S^{n-1} \times \frac{\pi}{2}$$

and

$$\pi\varphi_{|M^{3\pi/2}_{n-1}}: M^{3\pi/2}_{n-1} \to S^{n-1} \times \frac{3\pi\ \pi}{2}$$

are diffeomorphisms.

Moreover, for each $M^{\pi/2}_{n-1}$ *(*$M^{3\pi/2}_{n-1}$ *respectively) there are unique connected components* C^+ *and* C^- *of* $(\pi \cdot \varphi)^{-1}(U^+)$ *and* $(\pi \cdot \varphi)^{-1}(U^-)$*, such that*

$$\pi \cdot \varphi_{|C^- \cup M^{\pi/2}_{n-1} \cup C^+}: C^- \cup M^{\pi/2}_{n-1} \cup C^+ \to U^- \cup \left(S^{n-1} \times \frac{\pi}{2}\right) \cup U^+$$

$$\left(C^+ \cup M^{3\pi/2}_{n-1} \cup C^- \to U^+ \cup \left(S^{n-1} \times \frac{3\pi}{2}\right) \cup U^- \ \textit{respectively}\right)$$

is a diffeomorphism.

PROOF. The first statement of this lemma results from $\pi\varphi$ being a local diffeomorphism and from $n > 2$.

The second statement follows from Lemma 11 and the compactness of $M^{\pi/2}_{n-1}$ ($M^{3\pi/2}_{n-1}$ respectively). □

Remark 8. If $p \in C^+$ (C^- respectively) we have the relations

$$\sum_{i=1}^{n} f_i^2(p) \neq 0 \quad \text{and} \quad f(p) \neq 0,$$

(see Fig. 1). Hence the closure $\bar{C}^+$ ($\bar{C}^-$ respectively), according to Lemmas 11 and 12, is one and only one of the following sets:

$$p \cup C^+ \cup q; \qquad p \cup C^+ \cup M^{3\pi/2}_{n-1}; \qquad M^{\pi/2}_{n-1} \cup C^+ \cup M^{3\pi/2}_{n-1}$$
$$(r \cup C^- \cup s;\ r \cup C^- \cup M^{\pi/2}_{n-1};\ M^{\pi/2}_{n-1} \cup C^- \cup M^{3\pi/2}_{n-1} \text{ respectively})$$

where $\pi\varphi(p) = p_1$, $\pi\varphi(q) = p_2$ ($\pi\varphi(r) = p_4$, $\pi\varphi(s) = p_3$ respectively).

The final part of the proof of *b*) in Theorem 3 now follows from previous lemmas.

Let us assume that $\pi\varphi(p) = p_1$ and let C^+ be the unique connected component of $(\pi\varphi)^{-1}(U^+)$ such that $p \cup C^+$ is open and $\pi\varphi_{|p\cup C^+}: p \cup C^+ \to p_1 \cup U^+$ is a diffeomorphism (Lemma 9). In the closure of C^+ there can be either points or spheres (Remark 8). If there are only points in $\bar{C}^+$, then $\bar{C}^+ = p \cup C^+ \cup q$. This case has already been considered and hence M_n is diffeomorphic to S^n.

Let us assume that $\bar{C}^+ = p \cup C^+ \cup M^{3\pi/2}_{n-1}$ where

$$\pi\varphi_{|p\cup C^+ \cup M^{3\pi/2}_{n-1}}: p \cup C^+ \cup M^{3\pi/2}_{n-1} \to p_1 \cup U^+ \cup \left(S^{n-1} \times \frac{3\pi}{2}\right)$$

is a diffeomorphism and let C^- be the unique component of $(\pi\varphi)^{-1}(U^-)$ (Lemma 10) such that

(*a*) $C^+ \cup M_{n-1}^{3\pi/2} \cup C^-$ is open

(*b*) $\pi \cdot \varphi_{|C^+ \cup M_{n-1}^{3\pi/2} \cup C^-}: C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \to U^+ \cup \left(S^{n-1} + \frac{3\pi}{2}\right) \cup U^-$ is a diffeomorphism.

If

$$\bar{C}^- = M_{n-1}^{3\pi/2} \cup C^- \cup q,$$

all previous diffeomorphisms produce a single diffeomorphism

$$\pi\varphi: p \cup C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \cup q \to p_1 \cup U^+ \cup \left(S^{n-1} \times \frac{3\pi}{2}\right) \cup U^- \cup p_3$$

(see Fig. 1) where the second member is a manifold diffeomorphic to the sphere S^n. As M_n is connected it is diffeomorphic to S^n.

If

$$\bar{C}^- = M_{n-1}^{3\pi/2} \cup C^- \cup M_{n-1}^{\pi/2},$$

Lemma 12 proves that here is a component C^{+1} of $(\pi\varphi)^{-1}(U^+)$ such that $C^- \cup M_{n-1}^{\pi/2} \cup C^{+1}$ is open and

$$\pi\varphi_{|C^- \cup M_{n-1}^{\pi/2} \cup C^{+1}}: C^- \cup M_{n-1}^{\pi/2} \cup C^{+1} \to U^- \cup \left(S^{n-1} \times \frac{\pi}{2}\right) \cup U^{+1}$$

is a diffeomorphism ($U^{+1} = U^+$ but it is now diffeomorphic to C^{+1}).

Let us assume that $\bar{C}^{+1} = M_{n-1}^{\pi/2} \cup C^{+1} \cup q$ where $\pi\varphi(q) = p_2$. We have the diffeomorphisms

$$p \cup C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \xrightarrow{\pi\varphi} p_1 \cup U^+ \cup \left(S^{n-1} \times \frac{3\pi}{2}\right) \cup U^- = U_1.$$

$$C^- \cup M_{n-1}^{\pi/2} \cup C^{+1} \cup q \xrightarrow{\pi\varphi} U^- \cup \left(S^{n-1} \times \frac{\pi}{2}\right) \cup U^{+1} \cup p_2 = U_2.$$

By pasting the manifolds U_1 and U_2 through the identification of $U^- \subset U_1$ with $U^- \subset U_2$, the sphere S^n is obtained and consequently $\pi\varphi$ is a diffeomorphism

$$\pi\varphi \quad p \cup C^+ \cup M_{n-1}^{3\pi/2} \cup C^- \cup M_{n-1}^{\pi/2} C^{+1} \cup q \to S^n.$$

Because the number of C^+ and C^- components is finite, the above process would use up these components and by Lemmas 9 and 10 would end in a single point. As M_n is connected, it would be diffeomorphic to S^n.

The proof of the theorem is complete. □

References

[1] Gonzalo, J. Classification results for contact forms, *Indag. Math.* **48**(1986), 289-312.
[2] Gonzalo, J., Varela, F. Some surfaces that are distinguishable by the global expression of their volume elements, *Math. Ann.* **267** (1984), 199-212.
[3] Gromov, M. Partial Differential Relations, Springer Verlag, Heildelberg, 1986.
[4] Varela, F. Formes de Pfaff, classe et perturbations, *Annales Inst. Fourier*, XXVI, **4**(1976), 239-271.

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Compact Hypersurfaces with Constant Higher Order Mean Curvatures

Antonio Ros

A fundamental question about hypersurfaces in the Euclidean space is to decide if the sphere is the only compact hypersurface (embedded or immersed) with constant higher order mean curvature H_r, for some $r = 1, \ldots, n$.

If the hypersurface M^n is star-shaped, Hsiung [3] solved afirmatively the problem for any r. In particular if the Gauss-Kronecker curvature H_n is constant, then M^n is a sphere, because in this case the Hadamard theorem asserts that M^n is convex. The convex case was studied previously by Liebmann [5] and Süss [9]. If the mean curvature H_1 is constant and M^n is embedded, Aleksandrov [1] proved that M^n is a sphere. In the immersed case Hsiang, Teng and Yu [4] and Wente [10] constructed non-spherical examples in higher dimension and in $\mathbb{R}^3$ respectively. If the scalar curvature H_2 is constant and the hypersurface is embedded we proved in [8] that it must be a sphere. In this paper we extend this result to higher order mean curvatures. In particular we prove that

«*The sphere is the only embedded compact hypersurface in the Euclidean space with H_r constant for some $r = 1, \ldots, n$.*»

In this paper we use as in [8] a method of Reilly [7]. Recently with S. Montiel [6] we obtained a different proof of the above theorem. Another proof has been published by N. Korevaar [11].

1. Preliminaries

Let $\psi: M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface immersed in the Euclidean space. Let N be an unit normal vector field on M, σ the second fundamental form of M with respect to N and k_i, $i = 1, \dots, n$ the principal curvatures of M. For any $r = 1, \dots, n$ we define the *mean curvature of order r*, H_r, by the identity

$$(1) \qquad (1 + tk_1) \cdots (1 + tk_n) = 1 + \binom{n}{1} H_1 t + \binom{n}{2} H_2 t^2 + \cdots + \binom{n}{n} H_n t^n$$

for any real number t. For instance, H_1 is simply the mean curvature $H_1 = H = (k_1 + \cdots + k_n)/n$. H_2 is, up a constant, the scalar curvature and $H_n = k_1 k_2 \dots k_n$ is the Gauss-Kronecker curvature. From the Gauss equation we have that if r is even, then H_r is an intrinsic invariant of M. Note that for the unit sphere, and with respect to the unit inner normal, we have $H_r = 1$ for any r. By convenience we put $H_0 = 1$. These curvatures satisfy a basic relation in global hypersurface theory, which is stated in the following lemma.

Lemma (Minkowski Formulae [3]). *Let $\psi: M^n \to \mathbb{R}^{n+1}$ be a compact orientable hypersurface immersed in the Euclidean space. Then for any $r = 1, \dots, n$ we have*

$$(2) \qquad \int_M H_{r-1}\, dA + \int_M H_r \langle \psi, N \rangle\, dA = 0.$$

Let Ω^{n+1} be a compact Riemannian manifold with smooth boundary $M^n = \partial\Omega$. Let dV and dA be the canonical measures on Ω and M respectively and V and A the volumen of Ω and the area of M. Given f in $C^\infty(\bar{\Omega})$ we denote $z = f_{|M}$ and $u = \partial f/\partial N$, where N is the inner unit normal on M. Reilly's formula [7] states that

$$(3) \qquad \int_\Omega [(\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \operatorname{Ric}(\bar{\nabla} f, \bar{\nabla} f)]\, dV = \int_M [-2(\Delta z)u + nHu^2 + \sigma(\nabla z, \nabla z)]\, dA,$$

$\bar{\nabla} f$, $\bar{\Delta} f$ and $\bar{\nabla}^2 f$ being the gradient, the Laplacian and the Hessian of f in Ω, Ric the Ricci cuvature of Ω, ∇z and Δz the gradient and the Laplacian of z in M, and σ and H the second fundamental form and the mean curvature of M with respect to N.

If M is a compact hypersurface embedded in $\mathbb{R}^{n+1}$, then M is the boundary of a compact domain $\Omega \subset \mathbb{R}^{n+1}$. So if x denotes the position vector in $\mathbb{R}^{n+1}$, then $\bar{\Delta}|x|^2 = 2(n+1)$, and from the divergence theorem we have

$$(4) \qquad (n+1)V + \int_M \langle \psi, N \rangle\, dA = 0,$$

ψ being the immersion of M in $\mathbb{R}^{n+1}$.

2. An inequality

For the next result we will follow closely the ideas of Reilly [7].

Theorem 1. *Let Ω^{n+1} be a compact Riemannian manifold with smooth boundary M and non-negative Ricci curvature. Let H be the mean curvature of M. If H is positive everywhere, then*

$$\int_M \frac{1}{H} dA \geqq (n+1)V. \tag{5}$$

The equality holds if and only if Ω is isometric to an Euclidean ball.

PROOF. Let f in $C^\infty(\bar{\Omega})$ be the solution of the Dirichlet problem

$$\begin{cases} \bar{\Delta} f = 1 \text{ in } \Omega, \\ z = 0 \quad \text{on} \quad M. \end{cases}$$

From the divergence theorem we have

$$V = \int_\Omega \bar{\Delta} f \, dV = -\int_M u \, dA. \tag{6}$$

Combining Schwarz inequality $(\bar{\Delta} f)^2 \leqslant (n+1)|\bar{\nabla}^2 f|^2$ with (3) we obtain

$$\frac{V}{n+1} \geqq \int_M H u^2 \, dA. \tag{7}$$

Finally, from (6), Schwarz inequality and (7) it follows that

$$\begin{aligned} V^2 &= \left(\int_M u \, dA \right)^2 = \left(\int_M (u H^{1/2}) H^{-1/2} \, dA \right)^2 \\ &\leqq \int_M u^2 H \, dA \int_M H^{-1} \, dA \\ &\leqq \frac{V}{n+1} \int_M \frac{1}{H} dA, \end{aligned}$$

and we have proved inequality (5).

If equality holds, then $\bar{\nabla}^2 f$ is proportional to the metric everywhere. As $\bar{\Delta} f = 1$, we conclude that

$$\bar{\nabla}^2 f = \frac{1}{n+1} \langle \, , \rangle. \tag{8}$$

Deriving covariantly we obtain $\bar{\nabla}^3 f = 0$ and from the Ricci-identity,

$$R(X, Y)\bar{\nabla} f = 0, \tag{9}$$

for any X, Y tangent vector to Ω, where R is the curvature of Ω. From the maximum principle f attains its minimum at some point x_0 in the interior of Ω. From (8) it follows that

$$\bar{\nabla} f = \frac{1}{n+1} r \frac{\partial}{\partial r},$$

where r is the distance to the point x_0, which combined with (9), Cartan's Theorem and the fact that f vanishes at the boundary of Ω implies that Ω is an Euclidean ball whose center is x_0, and f is given by

$$f(x) = [2(n+1)]^{-1}(|x - x_0|^2 - a^2)$$

in Ω, a being the radius of the ball.

3. Hypersurfaces with H_r Constant

In this section we prove the main result of this paper. Given $k = 1, \ldots, n$ we consider the function $\sigma_k \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$\sigma_k(x_1, \ldots, x_n)$ = elementary symmetric polynomial of degree k in the variables $x_1, \ldots, x_n$.

Thus $H_k = \sigma_k(k_1, \ldots, k_n)$. We denote by C_k the connected component of the set $\{x \in \mathbb{R}^n / \sigma_k(x) > 0\}$ which contains the vector $(1, \ldots, 1)$. From Gärding [2] we have that if $k < r$, then $C_k \supset C_r$. Moreover if $x \in C_r$ we have for $k \leqq r$ that $\sigma_k(x)^{(k-1)/k} \leqq \sigma_{k-1}(x)$. For $k \geqq 2$ equality holds if and only if x is proportional to $(1, \ldots, 1)$. Clearly if $x_i > 0$ for any i, then $(x_1, \ldots, x_n) \in C_k$.

Theorem 2. *Let M^n be a compact hypersurface embedded in the Euclidean space $\mathbb{R}^{n+1}$. If H_r is constant for some $r = 1, \ldots, n$, then M^n is a sphere.*

PROOF. As M has an elliptic point, H_r must be a positive constant. As the principal curvatures are continuous functions we have that $(k_1, \ldots, k_n) \in C_r$ at any point. Hence $(k_1, \ldots, k_n) \in C_k$ for k smaller than r. In particular $H_k > 0$ for $k < r$. Moreover

$$H_k^{(k-1)/k} \leqq H_{k-1} \qquad k = 1, \ldots, r, \tag{10}$$

As consequence

$$H_r^{1/r} \leqq H \quad \text{in} \quad M. \tag{11}$$

Now we use Minkowski formulae and (3):

$$\begin{aligned} 0 &= \int_M H_{r-1}\,dA + \int_M H_r\langle\psi, N\rangle\,dA \\ &= \int_M H_{r-1}\,dA + H_r\int_M\langle\psi, N\rangle\,dA \\ &= \int_M H_{r-1}\,dA - (n+1)H_r V. \end{aligned}$$

Combining this equality with (10) we have

$$(n+1)H_r V = \int_M H_{r-1}\,dA \geqslant AH_r^{(r-1)/r}$$

and so

$$H_r^{1/r} \geqq \frac{A}{(n+1)V}. \tag{12}$$

On the other hand from (5) and (11) we obtain

$$(n+1)V \leqq \int_M \frac{dA}{H} \leqq AH_r^{-1/r},$$

which is

$$H_r^{1/r} \leqq \frac{A}{(n+1)V}, \tag{13}$$

and the equality holds if and only if M is a sphere. The theorem follows from (12) and (13).

4. An Extensión of the Aleksandrov Theorem

First we observe that a compact hypersurface embedded in the Euclidean space is a critical point of the isoperimetric functional if and only if it has constant mean curvature.

Theorem 3. *Let $\bar{M}^{n+1}$ be a Riemannian manifold with non-negative Ricci curvature, and let Ω be a compact domain in $\bar{M}$ with smooth boundary. If Ω is a critical point of the isoperimetric functional*

$$\Omega \to \frac{A(\partial\Omega)^{n+1}}{V(\Omega)^n},$$

then Ω is isometric to an Euclidean ball.

PROOF. Given a smooth function f on $\partial\Omega$, we consider the normal variation of $\partial\Omega$ defined by $\psi_t: \partial\Omega \to \bar{M}$, $\psi_t(p) = \text{Exp}_p(-tf(p)N(p))$, where Exp is the exponential map of $\bar{M}$. ψ_t determine a variation of Ω, Ω_t for $|t| < \epsilon$. We put $V(t) = V(\Omega_t)$ and $A(t) = A(\partial\Omega_t)$. The first variation formulae of the functionals above are given by

$$A'(0) = n \int_{\partial\Omega} fH\,dA,$$
$$V'(0) = \int_{\partial\Omega} f\,dA.$$

By hypothesis we have

$$\frac{d}{dt}\bigg|_{t=0} \frac{A(t)^{n+1}}{V(t)^n} = 0,$$

or equivalently

$$\int_{\partial\Omega} f[(n+1)VH - A]\,dA = 0, \quad \text{for any} \quad f.$$

Then $H = A/(n+1)V$ and we have equality in (5). Therefore Ω is isometric to an Euclidean ball.

Remark. Let $\psi: M^n \to \mathbb{R}^{n+1}$ be an *immersed* compact hypersurface. Suppose that M is the boundary of a certain manifold Ω^{n+1}, and that the immersion ψ extends to an immersion of Ω, $\bar{\psi}: \Omega^{n+1} \to \mathbb{R}^{n+1}$. It is easy to see that Aleksandrov proof [1] can be adapted to this situation: If M^n has constant mean curvature, it must be a sphere.

U. Pinkall pointed out to me that Reilly's method can also be used in this case. In fact, taking on Ω the pull-back of the Euclidean metric, inequality (5) remains true and the same holds for identity (3). On the other hand, Minkowski formulae hold for any immersed hypersurface. So theorem 2 extends to the above type of immersed hypersurfaces.

References

[1] Aleksandrov, A. D. Uniqueness theorems for surfaces in the large, *Vestnik Leningrad Univ.* **13**(1958), 5-8.

[2] Gärding, L. An inequality for hyperbolic polynomials, *J. Math. Mech.* **8**(1959), 957-965.

[3] Hsiung, C. C. Some integral formulas for closed hypersurfaces, *Math. Scand.* **2**(1954), 286-294.

[4] Hsiang, W. Y., Teng, Z. H. and Yu New examples of constant mean curvature immersion of $(2k-1)$-spheres into Euclidean $2k$-space. *Ann. of Math.* **117**(1983), 609-625.

[5] Liebmann, H. Eine neue Eigenschaft der Kugel, *Nachr. Kgl. Ges. Wiss. Göttingen Math.-Phys. Klasse,* (1899), 44-55.
[6] Montiel, S., Ros, A. In preparation.
[7] Reilly, R. Applications of the Hessian operator in a Riemannian manifold, *Indiana Univ. Math. J.* **26**(1977), 459-472.
[8] Ros, A. Compact hypersurfaces with constant scalar curvature and a congruence theorem, *J. Diff. Geom.* **27**(1988), 215-220.
[9] Süss, W. Uber kennzeichnungen der Kugeln un affinsphären durch Herrn K. P. Grotemeyer, *Arch. Math.* **3**(1952), 311-313.
[10] Wente, H. C. Counterexample to a conjecture of H. Hopf, *Pac. J. Math.* **121**(1986), 193-243.
[11] Korevaar, N. J. Sphere theorems via Aleksandrov for constant Weingarten curvature hypersurfaces—Appendix to a note of A. Ros, *J. Diff. Geom.* **27**(1988), 221-223.

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Oblique Derivative Problems for the Laplacian in Lipschitz Domains

Jill Pipher

Introduction

The aim of this paper is to extend the results of Calderón [1] and Kenig-Pipher [12] on solutions to the oblique derivative problem to the case where the data is assumed to be BMO or Hölder continuous. Suppose D is a bounded Lipschitz domain in $\mathbb{R}^n$, $N(Q)$ is the unit normal at a point Q on the boundary of D and $V(Q)$ is a continuous bounded vector field defined on the boundary of D such that for some constant c, $\langle V, N \rangle \geqslant c > 0$. At each point $Q \in \partial D$, the cone $\Gamma(Q) \subseteq D$ is defined by

$$\Gamma(Q) = \{P \in D: |P - Q| \leqslant (1 + \alpha)\,\mathrm{dist}\,(P, \partial D)\} \quad \text{for some} \quad \alpha > 0.$$

For $v \in C^2(D)$, one can define the nontangential maximal function of v and the square function of v by

$$Nv(Q) = \sup_{X \in \Gamma(Q)} |v(X)|$$

$$S^2 v(Q) = \int_{\Gamma(Q)} d(X)^{2-n} |\nabla v(X)|^2 \, dX$$

where $d(X) = \mathrm{dist}\,(X, \partial D)$ is comparable to $|X - Q|$ for $X \in \Gamma(Q)$. Consider the boundary value problem

$$(1.1) \qquad \begin{cases} \Delta u = 0 & \text{in} \quad D \\ V \cdot \nabla u = g & \text{on} \quad \partial D \end{cases}$$

with $g \in L^p(\partial D)$. Because V is continuous and never tangent to the boundary of D, $V \cdot \nabla u$ may be regarded (locally) as a perturbation of $\partial u/\partial y$ and the following results are known.

Theorem A (Calderón [1]). *There exists a solution* $u(x)$ *to* (1.1) *with*

$$\|N(\nabla u)\|_{L^p} \leqslant \|g\|_{L^p} \quad \text{for} \quad 2 - \epsilon < p < 2 + \epsilon$$

where ϵ *depends on the domain and* g *is assumed to satisfy finitely many linear conditions.*

Theorem B (Kenig-Pipher [12]). *If* $g \in L^p(\partial D)$ *for* $2 < p < \infty$ *and satisfies finitely many linear conditions, then the solution* $u(x)$ *to* (1.1) *satisfies*

$$\|N(\nabla u)\|_{L^p} \lesssim \|g\|_{L^p}.$$

This last theorem indicates that the oblique derivative problem (1.1) behaves more like the Dirichlet problem than the Neumann problem on a Lipschitz domain. (Indeed, only where the domain is C^1 will (1.1) contain the Neumann problem as a special case.) In light of the results of Fefferman-Stein [6] for harmonic functions in $\mathbb{R}^n_+$ and the work of Fabes-Neri [7] on the Dirichlet problem with BMO data on Lipschitz domains, it is natural to ask for BMO solutions to the oblique derivative problem for data in BMO. This problem is addressed in section 2. Again in analogy with the Dirichlet problem it is of interest to consider behavior of solutions to this problem when the data g is assumed to be Hölder continuous and this result is formulated in section 3. In both cases, the method of proof closely follows that of [12], hence some aspects of the proof presented here are deliberately brief.

I would like to thank Robert Fefferman for queries and conversations which led to this work and I thank Carlos Kenig for several helpful discussions while this work was in progress.

2. The Boundary Value Problem (1.1) with $g \in \mathrm{BMO}(\partial D)$

If $g \in L^1(\partial D)$ then g belongs to $\mathrm{BMO}(\partial D)$ if there exists a constant C such that

$$\sup_{\Delta} \frac{1}{\sigma(\Delta)} \int_{\Delta} |g - g_{\Delta}| \, d\sigma < C, \tag{2.1}$$

where Δ is a surface ball contained in ∂D, $d\sigma$ is surface measure on ∂D and

$$g_{\Delta} = \frac{1}{\sigma(\Delta)} \int_{\Delta} g \, d\sigma.$$

The norm of g in this space is

$$\|g\|_* = \left| \int_{\partial D} g\, d\sigma \right| + \inf \{ C\colon (2.1) \text{ holds for } C \}.$$

Equivalent definitions (and norms) are given by the L^p conditions for $1 < p < \infty$:

$$\sup_{\Delta} \left\{ \frac{1}{\sigma(\Delta)} \int_{\Delta} |g - g_{\Delta}|^p \, d\sigma \right\}^{1/p} < \infty.$$

(See John-Nirenberg [10].) Our question is the following: when can (1.1) be solved with $g \in \mathrm{BMO}(\partial D)$ to yield a solution u with $\nabla u \in \mathrm{BMO}$, *i.e.*, all derivatives of u belong to BMO?

Consider first the following simple problem in the upper half plane. Let $V = (0, a(x))$ with $c \leqslant a \leqslant 1$ and suppose $g \in \mathrm{BMO}(\mathbb{R}, dx)$. Then $V \cdot \nabla u = g$ on $\mathbb{R}$ with u harmonic in $\mathbb{R}^2_+$ is the same as $\partial u/\partial y = a^{-1} g$ and one expects a to satisfy a further condition to insure that u_y belong to BMO, namely that a^{-1} be a BMO multiplier. Since $a \in L^\infty$ this is equivalent to demanding that a itself be a BMO multiplier (that is, for all $g \in \mathrm{BMO}$, $ag \in \mathrm{BMO}$). Necessary and sufficient conditions for a function to be a BMO multiplier were found by Stegenga [14] and we impose this additional condition on the components V_i of V. However, we require slightly more smoothness on V which is a VMO version of this multiplier condition (see Sarason [13] for the properties of VMO functions) and from now on each V_i will satisfy (2.2). Given $\epsilon > 0$, there exists $\delta > 0$ such that if the radius $r(\Delta)$ of $\Delta \subseteq \partial D$ is less than δ, then

$$\log\left(\frac{1}{r(\Delta)}\right) \int_{\Delta} |V_i - (V_i)_{\Delta}| \frac{d\sigma}{\sigma(\Delta)} < \epsilon. \tag{2.2}$$

Further comments about this VMO requirement will be made at the end of this section but it should be noted that continuity of V does not imply (2.2) (see S. Janson [8]). When D is a C^1 domain, the boundary value problem (1.1) could be taken to be the Neumann problem. In this case however it is possible to construct a C^1domain in $\mathbb{R}^2$ for which there exists a harmonic u with $\partial u/\partial n$ in BMO but $\nabla_T u$ (the tangential derivative) not exponentially integrable, and hence not in BMO. Construction of such a C^1 domain is essentially given in Kenig [11].

The main result for $g \in \mathrm{BMO}(\partial D)$ will be formulated in terms of the Fefferman-Stein «sharp» function ([6]). For $f(Q)$ defined on ∂D, set

$$T^{\#,q}(f)(Q) = \sup_{\Delta \ni Q} \left\{ \frac{1}{\sigma(\Delta)} \int_{\Delta} |f(Q) - f_{\Delta}|^q \, d\sigma \right\}^{1/q}$$

for $1 \leqslant q < \infty$. Then a function f belongs to $\mathrm{BMO}(\partial D)$ if, for some q, $T^{\#,q}(f)$ belongs to L^∞, and if f belongs to L^p for $p > q$, $\|T^{\#,q}(f)\|_{L^p} \lesssim \|f\|_{L^p}$.

Theorem 2.3. *Suppose $V = (V_1, \ldots, V_n)$ is continuous, $(V, N) \geqslant c$ and V has property* (2.2), *and suppose $g \in \mathrm{BMO}(\partial D)$ and satisfies finitely many linear conditions. Let u be the harmonic solution to $V \cdot \nabla u = g$ on ∂D given by Theorem A. Then, given $\epsilon > 0$, there exists C_ϵ such that for a.e. $Q \in \partial D$,*

$$T^{\#,2}(\nabla u)(Q) < C \cdot \epsilon (M_\sigma[T^{\#,4}(\nabla u)]^2(Q))^{1/2} + C_\epsilon, \tag{2.4}$$

where M_σ is the Hardy-Littlewood maximal function on ∂D with respect to $d\sigma$.

The expression $T^{\#,2}(\nabla u)$ denotes $\sum_i T^{\#,2}(D_i u)$ and this sort of abbreviation will appear in other contexts later on. The constant C_ϵ in (2.4) depends on everything: the Lipschitz character of D, the modulus of continuity of V, the constant in (2.2), the diameter of D, the transversality constant c and the apertures of cones of square functions appearing in the proof of the theorem. The desired property of $|\nabla u|$ follows immediately.

Corollary 2.4. *All derivatives of u belong to* BMO (∂D).

PROOF. By Stromberg ([16], Lemma 3.7), $\| T^{\#,4}(\nabla u)\|_{L^p} \leqslant C \| T^{\#,2}(\nabla u)\|_{L^p}$ with a constant independent of p for $p > 4$. By Theorem B, $\| T^{\#,2}(\nabla u)\|_p$ is finite for large p and hence if ϵ is sufficiently small, $\| T^{\#,2}(\nabla u)\|_{L^p} \leqslant C_\epsilon$. Letting p tend to ∞ gives $T^{\#,2}(\nabla u) \in L^\infty$. □

The proof of Theorem 2.3 consists of a series of lemmas. The first of these quantifies the intuition that $V \cdot \nabla u$ is locally a perturbation of $\partial u / \partial y$. We fix a regular family of cones (see Dahlberg [3]) $\Gamma(Q) \subset \bar{\Gamma}(Q) \subseteq \bar{\bar{\Gamma}}$ and $\bar{S}v$ will denote the square function taken with respect to $\bar{\Gamma}(Q)$. The cone $\Gamma_h(Q)$ (or $\bar{\Gamma}_h(Q)$) is the truncated cone $\Gamma(Q) \cap \{X \in D: |X - Q| < h\}$ and $S_h v$ (or $\bar{S}_h v$) denotes the square function with integration taken over $\Gamma_h(Q)$ (or $\bar{\Gamma}_h(Q)$). Let $Hu(X)$ denote the Hessian of u and set

$$S^2(\nabla u)(Q) = \int_{\Gamma(Q)} d(X)^{1-n} |Hu(X)|^2 \, dX.$$

Lemma 2.5. (Stein [15], p.213). *Suppose $S_h(\nabla u)(Q_0) < 0$ and choose coordinates so that B is a neighborhood of $Q_0 = (0, \ldots, 0)$ with $D \cap B = \{y: y > \Phi(X)\}$ for Φ Lipschitz. Then there exists a constant C such that*

$$S_h(\nabla u)(Q_0) \leqslant C \bar{S}_{\bar{h}}\left(\frac{\partial u}{\partial y}\right)(Q_0) + Ch^2 |\nabla u(0, h)|^2$$

for $\bar{h} > h$.

The following two lemmas will be needed to obtain $T^{\#}$ estimates from square function estimates, and vice versa.

Lemma 2.6. *If $\Delta = \Delta(Q_0, r_0)$ is a surface ball contained in ∂D, let*

$$T(\Delta) = \{X \in D: |X - Q_0| < r_0\}$$

be the Carleson region associated to Δ. Then, if $d(X) = \operatorname{dist}(X, \partial D)$,

$$\frac{1}{\sigma(\Delta)} \int_\Delta |f - f_\Delta|^2 \, d\sigma(Q) \lesssim \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X) |\nabla f|^2 \, dX$$

for f harmonic in D and $f \in L^2(\partial D)$.

PROOF. If Ω is a Lipschitz domain, let $S_\Omega f$ denote the square function of f with respect to the domain Ω. Then

$$\begin{aligned}\int_{T(\Delta)} d(X)|\nabla f|^2 \, dX &\geqslant \int_{T(\Delta)} \operatorname{dist}(X, \partial T(\Delta))^{2-n} |\nabla f|^2 \, dX \\ &\approx \int_{\partial T(\Delta)} S^2_{T(\Delta)}(f)(Q) \, d\sigma(Q) \\ &\geqslant \int_\Delta |f(Q) - f_\Delta|^2 \, d\sigma(Q)\end{aligned}$$

by Dahlberg's area integral theorem [3]. □

Lemma 2.7. *If $f \in L^2(\partial D)$ and $\Delta f = 0$ in D, $\Delta = \Delta(Q_0, r_0)$, then*

$$\left\{\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X)|\nabla f|^2 \, dX\right\} \lesssim T^{\#,2}(f)(Q_0).$$

PROOF. Let

$$f_1(X) = \int_{\Delta(Q_0, 2r_0)} (f(Q) - f_\Delta) \, d\omega^X(Q) \quad \text{and} \quad f_2(X) = f(X) - f_1(X)$$

so that

$$\nabla f = \nabla f_1 + \nabla f_2.$$

Then

$$\begin{aligned}\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X)|\nabla f_1|^2 \, dX &\lesssim \frac{1}{\sigma(\Delta)} \int S^2 f_1(Q) \, d\sigma(Q) \\ &\leqslant \frac{1}{\sigma(\Delta)} \int_\Delta |f(Q) - f_\Delta|^2 \, d\sigma(Q) \\ &\lesssim (T^{\#,2}(f)(Q_0))^2.\end{aligned}$$

If $G(X)$ denotes the Green's function for D with pole at $P_0 \in D$ then since

$$f_2^*(X) \equiv \int_{c_{\hat{\Delta}}} |f(Q) - f_\Delta| \, d\omega^X(Q)$$

is a positive harmonic function which vanishes on Δ, the comparison theorem yields, for $X \in T(\Delta)$,

$$|\nabla f_2(X)| \lesssim |f_2^*(X)|/d(X) \lesssim G(X)/d(X) \cdot f_2^*(X_h)/G(X_h)$$

where $X_h \in T(\Delta)$ satisfies $\operatorname{dist}(X_h, \partial D) \geqslant ch$. By Hölder continuity of the Green's function, there is some $\alpha > 0$ such that $G(X)/G(X_h) \lesssim (d(X)/h)^\alpha$ (see [9]) and so

$$\frac{1}{\sigma(\Delta)} \int_\Delta d(X)|\nabla f_2(X)|^2\,dX \lesssim \left(\frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d(X)^{2\alpha-1} \cdot h^{-2\alpha}\,dX\right)(f_2^*(X_h))^2$$
$$\leqslant C(f_2^*(X_h))^2.$$

Estimates for $|f_2^*(X_h)|$ as in [5] or [7] show that

$$|f_2^*(X_h)| \leqslant CT^{\#,2}(\nabla u)(Q_0). \qquad \square$$

Fix a $Q_0 \in \partial D$. Then $\langle V(Q_0), N(Q_0)\rangle \geqslant c$ and since $V(Q)$ is continuous there is a neighborhood Δ of Q_0 such that $\langle V(Q), N(Q_0)\rangle \geqslant c/2$ for all $Q \in \Delta$. Given $\epsilon > 0$, choose a coordinate chart for ∂D near Q_0 with neighborhood $\Delta(Q_0, \delta)$ such that $V(Q_0)$ points in the direction of the y-axis and $|V(X) - V(Q)|^2 < \epsilon$ for all $X \in \bigcup_{Q\in\Delta(Q_0,\delta)} \bar\Gamma_\delta(Q)$, where $V_i(X)$ is the harmonic extension of $V_i(Q)$ to D. By Lemma 2.6, choose $h > 0$ so that

$$T^{\#,2}(\nabla u)(Q_0) \leqslant \frac{1}{\sigma(\Delta_h)} \int_{\Delta_h} S_h^2(\nabla u)(Q)\,d\sigma(Q)$$

and we can assume $h < \delta$. By Lemma 2.5 and the continuity of V, for all $Q \in \Delta_h$,

$$S_h^2(\nabla u)(Q) \leqslant \int_{\bar\Gamma_h(Q)} d(X)^{2-n} \sum_j \left|\sum_i V_i(Q) \frac{\partial^2 u}{\partial x_i\,\partial x_j}(X)\right|^2 dX + h^2|\nabla u(X_h)|^2$$

(2.8)

$$\leqslant C_\epsilon \int_{\bar\Gamma_h(Q)} d(X)^{2-n} \sum_j \left|\sum_i V_i(X) \frac{\partial^2 u}{\partial x_j\,\partial x_i}(X)\right|^2 dX + h^2|\nabla u(X_h)|^2.$$

Lemma 2.9. *If $X_h \in T(\Delta(Q_0, h))$ with $\operatorname{dist}(X_h, \partial D) \geqslant ch$, then*

$$|\nabla u(X_h)| \leqslant C\log(1/h)T^{\#,2}(\nabla u)(Q_0) + C.$$

PROOF. Let $\delta > h$ be fixed with $D \cap \{P\colon |P - Q_0| < 2\delta\} = \{(x,y)\colon y > \Phi(x)\}$, a coordinate chart. Then

$$|\nabla u(x, \Phi(x) + h)| \lesssim |\nabla u(x, \Phi(x) + \delta)| + \int_h^\delta |Hu(x, \Phi(x) + r)|\,dr.$$

By Calderón's theorem,

$$|\nabla u(x, \Phi(x) + \delta)| \leqslant C_\delta \int N(\nabla u)\, d\sigma \lesssim C_\delta \|g\|_{L^2} \leqslant C_\delta \|g\|_*.$$

Let $X_r = (x, \Phi(x) + r)$ and $B_r = \{P \in D: |P - X_r| < r/2\}$. The mean value theorem gives

$$\int_h^\delta |Hu(X_r)|\, dr \leqslant \int_h^\delta \left(\frac{1}{r^n}\int_{B_r} |Hu(Z)|^2\, dZ\right)^{1/2} dr$$

$$\leqslant \log\left(\frac{1}{h}\right) \sup_r \left\{\int_{T(\Delta(Q_0, r))} d(X)|Hu(X)|^2\, dX\right\}^{1/2}$$

$$\leqslant c\log\left(\frac{1}{h}\right) T^{\#,2}(\nabla u)(Q_0).$$

by Lemma 2.7. □

Set $v(x) = V(X) \cdot \nabla u(X)$. Inequality (2.8) together with Lemma 2.9 shows that, for $Q \in \Delta_h$,

$$S_h^2(\nabla u)(Q) \lesssim ch^2 \log^2\left(\frac{1}{h}\right)(T^{\#,2}(\nabla u)(Q_0))^2 + \bar{S}_h^2(v)(Q)$$

$$+ \int_{\bar{\Gamma}_h(Q)} d(X)^{2-n} \sum_{i,k} \left|\frac{\partial V_i}{\partial X_k}(X)\frac{\partial u}{\partial X_i}(X)\right|^2 dX + C_\delta.$$

Since $h < \delta$, the first term above will be smaller that $[T^{\#,2}(\nabla u)(Q_0)]^2/2$ for appropriate choice of ϵ and this proves

Lemma 2.10.

$$[T^{\#,2}(\nabla u)(Q_0)]^2 \lesssim C_\delta + \frac{1}{\sigma(\Delta_h)} \int_{\Delta_h} \bar{S}_h^2(v)(Q)\, d\sigma(Q)$$

$$+ \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V|^2 |\nabla u|^2\, dX.$$

Lemma 2.11. *If V satisfies (2.2), then given $\epsilon > 0$, there exists $C = C(\epsilon)$ such that*

$$\left\{\frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2\, dX\right\}^{1/2} \leqslant C + C'\epsilon T^{\#,2}(\nabla u)(Q_0).$$

PROOF. Let $F(X) = \nabla u(X) - \nabla u(X_h)$ where $X_h \in T(\Delta_h)$ with dist $(X_h, \partial D) \approx h$.

As in the argument for Lemma 2.7, split F into two components F_1 and F_2 where

$$F_1(X) = \int_{\Delta(Q_0, Mh)} F(Q)\, d\omega^X(Q) \quad \text{and} \quad F_2 = F - F_1$$

and M is a constant depending on the Lipschitz character of D. Because $d(X)|\nabla V(X)|^2\, dX$ is a Carleson measure and h is small,

$$\frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2 |F_1(X)|^2\, dX \leqslant \|V\|_{\mathrm{BMO}} \int_{\partial D} N^2(F_1) \frac{d\sigma}{\sigma(\Delta_h)}$$

$$< C'\epsilon \int_{\Delta_{Mh}} |\nabla u(Q) - \nabla u(X_h)|^2 \frac{d\sigma(Q)}{\sigma(\Delta_h)}$$

which is bounded by

$$C'\epsilon[T^{\#,2}(\nabla u)(Q_0)]^2$$

as

$$|\nabla u(X_h) - (\nabla u)_\Delta| \leqslant CT^{\#,2}(\nabla u)(Q_0)$$

(see [5]). To handle the term involving $|F_2(X)|$, one uses the usual estimates obtained by comparison with the Green's function and the fact that $|\nabla V(X)| \leqslant \epsilon/d(X)$ since $V \in \mathrm{VMO}$. Hence

$$\left\{ \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2 |F_2(X)|^2\, dX \right\}^{1/2} \leqslant C\epsilon T^{\#,2}(\nabla u)(Q_0)$$

and it remains to bound

$$|\nabla u(X_h)|^2 \cdot \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2\, dX.$$

By Lemma 2.9 and the multiplier condition (2.2) for V (which has an equivalent L^2 expression),

$$|\nabla u(X_h)|^2 \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V(X)|^2\, dX$$

$$\lesssim C_\delta + \log^2\left(\frac{1}{h}\right) \frac{1}{\sigma(\Delta_h)} \int_{T(\Delta_h)} d(X)|\nabla V|^2\, dX [T^{\#,2}(\nabla u)(Q_0)]^2$$

$$\lesssim C_\delta + C\epsilon[T^{\#,2}(\nabla u)(Q_0)]^2. \quad \square$$

By Lemma 2.11 we may choose ϵ sufficiently small depending only on V, D so that

(2.12) $$T^{\#,2}(\nabla u)(Q_0) \lesssim C(\epsilon) + \left\{\frac{1}{\sigma(\Delta_h)} \int_{\Delta_h} \bar{S}_h^2(v)\, d\sigma(Q)\right\}^{1/2}.$$

From now on, we write

$$F(X) = \int_D G(X, Y)\, \Delta v(Y)\, dY$$

for the Green's potential of v. Then if $g(X)$ is the harmonic extension of g to D, we have $v(x) = -F(x) + g(x)$. Because $d(x)|\nabla g(x)|^2\, dx$ is a Carleson measure ([7]), the only term to be controlled involves $\bar{S}_h^2(F)$. The following good-λ inequality is a modification (and simplification) of Lemma 7 of [12] and details of the proof are provided only where they essentially differ from those of [12].

Lemma 2.13. *Let*

$$N_h(F)(Q) = \sup_{X \in \Gamma_h(Q)} |F(X)|$$

be the truncated nontangential maximal function of F. For γ sufficiently small, there exists constants C and η depending on D such that (if $d|\nabla F|$ abbreviates $d(x)|\nabla F(x)|$),

$$\sigma\{Q \in \Delta_h : S_h(F) > 2\lambda, N_h(F) \leqslant \gamma\lambda, N_h(d|\nabla F|) \leqslant \gamma\lambda, S_h(\nabla u) \cdot N_h(F) \leqslant (\gamma\lambda)^2\} \leqslant C\gamma^\eta \sigma\{Q \in \Delta_h : S_h(F) > \lambda\}.$$

The inequality of Lemma 2.13 remains true for $v(x) = V(X) \cdot \nabla u(X)$ replacing $F(X)$ and consequently should be regarded as a good-λ inequality for the product of harmonic extensions of a BMO function and an L^2 *function.*

Corollary 2.13.

$$(T^{\#,2}(\nabla u)(Q_0))^2 \leqslant C + C \int_{\Delta_h} N_h^2(F) + N_h^2(d|\nabla F|) \frac{d\sigma}{\sigma(\Delta_h)}.$$

PROOF. Integrating the good-λ inequality and using (2.12) gives

$$\begin{aligned}(T^{\#,2}(\nabla u)(Q_0))^2 &\leqslant C_\delta + \|g\|_*^2 \\ &+ C \int_{\Delta_h} N_h^2(F) + N_h^2(d|\nabla F|) \frac{d\sigma}{\sigma(\Delta_h)} \\ &+ \int_{\Delta_h} S_h(\nabla u) N_h(F) \frac{d\sigma}{\sigma(\Delta_h)}.\end{aligned}$$

The last term above is bounded by

$$\epsilon_0 \int_{\Delta_h} S_h^2(\nabla u) \frac{d\sigma}{\sigma(\Delta_h)} + C_{\epsilon_0} \int_{\Delta_h} N_h^2(F) \frac{d\sigma}{\sigma(\Delta_h)}$$

which in turn, by Lemma 2.7, is less than

$$\frac{1}{2}(T^{\#,2}(\nabla u)(Q_0))^2 + C \int_{\Delta_h} N_h^2(F) \frac{d\sigma}{\sigma(\Delta_h)}$$

for suitable choice of ϵ_0.

SKETCH OF PROOF OF LEMMA 2.13. Observe that Lemma 7 of [12] was stated in terms of v, not F. However $\Delta F = \Delta v$ and the only important property of $v(x)$ used here is the fact that $|\Delta v| \leqslant |\nabla V|\,|Hu|$.

Let B_j be a Whitney cube of $\{S_h(F) > \lambda\}$ and let

$$F_j = B_j \cap \{S_h(F) > 2\lambda, N_\eta(F) \leqslant \gamma\lambda, N_\eta(d|\nabla F|) \leqslant \gamma\lambda, S_h(\nabla u)N_h(F) \leqslant (\gamma\lambda)^2\}.$$

Let Ω be the sawtooth region over F_j (see [4]) and fix an $X \in \Omega$ away from ∂D for the pole of $G_\Omega(X, Y)$, the Green's function for Ω. If $d\omega_\Omega$ is harmonic measure for Ω evaluated at X, then the estimate $\sigma(F_j) \leqslant C\gamma^n\sigma(B_j)$ follows from the estimate $\omega_\Omega(F_j) \leqslant \gamma^2$ (see [3]). At this point the proof in [12] carries over once one shows that

$$\int_\Omega G_\Omega(X, Y)|F(Y)|\,|\nabla V(Y)|\,|Hu(Y)|\,dY \leqslant (\gamma\lambda)^2.$$

The integral will be estimated first over the region $B_0(X)$, a ball of radius roughly $d(X) = \operatorname{dist}(X, \partial D)$ centered at the pole X of G_Ω. For $Y \in B_0(X)$,

$$|Hu(Y)| \lesssim \frac{1}{d(X)} S_h(\nabla u)(Q)$$

for any $Q \in F_j$ and $|F(Y)| \leqslant N_h(F)(Q)$, hence

$$\begin{aligned}\int_{B_0(X)} G_\Omega(X, Y)|F(Y)|\,|\nabla V(Y)|\,|Hu(Y)|\,dY \\ \lesssim N_h(F)(Q)S_h(\nabla u)(Q)\|V\|_\infty \int_{B_0(X)} G_\Omega(X, Y)d(X)^{-2}\,dX \\ \leqslant C(\gamma\lambda)^2.\end{aligned}$$

For $Y \in \Omega B_0(X)$,

$$G_\Omega(X, Y) \lesssim G_D(P_0, Y)/\omega^{P_0}(\Delta_h)$$

where $G_D(P_0, \cdot)$ is the Green's function for D with pole at $P_0 \in D$. Hence

$$\begin{aligned}
&\int_{\Omega\setminus B_0(X)} G_\Omega(X, Y)|F|\,|\nabla V|\,|Hu|\,dY \\
&\qquad \leqslant \frac{1}{\omega(\Delta_h)}\int_{\Delta_h\cap F_j}\int_{\Gamma h(Q)} d(Y)^{2-n}|F|\,|Hu|\,|\nabla V|\,dY\,d\omega(Q) \\
&\qquad \leqslant \frac{1}{\omega(\Delta_h)}\int_{\Delta_h\cap F_j} N_h(F)S_h(\nabla u)S_h(V)\,d\omega\,(Q) \\
&\qquad \lesssim (\gamma\lambda)^2\left[\frac{1}{\omega(\Delta_h)}\int_{\Delta_h} S_h^2(V)\,d\omega\,(Q)\right]^{1/2} \\
&\qquad \lesssim C(\gamma\lambda)^2\left\{\frac{1}{\omega(\Delta_h)}\int_{T(\Delta_h)} G_D(P_0, Y)|\nabla V(Y)|^2\,dY\right\}^{1/2} \\
&\qquad \leqslant C(\gamma\lambda)^2.
\end{aligned}$$

This last inequality follows from the fact that a function in BMO $(d\sigma)$ is also in

$$\mathrm{BMO}\,(d\omega) = \left\{g \in L^2(d\omega)\colon \sup\left(\frac{1}{\omega(\Delta)}\int_\Delta \left|g - \int_\Delta g\frac{d\omega}{\omega(\Delta)}\right|^2 d\omega\right)^{1/2} < \infty\right\}$$

and that $G(P_0, Y)|\nabla V(Y)|^2\,dY$ is a Carleson measure with respect to $d\omega$. (See Jerison-Kenig [9].) □

The following lemma provides a pointwise estimate for $N_h(F)(Q)$ which proves Theorem 2.3.

Lemma 2.14. *Given* $\epsilon > 0$ *and* $X \in \Gamma_\delta(Q)$, $\delta = \delta(\epsilon)$,

$$|F(X)| + \mathrm{dist}\,(X)|\nabla F(X)| \leqslant C\epsilon T^{\#,4}(\nabla u)(Q) + C\|g\|_* + C\epsilon S_h(\nabla u)(Q).$$

From the lemma one obtains

$$\begin{aligned}
\int_{\Delta_h} N_h^2(F) + N_h^2(\mathrm{dist}\,|\nabla F|)\frac{d\sigma}{\sigma(\Delta_h)} &\lesssim \|g\|_*^2 + C\epsilon M_\sigma(T^{\#,4}(\nabla u))^2(Q_0) \\
&\quad + C\epsilon\frac{1}{\sigma(\Delta_h)}\int_{\Delta_h} S_h^2(\nabla u)\,d\sigma,
\end{aligned}$$

which by (2.13) yields (2.4), for sufficiently small ϵ.

PROOF OF (2.14). Let us consider only the term $|F(X)|$; the estimate for $d(X)|\nabla F(X)|$ is exactly the same. If $B_0(X)$ is the ball centered at X with radius comparable to $d(X)$, and $G(X, \cdot)$ is the Green's function for the domain D,

$$|F(X)| \leqslant \int_D G(X, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY = \int_{B_0(X)} + \int_{D\setminus B_0(X)}.$$

As in Lemma 2.13, using $|Hu(Y)| \leqslant d(X)^{-1}S_h(\nabla_h(Q))$,

$$\int_{B_0(X)} G(X, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY$$

$$\leqslant S_h(\nabla u)(Q)\left\{\int_{B_0(X)} G(X, Y)|\nabla V(Y)|^2\,dY\right\}^{1/2}\left\{\frac{1}{d(X)}\int_{B_0(X)} G(X, Y)\,dY\right\}^{1/2},$$

For $Y \in B_0(X)$, $G(X, Y) \approx |X - Y|^{2-n}$, so that $G(X, Y)$ is comparable to $G_{\tilde{B}_0}(X, Y)$, the Green's function for $\tilde{B}_0 = B_0(X, 2d(X))$. Hence the third term in the product above is finite and

$$\left\{\int_{B_0(X)} G(X, Y)|\nabla V(Y)|^2\,dY\right\}^{1/2} \leqslant \left\{\int_{\partial\tilde{B}_0} |V(P) - V(X)|^2\,d\omega_{\tilde{B}_0}(P)\right\}^{1/2}$$

which is less than ϵ since V is continuous and $h < \delta(\epsilon)$.

To bound that part of the Green's potential over $D\setminus B_0(X)$ we introduce the regions Ω_j. For $j = 0, \ldots, N$, set

$$\Omega_j = \{Y \in D: |Y - Q| \leqslant 2^{j-1}d(X)\}, \quad \text{and} \quad R_j = \Omega_{j+1}\setminus\Omega_j,$$

where $2^N d(X) = \delta$. Thus the regions Ω_j form a nested sequence of domains which fills out $\Omega_\delta = T\Delta(Q, \delta))$. We assume that δ has been chosen so that

$$\frac{1}{\omega(\Delta)}\int_{T(\Delta)} G_D(Y)|\nabla V(Y)|^2\,dY < \epsilon$$

whenever radius $(\Delta) < \delta$. Choose a sequence of points $\{X_j\}$ such that $X_j \in \Omega_j$ and $d(X_j) \approx 2^j d(X)$. Then if $Y \in R_j$, the comparison theorem yields (see [2])

$$G(X, Y)/G_{\Omega_{j+1}}(X_{j+1}, Y) \lesssim G(X, X_j)/G_{\Omega_{j+1}}(X_{j+1}, X_j).$$

By Hölder continuity of G, there is some $\alpha > 0$ such that

$$G(X, X_j) \lesssim 2^{-j\alpha}G(X_{j+1}, X_j)$$

and altogether, for $Y \in \Omega_j$,

$$G(X, Y) \leqslant 2^{-j\alpha}G_{\Omega_{j+1}}(X_{j+1}, Y) \leqslant \frac{2^{-j\alpha}}{\omega(\Delta_{j+1})}G_D(P_0, Y)$$

where $\Delta_{j+1} = \partial D \cap \Omega_{j+1}$. It follows that

$$\int_{\Omega_\delta \setminus B_0(X)} G(X, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY$$

$$\lesssim \sum_{j=0}^{N} 2^{-j\alpha} \frac{1}{\omega(\Delta_{j+1})} \int_{T(\Delta_{j+1})} G_D(P_0, Y)|\nabla V(Y)|\,|Hu(Y)|\,dY$$

$$\leqslant \sum_{j \geq 0} 2^{-j\alpha} \left\{ \frac{1}{\omega(\Delta_{j+1})} \int_{T(\Delta_{j+1})} G(P_0, Y)|\nabla V|^2\,dY \right\}^{1/2}$$

$$\left\{ \frac{1}{\omega(\Delta_{j+1})} \int_{T(\Delta_{j+1})} G(P_0, Y)|Hu|^2\,dY \right\}^{1/2}$$

$$\leqslant \sum_{j} 2^{-j\alpha} \epsilon \sup_{\Delta \ni Q} \left\{ \frac{1}{\omega(\Delta)} \int_{\Delta} |\nabla u - C_\Delta|^2\,d\omega \right\}^{1/2}$$

$$\leqslant \sum_{j} 2^{-j\alpha} \epsilon T^{\#,4}(\nabla u)(Q) \leqslant C\epsilon T^{\#,4}(\nabla u)(Q).$$

The last inequality follows from the fact that $d\omega/d\sigma$ satisfies a reverse Hölder inequality of exponent two (Dahlber, [2]).

Let $D_\delta = \{Y \in D: \operatorname{dist}(Y, \partial D) \leqslant \delta\}$. We have estimated that part of the Green's potential over the region $D_\delta \cap \Omega_\delta$. The remainder, $D_\delta \setminus \Omega_\delta$, consists of a union of regions $\Omega_\delta^{(k)}$, which are, roughly speaking, translates of $\Omega_\delta^{(k)}$ by a factor of $2^k\delta$. The estimates on each of these are similar to the above, and similar to those of [12], Lemma 10, so the details are omitted.

Consider now the region $D \setminus D_\delta$. Away from the pole X of $G(X, Y)$ and at a distance at least δ from ∂D, one simply uses the pointwise estimates $|G(X, Y)| \leqslant C_\delta$, $|\nabla V(Y)| \leqslant C_\delta$ and $|Hu(Y)| \leqslant C_\delta |\nabla u(Y)|$ to obtain

$$\int_{D \setminus D_\delta} G(X, Y)|\nabla V|\,|Hu|\,dY \leqslant C(\delta) \sup_{Y \in D \setminus D_\delta} |\nabla u(Y)| \leqslant C \left\{ \int_{\partial D} N^2(\nabla u)\,d\sigma \right\}^{1/2}$$

$$\leqslant C\|g\|_*. \quad \square$$

3. The boundary value problem (1.1) with Hölder continuous data

The Green's function $G(P_0, \bullet)$ for D is Hölder continuous of some order γ_0 where γ_0 depends on the Lipschitz character of D. Thus, it can be shown that if $g(Q)$ on ∂D is Hölder continuous of order $\gamma < \gamma_0$, then the solution of the Dirichlet problem with boundary values $g(Q)$ satisfies $|\nabla g(x)| \lesssim d(x)^{\gamma - 1}$. Solutions to the boundary value problem (1.1) will also have this property when the vector field $\vec{V}$ is smooth enough. The proof requires the information established in §2, namely that the solution is known to be of bounded mean oscillation.

Theorem 3.1. *If the components $V_i(Q)$ of $\vec{V}(Q)$ and the boundary data $g(Q)$ are all Hölder continuous of order $\gamma < \gamma_0$, γ_0 as above, then if $u(x)$ is the solution to* (1.1) *given by Calderón's Theorem, there exists a constant C depending on D, $\|g\|_\gamma$ and $\|V\|_\gamma$ such that $|Hu(X)| \leqslant Cd(X)^{\gamma-1}$.*

Fix such a $\gamma < \gamma_0$. It will first be shown that ∇u is Hölder continuous of some order $\alpha < \gamma$; this information will then be used to modify some estimates and obtain the desired result. The results of section 2 guarantee that $\nabla u \in \mathrm{BMO}\,(d\sigma)$. Hence $T^{\#,2}(\nabla u)(Q)$ is bounded by $\|g\|_\infty$. Fix $Q_0 \in \partial D$ and set

$$H(r) = \left\{\frac{1}{\sigma(\Delta(Q_0, r))}\int_{T(\Delta_r)} d(X)|Hu(X)|^2\,dX\right\}^{1/2}.$$

A perturbation argument like that of section 2 will be used to prove

Theorem 3.2. *Given $\alpha < \gamma$ and $\epsilon > 0$, there exists a constant C_ϵ such that*

$$H(r) < \epsilon H(2r) + C_\epsilon r^\alpha. \tag{3.3}$$

Because $H(s)$ is finite for all $s > 0$, inequality (3.3) gives Hölder continuity of ∇u by repeated iteration of itself and by choosing $\epsilon \leqslant 2^{-\gamma-1}$. The notation and terminology of section 2 will be used in the subsequent lemmas. Fix a small $\epsilon > 0$, a $Q_0 \in \partial D$ and assume $r < \epsilon$. Recall that $v(x) = V(X)\cdot\nabla u(X)$, where $V(X)$ is the harmonic extension of $V(Q)$, and that Δ_r denotes the ball of radius r centered at Q_0.

Lemma 3.4. *If $X_r \in T(\Delta_r)$ with $\operatorname{dist}(X_r, \partial D) \geqslant cr$, then*

$$H(r) \leqslant C\left[r|\nabla u(X_r)| + \left\{\frac{1}{\sigma(\Delta_r)}\int_{\Delta_r} \bar{S}^2_{2r}(v)(Q)\,d\sigma(Q)\right\}^{1/2} + \left\{\frac{1}{\sigma(\Delta_r)}\int_{T(\Delta_r)} d(X)|\nabla V(X)|^2|\nabla u(X)|^2\,dX\right\}^{1/2}\right].$$

PROOF. The proof is identical to the argument leading up to Lemma 2.10, given that $T^{\#,2}(\nabla u)(Q_0) \leqslant \|g\|_\infty$. □

Lemma 3.5. *For any $\alpha < \gamma$,*

$$\frac{1}{\sigma(\Delta_r)}\int_{T(\Delta_r)} d(X)|\nabla V(X)|^2|\nabla u(X)|^2\,dX \leqslant Cr^{2\alpha}.$$

PROOF. The proof of Lemma 2.11 and the fact that $\nabla u \in \mathrm{BMO}\,(d\sigma)$ yields the estimate, for $X \in T(\Delta_r)$, (3.6) $$|\nabla u(X)| \leqslant C \log\left(\frac{1}{r}\right).$$

Since $|\nabla V(X)| \leqslant d(X)^{\gamma-1}$ we have

$$\frac{1}{\sigma(\Delta_r)} \int_{T(\Delta_r)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2 \, dX \leqslant C \log^2\left(\frac{1}{r}\right) r^{2\gamma},$$

which proves the lemma for any $\alpha < \gamma$. □

It should be observed that this is the only point in the proof where the expected order of Hölder continuity is not achieved. The reason is that the bound (3.6) for $|\nabla u(X)|$ is not sharp. However, once some order of continuity of ∇u is established, the bound (3.6) can be replaced by

(3.6)′ $$|\nabla u(X)| \leqslant C$$

and the inequality of Lemma 3.5 can be replaced by

(3.5)′ $$\frac{1}{\sigma(\Delta_r)} \int_{T(\Delta_r)} d(X)|\nabla V(X)|^2 |\nabla u(X)|^2 \, dX \leqslant C r^{2\gamma}.$$

which yields the right order of Hölder continuity.

Moreover, (3.6) and Lemmas 3.4 and 3.5 now show that

(3.7) $$H(r) \leqslant C r^{\alpha} + \left\{ \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} \bar{S}_{2r}^2(v)(Q) \, d\sigma(Q) \right\}^{1/2}$$

Write

$$v(x) = g(x) - F(x)$$

where

$$F(x) = \int_D G(X, Y) \, \Delta v(Y) \, dY$$

and since

$$g(x) = \int_{\partial D} g(Q) \, d\omega^X(Q)$$

satisfies $|\nabla g(x)| \leqslant d(X)^{\gamma-1}$ it suffices to consider the term involving $\bar{S}_{2r}^2(F)$. The good-λ inequality (2.13) yields

(3.8)
$$H(r) \leqslant C r^{\alpha} + \left\{ \frac{C_\epsilon}{\sigma(\Delta_r)} \int_{\Delta_r} N_r^2(F) \, d\sigma(Q) \right\}^{1/2} + \epsilon \left\{ \frac{1}{\sigma(\Delta_r)} \int_{\Delta_r} S_{2r}^2(\nabla u)(Q) \, d\sigma(Q) \right\}^{1/2}$$

and the last term in the summand above is bounded by $\epsilon H(2r)$. The next lemma will be used to handle the second term above; it is stated in a more general form.

Lemma 3.9. *Let γ_0 be the order of Hölder continuity of the Green's function $G(P_0, \cdot)$ of D. Suppose V and U are harmonic in D with $|\nabla V(X)| \leqslant d(X)^{\gamma-1}$ for $\gamma < \gamma_0$ and $U|_{\partial D}$ in* BMO $(d\sigma)$. *Set $v(x) = V(X)U(X)$, and*

$$F(X) = \int_D G(X, Y)\,\Delta v(Y)\,dY.$$

Then if $d(X) \leqslant r$, $|F(X)| \leqslant Cr^{\gamma}$.

PROOF. Fix X with $d(X) \leqslant r$ and observe that $|\nabla v(Y)| \leqslant |\nabla V(Y)|\,|\nabla U(Y)|$. Let $B_0(X) = \{Y \in D: |Y - X| < \frac{1}{2}d(X)\}$ be the ball centered at X containing the pole of $G(X, Y)$. Then

$$\int_{B_0(X)} G(X, Y)|\nabla V|\,|\nabla U|\,dY \leqslant \frac{r^{\gamma}}{d(X)} \int_{B_0(X)} G(X, Y)\left[\frac{1}{d(X)}\|U\|_{\mathrm{BMO}}\right] dX$$
$$\leqslant Cr^{\gamma}.$$

Choose N such that $2^N r = 1$, and for each $j = 1, \ldots, N$ introduce the regions Ω_j as in the proof of Lemma 2.14, and let Δ_j denote $\partial D \cap \Omega_j$. As before, one applies the estimates for $G(X, Y)$ in the annular regions $\Omega_j \setminus \Omega_{j+1}$ to obtain

$$\int_{\Omega\delta} G(X, Y)|\nabla V|\,|\nabla U|\,dY$$
$$\leqslant \sum_{j=1}^{N} 2^{-j\gamma_0} \frac{1}{\omega(\Delta_j)} \int_{\Omega_j} G(P_0, Y)|\nabla V(Y)|\,|\nabla U(Y)|\,dY$$
$$\leqslant \sum_{j=1}^{N} 2^{-j\gamma_0} \left\{\frac{1}{\omega(\Delta_j)} \int_{\Omega_j} G(P_0, Y)|\nabla V|^2\,dY\right\}^{1/2} \left\{\frac{1}{\omega(\Delta_j)} \int_{\Omega_j} G(P_0, Y)|\nabla U|^2\,dY\right\}^{1/2}$$
$$\leqslant \sum_{j=1}^{N} 2^{-j\gamma_0} \left\{\frac{1}{\omega(\Delta_j)} \int_{\Delta_j} S^2_{2^j r}(V)\,d\omega\right\}^{1/2}$$

where the last inequality follows from the fact that $U \in \mathrm{BMO}\,(d\omega)$. When $|\nabla V(Y)| \leqslant d(Y)^{\gamma-1}$, a pointwise estimate on the truncated square function of V follows; namely $S_{2^j r}(V)(Q) \lesssim (2^j r)^{\gamma}$. For $\gamma < \gamma_0$, the above inequality can be summed, obtaining

$$\int_{\Omega_j} G(X, Y)|\nabla V|\,|\nabla A U|\,dY \lesssim Cr^{\gamma}.$$

Let $D_\delta = \{X \in D: d(X) < \delta\}$ and then $D_\delta = \bigcup_k \Omega_\delta^{(k)}$ where $\Omega_\delta^{(k)}$ is a region like Ω_δ at a distance roughly $2^k\delta$ from Ω_δ. One achieves similar estimates for the

Green's potential when the region of integration is $\Omega_\delta^{(k)}$ and these in turn may be summed on k (see for example [12]). The term involving integration over the remaining region D_δ is easily handled by simple pointwise estimates:

$$\begin{aligned}\int_{D_\delta} G(X, Y)|\nabla V|\,|\nabla U|\,dY &\leqslant C_\delta d(X)^{\gamma_0} \sup_{Z\in D_\delta} |\nabla u(Z)| \\ &\leqslant Cr^\gamma \|N(\nabla u)\|_{L^2} \\ &\leqslant Cr^\gamma \|g\|_{\mathrm{BMO}}. \quad \square\end{aligned}$$

Lemma 3.9 is applied to estimate $N_r(F)(Q)$ at any $Q \in \Delta_r$ and together with (3.8) shows that $H(r) \leqslant Cr^\alpha + \epsilon H(2r)$, *i.e.*, that $H(r) \leqslant Cr^\alpha$ for all r. This estimate implies an immediate improvement of itself (see the argument following Lemma 3.5) and so $H(r) \leqslant Cr^\gamma$ for any $\gamma < \gamma_0$. From the fact that $H(r) \leqslant Cr^\gamma$ it follows, by the mean value property, that $|Hu(X)| \lesssim d(X)^{\gamma-1}$, which implies that u is of class $C^{1,\gamma}$.

References

[1] Calderón, A. P. Boundary value problems for the Laplace equation in Lipschitzian domains, *Recent Progress in Fourier Analysis*, North-Holland Mathematics Studies, **111**(1983), 33-48.

[2] Dahlberg, B. E. J. On estimates of harmonic measure, *Arch. Rational Mech. Anal.* **65**(1977), 272-288.

[3] Dahlberg, B. E. J. Weighted norm inequalities for the Lusin area integral and the nontangential maximal function for functions harmonic in a Lipschitz domain, *Studia Math.* **64**(1980), 297-314.

[3] Dahlberg, B. E. J., Jerison, D. S. and Kenig, C. E. Area integral estimates for elliptic differential operators with non-smooth coefficients, *Arkiv. Mat.,* **22**(1984), 97-108.

[5] Fabes, E. B., Kenig, C. E. and Neri, U. Carleson measures, H^1 duality, and weighted BMO in non-smooth domains, *Indiana J. Math.* **30**(1981), 547-581.

[6] Fefferman, C. and Stein, E. H^p spaces of several variables, *Acta Math.* **129**(1972), 137-193.

[7] Fabes, E. B. and Neri, U. Dirichlet problem in Lipschitz domains with BMO data, *Proc. Amer. Math. Soc.* **78**(1980), 33-39.

[8] Janson, S. On functions with conditions on the mean oscillation, *Arkiv Mat.* **14**(1976), 189-196.

[9] Jerison, D. S. and Kenig, C. E. Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46**(1) (1982), 80-147.

[10] John, F. and Nirenberg, L. On functions of bounded mean oscillation, *Comm. Pure Applied Math.* **14**(1961), 415-426.

[11] Kenig, C. E. Weighted Hardy spaces in Lipschitz domains, *Amer. J. Math.* **102**(1980), 129-163.

[12] Kenig, C. E. and Pipher, J. The oblique derivative problem on Lipschitz domains with L^p data, to appear, *Amer. J. Math.*

[13] Sarason, D. Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207**(1975), 391-405.

[14] Stegeniga, D. A. Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation, *Amer. J. Math.* **98**(1976), 573-589.

[15] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[16] Strömberg, J. O. Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana J. Math.,* **28**(1979), 511-544.

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