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Regularity of p-Harmonic Functions on the Plane

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1. Introduction

Given an open set Ω in $\mathbb{R}^n$, a real-valued function u in the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, is said to be p-harmonic if it is a weak solution of the p-harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1}$$

or equivalently, for all test functions $\phi \in W^{1,p}(\Omega)$ with compact support in Ω

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u(x), \nabla\phi(x)\rangle\, dx = 0.$$

Note that p-harmonic functions are free extremals of the variational integral

$$I_G[u] = \int_G |\nabla u(x)|^p\, dx$$

for each relatively compact open subset $G \subset \Omega$. Two-harmonic functions are real analytic since they are harmonic in the usual sense. When $p \neq 2$ note that equation (1) is non-linear and it degenerates at the zeros of the gradient of u. In consequence of this fact p-harmonic functions with $p \neq 2$ need not be C^∞-smooth. They are however in the Hölder class $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ with some $\alpha = \alpha(n,p)$, $0 < \alpha \leqslant 1$; see Ural'tseva [9], Evans [3] and Lewis [6].

In this note we determine the optimal regularity of p-harmonic functions defined on a plane domain for each exponent p, $1 < p < \infty$. To state the results we use the following function spaces.

The Hölder space $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$, $k = 0, 1, \ldots, 0 < \alpha \leqslant 1$, is the space of complex-valued functions $u \in C^k(\Omega)$ whose k-th order partial derivatives $D^\nu u$, $|\nu| = k$, are locally Hölder continuous with exponent α, $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$ is a locally convex linear space with topology determined by the seminorms

$$\|u\|_{C^{k,\alpha}(F)} = \sup_{x\in F} |u(x)| + \sup_{x,y\in F} \sum_{|\nu|=k} \frac{|D^\nu u(x) - D^\nu u(y)|}{|x-y|^\alpha},$$

where F is any compact subset of Ω. The completion of $C^\infty(\Omega)$ in this topology is a proper subspace of $C^{k,\alpha}_{\mathrm{loc}}(\Omega)$ which we denote by $C^{k+\alpha}_{\mathrm{loc}}(\Omega)$. For $\alpha = 1$ the space $C^{k+\alpha}_{\mathrm{loc}}(\Omega)$ coincides with $C^{k+1}(\Omega)$ whereas functions in $C^{k+\alpha}_{\mathrm{loc}}(\Omega)$ with $0 < \alpha < 1$ are characterized by the condition

$$\sum_{|\nu|=k} |D^\nu u(x) - D^\nu u(y)| = o(|x-y|^\alpha)$$

uniformly on compact subsets of $\Omega \times \Omega$. Hence, we have the following imbeddings

$$C^{k+\alpha}_{\mathrm{loc}}(\Omega) \subsetneqq C^{k,\alpha}_{\mathrm{loc}}(\Omega) \subsetneqq C^{k+\beta}_{\mathrm{loc}}(\Omega). \tag{2}$$

whenever $k = 0, 1, \ldots$ and $0 < \beta < \alpha \leqslant 1$. We denote by $W^{k,s}_{\mathrm{loc}}(\Omega)$, $k = 1, 2, \ldots$ $1 \leqslant s \leqslant \infty$, the space of functions $u\colon \Omega \to \mathbb{C}$ whose distributional derivatives $D^\nu u$, $|\nu| \leqslant k$, belong to $L^s_{\mathrm{loc}}(\Omega)$. By Sobolev theorem

$$W^{k+2,q}_{\mathrm{loc}}(\Omega) \subsetneqq W^{k+1,s}_{\mathrm{loc}}(\Omega) \subsetneqq C^{k+\alpha}_{\mathrm{loc}}(\Omega) \tag{3}$$

for $0 < \alpha < 1$ and $k = 0, 1, 2, \ldots$, where

$$s = \frac{n}{1-\alpha} \in (n, \infty), \tag{4}$$

and

$$q = \frac{ns}{n+s} = \frac{n}{2-\alpha} \in \left(\frac{n}{2}, n\right). \tag{5}$$

Furthermore

$$W^{k+1,\infty}_{\mathrm{loc}}(\Omega) = C^{k,1}_{\mathrm{loc}}(\Omega), \tag{6}$$

and

$$W^{k+2,n}_{\mathrm{loc}}(\Omega) \subsetneqq W^{k+1,s}_{\mathrm{loc}}(\Omega) \tag{7}$$

for every $1 \leqslant s < \infty$. Our main result is the following:

Theorem 1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, be a p-harmonic function defined on a plane domain $\Omega \subset \mathbb{R}^2$. Then*

$$u \in C^{k,\alpha}_{\mathrm{loc}}(\Omega) \cap W^{k+2,q}_{\mathrm{loc}}(\Omega), \tag{8}$$

where the integer $k \geqslant 1$ and the exponent $\alpha \in (0, 1]$ are determined by the equation

$$k + \alpha = \frac{1}{6}\left(7 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right). \tag{9}$$

The integrability exponent q is any number such that

$$1 \leqslant q < \frac{2}{2-\alpha} \leqslant 2. \tag{10}$$

For $p \neq 2$ the regularity class in (8) *is optimal. More precisely, for each $1 < p < \infty$, $p \neq 2$, there is a p-harmonic function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ which is not in the class $C^{k+\alpha}_{\mathrm{loc}}(\Omega) \cup W^{k+2,\frac{2}{2-\alpha}}_{\mathrm{loc}}(\Omega)$.*

Interest in the regularity problems for p-harmonic functions arises from several considerations. One particular compelling connection is with an open problem of Gehring and Reich [4] concerning the degree of summability of the derivatives of a quasiconformal mapping.

Our proof of Theorem 1 substantially exploits and extends the ideas from [2]. A key is the hodograph transformation that converts the p-harmonic equation onto a linear first order elliptic system. We solve this system by using Fourier series method. A careful examination of the Fourier expansion formula for the solution of the system leads us to the regularity statement in Theorem 1. This formula provides non-trivial examples of p-harmonic functions. Among them there is one showing that our regularity result is the best possible.

2. Complex Gradient

We are going to use some properties of plane quasiregular mappings. For this it is convenient to introduce the complex variable

$$z = x + iy \in \mathbb{C}, \qquad (x, y) \in \mathbb{R}^2,$$

and the operators of complex differentiation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Let Ω be an open subset of $\mathbb{C}$. A mapping $f\colon \Omega \to \mathbb{C}$ is said to be K-quasiregular, $1 \leqslant K < \infty$, if $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\left|\frac{\partial f(z)}{\partial \bar{z}}\right| \leqslant \frac{K-1}{K+1}\left|\frac{\partial f(z)}{\partial z}\right| \tag{11}$$

for almost every $z \in \Omega$.

Given a real-valued function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 \leqslant p \leqslant \infty$, the complex gradient of u is defined by

$$f = \frac{\partial u}{\partial z} = \frac{1}{2}(u_x - iu_y).$$

The complex gradient $f = \partial u/\partial z$ of a p-harmonic function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, turns out to be a quasiregular mapping. An essential part of this statement is that $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$. For $p \geqslant 2$ this follows from estimates of the L^2-modulus of continuity of ∇u, see [2]. The case $1 < p < 2$ is somewhat delicate and requires an approximation argument, see [8]. The differential inequality (11) follows from (1) without any difficulty. Actually the p-harmonic equation can be given the form of one complex equation

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{1}{p} - \frac{1}{2}\right)\left[\frac{\bar{f}}{f}\,\frac{\partial f}{\partial z} + \frac{f}{\bar{f}}\,\overline{\frac{\partial f}{\partial z}}\right]. \tag{12}$$

Hence, (11) holds with

$$K = \max\left\{p-1, \frac{1}{p-1}\right\}. \tag{13}$$

Another interesting equation arises for the function $g(z) = |f(z)|^{\sqrt{p-1}-1} f(z)$. It takes the form of Beltrami's equation

$$\frac{\partial g}{\partial \bar{z}} = \frac{1-\sqrt{p-1}}{1+\sqrt{p-1}}\,\frac{\bar{g}}{g}\,\frac{\partial g}{\partial z}, \tag{14}$$

see [2].

In this way the study of p-harmonic functions reduces to the study of solutions $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of the quasilinear elliptic system (12).

From the theory of quasiregular mappings [1], [5] we find that the complex gradient $f = \partial u/\partial z$ of a p-harmonic function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$ is continuous. Moreover, the set $f^{-1}(0) = \{z \in \Omega\colon \nabla u(z) = 0\}$ is discrete provided u is not identically constant. We also know from the general regularity theory concerning elliptic equations that f is C^∞-smooth outside the singular set $f^{-1}(0)$. We refer to this result only insofar as it simplifies the arguments used. It is not difficult to dispense with this result since it follows from formula (63).

For the proof of Theorem 1 it is sufficiently general to examine the regularity of f near one of its zeros, that we may assume is the origin. From now on, f is a $W^{1,2}_{\mathrm{loc}}(\Omega)$-solution of (12), $0 \in \Omega$, $f(0) = 0$.

3. Hodograph Transformation

One of the remarkable methods developed for the study of non-linear equations is the hodograph transformation. The idea was originated in an intensive work on non-linear problems in hydrodynamics due mainly to L. Bers and M. A. Lavrentieff. Roughly speaking, given a system of first order differential equations and given a solution f to this system, the hodograph transformation is to write the given system in the hodograph plane in which the independent variable is the inverse of f. If the system is quasilinear this simple trick converts it into a linear system with variable coefficients.

Let f be a solution of system (12), $f(0) = 0$. Using the factorization theorem for quasiregular mappings [5] we write

$$f(z) = [\chi(z)]^n, \tag{15}$$

where χ is a quasiconformal homeomorphism defined in a neighborhood of $z = 0$, $\chi(0) = 0$, and n is a positive integer.

From (12) we find that

$$\frac{\partial \chi}{\partial \bar{z}} = \left(\frac{1}{p} - \frac{1}{2}\right)\left[\frac{\chi}{\bar{\chi}}\frac{\overline{\partial \chi}}{\partial z} + \frac{\overline{\chi^n}}{\chi^n}\frac{\partial \chi}{\partial z}\right]. \tag{16}$$

Denote by $H = H(\xi)$ the inverse of $\chi = \chi(z)$ in a neighborhood of $z = 0$, $z = H(\chi(z))$, $\xi = \chi(H(\xi))$, $H(0) = 0$. We have

$$\frac{\partial \chi}{\partial z} = J^{-1}\overline{H_\xi}, \qquad \frac{\partial \chi}{\partial \bar{z}} = -J^{-1}H_{\bar{\xi}}, \tag{17}$$

where $J(\xi) = |H_\xi|^2 - |H_{\bar{\xi}}|^2 > 0$ a.e. since H is quasiconformal in a neighborhood of $\xi = 0$. From (16) we see that H satisfies the linear equation

$$H_{\bar{\xi}} = \left(\frac{1}{2} - \frac{1}{p}\right)\left[\frac{\xi}{\bar{\xi}}H_\xi + \frac{\bar{\xi}^n}{\xi^n}\bar{H}_\xi\right]. \tag{18}$$

Our immediate goal is to solve system (18) in a neighborhood of $\xi = 0$. There is no loss of generality assuming that H is defined on the unit disk $\mathbb{B} = \{\xi : |\xi| < 1\}$, that $H \in W^{1,2}(\mathbb{B})$ and that $H \in C^\infty(\mathbb{B} - \{0\})$. This can be done by rescaling the variable ξ, since the function $H(t\xi)$, $t \in \mathbb{R}$, is also a solution to (18).

4. Series Expansion for $H(\xi)$

Theorem 2. *Every solution $H \in W^{1,2}(\mathbb{B})$ of equation (18) on the unit disk $\mathbb{B}$ expands into an infinite series of the form*

$$H(\xi) = \sum_{k=n}^{\infty} (A_k \xi^k + \epsilon_k \bar{A}_k \bar{\xi}^k) |\xi|^{\lambda_k + n - k} \xi^{-n}, \tag{19}$$

where the numbers $\lambda_k = \lambda_k(n,p)$ and $\epsilon_k = \epsilon_k(n,p)$ are defined by

$$2\lambda_k = -np + \sqrt{4k^2(p-1) + n^2(p-2)^2}, \tag{20}$$

$$\epsilon_k = \frac{\lambda_k + n - k}{\lambda_k + n + k} \tag{21}$$

and the complex coefficients A_k, $k = n, n+1, \ldots$ satisfy

$$\sum_{k=n+1}^{\infty} k|A_k|^2 \leqslant C(n,p) \iint_{\mathbb{B}} (|H_\xi|^2 + |H_{\bar{\xi}}|^2)\, d\sigma(\xi) < \infty. \tag{22}$$

The series converges in $W^{1,2}(\mathbb{B})$.

Con*versely, given complex numbers A_k, $k = n, n+1, \ldots$ satisfying*

$$\sum_{k=n}^{\infty} k|A_k|^2 < \infty$$

the series (19) *converges in $W^{1,2}(\mathbb{B})$ to a solution of equation* (18) *for which we have*

$$\iint_{\mathbb{B}} (|H_\xi|^2 + |H_{\bar{\xi}}|^2)\, d\sigma(\xi) \leqslant C(n,p) \sum_{k=n+1}^{\infty} k|A_k|^2.$$

Remarks. By elementary considerations the following estimates follow from formulas (20) and (21):

$$\epsilon_n(n,p) = \lambda_n(n,p) = 0, \tag{23}$$

$$|\epsilon_k(n,p)| < 1, \quad \text{for} \quad k = n+1, n+2, \ldots, \tag{24}$$

$$\frac{p-1}{np} \leqslant \lambda_k(n,p) \leqslant k\sqrt{p-1}, \quad \text{for} \quad k = n+1, n+2, \ldots \tag{25}$$

If $p = 2$, $\lambda_k(n,2) = k - n$ and $\epsilon_k(n,2) = 0$, $k = n, n+1, \ldots$, thus

$$H(\xi) = \sum_{k=n}^{\infty} A_k \xi^{k-n},$$

From now on, we assume that $p \neq 2$.

PROOF. It is desirable to use polar coordinates for this problem. Let $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$, $\xi = re^{i\theta}$, $H(r, \theta) = H(re^{i\theta})$, we have the following transformation formula:

$$H_{\bar{\xi}} = \frac{1}{2}\left(H_r + \frac{i}{r}H_\theta\right)e^{i\theta}, \qquad H_\xi = \frac{1}{2}\left(H_r - \frac{i}{r}H_\theta\right)e^{-i\theta}.$$

Since H is K-quasiregular,

$$K = \max\left\{p - 1, \frac{1}{p-1}\right\},$$

then

$$K^{-1}|H_r|^2 \leqslant |H_\xi|^2 - |H_{\bar{\xi}}|^2 \leqslant K|H_r|^2, \tag{26}$$

and

$$(1 + K^{-2})|H_r|^2 \leqslant 2|H_\xi|^2 + 2|H_{\bar{\xi}}|^2 \leqslant (1 + K^2)|H_r|^2. \tag{27}$$

Write equation (18) in polar coordinates, conjugate it and eliminate the $\bar{H}_r$ term to obtain

$$2rH_r(r, \theta) = -ipH_\theta(r, \theta) + (p - 2)ie^{-2in\theta}\overline{H_\theta(r, \theta)}. \tag{28}$$

We expand $H(r, \theta)$ into Fourier series with respect to θ, $0 \leqslant \theta < 2\pi$.

$$H(r, \theta) = \sum_{k=-\infty}^{\infty} a_k(r)e^{i(k-n)\theta}, \tag{29}$$

where

$$a_k(r) = \frac{1}{2\pi}\int_0^{2k} H(r, \theta)e^{i(n-k)\theta}\, d\theta, \qquad k \in \mathbb{Z}. \tag{30}$$

From these formulas, since $H \in C(\mathbb{B}) \cap C^\infty(\mathbb{B} - \{0\})$, we see that

$$a_k \in C[0, 1) \cap C^\infty(0, 1), \qquad k \in \mathbb{Z}. \tag{31}$$

Furthermore, since $H \in W^{1,2}(\mathbb{B})$ we are justified to differentiate (29) term by term

$$\begin{cases} H_r(r, \theta) = \sum_{k=-\infty}^{\infty} a_k'(r)e^{i(k-n)\theta} \\ H_\theta(r, \theta) = i\sum_{k=-\infty}^{\infty} (k - n)a_k(r)e^{i(k-n)\theta} \end{cases}. \tag{32}$$

These series converge in $L^2(\mathbb{B})$.

Formula (30) and equation (28) yield

$$
\begin{aligned}
2ra_k'(r) &= \frac{1}{2\pi}\int_0^{2\pi} 2rH_r(r,\theta)e^{i(n-k)\theta}\,d\theta \\
&= -ip\frac{1}{2\pi}\int_0^{2\pi} H_\theta(r,\theta)e^{i(n-k)\theta}\,d\theta + (p-2)i\frac{1}{2\pi}\int_0^{2\pi} \overline{H_\theta(r,\theta)}e^{-i(k+n)\theta}\,d\theta \\
&= -p(n-k)\frac{1}{2\pi}\int_0^{2\pi} H(r,\theta)e^{i(n-k)\theta}\,d\theta \\
&\quad - (p-2)(n+k)\overline{\left(\frac{1}{2\pi}\int_0^{2\pi} H(r,\theta)e^{i(n+k)\theta}\,d\theta\right)}\cdot
\end{aligned}
$$

The last equality follows from integration by parts. In view of formulas (30) the right hand integrals are equal to $a_k(r)$ and $a_{-k}(r)$, respectively. It leads us to an infinite system of ordinary differential equations for the Fourier coefficients

$$
2ra_k'(r) = -p(n-k)a_k(r) - (p-2)(n+k)\overline{a_{-k}(r)} \tag{33}
$$

for all $k \in \mathbb{Z}$. Replace k by $-k$ in (33), conjugate it and multiply by $(p-2)$

$$
2(p-2)r\overline{a'_{-k}(r)} = -p(p-2)(n+k)\overline{a_{-k}(r)} - (p-2)^2(n-k)a_k(r).
$$

We eliminate the term $(p-2)(n+k)\overline{a_{-k}}$ by using (33) again

$$
\begin{aligned}
2(p-2)r\overline{a'_{-k}} &= p[2ra_k' + p(n-k)a_k] - (p-2)^2(n-k)a_k \\
&= 2pra_k' + 4(p-1)(n-k)a_k.
\end{aligned} \tag{34}
$$

Finally, we differentiate (33) and, by (34), we obtain

$$
2r(2ra_k')' = -2pr(n-k)a_k' - (n+k)[2pra_k' + 4(p-1)(n-k)a_k].
$$

In this way we arrive at the following uncoupled system of linear (over complex numbers) equations of second order.

$$
r(ra_k')' + pnra_k' + (n^2-k^2)(p-1)a_k = 0 \tag{35}
$$

for $k \in \mathbb{Z}$. Fix k, the general solution of (35) must take the form

$$
a_k(r) = A^+ r^{\lambda^+} + A^- r^{\lambda^-},
$$

where λ^+ and λ^- are distict roots of the quadratic equation

$$
\lambda^2 + pn\lambda + (n^2-k^2)(p-1) = 0,
$$

that is

$$2\lambda^{+} = -pn + \sqrt{4k^2(p-1) + n^2(p-2)^2}$$
$$2\lambda^{-} = -pn - \sqrt{4k^2(p-1) + n^2(p-2)^2}$$

Since $a_k \in C[0, 1)$ and $\lambda^{-} < 0$, the constant A^{-} must vanish. By the same reasoning, the constant A^{+} has to be zero whenever λ^{+} is negative, which happens if $|k| < n$. Hence,

$$a_k(r) = \begin{cases} 0 & \text{if } |k| < n \\ A_k r^{\lambda_k} & \text{if } |k| \geqslant n \end{cases}, \tag{36}$$

where

$$\lambda_k = \frac{1}{2}(-pn + \sqrt{4k^2(p-1) + n^2(p-2)^2}), \tag{37}$$

for $k = \pm n, \pm(n+1), \ldots$ Note that $\lambda_k = \lambda_{-k}$ and $\lambda_n = 0$.

We still should verify system (33) because this is not equivalent to (35). Inserting (36) into (33) we find conditions for the coefficients A_k, $k = \pm n$, $\pm(n+1), \ldots,$

$$[2\lambda_k + p(n-k)]A_k = (2-p)(n+k)\overline{A_{-k}}. \tag{38}$$

For $k = \pm n$, in view of $\lambda_{-n} = \lambda_n = 0$, we obtain $A_{-n} = 0$. There are no restrictions for A_n. For $|k| > n$ we write (38) as follows

$$A_{-k} = \epsilon_k \overline{A_k}, \tag{39}$$

where

$$\epsilon_k = \frac{2\lambda_k + p(n-k)}{(2-p)(n+k)} = \frac{\lambda_k + n - k}{\lambda_k + n + k}.$$

The later identity is computed from formula (37). Observe that $\epsilon_k \epsilon_{-k} = 1$, thus the change of sign of the index k in (39) leads to the same condition. In conclusion, we may take arbitrary numbers for $A_n, A_{n+1}, \ldots,$ and determine $A_{-n}, A_{-n-1}, \ldots$ from formula (39).

Returning to the Fourier expansion (29) we write:

$$H(r,\theta) = \sum_{k=n}^{\infty} (A_k e^{ik\theta} + \epsilon_k A_k e^{-ik\theta}) r^{\lambda_k} e^{-in\theta}, \tag{40}$$

This is the polar form of (19). A standard computation gives

$$\int_0^1 r\left(\int_0^{2\pi} |H_r(r,\theta)|^2 \, d\theta\right) dr = \pi \sum_{k=n+1}^{\infty} \lambda_k (1 + \epsilon_k^2) |A_k|^2.$$

This, together with estimates (24), (25) and (27), implies (22) with a constant $C(n,p)$ depending only on n and p.

Our arguments can be turned back proving the converse statement of Theorem 2.

Corollary 1. *Let $H(\xi)$ be a local solution of equation* (18) *which is quasiconformal in a neighborhood of the origin, $H(0)=0$. Then there is $\rho>0$ and there are constants $0<c<C$ and C_m, $m=0,1,2,\dots$, such that*

$$c|\xi|^{\gamma_n} \leqslant |H(\xi)| \leqslant C|\xi|^{\gamma_n} \quad \textit{for} \quad 0 \leqslant |\xi| \leqslant \rho, \tag{41}$$

$$c|\xi|^{2\gamma_n-2} \leqslant J(\xi) = |H_\xi|^2 - |H_{\bar\xi}|^2 \leqslant C|\xi|^{2\gamma_n-2} \quad \textit{for} \quad 0<|\xi|\leqslant\rho, \tag{42}$$

and

$$\sum_{|\nu|=m} |D^\nu H(\xi)| \leqslant C_m|\xi|^{\gamma_n-m} \quad \textit{for} \quad 0<|\xi|\leqslant\rho, \tag{43}$$

where

$$\gamma_n = \lambda_{n+1}(n,p) = \frac{n}{2}\left\{-p+\sqrt{4\left(1+\frac{1}{n}\right)^2(p-1)+(p-2)^2}\right\}. \tag{44}$$

PROOF. By rescaling the variable ξ one may assume without loss of generality that H is a quasiconformal solution of (18) on the unit disk, thus it expands into infinite series of the form (19) with $A_n=0$. Therefore, for each $0\leqslant|\xi|\leqslant\rho<1$, we have the following estimates

$$\begin{aligned}|H(\xi)| &\leqslant \sum_{k=n+1}^{\infty}(1+|\epsilon_k|)|A_k|\,|\xi|^{\lambda_k} \leqslant 2\rho^{-\lambda_{n+1}}|\xi|^{\lambda_{n+1}}\sum_{k=n+1}^{\infty}|A_k|\rho^{\lambda_k}\\ &\leqslant 2\rho^{-\lambda_{n+1}}|\xi|^{\lambda_{n+1}}\left(\sum_{k=n+1}^{\infty}k|A_k|^2\right)^{1/2}\left(\sum_{k=n+1}^{\infty}k^{-1}\rho^{2\lambda_k}\right)^{1/2} \leqslant C|\xi|^{\gamma_n}.\end{aligned}$$

The convergence of the last two series is verified by (22) and (25). These arguments also show that the series (19) converges uniformly, on $\{\xi\colon|\xi|\leqslant\rho\}$. Applying argument principle, we find that $A_{n+1}\neq0$. Hence for $0\leqslant|\xi|\leqslant\rho<1$, we obtain

$$\begin{aligned}|H(\xi)| &\geqslant (1-|\epsilon_{n+1}|)|A_{n+1}|\,|\xi|^{\lambda_{n+1}} - \sum_{k=n+2}^{\infty}(1+|\epsilon_k|)|A_k|\,|\xi|^{\lambda_k}\\ &\geqslant |\xi|^{\lambda_{n+1}}\left[(1-|\epsilon_{n+1}|)|A_{n+1}| - 2\rho^{\lambda_{n+2}-\lambda_{n+1}}\sum_{k=n+2}^{\infty}|A_k|\rho^{\lambda_k-\lambda_{n+2}}\right].\end{aligned}$$

For suffiently small ρ, the expression in rectangular parantheses is positive. Hence (41) follows. To prove (43) we note that $\lim\lambda_k=\infty$. Therefore, we can perform any finite number of term-by-term differentiations in formula (19) for $0 < |\xi| < 1$. This gives the required estimate

$$\sum_{|\nu|=m} |D^\nu H(\xi)| \leqslant C(n,p,m) \sum_{k=n+1}^{\infty} \lambda_k^m |A_k| |\xi|^{\lambda_k - m}$$

$$\leqslant C(n,p,m)|\xi|^{\lambda_{n+1}-m} \sum_{k=n+1}^{\infty} k^m |A_k| \rho^{\lambda_k - \lambda_{n+1}}$$

$$\leqslant C(n,p,m)|\xi|^{\lambda_{n+1}-m} \left[\sum_{k=n+1}^{\infty} k|A_k|^2 \right]^{1/2}$$

$$\times \left[\sum_{k=n+1}^{\infty} k^{2m-1} \rho^{2\lambda_k - 2\lambda_{n+1}} \right]^{1/2}.$$

For $m = 1$, this implies the upper bound of $J(\xi)$ as stated in (42). To prove the lower bound for $J(\xi)$, we use expansion (40) from which it follows that

$$H_r(r,\theta) = \sum_{k=n+1}^{\infty} \lambda_k (A_k e^{ik\theta} + \epsilon_k \overline{A_k} e^{-ik\theta}) e^{-in\theta} r^{\lambda_k - 1}.$$

Hence, for $0 < r = |\xi| \leqslant \rho < 1$ we find

$$|H_r| \geqslant \lambda_{n+1}(1 - |\epsilon_{n+1}|)|A_{n+1}| r^{\lambda_{n+1}-1} - \sum_{k=n+2}^{\infty} \lambda_k (1 + |\epsilon_k|)|A_k| \rho^{\lambda_k - 1}$$

$$\geqslant r^{\lambda_{n+1}-1}\left(\lambda_{n+1}(1 - |\epsilon_{n+1}|)|A_{n+1}| - 2\rho^{\lambda_{n+2}-\lambda_{n+1}} \sum_{K=n+2}^{\infty} \lambda_k |A_k| \rho^{\lambda_k - \lambda_{n+2}} \right)$$

$$= c(n,p,\rho) r^{\gamma_n - 1},$$

where, in view of (22) and (25), the constant $c(n,p,\rho)$ is positive as ρ gets small. This together with inequality (26) implies

$$J(\xi) = |H_\xi|^2 - |H_{\bar{\xi}}|^2 \geqslant c|\xi|^{2\gamma_n - 2} \quad \text{for} \quad 0 < |\xi| \leqslant \rho.$$

5. Estimates of the Derivatives of $f(z)$

We return to the function $f(z) = [\chi(z)]^n$, where $\chi = \chi(z)$ is the inverse to $H(\xi)$, that is

$$\chi(H(\xi)) = \xi \tag{45}$$

for $|\xi| \leqslant \rho$. We shall express the partial derivatives $D^\nu \chi(z)$, $|\nu| = 0, 1, \ldots$, at $z = H(\xi)$ in terms of $D^\mu H(\xi)$, $|\mu| = 0, 1, \ldots$ The first order derivatives of χ

are easy to derive from (45):

$$\begin{cases} \chi_z(z) = J^{-1}(\xi)\overline{H_\xi} \\ \chi_{\bar z}(z) = -J^{-1}(\xi)H_{\bar\xi} \end{cases}. \tag{46}$$

To describe the formulas for higher derivatives of $\chi(z)$ we need the following convention. Given a positive integer s let H^s denote one of the partials

$$\frac{\partial^s H}{\partial\xi^i\partial\bar\xi^j} \quad \text{or} \quad \overline{\frac{\partial^s H}{\partial\xi^i\partial\bar\xi^j}},$$

where i and j run over non-negative integers such that $i+j=s$. For example, the symbol H^1 stands for one of the first order derivatives H_ξ, $H_{\bar\xi}$, $\overline{H_\xi}$ or $\overline{H_{\bar\xi}}$ and we do not specify which one of them. Any expression of the form $H^{s_1}H^{s_2}\cdots H^{s_k}$, where $s_1+s_2+\cdots+s_k=s$ will be called a monomial of type (s,k).

The set of all linear combinations of monomials of type (s,k) is denoted by $\mathbb{P}(s,k)$, $s=1,2,\ldots$, $k=1,2,\ldots$ For instance, the Jacobian determinant $J(\xi)=H_\xi\overline{H_\xi}-H_{\bar\xi}\overline{H_{\bar\xi}}$ is a member of $\mathbb{P}(2,2)$. Let us remark that

(i) If $P\in\mathbb{P}(s,k)$, then P_ξ, $P_{\bar\xi}\in\mathbb{P}(s+1,k)$,
(ii) $J_\xi=H_{\xi\xi}\overline{H_\xi}+\overline{H_{\xi\bar\xi}}H_\xi-H_{\xi\bar\xi}\overline{H_{\bar\xi}}-\overline{H_{\xi\xi}}H_{\bar\xi}\in\mathbb{P}(3,2)$,
(iii) if $P\in\mathbb{P}(s,k)$ and $Q\in\mathbb{P}(t,l)$, then $PQ\in\mathbb{P}(s+t,k+l)$.

Now we generalize formula (46).

Lemma 1. *Let* $|\nu|=m$, $m=1,2,\ldots$, *then there exists*

$$P=P_\nu\in\mathbb{P}(4m-3,3m-2)$$

such that

$$D^\nu\chi(z)=J(\xi)^{1-2m}P_\nu(\xi), \tag{47}$$

where $z=H(\xi)$.

PROOF. We perform induction with respect to m. For $m=1$ formula (47) follows from (46). Suppose that (47) holds for some integer $m\geqslant 1$. Applying chain rule, we obtain

$$\begin{cases} (D^\nu\chi)_z H_\xi+(D^\nu\chi)_{\bar z}\overline{H_{\bar\xi}}=J^{-2m}[JP_\xi+(1-2m)PJ_\xi] \\ (D^\nu\chi)_z H_{\bar\xi}+(D^\nu\chi)_{\bar z}\overline{H_\xi}=J^{-2m}[JP_{\bar\xi}+(1-2m)PJ_{\bar\xi}] \end{cases}$$

This system can be solved for partials $(D^\nu\chi)_z$ and $(D^\nu\chi)_{\bar z}$:

$$(D^\nu\chi)_z=J^{-1-2m}[JP_\xi\overline{H_\xi}-JP_{\bar\xi}\overline{H_{\bar\xi}}+(1-2m)PJ_\xi\overline{H_\xi}-(1-2m)PJ_{\bar\xi}\overline{H_{\bar\xi}}],$$
$$(D^\nu\chi)_{\bar z}=J^{-1-2m}[JP_{\bar\xi}H_\xi-JP_\xi H_{\bar\xi}+(1-2m)PJ_{\bar\xi}H_\xi-(1-2m)PJ_\xi H_{\bar\xi}].$$

What remains to be established is that the expressions in the brackets [...] are members of $\mathbb{P}(4(m+1)-3,\ 3(m+1)-2) = \mathbb{P}(4m-1, 3m+1)$. One can verify this fact by using the induction hypothesis and properties (i-iii). Thus formula (47) holds for $m+1$.

Combining Lemma 1 together with Corollary 1 we obtain estimates for the derivatives of the complex gradient $f = f(z)$.

Corollary 2. *Suppose $f(z)$ is a local solution to the system* (12) *which has the form* (15), $f(0) = 0$. *Then there is $\delta > 0$ and there are constants B_m*, $m = 0, 1, 2, \ldots$, such that

$$\sum_{|\nu| = m} |D^\nu f(z)| \leqslant B_m |z|^{n/\gamma_n - m}, \tag{48}$$

for $0 < |z| < \delta$, *where γ_n is defined by* (44).

PROOF. First we examine the quasiconformal map $\chi = \chi(z)$. Fix $|\nu| = l$, $l = 1, 2, \ldots, m$. By Lemma 1,

$$D^\nu \chi(z) = J(\xi)^{1-2l} P_\nu(\xi), \tag{49}$$

for $0 < |\xi| < \rho$, where $P_\nu \in \mathbb{P}(4l-3, 3l-2)$. In view of inequality (43) any monomial $Q = H^{s_1} H^{s_2} \cdots H^{s_k}$, $s_1 + \cdots + s_k = s$, of type (s, k) admits an estimate

$$|Q(\xi)| \leqslant M |\xi|^{\gamma_n - s_1} |\xi|^{\gamma_n - s_2} \cdots |\xi|^{\gamma_n - s_k} = M |\xi|^{k\gamma_n - s},$$

for $0 < |\xi| < \rho$. Here and below the letter M stands for a constant independent of ξ, not necessarily the same in each appearance. Since $P_\nu(\xi)$ is a linear combination of monomials of type $(4l-3, 3l-2)$, we then find that

$$|P_\nu(\xi)| \leqslant M |\xi|^{(3l-2)\gamma_n - 4l + 3},$$

for $0 < |\xi| < \rho$. On the other hand, using (42), we see that

$$J(\xi)^{1-2l} \leqslant M |\xi|^{(1-2l)(2\gamma_n - 2)}.$$

The above estimates together with formula (49) imply

$$|D^\nu \chi(z)| \leqslant M |\xi|^{1 - l\gamma_n},$$

where $|\nu| = l$, $l = 0, 1, 2, \ldots, m$. This also holds true for $l = 0$ since $\chi(z) = \xi$.

According to (41), $c|\xi|^{\gamma_n} \leqslant |z| \leqslant C|\xi|^{\gamma_n}$, so for each $|\nu| = l$, $l = 0, 1, \ldots, m$ we have

$$|D^\nu \chi(z)| \leqslant M |z|^{1/\gamma_n - l}$$

for $0 < |z| < \delta$, with sufficiently small $\delta > 0$.

Finally, applying Leibniz formula for the derivatives of $f(z) = [\chi(z)]^n$ we come to the required estimate $|D^\nu f(z)| \leqslant M|z|^{n/\gamma_n - m}$ for each $|\nu| = m$.

6. Proof of the Regularity Statement

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, $p \neq 2$, be a p-harmonic function. As mentioned before, it suffices to establish the regularity of u near one of its singular points, that we may assume is the origin. Denote by $f = f(z)$ the complex gradient of u, $f(0) = 0$. Corollary 2 applies to f for some positive integer n. By formula (44) we see that the smallest value of the numbers n/γ_n, $n = 1, 2, \ldots$, occurs for $n = 1$. Let this minimum value be denoted by $d = d(p)$, $1 < p < \infty$.

$$d = \frac{1}{6}\left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-2)^2}}\right). \tag{50}$$

Hence, we obtain estimates which are independent of n

$$\sum_{|\nu| = m} |D^\nu f(z)| \leqslant B_m |z|^{d-m}, \tag{51}$$

or equivalently

$$\sum_{|\nu| = m+1} |D^\nu u(z)| \leqslant B_m |z|^{d-m}, \tag{52}$$

for $0 < |z| < \delta$ and $m = 0, 1, 2, \ldots$

Let the integer k and the exponent $\alpha \in (0, 1]$ be determined by (9), so $d = k - 1 + \alpha$. Take m in (52) equal to $k + 1$

$$\sum_{|\nu| = k+2} |D^\nu u(z)| \leqslant B_{k+1} |z|^{\alpha - 2} \quad \text{for} \quad 0 < |z| < \delta.$$

The function $|z|^{\alpha-2}$ is integrable over the disk $0 \leqslant |z| < \delta$ with every exponent q such that $1 \leqslant q < 2/(2 - \alpha)$. Hence $u \in W^{k+2,q}_{\mathrm{loc}}(\Omega)$.

These two estimates imply Hölder's condition of exponent α for functions $D^\nu u(z)$, $|\nu| = k$. This fact follows from an elementary lemma.

Lemma 2. *Let U be a convex open subset in $\mathbb{R}^n$ containing the origin and let F be a function of class $C(U) \cap C^1(U - \{0\})$ such that*

(i) $|F(x)| \leqslant M|x|^\alpha$ *for* $x \in U$,

(ii) $|\nabla F(x)| \leqslant M|x|^{\alpha - 1}$ *for* $x \in U - \{0\}$,

where $\alpha \in (0, 1]$. *Then*

$$|F(x) - F(y)| \leqslant 5M|x - y|^\alpha, \quad \textit{for all} \quad x, y \in U. \tag{53}$$

PROOF. Two cases are possible.

Case 1. Suppose that $|y| \leqslant 2|x-y|$, then

$$\begin{aligned}|F(x)-F(y)| &\leqslant |F(x)|+|F(y)| \leqslant 2^{1-\alpha}(|F(x)|^{1/\alpha}+|F(y)|^{1/\alpha})^{\alpha}\\ &\leqslant 2^{1-\alpha}M(|x|+|y|)^{\alpha} \leqslant 2^{1-\alpha}M(|x-y|+2|y|)^{\alpha}\\ &\leqslant 2^{1-\alpha}5^{\alpha}M|x-y|^{\alpha} \leqslant 5M|x-y|^{\alpha}.\end{aligned}$$

Case 2. Suppose that $|y| \geqslant 2|x-y| > 0$, then for each $0 \leqslant t \leqslant 1$ we have

$$|tx+(1-t)y| \geqslant |y| - t|x-y| \geqslant |x-y|.$$

Thus

$$|tx+(1-t)y|^{\alpha-1} \leqslant |x-y|^{\alpha-1}.$$

Using this estimate, we conclude

$$|F(x)-F(y)| = \left|\int_0^1 \frac{d}{dt}F(tx+y-ty)\,dt\right| = \left|\int_0^1 \langle x-y, \nabla F(tx+y-ty)\rangle\,dt\right|$$

$$\leqslant M\int_0^1 |x-y|\,|tx+(1-t)y|^{\alpha-1}\,dt \leqslant M|x-y|^{\alpha}.$$

7. The Extremal Function

Take $n=1$ in (15) and $A_1=0$, $A_2=1$, $A_3=A_4=\cdots=0$ in the series (19). In this case

$$\lambda_2=\gamma_1=\frac{1}{d} \quad \text{and} \quad \epsilon_2=\frac{\lambda_2-1}{\lambda_2+3}=\frac{1-d}{1+3d}.$$

We then have

$$H(\xi)=\left(\frac{\xi}{|\xi|}+\epsilon\frac{|\xi|^3}{\xi^3}\right)|\xi|^{1/d}, \qquad \epsilon=\frac{1-d}{1+3d}. \tag{54}$$

Note that for $p=2$, in view of formula (44), we have $d=1$, $\epsilon=0$, and $H(\xi)=\xi$. It is of crucial importance that $\epsilon \neq 0$ for $p \neq 2$. In this case H is a quasiconformal homeomorphism on the whole plane.

Let $f=f(z)$ denote the inverse of $H=H(\xi)$

$$f(H(\xi))=\xi, \qquad z=H(\xi), \qquad \xi=f(z). \tag{55}$$

This defines a p-harmonic function $v = W^{1,p}_{\mathrm{loc}}(\mathbb{C})$ of complex gradient equal to f, $\partial v/\partial z = f(z)$. We complete the proof of Theorem 1 by showing that

$$f \notin C^{k-1+\alpha}_{\mathrm{loc}}(\mathbb{C}) \cup W^{k+1,q}_{\mathrm{loc}}(\mathbb{C}), \qquad q = \frac{2}{2-\alpha}. \tag{56}$$

For this effect let us remark that f is a homogeneous function of degree d,

$$f(tz) = t^d f(z), \qquad z \in \mathbb{C} \quad \text{and} \quad t \geqslant 0. \tag{57}$$

Indeed, by (54), $tH(\xi) = H(t^d \xi)$, and by (55)

$$f(tz) = f(tH(\xi)) = f(H(t^d\xi)) = t^d \xi = t^d f(z).$$

Hence, partial derivatives $D^\nu f$ are homogeneous functions of degree $d - |\nu|$. In particular, since $d = k - 1 + \alpha$, we have

$$(D^\nu f)(tz) = t^\alpha D^\nu f(z), \tag{58}$$

for $z \neq 0$, $t > 0$ and $|\nu| = k - 1$, and

$$D^\nu f(z) = t^{2-\alpha}(D^\nu f)(tz), \tag{59}$$

for $z \neq 0$, $t > 0$ and $|\nu| = k + 1$. Now, suppose to the contrary that (56) fails. We have two cases:

Case 1. If $f \in W^{k+1,q}_{\mathrm{loc}}(\mathbb{C})$, then integration of (59) gives

$$\iint_{|z|<R} |D^\nu f(z)|^q \, d\partial\,(z) = \iint_{|z|<tR} |D^\nu f(z)|^q \, d\partial\,(z)$$

for all positive t and R. Hence $D^\nu f(z) = 0$ for each $|\nu| = k + 1$.

Case 2. If $f \in C^{k-1+\alpha}_{\mathrm{loc}}(\mathbb{C})$, then, by (58) and by the definition of the class $C^{k-1+\alpha}_{\mathrm{loc}}(\mathbb{C})$, we find that

$$D^\nu f(z) = \lim_{t\to 0} t^{-\alpha} D^\nu f(tz) = \begin{cases} 0 & \text{if} \quad 0 < \alpha < 1 \\ z(D^\nu f)_z(0) + \bar{z}(D^\nu f)_{\bar{z}}(0) & \text{if} \quad \alpha = 1 \end{cases}$$

for each $|\nu| = k - 1$.

In each case, we conclude that f must be a homogeneous polynomial. The number d in (57) is therefore a positive integer and

$$f(z) = \sum_{m+n=d} a_{mn} z^m \bar{z}^n,$$

or, equivalently,

$$\xi = \sum_{m+n=d} A_{mn} H(\xi)^m \overline{H(\xi)}^n.$$

We use this formula only for $|\xi| = 1$, in which case $H(\xi) = \xi + \epsilon\xi^{-3}$ and $\bar{H}(\xi) = \xi^{-1} + \epsilon\xi^3$. Thus,

$$\xi = \sum_{m+n=d} A_{mn}(\xi + \epsilon\xi^{-3})^m(\xi^{-1} + \epsilon\xi^3)^n. \tag{60}$$

This identity admits analytic continuation to all $\xi \neq 0$. In particular, we may evaluate (60) at each of the four possible values of the root $\sqrt[4]{-\epsilon}$ getting $(\sqrt[4]{-\epsilon})^{d+1} = A_{0d}(1 - \epsilon^2)^d$. This yields $d + 1 \equiv 0 \pmod 4$. On the other hand (60) evaluated at each of the four values of $\sqrt[4]{-1/\epsilon}$ gives

$$\left(\sqrt[4]{\frac{-1}{\epsilon}}\right)^{3d+1} = A_{d0}\left(\epsilon - \frac{1}{\epsilon}\right)^d,$$

which yields $3d + 1 = 0 \pmod 4$. Contradiction arises since $(d + 1) + (3d + 1) \equiv 2 \pmod 4$.

The proof of Theorem 1 is complete.

8. Notes and Remarks

Let us perform hodograph transformation of system (14) for the function $g(z) = |f(z)|^{\sqrt{p-1}-1} f(z)$, $f(0) = 0$. The local solutions of (14) take the form

$$g(z) = [y(z)]^n,$$

where n is a positive integer and y is a quasiconformal mapping in a neighborhood of the origin. The corresponding system for $y(z)$ can be written as follows:

$$\frac{\partial y}{\partial \bar{z}} = \frac{1 - \sqrt{p-1}}{1 + \sqrt{p-1}} \frac{\bar{y}^n}{y^n} \frac{\partial y}{\partial z}. \tag{61}$$

Denote by $F = F(\xi)$ the inverse of $y = y(z)$. Then (61) converts onto a linear system

$$F_{\bar{\xi}}(\xi) = \frac{\sqrt{p-1} - 1}{\sqrt{p-1} + 1} \frac{\bar{\xi}^n}{\xi^n} \overline{F_\xi(\xi)}.$$

An advantage in studying this system is that after an elementary substitution

$$W(\xi) = \left(\xi^n F - \frac{\sqrt{p-1} - 1}{\sqrt{p-1} + 1} \bar{\xi}^n \bar{F}\right) |\xi|^{-1 + p/2\sqrt{p-1}},$$

one obtains a very simple system of Cauchy-Riemann type

$$W_{\bar{\xi}}(\xi) = \frac{p-2}{4\sqrt{p-1}}\,\frac{1}{\xi}\,\overline{W(\xi)}. \tag{63}$$

In particular, successive differentiation of (63) leads to the C^{∞}-regularity result outside $\xi = 0$. To finish, we remark that the complex gradient $f = u_z$ of a p-harmonic function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a K-quasiregular mapping with

$$K = \max\left\{p-1, \frac{1}{p-1}\right\}.$$

It is known that K-quasiregular mappings are locally Hölder continuous with exponent K^{-1}, [1], [5]. In our case, it would give $f \in C^{\beta}_{\mathrm{loc}}(\Omega)$ with

$$\beta = \min\left\{p-1, \frac{1}{p-1}\right\}.$$

However, for the complex gradient $f = u_z$ we have a better result since $f \in C^{k-1,\alpha}_{\mathrm{loc}}(\Omega)$ and by (9) and (50)

$$k - 1 + \alpha = d(p) = \frac{1}{6}\left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right)$$

$$> \min\left\{p-1, \frac{1}{p-1}\right\} \quad \text{for} \quad 1 < p < \infty, \qquad p \neq 2.$$

A well known conjecture in quasiconformal analysis [4] asserts that plane K-quasiregular mappings belong to $W^{1,s}_{\mathrm{loc}}(\Omega)$ for any $2 < s < 2 + 2/(K-1)$. For the complex gradient f of a p-harmonic function, $p \neq 2$, we would obtain

$$f \in W^{1,s}_{\mathrm{loc}}(\Omega),$$

for any

$$2 < s < \begin{cases} \dfrac{2p-2}{p-2} & \text{if} \quad 2 < p < \infty \\[2ex] \dfrac{2}{2-p} & \text{if} \quad 1 < p < 2 \end{cases}.$$

However, for this special class of quasiregular mappings, Theorem 1 states a better result.

NOTE ADDED IN PROOF. We have received a preprint of G. Aronsson «Representation of a p-harmonic function in the plane», in which he obtains, via stream functions, representation of a p-harmonic function u near a singular point. As a corollary of this representation he also obtains a Hölder regularity result. This result, according to Theorem 1, is sharp if $p > 2$.

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A Microlocal F. and M. Riesz Theorem with Applications

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Introduction

Consider, by way of example, the following F. and M. Riesz theorem for $\mathbb{R}^n$: Let μ be a finite measure on $\mathbb{R}^n$ whose Fourier transform $\hat{\mu}$ is supported in a closed convex cone which is *proper*, that is, which contains no entire line. Then μ is absolutely continuous (*cf.* Stein and Weiss [SW]). Here, as in the sequel, «absolutely continuous» means with respect to Lebesque measure. In this theorem one can replace the condition on the support of $\hat{\mu}$ by a similar condition on the wave front set $WF(\mu)$ of μ, while keeping the same conclusion. The resulting «microlocal F. and M. Riesz theorem» can be applied with great flexibility to derive F. and M. Riesz theorems for measures on Lie groups, measures satisfying partial differential equations, etc. This is, essentially, the program of this paper.

Actually, the microlocal F. and M. Riesz theorem which we are going to use is much stronger than the one indicated above: it states that μ is absolutely continuous if $WF(\mu) \cap (-WF(\mu)) = \varnothing$; and, in fact, such a μ will be in the local H^1-space of Goldberg [G]. An important tool for the proof of this result is Uchiyama's characterization of the real Hardy space $H^1(\mathbb{R}^n)$, *cf.* [U]. This will be done in Section 1. In the remainder of this paper we give two applications, which we now describe.

In [B1] the author proved an F. and M. Riesz theorem for the unit sphere $S_{2n-1} \subseteq \mathbb{C}^n$ by completely different (group theoretic and functional analytic) methods. (The result in [B1] was actually for homogeneous spaces of compact groups whose center contains a circle group.) In Section 2 we prove a new theorem of this type for S_{2n-1}, which greatly extends some important special cases of the result of [B1]. An interesting question is whether one can regain the full F. and M. Riesz theorem of [B1] by the methods of the present paper.

It should be noted that the reasoning used in Section 2 can also be applied in more general situations. However, it seemed preferable first to treat a typical example rather than trying to formulate the most general result, *e.g.*, for compact Lie groups. (*Cf.* also [B2], where the main result of [B1] is extended to compact Lie groups.)

To motivate the second application, treated in Section 3, we consider the following formulation of the classical F. and M. Riesz theorem for $\mathbb{R}^n$. Let μ be a finite measure on $\mathbb{R}$ which is boundary value (in the weak-$*$ sense, say) of a holomorphic function $F(x+iy)$ defined in the upper half plane $\{x+iy : y>0\}$. Then μ is absolutely continuous. Holomorphic functions are solutions of the Cauchy-Riemann equations

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)F=0$$

and it is natural to ask whether one can replace the Cauchy-Riemann operator here by other vector fields

$$X=a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}\cdot$$

It turns out that the answer is «yes» if $b(x,0)\neq 0$ for all $x\in\mathbb{R}$, that is, if $\mathbb{R}\times\{0\}$ is not characteristic for X. If a/b is real, this is quite easy to see; for the case that $\operatorname{Im}(a/b)\neq 0$, we use the microlocal F. and M. Riesz theorem together with estimates on $WF(\mu)$; *cf.* Section 3 below for details.

More generally, let $P_1,\dots,P_N$ be N vector fields (with complex-valued coefficients) on $\mathbb{R}^{n+1}$, and let μ be a measure on $\mathbb{R}^n$ which is the boundary value (in distributional sense) of a function f on $\mathbb{R}^{n+1}_+ = \{(x,t) : x\in\mathbb{R}^n, t>0\}$ satisfying $P_j f=0$, $1\leqslant j\leqslant N$. For which P_j is such a μ necessarily absolutely continuous? In Section 3 we give a sufficient condition whose proof uses the microlocal F. and M. Riesz theorem. As a corollary we show that a measure μ on a hypersurface S in $\mathbb{C}^n$ which is the boundary value of a holomorphic function defined on one side of S is absolutely continuous.

Finally, I would like to thank J. Korevaar and J. Wiegerinck for their comments upon an earlier version of this paper, and M. Christ and D. Geller for some conversations on Uchiyama's theorem.

1. A Microlocal F. and M. Riesz Theorem

We use Uchiyama's powerful characterization of $H^1(\mathbb{R}^n)$ to derive an F. and M. Riesz theorem for $\mathbb{R}^n$, which we then microlocalize. We first recall the definitions of $H^1(\mathbb{R}^n)$ and of Goldberg's local Hardy space h^1.

Definition. *A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is in the real Hardy space $H^1(\mathbb{R}^n)$ if for some $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi}(0) \neq 0$,*

$$\|f\|_{H^1} \equiv \left\| \sup_{t>0} |\psi_t * f(\cdot)| \right\|_{L^1(\mathbb{R}^n)} < \infty \tag{1.1}$$

(where, as usual, $\psi_t(x) = t^{-n}\psi(x/t)$).

f is in $h^1(\mathbb{R}^n)$, Goldberg's local Hardy space, if

$$\|f\|_{h^1} \equiv \left\| \sup_{0<t<1} |\psi_t * f(\cdot)| \right\|_{L^1(\mathbb{R}^n)} < \infty. \tag{1.2}$$

For equivalent definitions and further properties of these spaces, *cf.* Fefferman and Stein [FS], Goldberg [G]. Note that both H^1 and h^1 are contained in $L^1(\mathbb{R}^n)$. In (1.1), one may replace $\psi_t(x)$ by the Poisson-Kernel for the upper half space. The interest of h^1 is that it can also be defined on manifolds (in the usual way, using coordinate charts), *cf.* [G, Proposition 3]. We will also use the following two properties of $h^1(\mathbb{R}^n)$, *cf.* [G]:

$$\mathcal{S} \subseteq h^1(\mathbb{R}^n) \tag{1.3}$$

$$\mathcal{S}(\mathbb{R}^n) \cdot h^1(\mathbb{R}^n) \subseteq h^1(\mathbb{R}^n) \tag{1.4}$$

(and more generally, $h^1(\mathbb{R}^n)$ is closed under 0-th order pseudo-differential operators).

The following notation will be useful:

$$h^1_{\text{loc}}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\text{: for every } \phi \in C_c^\infty(\mathbb{R}^n)\ \phi \cdot f \in h^1(\mathbb{R}^n)\}.$$

We now recall Uchiyama's characterization of $H^1(\mathbb{R}^n)$ (*cf.* [U]):

Theorem 1.1. *Let $\phi_1, \dots, \phi_k \in C^\infty(\mathbb{R}^n \setminus 0)$ be homogeneous of degree 0 such that*

$$\operatorname{rank}\begin{pmatrix} \phi_1(\xi) & \dots & \phi_k(\xi) \\ \phi_1(-\xi) & \dots & \phi_k(-\xi) \end{pmatrix} = 2, \quad \textit{for every} \quad \xi \in S. \tag{1.5}$$

Let K_j be the multiplier operator associated to ϕ_j: $(K_j f)^\wedge(\xi) = \phi_j(\xi)\hat{f}(\xi)$. Then, for $f \in L^1(\mathbb{R}^n)$

$$C_1 \| f \|_{H_1} \leqslant \sum_{j=1}^{k} \| K_j f \|_1 \leqslant C_2 \| f \|_{H_1}$$

(*with constants* C_1, C_2 *only depending on the* ϕ_j *and on* n).

If X is a manifold we let $M(X)$ and $M_{\text{loc}}(X)$ denote the spaces of finite and of locally finite measures on X. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, $\hat{u}$ denotes the Fourier transform of u.

From Theorem 1.1 one can derive the following F. and M. Riesz theorem.

Theorem 1.2. *Let* $F \subseteq \mathbb{R}^n$ *be a closed conic subset such that* $F \cap (-F) = \{0\}$. *Let* $\mu \in M(\mathbb{R}^n)$ *be such that* $\operatorname{supp} \hat{\mu} \subseteq F$. *Then* μ *is in* $H^1(\mathbb{R}^n)$ (*and in particular,* μ *is absolutely continuous*).

PROOF. Let $F' = F \cap S$, $S = S_{n-1}$ the unit sphere. Since $F' \cap (-F') = \emptyset$, there exists an open set $U \supseteq F'$, $U \subseteq S$, such that $U \cap (-U) = \emptyset$. Let

$$W = S \setminus (F' \cup -F')$$

and let Q_j denote the j-th «quadrant» in $\mathbb{R}^n$:

$$Q_1 = \{\xi = (\xi_1, \dots, \xi_n) : \xi_1 \geqslant 0, \xi_2 \geqslant 0, \dots, \xi_n \geqslant 0\},$$
$$Q_2 = \{\xi_1 \leqslant 0, \xi_2 \geqslant 0, \dots, \xi_n \geqslant 0\},$$

etc. Let $U_j = W \cap (\epsilon\text{-neighborhood of } Q_j)$, where ϵ is so small that $U_j \cap (-U_j) = \emptyset$. Then $\{U, -U, U_1, \dots, U_{2^n}\}$ is an open cover of S. Relabel the elements of this cover as $\{V_1, \dots, V_k\}$, with $V_1 = U$ (and $k = 2^n + 2$).

Let $\{\phi_1, \dots, \phi_k\}$ be a partition of unity subordinate to this cover, such that $\phi_1 \equiv 1$ on $F' \subseteq V_1 = U$. Then these ϕ_j's satisfy (1.5).

Now consider a $\mu \in M(\mathbb{R}^n)$ satisfying $\operatorname{supp} \hat{\mu} \subseteq F$. Let

$$P_y(x) = c_n \frac{y}{(|x|^2 + y^2)^{(n+1)/2}} \qquad (x \in \mathbb{R}^n, \quad y > 0)$$

denote the Poisson-kernel and define $f_\epsilon = P_\epsilon * \mu$. Then, with K_j the singular integral operator associated to ϕ_j as in Theorem 1.1,

$$K_1 f_\epsilon = f_\epsilon, \qquad K_j f_\epsilon = 0 \quad \text{for} \quad j \neq 1.$$

By Theorem 1.1, $\| f_\epsilon \|_{H^1} \leqslant C \| f_\epsilon \|_1 \leqslant C \| \mu \|$. By taking $\psi(x) = P_1(x)$ in (1.1) and letting $\epsilon \downarrow 0$ it now follows that

$$\left\| \sup_{y > 0} |P_y * \mu(\cdot)| \right\|_1 \leqslant C \| \mu \|$$

(we use that $P_{t_1} * P_{t_2} = P_{t_1 + t_2}$), which implies that $\mu \in H^1(\mathbb{R}^n)$. □

The following corollary will be needed below

Corollary 1.3. *Let $\nu \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered measure on $\mathbb{R}^n$ such that* $\operatorname{supp} \hat{\nu} \subseteq F$, *F as in Theorem* 1.2. *Then* $\nu \in h^1_{\text{loc}}(\mathbb{R}^n)$.

PROOF. We may suppose, without loss of generality, that $\operatorname{supp} \hat{\nu} \cap \{|\xi| \leqslant 1\} = \varnothing$. For if $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ for $|\xi| \leqslant 2$, $\chi(\xi) = 1$ for $|\xi| \geqslant 3$, then $(1 - \chi)\hat{\nu}$ is the Fourier transform of a tempered C^∞-function g, and $\nu - g$ is a tempered measure such that $\operatorname{supp}(\nu - g)^\wedge \cap \{|\xi| \leqslant 1\} = \varnothing$. By (1.3), $g \in h^1_{\text{loc}}(\mathbb{R}^n)$.

Now let $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\psi} \geqslant 0$, $\operatorname{supp} \hat{\psi} \subseteq \{|\xi| \leqslant 1\}$, $\psi(0) = 1$. Write $\psi^\epsilon(x) = \psi(\epsilon x)$. Then $\nu_\epsilon = \psi^\epsilon \cdot \nu$ is a finite measure on $\mathbb{R}^n$ such that $\operatorname{supp} \hat{\nu}_\epsilon = \operatorname{supp}(\psi^\epsilon * \hat{\nu})$ is contained in a conic ϵ-neighbourhood of F. By Theorem 1.2, $\nu_\epsilon \in H^1(\mathbb{R}^n) \subseteq h^1(\mathbb{R}^n)$ for sufficiently small ϵ. Now take $\phi \in C^\infty_c(\mathbb{R}^n)$ and let $\epsilon > 0$ be so small that $\nu_\epsilon \in h^1(\mathbb{R}^n)$ and $\psi^\epsilon(x) \neq 0$ on $\operatorname{supp} \phi$. Then $\phi \cdot \nu = (\phi/\psi^\epsilon) \cdot (\psi^\epsilon \nu)$ is in $h^1(\mathbb{R}^n)$ by (1.4). $\square$

We now microlocalize Theorem 1.2. If $X \subseteq \mathbb{R}^n$ is open and $u \in \mathcal{D}'(X)$ is a distribution on X, we let $WF(u) \subseteq X \times \mathbb{R}^n \backslash 0 = T^*(X) \backslash 0$ denote the wave front set of u (*cf.* Hörmander [H] for the definition). For $x \in X$ let $WF_x(u) = \{\xi \in \mathbb{R}^n \backslash 0 : (x, \xi) \in WF(u)\} = WF(u) \cap T^*_x(X)$. All this also makes sense if X is a manifold.

Theorem 1.4. *Let X be a manifold and $\mu \in M_{\text{loc}}(X)$ a locally finite measure such that*

$$WF_x(\mu) \cap -WF_x(\mu) = \varnothing, \quad \textit{for every} \quad x \in X. \tag{1.6}$$

Then μ is in $h^1_{\text{loc}}(X)$. In particular, μ is absolutely continuous with respect to any Lebesque measure on X.

Here $\mu \in h^1_{\text{loc}}(X)$ means that $\phi \cdot \mu \in h^1(\mathbb{R}^n)$ for all $\phi \in C^\infty_c(X)$ supported in a coordinate neighborhood.

PROOF. It suffices to prove the theorem for X an open subset of $\mathbb{R}^n$. We show that

(1.7) Given $x \in X$ there exists a neighborhood U_x of x such that for any $\phi \in C^\infty_c(U_x)$ $\phi \cdot \mu \in h^1(\mathbb{R}^n)$.

If $v \in \mathcal{E}'(\mathbb{R}^n)$ is a compactly supported distribution, let $\Sigma(v) \subseteq \mathbb{R}^n \backslash 0$ be the closed conic subset defined as follows: $\xi \notin \Sigma(v)$ if and only if there exists a conic neighborhood Γ of ξ such that for $\eta \in \Gamma$ and $N \in \mathbb{N}$

$$|\hat{v}(\eta)| \leqslant C_N(1 + |\eta|)^{-N}. \tag{1.8}$$

Then $WF_x(u) = \bigcap \{\Sigma(\phi u): \phi \in C_c^\infty(X),\ \phi(x) \neq 0\}$ and $\Sigma(\phi u) \to WF_x(u)$ as $\operatorname{supp}\phi \to \{x\}$, ϕ ranging over $C_c^\infty(X)$-functions for which $\phi(x) \neq 0$ (*cf.* [H], Section 8.1).

Now suppose that $\mu \in M_{\mathrm{loc}}(X)$ satisfies (1.6) and let $x \in X$. There exists a conic open $\Gamma \subseteq \mathbb{R}^n \backslash 0$ such that $\Gamma \supseteq WF_x(\mu)$ and such that $\Gamma \cap (-\Gamma) = \varnothing$.

Let $\Delta \subseteq \mathbb{R}^n \backslash 0$ be an open conic subset such that $WF_x(\mu) \subseteq \Delta \subseteq \bar{\Delta} \subseteq \Gamma$ and let $U = U_x$ be an open neighborhood of x such that for $\phi \in C_c^\infty(U)$ with $\phi(x) \neq 0$. $\Sigma(\phi\mu) \subseteq \Delta$. Take such a ϕ. Then $\widehat{\phi\mu}$ is rapidly decreasing (in the sense of (1.8)) on $\mathbb{R}^n \backslash \bar{\Delta}$. Let $\chi \in C^\infty(\mathbb{R}^n)$, $0 \leqslant \chi \leqslant 1$, such that $\operatorname{supp} \chi \subset \Gamma \backslash \{|\xi| \leqslant 1\}$ and such that $\chi = 1$ on $\Delta \backslash \{|\xi| \leqslant 2\}$. The inverse Fourier transform g of $(1 - \chi) \cdot \widehat{\phi\mu}$ is then in $C^\infty(\mathbb{R}^n)$ and $\nu = \phi\mu - g\,dx$ is a tempered measure such that $\operatorname{supp} \hat{\nu} \subseteq \Gamma$. By Corollary 1.3, $\nu \in h^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence $\phi \cdot \mu \in h^1_{\mathrm{loc}}(\mathbb{R}^n)$. This proves (1.7) for those ϕ with $\phi(x) \neq 0$, which obviously suffices. □

2. F. and M. Riesz for the Unit Sphere in $\mathbb{C}^n$

We now use Theorem 1.4 to prove an F. and M. Riesz theorem for the unit sphere S in $\mathbb{C}^n$. For the statement we need some notation from the theory of spherical harmonics on S, *cf.* Rudin [R], Chapter 12. Let σ denote the rotation invariant measure on S, normalized by $\sigma(S) = 1$, say. Let $H(p, q)$ be the set of restrictions to S of harmonic functions u on $\mathbb{C}^n$ which are homogeneous of degree p in z and of degree q in $\bar{z}$. Then $L^2(S, \sigma) = \Sigma_{p,q} H(p, q)$ (orthogonal direct sum). Let π_{pq} denote the orthogonal projection onto $H(p, q)$; π_{pq} can be extended to distributions.

For a finite measure μ we let the spectrum of μ be $\operatorname{spec}\mu = \{(p, q): \pi_{pq}\mu \neq 0\}$. If $F \subseteq \mathbb{R}_+ \cup \{\infty\}$, let $i(F) = \left\{\frac{1}{\alpha}: \alpha \in F\right\}$ $\left(\text{where } \frac{1}{0} = \infty,\ \frac{1}{\infty} = 0, \text{ as usual}\right)$. Also, let $\Sigma(F) \subseteq \mathbb{N} \times \mathbb{N}$ be defined by

$$\Sigma(F) = \left\{(p, q) \in \mathbb{N} \times \mathbb{N}: \frac{q}{p} \in F\right\}.$$

Our F. and M. Riesz theorem for S is the following.

Theorem 2.1. *Suppose that $F \subseteq \mathbb{R}_+ \cup \{\infty\}$ is a closed subset such that $F \cap i(F) = \varnothing$. Let μ be a finite measure on S such that* $\operatorname{spec}\mu \subseteq \Sigma(F)$. *Then μ is in $h^1(S)$. In particular, μ is absolutely continuous with respect to σ.*

EXAMPLES.

(i) $F = [0, \alpha]$, $\alpha < 1$, and $\Sigma(F) = \{(p, q): q \leqslant \alpha p\}$. This special case of Theorem 2.1 is also contained in [B1, Theorem 1.1].

(ii) $F = [0, \alpha] \cup [\beta, \gamma]$ with $\alpha < 1 < \beta \leqslant \gamma$ and $\gamma < 1/\alpha$. In this case $\Sigma(F)$ looks like the union of two cones such that the reflection of one with respect to the line $p = q$ has zero intersection with the other. Note that, contrary to one of the conditions of Theorem 1.1 of [B1], $\{p - q: (p, q) \in \Sigma(F)\}$ is not bounded from below or from above anymore.

PROOF OF THEOREM 2.1. We are going to exploit the fact that $H(p, q)$ is the simultaneous eigenspace of two commuting self-adjoint differential operators on S, namely the Laplace-Beltrami operator Δ_S and the tangential vector field T defined by

$$Tf(\zeta) = \frac{1}{i} \frac{d}{d\theta} f(e^{i\theta}\zeta)\bigg|_{\theta=0}.$$

In fact, if we write

$$\nu = (-\Delta_S + (n-1)^2)^{1/2} - (n-1),$$

then ν is a first order pseudo-differential operator on S with eigenvalue k on the eigenspace $\mathcal{H}(k) = \{u: \Delta u = 0 \text{ on } \mathbb{C}^n,\ u(rz) = r^k u(z) \text{ for } r \geqslant 0\}$ (*cf.* Taylor [T2, Chapter 4]). Since

$$\mathcal{H}(k) = \sum_{p+q=k} H(p, q),$$

it follows that

$$H(p, q) = \{u \in L^2(S): \nu(u) = (p+q)u,\ T(u) = (p-q)u\}. \tag{2.1}$$

Let $F \subseteq \mathbb{R}_+ \cup \{\infty\}$ be a closed subset satisfying $F \cap i(F) = \emptyset$, and let $a(x, y) \in C^\infty(\mathbb{R}^2 \setminus 0)$ be homogeneous of degree 0 such that

$$F = \{y/x: a(x, y) = 0\}. \tag{2.2}$$

Now if spec $\mu \subseteq \Sigma(F)$, μ is annihilated by the operator

$$\sum_{p,q} a(p, q)\pi_{p,q} \tag{2.3}$$

(because of (2.2)). Writing

$$\tilde{a}(x, y) = a\left(\frac{x+y}{2}, \frac{x-y}{2}\right),$$

we see that (2.3) is equal to the operator $\tilde{a}(\nu, T)$ as defined using the spectral theorem, *cf.* (2.1). By a result of Strichartz [S] (*cf.* also Section 12.1 in Taylor [T1]), $\tilde{a}(\nu, T)$ is a first order pseudo-differential operator on S and, if we denote the principal symbol of a pseudo-differential operator A by $\sigma(A)$,

$$\sigma(\tilde{a}(\nu, T)) = \tilde{a}(\sigma(\nu), \sigma(T)). \tag{2.4}$$

(Strictly speaking, we should have made $\tilde{a}$ smooth in 0, but this would only change $\tilde{a}(\nu, T)$ by a smoothing operator.)

Now $\tilde{a}(\nu, T)\mu = 0(\mathrm{mod}\, C^\infty)$ implies that

$$WF(\mu) \subset \mathrm{Char}\, \tilde{a}(\nu, T) = \{(z, \xi) \in T^*(S): \sigma(\tilde{a}(\nu, T))(z, \xi) = 0\}.$$

To finish the proof we compute $\sigma(\tilde{a}(\nu, T))$. If $z \in S$ we let

$$T_z(S) = T_z^{\mathbb{C}}(S) + \mathbb{R} \cdot iz$$

be the usual splitting of the tangent space in $\mathbb{C}^n$, with

$$T_z^{\mathbb{C}}(S) = \{\xi \in \mathbb{C}^n: \langle z, \xi \rangle = 0\}$$

the maximal complex subspace of $T_z(S)$ ($\langle \bullet, \bullet \rangle$ being the standard Hermitian inner product on $\mathbb{C}^n$). Identify $T_z(S)$ and $T_z^*(S)$, using the Riemannian metric on S induced by $\mathbb{C}^n$. If $\xi \in T_z(S)$, $\xi = \xi' + \theta \cdot iz$ with $\xi' \in T_z^{\mathbb{C}}(S)$, $\theta \in \mathbb{R}$, then

$$\begin{aligned} \sigma(\nu)(z, \xi) &= c \cdot (|\xi'|^2 + \theta^2)^{1/2} \\ \sigma(T)(z, \xi) &= \theta. \end{aligned} \tag{2.5}$$

By (2.4),

$$\sigma(\tilde{a}(\nu, T))(z, \xi) = a\left(\frac{c(|\xi'|^2 + \theta^2)^{1/2} + \theta}{2}, \frac{c(|\xi'|^2 + \theta^2)^{1/2} - \theta}{2}\right).$$

Now suppose that there exists a $(z, \xi) \in \mathrm{Char}\,(\tilde{a}(\nu, T))$ such that also $(z, -\xi) \in \mathrm{Char}\,(\tilde{a}(\nu, T))$, $\xi = \xi' + \theta \cdot iz$ as above, $\xi \neq 0$. Then

$$\frac{c(|\xi'|^2 + \theta^2)^{1/2} - \theta}{c(|\xi'|^2 + \theta^2)^{1/2} + \theta} \in F \cap i(F),$$

contradicting the assumption on F. Hence $WF(\mu)$ satisfies the condition of Theorem 1.4 and hence μ is in $h^1(S)$. □

Probably this type of argument can be used in more general situations, *e.g.*, measures on homogeneous spaces of compact Lie groups. However, the formulation of an analogon of Theorem 2.1 is likely to become much more complicated. *Cf.* for example [B2], where an F. and M. Riesz theorem for arbitrary compact Lie groups can be found which generalizes the one of [B1]. The reason for these complications is that one has to refine the notion of spectrum.

3. Absolute Continuity of Measures Arising as Boundary Values of Solutions of Partial Differential Equations

Let $X \subseteq \mathbb{R}^n$ be open and let U be an open neighborhood of $X \times \{0\}$ in $\mathbb{R}^{n+1}$; let $U_+ = U \cap \{(x,t): x \in \mathbb{R}^n, t > 0\}$. Let $P_1, \ldots, P_N$ be a set of first order linear partial differential operators with C^∞-coefficients defined on the closure of U_+. Consider measures μ on X which arise in the following way: there is an $f \in C^1(U_+)$ satisfying $P_j f = 0$, $1 \leqslant j \leqslant N$, such that μ is the limit, in $\mathfrak{D}'(X)$, of $f(x,t)$ as $t \downarrow 0$. The question with which we concern ourselves here is for which P_j such a μ necessarily is absolutely continuous. We will give a sufficient condition for P's which are vectorfields:

$$P_j = c_j \partial_t + \langle a_j, \partial_x \rangle = c_j(x,t)\frac{\partial}{\partial t} + \sum_\mu a_{j\mu}(x,t)\frac{\partial}{\partial x_\mu}. \tag{3.1}$$

For x in X let $J(x) = \{j: 1 \leqslant j \leqslant N, c_j(x,0) \neq 0\}$. Then the main result of this section is the following:

Theorem 3.1. *All notation as above. Let P_j be given by* (3.1). *Suppose that for all $x \in X$ the following closed convex cone is proper* (*i.e. contains no straight lines*):

$$\bigcap_{j \in J(x)} \{\xi \in \mathbb{R}^n: \operatorname{Im}(c_j(x,0)^{-1}\langle a_j(x,0), \xi\rangle) \leqslant 0\} \tag{3.2}$$

$$\cap \bigcap_{j,k} \{\xi \in \mathbb{R}^n: c_j(x,0)\langle a_k(x,0), \xi\rangle = c_k(x,0)\langle a_j(x,0), \xi\rangle\}.$$

Let μ be a locally finite measure on X which is the distributional boundary value $\lim_{t \downarrow 0} f(\cdot, t) = \mu$ of an $f \in C^1(U_+)$ that satisfies $P_j f = 0$, $1 \leqslant j \leqslant N$, and for which there exists an $M \in \mathbb{N}$ such that

$$|f(x,t)|, |\partial_x f(x,t)| = O(t^{-M}), \tag{3.3}$$

uniformly on compacta of X. Then μ is in $h^1_{\mathrm{loc}}(X)$.

If $a_j(x,0) = \partial_t a_j(x,0) = \cdots = \partial_t^{k-1} a_j(x,0) \equiv 0$ on X one may replace $a_j(x,0)$ in the first line of (3.2) *by $\partial_t^k a_j(x,0)$.*

Before giving the proof of Theorem 3.1 let us make some remarks. If one takes $n = N = 1$ and P_1 to be the Cauchy-Riemann operator on $\mathbb{C} = \mathbb{R}^2$, one obviously obtains (a local version of) the classical F. and M. Riesz theorem for $\mathbb{R}$. More generally, one can show using Theorem 3.1 that a measure on a hypersurface S in $\mathbb{C}^n$ which is boundary value (in distribution sense) of a holomorphic function defined on one side of S is absolutely continuous with respect to surface measure (just straighten out S locally and then apply Theorem 3.1).

It is clear that Theorem 3.1 is meaningless if the hypersurface $\{t = 0\}$ is characteristic for all P_j. The following easy example shows that Theorem 3.1 is false in this case: Take $n = N = 1$ and let

$$f(x, t) = \pi^{-1/2} t^{-1} e^{-x^2/2t^2}, \qquad x \in \mathbb{R}, \quad t > 0.$$

Then $f(x, t) \to \delta(x)$ as $t \downarrow 0$ and f satisfies the partial differential equation

$$(x^2 - t^2)\frac{\partial f}{\partial x} + xt\frac{\partial f}{\partial t} = 0.$$

Also, the cone on the right hand side of (3.1) can not be proper if $c_j^{-1} a_j$ is real on $\{t = 0\}$ for all j. But if one of the P's, say P_1, has real coefficients and $\{t = 0\}$ is not characteristic for P_1, μ is absolutely continuous for trivial reasons: $P_1 f = 0$ means that f is constant along the characteristics of P_1, which intersect $\{t = 0\}$ transversally. Hence one can extend f in a C^1-way to a neighborhood of $X \times \{0\}$ in $\mathbb{R}^{n+1}$ in such a way that this extension of f is still annihilated by P. If follows that Theorem 3.1 is only interesting in case all non-characteristic P's have complex coefficients.

Finally note that if $n > 1$, f has to satisfy an *overdetermined* system for the cone (3.1) to be proper.

The proof of Theorem 3.1 consists of showing that $WF(\mu)$ is contained in the cone (3.1). The conclusion then follows by Theorem 1.4. To estimate $WF(\mu)$ we first estimate the wave front set of a distribution which is the boundary value of a function annihilated by a single vector field. The arguments we will use have been inspired by Hörmander's treatment of this problem for the Cauchy-Riemann operator (*cf.* [H]) but are more involved since we are dealing with variable coefficient operators.

Let P be a vector field of the form $P = \partial_t + \langle a, \partial_x \rangle$. (This could be relaxed at times.) Let P^* denote the formal (real) adjoint of P: $P^* = -\partial_t - \langle \partial_x, a \rangle$.

Lemma 3.2. *Let $T > 0$ and let $u(x, t)$, $v(x, t) \in C^1(\mathbb{R}^n \times [0, T])$ be such that* supp $v(\cdot, t)$ *is contained in a fixed compactum $K \subset \mathbb{R}^n$ for all t, $0 \leqslant t \leqslant T$. Then*

$$\int_{\mathbb{R}^n} u(x, T) v(x, T)\, dx - \int_{\mathbb{R}^n} u(x, 0) v(x, 0)\, dx = \int_0^T \int_{\mathbb{R}^n} ((Pu)v - u(P^* v))\, dx\, dt.$$

PROOF. Integration by parts. □

For simplicity we first consider the case where $P = \partial_t + \langle a, \partial_x \rangle$ with a not depending on t.

Theorem 3.3. *Let $X \subseteq \mathbb{R}^n$ be open, U an open neighborhood of $X \times \{0\}$ in $\mathbb{R}^{n+1}$, $U_+ = U \cap \mathbb{R}^{n+1}_+$. Let $P = \partial_t + \langle a, \partial_x \rangle, a = a(x)$ C^∞ on X. Let $f \in C^1(U_+)$ be such that $Pf \in L^\infty(U_+)$ while for some $N \in \mathbb{N}$,*

$$|f(x, t)| = O(t^{-N}) \quad \text{as} \quad t \downarrow 0,$$

uniformly on compacta of X. Then $\lim_{t \downarrow 0} f(x, t) = f(x, 0+)$ exists in $\mathcal{D}'(X)$.

PROOF. Let $\phi \in C_c^\infty(X)$. Let $T > 0$ be such that $\operatorname{supp} \phi \times [0, 2T] \subseteq X \cup U_+$. Let $k \in \mathbb{N}$. Determine $\phi_0, \phi_1, \ldots, \phi_k \in C^\infty(\bar{U}_+)$ such that

$$\Phi(x, t) \equiv \Phi^{(k)}(x, t) = \sum_{j=0}^{k} \phi_j(x, t) \frac{t^j}{j!}$$

satisfies the conditions

(i) $\Phi(x, 0) = \phi(x)$.
(ii) $|P^*\Phi(x, t)| \leqslant Ct^k$, $x \in \operatorname{supp} \phi$, $0 \leqslant t \leqslant T$.

The constant C here depends on the derivatives of ϕ up till order $N + 1$.

To prove the existence of Φ, write $P = \partial_t + Q(x, \partial_x)$. Then

$$P^*[\Phi] = \sum_{j=0}^{k-1} \left(-\frac{\partial \phi_j}{\partial t} + Q^*[\phi_j] - \phi_{j+1} \right) \frac{t^j}{j!} + \left(-\frac{\partial \phi_k}{\partial t} + Q^*[\phi_k] \right) \frac{t^k}{k!}$$

and one need only take $\phi_0(x, t) = \phi(x)$, $\phi_j = -\partial_t \phi_{j-1} + Q^*[\phi_{j-1}]$, $1 \leqslant j \leqslant k$. In the present case the ϕ_j do not depend on the variable t, but they will do it in the proof of the next theorem, when Q depends on t. Note, that $\operatorname{supp} \Phi(\cdot, t) \subseteq \operatorname{supp} \phi$, $0 \leqslant t \leqslant T$.

Let $0 < \epsilon < T$ and write $f_\epsilon(x, t) = f(x, t + \epsilon)$ $(t < T)$. Apply Lemma 3.2 with $u = f_\epsilon$, $v = \Phi^{(k)} = \Phi$. Then

$$\int_X f(x, \epsilon)\phi(x)\, dx = \int_X f(x, T + \epsilon)\Phi(x, T)\, dx - \int_{X \times (0, T)} (Pf_\epsilon)\Phi \, dx\, dt + \int_{X \times (0, T)} f_\epsilon(P^*\Phi)\, dx\, dt.$$

Now

$$|f_\epsilon(x, t) \cdot P^*\Phi(x, t)| \leqslant Ct^{k-N}, \qquad (x, t) \in \operatorname{supp} \phi \times (0, T),$$

C independent of ϵ, and

$$\sup |P[f_\epsilon](x, t)| \leqslant \|Pf\|_{L^\infty(U_+)},$$

since $P[f_\epsilon](x, t) = P[f](x, t + \epsilon)$.

Take $k \geqslant N$ and let $\epsilon \to 0$. Then, by Lebesgue's dominated convergence theorem,

$$\begin{aligned} \langle f(\cdot, 0+), \phi \rangle &= \lim_{\epsilon \downarrow 0} \langle f(\cdot, \epsilon), \phi \rangle \\ &= \int_X f(x, T)\Phi^{(k)}(x, T)\, dx - \int_{X \times (0,T)} Pf \cdot \Phi^{(k)}\, dx\, dt \\ &\quad + \int_{X \times (0,T)} fP^*\Phi^{(k)}\, dx\, dt. \end{aligned} \tag{3.4}$$

□

If a is allowed to depend on t in Theorem 3.3 the proof may fail: the main problem is that $\sup_{\epsilon > 0} |Pf_\epsilon|$ (where $f_\epsilon(x, t) = f(x, t + \epsilon)$) need not be in L^1. We now show how to modify the proof in this case in order to arrive at the following result.

Theorem 3.4. *Let X, U_+ be as in Theorem 3.3, $P = \partial_t + \langle a, \partial_x \rangle$, $a = a(x, t)$ be of class C^∞ on $X \cup U_+$. Let $f \in C^1(U_+)$ be such that $Pf \in L^\infty(U_+)$ while for some $N \in \mathbb{N}$,*

$$|f(x, t)|, |\partial_x f(x, t)| = O(t^{-N}) \quad \textit{as} \quad t \to 0, \tag{3.5}$$

uniformly on compacta of X. Then $\lim_{t \to 0} f(x, t) = f(x, 0+)$ exists in $\mathfrak{D}'(X)$ and formula (3.4) for $f(\cdot, 0+)$ remains valid.

PROOF. The idea is to replace $f(x, t + \epsilon)$ in the proof of Theorem 3.3 by

$$f_\epsilon(x, t) = f(x + \Psi_\epsilon(x, t), t + \epsilon),$$

where Ψ_ϵ is a C^∞-function on $X \cup U_+$ which is to be determined such that

$$P[f_\epsilon](x, t) = P[f](x + \Psi_\epsilon, t + \epsilon) + O(1), \tag{3.6}$$

uniformly in ϵ and x, t. Furthermore, Ψ_ϵ has to satisfy

$$\Psi_\epsilon(x, 0) = 0, \tag{3.7a}$$

$$\Psi_\epsilon(x, t), \partial_t \Psi_\epsilon(x, t), \partial_x \Psi_\epsilon(x, t) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.7b}$$

Retracing the steps of the previous proof with this f_ϵ one sees that Theorem 3.4 is true: Lebesgue's dominated convergence theorem can be applied as at the end of the proof of Theorem 3.3 because of (3.6). Finally, because of (3.7b), $P[f_\epsilon](x, t) \to P[f](x, t)$ as $\epsilon \to 0$ so that formula (3.4) remains valid also.

A straightforward calculation shows that, with the notations $\tilde{x} = x + \Psi_\epsilon(x, t)$, $\tilde{t} = t + \epsilon$, $\Psi = \Psi_\epsilon$, Ψ'_x = Jacobian of Ψ with respect to x,

$$P[f_\epsilon](x, t) - P[f](\tilde{x}, \tilde{t}) = \langle \partial_t \Psi + (Id + \Psi'_x)a(x, t) - a(\tilde{x}, \tilde{t}), \partial_x f(\tilde{x}, \tilde{t}) \rangle.$$

Beacuse of (3.5) it suffices to determine $\Psi = \Psi_\epsilon$ in such a way that

$$\partial_t \Psi + a(x, t) + \Psi'_x \cdot a(x, t) - a(\tilde{x}, \tilde{t}) = O(t^N). \tag{3.8}$$

We try a solution of the form

$$\Psi_\epsilon = \sum_{j=1}^{N} \Psi_{j,\epsilon}(x) \frac{t^j}{j!}.$$

With such a Ψ_ϵ equation (3.7*a*) is automatically satisfied. Expand the left hand side of (3.8) in a Taylor series in t, up till order N, while treating $a(x, t)$ in the following way:

$$\begin{aligned} a(x + \Psi_\epsilon, t + \epsilon) &\approx \sum_{\substack{\alpha \in \mathbb{N}^n \\ k \in \mathbb{N}}} \frac{1}{\alpha!\, k!} (\partial_x^\alpha \partial_t^k a)(x, \epsilon)(\Psi_\epsilon)^\alpha t^k \\ &= a(x, \epsilon) + (\partial_t a(x, \epsilon) + \partial_x a(x, \epsilon) \cdot \Psi_{1,\epsilon}) t \\ &\quad + \Bigg(\partial_t^2 a(x, \epsilon) + \partial_x a(x, \epsilon) \cdot \Psi_{2,\epsilon} + \sum_{|\alpha| = 2} \frac{2!}{\alpha!} \partial_x^\alpha a(x, \epsilon)(\Psi_{1,\epsilon})^\alpha \\ &\quad + \sum_{|\alpha| = 1} 2!\, \partial_x^\alpha \partial_t a(x, \epsilon)(\Psi_{1,\epsilon})^\alpha \Bigg) \frac{t^2}{2!} + \cdots \end{aligned}$$

If one sets the coefficient of t^j in (3.8) equal to 0 for $j < N$ one obtains a system of equations for $\Psi_{j,\epsilon}(x)$ of the form

$$\Psi_{j,\epsilon}(x) = \{\text{expression in } \Psi_{1,\epsilon}, \ldots, \Psi_{j-1,\epsilon} \text{ and their derivatives}\}$$

which can be solved recursively in a unique way.

The first equation yields

$$\Psi_{1,\epsilon}(x) = a(x, \epsilon) - a(x, 0).$$

It is clear that $\Psi_{j,\epsilon}(x)$ is a C^∞-function of $\epsilon \geqslant 0$ and x and that $\Psi_{j,0}(x) \equiv 0$. Hence equations (3.7*b*) are satisfied. □

We now estimate $WF(f(\cdot, 0+))$ for solutions f of $Pf = 0$.

Theorem 3.5. *All notations as in Theorem* 3.4. *Suppose that* $f \in C^1(U_+)$ *satisfies* (3.5) *and*

$$|Pf(x, t)| = O(t^k), \qquad k = 1, 2, 3, \ldots, \tag{3.9}$$

uniformly on compacta of X. *Then*

$$WF(f(\cdot, 0+)) \subseteq \{(x, \xi) \in X \times \mathbb{R}^n \backslash 0 : \operatorname{Im} \langle a(x, 0), \xi \rangle \leqslant 0\}.$$

PROOF. We have to show that if $\operatorname{Im}\langle a(x_0, 0), \xi_0\rangle > 0$, there exists a $\phi \in C_c^\infty(X)$, $\phi(x_0) \neq 0$, such that

$$|\langle f(\cdot, 0+), e^{-i\langle\cdot,\xi\rangle}\phi\rangle| \leqslant C_k(1+|\xi|)^{-k}, \qquad k = 1, 2, \ldots$$

for ξ in a conic neighborhood of ξ_0.

Fix $k \in \mathbb{N}$. We are going to determine functions $b_1, \ldots, b_k$ of (x, ξ) such that

$$u_{\xi,k}(x,t) = \exp\left(-i\langle x, \xi\rangle + \sum_{j=1}^{k} b_j(x,\xi)\frac{t^j}{j!}\right)$$

is an approximate solution of $Pu = 0$ in the sense that on compacta of X and for small t,

$$|Pu_{\xi,k}(x,t)| \leqslant C|\xi|t^k.$$

Note that $u_{\xi,k}(x, 0) = e^{-i\langle x,\xi\rangle}$.

A computation shows that

$$\text{(3.10)} \qquad Pu_{\xi,k} = \left(\sum_{j=1}^{k} b_j \frac{t^{j-1}}{(j-1)!} - i\langle a, \xi\rangle + \sum_{j=1}^{k}\langle a, \partial_x b_j\rangle \frac{t^j}{j!}\right)u_{\xi,k}.$$

Write

$$a(x,t) = \sum_{l=0}^{k-1} a^{(l)}(x)\frac{t^l}{l!} + a^{(k)}(x,t)\frac{t^k}{k!},$$

where $a^{(j)} = \partial^j{}_t a$, and $a^{(j)}(x) = a^{(j)}(x, 0)$. Then

$$\sum_{j=1}^{k}\langle a, \partial_x b_j\rangle\frac{t^j}{j!} = \sum_{j=1}^{2k}\left(\sum_{l=\max(1,j-k)}^{\min(j,k)}\binom{j}{l}\langle a^{(j-l)}, \partial_x b_l\rangle\right)\frac{t^j}{j!}.$$

Substitute this expression in (3.10) and put the coefficient of t^j equal to 0 for $j \leqslant k-1$. Then

$$\text{(3.11)} \qquad b_{j+1} = i\langle a^{(j)}, \xi\rangle - \sum_{l=1}^{j}\binom{j}{l}\langle a^{(j-l)}, \partial_x b_l\rangle, \qquad 0 \leqslant j \leqslant k-1.$$

In particular, $b_1(x,\xi) = i\langle a(x,0), \xi\rangle$. Note that $b_1, \ldots, b_k$, as defined by (3.11), do not depend on t, since $a^{(0)}, \ldots, a^{(k-1)}$ only depend on x. It follows from (3.10) that the $b_j(x,\xi)$ are all linear in ξ.

Let $(x_0, \xi_0) \in X \times \mathbb{R}^n\backslash 0$ be such that $\operatorname{Re}(i\langle a(x_0,0), \xi_0\rangle) = -\operatorname{Im}\langle a(x_0, 0), \xi_0\rangle < 0$. Then there exist a neighborhood $U(x_0)$ of x_0, a conic neighborhood $V(\xi_0)$ of ξ_0 and a $T > 0$ such that for $x \in U(x_0)$, $\xi \in V(\xi_0)$ and $t \leqslant T$,

$$\text{(3.12)} \qquad \operatorname{Im}\langle a(x,0), \xi\rangle > 2c|\xi| > 0$$

(c a suitable constant) and

$$\operatorname{Re} \sum_{j=2}^{k} b_j(x, \xi) \frac{t^j}{j!} \leqslant \frac{1}{2} \operatorname{Im} \langle a(x, 0), \xi \rangle t.$$

Hence for $x \in U(x_0)$, $\xi \in V(\xi_0)$, $t \leqslant T$:

$$|u_{\xi, k}(x, t)| \leqslant e^{-(1/2) \operatorname{Im} \langle a(x, 0), \xi \rangle t} \tag{3.13}$$

and

$$|Pu_{\xi, k}(x, t)| \leqslant C(k) |\xi| t^k e^{-(1/2) \operatorname{Im} \langle a(x, 0), \xi \rangle t} \tag{3.14}$$

Let $\phi \in C_c^\infty(U(x_0))$ be arbitrary. We now apply formula (3.4) to

$$f(x, 0+) e^{-i\langle x, \xi \rangle} = \lim_{t \downarrow 0} f(x, t) u_{\xi, k}(x, t).$$

Since $P(fu_{\xi,k}) = Pf \cdot u_{\xi,k} + f \cdot Pu_{\xi,k}$ and since Pf and $P^*[\Phi]$ are both $O(t^k)$, $\Phi = \Phi^{(k)}$ as in (3.4), the inequalities (3.12), (3.13) and (3.14) lead to the following estimate for $|\xi| \geqslant 1$ and $k \geqslant N$ (we assume that $T < 1$):

$$|\langle f(\cdot, 0+) e^{-i\langle \cdot, \xi \rangle}, \phi \rangle| \leqslant C \cdot \left(e^{-c|\xi| T} + \int_0^T |\xi| t^{k-N} e^{-c|\xi| t} \, dt \right) \leqslant \frac{C(k)}{|\xi|^{k-N}}.$$

where the constants depend on the supremum norms of ϕ and its derivatives up till order $k + 1$. Since k is otherwise arbitrary, this proves the theorem. □

Remark 3.6. The proof also shows that if

$$a(x, 0) = \partial_t a(x, 0) = \cdots = \partial_t^{l-1} a(x, 0) \equiv 0$$

on X Theorem 3.5 holds with $a(x, 0)$ replaced by $\partial_t^l a(x, 0)$.

PROOF OF THEOREM 3.1. After these preparations we can now easily prove Theorem 3.1. Let $f \in C^1(U_+)$ and μ be as in the theorem. Let $x \in X$ and $j \in J(x)$. Then for y in a neighborhood of x and t small,

$$\partial_t f(y, t) + c_j(y, t)^{-1} \langle a_j(y, t), \partial_x f(y, t) \rangle = 0.$$

By Theorems 3.4 and 3.5,

$$WF(\mu) \subseteq \{(x, \xi) \in X \times \mathbb{R}^n \setminus 0 \colon \text{for every } j \in J(x) \colon \operatorname{Im} (c_j(x, 0)^{-1} \langle a_j(x, 0), \xi \rangle) \leqslant 0\}. \tag{3.15}$$

Now fix j and k and eliminate $\partial_t f$ from $P_j f = P_k f = 0$. It follows that $c_j \langle a_k, \partial_x f \rangle - c_k \langle a_j, \partial_x f \rangle = 0$ on U_+. Hence $\mu = f(\cdot, 0+)$ satisfies the *induced* *equations*

$$L_{jk}\mu = \langle c_j(x,0)a_k(x,0) - c_k(x,0)a_j(x,0), \partial_x \mu \rangle = 0.$$

By [H, Theorem 8.3.1],

$$(3.16) \quad WF(\mu) \subseteq \bigcap_{j,k} \operatorname{Char}(L_{jk})$$

$$= \bigcap_{j,k} \{(x,\xi) \in X \times \mathbb{R}^n \setminus 0 : c_j(x,0)\langle a_k(x,0), \xi\rangle = c_k(x,0)\langle a_j(x,0), \xi\rangle \}.$$

By (3.15), (3.16) and the hypothesis of Theorem 3.1, $WF(\mu)$ is proper. Hence $\mu \in h^1_{\mathrm{loc}}(X)$. □

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An Extremal Property of Entire Functions with Positive Zeros

Daniel F. Shea and Allen Weitsman

1. Introduction

Let $f(z)$ be a Weierstrass product of finite genus q with zeros $z_\nu \neq 0$ so that

$$f(z) = \prod_{\nu=1}^{\infty} E_q(z/z_\nu) \tag{1.1}$$

where

$$E_q(u) = \begin{cases} 1 - u & q = 0 \\ (1-u)\exp(u + u^2/2 + \cdots + u^q/q) & q > 0 \end{cases} \tag{1.2}$$

is the usual Weierstrass primary factor and

$$\sum_{\nu=1}^{\infty} |z_\nu|^{-q-1} < \infty.$$

Put

$$\hat{f}(z) = \prod_{\nu=1}^{\infty} E_q(z/|z_\nu|), \tag{1.3}$$

and define

$$u(re^{i\varphi}, f) = \sup_{\theta} \{\log|f(re^{i(\theta+\varphi)})| + \log|f(re^{i(\theta-\varphi)})|\}. \tag{1.4}$$

Let $n(r, 0)$ and $N(r, 0)$ be the counting functions for the zeros of f [H; p. 6].

Theorem 1. *For*

$$\pi/2(q+1) \leqslant \varphi \leqslant \pi/2q \tag{1.5}$$

we have

$$u(re^{i\varphi}, f) \leqslant u(re^{i\varphi}, \hat{f}) = 2\log|\hat{f}(re^{i\varphi})|. \tag{1.6}$$

(*When* $q = 0$ *we interpret* (1.5) *as* $\pi/2 \leqslant \varphi \leqslant \pi$.)

Further, the convolution inequalities

$$u(re^{i\varphi}, f) \leqslant \int_0^\infty n(t, 0) J(r/t, \varphi) t^{-1}\, dt, \tag{1.7}$$

$$u(re^{i\varphi}, f) \leqslant \int_0^\infty N(t, 0) K(r/t, \varphi) t^{-1}\, dt \tag{1.8}$$

both hold for φ *in the range* (1.5), *where*

$$J(s, \varphi) = \frac{2s^{q+1}(s\cos q\varphi - \cos(q+1)\varphi)}{1 + s^2 - 2s\cos\varphi} \tag{1.9}$$

and

$$K(s, \varphi) = s\, \partial J(s, \varphi)/\partial s \tag{1.10}$$

satisfy

$$J(s, \varphi) \geqslant 0, \qquad K(s, \varphi) \geqslant 0 \qquad (0 < s < \infty). \tag{1.11}$$

Recall that, for a nondecreasing function $S(r)$ $(0 < r < \infty)$, a sequence $\{r_m\}$ tending to ∞ is a sequence of Pólya peaks of order λ for S if for every $\epsilon > 0$

$$S(u) \leqslant \left(\frac{u}{r_m}\right)^{\lambda - \epsilon} S(r_m) \qquad (1 < u \leqslant r_m)$$

$$S(u) \leqslant \left(\frac{u}{r_m}\right)^{\lambda + \epsilon} S(r_m) \qquad (r_m < u)$$

whenever $m \geqslant m_0(\epsilon)$, from which it follows that $S(r_m) r_m^{-\lambda+\delta} \to \infty$ as $m \to \infty$ for any $\delta > 0$ (cf. [F; p. 136]).

If g is an entire function of nonintegral order λ, then $g(z) = z^k e^{P(z)} f(z)$ where f has the representation (1.1) with $q = [\lambda]$, P is a polynomial of degree at most q, and, by known existence theorems ([H; p. 103], [DS]), $N(r, 0)$ and $n(r, 0)$ each have Pólya peaks of order λ.

The inequalities (1.7) and (1.8) along with the positivity (1.11) of J and K allow for very precise estimates of $u(re^{i\varphi}, f)$ near the Pólya peaks of the counting functions. These estimates will be carried out in Theorem 2.

There are numerous known results on the distribution of values of entire and meromorphic functions of orders $\lambda < 1$ for which the extremal functions have positive zeros, and whose counterparts for $\lambda > 1$ are unknown (cf. [H; pp. 109-119] and [P]). This is due to the particularly simple behavior of $|E_0(re^{i\theta})|$, which for every $r > 0$ is increasing on $(0, \pi)$ and then decreases symmetrically on $(\pi, 2\pi)$. When $q \geqslant 1$, the intervals on which $|E_q(re^{i\theta})|$ increase and decrease depend upon r.

Our Theorem 1 presents a rare instance when an inequality on primary factors is sharp for a range of θ, independent of r, and hence leads directly to extremal properties of $\hat{f}$.

Theorem 2. *Let g have nonintegral order* λ *and* $\{r_m\}$ *be a sequence of Pólya peaks for N of order* λ. *Then*

$$\limsup_{m \to \infty} \frac{u(tr_m e^{i\varphi}, g)}{N(r_m, 0)} \leqslant \frac{2\pi\lambda t^\lambda}{\sin \pi\lambda} \cos((\pi - \varphi)\lambda) \tag{1.12}$$

for φ *satisfying* (1.5), *uniformly for t in compact subsets of* $0 < t < \infty$.

Similarly,

$$\limsup_{m \to \infty} \frac{u(tR_m e^{i\varphi}, g)}{n(R_m, 0)} \leqslant \frac{2\pi t^\lambda}{\sin \pi\lambda} \cos((\pi - \varphi)\lambda) \tag{1.13}$$

for $\{R_m\}$ *a sequence of Pólya peaks of order* λ *for n, with* φ *in the range* (1.5) *and uniformly for t in compact subsets of* $0 < t < \infty$.

Theorem 2 is sharp and extends a theorem of Fuchs [F] who proved (1.12) for $t = 1$, $\lambda > 1/2$, and φ restricted to the range $\pi/2(q+1) \leqslant \varphi \leqslant \pi/2\lambda$.

Inequality (1.12) still holds for entire g of finite lower order μ, provided λ is replaced in (1.12) by any finite nonintegral $\rho \in [\mu, \lambda]$ and the r_m are chosen to be *strong peaks* of $N(r, 0)$ in the sense of [MS]. The corresponding remark applies also to (1.13). For proof, combine the arguments used here for Theorem 2 with those of [MS].

2. A Preliminary Lemma

Put $k(z, \varphi) = \log |E_q(ze^{i\varphi})E_q(ze^{-i\varphi})|$. For (1.6)-(1.8) we require

Lemma 1. *For* φ *in the range* (1.5) *and* $|z| = r$ *we have*

$$k(z, \varphi) \leqslant k(r, \varphi) \qquad (0 < r < \infty). \tag{2.1}$$

When $r < 1$, (2.1) is equivalent to

$$-\sum_{k=q+1}^{\infty} (r^k/k)(1 - \cos k\theta)\cos k\varphi \geqslant 0,$$

but a direct proof of this seems difficult. Exponentiating (2.1) leads however to an easy proof.

PROOF OF LEMMA 1. Put

$$G(z) = E_q(ze^{-i\varphi})E_q(ze^{i\varphi}) = \sum_{n=0}^{\infty} g_n z^n. \tag{2.2}$$

To prove (2.1) it then suffices to show that

$$g_n \geqslant 0, \qquad n = 0, 1, \ldots \tag{2.3}$$

for φ in the range (1.5).

Let

$$E(z) = E_q(z) = (1 - z)e^{R(z)}, \qquad R(z) = \sum_{j=1}^{q} \frac{z^j}{j} \quad (= 0 \text{ if } q = 0)$$

so that $E'(z) = -z^q e^{R(z)}$. Thus,

$$\begin{aligned} G'(z) &= e^{i\varphi}E'(ze^{i\varphi})E(ze^{-i\varphi}) + e^{-i\varphi}E(ze^{i\varphi})E'(ze^{-i\varphi}) \\ &= -e^{i\varphi}(ze^{i\varphi})^q e^{R(ze^{i\varphi})}E(ze^{-i\varphi}) - e^{-i\varphi}(ze^{-i\varphi})^q e^{R(ze^{-i\varphi})}E(ze^{i\varphi}) \\ &= -z^q e^{R(ze^{i\varphi})}e^{R(ze^{-i\varphi})}(e^{i(q+1)\varphi}(1 - ze^{-i\varphi}) + e^{-i(q+1)\varphi}(1 - ze^{i\varphi})) \\ &= e^{R(ze^{i\varphi}) + R(ze^{-i\varphi})}(\alpha z^q + \beta z^{q+1}) \end{aligned}$$

where $\alpha = -2\cos(q+1)\varphi$ and $\beta = 2\cos q\varphi$. Since

$$R(ze^{i\varphi}) + R(ze^{-i\varphi}) = 2\sum_{j=1}^{q} \frac{\cos j\varphi}{j} z^j,$$

we have that $R(ze^{i\varphi}) + R(ze^{-i\varphi})$ and thus $\exp[R(ze^{i\varphi}) + R(ze^{-i\varphi})]$ have nonnegative Taylor coefficients for φ in the range (1.5). Now, $\alpha \geqslant 0$ and $\beta \geqslant 0$ for φ satisfying (1.5) so that G' has nonnegative Taylor coefficients. Finally since $G(0) = 1$ it follows that (2.3) and consequently (2.1) hold for the range (1.5). □

3. Proof of Theorem 1

By Lemma 1 and (1.1) it follows that for any z with $|z| = r$ and φ satisfying (1.5) we have

(3.1)
$$\begin{aligned}\log|f(ze^{i\varphi})f(ze^{-i\varphi})| &= \sum_{\nu=1}^{\infty} k(z/z_\nu, \varphi)\\ &\leqslant \sum_{\nu=1}^{\infty} k(r/|z_\nu|, \varphi)\\ &= \int_0^\infty k(r/t, \varphi)\, dn(t, 0)\\ &= 2\log|\hat{f}(re^{i\varphi})|\\ &= \int_0^\infty n(t, 0)k_1(r/t, \varphi)rt^{-2}\, dt\end{aligned}$$

where

(3.2)
$$k_1(s, \varphi) = \frac{\partial k(s, \varphi)}{\partial s}.$$

Thus (1.6) holds and if we put

(3.3)
$$J(s, \varphi) = sk_1(s, \varphi)$$

a direct computation with (1.2) shows that J is also given by (1.9) and that $J(s, \varphi) \geqslant 0$ for φ satisfying (1.5). From (1.4) and (3.1) we then obtain (1.7).

Continuing on from (3.1) with another integration by parts and K as in (1.10), we get

$$\log|f(ze^{i\varphi})f(ze^{-i\varphi})| \leqslant \int_0^\infty N(t, 0)K(r/t, \varphi)t^{-1}\, dt$$

which implies (1.8).

It remains only to verify that $K(s, \varphi) \geqslant 0$. In fact with G again as in (2.2), then $k = \log G$ so that from (3.2), (3.3), and (1.10) we have

$$\begin{aligned}K(s, \varphi) &= s\frac{\partial}{\partial s}\left(\frac{s\partial \log G}{\partial s}\right)\\ &= G(s)^{-2}[sG'(s)G(s) + s^2G''(s)G(s) - (sG'(s))^2]\\ &= G(s)^{-2}\sum_{n=0}^{\infty} b_n s^n\end{aligned}$$

where $G(s)^{-2} > 0$ and $\sum b_n s^n$ has infinite radius of convergence. Thus, it suffices to show that $b_n \geqslant 0$, $n = 0, 1, \ldots$.

With the notation of (2.2) we have

$$\begin{aligned} b_n &= \sum_{k=0}^{n} k g_k g_{n-k} + \sum_{k=0}^{n} k(k-1) g_k g_{n-k} - \sum_{k=0}^{n} k g_k (n-k) g_{n-k} \\ &= \sum_{k=0}^{n} (2k^2 - kn) g_k g_{n-k} \\ &= \sum_{j=0}^{n} (2(n-j)^2 - n(n-j)) g_{n-j} g_j. \end{aligned}$$

Thus, for φ satisfying (1.5) it follows from (2.3) that

$$\begin{aligned} 2b_n &= \sum_{k=0}^{n} (2k^2 - kn + 2(n-k)^2 - n(n-k)) g_k g_{n-k} \\ &= \sum_{k=0}^{n} (n-2k)^2 g_k g_{n-k} \geqslant 0. \end{aligned}$$

4. Proof of Theorem 2

Since $\{r_m\}$ is a sequence of Pólya peaks of order λ of N we have from (1.8) that

$$u(tr_m e^{i\varphi}, g) \leqslant N(r_m, 0)\left[\int_0^\infty K(t/\sigma, \varphi)\sigma^{\lambda-1}\, d\sigma + \eta_m(t)\right] \tag{4.1}$$

where

$$\begin{aligned} \eta_m(t) = \frac{1}{N(r_m, 0)}\Bigg[&\int_0^1 N(\sigma, 0) K(tr_m/\sigma, \varphi)\, d\sigma \\ &+ \int_{r_m^{-1}}^1 K(t/\sigma, \varphi)(\sigma^{\lambda-1-\epsilon} - \sigma^{\lambda-1})\, d\sigma \\ &+ \int_1^\infty K(t/\sigma, \varphi)(\sigma^{\lambda-1+\epsilon} - \sigma^{\lambda-1})\, d\sigma \\ &+ k \log(tr_m) + C(tr_m)^q\Bigg]. \end{aligned} \tag{4.2}$$

Using (1.9) and (1.10) we find that for $\varphi \neq 0$,

$$\begin{aligned} |K(s, \varphi)| &\leqslant C_1(q, \varphi) s^{q+1} \qquad s < 1 \\ |K(s, \varphi)| &\leqslant C_2(q, \varphi) s^{q} \qquad s \geqslant 1. \end{aligned}$$

These inequalities along with the fact that $N(r_m, 0) r_m^{-\lambda+\delta} \to \infty (\delta > 0)$ as $m \to \infty$ imply that for φ in the range (1.5) and t in a compact subset of $(0, \infty)$ we may take η_m arbitrarily small, for sufficiently small ϵ and large m in (4.2).

The integral in (4.1) can now be explicitly evaluated as follows, using the notations of (3.1)-(3.3).

$$\begin{aligned}
\int_0^\infty K(t/\sigma, \varphi)\sigma^{\lambda-1}\,d\sigma &= t^\lambda \int_0^\infty K(r, \varphi) r^{-\lambda-1}\,dr \\
&= t^\lambda \int_0^\infty \frac{\partial J(r, \varphi)}{\partial r} r^{-\lambda}\,dr = \lambda t^\lambda \int_0^\infty J(r, \varphi) r^{-\lambda-1}\,dr \\
&= \lambda t^\lambda \int_0^\infty k_1(r, \varphi) r^{-\lambda}\,dr = \lambda^2 t^\lambda \int_0^\infty k(r, \varphi) r^{-\lambda-1}\,dr \\
&= \lambda^2 t^\lambda \int_0^\infty (\log|E_q(re^{i\varphi})| + \log|E_q(re^{-i\varphi})|) r^{-\lambda-1}\,dr \\
&= 2\lambda^2 t^\lambda \int_0^\infty \log|E_q(-re^{i(\pi-\varphi)})| r^{-\lambda-1}\,dr \\
&= \frac{2\lambda t^\lambda \pi \cos((\pi-\varphi)\lambda)}{\sin \pi\lambda}.
\end{aligned}$$

The computation of this last integral is done in [HS; p. 222]. This completes the proof of (1.12). The proof of (1.13) is similar and is thus omitted.

5. Estimates of $\log M(r)/N(r, 0)$

Let g be as in Theorem 2. Then $u(re^{i\varphi}, g)$ is easily seen to be subharmonic in $\mathbb{C}$. We may therefore form a *local indicator* $h(\theta)$ as in [E] where the details are carried out for the case $u = \log|g|$, but they go through without essential change for $u = u(re^{i\varphi}, g)$. The functions

$$V(r) = N(r_m, 0)(r/r_m)^\lambda \qquad (\sigma^{-1} r_m < r < \sigma r_m, \quad \sigma > 1)$$

serve as valid comparison functions in the sense of [E], since for each fixed $t > 0$

$$\limsup_{m\to\infty} \frac{\log M(tr_m, g)}{t^\lambda N(r_m, 0)} \leqslant B = B(\lambda) < \infty. \tag{5.1}$$

To verify (5.1) we observe that the argument of [H; *p.* 102] proves

$$\begin{aligned}
\log M(r, g) &\leqslant c_1(q)\left(q \int_0^r (r/s)^q N(s, 0)\,ds/s + (q+1) \int_r^\infty (r/s)^{q+1} N(s, 0)\,ds/s \right) \\
&\quad + O(r^q) + O(\log r) \\
&\leqslant 2(q+1)c_1(q) \int_0^\infty \frac{(r/s)^{q+1}}{1 + r/s} N(s, 0) \frac{ds}{s} + O(r^q) + O(\log r)
\end{aligned}$$

where $c_1(q) = 2(q + 1)\{2 + \log(q + 1)\}$. Since the r_m are Pólya peaks, we have for fixed $t > 0$ and $0 < \epsilon < \min(\lambda - q, q + 1 - \lambda)$ that

$$\begin{aligned}\log M(tr_m, g) &\leqslant c_2(q) \int_0^\infty \frac{(tr_m/s)^{q+1}}{1 + tr_m/s} N(s, 0) \frac{ds}{s} + O((tr_m)^q) + O(\log r_m) \\ &\leqslant c_2(q) N(r_m, 0) \left\{ t^{\lambda - \epsilon} \int_t^\infty \frac{u^{q - \lambda + \epsilon}}{1 + u} du + t^{\lambda + \epsilon} \int_0^t \frac{u^{q - \lambda - \epsilon}}{1 + u} du \right\} \\ &\quad + O((tr_m)^q) + O(\log r_m)\end{aligned}$$

where $c_2(q) = 2(q + 1)c_1(q)$. Letting $m \to \infty$ and then $\epsilon \to 0$ yields

$$\limsup_{m \to \infty} \frac{\log M(tr_m, g)}{t^\lambda N(r_m, 0)} \leqslant c_2(q) \int_0^\infty \frac{u^{q - \lambda} du}{1 + u} = c_2(q) \frac{\pi}{|\sin \pi\lambda|}$$

as claimed in (5.1).

We may then define

$$h_\sigma^{(m)}(\varphi) = \sup_{\sigma - 1 \leq t \leq \sigma} \frac{u(tr_m e^{i\varphi}, g)}{t^\lambda N(r_m, 0)},$$

$$h_\sigma(\varphi) = \limsup_{m \to \infty} h_\sigma^{(m)}(\varphi),$$

and finally

$$h(\varphi) = \lim_{\sigma \to \infty} h_\sigma(\varphi).$$

Then,

(i) $h(\varphi)$ is subtrigonometric (see [E]);
(ii) for $\beta = 0$ or $\beta = \pi$,

$$h(\beta) \geqslant \limsup_{m \to \infty} \frac{2 \log M(r_m, g)}{N(r_m, 0)} \tag{5.2}$$

and for $|\beta - \varphi| < \pi/\lambda$,

$$h(\beta) \cos((\beta - \varphi)\lambda) \leqslant h(\varphi); \tag{5.3}$$

(iii) for φ in the range (1.5),

$$h(\varphi) \leqslant \frac{2\pi\lambda}{\sin \pi\lambda} \cos((\pi - \varphi)\lambda). \tag{5.4}$$

Here (5.4 follows from Theorem 2, (5.3) from [L, p. 56], and (5.2) is immediate from the definitions of u and h.

Following Pólya [P], we seek estimates for

$$C(g) = \liminf_{r\to\infty} \frac{\log M(r,g)}{N(r,0)}.$$

When $\lambda < 1$ we can take $\beta = \pi$ and $\varphi \in [\pi/2, \pi)$ in (5.2)-(5.4) to deduce $C(g) \leqslant \pi\lambda/\sin \pi\lambda$, a classical result due to Valiron [V] and Pólya [P].

For $\lambda > 1$, good bounds on $C(g)$ are not yet known. To see what (5.2)-(5.4) can tell us, we take $\beta = 0$ and $\varphi = \pi/2(q+1)$ in (5.3) to deduce

$$C(g) \leqslant \frac{\pi\lambda}{|\sin \pi\lambda|} A(\lambda)$$

where the estimate

$$A(\lambda) \leqslant (-1)^q \frac{\cos((\pi-\varphi)\lambda)}{\cos\varphi\lambda} = \frac{\sin((2q+1)\gamma)}{\sin\gamma} \qquad \left(\gamma = \frac{\pi}{2} - \varphi\lambda\right) \tag{5.5}$$

is far from sharp for large λ.

When g has order $1 < \lambda < 2$ we have two explicit estimates:

$$\begin{aligned} A(\lambda) &\leqslant 1 + 2|\cos(\pi\lambda/2)|, \\ A(\lambda) &\leqslant 2|\cos(2\pi\lambda/3)|. \end{aligned} \tag{5.6}$$

The first is equivalent to (5.5) when $q = 1$; the second uses

$$\log M(r,f) \leqslant u(re^{i\pi/3}, \hat{f}) \qquad (0 < r < \infty), \tag{5.7}$$

with f, $\hat{f}$ as in (1.1), (1.3), together with an application of Theorem 2.

The inequality (5.7) follows in case $f = E_1$ from the calculation

$$\max_\theta \log|E_1(re^{i\theta})| = \begin{cases} r^2/2 & (0 < r \leqslant 2) \\ r + \log(r-1) & (2 \leqslant r) \end{cases}$$

together with

$$\begin{aligned} u(re^{i\pi/3}, E_1) &= 2\log|E_1(re^{i\pi/3})| \\ &= r + \log(r^2 - r + 1) \quad (0 < r < \infty). \end{aligned}$$

For f of the form (1.1) we deduce

$$\begin{aligned} \log M(r,f) &\leqslant \sum_{\nu=1}^{\infty} \log M(r, E_1(z/z_\nu)) \\ &\leqslant \sum_{\nu=1}^{\infty} u(re^{i\pi/3}, E_1(z/|z_\nu|)) = u(re^{i\pi/3}, \hat{f}). \end{aligned}$$

Thus the second inequality in (5.6) follows from Theorem 2:

$$C(g) \leqslant \limsup_{m \to \infty} \frac{\log M(r_m, f) + O(r)}{N(r_m, 0)}$$

$$\leqslant \limsup_{m \to \infty} \frac{u(r_m e^{i\pi/3}, \hat{f})}{N(r_m, 0)}$$

$$\leqslant \frac{2\pi\lambda}{\sin \pi\lambda} \cos\left(\frac{2\pi\lambda}{3}\right).$$

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Domains with Strong Barrier

José L. Fernández

Dedicated to the Memory of J. L. Rubio de Francia

Introduction

The level sets of any Riemann mapping f can not be arbitrarily long. More precisely, there exists an absolute constant P so that if Ω is a plane simply connected domain, f a Riemann mapping onto Ω and L is an straight line then

$$\text{length}\,(f^{-1}(\Omega \cap L)) \leqslant P.$$

This beautiful result was first proved by Hayman and Wu [HW], and a bit later by Garnett, Gehring and Jones, [GGJ]. See [FHM] for a simple proof, where it is shown that one can take $P = 4\pi^2$ and a conjecture as to the correct value of P is offered.

One wonders as to what is the role of simple connectivity in the Hayman-Wu theorem. Let us call a domain Ω in the plane a *Hayman-Wu domain* if there exists a constant $C(\Omega)$ so that

$$\text{length}\,(F^{-1}(\Omega \cap L)) \leqslant C(\Omega) \tag{0.1}$$

for any straight line L and universal cover F from the unit disk Δ onto Ω.

It was shown in [FH] that domains of finite connectivity with no complementary components consisting of a single point are Hayman-Wu domains. A word of caution: in [FH] one is not concerned with the dependence of the constant of (0.1) upon F, but the argument applies. Moreover, it is easy to see that the punctured disk, Δ^*, is not a Hayman-Wu domain, so that the non-degeneracy condition on the complementary components is essential.

Let Γ denote a covering group of the domain Ω, *i.e.*, a fix-point free discrete group of Möbius transformations of Δ with quotient Δ/Γ conformally equivalent to Ω. Any two covering groups of Ω are conjugate, and conversely.

With Γ we associate the invariant function

$$U_{\Gamma,t}(z) = U_t(z) = \sum_{t \in \Gamma} (1 - [z, Tz]^2)^t$$

where

$$[a, b] = \left| \frac{a - b}{1 - \bar{a}b} \right|.$$

In [FH] it was shown (see Section 6 for the proof).

Theorem A. *If $U_{1/2}$ is bounded in Δ then Ω is Hayman-Wu.*

As a consequence of Theorem 2 one also has that if Ω is Hayman-Wu then U_1 is bounded. Notice that $U_1 \leqslant U_{1/2}^2$.

The exponent of Γ is defined as the exponent of convergence of the Dirichlet series

$$\sum_{\gamma \in \Gamma} \exp(-s\rho(0, \gamma(0)))$$

i.e. the smallest number s which makes the series convergent. Here, and hereafter, $\rho(a, b)$ denotes the Poincaré distance between the points a and b on the unit disk; namely,

$$\rho(a, b) = h([a, b]),$$

with

$$h(t) = \log \frac{1 + t}{1 - t}, \qquad 0 \leqslant t \leqslant 1.$$

Since conjugate groups have the same exponent we may also speak of the exponent of Ω. We shall use the notation $\delta(\Gamma)$, $\delta(\Omega)$ to denote exponents.

It is an elementary fact that $\delta(\Omega) \leqslant 1$. Also, $\delta(\Omega) \geqslant 1/2$, if Γ contains parabolic elements, or, equivalently if $\partial\Omega$ has isolated points.

Notice that the groups satisfying the hypothesis of Theorem A have exponent at most $1/2$.

Here we shall show the following somehow surprising result.

Theorem 1. *If Ω is a Hayman-Wu domain then $\delta(\Omega) < 1$.*

Since there are domains of finite connectivity (with no point-boundary components) with exponent arbitrarily close to 1 we see that Theorem 1 is in a certain sense sharp. The exponent of the domain

$$\Omega_\epsilon = \Delta\left(0, \frac{1}{\epsilon}\right)\backslash\bar{\Delta}(0, \epsilon)\backslash\bar{\Delta}(1, \epsilon), \qquad \epsilon \in \left(0, \frac{1}{2}\right)$$

increases to 1 as ϵ decreases to 0.

We shall deduce Theorem 1 from combining two results about domains with strong barriers.

Definition. *Let Ω be a plane domain. A non-constant positive superharmonic function U of Ω is called a* ***strong barrier*** *if there exists a positive number ϵ such that*

$$\Delta U + \frac{\epsilon \cdot U}{\operatorname{dist}(\cdot, \partial\Omega)^2} \leqslant 0,$$

(*where this inequality is meant in the weak sense*).

If Ω has a strong barrier then Ω has a Green's function and moreover every boundary point is regular for the Dirichlet problem, and thus Ω has no point-boundary components.

Domains with strong barriers can be characterized in a variety of ways, and we shall use the rich knowledge about them to prove the following two results which yield Theorem 1 immediately.

Theorem 2. *If Ω is a Hayman-Wu domain then Ω posseses a strong barrier.*

The reciprocal of Theorem 2 does not hold. This follows from Theorem 4 below.

Theorem 3. *If Ω posseses a strong barrier then $\delta(\Omega) < 1$.*

In this case it is easy to see that the reciprocal does not hold; simply take $\Omega = \Delta^*$, then $\delta(\Omega) = 1/2$, but Ω does not have a strong barrier.

It should be remarked that in [Po2] an example is offered of a domain with a strong barrier but $\delta = 1$. There is an error in the calculations there.

A *Denjoy domain* is a domain in the sphere whose complement is a compact set of the real line. Thus $\Omega = \hat{\mathbb{C}}\backslash E$, $E \subset \mathbb{R}$, E compact. Denjoy domains have been recently studied by several authors in connection mostly with the Corona problem. See [RR], [C], [JM], [GJ]. They provide a test case for problems about multiply connected domains.

A compact set $E \subset \mathbb{R}$ is called *homogeneous* if there exists a constant c_E so that if $x \in \mathbb{R}$ and $\delta > 0$.

$$\frac{|(x-\delta, x+\delta) \cap E|}{\delta} \geqslant c_E.$$

Carleson introduced this condition in [C] where he showed that the associated Denjoy domain satisfies the Corona theorem.

Garnett and Jones [GJ] later showed this with no restriction on the set E. More recently, Zinsmeister has shown that E is homogeneous if and only if $H^1(E) = H^1(\mathbb{R})$ (see [Z] for definitions and results).

If E is homogeneous then $\hat{\mathbb{C}} \backslash E$ has an strong barrier.

For Denjoy domains the homogeneity of the boundary is the key for being a Hayman-Wu domain.

Theorem 4. *If Ω is a Denjoy domain, then Ω is a Hayman-Wu domain if and only if $\partial\Omega$ is homogeneous.*

The proof of Theorem i is in Section i, $i = 2, 3, 4$. In Section 5 we consider another notion of a domain being almost simply connected and relate that to the results above. In Section 6 we give the proof of Theorem A for the sake of completeness.

I wish to thank A. Ancona for pointing out the example in the Remark in Section 7. I am most grateful to Juha Heinonen for very stimulating conversations which motivated this paper.

1. Domains with Strong Barrier

Here we collect the relevant features of domains with strong barrier.

Let Ω be a plane domain other than the plane or a punctured plane. The universal covering Riemann surface is the unit disk. Consider the Poincaré metric in the unit disk. Via the universal covering map, π, it can be projected onto a metric in Ω so that π is a local isometry. This projected metric is conformal with the euclidean metric and the scale factor, denoted by λ_Ω, is determined by the equation

$$\lambda_\Omega(\pi(z))|\pi'(z)| = \frac{2}{1-|z|^2}, \qquad z \in \Delta.$$

The volume form of this metric will be denoted by ω_Ω; it is simply

$$\omega_\Omega = \lambda_\Omega^2 \, dx \wedge dy.$$

It is always the case and follows from Schwarz's lemma that

$$\lambda_\Omega \leqslant \frac{2}{\operatorname{dist}(\cdot, \partial\Omega)}.$$

To have a reversed inequality, *i.e.*, to have $0 < \inf_{z \in \Omega} \lambda_\Omega(z) \operatorname{dist}(z, \partial\Omega)$ is equivalent to the existence of a strong barrier, [BP], [Po1]. Also in terms of the group Γ we have that Ω has a strong barrier if and only if there exists $\tau_0 > 0$ so that the translation length of every element of Γ is at least τ_0, [P1]. (The translation length of a parabolic element is defined to be zero.) In geometric terms this translates into having no punctures plus the existence of a positive lower bound for the length of closed simple Poincaré geodesics of Ω.

We shall need another characterization. A domain Ω has a strong barrier if and only if $\partial\Omega$ verifies the following capacity condition: there exists a constant $C_0 > 0$

$$\operatorname{cap}(\Delta(b, r) \cap \partial\Omega) > C_0 r \tag{1.1}$$

for every $b \in \partial\Omega$, and r, $0 < r \leqslant \operatorname{diam}(\partial\Omega)$.

The strong barrier condition is also equivalent to $U_{\Gamma, 1}$ being bounded [Po2]. Recall that the condition appearing in Theorem A is that $U_{\Gamma, 1/2}$ is bounded.

All this can be found in [A], [BP], [Po1], [Po2].

2. Proof of Theorem 2

We will check that if Ω is a Hayman-Wu domain then ***(a)***, there is a constant $\tau_0 > 0$ so that all closed simply geodesic have Poincaré length at least τ_0, and ***(b)***, there are no punctures. We need a simple lemma:

Lemma. *Let T be hyperbolic Möbius transformation of the unit disk onto itself whose axis passes through* 0. *Then*

$$\frac{1}{|T(0)|} \leqslant \sum_k (1 - |T^k(0)|^2) \leqslant \frac{2}{|T(0)|^2}.$$

PROOF. We may assume that the fixed points of T are -1 and 1, and that $T(0) = a \in (0, 1)$. Let $b_n = T^n(0)$, $n \geqslant 0$. Then

$$1 - |b_n|^2 = 1 - |T(b_{n-1})|^2 = \frac{(1 - |b_{n-1}|^2)(1 - |a|^2)}{|1 + b_{n-1}a|^2}, \qquad n \geqslant 1.$$

For $n \geqslant 1$ we have:

$$1 \leqslant |1 + b_{n-1}a| \leqslant 1 + a,$$

and so

$$(1-a^2)^n \geqslant 1-|b_n|^2 \geqslant \left(\frac{1-a}{1+a}\right)^n, \qquad n \geqslant 1.$$

Therefore

$$\frac{2}{a^2} > \sum_{k\in\mathbb{Z}} (1-|T^k(0)|^2) = 1 + 2\sum_{n=1}^{\infty}(1-|b_n|^2) \geqslant \frac{1}{a}.$$

(*a*) Let σ be a closed simple geodesic in Ω.

The Jordan curve σ contains points of $\partial\Omega$ in its Jordan interior. Let $s \in \sigma$ and $b \in \partial\Omega$ be such that

$$|b-s| = \operatorname{dist}(\sigma, \partial\Omega \cap \operatorname{interior}(\sigma)).$$

Let F be a universal covering map which takes 0 to s.

Lift σ to a geodesic segment in Δ through 0. The lift is part of a diameter $\tilde{\sigma}$ of Δ. Let T be the Möbius covering transformation ($F \circ T = F$) corresponding to σ. Then the axis of T is $\tilde{\sigma}$ since σ is smooth. Moreover the length L of σ safisfies

$$\frac{1}{\tanh\left(\frac{L}{2}\right)} \leqslant \sum_{k\in\mathbb{Z}} 1-|T^k(0)|^2. \tag{2.1}$$

The segment from s to b is contained in Ω and its preimage under F contains a collection of curves each one of them emanates from a point of the orbit of 0 and goes all the way to $\partial\Delta$, therefore the total length of these curves is at least $\sum_{\gamma\in\Gamma} 1-|\gamma(0)|$. And consequently we have that

$$\sum_{\gamma\in\Gamma}(1-|\gamma(0)|) \leqslant c_\Omega. \tag{2.2}$$

Therefore,

$$\tanh\left(\frac{L}{2}\right) \geqslant \frac{1}{2c_\Omega}.$$

and so L is bounded below by a constant depending only on c_Ω.

(*b*) It remains to deal with the possible isolated points of the boundary of Ω. We may assume that $0 \in \partial\Omega$ and $\Delta^* \subset \Omega$. Let F be a universal covering map which takes 0 to 1/2. The circle $|z| = 1/2$ is lifted to a curve joining 0 to $T(0)$ where $T \in \Gamma$ is parabolic. We may assume that the unique fixed point of T is 1. Now the segment σ from 1/2 to 0 lifts to a curve $\tilde{\sigma}$ in

Δ which joins 0 to 1. Notice that $F\big(\bigcup_{k\in\mathbb{Z}} T^k(\tilde{\sigma})\big) \subset (0, 1/2]$, and therefore since $T(\tilde{\sigma})$ joins $\gamma(0)$ to 1, we see that

$$\sum_{k\in\mathbb{Z}} |1 - T^k(0)| \leqslant \operatorname{length}(F^{-1}(\Omega \cap \mathbb{R})) \leqslant c(\Omega). \tag{2.3}$$

But it is easy to see that $|1 - T^k(0)||k| \to t_0$ as $|k| \to \infty$ where t_0 is a positive number. Therefore the sum on the left is actually infinite. Thus we have shown that $\partial\Omega$ has no isolated points and so the proof is complete.

3. Proof of Theorem 3

Our proof of Theorem 3 is actually a combination of results which appear in papers by Ancona [A] and Sullivan [S1]. Ancona shows that in domains with strong barrier the following form of Hardy's inequality holds: there exists a constant c_1 so that for every smooth function φ compactly supported in Ω

$$\iint_\Omega |\varphi(z)|^2 \frac{dx\,dy}{\operatorname{dist}(z, \partial\Omega)^2} \leqslant c_1 \iint_\Omega |\nabla\varphi(z)|^2\,dx\,dy, \qquad (z = x + iy). \tag{3.1}$$

The constant c_1 depends only on the ϵ in the definition of strong barrier. As a matter of fact the existence of strong barrier is equivalent to (3.1).

Recall that the density of the Poincaré metric is denoted by λ_Ω, while its volume form is denoted by ω_Ω.

The Dirichlet integral is a conformal invariant. Therefore the integral on the right hand side of the inequality (3.1) equals

$$\iint_\Omega |\nabla_\Omega \varphi|_\Omega^2\, \omega_\Omega \tag{3.2}$$

where ∇_Ω denotes the gradient with respect to the Poincaré metric of Ω, and $|\ \ |_\Omega$ denotes length in the tangent space with respect to the Poincaré metric of Ω.

Moreover, it is always the case that

$$\lambda_\Omega(z) \leqslant \frac{2}{\operatorname{dist}(z, \partial\Omega)}, \quad \text{for every} \quad z \in \Omega. \tag{3.3}$$

Using (3.2) and (3.3) we see that inequality (3.1) implies that

$$\iint_\Omega |\varphi|^2 \omega_\Omega \leqslant c_1 \iint_\Omega |\nabla_\Omega \varphi|^2 \omega_\Omega, \quad \text{for every} \quad \varphi \in C_0^\infty(\Omega). \tag{3.4}$$

But this means that the Poincaré inequality holds in the Riemannian manifold Ω and therefore the spectrum of the Laplace-Beltrami operator of Ω is contained in $(-\infty, -1/C_1)$.

And now the theorem of Elstrodt-Patterson-Sullivan (see [S1, p. 333]) provides the final stroke because if $\delta = \delta(\Gamma)$ then it claims in our case that

$$\delta(1-\delta) \geqslant \frac{1}{C_1},$$

if $\delta \geqslant 1/2$. In particular,

$$\delta \leqslant \max\left\{1 - \frac{1}{C_1}, \frac{1}{2}\right\} < 1.$$

Remark. One can use the argument of Lemma 1 of [Su] to show directly that if a domain posseses strong barrier then the isoperimetric inequality, $A < cL$, holds (for its Poincaré metric), and combine this with Cheeger's inequality to give the result.

4. Proof of Theorem 4

Sufficiency

Here we assume that $\partial\Omega$ is homogeneous.

First of all we reduce the proof to the case $L = \mathbb{R}$. Let a universal covering map F be given and assume that we have seen that

$$\text{length}\,(F^{-1}(\Omega \cap \mathbb{R})) \leqslant M \tag{4.1}$$

where M depends on Ω but not on F. Let L be any other straight line and L^+ be the part of L above $\mathbb{R}$. Let G be any branch of F^{-1} defined on the upper half plane. By the Hayman-Wu theorem (see [GGJ]) we have that

$$\text{length}\,(G(L^+)) \leqslant \tilde{P}\,\text{length}\,(\partial G(U)) \tag{4.2}$$

where $\tilde{P}$ is an absolute constant. Adding up (4.2) over all branches G and using (4.1) we see that

$$\text{length}\,(F^{-1}(L^+)) \leqslant \tilde{M},$$

where $\tilde{M}$ depends only on Ω. Similarly, length $(F^{-1}(L^-)) \leqslant \tilde{M}$ and so

$$\text{length}\,(F^{-1}(L)) \leqslant 2\tilde{M}$$

Choose now a universal covering map F. We will check that (4.1) holds.

Let us denote by I_j the complementary intervals of E in $\mathbb{R}$.

In each I_j we select points $z_k^{(j)}$ as follows: if $I_j = (a, b)$, with a, b finite then

$$z_0^{(j)} = \frac{a+b}{2},$$

and

$$z_k^{(j)} = z_0^{(j)} + \operatorname{sign}(k)\frac{|I_j|}{2}(1 - 1/2^k), \qquad k \in \mathbb{Z}.$$

If the interval contains ∞, we select ∞ as $z_0^{(j)}$ and, if $q = \sup E$ and $p = \inf E$,
we let

$$z_k^{(j)} = q + \frac{1}{2^k}\operatorname{diam}(E), \qquad k \geqslant 1,$$

$$z_k^{(j)} = p - 2^k \operatorname{diam}(E), \qquad k \leqslant -1.$$

Let $Z = \{z_k^{(j)}: j, k\}$. We shall check that $F^{-1}(Z)$ is an interpolating sequence whose constants are independent of the choice of the universal covering F of Ω. Assume this for the moment and let us show how to finish the proof.

Denote by $I_{j,k}$ the interval $(z_k^{(j)}, z_{k+1}^{(j)})$, $k \in \mathbb{Z}$. Let G be any branch of F^{-1} defined on the whole interval $3I_{j,k}$ (which is the interval with same center and triple the length). Then $G(I_{j,k})$ is a curve in Δ whose Poincaré diameter is bounded by an absolute constant (log 4); this follows from Schwarz' Lemma. In particular if $x \in I_{j,k}$ we have

$$1 - |G(x)|^2 \leqslant A(1 - |G(z_k^{(j)})|^2), \tag{4.3}$$

where A is an absolute constant. Thus, if $x \in I_{j,k}$,

$$|G'(x)| = (1 - |G(x)|^2)\lambda_\Omega(x) \leqslant \frac{1 - |G(x)|^2}{\operatorname{dist}(x, \partial\Omega)} \leqslant \frac{1 - |G(x)|^2}{|I_{j,k}|} \leqslant A\frac{1 - |G(z_k^{(j)})|^2}{|I_{j,k}|}.$$

Consequently,

$$\int_{I_{j,k}} |G'(x)|\, dx \leqslant A(1 - |G(z_k^{(j)})|^2). \tag{4.4}$$

And, in particular, adding up (4.4) over all j, k and G we obtain that

$$\operatorname{length}(F^{-1}(\Omega \cap \mathbb{R})) \leqslant A \sum_{w \in F^{-1}(Z)} (1 - |w|^2).$$

But we are assuming that we have already shown that $F^{-1}(Z)$ is an interpolating sequence and so, in particular, that the measure

$$\mu = \sum_{w \in F^{-1}(Z)} (1 - |w|^2)\delta_w$$

is finite (as a matter of fact, that μ is a Carleson measure). The interpolation constants of $F^{-1}(Z)$ depend only on Ω and thus so does the mass of μ; this implies that (4.1) holds.

All that remains is to show that $F^{-1}(Z)$ is an interpolating sequence. But before doing so let us remark that the argument above (which appears in [GGJ]) is general. In fact, given Ω (not necessarily Denjoy), split the intersection with Ω of a given line L into disjoint intervals J_k so that in each interval J_k

$$\frac{1}{100} \leqslant \frac{\operatorname{dist}(z, \partial\Omega)}{\operatorname{length}(J)} \leqslant 100.$$

Let z_k be the center of J_k. Then if $F^{-1}(\{z_k\})$ is interpolating with contants depending on Ω alone one deduces that Ω is a Hayman-Wu domain. Conversely, if Ω is Hayman-Wu then using that Ω has strong barrier one may show that the inverse image of such a sequence is interpolating.

There is an argument introduced by Garnett-Gehring-Jones for checking whether $F^{-1}(Z)$ is interpolating or not by transfering the problem to a harmonic measure estimate on Ω itself. If we assume that Ω has an strong barrier then we have that $F^{-1}(Z)$ is interpolating if and only if there is $\epsilon < 1/4$ and $a > 0$ so that if for $z \in Z$ we define

$$H_\epsilon(z) = \sum_{z' \in Z \setminus \{z\}} \bar{\Delta}(z', \epsilon \operatorname{dist}(z', \partial\Omega)) \cap \mathbb{R}.$$

Then

$$\omega(z, \partial\Omega, \Omega \setminus H_\epsilon(z)) \geqslant a, \quad \text{for all} \quad z \in Z. \tag{4.5}$$

This appears in [Po2] and in [JM]. If $z' = \infty$ by $\bar{\Delta}(\infty, \epsilon \operatorname{dist}(\infty, \partial\Omega))$ we mean $\bar{\mathbb{R}} \setminus (p - (1/\epsilon) \operatorname{diam} \partial\Omega, q + (1/\epsilon) \operatorname{diam} \partial\Omega)$. It turns out that for Denjoy domains with homogeneous complement (4.5) can be easily checked. This could be done as follows: if $z \in Z \setminus \{\infty\}$, then $\Delta(z, (1/8) \operatorname{dist}(z, \partial\Omega)) \subset \Omega \setminus H_\epsilon(z)$; by Harnack's inequality it is enough to estimate

$$\omega(z + id, \partial\Omega, \Omega \setminus H_\epsilon(z))$$

from below, where

$$d = \frac{1}{16} \operatorname{dist}(z, \partial\Omega),$$

But

$$\omega(z + id, \partial\Omega, \Omega \setminus H_\epsilon(z)) \geqslant \omega(z + id, \partial\Omega, U),$$

(where U is the upper half plane).

Let $b \in \partial\Omega$ be such that $|z - b| = \operatorname{dist}(z, \partial\Omega)$, using again Harnack's inequality we see that we just need to estimate $\omega(b + id, E, U)$ form below. But from the explicit expression of the Poisson kernel of the upper half plane we readily see that

$$\omega(b + id, E, U) \geqslant C \frac{|(b - 10d, b + 10d)|}{d}$$

where C is an absolute constant. And this gives the desired result. (For $z = \infty$ one needs a minor variation of the argument.)

Necessity

Assume that Ω is a Hayman-Wu domain. We want to check that $\partial\Omega$ is homogeneous. Write $E = \partial\Omega$.

We already know that E satisfies the capacity condition (1.1).

We use the notation of the proof of the sufficiency.

We know that for some $\epsilon > 0$ and $a = a(\epsilon) > 0$

$$\omega(z, E, \Omega \backslash H_\epsilon(z)) \geqslant a, \quad \text{for every} \quad z \in A.$$

It is easy to check that $E \cup \hat{H}_\epsilon(z)$ is homogeneous with a constant depending only on ϵ (and not on E). Here $\hat{H}_\epsilon(z)$ is the part of $H(z)$ not lying in the component of ∞ of $\bar{\mathbb{R}} \backslash E$. Clearly

$$\omega(z, E, \Omega \backslash \hat{H}_\epsilon(z)) \geqslant a, \quad \text{for every} \quad z \in Z.$$

Let $V = [p, q]$ be the smallest interval containing E. We shall check that for an appropriate constant $M = M(\epsilon)$ we have for all $y \in V \backslash E$ that

$$|\Delta(y, M \operatorname{dist}(y, E)) \cap E| \geqslant C \operatorname{dist}(y, E) \tag{4.7}$$

where $C = C(\epsilon)$.

This will be enough as the following simple lemma shows.

Lemma. *Let $A \subset [0, 1]$ be a closed set and assume that there exist constants η, N such that if $y \in [0, 1] \backslash A$*

$$|(y - Nd(y), y + Nd(y)) \cap A| \geqslant \eta\, d(y),$$

where $d(y) = \operatorname{dist}(y, A)$ then

$$|A| \geqslant \eta / 8N.$$

PROOF OF LEMMA. Let $J_y = (y - Nd(y), y + Nd(y))$.

Consider

$$B = \bigcup_{y \in [0,1] \setminus A} J_y .$$

We may choose points y_j so that

$$B = \bigcup_j J_{y_j}$$

and

$$\sum_j \chi_{y_j} \leqslant 2\chi_I$$

(*i.e.* no point of B is in more than two of the J_{y_j}). Then

$$|A \cap B| = \int_A \chi_B \geqslant \frac{1}{2} \sum_j |A \cap J_{y_j}| \geqslant \frac{\eta}{2} \sum_j d(y_j) \geqslant \frac{\eta}{4N} |B| .$$

Now, $A \cap B \subset A$, and $B \supset [0,1] \setminus A$ so that

$$|A| \geqslant \frac{\eta}{4N} (1 - |A|)$$

and so

$$|A| \geqslant \frac{\eta}{8N} .$$

It is clear that in order to check (4.7) for all $y \in V \setminus E$ it is enough to do so when y is one of the points $z_k^{(j)}$.

Since both the data and the desired conclusion are translation and scale invariant, we may assume that $z_k^{(j)} = 0$, $1 \in E$, and $\operatorname{dist}(z_k^{(j)}, E) = 1$.

Around $1/2$ there is an interval of length 2ϵ which lies in $\partial H_\epsilon(0)$. Then there exists $M = M(\epsilon)$ so that

$$\omega(0, \mathbb{R} \setminus (-M(\epsilon), M(\epsilon)), \Omega \setminus \hat{H}_\epsilon(z) \setminus [-M(\epsilon), M(\epsilon)]) \leqslant a/2.$$

Therefore we see that

$$\omega(0, E \cap [-M(\epsilon), M(\epsilon)], \Omega \setminus \hat{H}_\epsilon(z)) \geqslant a/2. \tag{4.8}$$

We define two sets $\tilde{E}, \tilde{K}$ as follows: we let $\tilde{E}$ be the set $E \cap [-M(\epsilon), M(\epsilon)]$ and $\tilde{K}$ be the set $E \cup ([-M(\epsilon), M(\epsilon)] \cap \hat{H}_\epsilon(0))$. Consider $\tilde{\Omega} = \hat{\mathbb{C}} \setminus \tilde{K}$. We know from (4.8) that

$$\omega(0, \tilde{E}, \tilde{\Omega}) \geqslant a/2.$$

Again $\tilde{K}$ is homogeneous with a constant depending only on ϵ, and since $\tilde{K} \subset [-M(\epsilon), M(\epsilon)]$ then we know that $\omega(\infty, \bullet, \tilde{\Omega})$ is absolutely continuous with respect to length and in fact, that the Radon-Nikodym derivative h is in L^p, for some $p > 1$. More precisely.

$$\omega(\infty, \bullet, \tilde{\Omega}) = h\,dx$$

and for $p = p(\epsilon) > 1$ and $T = T(\epsilon)$ we have

$$\int_{\partial\tilde{\Omega}} |h(x)|^p\,dx \leqslant T(\epsilon).$$

This is the heart of the matter. It is due to Jones and Marshall ([JM]).

From Harnack's inequality (and a bit of Poincaré geometry), we have

$$\omega(\infty, \tilde{E}, \tilde{\Omega}) \geqslant a'.$$

$(a' = a'(a, \epsilon) = a'(\epsilon))$. Therefore

$$a' \leqslant \int_E |h(x)|\,dx \leqslant T(\epsilon)^{1/p} |\tilde{E}|^{1-1/p}.$$

And so

$$|E \cap [-M(\epsilon), M(\epsilon)]| \geqslant c = c(\epsilon)$$

and we are done.

5. Fully Accessible Domains

This is a notion that has been introduced and studied by Patterson, [Pa1], [Pa2], Pommerenke [Po3], [Po4], [Po5], and Sullivan [S2]. A Fuchsian group Γ is called *fully accessible* if the action of Γ on $\partial\Delta$ is fully dissipative *i.e.* if there is a measurable set $B \subset \partial\Delta$ so that if $\gamma \in \Gamma \setminus \{id\}$, $|\gamma(B) \cap B| = 0$ and $|\partial\Delta \setminus \bigcup_{\gamma\in\Gamma} \gamma(B)| = 0$, or in other terms that the action of Γ on $\partial\Delta$ has a measurable fundamental set.

A domain is called *fully accessible* if its covering group is fully accessible.

Patterson showed in [Pa1] that if $\delta(\Gamma) < 1/2$ then Ω is fully accesible. On the other hand fully accesible domains may have $\delta(\Omega) = 1$. One such example is provided by $\Omega = \Delta^* \setminus \{a_n\}$, where $a_n \to 0$. It is easy to see that Ω is fully accessible (see Theorem 3 or Example 1 in [Po4]) but $\delta(\Omega) = 1$. See Remark 1.

It is reasonable to expect that Hayman-Wu domains must be fully accesible. We can only show this for Denjoy domains. In that case a Hayman-Wu domain satisfies that if F is the symmetric universal covering map with $F(0) = \infty$, Γ its covering group, and D_0 the associated Dirichlet region at 0 then

$$\sum_{\gamma \in \Gamma} \text{length}\,(\partial(\gamma(D_0))) < \infty$$

(see [FH]).

This clearly implies that

$$\left| \partial\Delta \setminus \bigcup_{\gamma} \gamma(\partial D_0 \cap \partial\Delta) \right| = 0,$$

which gives that Ω is fully accessible. Another argument to show this is provided by two characterizations. Assume that Ω is a Denjoy domain. We have seen that Ω is Hayman-Wu if and only if Ω is homogeneous; on the other hand it has been shown by D. Hamilton and the author that Ω is fully-accesible if and only if harmonic measure in $\partial\Omega$ is absolutely continuous with respect to arc length (see Remark 2). But Carleson, [C], showed that for homogeneous sets harmonic measure is in fact absolutely continuous.

Remark 1. Let a_n be a sequence of numbers converging to zero. Let

$$\Omega = \Delta^* \setminus \{a_k\}_{k=1}^{\infty}.$$

Now

$$\delta(\Omega) \geqslant \delta(\Delta \setminus \{0, a_n\}).$$

This follows from the results about signatures in [Pa2], but in [F] it is shown that $\delta(\Delta \setminus \{0, a_n\}) \to 1$ as $n \to \infty$ therefore $\delta(\Omega) = 1$.

Remark 2. We simply sketch the argument. It is based on the special form of the Dirichlet's, D_0, and Green's, G_0, fundamental region associated to the covering map F with takes 0 to ∞ and is symmetric under complex conjugation. The Dirichlet region is mapped under F on $\mathbb{C} \setminus [p, q]$ where $[p, q]$ is the smallest closed interval which contains $\partial\Omega$. Since ∂D_0 is rectifiable it follows that if Γ is fully accessible then $\omega(\infty, \cdot, \partial\Omega)$ is absolutely continuous with respect to length. Conversely, since the Green's region is mapped onto $\mathbb{C} \setminus [p, q]$ one sees that if $\omega(\infty, \cdot, \partial\Omega)$ is absolutely continuous with respect to length then the Green's measure is absolutely continuous with respect to $d\theta$, and this is equivalent to full accessibility; (see [Po3] for definitions and this last result).

6. Proof of Theorem A

We start with

Lemma. *Let G be a Fuchsian group and denote by $D_0(G)$ the Dirichlet region of G at* 0. *Then*

$$\sum_{g\in G} \operatorname{length}(\partial g(D_0(G)) \leqslant \pi^2 \sum_{g\in G} (1 - |g(0)|^2)^{1/2}.$$

PROOF. The domain $g(D_0(G))$ is contained in

$$\{z: \rho(z, g(0)) \leqslant \rho(z, 0)\} = H(g(0)).$$

By a result of B. Brown, [B], we have that

$$\operatorname{length}(\partial g(D_0(G))) \leqslant \frac{\pi^2}{2} \operatorname{diam}(g(D_0(G))).$$

But

$$\operatorname{diam}(H(g(0))) = 2(1 - |g(0)|^2)^{1/2},$$

and so the result follows.

If Γ satisfies that $U_{1/2}$ is bounded then for any group G conjugate to Γ we have

$$\sum_{g\in G} \operatorname{length}(g(\partial D_0(G))) \leqslant \pi^2 \| U_{1/2} \|_\infty. \tag{6.1}$$

For $G = \omega^{-1}\Gamma\omega$, where $\omega \in \text{Möb}(\Delta)$, and then

$$\sum_{g\in G} (1 - |g(0)|^2)^{1/2} = \sum_{\gamma\in G} (1 - [\omega(0), \gamma(\omega(0))]^2)^{1/2} = U_{1/2}(\omega(0)).$$

Assume that a covering group Γ of Ω (and hence all) has $\| U_{1/2} \|_\infty < \infty$.
Let F be any universal covering map from Δ onto Ω. The group of deck transformations of Γ is a group G conjugate to Γ.
We want to estimate the length of the set $V = F^{-1}(\Omega \cap L)$ where L is an straight line. Since F is one-to-one on $g(D_0(G))$ we deduce form the Hayman-Wu theorem (see [GGJ]) that

$$\operatorname{length}(V \cap g(D_0(G))) \leqslant C \operatorname{length}(g(\partial D_0(G))) \tag{6.2}$$

where C is an absolute constant. But then using (6.1) and (6.2) we deduce that

$$\begin{aligned}\operatorname{length}(V) &\leqslant \sum_{g\in G} \operatorname{length}(\partial g(D_0(G))) + \sum_{g\in G} \operatorname{length}(V \cap g(D_0(G)))\\ &\leqslant (1 + c) \sum_{g\in G} \operatorname{length}(g(\partial D_0(G)))\\ &\leqslant (1 + c)\pi^2 \| U_{1/2} \|_\infty.\end{aligned}$$

7. An Example

We know that for a domain Ω, U_1 is bounded if and only if Ω posseses a strong barrier. Possesing a strong barrier means that Ω contains no doubly connected domains (separating $\partial\Omega$) of arbitrarily large modulus, or equivalently, in view of a theorem of Teichmüller ([Ah, p. 74]), that contains no ring (separating $\partial\Omega$) of arbitrarily large modulus (see [BP], [Po1]).

Let us define the modulus of a domain Ω as

$$M(\Omega) = \sup\{\operatorname{mod}(R): R, \text{ring}, R \subset \Omega, R \text{ separating } \partial\Omega\}.$$

The constant $M(\Omega)$ and the reciprocal of the ϵ in the definition of strong barrier are bounded by functions of each other.

Since $\delta(\Omega) < 1/2$ guarantees that there are no isolated boundary points it is tempting to guess that $\delta(\Omega) < 1/2$ implies that Ω posseses an strong barrier. Theorem A also points in that direction. Unfortunately

Example. *Given $\delta_0 > 0$ there exist a domain Ω with $\delta(\Omega) \leqslant \delta_0$ but $M(\Omega) = \infty$.*

In order to show that the exponent of a domain is close to 1 one only has to provide an example of a function $\varphi \in C_0^\infty(\Omega)$ with small

$$\frac{\iint_\Omega |\nabla\varphi|_\Omega^2\, dx\, dy}{\iint_\Omega |\varphi|^2 \omega_\Omega}.$$

But the Rayleigh quotient is of no help here since at most it can be used to show that $\delta(\Omega) \leqslant 1/2$. We do have to look into the geometry of Ω.

Given a sequence ϵ_j of positive numbers tending to zero consider the domain

$$\Omega = \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z}} \bar{\Delta}(2n+1, \epsilon_{|n|}) \setminus \bigcup_{n\in\mathbb{Z}} T(2n, \eta_n).$$

where if $a \in \mathbb{R}$ and $\eta > 0$

$$T(a, \eta) = \{a + iy: |y| \geqslant \eta\} \cup \{x + iy: |x - a| \leqslant 1/2, |y| = \eta\}.$$

If δ_0 is given we can choose the numbers η_n converging to zero so fast that $\delta(\Omega) \leqslant \delta_0$. Of course, $M(\Omega) = \infty$.

We content ourselves with giving a proof of the following

Lemma. *Given $\delta_0 > 0$ and M_0 there exists a triply connected domain Ω with*

$$\delta(\Omega) \leqslant \delta_0 \quad \text{and} \quad M(\Omega) \geqslant M_0.$$

Consider the domain

$$\Omega = \mathbb{C} \setminus \bar{\Delta}(1, \epsilon) \setminus \bar{\Delta}(-1, \epsilon) \setminus T(0, \eta).$$

The set $\{x + iy: |y| \leqslant \eta/2, |x| \leqslant 1/2\}$ will be called the tunnel. It is clear that if ϵ is small enough then $M(\Omega) \geqslant M_0$ (recall that $\eta \leqslant \epsilon$). We now fix ϵ and show that $\delta(\Omega)$ tends to zero as $\eta \to 0$.

Notice that Ω is symmetric under reflection on the imaginary axis ($(x + iy)^* = -x + iy$). Choose F so that $F(0) = 1$ and $F(\bar{z}) = F(z)^*$. We have to check that for s small (assuming η small) we have

$$\sum_{\gamma \in \Gamma} e^{-s\rho(0, \gamma(0))} < \infty \tag{7.1}$$

where Γ is the covering group of F. The group Γ is a free group in two generators. One generator, α, corresponds to the loop with base at 1, which surrounds $\bar{\Delta}(1, \epsilon)$ the other one, β, corresponds to the $*$-symmetric loop. We decompose the sum in (7.1) as follows

$$1 + \sum_{k=1}^{\infty} \sum_{\gamma \in A_k} e^{-s\rho(0, \gamma(0))}, \tag{7.2}$$

where A_k denotes the collection of those elements of Γ of the form

$$\sigma = w_1^{p_1} w_2^{p_2} \cdots w_k^{p_k}$$

where w_i is α or β but $w_i \neq w_{i+1}$, $i = 1, \ldots, k-1$, and $p_i \in \mathbb{Z} \setminus \{0\} = \mathbb{Z}^*$. Consider $\sigma \in A_k$, we will estimate $\rho(0, \sigma(0))$ from below. Let h denote the length of the shortest geodesic in Ω which surrounds $\bar{\Delta}(1, \epsilon)$. This number h depends on ϵ and η but there exist $h_0 = h_0(\epsilon)$ which depends only on ϵ so that $h \geqslant h_0$. (This could be seen by using the convergence results in [H]).

The segment from 0 to $\sigma(0)$ is mapped by F onto a curve $\hat{\sigma}$ which is locally a geodesic and

$$\rho(0, \sigma(0)) = l_\Omega(\hat{\sigma}) \qquad (= \text{the Poincaré length of } \hat{\sigma}).$$

With this information we may estimate $l_\Omega(\hat{\sigma})$ from below as follows:

$$l_\Omega(\hat{\sigma}) \geqslant \left(\sum_{j=1}^{k} |p_j| - k \right) h_0 + k\left(\frac{1}{\eta}\right).$$

For the length of a curve connecting the short sides of the tunnel is at least $1/\eta$ and $\hat{\sigma}$ «contains» k arcs connecting these short sides.

For a vector v in $(\mathbb{N} - \{0\})^k$ we write

$$\|v\| = \sum_{j=1}^{k} |v_j|.$$

Then we have that

$$\sum_{\gamma\in A_k} e^{-s\rho(0,\gamma(0))} \leqslant 2^k \sum_{v\in(\mathbb{N}-\{0\})^k} e^{-s(\|v\|h_0 + k(1/\eta - h_0))}$$

$$= 2^k e^{-sk(1/\eta - h_0)} \sum_{v\in(\mathbb{N}-\{0\})^k} e^{-sh_0\|v\|}$$

$$= 2^k e^{-s(1/\eta - h_0)k}\left[\frac{e^{-sh_0}}{1-e^{-sh_0}}\right]^k$$

and given s if we choose η small enough we have that

$$e^{-s(1/\eta - h_0)} < \frac{1}{4}(e^{sh_0} - 1)$$

and then the sum (7.2) is majorized by

$$1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = 2.$$

Remark. The example shows that one can have δ small while U_1 is unbounded. On the other hand Theorem 3 shows that for plane domains if U_1 is bounded then $\delta < 1$. This last fact *does not hold for Riemann surfaces.* Consider a Z^3-cover R of a compact Riemann surface S. Now R has a Green's function (see, *e.g.*, [T, p. 484]) and since U_1 is invariant under the Z^3-action we have that U_1 of R is bounded. On the other hand it is easy to see that the infimum of the Rayleigh's quotient is zero, and so $\delta = 1$. This example was pointed out by A. Ancona.

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Polyèdre Caractéristique et Éclatements Combinatoires

V. Cossart

Introduction

Soient R un anneau local régulier, J un idéal de R et $u = (u_1, \dots, u_d)$ une suite R/J-régulière. A cette situation, Hironaka [8] attache un polyèdre $\Delta(J; u) \subset \mathbb{R}^d$ dont les côtés renferment des informations cruciales pour la désingularisation de R/J.

Le côté le plus intéressant est le «premier côté», c'est-à-dire celui qui a une équation du type

$$x(1) + \cdots + x(d) = \delta(J; u).$$

En effet, $\delta(J; u)$ est ce qu'on appelle l'exposant caractéristique [8, Introduction].

De plus, à chaque côté de $\Delta(J; u)$, Hironaka attache un gradué. Par exemple, en caractéristique 0, le gradué respectif au «premier côté» contient des informations concernant l'espace tangent strict de l'exposant idéaliste que Hironaka considère dans sa récurrence. En caractéristique p, ce gradué est essentiel ([4] pour la dimension 2 et [3] pour la dimension 3).

D'autres côtés de $\Delta(J; u)$ sont intéressants, ce sont ceux qui ont des équations

$$\lambda(1)x(1) + \cdots + \lambda(d)x(d) = c$$

vérifiant la condition suivante

$$(*) \qquad 0 < \lambda(i) \leqslant c, \qquad 1 \leqslant i \leqslant d.$$

En effect, on verra en A.6 que, lorsque l'on effectue une suite d'éclatements permis, c'est parmi ces côtés que l'on trouvera les informations quant aux «premiers côtés» des points proches combinatoires.

Nous montrons ici que le polyèdre $\Delta(J; u)$ et les gradués associés a ses côtés peuvent être construits en n'utilisant que la seule donnée de $A = R/J$ et de la suite d'idéaux $u_1A, \ldots, u_dA$.

De plus dans B, nous donnons des interprétations géométriques des côtés de $\Delta(J; u)$ satisfaisant à (*) ainsi que de leurs gradués, simplement en considérant des suites *d'éclatements permis* de $\operatorname{Spec} A[t]$. On peut en déduire que les algorithmes de désingularisation définis à l'aide de végétations idéalistes sont indépendants du plongement de la singularité.

Nous avons fait les démonstrations cruciales des sections B, C et D en nous plaçant dans le cas de la caractéristique 0, car on peut utiliser les résultats de [1] ce qui simplifie considérablement les calculs, mais les résultats annoncés sont vrais en toute généralité.

On peut noter que l'idée utilisée ici, qui est d'étudier une déformation équisingulière de la singularité est classique ([10] Appendice de B. Teissier, [7] où Hironaka utilise de telles déformations pour étudier l'exposant de contact).

A. VALUATION SUR UN ANNEAU LOCAL RÉGULIER

A.1. Décomposition barycentrique

Soit L une forme linéaire sur $\mathbb{R}^q$ notée

$$(1)\quad L(x(1), \ldots, x(q)) = a(1)x(1) + \cdots + a(q)x(q), \quad a(i) \in \mathbb{N}^*, \quad 1 \leqslant i \leqslant q.$$

Soit $l(L)$ le cardinal de l'ensemble des valeurs prises par les $a(i)$. En général, on écrira l au lieu de $l(L)$. On définit par récurrence une famille décroissante de parties I_k, $1 \leqslant k \leqslant l(L)$ de $\{1, \ldots, q\}$ affectées d'un poids $m(k)$ par

$$(2)\quad \begin{aligned} I_1 &= \{1, \ldots, q\}, & m(1) &= \inf\{a(i) : 1 \leqslant i \leqslant q\}, \\ I_k &= \{i \in I_1 : a(i) > m(k-1)\}, & m(k) &= \inf\{a(i) : i \in I_k\}, \end{aligned}$$

$1 \leqslant k \leqslant l(L)$.

On pose alors

$$(3)\quad \begin{aligned} b(1) &= m(1), \\ b(k) &= m(k) - m(k-1), \qquad 2 \leqslant k \leqslant l, \end{aligned}$$

ce qui nous donne

$$L(x(1), \dots, x(q)) = \sum_{1 \leq k \leq l} b(k)\Big(\sum_{i \in I_k} x(i)\Big).$$

Désormais, pour simplifier, nous supposons que l'indexation est choisie de façon que

$$(4) \qquad 1 \leqslant a(1) \leqslant a(2) \leqslant \cdots \leqslant a(q).$$

On a donc

$$(5) \quad L(x(1), \dots, x(q)) = b(1)(x(1) + \cdots + x(q)) + b(2)(x_{(\alpha(2))} + \cdots + x_{(q)}) + \cdots + b(l)(x_{(\alpha(l))} + \cdots + x_{(q)})$$

où

$$I_k = (\alpha(k), \dots, q), \qquad 1 \leqslant k \leqslant l,$$

et

$$\begin{gathered} b(1) = a(1) \\ b(1) + b(2) = a(\alpha(2)) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ b(1) + b(2) + \cdots + b(l) = a(\alpha(l)) = a(q). \end{gathered}$$

En un mot, $b(i)$ est la i-ème composante non nulle de L sur la base $(x(1) + \cdots + x(q), x(2) + \cdots + x(q), \dots, x(q))$ de $\mathbb{Z}^{q*}$.

A.2. Valuations combinatoires

Soit R un anneau local régulier, M est son idéal maximal, $s = (s_1, \dots, s_q)$ est un système régulier de paramètres (s.r.p.), $q = \dim R$, L est une forme linéaire sur $\mathbb{R}^q$ qui satisfait aux conditions A.1. (1)(4).

Alors (L, s) définit une valuation $v_{L,s}$ sur R par

$$(1) \qquad v_{L,s}(\lambda s_1^{x(1)} \cdots s_q^{x(q)}) = L(x(1), \dots, x(q))$$

où $\lambda \in R \backslash M$.

On note $gr_{L,s}(R)$ le gradué associé. D'après [8, Section 1], on a

$$(2) \qquad gr_{L,s}(R) = R/M[\mathrm{in}_{L,s}(s_1), \dots, \mathrm{in}_{L,s}(s_q)].$$

A.3. Arbre combinatoire attaché à L et s

Posons

$$(1) \qquad Z(0,0) = \operatorname{Spec} R[t].$$

Soit

$$(2) \qquad \mathcal{E} = \{(i,j) : (i,j) = (0,0) \quad \text{ou} \quad 1 \leqslant i \leqslant l, \quad 1 \leqslant j \leqslant b(i)\}.$$

Nous munissons $\mathcal{E}$ de l'ordre lexicographique. Pour tour couple (i,j) de $\mathcal{E}$, on désigne par

(3) $(i,j)_-$ = l'élément de $\mathcal{E}$ qui précède (i,j) pour l'ordre lexicographique,
$(i,j)_+$ = l'élément de $\mathcal{E}$ qui suit (i,j) pour l'ordre lexicographique,

s'il en existe.

Par exemple, on a

$$(1,1)_- = (0,0) \quad \text{et} \quad (0,0)_+ = (1,1).$$

Posons

$$(4) \qquad R(i,j) = R[t, s_1/t^{a(1)}, s_2/t^{a(2)}, \ldots, s_{\alpha(i)-1}/t^{a(\alpha(i)-1)}, s_{\alpha(i)}/t^{a(\alpha(i)-1)+j}, \ldots, s_q/t^{a(\alpha(i)-1)+j}]$$

et

$$Z(i,j) = \operatorname{Spec} R(i,j).$$

Posons $a(0) = 0$ et considérons l'idéal

$$(5) \qquad I((i,j)_-) = (t, s_{\alpha(i)}/t^{a(\alpha(i)-1)+j-1}, \ldots, s_q/t^{a(\alpha(i)-1)+j-1})R((i,j)_-),$$

$(i,j) \geqslant (1,1)$.

A.3.1. Alors, $Z(i,j)$ est l'ouvert affine complémentaire du transformé strict de $\operatorname{div}(t)$ dans l'éclaté de $Z((i,j)_-)$ le long de l'idéal $I((i,j)_-)$.

A.3.2. On a donc construit un arbre

$$Z(0,0) \leftarrow Z(1,1) \leftarrow \cdots \leftarrow Z(i,j) \leftarrow \cdots \leftarrow Z(l, b(l))$$

de longueur

$$a(q) = b(1) + b(2) + \cdots + b(l).$$

A.4. Interprétation du gradué associé. Cas d'une seule forme linéaire.

Théorème. *Avec les hypothèses et notations de* A.1, A.2 *et* A.3, *notons* $E(i,j)$ *le diviseur exceptionnel de*

$$Z(0,0) \leftarrow Z(i,j), \qquad 1 \leqslant i \leqslant d, \quad 1 \leqslant j \leqslant b(i).$$

Alors

(i) $E(i,j)$ *est irréductible dans* $Z(i,j)$, t *est une équation de* $E(i,j)$ *dans* $Z(i,j)$.

(ii) *Pour tout* g *dans* R, *on a*

$$\mathrm{ord}_{\nu(l,b(l))}(g) = v_{L,s}(g),$$

où $\nu(l,b(l))$ *est le point générique de* $E(l,b(l))$.

(iii) *L'application*

$$\mathrm{in}_{L,s}(g) \to t^{-N} g \bmod (t)$$

où $N = v_{L,s}(g)$ *définit un isomorphisme entre* $gr_{L,s}(R)$ *et l'anneau de fonctions de* $E(l,b(l))$.

PREUVE. L'assertion (i) découle de A.3.1. Prouvons (ii). Si

$$g = s_1^{x(1)} \cdots s_q^{x(q)},$$

on a

$$g = t^{a(1)x(1) + \cdots + a(q)x(q)} (s_1 t^{-a(1)})^{x(1)} \cdots (s_q t^{-a(q)})^{x(q)}$$

et

$$\begin{aligned} N &= v_{L,s}(g) \\ &= a(1)x(1) + \cdots + a(q)x(q), \end{aligned}$$

on a (ii) en remarquant que

$$\mathrm{ord}_{\nu(l,b(l))}[(s_1 t^{-a(1)})^{x(1)} \cdots (s_q t^{-a(q)})^{x(q)}] = 0.$$

Pour prouver (iii), on remarque que notre application peut être ainsi définie

$$R/M[\mathrm{in}\, s_1, \ldots, \mathrm{in}\, s_q] \to R[t, st^{-a(1)}, \ldots, s_q t^{-a(q)}]/(t) = R/M[\bar{s}_1, \ldots, \bar{s}_q]$$

$$\bar{\lambda} \in R/M \to \lambda \bmod (t)$$

$$\mathrm{in}_{L,s}(s_i) \to s_i t^{-a(i)} \bmod (t) = \bar{s}_i.$$

A.5. Interprétation du gradué associé: cas général.

Le théorème A.4 se généralise au cas où R est muni d'une filtration définie par $s = (s_1, \ldots, s_q)$ et une famille de formes linéaires L_i, $i \in I$, I fini,

$L_i(x_1, \ldots, x_q) = a(1, i)x_1 + \cdots + a(q,i)x_q$, $a(j, i) \in \mathbb{N} \setminus \{0\}$, $i \in I$, $1 \leqslant j \leqslant q$,

on pose:

$$\Delta = \{(x_1, \ldots, x_q) \colon L_i(x) \geqslant 1, i \in I\} \neq \varnothing, \qquad \Delta \subset \mathbb{R}^q_+,$$

oú

$$\mathbb{R}_+ = \{x \geqslant 0 \colon x \in \mathbb{R}\}.$$

Chaque L_i définit un anneau

(1) $$R(L_i) = R[t, s_1/t^{a(1,i)}, \ldots, s_q/t^{a(q,i)}] = R[t, s^A t^{-\alpha}]$$

où $L_i(A) + \alpha \geqslant 0$ qui est l'anneau que nous avons noté $R(l, b(l))$ en A.3 (4).
Posons

(2) $$R(I) = \bigcap_{i \in I} R(L_i).$$

Alors on a

(3) $$R(I) = R[t, s^A t^{-\alpha}] \quad \text{où} \quad L_i(A) + \alpha \geqslant 0, \quad \text{pour tout} \quad i \in I.$$

Alors, Δ et s définissent une filtration sur R (*cf.* [8, Section 1]), pour tout $b \in \mathbb{R}^+$, on pose

(4) $$I(\Delta, b)_s = \{g \in R \colon v_{L_i, s}(g) \geqslant b, i \in I\},$$
$$I^+(\Delta, b)_s = \{g \in R \colon v_{L_i, s}(g) > b, i \in I\}.$$

On définit

$$gr_{\Delta, s}(R) = \bigoplus_{b \in \mathbb{R}+} I(\Delta, b)_s / I^+(\Delta, b)_s.$$

la composante homogène $I(\Delta, b)_s / I^+(\Delta, b)_s$ n'étant différente de 0 que pour un ensemble discret de réels b qui sont d'ailleurs entiers puisque les L_i sont à coefficients entiers.

Pour tout $g \in R$, on note

(5) $$v_{\Delta, s}(g) = \inf_{i \in I} (v_{L_i, s}(g))$$
$$= \inf \{b \colon g \in I(\Delta, b)_s\}.$$

Théorème A.5.1.

1. *Soit* $g \in R$ *et soit* $N = v_{\Delta,s}(g)$ *alors* $t^{-N}g \in R(I)$.
2. *L'application* Φ

$$\mathrm{in}_{\Delta,s}(g) \to t^{-N}g \bmod (t), \qquad g \in R, \quad \textit{où} \quad N = v_{\Delta,s}(g)$$

définit un isomorphisme entre $gr_{\Delta,s}(R)$ *et* $R(I)/tR(I)$.

PREUVE. Pour tout L_i, on a $v_{L_i,s}(g) \geqslant N$ donc $t^{-N}g \in R(L_i)$, donc

$$t^{-N}g \in \bigcap_{i \in I} R(L_i) = R(I),$$

ce qui est 1.

Prouvons 2. Soit $g \in R$ avec $v_{\Delta,s}(g) \geqslant N$, $N \in \mathbb{N}$. Alors on a

$$g = \sum_{A \in N\Delta} \nu_A s^A, \qquad \nu_A \in R$$

et

$$\Phi[cl^N_{\Delta,s}(g)] = \sum \nu_A s^A t^{-N} \bmod (t).$$

On en déduit

$$\Phi[cl^N_{\Delta,s}(g)] = \sum_{A \in N\Delta} \Phi[cl^N_{\Delta,s}(\nu_A s^A)]. \tag{1}$$

De (1) on tire facilement que Φ est un morphisme. Rappelons cependant une difficulté. On a, pour g, g' dans R

$$v_{\Delta,s}(gg') \geqslant v_{\Delta,s}(g) + v_{\Delta,s}(g')$$

avec parfois une inégalité stricte, ce qui implique alors

$$cl^{N+N'}_{\Delta,s}(gg') = 0 = cl^N_{\Delta;s}(g)cl^{N'}_{\Delta,s}(g')$$

où $N = v_{\Delta,s}(g)$, $N' = v_{\Delta,s}(g')$.

Avant de montrer la bijectivité de Φ, montrons le lemme suivant.

Lemme A.5.2.

$$(t^a R(I)) \cap R[t] = \sum_{b \leq a} t^{a-b} I(\Delta, b)_s R[t], \qquad a \in \mathbb{N}, \quad b \in \mathbb{N}.$$

On a $I(\Delta, b)_s R(I) \subset t^b R(I)$, donc l'inclusion du deuxième membre dans le premier est claire. Voyons l'inclusion inverse.

Soit

$$g \in (t^a R(I)) \cap R[t], \qquad g = t^a \sum \nu_A s^A t^{-v(A)}, \quad \nu_A \in R, \quad v(A) = v_{\Delta,s}(s^A).$$

Soit M un entier tel que $M \geqslant v(A)$ pour tout A avec $\nu_A \neq 0$.

$$t^M g = t^a \sum \nu_A s^A t^{M-v(A)}.$$

On a

$$t^M g \in t^M R[t] \cap \sum_A t^{a+M-v(A)} I(\Delta, v(A))_s.$$

En appliquant [8, (2.1)] à $R[t]$ et aux ensembles

$$E_i = \{(x_0, x_1, \ldots, x_q) \in \mathbb{N}^{q+1} : x_0 + L_i(x) \geqslant a + M\}$$

et

$$E = \{(x_0, x_1, \ldots, x_q) \in \mathbb{N}^{a+1} : x_0 \geqslant M\},$$

on a dans $R[t]$

$$t^M g \in I\Big(\bigcap_i E_i \cap E\Big) = \sum_{a - v(A) \geq 0} t^{a+M-v(A)} I(\Delta, \alpha(A))_s.$$

D'où le résultat en divisant par t^M.

A.5.3. Montrons l'injectivité de Φ.

Soit $G \in gr_{\Delta,s}(R)$ avec $\Phi(G) = 0$. On a

$$G = \sum_{i \in N} cl^i_{\Delta,s}\Big(\sum \mu_{A(i)} s^{A(i)}\Big)$$

avec $\mu_{A(i)} \in R$, $v_{\Delta,s}(s^{A(i)}) = i$,

$$\Phi(G) = \sum t^{-i}\Big(\sum \mu_{A(i)} s^{A(i)}\Big) \bmod (t) = 0 \bmod (t).$$

Donc

$$\sum_{i, A(i)} t^{-i} \mu_{A(i)} s^{A(i)} = tg \in tR(I).$$

Soit M un entier tel que $t^M g \in R[t]$ et $M \geqslant i$ pour tout i tels que

$$\sum_{A(i)} \mu_{A(i)} s^{A(i)} \neq 0.$$

On a

$$\sum t^{M-i} \mu_{A(i)} s^{A(i)} \in (t^{M+1} R(I)) \cap R[t].$$

Le lemme nous donne:

$$\sum_{i,A(i)} t^{M-i}\mu_{A(i)}s^{A(i)} = \sum \nu_c s^c t^{M+1-\alpha(c)}$$

où $\nu_c \in R$, $\alpha(c) = v_{\Delta,s}(s^c)$.

Comme $R[t]$ est une algèbre de polynômes, on a, pour tout i

$$\sum_{A(i)} t^{M-i}\mu_{A(i)}s^{A(i)} = \sum_{\alpha(c)=1+i} \nu_c s^c t^{M-i}$$

d'où

$$\sum_{A(i)} \mu_{A(i)}s^{A(i)} = \sum_{\alpha(i)=1+i} \nu_c s^c,$$

donc

$$cl^i_{\Delta,s}\Big(\sum_{A(i)} \mu_{A(i)}s^{A(i)}\Big) = 0, \qquad i \in \mathbb{N},$$

donc $G = 0$.

A.5.4. Montrons la surjectivité.

Soit g dans $R(I)$, cherchons un antécédent à $g \bmod (t)$. Comme $R(I)$ est un $R[t]$-module engendré par les $s^A t^{-v(A)}$ où $v(A) = v_{\Delta,s}(s^A)$, on a

$$g = \sum_{\mu A \in R} \mu_A s^A t^{-v(A)} + tg'$$

On a

$$g \bmod (t) = \sum \Phi(cl^{v(A)}(\mu_A s^A)).$$

A.5.5. Remarquons que, puisque I est fini, Δ est effectif, c'est-à-dire

$$\bigcup_{b\in\mathbb{R}^+} b\Delta = \mathbb{R}^{+q}\setminus\{0\},$$

on a

(1) $$tR(I) = (t, s_1, \ldots, s_q)R(I).$$

C'est-à-dire que

$$\operatorname{div}(t) = f(I)^{-1}(M, t)$$

où $f(I)$ est le morphisme

$$\operatorname{Spec} R(I) \to \operatorname{Spec} R[t],$$

donné par l'inclusion des anneaux.

A.6. Polyèdre caractéristique

Soient R un anneau local régulier d'idéal maximal M et J un idéal non trivial de R. Alors, pour tout s.r.p. de R que l'on note

$$s = (s_1, \ldots, s_q) = (y_1, \ldots, y_r, u_1, \ldots, u_d) = (u, y)$$

et tel que $\mathrm{in}_M(y)$ est l'idéal de la directrice (espace tangent strict) de Spec R/J, Hironaka construit un polyèdre $\Delta(J; u; y) \subset \mathbb{R}^d$, [8]. Ce polyèdre est une projection du nuage de points de certains générateurs de J. Hironaka montre que, quitte à passer au complété $\hat{R}$ de R, on peut choisir $(y_1, \ldots, y_r)$ tel que $\Delta(J; u; y)$ soit minimal pour l'inclusion. On note $\Delta(J; u)$ ce polyèdre minimal et on l'appelle polyèdre caractéristique.

Dans les paragraphes qui suivent, nous allons construire $\Delta(J; u)$ avec la simple donnée de $R/J = A$ et de la suite d'idéaux $(u_1R/J, \ldots, u_dR/J)$. Faisons d'abord la remarque suivante.

A.7. La condition (*) est héréditaire

Avec les notations de A.5, supposons que le fermé

$$Y = V(y_1, \ldots, y_r, u_1, \ldots, u_k), \qquad 1 \leqslant k \leqslant d,$$

est permis pour J et que $\Delta(J; u; y) = \Delta(J; u)$. Effectuons l'éclatement centré en Y et plaçons nous à l'origine d'un des ouverts affines de l'éclaté. Posons par exemple

$$y_i = u_1 y'_i, \quad 1 \leqslant i \leqslant r; \qquad u_j = u_1 u'_j, \quad 2 \leqslant j \leqslant k,$$
$$u'_l = u_l, \quad k+1 \leqslant l \leqslant d; \qquad u_1 = u'_1.$$

Notons J' le transformé strict de J. Un calcul classique montre que le polyèdre $\Delta(J'; u')$ est obtenu à partir de $\Delta(J; u)$ en effectuant la transformation affine

$$(x(1), \ldots, x(d)) \to (x(1) + \cdots + x(k) - 1, x(2), \ldots, x(d))$$

et en prenant le monoidéal convexe engendré dans $\mathbb{R}^d_+$. Soit

$$\lambda(1)x(1) + \cdots + \lambda(d)x(d) = c$$

l'équation d'un côté de $\Delta(J'; u')$, ce côté est le transformé affine du côté de $\Delta(J; u)$ d'équation

$$\lambda(1)(x(1) + \cdots + x(k) - 1) + \cdots + \lambda(d)x(d) = c,$$

ce qui peut s'écrire:

$$\lambda(1)x(1) + (\lambda(1) + \lambda(2))x(2) + \cdots + (\lambda(1) + \lambda(k))x(k) + \cdots + \lambda(d)x(d) = c + \lambda(1).$$

On vérifie que, si on a $(*)$ pour le côté de $\Delta(J', u')$, c'est-à-dire

$$(*) \qquad 0 < \lambda(i) \leqslant c, \qquad 1 \leqslant i \leqslant d,$$

alors on a $(*)$ pour le côté de $\Delta(J; u)$ dont il est le transformé.

Comme le «premier côté» de $\Delta(J'; u')$ vérifie $(*)$, on déduit de cette remarque que, dans une suite d'éclatements combinatoires (*i.e.* en restant aux origines des ouverts affines des éclatés, sans effectuer de translations sur les variables), les premiers côtés des polyèdres associés sont des transformés affines de certains côtés de $\Delta(J; u; y)$ satisfaisant à $(*)$.

B. COTÉS SATISFAISANT À $(*)$

B.1. Modifications attachées à un diviseur pondéré

B.1.1. Soit A un anneau local noethérien (qu'on ne suppose pas régulier) d'idéal maximal $\mathfrak{M}$; soient $v_1A, \ldots, v_dA$ des idéaux monogènes de A tels que,

$$(1) \qquad v_i \in \mathfrak{M} \setminus \mathfrak{M}^2$$

et, en posant

$$V_i = cl^1_{\mathfrak{M}}(v_i),$$

$k[V_1, \ldots, V_d]$ est l'anneau de la directrice du cône tangent de Spec A.

Soit Λ une forme linéaire sur $\mathbb{R}^d$

$$(2) \qquad \Lambda(x(1), \ldots, x(d)) = \lambda(1)x(1) + \cdots + \lambda(d)x(d),$$

$0 < \lambda(i)$, $\lambda(i) \in \mathbb{Q}$, $1 \leqslant i \leqslant d$.

Soit N un entier strictement positif multiple du dénominateur commun des $\lambda(i)$. Pour plus de commodité, on posera

$$(3) \qquad a(i) = N\lambda(i), \qquad 1 \leqslant i \leqslant d.$$

Les v_i définissent un diviseur $V(v_1, \ldots, v_d)$ de Spec$A$ et l'on a des poids $a(1), \ldots, a(d)$ pour chaque composante.

Etant donné (A, v, Λ, N), nous allons définir et étudier un arbre $X(i,j)$ sur $X(0,0) = \operatorname{Spec} A[t]$, la hauteur de l'arbre sera notée $L(A, v, \Lambda, N)$ ou plus simplement $l(\Lambda, N)$.

Dans un cas extrême, nous pourrons avoir

$$l(\Lambda, N) = \infty.$$

B.1.2. Effectuons la décomposition barycentrique de $N\Lambda$ et reprenons les notations de A.1. c'est-à-dire

$$\begin{aligned}(1)\quad N\Lambda(x(1), \ldots, x(d)) &= a(1)x(1) + \cdots + a(d)x(d)\\ &= b(1)(x(1) + \cdots + x(d)) + b(2)(x(\alpha(2)) + \cdots\\ &\quad + x(d)) + \cdots + b(l)(x(\alpha(l)) + \cdots + x(d)).\end{aligned}$$

Donc $b(i)$ est la i-ème composante non nulle de $N\Lambda$ sur la base

$$(x(1) + \cdots + x(d), x(2) + \cdots + x(d), \ldots, x(d))$$

de $\mathbb{Z}^*$, l est le nombre de composantes non nulles.

On pose, comme plus haut

$$(2)\qquad I_i = \{k : k \geqslant \alpha(i)\}, \qquad 1 \leqslant i \leqslant l,$$

et

$$(3)\qquad \begin{aligned} I_{l+1} &= \varnothing \\ b(l+1) &= +\infty. \end{aligned}$$

Soit

$$(4)\qquad \mathcal{E}(\Lambda, N, \infty) = \{(i,j) \in \mathbb{N}^2 : (i,j) = (0,0) \text{ ou } 1 \leqslant i \leqslant l+1,\ 1 \leqslant j \geqslant b(i)\}.$$

On munit $\mathcal{E}(\Lambda, N, \infty)$ de l'ordre lexicographique.

La suite $X(i,j)$ d'éclatements modifiés que nous allons construire est indexée par le segment $\mathcal{E}(A, v, \Lambda, N)$ de $\mathcal{E}(\Lambda, N, \infty)$ d'origine $(0,0)$ et de longueur $l(\Lambda, N)$, ce qui fait $l(\Lambda, N) + 1$ indices si on compte $(0,0)$.

Pour tout (i,j) de $\mathcal{E}(\Lambda, N, \infty)$, on note $(i,j)_-$ le couple qui le précède et $(i,j)_+$ celui que le suit pour l'ordre lexicographique. Par exemple, on a

$$(1,1)_- = (0,0) \quad \text{et} \quad (0,0)_+ = (1,1).$$

B.1.3. La récurrence. Pour tout (i,j) de $\mathcal{E}(\Lambda, N)$, nous allons définir un heptuplet

$$(1)\qquad H(i,j) = (\tilde{X}(i,j), X(i,j), E(i,j), V(i,j), T(i,j), D(i,j), x(i,j))$$

où

$\tilde{X}(i,j)$ est un schéma ou un espace analytique complexe,

$X(i,j)$ est un ouvert affine de $\tilde{X}(i,j)$,

$E(i,j)$ est un diviseur de $X(i,j)$,

$V_.(i,j)$ est un drapeau, c'est-à-dire une suite croissante de sous-schémas fermés de $X(i,j)$ notée

$$V_1(i,j) \subset V_2(i,j) \subset \cdots \subset V_l(i,j) \subset V_{l+1}(i,j) = X(i,j),$$

$T(i,j)$ est un fermé vide ou irréductible de $E(i,j)$, on notera $\eta(i,j)$ le point générique de $T(i,j)$ si $T(i,j) \neq \emptyset$,

$D(i,j)$ est une courbe irréductible de $X(i,j)$,

$x(i,j)$ est un point fermé de $D(i,j)$.

Pour $(i,j) = (0,0)$, on pose

(2)
$$\begin{aligned} X(0,0) &= \tilde{X}(0,0) = \operatorname{Spec} A[t], \\ E(0,0) &= V(t), \\ V_0(0,0) &= V(\mathfrak{M}, t), \\ V_k(0,0) &= V(v_i A[t], i \in I_k), \quad 1 \leqslant k \leqslant l \quad (cf.\ \text{B.1.2(2)}), \\ V_{l+1}(0,0) &= X(0,0), \\ T(0,0) &= V(tA[t], v_i A[t], 1 \leqslant i \leqslant d), \\ D(0,0) &= V(\mathfrak{M}A[t]), \end{aligned}$$

$x(0,0)$ est le point fermé correspondant à l'idéal maximal $\mathfrak{M} + t$.

Remarquons que par construction chaque $E(i,j)$ pour $(i,j) > (0,0)$ se trouve être un schéma affine sur le corps résiduel $A/\mathfrak{M}$, ce qui nous autorise à parler de point générique pour un fermé irréductible de $E(i,j)$, même dans le cas analytique complexe.

Etant donné $H(i,j)$, définissons un test $OC(i,j)$ (OC signifie: on continue...), qui nous indique si (i,j) est le dernier indice pour lequel $H(i,j)$ est défini ou bien si $H((i,j)_+)$ existe; dans le premier cas on aura $OC(i,j) = F$ (on ne continue pas), dans le deuxième cas, on aura $OC(i,j) = V$ (on continue).

On pose, pour $(i,j) \neq (0,0)$

(3)
$$Y(i,j) = T(i,j) \cap V_{a(i,j)}(i,j), \qquad 1 \leqslant i \leqslant l+1.$$

où

$$a(i,j) = i \quad \text{si} \quad j \leqslant b(i) - 1,$$
$$a(i,j) = i+1 \quad \text{si} \quad j = b(i).$$

Si $x(i,j) \notin T(i,j)$ ou si $Y(i,j)$ n'est pas permis pour $X(i,j)$ en $x(i,j)$ ou si $Y(i,j)$ est vide, ou si $Y(i,j)$ n'est pas irréductible, alors $OC(i,j) = F$. Sinon $OC(i,j) = V$.

Nous verrons plus tard qu'en fait $Y(i,j)$ est toujours irréductible (B.2.7.3.).

Pour $(i,j) = (0,0)$, posons

(4) $$Y(0,0) = \{x(0,0)\}$$

et alors $OC(0,0) = V$.

Supposons $OC(i,j) = V$ et construisons $H((i,j)_+)$. On note

(5) $$\tilde{\pi}(i,j) = \tilde{X}((i,j)_+) \to X(i,j)$$

l'éclatement de $X(i,j)$ le long de $Y(i,j)$.

On définit $X((i,j)_+)$ comme l'ouvert affine de $\tilde{X}((i,j)_+)$ complémentaire du transformé strict de $E(i,j)$ et on note

$$\pi(i,j) \colon X((i,j)_+) \to X(i,j)$$

la restriction de $\tilde{\pi}(i,j)$ à $X((i,j)_+)$.

Bien sûr, $E((i,j)_+)$ est le diviseur exceptionnel de $\pi(i,j)$, c'est-à-dire

$$E((i,j)_+) = \pi(i,j)^{-1}(Y(i,j)).$$

Pour tout k, $0 \leqslant k \leqslant l+1$, $V_k((i,j)_+)$ est le transformé strict de $V_k(i,j)$. On a donc bien

$$V_{l+1}((i,j)_+) = X((i,j)_+),$$

on remarque que, pour $(i,j) \geqslant (1,1)$, on a $V_0(i,j) = \emptyset$.

Désignons par $\epsilon(i,j)$ le point générique de $Y(i,j)$. Alors $\tilde{\pi}(i,j)^{-1}(\epsilon(i,j))$ est canoniquement isomorphe à Proj $C_{\epsilon(i,j)}(X(i,j))$. Notons $\Gamma(i,j)$ la directrice de $C_{\epsilon(i,j)}(X(i,j))$, soit $\eta((i,j)_+)$ le point générique du fermé de $\pi(i,j)^{-1}(\epsilon(i,j))$ qui correspond au point générique de $\mathrm{Proj}_{\epsilon(i,j)}\Gamma(i,j)$ par l'isomorphisme canonique.

Si $\eta((i,j)_+) \notin X((i,j)_+)$, on pose $T((i,j)_+) = \emptyset$, sinon

$$T(i,j) = \{\overline{\eta(i,j)_+}\} \subset X(i,j)_+ .$$

$D((i,j)_+)$ est le transformé strict de $D(i,j)$ et $x((i,j)_+)$ est le point fermé de $D((i,j)_+)$ au dessus de $x(i,j)$. (On verra en B.2 que $D(i,j)$ est transverse à $Y(i,j)$ en $x(i,j)$ et donc que $x((i,j)_+)$ est bien défini).

B.1.4. Remarquons qu'on a toujours

$$V_{l+1}(i,j) = X(i,j)$$

et donc, pour $l + 1 \leqslant i$, on a

$$Y(l+1,j) = T(l+1,j) \qquad 1 \leqslant j,$$

bien sûr, ceci sous réserve que $H(l+1,j)$ existe.

B.2. Explicitation

B.2.1. Nous nous plaçons dans le cas où $A = R/J$, R anneau local régulier, soit $u = (u_1, \ldots, u_d)$ un relèvement de $v = (v_1, \ldots, v_d)$ (*cf.* B.1.1.). Le théorème B.2.2. énoncé ci-dessous est alors vrai sans autre hypothèse. Nous ne ferons la démonstration que dans le cas où

$$R = \mathbb{C}\{z_1, \ldots, z_r, u_1, \ldots, u_d\} \quad \text{ou} \quad \mathbb{C}[[z_1, \ldots, z_r, u_1, \ldots, u_d]].$$

Par [7, Ch. I (6.4)(6.5)(6.6.1)], on peut trouver

$$y = (y_1, \ldots, y_d) \quad \text{et} \quad f = (f_1, \ldots, f_m)$$

tels que (f, u, y) est une donnée distinguée pour R, J et de plus

(1) $$W = V(y_1, \ldots, y_r)$$

contient la strate de Samuel relative à $\mathbb{C}\{u_1, \ldots, u_d\} \hookrightarrow R$ (*cf.* [6, (2.1)])

(2) $$\Delta(J; u; y) = \Delta(J; u) \quad (\textit{cf.}\ [1, \text{p. } 18])$$

et même

(3) $$\Delta(J; u) = \Delta(f; u; y) \quad (\textit{cf.}\ [1, \text{lemme } 3]).$$

Théorème B.2.2. *Reprenons les hypothèses et notations de* B.1. *Soit* $a \in \mathbb{Q}^+ \cup \{\infty\}$ *donné par*

(1) $$\Lambda[\Delta(J; u)] = [a, +\infty[.$$

1. *On a les équivalences*

$$a \geqslant \sup\{\lambda(i) : 1 \leqslant i \leqslant d\}$$

si et seulement si il existe $N \in \mathbb{N}$ *avec* $N\lambda(i) \in \mathbb{N}$ *et*

$$l(A, v, \Lambda, N) \geqslant \sup\{N\lambda(i) : 1 \leqslant i \leqslant d\}$$

si et seulement si pour tout $N \in \mathbb{N}$ *avec* $N\lambda(I) \in \mathbb{N}$ *on a*

$$l(A, v, \Lambda, N) \geqslant \sup\{N\lambda(i) : 1 \leqslant i \leqslant d\}.$$

2. *Si*

$$a \geqslant \sup \{\lambda(i) : 1 \leqslant i \leqslant d\}$$

alors

(*a*) $l(A, v, \Lambda, N) = \lfloor Na \rfloor$ (*en posant* $\infty = \lfloor \infty \rfloor$),

(*b*) $a = \lim_{N \to \infty} N^{-1} l(A, v, \Lambda, N)$,

(*c*) *Pour* $(i,j) \leqslant (l, b(l))$ *l'algorithme de* B.1.3. *défini par* (A, v, Λ, N) *est la restriction à* $X(0,0) = \operatorname{Spec}(R[t]/J)$ *et à ses transformés stricts* $X(i,j)$ *de la suite d'éclatements combinatoires définis en* A.3. *pour* $Z(0,0) = \operatorname{Spec} R[t]$,

$$s = (u, y),$$
$$a(i) = N\lambda(I), \quad 1 \leqslant i \leqslant d,$$
$$a(j) = N, \quad d + 1 \leqslant j \leqslant q = r + d,$$

(*d*) *Fixons N, posons*

$$(\alpha, \beta) = \sup(\mathcal{E}(A, v, \Lambda, N)),$$

$\Lambda' = (N/\lfloor Na \rfloor)\Lambda$ *forme linéaire sur* $\mathbb{R}^d$, *alors pour tout* $y \in A = R/J$, *on a*

$$v_{\Lambda', u, y}(g) = \sup\{M/\lfloor Na \rfloor : t^{-M} g \in \Gamma(X(\alpha, \beta))\}$$

ou $\Gamma(X(\alpha, \beta))$ *est l'anneau de fonctions de* $X(\alpha, \beta)$. *De plus, l'application de* $gr_{\Lambda', u, y}(A)$ *vers l'anneau de fonctions de* $E(\alpha, \beta)$ *donné par* $\mathrm{in}_{\Lambda', u, y}(g) \to t^{-M} g$ *est un isomorphisme d'anneaux.*

B.2.3. Rappelons que si Λ' est une forme linéaire sur $\mathbb{R}^d$ à coefficients strictement positifs, (u, y) étant fixé, on définit

$$v_{\Lambda', u, y}(u^D y^E) = |E| + \Lambda'(D).$$

Montrons que B.2.2.2.*d* est une conséquence des autres résultats su théorème B.2.2.

Puisque $a \geqslant \sup\{\lambda(i) : 1 \leqslant i \leqslant d\}$, par B.2.2.1., on a

$$\Lambda'(D) + |E| = \lfloor Na \rfloor^{-1}(N\Lambda(D) + N|E|) = \lfloor Na \rfloor^{-1} L(\alpha, \beta)(D, E)$$

où $L(\alpha, \beta)$ est le forme linéaire sur $\mathbb{R}^{d+r}$ de coefficients

$$a(i) = N\lambda(i), \quad 1 \leqslant i \leqslant d; \qquad a(j) = N, \quad d + 1 \leqslant j \leqslant q = r + d.$$

Alors B.2.2.c et A.4 appliqués avec $L = L(\alpha, \beta)$ et $s = (u, y)$ nous donnent B.2.2.*d*.

B.2.4. Pour montrer les autres assertions de B.2.2., nous avons besoin de quelques notations.

Pour tout (i,j), $1 \leqslant i \leqslant l+1$, $1 \leqslant j \leqslant b(i)$ (*cf.* B.1.2), on note $L(i,j,\Lambda,N)$ ou plus simplement $L(i,j)$ la forme linéaire

$$(1) \qquad L(i,j)(x(1),\dots,x(q)) = a(1)(x(1)+\cdots+x(q))+\cdots \\ + b(\alpha(i)-1)(x(\alpha(i)-1)+\cdots+x(q)) \\ + (b(\alpha(i)-1)+j)(x(\alpha(i))+\cdots+x(q)).$$

Pour simplifier, on pose

$$(2) \qquad h(i,j) = a(\alpha(i)-1)+j$$

ce qui, par définition de $\alpha(i)$ (*cf.* B.1.2) donne

$$(3) \qquad h(i,j) = \#\left\{(a,b) : 1 \leqslant a \leqslant l+1, 1 \leqslant b \leqslant b(a), (a,b) \underset{lex}{\leqslant} (i,j)\right\}$$

Alors, pour $i \geqslant l+1$, (1) devient

$$(4) \qquad L(l+1,j)(x(1),\dots,x(q)) = N\Lambda(x(1),\dots,x(q)) \\ + h(l+1,j)(x(d+1)+\cdots+x(q)).$$

Lemme B.2.5. *Soit*

$$X = (x(1),\dots,x(d),x(d+1),\dots,x(q)) \in \mathbb{R}^{+q}$$

avec

$$(1) \qquad x(1)+\cdots+x(q) > 1 \quad et \quad x(d+1)+\cdots+x(q) < 1.$$

Alors la suite

$$v(i,j)(X) = L(i,j)(X) - h(i,j), \qquad (i,j) \in \mathcal{E}(\Lambda,N,\infty)$$

est d'abord strictement croissante, puis éventuellement stationnaire, puis strictement décroissante.

PREUVE. Par B.2.4 (1)(2), on a

$$v(i,j)(X) = a(1)(x(1)+\cdots+x(q)-1)+\cdots \\ + b(\alpha(i)-1)(x(\alpha(i)-1)+\cdots+x(q)-1) \\ + (b(\alpha(i)-1)+j)(x(\alpha(i))+\cdots+x(q)-1)$$

$$v(i,j)(X) - v((i,j)_-)(X) = x(\alpha(i))+\cdots+x(q)-1,$$

où $(i,j)_-$ est l'élément de $\mathcal{E}(\Lambda, N, \infty)$ qui précède (i,j) pour l'ordre lexicographique.

On constate qu'on a croissance stricte de $v(i,j)(X)$ tant que $x(\alpha(i)) + \cdots + x(q) > 1$, ce qui se réalise au moins pour $i = 1$ et qu'on a décroissance stricte dès que $x(\alpha(i)) + \cdots + x(q) < 1$, ce qui se réalisera pour $i > d$.

Lemme B.2.6. *Reprenons les notations de* B.2.1. *Notons M l'idéal maximal de R.*

1. *Si* $\Delta(f; u; y) = \emptyset$, *pour tout* k, $1 \leqslant k \leqslant m$ *et pour tout* $(i,j) \in \mathcal{E}(\Lambda, N, \infty)$, *on a*

$$v_{L(i,j),u,y}(f_k) - n(k)h(i,j) = 0,$$

où

$$n(k) = \mathrm{ord}_M(f_k), \qquad cl^{n(k)h(i,j)}_{L(i,j),u,y}(f_k) \in \mathbb{C}[Y_1, \ldots, Y_r],$$

où $Y_a = cl^{h(i,j)}_{L(i,j),u,y}(y_a)$, $1 \leqslant a \leqslant r$.

2. *Si* $\Delta(f; u; y) \neq \emptyset$ *alors il existe* $(\alpha', \beta') \in \mathcal{E}(\Lambda, N, \infty)$, $(\alpha', \beta') \geqslant (1,1)$ *et* $\beta < \infty$ *tel que*

$$\begin{aligned} (i,j) \leqslant (\alpha',\beta') \quad &\textit{implique} \quad v_{L(i,j),u,y}(f_k) - n(k)h(i,j) = 0, \quad 1 \leqslant k \leqslant m,\\ (i,j) < (\alpha',\beta') \quad &\textit{implique} \quad cl^{n(k)h(i,j)}_{L(i,j),u,y}(f_k) \in \mathbb{C}[Y_1, \ldots, Y_r],\\ (\alpha',\beta') < (i,j) \quad &\textit{implique} \quad v_{L(i,j),u,y}(f_k) - n(k)h(i,j) < 0. \end{aligned}$$

B.2.6.1. Développons f_k, $1 \leqslant k \leqslant m$

$$f_k = \sum_{|c| = n(k)} \lambda_{c,k} y^c + \sum_{|E| < n(k)} \lambda_{D,E,k} y^E u^D \bmod (y)^{1+n(k)} \tag{1}$$

$\lambda_{c,k} \in \mathbb{C}\{u, y\}$ ou $\mathbb{C}[[u, y]]$, $\lambda_{D,E,k} \in \mathbb{C}\{u, y\}$ ou $\mathbb{C}[[u, y]]$, avec $\lambda_{c,k}$ nul our inversible, $\lambda_{D,E,k}$ nul ou inversible, avec, par définition de $\Delta(f; u; y)$

$$\lambda_{D,E,k} \neq 0 \quad \text{implique} \quad (n(k) - |E|)^{-1} D \in \Delta(f; u; y).$$

Si $\Delta(f; u; y) = \emptyset$ alors tous les $\lambda_{D,E,k}$ sont nuls et la première assertion est claire.

B.2.6.2. Pour tout D, E, k avec $\lambda_{D,E,k} \neq 0$, on applique B.2.5 pour

$$X = n(k)^{-1}(D, E).$$

On a

$$v_{L(1,1)}(y^E u^D) = |E| + |D|$$

et, par définition des données distinguées, on a

(2) $$\mathrm{in}_M(f_k) = \sum_{|c| = n(k)} \bar{\lambda}_{c,k} Y^c, \qquad \bar{\lambda}_{c,k} = \mathrm{in}_M \lambda_{c,k}$$

donc $v_{L(1,1),u,y}(\lambda_{D,E,k} y^E u^D) - n(k) \times 1 > 0$,

$$v_{L(i,j),u,y}(f_k) = n(k).$$

De plus, pour $(i,j) \in \mathcal{E}(\Lambda, N, \infty)$, on a $v_{L(i,j),u,y}(Y^c) = |c| h(i,j)$, les variations de $v(i,j)[n(k)^{-1}(D,E)]$ donnés en B.2.5. montrent la deuxième assertion.

Lemme B.2.7. *Avec les hypothèses et notations de* B.2.2. *et* B.2.6., *on a les assertions suivantes, en posant*

$$(\alpha, \beta) = \sup \mathcal{E}(A, v, \Lambda, N).$$

1. $$(\alpha, \beta) = (\alpha', \beta') = \sup \mathcal{E}(A, v, \Lambda, N) \quad si \quad \Delta(J; u) \neq \varnothing,$$
$$(\alpha, \beta) = (\alpha(l), \infty) \qquad si \quad \Delta(J; u) = \varnothing.$$

2. *De plus, pour tour* $(i,j) \leqslant (\alpha', \beta')$, *on a*

 (*a*) $X(i,j) = \mathrm{Spec}\,[R(i,j)/J(i,j)]$ *où*

 $R(i,j) = R[t, s_1 t^{-a(1)}, \ldots, s_{\alpha(i)-1} t^{-a(\alpha(i)-1)}, s_{\alpha(i)} t^{-h(i,j)}, \ldots, s_q t^{-h(i,j)}]$,

 $J(i,j) = (f_k t^{-n(k)h(i,j)}, 1 \leqslant k \leqslant m)$,

 $s = (s_1, \ldots, s_q) = (u_1, \ldots, u_d, y_1, \ldots, y_r)$,

 (*b*) $E(i,j) = V(t + J(i,j)) \subset Z(i,j) = \mathrm{Spec}\, R(i,j)$.

 (*c*) *pour* $1 \leqslant r \leqslant l+1$,

 $$V_r(i,j) = V(J(i,j) + (s_a(i,j), a \in I_r))$$

 où

 $s(i,j){:} = (s_1(i,j), \ldots, s_q(i,j))$

 $= (s_1 t^{-a(1)}, \ldots, s_{\alpha(i)-1} t^{-a(\alpha(i)-1)}, s_{\alpha(i)} t^{-h(i,j)}, \ldots, s_q t^{-h(i,j)})$,

 (*d*) $x(i,j)$ *est le point fermé de* $Z(i,j)$ *de paramètres* $(t, s_a(i,j), 1 \leqslant a \leqslant q)$.

 (*e*) $D(i,j) = V(s_a(i,j), 1 \leqslant a \leqslant q)$.

 (*f*) $\mathrm{ord}_{x(i,j)}(f_k t^{-n(k)h(k,j)}) = n(k)$,

 $$\mathrm{in}_{x(i,j)}(J(i,j)) = \mathrm{in}_{x(i,j)}(f_k t^{-n(k)h(i,j)}; 1 \leqslant k \leqslant m).$$

3. *Pour* $(i,j) < (\alpha', \beta')$, *on a*

 (*a*) $$\mathrm{ord}_{\epsilon(i,j)}(f_k t^{-n(k)h(i,j)}) = n(k),$$
 $$\mathrm{in}_{\epsilon(i,j)}(J(i,j)) = \mathrm{in}_{\epsilon(i,j)}(f_k(i,j)),$$

4. *Pour* $(i,j) \leqslant (\alpha', \beta')$

(*a*) *si on a*

(1) $\quad$ *pour tout* $k, \quad 1 \leqslant k \leqslant m, \quad cl^{n(k)h(i,j)}_{L(i,j),u,y}(f_k) \in \mathbb{C}[Y_1, \ldots, Y_r],$

où

$$Y_a = cl^{h(i,j)}_{L(i,j),u,y} y_a, \qquad 1 \leqslant a \leqslant r,$$

alors

$$T(i,j) = V(t, y_1 t^{-h(i,j)}, \ldots, y_r t^{-h(i,j)}),$$
$$Y(i,j) = V(t, s_k(i,j); a(k) > h(i,j)).$$

(*b*) *Si on n'a pas* (1) *pour* (i,j), *alors* $T(i,j) = \varnothing$.

B.2.7.1. Remarquons que, pour $i = 1$, on a $h(i,j) = j$, on déduit de B.2.7 3.(*c*) que les éclatements

$$X(0,0) \leftarrow X(1,1) \leftarrow \cdots \leftarrow X(1, a(1))$$

sont centrés en les $\{x(1,j)\}$, $0 \leqslant j \leqslant a(1) - 1$.

B.2.7.2. Montrons que B.2.7 termine la preuve de B.2.2. Remarquons d'abord que pour $h(i,j) \geqslant \sup\{N\lambda(i): 1 \leqslant i \leqslant d\}$, c'est-à-dire, pour $(i,j) \geqslant (l, b(l))$, on a

$$L(i,j)(D,E) = N\Lambda(D) + h(i,j)|E|,$$

d'où

(1) $$v_{L(i,j),u,y}(y^E u^D) - n(k)h(i,j) = N\Lambda(D) + h(i,j)(|E| - n(k)).$$

Or, par définition de $a = \inf\{\Lambda(\Delta(f; u; y))\}$ dans le dévelopement de f_k, on a pour $\lambda_{D,E,k} \neq 0$ avec $\Lambda(D) \geqslant a(n(k) - |E|)$,

(2) $$v_{L(i,j),u,y}(y^E u^D) - n(k)h(i,j) \geqslant (Na - h(i,j))(n(k) - |E|)$$

avec égalité pour au moins un (D, E, k).

Donc, si $a < \sup\{\lambda(i)\}$, on a $Na < h(i,j)$, d'où en appliquant (2), pour au moins un $n(k)$, on a

$$v_{L(i,j),u,y}(f_k) - n(k)h(i,j) < 0,$$

d'où

$$a < \sup\{\lambda(i)\} \quad \text{implique} \quad ((l, b(l)) > (\alpha, \beta)).$$

Réciproquement, si $a \geqslant \sup\{\lambda(i)\}$, d'après (2),

$$v_{L(i,j),u,y}(f_k) - n(k)h(i,j) \geqslant 0 \quad \text{pour} \quad h(i,j) \leqslant Na,$$

donc en ce cas

$$l(A, v, \Lambda, N) \geqslant h(l, b(l)) = \sup\{N\lambda(i) : 1 \leqslant i \leqslant d\}$$

ce qui prouve B.2.2.1.

Si $a \geqslant \sup\{\lambda(i) : 1 \leqslant i \leqslant d\}$, alors on remarque que (2) implique

$$(\alpha, \beta) = \sup\{(i,j) : h(i,j) < Na\},$$

comme $h(\alpha, \beta) = l(A, v, \Lambda, N)$, B.2.2. 1.(*a*) est clair et il implique B.2.2. 2.(*b*). Quant à B.2.2. 2.(*c*), c'est une conséquence de B.2.7. 2.(*a*). On a déjà vu (B.2.3) que B.2.2. 2.(*d*) est une conséquence des autres assertions de B.2.2.

B.2.7.3. Remarquons que la condition (1) de B.2.7 est vérifiée pour $(i,j) < (\alpha', \beta')$, en effet, d'après B.2.5 appliqué à $X = n(k)^{-1}(D, E)$, pour $(i,j) < (\alpha', \beta')$, $v_{L(i,j),u,y}(y^E u^D) > n(k)h(i,j)$ pour tout (D, E) tel qu'un $\lambda_{D,E,k} \neq 0$.

B.2.8. Prouvons B.2.7.

Nous allons d'abord prouver 2,3 et 4 par récurrence sur (i,j), ce qui entraînera l'existence de $H(i,j)$ pour $(i,j) \leqslant (\alpha', \beta')$. On prouvera ensuite 1.

B.2.9. Supposons 2, 3 et 4 vérifiés à l'étage (i,j) et prouvons 2, 3 et 4 à l'étage $((i,j)_+)$ si $((i,j)_+) < (\alpha', \beta')$, et 2 et 4 si $((i,j)_+) = (\alpha', \beta')$. Remarquons qu'on a

$$(1) \quad f_k(i,j) = \sum_{|c| = n(k)} \lambda_{c,k}\, y(i,j)^c + \sum_{|E| < n(k)} \lambda_{D,E,k}\, y(i,j)^E u(i,j)^D t^{L(i,j)(D,E) - n(k)h(i,j)} \bmod (y(i,j))^{1+n(k)},$$

$1 \leqslant k \leqslant m$, $(1, 1) \leqslant (i,j)$, $\lambda_{c,k}$ et $\lambda_{D,E,k}$ étant définis en B.2.6.1 (1).

Par B.2.7. 2.(*f*) (i,j), $x(i,j)$ est un point proche de $x(0, 0)$, de plus, on voit que $\mathrm{in}_{x(i,j)}(f_k(i,j)) \in \mathbb{C}[Y_1, \ldots, Y_r]$, donc $W((i,j)_-)$, le transformé strict de

$$W(0, 0) = V(y_1, \ldots, y_r) \subset Z(0, 0)$$

épouse $X(i,j)$ en $x(i,j)$ [6, (3.9)], donc $x(i,j)$ est très proche de $x(0, 0)$ et $(f(i,j), u(i,j), y(i,j))$ est une donnée distinguée de $J(i,j)$. Si $(i,j) < (\alpha', \beta')$, alors par $3(i,j)$, $Y(i,j)$ est permis pour $X(i,j)$ en $x(i,j)$ (rappelons qu'on a (1) pour (i,j) d'après B.2.7.3).

Effectuons l'éclatement $\tilde{\pi}(i,j)$: $\tilde{X}((i,j)_+) \to X(i,j)$ le long de $Y(i,j)$. Les règles usuelles de calcul des transformées stricts nous donnent 2.(*a*)(*b*)(*c*)(*d*)(*e*) pour $((i,j)_+)$. On obtient 2.(*f*) en regardant l'expression de $f_k((i,j)_+)$.

Maintenant, remarquons que, si on a (1) en $((i,j)_+)$ alors, pour tout D, E, k avec $\lambda_{D,E,k} \neq 0$,

$$L((i,j)_+)(B,E) - h((i,j)_+)n(k) > 0$$
$$L(i,j)(B,E) + b_{k'} + \cdots + b_d + |E| - h((i,j)_+)n(k) > 0$$

où $k' = \inf\{x : a(x) \geqslant h((i,j)_+)\}$ or $h((i,j)_+) = 1 + h(i,j)$, d'où

$$L(i,j)(B,E) - h(i,j)n(k) + b_{k'} + \cdots + b_d + |E| > n(k)$$

on remarque que

$$\operatorname{ord}_{\epsilon(i,j)}(u^B y^E) = L(i,j)(B,E). \tag{2}$$

$$L(i,j)(B,E) - h(i,j)n(k) + b_{k'} + \cdots + b_d + |E|$$
$$= L((i,j)_+)(B,E) - h((i,j)_+)n(k) + n(k).$$

donc

$$\operatorname{in}_{\epsilon(i,j)}(f_k(i,j)) \in K[Y_1, \ldots, Y_r],$$
$$Y_a = \operatorname{in}_{\epsilon(i,j)}(y_a(i,j)), \qquad 1 \leqslant a \leqslant r,$$

où K est le corps résiduel de $\epsilon(i,j)$, donc

$$T_{X(i,j),\epsilon(i,j)} \supset V[Y_1, \ldots, Y_r]$$

mais par semi-continuité de $\dim T_{X,\epsilon} - \dim O_{X(i,j),\epsilon}$ le long de la strate de Samuel [5, I.5.3.3.(9)] on a égalité. Alors, $\eta((i,j)_+)$ étant le point au-dessus de $\epsilon(i,j)$ qui correspond au point générique de $\operatorname{Proj}_{\epsilon(i,j)}(T_{X(i,j)\epsilon(i,j)})$, 4.(*a*) (i,j) est clair.

Prouvons 4.(*b*) (i,j). Si on n'a pas B.2.7 (1) alors par (2) on a

$$\operatorname{in}_{\epsilon(i,j)}(f_k(i,j)) \in K[T, Y_1, \ldots, Y_r]$$
$$\notin K[Y_1, \ldots, Y_r]$$

où $T = \operatorname{in}_{\epsilon(i,j)}(t)$.

Alors $W(i,j)$ n'épouse pas $X(i,j)$ en $\epsilon(i,j)$ [6, (3.9)] et

$$T_{X(i,j),\epsilon(i,j)} = V(T, Y_1, \ldots, Y_r)$$

et le point de $\tilde{X}((i,j)_+)$ qui correspond au point générique de

$$\operatorname{Proj}_{\epsilon(i,j)}(T_{X(i,j),\epsilon(i,j)})$$

est sur le transformé strict de $E(i,j)$, donc pas dans $X((i,j)_+)$, ce qui prouve 4.(*b*) $((i,j)_+)$. Maintenant, prouvons 3 à l'étage $((i,j)_+)$, dans le cas où $((i,j)_+) < (\alpha', \beta')$. Par (2), pour tout (D, E, k) tel que $\lambda_{D,E,k} \neq 0$,

$$\operatorname{ord}_{\epsilon((i,j)_+)} u^D y^E t^{L((i,j)_+)(D,E) - n(k)h((i,j)_+)} = L(((i,j)_+)_+)(D,E) - h(((i,j)_+)_+)n(k) + n(k).$$

Or $(((i,j)_+)_+) \leqslant (\alpha', \beta')$, donc, par définition de (α', β') (*cf.* B.2.6), et par B.2.5., on a $\operatorname{ord}_{\epsilon((i,j)_+)}(f_k(i,j)_+)) = n(k)$.

B.2.10. Prouvons 1. Il n'y a plus qu'à montrer que l'algorithme s'arrête en (α', β').

Si on n'a pas B.2.7 (1) en (α', β') alors par B.2.7. 4.(*b*) déjà prouvé,

$$T(\alpha', \beta') = Y(\alpha', \beta') = \varnothing \quad \text{et} \quad OC(\alpha', \beta') = F.$$

Si on a B.2.7 (1) en (α', β'), il existe k, $1 \leqslant k \leqslant m$ tel que

$$\operatorname{ord}_{\epsilon(\alpha', \beta')}(f_k(\alpha', \beta')) < n(k).$$

Alors par [9, Ch. III, Sect. 4, Lemma 14], $\epsilon(\alpha', \beta')$ n'est pas dans la strate de Samuel de $x(\alpha', \beta')$, donc n'est pas permis et $OC(\alpha', \beta') = F$.

B.3. Comment se ramener à un algorithme fini

B.3.1. Soit (A, v, Λ) vérifiant la condition $(*)$, soit $a = \inf \Lambda(\Delta(J; u))$. Le théorème B.2.2 nous permet de calculer a par un passage à la limite car $a = \lim (L(A, v, \Lambda, N)/N)$.

Cepedant par B.2.2., pour les entiers N tels que $Na \in \mathbb{N}$, on a

$$a = l(A, v, \Lambda, N)/N.$$

Grâce à la proposition suivante, on peut aussi déterminer a en testant seulement un nombre fini de valeurs de N.

Proposition B.3.2. *Soit N tel que $N\lambda(i)$ soit entier pour $1 \leqslant i \leqslant d$. Soit*

$$(\alpha, \beta) = \sup \mathcal{E}(A, v, \Lambda, N)$$

(l'indice qui correspond au sommet de l'arbre). Les conditions suivantes sont équivalentes.

(i) $a = l(A, v, \Lambda, N)/N$.

(ii) *Il n'y a pas d'isomorphisme entre $E(\alpha, \beta)$ et $C_x(X)$ qui envoie $x(\alpha, \beta)$ sur le sommet de $C_x(X)$.*

Lemme B.3.2.1.

(i) *Pour tout couple* $(i,j) \in \mathcal{E}(A, v, \Lambda, N)$, $(i,j) \neq (0,0)$ *avec* $h(i,j) < Na$, *il y a un isomorphisme entre* $E(i,j)$ *et* $C_x(X)$, *l'image de* $x(i,j)$ *étant le sommet de* $C_x(X)$.

(ii) *Si, pour* $(\alpha, \beta) = \sup \mathcal{E}(A, v, \Lambda, N)$ *on a* $h(\alpha, \beta) = Na$, *alors il n'y a pas d'isomorphisme entre* $E(\alpha, \beta)$ *et* $C_x(X)$ *tel que l'image de* $x(\alpha, \beta)$ *est le sommet de* $C_x(X)$.

B.3.2.2. Montrons que ce lemme implique B.4.2. L'implication (i) $\Rightarrow$ (ii) de la proposition est une conséquence de B.4.2.1. (ii). Par B.2.2., on a $l(A, v, \Lambda, N) = \lfloor Na \rfloor$, ce qui s'écrit aussi $h(\alpha, \beta) = \lfloor Na \rfloor$. Donc, si B.3.2. (ii) est vrai, par B.3.2.1 (i), c'est que $h(\alpha, \beta) \geqslant Na$ et donc $h(\alpha, \beta) = Na$, ce qui est B.3.2.(i). Il n'y a plus qu'à prouver le lemme.

B.3.3. Prouvons (i). Choisissons (R, f, u, y) tel que (f, u, y) est une donnée distinguée pour $R/J = A$. Alors nous pouvons appliquer B.2.7. et donc pour $(i,j) \leqslant (\alpha, \beta)$

$$E(i,j) = \operatorname{Spec}\, [R(i,j)/(t) + (f_k t^{-n(k)h(i,j)}), 1 \leqslant k \leqslant m]. \tag{1}$$

Par B.2.9. (1) on a

$$\begin{aligned} f_k t^{-n(k)h(i,j)} = &\sum_{|c| = n(k)} \lambda_{c,k} y(i,j)^c \\ &+ \sum_{|E| < n(k)} \lambda_{D,E,k} y(i,j)^E u(i,j)^E t^{L(i,j)(D,E) - h(i,j)n(k)} \\ &\cdot \operatorname{mod}\, (y_1(i,j), \dots, y_r(i,j))^{1+n(k)} \end{aligned} \tag{2}$$

et $\lambda_{D,E,k} \neq 0$ implique $\Lambda(D) \geqslant a(n(k) - |E|)$ puisque $\Lambda(x) = a$ est l'équation d'un côté de $\Delta(J; u)$. D'après les variations de $L(i,j)(D,E) - h(i,j)n(k)$ décrites en B.2.9, pour $(i,j) < (\alpha, \beta)$, on a

$$L(i,j)(D,E) - h(i,j)n(k) > 0 \quad \text{pour} \quad \lambda_{D,E,k} \neq 0. \tag{3}$$

Pour $i = \alpha$, on remarque que, par définition de $L(i,j)$, on a

$$L(\alpha, j)(D,E) = N\Lambda(D) + h(\alpha, j)|E|.$$

Or $\Lambda(D) \geqslant a(n(k) - |E|)$ d'où

$$N\Lambda(D) + h(\alpha,j)|E| - h(\alpha,j)n(k) \geqslant (Na - h(\alpha,j))(n(k) - |E|) \geqslant 0 \tag{4}$$

avec positivité stricte si $h(\alpha, j) < Na$ ou si $\Lambda(D) > a(n(k) - |E|)$. De (3) et (4), on déduit que si $h(i,j) < Na$

(5) $$f_k t^{-n(k)h(i,j)} = \sum_{|c|=n(k)} \lambda_{c,k} y(i,j)^c \bmod (t).$$

Maintenant, remarquons que

(6) $C_x(X) = \operatorname{Spec}[(R/M)[\bar{u},\bar{y}]/\mathrm{in}_M(f)]$ où $\bar{u} = \mathrm{in}_M(u)$, $\bar{y} = \mathrm{in}_M(y)$.

Par (1) et (5) on a un isomorphisme

$$\begin{aligned}
&(R/M)[\bar{u},\bar{y}]/\mathrm{in}_M(f) \to \Gamma(E(i,j))\\
&\bar{u} = \mathrm{in}_M(u) \to u(i,j) \bmod (t, f(i,j))\\
&\bar{y} = \mathrm{in}_M(y) \to y(i,j) \bmod (t, f(i,j))\\
&\bar{\lambda} \in R/M \to \lambda \bmod (t, f(i,j)).
\end{aligned}$$

B.3.4. Prouvons (ii). D'après (4) et puisque $h(\alpha,\beta) = Na$, on a

(7) $$\begin{aligned} f_k t^{-n(k)k(\alpha,\beta)} = &\sum_{|c|=n(k)} \lambda_{c,k} y(\alpha,\beta)^c \\ &+ \sum_{(D,E)\in\mathcal{E}} \lambda_{D,E,k} y(\alpha,\beta)^E u(\alpha,\beta)^D \bmod (t) \end{aligned}$$

où

$$\begin{aligned} \mathcal{E} &= \{(D,E) | \Lambda(D) = a(n(k) - |E|)\} \\ &= \{(D,E) | L(\alpha,\beta)(D,E) = n(k)h(\alpha,\beta)\}. \end{aligned}$$

Posons $g_k = f_k t^{-n(k)h(\alpha,\beta)} \bmod (tR(\alpha,\beta))$, $g_k \in R(\alpha,\beta)/tR(\alpha,\beta)$, $1 \leqslant k \leqslant m$,

$$\begin{aligned} \tilde{u} &= u(\alpha,\beta) \bmod (tR(\alpha,\beta)) \\ \tilde{y} &= y(\alpha,\beta) \bmod (tR(\alpha,\beta)). \end{aligned}$$

Supposons que l'on ait un isomorphisme

$$\theta\colon (R/M)[\bar{u},\bar{y}]/\mathrm{in}_M(f) = \Gamma(C_x(X)) \to (R/M)[\tilde{u},\tilde{y}]/(\tilde{y}) = \Gamma(E(\alpha,\beta))$$

envoyant l'idéal $(\bar{u},\bar{y})$ sur l'idéal $(\tilde{u},\tilde{y})$.

Un tel isomorphisme respecte les directrices des cônes tangents aux points ω et $x(\alpha,\beta)$ correspondants aux idéaux $(\bar{u},\bar{y})$ et $(\tilde{u},\tilde{y})$. On a donc

$$\theta(\bar{y}_j) = l_j(\tilde{y}) + \varphi_j(\tilde{u},\tilde{y}), \qquad 1 \leqslant j \leqslant r$$

où l_j est une forme linéaire et $\varphi_j(\tilde{u},\tilde{y}) \in (\tilde{u},\tilde{y})^2$, on a donc

(8) $$g_k = \sum_{|c|=n(k)} \lambda_{c,k} [l(\tilde{y}) + \varphi(\tilde{u},\tilde{y})]^c,$$

or par B.2.2.c, g_k est la forme initiale de f_k pour la graduation définie par $(N\Lambda, u, y)$ (ou (Λ, u, y), le fait de changer Λ en $N\Lambda$ ne modifie pas le gradué) et (8) veut dire que le côté d'équation $\Lambda(x) = a$ de $\Delta(f; u; y)$ est soluble [8, (3,9)], c'est-à-dire que $\Delta(f; u; y)$ n'est pas minimal, ce qui contredit B.2.1 (2), donc θ n'existe pas. Q.E.D.

C. LE PREMIER CÔTÉ EST DETERMINÉ PAR LA SEULE DONNÉE DE $A = R/J$

Le premier côté de $\Delta(J; u)$ est le côté d'équation

$$x(1) + \cdots + x(d) = \delta(J; u)$$

c'est-à-dire que tous nos $\lambda(i)$ sont égaux à 1. Ce côté satisfait à (*) puisque $\delta(J; u) > 1$ (*cf.* [7, p. 3]).

L'ensemble $\mathcal{E}(\Lambda, N, \infty)$ de B.1.2. se réduit à

$$\mathcal{E}(\Lambda, N, \infty) = \{(0,0)\} \cup \{(1,j): 1 \leqslant j \leqslant N\} \cup \{(2,j): 1 \leqslant j\}.$$

Pour $i = 1$, $1 \leqslant j \leqslant N-1$, l'éclatement

$$\pi(i,j): \tilde{X}((i,j)_+) \to X(i,j)$$

est l'éclatement centré en $Y(i,j) = \{\overline{x(i,j)}\}$ (B.2.7. 1.).

Pour $i = 2$, $1 \leqslant j$, on a $Y(2,j) = T(2,j)$ (B.1.4).

Bref, pour définir $Y(i,j)$, il n'y a nul besoin de $V_.(i,j)$. Dans ce cas particulier, la connaissance de $V_.(i,j)$ est superflue pour construire l'algorithme de B.1.3, or les v_i ne servent qu'à définir $V_.(i,j)$, donc ils sont inutiles pour connaître le premier côté.

D. LES AUTRES CÔTÉS

D.1. Soit Λ une forme linéaire sur $\mathbb{R}^d$ à coefficients rationnels strictement positifs

$$\Lambda(x(1), \ldots, x(d)) = \lambda(1)x(1) + \cdots + \lambda(d)x(d).$$

(A, v) étant donné (B.1.1), supposons que l'algorithme de B nous dise que le côté du polyèdre $\Delta(J; u)$ associé à Λ pour R, J, u satisfaisant à (B.2) ne vérifie pas la condition (*) c'est-à-dire que

$$\Lambda[\Delta(J; u)] = [c(J, u, \Lambda), +\infty[$$

avec

$$c(J, u, \Lambda) < \mathrm{Sup}\,\{\lambda(i): 1 \leqslant i \leqslant d\}.$$

Choisissons alors $M \in \mathbb{N}^*$ tel que $M\lambda(i) \in \mathbb{N}^*$, $1 \leqslant i \leqslant d$. Posons

$$S_{M\Lambda} = \mathbb{Z}[T, X_1, \ldots, X_d]$$
$$S'_{M\Lambda} = \mathbb{Z}[T, X_1 T^{-n\lambda(1)}, \ldots, X_d T^{-M\Lambda(d)}].$$

Bien sûr $A[t]$ est une $S_{M\Lambda}$-algèbre par

$$T \to t, \quad X_i \to v_i, \quad 1 \leqslant i \leqslant d,$$

et $R[t]$ est aussi une $S_{M\Lambda}$-algèbre par

$$T \to t, \quad X_i \to u_i, \quad 1 \leqslant i \leqslant d.$$

On a, en omettant d'écrire les indices $M\Lambda$

$$A[t] \underset{S}{\otimes} S' = A[t, v_1 t^{-M\lambda(1)}, \ldots, v_d t^{-M\lambda(d)}]$$

noté $A_{M\Lambda}$,

$$R[t] \underset{S}{\otimes} S' = R[t, u_1 t^{-M\lambda(1)}, \ldots, u_d t^{-M\lambda(d)}]$$

noté $R_{M\Lambda}$.

On a la suite exacte

$$J[t] \underset{S}{\otimes} S' \to R[t] \underset{S}{\otimes} S' \to A[t] \underset{S}{\otimes} S' \to 0.$$

Ce qui implique que le noyau du morphisme surjectif $R_{M\Lambda} \to A_{M\Lambda}$ est $JR[t, u_1 t^{-M\lambda(1)}, \ldots, u_d t^{-M\lambda(d)}]$ noté $J_{M\lambda}$.

Soit (f, u, y) tel que f est une base standard de J totalement préparée relativement à (u, y) [8, Section 4]. Ce qui implique que $\Delta(f; u; y) = \Delta(J; u)$. (Par exemple on peut prendre (f, u, y) donnée distinguée).

En reprenant les notations de B, on a

$$\begin{aligned} f_k &= \sum \lambda_{c,k} y^c + \sum \lambda_{D,E,k} y^E u^D \operatorname{mod} (y)^{1+n(k)} \\ &= \sum \lambda_{c,k} y^c + \sum \lambda_{D,E,k} y^E u'^D t^{M\Lambda(D)} \operatorname{mod} (y)^{1+n(k)} \end{aligned}$$

où $u'_i = u_i t^{-M\lambda(i)}$, $1 \leqslant i \leqslant d$.

Pour tout triplet (D, E, k), définissons $w \in \mathbb{R}^d_+$ par $D = (n(k) - |E|)w$. Alors, on a dans $\mathbb{R}^{d+1}_+$, $(D, M_\Lambda(D)) = (n(k) - |E|)(w, M\Lambda(w))$.

On en déduit que, pour tout sommet s de $\Delta(f; u', t; y)$, il existe un sommet w de $\Delta(J; u)$ tel que $s = (w, M\Lambda(w))$.

D.2. Montrons que $(f; u', t; y)$ est une donnée distinguée pour $R_{M\Lambda}$, $J_{M\Lambda}$. En fait, le seul point difficile est de montrer que les formes initiales de f relativement à $\mathfrak{M}'$ engendrent $gr_{\mathfrak{M}'}(J_{M\Lambda})$, où $\mathfrak{M}'$ est le maximal de $R_{M\Lambda}$.

Pour tout $(D, E, c) \in \mathbb{R}_+^r \times \mathbb{R}_+^r \times \mathbb{R}_+$

$$y^E u^D t^c = y^E u'^D t^{c + M\Lambda(D)},$$

donc $\text{ord}_{\mathfrak{M}'}(y^E u^D t^c) = |E| + |D| + c + M_\Lambda(D)$. Soit L la forme linéaire sur $\mathbb{R}^{1+d}$ donnée par $L(D, c) = c + |D| + M\Lambda(D)$, alors on a $v_{L,y,u,t}(f_i) = n(i)$, $1 \leqslant i \leqslant m$ et par [8, (2-21d)],

$$\text{in}_{L,y,u,t}(J) = \text{in}_{L,y,u,t}(f) \subset gr_{L,y,u,t}(R). \tag{1}$$

Soit $g \in J_M$, pour $n \in \mathbb{N}$ assez grand, $t^n g \in JR[t]$, donc par (1),

$$t^n g = \sum_{i=1}^{m} a(i) f_i,$$

$$v_L(a(i)) + v_L(f_i) = v_L(t^n g),$$
$$a(i) \in R[t].$$

On en déduit que

$$T^n_{\text{in}_{\mathfrak{M}'}}(g) = \text{in}_{\mathfrak{M}'}(t^n g) = \sum_{i=1}^{m} \text{in}_{\mathfrak{M}'}(a(i))\, \text{in}_{\mathfrak{M}'}(f_i).$$

On en déduit le résultat par le fait que R_M est une algèbre de polynômes en t.

D.3. On dèduit de D.2 que

$$\Delta(f; u', t; y) = \Delta(J_{M\Lambda};, u').$$

Soit Λ' la forme linéaire sur $\mathbb{R}^{1+d}$ donnée par

$$\Lambda'(x(0), x(1), \ldots, x(d)) = (M+1)^{-1}[x(0) + \Lambda(x(1), \ldots, x(d))].$$

On a

$$\Lambda'[\Delta(J_{M\Lambda}; u')] = \Lambda'[\Delta(f; u', t; y)] = [c(J, u, \Lambda), +\infty[.$$

Si on prend M suffisamment grand, c'est-à-dire tel que

$$(M+1)^{-1}\lambda(I) \leqslant c(J, u, \lambda), \qquad 1 \leqslant i \leqslant d,$$

le côté de $\Delta(J_{M\Lambda}; u', t)$ défini par Λ' vérifie $(*)$. D'où

$$\inf \Lambda'[\Delta(J_{M\Lambda}; u'; t)] = (M+1)c(J, u, \Lambda) = \lim_{N \to \infty} N^{-1} l(A_{M\Lambda}, (v', t), \Lambda', N)$$

ce qui nous donne une expression de $c(J, u, \Lambda)$ déterminée par (A, v) uniquement.

Maintenant, on remarque que

$$gr_{\Lambda, u, y}(A) = gr_{\Lambda'}(A_{M\Lambda, v', t})/(T - 1),$$

et pour tout $g \in A \subset A_{M\Lambda}$, on a

$$\mathrm{ord}_{\Lambda, v}(g) = \mathrm{ord}_{\Lambda', v'}(g).$$

Ainsi (A, v) nous permet de construire $A_{M'\Lambda}$ grâce auquel on peut reconstituer la filtration associée à (R, J, Λ, u, y).

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Quasiconformal Mappings Onto John Domains

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1. Introduction

In this paper we study quasiconformal homeomorphisms of the unit ball $\mathbb{B} = \mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$ onto John domains. We recall that John domains were introduced by F. John in his study of rigidity of local quasi-isometries [J]; the term John domain was coined by O. Martio and J. Sarvas seventeen years later [MS]. From the various equivalent characterizations we shall adapt the following definition based on diameter carrots, *cf.* [V4], [V5], [NV].

Let E be an arc in $\mathbb{R}^n$ with end points x_0 and x_1, and let $E[x_1, x]$ denote the subarc of E between $x_1, x \in E$. For $b \geqslant 1$ the open set

$$\operatorname{car}(E, b) = \bigcup\{B(x, b^{-1} \operatorname{diam} E[x_1, x]) : x \in E\}$$

is called a *b-carrot* (or *b-cone* [GHM]) with vertex x_1 joining x_1 to x_0. Here $B(x, r)$ denotes the open n-ball centered at x with radius r. A domain D in $\mathbb{R}^n$ is said to be a *b-John domain* with center x_0 if there is $x_0 \in D$ such that each $x_1 \in D$ can be joined to x_0 by a b-carrot in D. It follows that if $D \neq \mathbb{R}^n$ is b-John, then it is bounded; indeed, $D \subset B(x_0, b \operatorname{dist}(x_0, \partial D))$.

Among simply connected planar domains John domains can be recognized from a number of different geometric properties as well as from the properties of the Riemann mapping [P2], [NV], [GHM]. It is our purpose in this paper to show that certain analogues of those results can be found also in higher dimensions. In fact, if D is a bounded domain in $\mathbb{R}^n$ and quasiconformally equivalent to the unit ball, then our main theorem provides nine equivalent conditions for D to be John. Two of those conditions were previously known [V5], [NV]. It is interesting to note that in our main theorem, Theorem 3.1, the requirement «D is quasiconformally equivalent to the unit ball» cannot be replaced *e.g.* by «D is homeomorphic to the unit ball» or «D is a Jordan domain». Thus, among all John domains those which can be quasiconformally mapped to a ball lend themselves to more clear pictures.

The main theorem is stated in Section 3 after some preliminary discussion. Our proofs are mainly based on the modulus method but, at least implicitly, also the analytic aspects of the higher dimensional quasiconformal theory are present. We also feel that J. Väisälä's theory of quasisymmetric mappings has come to be an indispensable guide to the geometry of John domains.

The proof of the main theorem leads us to consider more general subinvariance properties of certain domains under quasiconformal mappings. These phenomena were previously studied in [FHM] and [V5]. In Section 6 we present a quite general theorem which describes the internal distortion of quasiconformal mappings and extends a recent result of J. Väisälä [V5, Theorem 2.20]. A few corollaries will be discussed in Section 7; we demonstrate, for instance, that broad domains are subinvariant under quasiconformal mappings.

There is a beautiful theorem due to F. W. Gehring and W. K. Hayman [GH] which states that in simply connected planar domains the hyperbolic geodesic essentially has the least length (or diameter, see [P1, pp. 136]) among all paths with same endpoints. In proving our main theorem we shall require a similar result which can be viewed as a quasiconformal analogue of the Gehring-Hayman Theorem and which as such may have some independent interest. This result, Theorem 4.1, is stated and proved in Section 4. Having seen the first draft of this paper, R. Näkki informed the author that Theorem 4.1 also follows from [HN, Theorem 2] after a simple limiting procedure.

I wish to thank J. Väisälä for generously showing me his unpublished work and P. Koskela whose question about the equivalence of I and VIII in Theorem 3.1 partly led me to investigate the problems in this paper. I also thank the referee for a meticulous reading of the paper and for many useful comments. Indeed, the proofs for the implications III $\Rightarrow$ IV and III $\Rightarrow$ VI in Theorem 3.1 are due to the referee; this route of reasoning substantially shortened my original arguments.

2. Some Definitions and Lemmas

Before stating and proving our main theorem we shall record in this section some definitions and results needed later on.

2.1. Notation

Our basic notation is fairly standard and generally as in [V1]. For example, D and D' will denote proper subdomains of $\mathbb{R}^n$, $n \geqslant 2$, and $f: D \to D'$ includes the assumption that f is a homeomorphism onto D'. Open balls and spheres in a metric space (X, e) are denoted by $B_e(x, r)$ and $S_e(x, r)$, respectively; whenever X is a subset of $\mathbb{R}^n$ with the euclidean metric in it, the subscript e is omitted. We abbreviate $B(r) = B(0, r)$, $S(r) = S(0, r)$ and $\mathbb{S} = \partial\mathbb{B}$, where $\mathbb{B} = B(0, 1)$ is the unit ball. By a boundary cap $I \subset \mathbb{S}$ we mean a set of the form $\bar{B}(x, r) \cap \mathbb{S}$ for some $x \in \bar{\mathbb{B}}$. The (euclidean) diameter of a set A is $d(A)$ and the (euclidean) distance between sets A and B is $d(A, B)$. For brevity, $d(\{x\}, A) = d(x, A)$. If E is an arc in $\mathbb{R}^n$ and $x, y \in E$, then $E[x, y]$ will denote the closed subarc of E between x and y. The closed line segment between points $x, y \in \mathbb{R}^n$ is denoted by $[x, y]$.

2.2. John domains and cigars

In addition to the definition given in the introduction we will need the following cigar property of John domains. Recall that if E is an arc in $\mathbb{R}^n$ with endpoints x_1 and x_2, then for $b \geqslant 1$ the open set

$$\operatorname{cig}(E, b) = \bigcup\left\{B\left(x, b^{-1} \min_{i=1,2} d(E[x_i, x])\right) : x \in E\right\}$$

is a *b-cigar* (or *double cone* [GHM]) joining x_1 and x_2.

Then a bounded domain D is a b-John domain if and only if each pair of points in D can be joined by a b'-cigar in D; the constants b and b' depend only on each other [NV, Theorem 2.16].

This equivalence allows us to define John domains in the compactified space $\bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$: a domain D in $\bar{\mathbb{R}}^n$ is a b-John domain if each pair of points in $D \cap \mathbb{R}^n$ can be joined by a b-cigar in D, see [NV]. John domains in $\bar{\mathbb{R}}^n$ are briefly discussed in Section 6.

2.3. The internal metric

The internal metric δ_D in D is defined by

$$\delta_D(x, y) = \inf d(E)$$

where the infimum is taken over all arcs joining x and y in D. We shall often abbreviate $\delta_D = \delta$. The internal distance between two sets $A, B \subset D$ is written as $\delta_D(A, B)$, and the internal diameter of $A \subset D$ is $\delta_D(A)$.

2.4. Broad domains

Let $\varphi\colon (0, \infty) \to (0, \infty)$ be a decreasing homeomorphism. We say that D is *φ-broad* if for each $t > 0$ and each pair (C_0, C_1) of continua in D the condition $\delta_D(C_0, C_1) \leqslant t \min\{d(C_0), d(C_1)\}$ implies $M(\Delta(C_0, C_1; D)) \geqslant \varphi(t)$. Recall that $\Gamma = \Delta(C_0, C_1; D)$ is the family of all paths joining C_0 and C_1 in D, and $M(\Gamma)$ denotes the modulus of Γ.

Broad domains were introduced in [V5] and it was later proved in [NV] that a simply connected planar domain is broad if and only if it is John. Broad domains also provide some new insight to internal distortion properties of quasiconformal mappings, *cf.* [V5, Theorem 2.20] and Theorem 6.1 below.

The definition for broad domains in $\bar{\mathbb{R}}^n$ is similar.

2.5. Linearly locally connected sets

Suppose that A is a subset of D and $b \geqslant 1$. We say that A is *b-LLC_2* (with respect to δ_D) in D if for all $x \in A$ and $r > 0$ the points in $A \backslash \bar{B}(x, br)$ (in $A \backslash \bar{B}_{\delta_D}(x, br)$) can be joined in $D \backslash \bar{B}(x, r)$ (in $D \backslash \bar{B}_{\delta_D}(x, r)$). If $A = D$, we say D is *b-LLC_2* or *b-LLC_2* with respect to δ_D.

The expression LLC_2 is used because the condition above is but the second of the two requirements placed on linearly locally connected domains (then $A = D$), *cf.* [G], [GM1], [V3].

It turns out that if D is LLC_2, then it is LLC_2 with respect to δ_D; see Lemma 5.12 below. However, the converse need not be true in general (the examples that we have found are somewhat complicated and irrelevant in this connection).

Note that if A is *b-LLC_2* in D, then it need not be connected.

2.6. Quasisymmetric mappings

Let X_1 and X_2 be metric spaces with distance written as $|x - y|$ and let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f\colon X_1 \to X_2$ is *η-quasisymmetric* if $|a - x| \leqslant t|a - y|$ implies $|f(a) - f(x)| \leqslant \eta(t)|f(a) - f(y)|$ for all $a, x, y \in X_1$. If there is $H \geqslant 1$ such that $|a - x| \leqslant |a - y|$ implies $|f(a) - f(x)| \leqslant H|f(a) - f(y)|$, then f is said to be *weakly H-quasisymmetric.* Clearly an η-quasisymmetric mapping is weakly quasisymmetric (with $H = \eta(1)$) but the converse is not true in general. For background information about quasisymmetric mappings and their role in Geometric Function Theory see [TV], [V2], [V3], [V5].

A quasisymmetric embedding $f\colon D \to D'$ is always quasiconformal whilst the converse is true only for certain domains [V3].

We also have the following

Lemma 2.7. [V2, Theorem 2.4]. *Suppose that $f: D \to D'$ is K-quasiconformal, $x \in D$, and $0 < \lambda < 1$. Then $f|B(x, \lambda d(x, \partial D))$ is η-quasisymmetric, where η depends only on n, K, and λ.*

The next lemma follows from Lemma 2.7; see also [V1, Theorem 18.1].

Lemma 2.8. *Suppose that $f: D \to D'$ is K-quasiconformal, $x \in D$, and $0 < \lambda < 1$. Then there are positive constants λ_1 and λ_2, depending only on n, K and λ, such that*

$$B(f(x), \lambda_1 d(f(x), \partial D')) \subset f(B(x, \lambda_2 d(x, \partial D))) \subset B(f(x), \lambda d(f(x), \partial D')).$$

2.9. The function a_f

Let $f: D \to D'$ be K-quasiconformal. For $x \in D$ write

$$B_x = B\left(x, \frac{1}{2} d(x, \partial D)\right)$$

and set

$$a_f(x) = \exp\left(\frac{1}{nm(B_x)} \int_{B_x} \log J_f dm\right), \tag{2.10}$$

where J_f is the Jacobian of f and $m(B_x)$ stands for the n-measure of the ball B_x.

It was observed by Astala and Gehring that for certain distortion properties of quasiconformal mappings the function a_f plays a role analogous to that played by $|f'|$ when f is planar and conformal [AG1], [AG2]. In particular,

Lemma 2.11. [AG2, Theorem 1.8]. *There is a constant $c = c(n, K)$ such that*

$$\frac{1}{c} \frac{d(f(x), \partial D')}{d(x, \partial D)} \leqslant a_f(x) \leqslant c \frac{d(f(x), \partial D')}{d(x, \partial D)}$$

for all $x \in D$.

The careful reader notices that in [AG1], [AG2] the integral in (2.10) is defined with $B_x = B(x, d(x, \partial D))$. However, as seen from (2.12) below, these two definitions prove to be equivalent and for our purposes (2.10) is more convenient.

The next lemma derives from Lemma 2.11 and from the n-dimensional version of [AG1, Lemma 5.10]:

$$\left|\frac{1}{m(B_1)} \int_{B_1} \log J_f dm - \frac{1}{m(B_2)} \int_{B_2} \log J_f dm\right| \leqslant c(n, K)\left(\log \frac{m(B_1)}{m(B_2)} + 1\right); \tag{2.12}$$

here $f: D \to D'$ is K-quasiconformal and $B_2 \subset B_1$ are balls in D.

Lemma 2.13. *Let $f\colon \mathbb{B} \to D$ be K-quasiconformal and $x, y \in \mathbb{B}$. Then there are constants c_1, c_2 which depend only on n, K, and the hyperbolic distance between x and y such that*

$$\frac{1}{c_1} a_f(y) \leqslant a_f(x) \leqslant c_1 a_f(y) \tag{2.14}$$

and

$$\frac{1}{c_2} d(f(y), \partial D) \leqslant d(f(x), \partial D) \leqslant c_2 d(f(y), \partial D). \tag{2.15}$$

Recall that the hyperbolic metric in $\mathbb{B}$ is given by the metric density

$$ds = \frac{2|dx|}{1 - |x|^2};$$

the hyperbolic geodesic joining two points x and y in $\mathbb{B}$ is an arc of a circle orthogonal to $\mathbb{S}$.

The final result we record in this section is the following consequence of a theorem due to M. Zinsmeister; see [Z, Theorem 2].

For $x \in \mathbb{B}$ we define the cap $I(x) = \bar{B}(x, 3(1 - |x|)) \cap \mathbb{S}$.

Lemma 2.16. *Let $f\colon \mathbb{B} \to D$ be K-quasiconformal and let $x, y \in \mathbb{B}$ be such that $I(y) \subset I(x)$. Then there is a hyperbolic geodesic L from x to $I(y)$ such that*

$$d(f(L)) \leqslant c\, d(f(x), \partial D), \tag{2.17}$$

where the constant c depends only on n, K, and the hyperbolic distance between x and y (or, equivalently, on the ratio $d(I(y))/d(I(x))$).

3. Main Theorem

Let f be a K-quasiconformal mapping from $\mathbb{B}$ onto a bounded domain D. We assume further that f has a continuous extension to $\bar{\mathbb{B}}$, which is true if and only if D is finitely connected on the boundary [V1, pp. 58], in particular if D is John [NV, 2.17].

The following is the main result of the paper.

Theorem 3.1. *The following are equivalent*

I. *D is b-John with center $f(0)$;*

II. *D is φ-broad;*

III. $f\colon \mathbb{B} \to (D, \delta_D)$ *is* η*-quasisymmetric*;

IV. $d(f(I(x))) \leqslant b d(f(x), \partial D)$ *for all* $x \in \mathbb{B}$ *and* $I(x) = \bar{B}(x, 3(1 - |x|)) \cap \mathbb{S}$;

V. $d(f([x, w])) \leqslant b d(f(x), \partial D)$ *for all* $w \in \mathbb{S}$ *and* $x \in [0, w]$;

VI. $a_f(rw)(1 - r)^{1-\alpha} \leqslant b a_f(\rho w)(1 - \rho)^{1-\alpha}$ *for all* $w \in \mathbb{S}$ *and* $0 \leqslant \rho \leqslant r < 1$;

VII. $\dfrac{d(f(I))}{d(f(J))} \leqslant b\left(\dfrac{d(I)}{d(J)}\right)^{\alpha}$ *for all boundary caps* $I \subset J \subset \mathbb{S}$;

VIII. D *is* b*-*LLC_2;

IX. D *is* b*-*LLC_2 *with respect to* δ_D;

X. $f\colon \mathbb{B} \to (D, \delta_D)$ *is weakly* H*-quasisymmetric.*

The constants b, α, H *(not necessarily the same at each occurrence) and the functions* φ, η *depend only on each other and the data*

$$v = \left(n, K, \frac{d(D)}{d(f(0), \partial D)}\right).$$

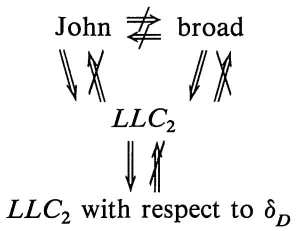
The equivalence of I, II, and III is known. The hard part is to show that I (or II) implies III [V5, Theorem 2.20] whereas it is considerably simpler to demonstrate that John domains and broadness are preserved under quasisymmetric mappings [NV, Theorems 3.6, 3.9]. We shall prove

$$\begin{array}{ccccccccccccc} \text{III} & \Rightarrow & \text{IV} & \Rightarrow & \text{V} & \Rightarrow & \text{VIII} & \Rightarrow & \text{IX} & \Rightarrow & \text{X} & \Rightarrow & \text{I} \\ \Downarrow & & \Uparrow & & & & & & & & & & \\ \text{VI} & \Rightarrow & \text{VII} & & & & & & & & & & \end{array}$$

We also provide a new proof for the implications I $\Rightarrow$ III and II $\Rightarrow$ III; see Remark 6.7 (*b*).

In the plane most of the implications are known for conformal mappings. In particular, the equivalence of IV, V, VI, and VII was proved by Ch. Pommerenke [P2].

Let it be remarked that the equivalence of I, II, VIII, and IX is not true for general domains when the picture is as follows



Proofs for the implications can be found below in Lemmas 6.2, 7.2, and 5.12. As for the counterexamples, it is clear that throwing in a countable set of points may destroy the carrot property of John domains whereas the modulus remains intact; on the other hand, by judiciously removing open intervals $\{I_i\}$ from $[0, 1]$, the John domain $D = (\mathbb{B}^2 \setminus [0, 1]) \cup \{I_i\colon\ i = 1, 2, \dots\}$ is not broad. Further, if $n \geqslant 3$, a Jordan domain with an outward directed wedge is LLC_2 but neither John nor, if the wedge is sharp enough, broad.

The reader is invited to compare Theorem 3.1 to results in [GM1], [V3] and how the concepts John, broad, LLC_2, and quasisymmetry in δ_D are related to their predecessors: uniform, QED, LLC, and quasisymmetry. The analogue is particularly patent in Theorem 6.1 below from which we derive the implications IX $\Rightarrow$ X, I $\Rightarrow$ III, and II $\Rightarrow$ III as a special case.

Astala and Gehring proved in [AG2] that if f is a bounded K-quasiconformal mapping in $\mathbb{B}$, then f is Hölder continuous in $\bar{\mathbb{B}}$ with the exponent $\alpha, 0 < \alpha \leqslant K^{1/(1-n)}$, if and only if $a_f(x) \leqslant b(1 - |x|)^{\alpha - 1}$. It has been proved by several authors [NP], [GM2], [MV] that quasiconformal mappings onto John domains are Hölder continuous. In light of the Astala-Gehring theorem, Theorem 3.1 VI above shows that slightly more is true.

4. A Distortion Theorem for Quasiconformal Mappings

In this section we establish the following theorem (see [HN, Theorem 2] for a similar result).

Theorem 4.1. *Let $f\colon \mathbb{B} \to D$ be K-quasiconformal and bounded. Let L be the line segment from* 0 *to a point $w \in \mathbb{S}$. If γ is any arc joining* 0 *to w in $\mathbb{B}$, then*

$$d(f(L)) \leqslant b d(f(\gamma))$$

where the constant b depends only on n and K.

PROOF. We denote the images by primes: $f(x) = x'$, $x \in \mathbb{B}$, $f(A) = A'$, $A \subset \mathbb{B}$. By normalizing we may assume that $d(\gamma') = 1$. Fix a point $z' \in \gamma'$ so that $\gamma' \subset \bar{B}(z', 1)$. For $k = 0, 1, 2, \dots$ let $L_k = [(1 - 2^{-k})w, (1 - 2^{-k-1})w]$. We shall first show that for each k there is a point $x_k \in L_k$ such that $f(x_k) \subset \bar{B}(z', c)$ for some $c = c(n, K) < \infty$. Indeed, if $L'_k \subset \mathbb{R}^n \setminus \bar{B}(z', c)$, then

$$M(\Delta(L'_k, \gamma'; D)) \leqslant \omega_{n-1} (\log c)^{1-n}$$

while the modulus estimate [V1, 10.12] implies

$$M(\Delta(L_k, \gamma; \mathbb{R}^n)) \geqslant c(n).$$

Since

$$\frac{1}{2}M(\Delta(L_k, \gamma; \mathbb{R}^n)) \leqslant M(\Delta(L_k, \gamma; \mathbb{B})) \leqslant KM(\Delta(L'_k, \gamma'; D)),$$

the desired upper bound for c can be found.

Next, fix $k \geqslant 1$ and $y \in L_k$. Let $x_{k-1} \in L_{k-1}$ and $x_{k+1} \in L_{k+1}$ be points whose images lie in $\bar{B}(z', c)$. Since

$$\frac{1 - |x_{k-1}|}{1 - |x_{k+1}|} \leqslant \frac{2^{-k+1}}{2^{-k-2}} = 8,$$

the points x_{k-1} and x_{k+1} lie in a hyperbolic ball with fixed radius; by Lemma 2.7 f is $\eta = \eta(n, K)$-quasisymmetric in that ball. Since

$$|y - x_{k+1}| \leqslant |x_{k-1} - x_{k+1}|,$$

then

$$|y' - x'_{k+1}| \leqslant \eta(1)|x'_{k-1} - x'_{k+1}| \leqslant 2\eta(1)c = c',$$

and hence

$$|y' - z'| \leqslant |y' - x'_{k+1}| + |x'_{k+1} - z'| \leqslant c' + c = c(n, K).$$

A similar reasoning shows that if $y \in L_0$, then $y' \in \bar{B}(z', c(n, K))$ as well. Consequently, the diameter of $f(L)$ is bounded by a number which depends only on n and K as required.

Remark 4.2. The above proof shows that the conclusion of Theorem 4.1 is retained if L is the hyperbolic geodesic joining two points $w_1, w_2 \in \mathbb{S}$ and γ is any arc joining those points in $\mathbb{B}$.

5. Proof of Theorem 3.1

Throughout the proof we let c, $c_1, \ldots$ denote positive constants, not necessarily the same at each occurrence, which depend only on the numbers b, α, H, the functions φ, η, and the data

$$v = \left(n, K, \frac{d(D)}{d(f(0), \partial D)}\right).$$

As usual, $c(a, \ldots)$ denotes a constant which depends only on $a, \ldots$.

III ⇒ IV. Fix $x \in \mathbb{B}$ and write $I = I(x)$. We denote images by primes, *i.e.* for $z \in \bar{\mathbb{B}}$, $z' = f(z)$. Choose $w_0 \in \mathbb{S}$ such that $|x' - w_0'| = d(x', \partial D)$. Then for any point w' on the half open line segment $[x', w_0')$ we have $\delta_D(x', w') = |x' - w'|$, and hence by quasisymmetry

$$\left|\frac{x' - z'}{x' - w'}\right| \leqslant \frac{\delta_D(x', z')}{\delta_D(x', w')} \leqslant \eta\left(\left|\frac{x - z}{x - w}\right|\right) \tag{5.1}$$

for $z \in \mathbb{B}$. In particular, if $w_j' \in [x', w_0')$ such that $w_j' \to w_0'$ and $z_j \in \mathbb{B}$ such that $z_j \to z_0 \in I$, then (5.1) implies

$$|x' - z_j'| \leqslant \eta\left(\frac{|x - z_j|}{|x - w_j|}\right)|x' - w_j'|.$$

By letting $j \to \infty$, we obtain

$$\begin{aligned} |x' - z_0'| &\leqslant \eta\left(\frac{|x - z_0|}{1 - |x|}\right) d(x', \partial D) \\ &\leqslant \eta(3)\, d(x', \partial D). \end{aligned}$$

Consequently,

$$d(f(I)) \leqslant 2\eta(3)\, d(x', \partial D)$$

as required.

IV ⇒ V. Fix $w \in \mathbb{S}$ and $x \in [0, w]$. Let L be a non euclidean segment from x to $I = I(x)$ such that $d(f(L)) \leqslant c d(f(x), \partial D)$, see (2.17). It then follows from Theorem 4.1 that

$$\begin{aligned} d(f([x, w])) &\leqslant c d(f(L) \cup f(I)) \\ &\leqslant c(d(f(L)) + d(f(I))) \\ &\leqslant c_1 d(f(x), \partial D) \end{aligned}$$

as required.

III ⇒ VI. Fix $w \in \mathbb{S}$ and $0 \leqslant \rho \leqslant r < 1$. By Lemma 2.11 it suffices to show

$$\frac{d(f(rw), \partial D)}{d(f(\rho w), \partial D)} \leqslant b\left(\frac{1 - r}{1 - \rho}\right)^{\alpha}, \tag{5.2}$$

where b and α depend only on η and the data v. It follows from [TV, 3.12] that η may be assumed to be of the form $\eta(t) = c \max(t^{\alpha}, t^{1/\alpha})$, where

$c = c(\eta) > 0$ and $\alpha = \alpha(\eta) \leqslant 1$. Thus for $r < s < 1$ we have

$$(5.3) \qquad \frac{\delta_D(f(rw), f(sw))}{\delta_D(f(\rho w), f(sw))} \leqslant \eta\left(\frac{s-r}{s-\rho}\right) \leqslant c\left(\frac{s-r}{s-\rho}\right)^{\alpha}.$$

Since

$$\delta_D(f(rw), f(sw)) \geqslant |f(rw) - f(sw)|$$

and since

$$\delta_D(f(\rho w), f(sw)) \leqslant d(f([\rho w, w])) \leqslant bd(f(\rho w), \partial D)$$

by V, (5.3) implies

$$(5.4) \qquad \frac{|f(rw) - f(sw)|}{bd(f(\rho w), \partial D)} \leqslant c\left(\frac{s-r}{s-\rho}\right)^{\alpha};$$

note that we have already established the implications III $\Rightarrow$ IV $\Rightarrow$ V. Finally, by letting $s \to 1$ in (5.4) establishes (5.2) and the proof of III $\Rightarrow$ VI is complete.

VI $\Rightarrow$ VII. Let $I \subset J \subset \mathbb{S}$ be two boundary caps. Choose distinct points $x_I, x_J \in \mathbb{B}\setminus\{0\}$ such that $|x_J| \leqslant |x_I|$ and that

$$(1 - |x_I|) \sim d(x_I, I) \sim d(I), \qquad (1 - |x_J|) \sim d(x_J, J) \sim d(J),$$

where $A \sim B$ means that the ratio A/B is bounded from above and below by an absolute constant. It then follows that the hyperbolic distance from x_I (or x_J) to a point $|x_I|x$, $w \in I$ (or $|x_J|w$, $w \in J$) is bounded by an absolute constant. In particular, Lemmas 2.7 and 2.11 yield

$$(5.5) \qquad \begin{aligned} |f(|x_I|w) - f(x_I)| &\leqslant c(n, K)d(f(x_I), \partial D) \\ &\leqslant c(n, K)(1 - |x_I|)a_f(x_I) \end{aligned}$$

for all $w \in I$. Likewise, by (2.14)

$$(5.6) \qquad \frac{1}{c}a_f(|x_I|w) \leqslant a_f(x_I) \leqslant ca_f(|x_I|w), \qquad c = c(n, K), \qquad w \in I.$$

Similar estimates hold for x_J. In the conformal case the conclusion follows from (5.5) and (5.6) by integrating $|f'|$ along a line, see [P2, pp. 81]. We need to make the following detour.

Fix $w \in I$ and let $x_1, x_2, \ldots$ be the points on $[|x_I|w, w]$ determined by

$$x_1 = |x_I|w, \qquad 1 - |x_j| = \frac{3}{4}(1 - |x_{j-1}|) \quad \text{for} \quad j \geqslant 2.$$

Write

$$B_j = B(x_j, |x_j - x_{j+1}|) = B\left(x_j, \frac{1}{4}(1 - |x_j|)\right).$$

It again follows from Lemmas 2.7 and 2.11 that

$$|f(x_j) - f(x_{j+1})| \leqslant cd(f(x_j), \partial D) \leqslant c(1 - |x_j|)a_f(x_j). \tag{5.7}$$

We obtain from (5.7) and VI that

$$\begin{aligned} |f(x_j) - f(x_{j+1})| &\leqslant c(1 - |x_j|)a_f(x_j) \\ &\leqslant c(1 - |x_j|)^{\alpha}(1 - |x_I|)^{1-\alpha}a_f(|x_I|w) \\ &= c\left(\frac{3}{4}\right)^{\alpha(j-1)}(1 - |x_I|)a_f(|x_I|w) \end{aligned}$$

whence

$$\begin{aligned} |f(|x_I|w) - f(w)| &\leqslant \sum_{j=1}^{\infty} |f(x_j) - f(x_{j+1})| \\ &\leqslant c(1 - |x_I|)a_f(|x_I|w). \end{aligned} \tag{5.8}$$

Thus, for $w_1, w_2 \in I$

$$\begin{aligned} |f(w_1) - f(w_2)| &\leqslant |f(|x_I|w_1) - f(w_1)| + |f(|x_I|w_1) - f(|x_I|w_2)| \\ &\quad + |f(|x_I|w_2) - f(w_2)| \\ &\leqslant c(1 - |x_I|)a_f(x_I); \end{aligned}$$

here (5.5), (5.6) and (5.8) were utilized. We conclude

$$d(f(I)) \leqslant c(1 - |x_I|)a_f(x_I). \tag{5.9}$$

Next, suppose that we have the lower bound

$$d(f(J)) \geqslant cd(f(|x_J|\,|x_I|^{-1}x_I), \partial D). \tag{5.10}$$

Then Lemma 2.11 implies $d(f(J)) \geqslant c(1 - |x_J|)a_f(x_0)$, where $x_0 = |x_J|\,|x_I|^{-1}x_I$. By combining this with (5.9) we arrive at

$$\frac{d(f(I))}{(1 - |x_I|)a_f(x_I)} \leqslant \frac{cd(f(J))}{(1 - |x_J|)a_f(x_0)}$$

which, in view of VI, is the desired inequality because $1 - |x_J| \sim d(J)$, $1 - |x_I| \sim d(I)$ and $a_f(x_0) \sim a_f(x_J)$.

It remains to establish (5.10) or generally

$$d(f(y), \partial D) \leqslant c(n, K) d(f(J)) \tag{5.11}$$

whenever $J \subset \mathbb{S}$ is a boundary cap and $y \in \mathbb{B}$ is such that $1 - |y| \sim d(y, J) \sim d(J)$. To see why this is true, consider the path family $\Gamma = \Delta(\bar{B}_y, J; \mathbb{B})$ where $B_y = B(y, (1 - |y|)/2)$. Then $0 < c(n) \leqslant M(\Gamma)$, and hence $0 < c(n, K) \leqslant M(f(\Gamma))$. It follows again from (2.15) that

$$\frac{1}{c} d(f(x), \partial D) \leqslant d(f(y), \partial D) \leqslant c d(f(x), \partial D), \qquad x \in B_y, \qquad c = c(n, K),$$

and hence, if

$$\frac{d(f(y), \partial D)}{d(f(J))} = R$$

is very large, we have

$$0 < c(n, K) \leqslant M(f(\Gamma)) \leqslant \omega_{n-1} (\log cR)^{1-n}.$$

This establishes (5.11) and, therefore, the implication VI $\Rightarrow$ VII.

VII $\Rightarrow$ IV. Let $I = I(x) \subset \mathbb{S}$ be a cap as in IV. We follow the idea presented in [P2, pp. 81-82].

Indeed, by VII we can choose a constant $c_1 = c_1(\alpha, b)$ such that if $J \subset I$ and $d(J) < c_1 d(I)$, then $d(f(J)) < d(f(I))/4$. Let c be the constant in Lemma 2.16 corresponding to the value c_1 above. That is, if $J \subset I$ and $d(J) \geqslant c_1 d(I)$, then there is an arc L from x to J satisfying (2.17).

Next consider the set

$$A = \{z \in \operatorname{int}_{\mathbb{S}} I \colon |f(z) - f(x)| > c d(f(x), \partial D)\}.$$

Then A is open in I and can be written as a countable union of boundary caps A_k with the property that $\bar{A}_k \cap (I \setminus A) \neq \varnothing$. Fix one A_k. Necessarily $d(A_k) < c_1 d(I)$, for otherwise one can find a curve L from x to A_k such that

$$d(f(L)) \leqslant c d(f(x), \partial D) < |f(z) - f(x)| \quad \text{for all} \quad z \in A_k,$$

which is absurd. Thus

$$d(f(A_k)) < \frac{1}{4} d(f(I))$$

by the choice of c_1. Let z_1, z_2 be two interior points of I. If $z_1 \in A_{k_1}$ for some k_1, choose a point $z_1' \in \bar{A}_{k_1} \cap (I \setminus A)$. If $z_1 \notin A$, set $z_1' = z_1$. Define z_2' similarly.

Then

$$\begin{aligned}|f(z_1)-f(z_2)| &\leqslant |f(z_1)-f(z_1')| + |f(z_1')-f(x)| \\ &\quad + |f(x)-f(z_2')| + |f(z_2')-f(z_2)| \\ &\leqslant d(f(A_{k_1})) + 2cd(f(x),\partial D) + d(f(A_{k_2})) \\ &\leqslant \frac{1}{2}d(f(I)) + 2cd(f(x),\partial D).\end{aligned}$$

Since $z_1, z_2 \in I$ were arbitrary, we have

$$d(f(I)) \leqslant bd(f(x), \partial D)$$

as required.

V ⇒ VIII. Fix $x \in D$ and $r > 0$. Suppose that two components, D_1 and D_2, of $D\backslash\bar{B}(x,r)$ meet $D\backslash\bar{B}(x,cr)$. We shall show that $c \leqslant 4b$.

Assume $f(0) = 0$. First observe that $M = d(D) \leqslant 2bd(0,\partial D)$ and therefore $B(0, 2c_1M) \subset D$ where $c_1 = 1/4b$. If $|x| < c_1M$, then $r > c_1M$, and therefore $D\backslash\bar{B}(x,cr) = \varnothing$ as soon as $c > 4b = 1/c_1$; a similar conclusion holds if $r \geqslant |x| \geqslant c_1M$. We may therefore assume that $0 \notin \bar{B}(x,r)$. Then 0 is not in one of the components D_1 and D_2, say $0 \notin D_1$. There is a boundary point $w \in \mathbb{S}$ such that $f(w) \in \partial D_1 \cap (\mathbb{R}^n\backslash\bar{B}(x,cr))$ and that the arc $\gamma = f([0,w])$ approaches $f(w)$ from D_1. In particular, since $0 \notin D_1$, there is $z \in \gamma \cap \bar{B}(x,r)$. Since $d(\gamma[f(w),z]) \geqslant (c-1)r$, V implies $(c-1)r \leqslant bd(z,\partial D) \leqslant 2br$. This shows that $c \leqslant 2b + 1 \leqslant 4b$ as required.

The implication VIII ⇒ IX is an immediate consequence of the following lemma.

Lemma 5.12. *Let A be an arcwise connected subset of a domain D. If A is b-LLC_2 in D, then A is b_1-LLC_2 with respect to δ_D in D with $b_1 = b_1(b)$.*

PROOF. Fix $x \in A$ and $r > 0$. Pick $z, y \in A\backslash\bar{B}_{\delta_D}(x, cr)$ and suppose they cannot be joined in $D\backslash\bar{B}_{\delta_D}(x,r)$. We shall show that $c \leqslant 2b + 1$.

Let α be an arc joining z and y in A. Choose $z_0 \in \alpha$ and $y_0 \in \alpha$ such that

$$\text{(i)} \qquad \min\{|x-z_0|, |x-y_0|\} > \frac{1}{2}(c-1)r$$

and

(ii) the subarcs $\alpha[z,z_0]$ and $\alpha[y,y_0]$ lie in $D\backslash\bar{B}_{\delta_D}(x,r)$.

This choice is possible since $\alpha \cap \bar{B}_{\delta_D}(x,r) \neq \varnothing$. Now (i) and the LLC_2-property imply that the points z_0 and y_0 can be joined in $D\backslash\bar{B}(x,(c-1)r/2b)$ by an arc

β. Then $\gamma = \alpha[z, z_0] \cup \beta \cup \alpha[y, y_0]$ joins z and y in D. Necessarily, because of (ii), β meets $\bar{B}_{\delta_D}(x, r)$. Therefore, for some $w \in \beta$, $(c-1)r/2b \leqslant |w - x| \leqslant \delta_D(w, x) \leqslant r$ which establishes the desired inequality $c \leqslant 2b + 1$.

We have two more implications to work out. The proof of IX $\Rightarrow$ X is somewhat long and, by the same token, we shall establish a more general result: Theorem 6.1 in Section 6. Assuming Theorem 6.1, we next show how to finish the proof of Theorem 3.1. First, the implication IX $\Rightarrow$ X follows directly from Theorem 6.1 by choosing $A = \mathbb{B}$. The final implication X $\Rightarrow$ I is essentially done in [NV, 3.5] but for convenience we incude a proof:

Fix $x_0' \in D$, $x_0 = f^{-1}(x_0')$. Let E be the line segment from 0 to x_0 and set $E' = f(E)$. We may assume that E is nondegenerate, for if $x_0 = 0$, there is nothing to prove. Then fix $x' \in E'$ and write $\rho = d(E'[x_0', x'])$. We need to show

$$B\left(x', \frac{\rho}{c}\right) \subset D, \qquad c = c(H). \tag{5.13}$$

For this, let $x = f^{-1}(x')$ and let $y \in S(x, |x - x_0|)$ be such that

$$|f(y) - x'| = \min_{|z-x| = |x-x_0|} |f(z) - x'|.$$

Then for all $z \in [x_0, x]$ we have $|x - z| \leqslant |x - y|$, and whence

$$\delta_D(f(z), x') \leqslant H\delta_D(f(y), x') = H|f(y) - x'| \leqslant Hd(x', \partial D).$$

It follows that

$$d(E'[x_0', x']) \leqslant 2Hd(x', \partial D),$$

which proves (5.13) with $c = 2H$.

Save Theorem 6.1, the proof of Theorem 3.1 is now complete.

6. Remarks on Internal Distortion and Subinvariance

The general subinvariance problem can be described as follows. Suppose that $\mathfrak{D}$ is a class of domains in $\mathbb{R}^n$ and $f: D \to D'$ is quasiconformal. When can one conclude that $A \in \mathfrak{D}$ implies $f(A) \in \mathfrak{D}$ for all subdomains A of D? It was shown in [FHM, pp. 120-121] that the conclusion holds if $\mathfrak{D}$ comprises all QED domains in $\mathbb{R}^n$ and $D' \in \mathfrak{D}$; moreover, in conjunction with [V3, Theorem 5.6] this implies that if D' is uniform, then so is $f(A)$ whenever A is a uniform subdomain of D. (The definitions for QED and uniform domains are recalled in Section 7 below.) Subsequently, J. Väisälä [V5, Theorem 2.20] proved that if

D' is broad, then every John subdomain of D is mapped onto a John subdomain of D'. This interesting phenomenon reflects certain internal distortion properties of quasiconformal mappings which, we believe, are worth deeper study.

In this section we first prove the following theorem which in the case of bounded domains generalizes [V5, Theorem 2.20]. Theorem 6.1 also establishes the missing link in the proof of Theorem 3.1. Some applications of Theorem 6.1 to subinvariance problems are discussed in Section 7.

Theorem 6.1. *Suppose that D, D' are bounded, that $f: D \to D'$ is K-quasiconformal, and that D is φ-broad. If $A \subset D$ is such that $f(A)$ is b-LLC_2 with respect to $\delta_{D'}$ in D', then $f|A: A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$ with H depending only on the data*

$$v = \left(n, K, b, \varphi, \frac{\delta_D(A)}{d(x_0, \partial D)}, \frac{\delta_{D'}(f(A))}{d(f(x_0), \partial D')} \right),$$

where x_0 is some fixed point in A.

Before we turn to the proof, let us indicate why Theorem 6.1 can be regarded as an extension of Väisälä's theorem [V5, 2.20]; the only drawback is that in Theorem 6.1 we require the domains to be bounded.

Theorem 2.20 in [V5] follows from Theorem 6.1 above as soon as the following two facts are established:

(i) if $f(A)$ has a b-carrot property in D', then $f(A)$ is b_1-LLC_2 with respect to $\delta_{D'}$ in D';
(ii) in the situation of [V5, 2.20] the weak quasisymmetry implies quasisymmetry.

The condition (ii) derives from [V5, Theorem 2.9] since A is pathwise connected and both A and $f(A)$ are HTB metric spaces by [V5, 2.14 and 2.18]. (The definition for HTB spaces is recalled below before Theorem 6.6.) The condition (i) is established in the following lemma.

Lemma 6.2. *Let D be a domain in $\mathbb{R}^n$ and let $A \subset D$ be such that each $x \in A$ can be joined to a fixed point $x_0 \in D$ by a b-carrot in D. Then A is both b_1-LLC_2 and b_1-LLC_2 with respect to δ_D in D, where $b_1 = b_1(b)$.*

PROOF. The proof is the same for both assertions. Fix $x \in A$ and $r > 0$. Suppose that there are two points $x_1, x_2 \in A \setminus \bar{B}(x, b_1 r)$ which cannot be joined in $D \setminus \bar{B}(x, r)$. We shall show that $b_1 \leqslant 2b + 1$.

Let E_1 and E_2 be the cores of two b-carrots joining x_1 and x_2, respectively, to x_0. Then $E = E_1 \cup E_2$ joins x_1 and x_2 in D. Necessarily E meets $\bar{B}(x, r)$. Pick

$z \in E \cap \bar{B}(x, r)$ and suppose $z \in E_1$. Since $d(E[x_1, z]) > (b_1 - 1)r$ and since E_1 is the core of a b-carrot, we have $B(z, b^{-1}(b_1 - 1)r) \subset D$. On the other hand,

$$d(z, \partial D) \leqslant |z - x| + d(x, \partial D) \leqslant 2r,$$

and hence $b^{-1}(b_1 - 1)r \leqslant 2r$ or $b_1 \leqslant 2b + 1$ as required.

PROOF OF THEOREM 6.1. It is no loss of generality to assume that $x_0 = 0 = f(0)$ and that $1 = d(0, \partial D) = d(0, \partial D')$. We shall denote images by primes: $f(x) = x'$, $x \in D$, $f(E) = E'$, $E \subset D$; and also for brevity $\delta = \delta_D$, $\delta' = \delta_{D'}$, $M = \delta(A)$, $M' = \delta'(A')$.

Thus, let a, x, y be three distinct points in A with $\delta(a, x) \leqslant \delta(a, y)$. (Note that the claim is vacuous if the cardinality of A is less than three.) We need to find an upper bound for

$$H = \frac{\delta'(a', x')}{\delta'(a', y')}.$$

For this we consider two cases.

Case 1.

$$\delta'(a', 0) > c_0 \delta'(a', y')$$

where $c_0 = c_0(v)$ is a constant to be determined later on. We shall show that with an appropriate choice of c_0 too large H generates a contradiction.

In Case 1 we separate to subcases.

Subcase 1*a*.

$$x \in B(0, c_1/2)$$

where

$$c_1 = c_1(n, K), \qquad c_2 = c_2(n, K) < \frac{1}{2}$$

are chosen to satisfy

$$\begin{cases} f^{-1}(B(0, 2c_2)) \subset B\left(0, \dfrac{1}{2}\right); \\ f(B(0, c_1)) \subset B\left(0, \dfrac{c_2}{2}\right) \quad \text{and} \quad f(S(0, c_1)) \cap S\left(0, \dfrac{c_2}{2}\right) \neq \varnothing, \end{cases}$$

see Lemma 2.8.

In proving the claim in the first subcase we again distinguish two possibilities:
(i) $a' \in B(0, c_2)$ or (ii) $a' \notin B(0, c_2)$. If (i) occurs, then

$$H = \frac{\delta'(a', x')}{\delta'(a', y')} = \frac{|a' - x'|}{\delta'(a', y')} \leqslant \left| \frac{a' - x'}{a' - y'} \right|,$$

and because f is $\eta = \eta(n, K)$-quasisymmetric in $B(0, 1/2)$, we obtain $H \leqslant \eta(1)$ provided that $y' \in B(0, 2c_2)$. On the other hand, if $y' \notin B(0, 2c_2)$, then $|a' - y'| \geqslant c_2$, and hence $H \leqslant M'/c_2$.

We may therefore suppose that $a' \notin B(0, c_2)$. Choose a point $z \in S(0, c_1)$ such that $z' \in S(0, c_2/2)$ and let α be the line segment from x to z. Then $\alpha' \subset B(0, c_2/2)$. Let next β' be an arc joining a' and y' in D' with $d(\beta') < 2\delta'(a', y')$, and let $\beta = f^{-1}(\beta')$ be its preimage in D. Then

$$\delta(\alpha, \beta) \leqslant \delta(a, x) \leqslant \delta(a, y) \leqslant d(\beta)$$

and

$$\delta(\alpha, \beta) \leqslant \frac{\delta(a, x) d(\alpha)}{d(\alpha)} \leqslant \frac{2M}{c_1} d(\alpha).$$

Because D is φ-broad, we thus obtain

$$M(\Delta(\alpha, \beta; D)) \geqslant c_3 = c_3(v) > 0,$$

and the quasiconformality of f yields

$$M(\Delta(\alpha', \beta'; D')) \geqslant c_4 = c_4(v) > 0. \tag{6.3}$$

Observe further, that if $\alpha' \cap \bar{B}_{\delta'}(a', c_0\delta'(a', y')) \neq \emptyset$, then for some $w' \in \alpha' \subset B(0, c_2/2)$

$$\frac{c_2}{2} \leqslant \delta'(a', w') \leqslant c_0\delta'(a', y')$$

whence

$$H = \frac{\delta'(a', x')}{\delta'(a', y')} \leqslant \frac{2c_0 M'}{c_2} = c(v) < \infty$$

and the proof is complete. We may therefore assume that

$$\alpha' \subset D' \backslash \bar{B}_{\delta'}(a', c_0\delta'(a', y')).$$

Let us leave the subcase $1a$ for a moment and consider

Subcase 1*b*.

$$x \notin B(0, c_1/2).$$

We still have $\delta'(a', 0) > c_0\delta'(a', y')$ and $\delta'(a', x') > c_0\delta'(a', y')$ for otherwise H is less than $c_0 = c_0(v)$, completing the proof. Because of the LLC_2-property, we can join x' to 0 by an arc α' in $D'\backslash \bar{B}_{\delta'}(a', c_5\delta'(a', y'))$ where $c_5 = c_0/b \to \infty$ as $c_0 \to \infty$. Let β' be an arc joining a' and y' as in the subcase 1*a*, and let again $\alpha = f^{-1}(\alpha')$, $\beta = f^{-1}(\beta')$. Then

$$\delta(\alpha, \beta) \leqslant \delta(a, x) \leqslant \delta(a, y) \leqslant d(\beta)$$

and

$$\delta(\alpha, \beta) \leqslant \frac{\delta(a, x)d(\alpha)}{d(\alpha)} \leqslant \frac{2M}{c_1} d(\alpha).$$

Consequently, as in the subcase 1a the broadness and quasiconformality imply the estimate (6.3) for the present continua α' and β' as well.

Thus in both subcases we have arrived at the situation where

(i) x' is joined to a point in D' by an arc α' which lies entirely outside the ball $\bar{B}_{\delta'}(a', c_6\delta'(a', y'))$, where the constant c_6 depends only on b and c_0, and $c_6 \to \infty$ as $c_0 \to \infty$,

(ii) a' is joined to y' by an arc β' which lies entirely inside the ball $B_{\delta'}(a', 2\delta'(a', y'))$, and

(iii) the modulus estimate (6.3) holds.

It is desirable that this leads to a contradiction if $c_0 = c_0(v)$ is large enough. But that is indeed the case, for if γ is a path joining α' and β' in D', then $d(\gamma) \geqslant (c_6 - 2)\delta'(a', y')$ which means that γ joins the spheres $S(a', 2\delta'(a', y'))$ and $S(a', c_7\delta'(a', y'))$, where $c_7 \to \infty$ as $c_0 \to \infty$. In conclusion,

$$M(\Delta(\alpha', \beta'; D')) \leqslant \omega_{n-1}\left(\log \frac{c_7}{2}\right)^{1-n}$$

contradicting (6.3) for too large c_0. We have thus shown that $H \leqslant c(v) < \infty$, and the proof is complete in Case 1.

Case 2.

$$\delta'(a', 0) \leqslant c_0\delta'(a', y')$$

where $c_0 = c_0(v)$ is the constant in Case 1.

We start by observing that because

$$\delta'(a', 0) \leqslant c_0\delta'(a', y') \leqslant \frac{c_0 M'}{H}, \tag{6.4}$$

we are allowed to assume that $\delta'(a', 0) = |a'|$. In the rest of the proof we let ϵ denote any function which depends only on the data v and satisfies $\epsilon(H) \to 0$ as $H \to \infty$. In particular, we have by (6.4)

$$|y'| \leqslant |a'| + |a' - y'| \leqslant \delta'(a', 0) + \delta'(a', y') \leqslant \epsilon(H).$$

Assuming that $\epsilon(H) < 1$, let α' be the line segment joining y' and a' in $B(0, \epsilon(H))$. Then

$$M(\Delta(\alpha', \partial D'; D')) \leqslant \omega_{n-1}\left(\log \frac{1}{\epsilon(H)}\right)^{1-n}$$

whence

$$M(\Delta(\alpha, \partial D; D)) \leqslant \epsilon(H),$$

where $\alpha = f^{-1}(\alpha')$. Since α joins a and y in D, it follows from the standard Teichmüller estimate [V1, Theorem 11.9] that

$$M(\Delta(\alpha, \partial D; D)) \geqslant \varkappa_n\left(\frac{d(a, \partial D)}{|a - y|}\right)$$

where $\varkappa_n$ is positive decreasing and $\varkappa_n(t) \to 0$ as $t \to \infty$. Therefore, by choosing H large enough, we infer that $f^{-1}(B(a', d(a', \partial D')))$ contains the ball $B(a, |a - y|)$ which in turn is contained in $B(a, d(a, \partial D)/2)$, see Lemma 2.8. In particular, $x \in B(a, |a - y|)$ in view of our initial assumption $\delta(a, x) \leqslant \delta(a, y) = |a - y|$. Hence $\delta'(a', x') = |a' - x'|$, and this finishes the proof because f is $\eta(n, K)$-quasisymmetric in $B(a, d(a, \partial D)/2)$.

The next theorem is a version of Theorem 6.1 for domains containing the point at infinity. We omit the proof of Theorem 6.5; it is similar to but simpler than that of Theorem 6.1.

Theorem 6.5. *Suppose that $f: D \to D'$ is K-quasiconformal and that $f(\infty) = \infty \in D \cap D'$. Suppose further that D is φ-broad, that $A \subset D$, $\infty \in A$, and $f(A)$ is a b-LLC_2 with respect to $\delta_{D'}$ in D'. Then $f|A: A \to f(A)$ is weakly H-quasisymmetric in the metrics δ_D and $\delta_{D'}$ with H depending only on n, K, φ, and b.*

It is not clear to us if one always could draw the more desirable conclusion «$f: A \to f(A)$ is quasisymmetric in the metrics δ_D and $\delta_{D'}$» in Theorem 6.1. The amenable HTB-criterion given in [V5, Theorem 2.9] is not automatically satisfied as there are domains D which are LLC_2 with respect to δ_D but which are not HTB.

What is more, one often wants to know when f is quasisymmetric in the euclidean metric in some subset A of D, which is indeed a much stronger conclusion than that in Theorem 6.1. The following theorem partially answers this question.

We say that $A \subset D$ is of *b-bounded turning* or b-BT in D if each pair of points $x, y \in A$ can be joined by an arc E in D such that $d(E) \leqslant b|x-y|$; if $A = D$, we say D is b-BT. A metric space (X, e) is said to be *k-homogeneously totally bounded* or k-HTB if $k\colon [1/2, \infty) \to \mathbb{N}$ is an increasing function and if, for each $\alpha \geqslant 1/2$, every closed ball $\bar{B}_e(x, r)$ can be covered by $k(\alpha)$ sets of diameter less than r/α; see [TV], [V5].

Theorem 6.6. *Suppose that $D, D' \subset \mathbb{R}^n$ are bounded domains and $f\colon D \to D'$ is K-quasiconformal. Suppose further that*

(i) *$A \subset D$ is pathwise connected, b_1-LLC$_2$ with respect to δ_D, and b_2*-BT *in D;*
(ii) *D' is φ-broad and b_3*-BT.

Then $f\colon A \to f(A)$ is η-quasisymmetric with η depending only on the data

$$v = \left(n, K, \varphi, b_1, b_2, b_3, \frac{d(A)}{d(x_0, \partial D)}, \frac{d(f(A))}{d(f(x_0), \partial D')}\right),$$

where x_0 is some fixed point in A.

PROOF. Theorem 6.1 implies that $g = f^{-1}|f(A)$ is weakly $H(v)$-quasisymmetric in the metrics $\delta_{D'}$ and δ_D. On the other hand, $f(A)$ as a subset of the broad domain D' is $k(n, \varphi)$-HTB in $\delta_{D'}$ by [V5, 2.18] and it is easy to see that the bounded turning condition in (i) implies that A is $k(n, b_2)$-HTB in δ_D. We may then deduce from [V5, 2.9] that g, and hence $f|A$, is $\eta(v)$-quasisymmetric in the internal metrics. The theorem follows from this since the bounded turning condition implies that both δ_D and $\delta_{D'}$ are bilipschitz equivalent to the euclidean metric in A and $f(A)$, respectively.

Remark. 6.7.

(*a*) Similarly to Theorem 6.5, Theorem 6.6 admits a formulation for domains containing ∞. Then the assumptions include $f(\infty) = \infty \in A$ whilst the dependence of η on $d(A)/d(x_0, \partial D)$, and $d(f(A))/d(f(x_0), \partial D')$ disappears.
(*b*) As discussed above, Theorem 6.1 implies [V5, Theorem 2.20] if D and D' are bounded. We therefore have obtained a somewhat different proof for the implications I $\Rightarrow$ III and II $\Rightarrow$ III in Theorem 3.1.
(*c*) If A is connected, then $\delta_D(A)$ and $\delta_{D'}(f(A))$ in Theorem 6.1 can be replaced by $d(A)$ and $d(f(A))$, respectively. See [V5, 2.13].

7. Applications of Theorem 6.1

Several interesting corollaries can be drawn from Theorem 6.1. In this final section we present three such results which we feel have some specific interest.

In [FHM] the following subinvariance property of QED domains was proved: if f is a quasiconformal mapping of a domain D onto a QED-domain D', then $f(A) \subset D'$ is a QED domain whenever $A \subset D$ is a QED domain. Recall that a domain A is b-QED if $M(\Delta(C_0, C_1; \mathbb{R}^n)) \leqslant bM(\Delta(C_0, C_1; A))$ for each pair of continua C_0, C_1 in A, see [GM1]. Next we establish an analogous subinvariance result for broad domains.

Theorem 7.1. *Suppose that D, D' are bounded, that $f: D \to D'$ is K-quasiconformal and that D' is φ-broad. If $A \subset D$ is φ_1-broad, then $f(A) \subset D'$ is φ_2-broad with φ_2 depending only on the data*

$$v = \left(n, K, \varphi, \varphi_1, \frac{d(A)}{d(x_0, \partial D)}, \frac{d(f(A))}{d(f(x_0), \partial D')}\right),$$

where x_0 is some fixed point in A.

Proof. Since broad domains are preserved under mappings which are quasisymmetric in the internal metrics [NV, Theorem 3.9], it thus suffices to show that $f: A \to f(A)$ is $\eta(v)$-quasisymmetric in the metrics δ_D and $\delta_{D'}$. Further, since A and $f(A)$ are pathwise connected $k(v)$-HTB metric spaces in δ_D and $\delta_{D'}$, we only need to show that $g = f^{-1}: f(A) \to A$ is weakly $H(v)$-quasisymmetric in $\delta_{D'}$ and δ_D, see [V5, 2.18 and 2.9]. This in turn follows from Theorem 6.1, Remark 6.7(c), and from the lemma below.

Lemma 7.2. *If A is a φ-broad subdomain of D, then A is b-LLC$_2$ in D with b depending only on n and φ. In particular, A is b_1-LLC$_2$ with respect to δ_D in D with $b_1 = b_1(n, \varphi)$.*

Proof. In view of Lemma 5.12, only the first assertion needs to be proved. Fix $x \in A$ and $r > 0$ and suppose that $z, y \in A \setminus \bar{B}(x, br)$ cannot be joined in $D \setminus \bar{B}(x, r)$. We shall show that $b \leqslant b_0(n, \varphi) < \infty$.

We may clearly assume that $b > 2$. Let E be an arc of finite length joining z and y in A, and let ζ_1 be the first point in E with $|\zeta_1 - x| = r$ when traveling from z to y. We define w_1 to be the first point in E with $|w_1 - x| = br$ when traveling from ζ_1 to z and w_1' to be the first point in E with $|w_1' - x| = br$ when traveling from ζ_1 to y. Next, let E_1 and E_1' be two disjoint subarcs of $E[w_1, w_1']$ joining the spheres $S(x, br)$, $S(x, br/2)$ in $\bar{B}(x, br) \setminus B(x, br/2)$. Then suppose that $E[w_i, w_i']$ and E_i, E_i' have been chosen for $i \geqslant 1$. If $E[w_i', y]$ does not meet

$\bar{B}(x,r)$, then stop. Otherwise let ζ_{i+1} be the first point in $\bar{B}(x,r)$ when traveling from w_i' to y, and define the points w_{i+1}, w_{i+1}' as above: w_{i+1} is the first point in E with $|w_{i+1} - x| = br$ when traveling from ζ_{i+1} to w_i' and w_{i+1}' is the first point in E with $|w_{i+1}' - x| = br$ when traveling from ζ_{i+1} to y. For E_{i+1} and E_{i+1}' we choose two disjoint subarcs of $E[w_{i+1}, w_{i+1}']$ which join $S(x, br)$ and $S(x, br/2)$ in $\bar{B}(x, br)\backslash B(x, br/2)$. Since E has finite length, the process stops at some integer $p \geqslant 1$.

Pick a pair of arcs E_i, E_i', $1 \leqslant i \leqslant p$. By the construction, $\delta_A(E_i, E_i') \leqslant 2br \leqslant 4\min\{d(E_i), d(E_i')\}$, and hence the broadness of A implies

$$M(\Delta(E_i, E_i'; A)) \geqslant \varphi(4).$$

Next write $\Gamma = \Delta(E_i, E_i'; A)$ and suppose that each path $\gamma \in \Gamma$ goes through $\bar{B}(x,r)$. Since $E_i, E_i' \subset \mathbb{R}^n\backslash B(x, br/2)$, we have by [V1, 6.4]

$$0 < \varphi(4) \leqslant M(\Gamma) \leqslant \omega_{n-1}\left(\log \frac{b}{2}\right)^{1-n}.$$

Therefore, by choosing $b = b(n, \varphi)$ large enough we infer that there is a path γ_i joining E_i and E_i' in $A\backslash\bar{B}(x,r)$. This being true for all $i = 1, \dots, p$, it is evident that by piecing together all γ_i's and parts of E we can construct a continuum which joins z and y in $A\backslash\bar{B}(x,r)$, hence in $D\backslash\bar{B}(x,r)$, contradicting our initial assumption. It follows that b is bounded by a number which depends only on n and φ, as required.

The proof of Lemma 7.2, and hence that of Theorem 7.1, is complete.

In general, if f maps $\mathbb{B}$ quasiconformally onto a John domain, one cannot hope for better distortion than described in Theorem 3.1. Our next application reveals however that the distortion improves when f is restricted to some Stolz cone.

We recall that a domain D is *b-uniform* if each pair of points $x, y \in D$ can be joined in D by a b-cigar cig (E, b) the core of which satisfies the additional turning condition $d(E) \leqslant b|x - y|$. The *Stolz cone $C_M(w)$ with vertex at $w \in \mathbb{S}$* is defined to be the interior of the closed convex hull of w and the hyperbolic ball centered at 0 with radius $M > 0$.

Theorem 7.3. *Let $C_M(w)$ be a Stolz cone in $\mathbb{B}$ and let $f\colon \mathbb{B} \to D$ be a K-quasiconformal mapping onto a b-John domain D with center $f(0)$. Then $f|C_M(w)$ is η-quasisymmetric with η depending only on the data $v = (n, K, b, M)$. In particular, $f(C_M(w))$ is $b_1(v)$-uniform.*

PROOF. It is well known that quasisymmetric mappings preserve uniform domains, see *e.g.* [V3], and therefore only the first assertion needs to be proved.

We shall show that Theorem 6.6 can be applied to f^{-1} with $D' = \mathbb{B}$ and $A = f(C_M(w))$. For this, observe first that the condition (ii) is immediately met, and clearly is A pathwise connected. Further, since D is φ-broad by Theorem 3.1, we obtain from Theorem 7.1 and Lemmas 7.2 and 5.12 that A is $b(v)$-LLC_2 with respect to δ_D. Note that $d(D) \leqslant bd(f(0), \partial D)$. Hence it remains to verify the bounded turning condition in Theorem 6.6(i).

To this end, let x' and y' be two points in A and denote by x and y their respective preimages in $C = C_M(w)$. Let x^* and y^* be the points on $[0, w]$ with $|x| = |x^*|$, $|y| = |y^*|$. Then the hyperbolic distance between x and x^*, or y and y^*, is bounded by a constant $c_0 = c_0(M)$. We may suppose that the hyperbolic balls $D(x, 3c_0)$ and $D(y, 3c_0)$ do not intersect; for if that were the case, f would be $\eta(n, K, M)$-quasisymmetric on the line segment $[x, y]$ by Lemma 2.7, whence $d(f[x, y]) \leqslant c(v)|x - y|$, proving the assertion.

Next suppose that $0 \in D(x, 2c_0)$. Then there are positive numbers $\lambda_1 = \lambda_1(M)$, $\lambda_2 = \lambda_2(M)$ such that $0 < \lambda_1 < \lambda_2 < 1$ and $x \in B(0, \lambda_1)$, $y \notin B(0, \lambda_2)$. Choose points z_1, $|z_1| = \lambda_1$, and z_2, $|z_2| = \lambda_2$ such that

$$|f(z_1) - f(z_2)| \leqslant |f(x) - f(y)| = |x' - y'|. \tag{7.4}$$

Since f is $\eta(n, K, M)$-quasisymmetric in $B(0, \lambda_2)$, we have

$$\frac{|f(0) - f(z_2)|}{|f(z_1) - f(z_2)|} \leqslant \eta\left(\frac{\lambda_2}{\lambda_2 - \lambda_1}\right)$$

whence

$$|f(z_1) - f(z_2)| \geqslant c_1 |f(0) - f(z_2)| \geqslant c_2 d(f(0), \partial D), \qquad c_2 = c_2(n, K, M),$$

where the last inequality again is a consequence of Lemma 2.8. This together with (7.4) insures that any arc E joining x' and y' in D satisfies

$$\begin{aligned} d(E) \leqslant d(D) &\leqslant c_3 \frac{d(D)}{d(f(0), \partial D)} |f(z_1) - f(z_2)| \\ &\leqslant c_4 |x' - y'|, \end{aligned}$$

where $c_4 = c_4(v)$.

The proof of the theorem is therefore complete if $0 \in D(x, 2c_0)$ or, by symmetry, if $0 \in D(y, 2c_0)$.

Next we suppose that $0 \notin D(x, 2c_0) \cup D(y, 2c_0)$ and invoke Lemma 3.1 in [GP]: there is a $K_1(M)$-quasiconformal self mapping g of $\mathbb{B}$ such that

$$g(x^*) = x, \qquad g(y^*) = y, \quad \text{and} \quad g(z) = z$$

for all z not in $D(x, 2c_0) \cup D(y, 2c_0)$. Then the $K_2(M)$-quasiconformal mapping

$h = f \circ g: \mathbb{B} \to D$ satisfies $h(0) = f(g(0)) = f(0)$, $h(x^*) = x'$ and $h(y^*) = y'$, and it is therefore no loss of generality to assume originally that x and y lie on the same ray $[0, w] \subset C_M(w)$. In fact, it suffices to show that under the conditions of Theorem 7.3

$$d(f[x, y]) \leqslant b_1 |f(x) - f(y)|, \qquad b_1 = b_1(v), \tag{7.5}$$

whenever x and y lies on a line $[0, w]$, $w \in \mathbb{S}$.

To establish (7.5) we may assume that $|x| \leqslant |y|$. Then the ball $B(x, |x - y|)$ is contained in $\mathbb{B}$, and its image $f(B(x, |x - y|))$ is $\varphi(v)$-broad by Theorem 7.1. The desired conclusion now follows from Theorem 3.1 II and V, applied to the ball $B(x, |x - y|)$.

The proof of Theorem 7.3 is complete.

Our final application divulges a property of conformal mappings, generalizing [FHM, Theorem 1]. J. Väisälä had proved the following theorem before this author in an unpublished manuscript.

Theorem 7.6. *Let D be a doubly connected domain in the Riemann sphere and let $f: D \to \mathcal{A}$ be a K-quasiconformal mapping onto an annulus $\mathcal{A} = B(0, R) \setminus \bar{B}(0, r)$. If A is a circle in D, then $f(A)$ is a quasicircle in $\mathcal{A}$ with constant depending only on K and R/r, the modulus of D.*

PROOF. Note that the theorem is trivial if $r = 0$ and $R = \infty$ so that we may assume $0 \leqslant r < R < \infty$; also if $r = 0$, the assertion follows from [FHM, Theorem 1] but we do not need that result.

By performing preliminary Möbius transformations, we may assume that $\infty \in A$ and that f maps D onto D' with $f(\infty) = \infty$, where D' is the image of $\mathcal{A}$ under a Möbius transformation. We apply Theorem 6.6 for unbounded domains, see Remark 6.7. Indeed, the line A clearly satisfies the assumptions in (i), and it is not difficult to see that D' is φ-broad with φ depending on the modulus R/r only. Thus f is $\eta(v)$-quasisymmetric on A, in particular $f(A)$ satisfies Ahlfors' three point condition whence it is a quasicircle [G]. Theorem 7.6 is proved.

We close the paper by two questions.

Question 1. (Question of J. Väisälä.) Are John domains subinvariant under quasiconformal mappings? In other words, if $f: D \to D'$ is a quasiconformal mapping onto a John domain D', is it then true that every John subdomain of D is mapped onto a John subdomain of D'?

Question 2. Suppose that D is a b-John domain with center x_0. It was shown in [GHM] that if D is planar and simply connected and if E is a quasihyperbolic geodesic joining a point x to x_0 in D, then $\operatorname{car}(E, b') \subset D$ for some $b' = b'(b)$. Is this property of John disks shared by John domains in $\mathbb{R}^n$ which are quasiconformally equivalent to the unit ball $\mathbb{B}$? Note that for general John domains the answer is no, [GHM].

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On the Angular Boundedness of Bloch Functions

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1. Introduction

Let D be the unit disk and $T = \partial D$. A Bloch function [12, p. 268] is a function f analytic in D such that

$$\|f\|_* = |f(0)| + \sup (1 - |z|^2)|f'(z)| < +\infty.$$

With this norm the Bloch functions form a Banach space $\mathcal{B}$. The closure in $\mathcal{B}$ of the polynomials is a subspace $\mathcal{B}_0$ that consists of all $f \in \mathcal{B}$ such that

$$(1 - |z|^2)|f'(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

For Bloch functions radial and angular limits are identical. Furthermore, a Bloch function is radially bounded at a point of T if and only if it is angularly bounded at this point [12, p. 269].

In this paper we deal with the size of the set

$$B_f = \{\zeta \in T: \overline{\lim}_{r \to 1} |f(r\zeta)| < +\infty\}.$$

There are Bloch functions [11] that do not have a radial limit at any point of T, but it is known [6] that for each Bloch function f, the set B_f is an uncountable dense set.

It was asked in [4] whether all $f \in \mathcal{B}$ satisfy $\dim B_f = 1$, where dim denotes the Hausdorff dimension and, as far as we know, this question remains open. One has to remark that it is not possible to replace dim 1 by positive (Lebesgue) measure, since there are Bloch functions f such that the corresponding set B_f has zero measure (see the comment after Theorem 4).

In this paper we develop a simplified version of a method of Noshiro [10] based in the Ahlfors'theory of covering surfaces, that may go further than used here, and apply it to prove that the set B_f has positive logarithmic capacity when $f \in \mathcal{B}$. In fact, using a sort of localization of Bloch functions we prove that $\operatorname{Cap}(B_f \cap I) > 0$ for every arc I, $I \subset T$. This is done in Sections 2 and 3.

The localization that we use, when applied to many functions in $\mathcal{B}_0$, gives a new way to obtain inner functions in $\mathcal{B}_0$. In connection with these functions T. Wolff asked [3] if the singular set of each inner function in $\mathcal{B}_0$ has Hausdorff dimension one and proved [16], by means of Noshiro's method, that this singular set has positive capacity. † In fact, the present paper is a development of Wolff's ideas.

In Section 4 we give a modification of Noshiro's method, using the equilibrium potential rather than the Evan's potential, that leads to a lower bound for the capacity of B_f. More precisely we prove that if $f \in \mathcal{B}$ and $f(z_0) = 0$, then there is a set $A_f \subset T$ such that

$$\overline{\lim}\, |f(r\zeta)| \leqslant k \| f \|_* \quad \text{for each} \quad \zeta \in A_f,$$

and

$$\operatorname{Cap} A_f \geqslant (1 - |z_0|^2) \exp\left[- \frac{k \| f \|_*^2}{(1 - |z_0|^2)^2 |f'(z_0)|^2} \right],$$

where k is an absolute constant.

2. The use of Ahlfors' Covering Theorem

Let f be a non-constant analytic function in D. For $z_0 \in D$ and $r > 0$ let $\Omega(z_0, r)$ denote the component of $\{z \in D: |f(z) - f(z_0)| < r\}$ that contains z_0. As a first result we can improve Theorem 1 of [6] by means of the following.

Theorem 1. *Let $f \in \mathcal{B}$. If $r > e\| f \|_*$, then* $\operatorname{Cap}(\partial\Omega(z_0, r) \cap T) > 0$ *for each* $z_0 \in D$.

To prove this theorem we use a consequence of the Ahlfors' Covering Theorem that we state in the following form [15, p. 255].

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† After this paper was submitted for publication we received the preprint «Boundaries of smooth sets and singular sets of Blaschke products in the little Bloch class» by G. Hungerford in which Wolff's conjecture is proved. (Thesis, California Institute of Technology, Pasadena, Cal., 1988.)

Lemma 1. *Let G be a simply connected domain, let H be a disk and let H_1, H_2 be disks contained in H with disjoint closures. Let f be an analytic function in G with $f(G) \subset H$. For $k = 1, 2$ let N_k be the number of domains $\Omega \subset G$ that satisfy* (i) $\bar{\Omega} \subset G$, (ii) $f(\Omega) = H_k$. *Then, for each domain $U \subset G$ such that ∂U is piecewise analytic and f is analytic on ∂U, one has*

$$N_1 + N_2 \geqslant \int_U |f'(z)|^2 \, dm(z) - k_0 \int_G |f'(z)| \, |dz|,$$

where k_0 is a constant that depends only on H, H_1 and H_2 and C is the relative boundary,

$$C = \{z \in \partial U : f(z) \in H\}.$$

PROOF OF THEOREM 1. We can suppose that $z_0 = 0$, $f(0) = 0$ and $\|f\|_* = 1$. Put $E = \partial\Omega(0, r) \cap T$ with $r > e$, suppose that $\operatorname{Cap} E = 0$ and let u be the Evan's potential for E [15, p. 75]. For each $\rho < \infty$ set

$$U_\rho = \{z \in \Omega(0, r) : u(z) < \rho\}, \qquad C_\rho = \{z \in \Omega(0, r) : u(z) = \rho\}.$$

We shall apply Lemma 1 with $G = \Omega(0, r)$ and $U = U_\rho$. We claim now that

$$\liminf_{\rho \to \infty} \frac{L_\rho}{S_\rho} = 0,$$

where

$$L_\rho = \int_{C_\rho} |f'(z)| \, |dz| \quad \text{and} \quad S_\rho = \int_{U_\rho} |f'(z)|^2 \, dm(z).$$

The claim is proved in [10], [15] but for later use we do the necessary calculations in a more direct way.

Assume that the claim is false, so that

$$L_\rho \geqslant c S_\rho \quad \text{for} \quad \rho \geqslant \rho_0, \quad c > 0.$$

The Schwarz inequality yields

$$L_\rho^2 = \left(\int_{C_\rho} \frac{|f'(z)|}{|\nabla u(z)|^{1/2}} |\nabla u(z)|^{1/2} |dz| \right)^2$$

$$\leqslant \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz| \int_{C_\rho} |\nabla u(z)| \, |dz|$$

$$\leqslant \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz| \cdot \int_{u=\rho} \frac{\partial u}{\partial n}(z) \, |dz| = 2\pi \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz|,$$

where $\partial u/\partial n$ is the derivative of u with respect to the normal to $\{u = \rho\}$. The crucial fact that the integral of this derivative along $\{u = \rho\}$ equals 2π was used. For S_ρ we can use the co-area formula [9, p. 37]

$$S_\rho - S_{\rho'} = \int_{\rho'}^{\rho} dr \int_{C_r} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz|,$$

for adequate $\rho' > 0$.

This shows that S_ρ is an absolutely continuous function of ρ and

$$S'_\rho = \frac{dS_\rho}{d\rho} = \int_{C_\rho} \frac{|f'(z)|^2}{|\nabla u(z)|} |dz| \quad \text{a.e. } (\rho).$$

So we get

$$L_\rho^2 \leqslant 2\pi S'_\rho \quad \text{a.e. for} \quad \rho \geqslant \rho_0$$

and

$$S_\rho^2 \leqslant \frac{1}{c^2} L_\rho^2 \leqslant c_1 S'_\rho \quad \text{a.e.}$$

Integrating from ρ_0 to ρ one gets

$$\frac{1}{c_1}(\rho - \rho_0) \leqslant \int_{\rho_0}^{\rho} \frac{S'_\rho}{S_\rho^2} d\rho \leqslant \frac{1}{S_{\rho_0}} \quad \text{for} \quad \rho \geqslant \rho_0,$$

which is impossible, and the claim is proved.

Let us take now for H_1, H_2 two discs of disjoint closures and radii s, $e < 2s < r$, contained in $\{|w| < r\}$. Acording to Lemma 1 we get

$$N_1 + N_2 \geqslant S_\rho - k_0 L_\rho = S_\rho\left(1 - k_0 \frac{L_\rho}{S_\rho}\right) > 0 \quad \text{for} \quad \rho \geqslant \rho_1.$$

Applying [6, Lemma 2] we obtain

$$1 \geqslant \sup\{(1 - |z|^2)|f'(z)| : z \in \Omega(0, r)\} \geqslant 2e^{-1}s,$$

a contradiction. □

Corallary. *Let $f \in \mathcal{B}$ and $f(0) = 0$. Then there is a set $A_f \subset T$ such that* $\operatorname{Cap} A_f > 0$ *and*

$$\overline{\lim} |f(r\zeta)| \leqslant k \|f\|_* \quad \textit{for each} \quad \zeta \in A_f,$$

where k is an absolute constant.

Proof. We can suppose that $\|f\|_* = 1$. Applying Theorem 1 with $r = 3$ and $A_f = \partial\Omega(0, r) \cap T$ we get $\operatorname{Cap} A_f > 0$. Moreover, for $\zeta \in A_f$ there is a curve γ in $\Omega(0, r)$ ending at ζ. So $|f(z)| \leqslant 3$, $z \in \gamma$ and, since $f \in \mathcal{B}$, we get $\overline{\lim}_{r\to 1} |f(r\zeta)| \leqslant k$, for each $\zeta \in A_f$, according to [1, Theorem 4.2]. □

3. The Local Results

In order to prove that the set B_f has locally positive capacity when $f \in \mathcal{B}$, we improve the conclusion of Theorem 1. We follow the notation of the beginning of Section 2.

Lemma 2. *Let $f \in \mathcal{B}$ and $r > 0$. Let I be an open arc in T and $z_0 \in D$. If $\partial\Omega(z_0, r) \cap I \neq \emptyset$, then* $\operatorname{Cap}(\partial\Omega(z_0, r') \cap I) > 0$ *for each $r' > r$.*

Proof. Write $G_s = \Omega(z_0, s)$ and $E_s = \partial G_s$ for each $s \geqslant r$. Let $I \subset T$ be an arc and suppose that $I \cap E_r \neq \emptyset$. We can assume that $I \cap E_r$ has zero Lebesgue measure. The boundary of G_r is formed by a sequence of Jordan arcs that end at two points of T [8, p. 10]. Since $I \cap E_r$ has no interior points we can take an arc $I_0 \subset I$ whose extreme points A_0, B_0 are the mid points of arcs in $T \backslash E_r$. We can join A_0 and B_0 to points A_r, B_r on E_r by means of arcs contained in $D \backslash G_r$. Take now a Jordan arc Γ_r from A_r to B_r with $\Gamma_r \subset G_r$. This arc separates G_r into two parts. Let F_r be the part of G_r with $\bar{F}_r \cap I_0 \neq \emptyset$. So $\partial F_r \cap T \subset I_0 \cap E_r$.

Let us assume that $\operatorname{Cap}(\partial F_r \cap T) = 0$. Using the notation of Theorem 1 with F_r instead of G we see that $\{z \in \partial U_\rho : |f(z) - f(z_0)| < r\}$ is formed by $C_\rho = \{z \in G_0 : u(z) = \rho\}$ plus the arc Γ_r for large enough ρ. The argument used in this proof shows that

$$\lim_{\rho\to\infty} \frac{L_\rho + \int_\Gamma |f'(z)|\,|dz|}{S_\rho} = 0$$

and Lemma 1 leads to a contradiction.

If g_r has no singularity in γ_r then for $r' > r$, close enough to r, we can repeat the argument with $F_{r'}$ instead of F_r. Now $g_{r'}$ has some singularity in $\gamma_{r'}$ because, if this is not the case, $g_{r'}$ would be analytic through γ_r'. Then the pre-image by f of an arc contained in $|z| = r$ would be a compact set in $F_{r'}$. But it contains some point of $T \cap \partial F_r \neq \emptyset$ and this is a contradiction. □

It is possible that $\operatorname{Cap}(\partial\Omega(z, r) \cap I) = 0$ even with the additional hypothesis $r > e\|f\|_*$. We thank the referee because his observation about the previous statement of Lemma 2 allowed us to correct it.

Theorem 2. *If $f \in \mathcal{B}$, there is a set $E_f \subset T$ such that* $\operatorname{Cap}(I \cap E_f) > 0$ *for each arc $I \subset T$ and $\overline{\lim}_{r\to 1} |f(r\zeta)| < +\infty$ for $\zeta \in E_f$.*

PROOF. Suppose $f(0) = 0$ and $\|f\|_* = 1$. Consider the components of $H = f^{-1}(\{|w| < 3\})$. Each of them touches T; see [13] or Theorem 1 in [6]. Put $E_f = E_1 \cup E_2$ where

$$E_1 = \{\zeta \in T: \zeta \text{ is in the closure of some component of } H\},$$
$$E_2 = \{\zeta \in T: \lim_{r\to 1} f(r\zeta) \neq \infty \text{ exists}\}.$$

Since $f \in \mathcal{B}$ we have $\lim |f(r\zeta)| < \infty$ at each point of E_f. Given an arc $I \subset T$, if $|I \cap E_2| = |I|$ it is clear that $\operatorname{Cap}(E_f \cap I) > 0$. So, let us assume that $|I \cap E_2| < |I|$. In this case there is some point $\zeta \in T$ which is a Plessner point for f [12, p. 324]. So there are points $z_n \to \zeta$, $z_n \in D$ with $f(z_n) \to 0$. If there are only a finite number of components of H some one has to touch T on I. If there are infinitely many, by a result of McLane [8, p. 10], it is not possible that all of these components do not touch I, because then their diameter would not go to zero. So, in this case some component has to touch T on I. Now we apply Lemma 2. □

Remark. If the function f has angular limits almost nowhere on T the above proof shows that, in this case, there is a set $E_f \subset T$ with $\operatorname{Cap}(I \cap E_f) > 0$ for each arc $I \subset T$ and

$$\overline{\lim} |f(r\zeta) - f(0)| < k\|f\|_*,$$

for all $\zeta \in E_f$, where k is an absolute constant.

Let f be an inner function and let $S(f)$ denote its singular set, the set of points of T at which f has no analytic continuation. We can improve the result of Wolff [16], which was the starting point of the present paper.

Theorem 3. *Let f be an inner function and suppose that $\|f\|_* < e^{-1}$ or that $f \in \mathcal{B}_0$. Then for each arc $I \subset T$ one has $I \cap S(f) = \emptyset$ or* $\operatorname{Cap}(I \cap S(f)) > 0$.

PROOF. Assume first $\|f\|_* < e^{-1}$ and $f(0) = 0$. Take r with $e\|f\|_* r < 1$ and consider the components of $f^{-1}(\{|w| < r\})$. It is clear that $\partial\Omega(z, r) \cap T \subset S(f)$. Now given an arc I with $I \cap S(f) \neq \emptyset$, there are points $\zeta \in I \cap S(f)$ and $z_n \to \zeta$ with $f(z_n) \to 0$ and we can follow the same argument as in the proof of Theorem 2. Consider now that $f \in \mathcal{B}_0$, by use of Frostman's Theorem we may assume without loss of generality that f is an infinite Blaschke product. Given an arc $I \subset T$ with $I \cap S(f) \neq \emptyset$ we can find components $\Omega(z, r)$ with $r < 1$ such that $\partial\Omega(z, r) \subset I \cap S(f)$. Now the domains U_ρ used in the proof of Theorem 1 satisfy also $S_\rho \to \infty$ as $\rho \to \infty$ and the assumption $\operatorname{Cap}(I \cap S(f)) = 0$ leads to

$N_1 + N_2 = +\infty$ in Lemma 1 for two discs H_1, H_2 contained in $\{|w| < r\}$. This contradicts the geometric characterization of $\mathcal{B}_0$ functions given by Theorem 1, (i) of [13]. $\square$

Let f be a non constant analytic function in $\boldsymbol{D}$ with $f(0) = 0$ and let $G = \Omega(z, r)$ be a component of $f^{-1}(\{|w| < r\})$. If φ is a conformal mapping from $\boldsymbol{D}$ onto G, the function $g = f \circ \varphi$ reproduces in the unit disk the local behaviour of f. This localization has been used in the proof of Lemma 2 and we will show now that it can be used to produce inner functions in $\mathcal{B}_0$ with some additional properties.

Theorem 4. *Suppose that f is a function in $\mathcal{B}_0$ that has angular limits almost nowhere. If φ maps $\boldsymbol{D}$ conformally onto a component of $f^{-1}(\boldsymbol{D})$ then $f \circ \varphi$ is an inner function in $\mathcal{B}_0$. Furthermore, if f has Hadamard gaps then $f \circ \varphi$ assumes every value in $\boldsymbol{D}$ infinitely often.*

A (lacunary) series with Hadamard gaps has the form

$$f(z) = \sum_{k=0}^{\infty} b_k z^{n_k}, \qquad \frac{n_{k+1}}{n_k} \geqslant \lambda > 1 \quad (k = 0, 1, \ldots).$$

This function belongs to $\mathcal{B}_0$ if and only if $b_k \to 0$ as $k \to \infty$, [1], and is radially bounded on a set of positive measure if and only if $\sum |b_k|^2 < +\infty$ [17, vol. 1, p. 203]. Hence

$$f_0 = \sum_{k=1}^{\infty} k^{-1/2} z^{2^k}$$

is an example that has all the properties required in the theorem; note that the angular limit ∞ occurs almost nowhere by the Privalov Uniqueness Theorem.

PROOF. We have

$$(1 - |z|^2)\left|\frac{d}{dz} f(\varphi(z))\right| = \frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}(1 - |\varphi(z)|^2)|f'(\varphi(z))|.$$

If $|\varphi(z_n)| \to 1$ then the last factor tends to 0 because $f \in \mathcal{B}_0$ while the quotient is bounded by 1. If however $\limsup |\varphi(z_n)| < 1$ then the quotient tends to 0 because φ is univalent while the last factor is bounded. The fact that g is inner is a consequence of Loewner's Lemma as used in the proof of Lemma 2, (see also [15, p. 323]).

Suppose now that f has Hadamard gaps. Let w_0 be a point with $|w_0| = r$ and $f(z) \neq w_0$ for each z with $f'(z) = 0$. We claim that there are infinitely many points $z_n \in \partial\varphi(D)$ with $f(z_n) = w_0$. If the number of these points were finite we could then draw a curve $\gamma \subset \partial\varphi(D)$ with $\bar{\gamma} \cap T \neq \emptyset$ and f would map γ in a finite-to-one manner on an arc of finite length in $|w| = r$. Since $\sum_{n=1}^{\infty} |b_n| = +\infty$ this would contradict Theorem 1 of [5].

Now take a point w, $|w| < r$, and let Γ be a rectifiable Jordan arc from w_0 to w, with $f'(z) \neq 0$ when $f(z) \in \Gamma \setminus \{w\}$. Let us consider the components Γ_n of $f^{-1}(\Gamma)$ with $z_n \in \Gamma_n$ since Γ has finite length it follows from [5] that Γ_n is a Jordan arc and $f(\Gamma_n) = \Gamma$. We conclude that there are distinct points $z_n' \in \Gamma_n$ with $f(z_n') = w$. $\square$

Remark. Previously, the known ways to construct inner functions in $\mathcal{B}_0$ were the use of a singular measure whose primitive is in the Zygmund class λ_* or the more geometric one of [14], by means of the Riemann surface of the function. T. Wolff asked if $\dim S(f) = 1$ when $f \in \mathcal{B}_0$ is inner. The corresponding result for inner functions omitting some values is true [3]. Theorem 4 shows that this gives no aid in order to answer Wolff's question. Theorem 4 shows that there is a close relation between the size of $\partial\Omega(z, r) \cap T$ for a function in $\mathcal{B}_0$ and the answer to Wolff's conjecture.

4. A Lower Bound for Capacities

In this Section we present a modification of the idea of Noshiro using the equilibrium potential rather than the Evans's potential. It leads to a lower bound for the capacity of the set B_f.

The following Lemma contains the basic estimate.

Lemma 3. *Let f be analytic in D and continuous in $\bar{D}$ and let $f(D) \subset D$, $f(0) = 0$, $f'(0) \neq 0$. We assume that $\|f\|_* < 1/3$ and that the set*

$$B = \{\zeta \in T: |f(\zeta)| < 1\}$$

consists of a finite number of arcs. Then

$$\operatorname{cap} B \geqslant \exp\left(-\frac{k}{|f'(0)|^2}\right),$$

k being an absolute constant.

PROOF. The set $A = \bar{B}$ is regular for the Dirichlet problem with respect to $C \setminus A$. If we take the equilibrium potential v, we have $v(z) \leqslant V$ and $v(z) = V$ if $z \in A$, where $\operatorname{Cap} A = \exp(-V)$.

For $\rho < V$ let

$$G_\rho = \{z \in D: v(z) < \rho\},$$
$$C_\rho = \{z \in \partial G_\rho: |f(z)| < 1\} = D \cap \partial G_\rho.$$

Also put

$$L_\rho = \int_{C_\rho} |f'(z)|\,|ds|, \qquad S_\rho = \int_{G\rho} |f'(z)|^2\,dm(z).$$

Take now two disks $H_1, H_2 \subset D$ with disjoint closures and radii 5/11. Then, by Lemma 2 of [6] and since $\|f\|_* < 1/3$, we see that here is no domain $\Omega, \bar{\Omega} \subset D$ with $f(\Omega) = H_1$ or $f(\Omega) = H_2$. Now, by the reflection principle, f has an analytic extension to $T \backslash B$, so we can apply Lemma 1, and we get $S_\rho \leqslant k_0 L_\rho$ for $\rho < V$, k_0 being some constant. Moreover the same calculation performed in the proof of theorem 1 yields

$$L_\rho^2 \leqslant 2\pi S'_\rho \quad \text{a.e.} \quad (\rho)$$

and so $S_\rho^2 \leqslant k_1 S'_\rho$ a.e. $\rho < V$. Now we remark that there is a number ρ_0, independent of f, such that $D(0, 1/2) \subset G_{\rho_0}$. To see this we can suppose $\operatorname{Cap} A \leqslant 1/2$ or $V \geqslant \log 2$. In this case for $|z| < 1/2$ and $\zeta \in A$ one has $|z - \zeta| \geqslant 1/2$ and

$$v(z) = \int_A \log \frac{1}{|z - \zeta|}\,d\mu(\zeta) \leqslant \log 2.$$

Then we can take $\rho_0 = \log 2$ to guarantee that G_{ρ_0} contains $D(0, 1/2)$.

Integrating now the inequality $S_\rho / S_\rho^2 \geqslant k_2$ a.e. from ρ_0 to V we get

$$\frac{1}{S_{\rho_0}} \geqslant \frac{1}{S_{\rho_0}} - \frac{1}{S_V} \geqslant k_2(V - \rho_0) \quad \text{or} \quad V \leqslant \rho_0 + \frac{k_3}{S_{\rho_0}}.$$

Taking into account that S_{ρ_0} is the area (counting multiplicities) of the image through f of G_{ρ_0} and the inclusion $G_{\rho_0} \supset D(0, 1/2)$, we get by means of Bloch's Theorem [15, p. 262] that

$$S_{\rho_0} \geqslant k_4 |f'(0)|^2.$$

This inequality and the previous one give the lemma. □

Lemma 4. *Let f be analytic in D with $f(z_0) = 0$ and let G be a component of $f^{-1}(\{|w| < 1\})$ containing z_0. We write $\delta(z) = \operatorname{dist}(z, \partial G)$ and assume*

$$\sup_{z \in G} \delta(z)|f'(z)| < \frac{1}{12}.$$

Then

$$\operatorname{Cap}(\partial G \cap T) \geqslant (1 - |z_0|) \exp\left(- \frac{k}{\delta(z_0)^2 |f'(z_0)|^2} \right),$$

where k is an absolute constant.

PROOF. For each r, $|z_0| < r < 1$ let $G(r)$ be the component of $G \cap \{|z| < r\}$ containing z_0. Since $G(r)$ is simply connected we can take a conformal mapping φ_r from $\boldsymbol{D}$ onto $G(r)$ with $\varphi_r(0) = z_0$. Writing $\tilde{f}_r = f \circ \varphi_r$ we see that $\tilde{f}_r$ is analytic in $\boldsymbol{D}$, continuous on $\bar{\boldsymbol{D}}$ and the set

$$B_r = \{ z \in \boldsymbol{D} \colon |\tilde{f}_r(z)| < 1 \}$$

consists of a finite number of arcs. Moreover

$$\tilde{f}_r(0) = 0, \qquad \tilde{f}'_r(0) = f'(z_0)\varphi'_r(0).$$

Also

$$\| \tilde{f}_r \|_* = \sup_{w \in \boldsymbol{D}} (1 - |w|^2) |\varphi'_r(w)| \, |f'(\varphi_r(w))|.$$

The Koebe distorsion Theorem [12, p. 22] gives

$$(1 - |w|^2)|\varphi'_r(w)| \leqslant 4 \operatorname{dist}(\varphi_r(w), \partial G(r)) \leqslant 4\delta(\varphi_r(w)),$$

and we conclude that $\| \tilde{f}_r \|_* < 1/3$. Furthermore

$$|\tilde{f}'_r(0)| \geqslant \operatorname{dist}(z_0, \partial G(r)) |f'(z_0)| \geqslant \frac{1}{2} \delta(z_0) |f'(z_0)|,$$

if r is big enough. So Lemma 3 implies

$$\operatorname{Cap} B_r \geqslant \exp\left(- \frac{k}{\delta(z_0)^2 |f'(z_0)|^2} \right).$$

If we write $L(r) = \partial G(r) \cap \{|z| = r\}$, one has $L(r) \supset \varphi_r(B_r)$. Furthermore the mapping $h = \tau_{z_0/r} \circ (\varphi_r / r)$ where $\tau_{z_0/r}$ is the automorphism of $\boldsymbol{D}$ sending z_0/r to 0, satisfies $h(B_r) \subset T$ and $h(0) = 0$. Then applying [12, p. 348] and Schwarz's Lemma we obtain $\operatorname{Cap} h(B_r) \geqslant \operatorname{Cap} B_r$.

Moreover, [7, p. 138] gives

$$\operatorname{Cap} \varphi_r(B_r) \geqslant \frac{1}{2} \left(1 - \frac{|z_0|}{r} \right) \operatorname{Cap} h(B_r) \geqslant \frac{1}{2} r \left(1 - \frac{|z_0|}{r} \right) \operatorname{Cap} B_r.$$

So

$$\operatorname{Cap} L(r) \geqslant \operatorname{Cap} \varphi_r(B_r) \geqslant \frac{1}{2} r \left(1 - \frac{|z_0|}{r} \right) \exp\left(- \frac{k}{\delta(z_0)^2 |f'(z_0)|^2} \right).$$

Considering now the compact sets $E_r = \{z \in \bar{G}: |z| \geqslant r\}$ we have $\operatorname{Cap} E_r \geqslant \operatorname{Cap} L(r)$ and since E_r decreases to $\partial G \cap T$ when $r \to 1$, we get the estimate of the lemma. □

The announced lower bound for the capacity of the set B_f is the following.

Theorem 5. *Let f be a Bloch function and $z_0 \in D$. Then there is a set $A_f \subset T$ such that*

$$\overline{\lim_{r \to 1}} |f(r\zeta) - f(z_0)| \leqslant k \|f\|_* \quad \textit{for each} \quad \zeta \in A_f,$$

and

$$\operatorname{Cap} A_f \geqslant (1 - |z_0|) \exp\left(-\frac{k\|f\|_*^2}{(1 - |z_0|^2)^2 |f'(z_0)|^2}\right),$$

where k in an absolute constant.

PROOF. We can assume $\|f\|_* = 1$ and $f(z_0) = 0$. Consider the analytic function $\tilde{f} = \lambda f \circ \tau$, where $\lambda = 1/12$ and $\tau(z) = (z + z_0)/(1 + z_0 z)$.

We have

$$\tilde{f}(0) = 0, \qquad \|\tilde{f}\|_* \leqslant \lambda \quad \text{and} \quad |\tilde{f}'(0)| = \lambda(1 - |z_0|^2)|f'(z_0)|.$$

Let G be the component of $\tilde{f}^{-1}(\{|w| < 1\})$ that contains 0. Then

$$\delta(z)|\tilde{f}'(z)| \leqslant (1 - |z|)|f'(z)| \leqslant \|f\|_* < \frac{1}{12},$$

where, as before, we write $\delta(z) = \operatorname{dist}(z, \partial G)$.

Furthermore $|z| < 1/2$ implies

$$|\tilde{f}(z)| \leqslant \|\tilde{f}\|_* \log \frac{1 + 1/2}{1 - 1/2} < 2\lambda < 1.$$

This means that $\{|z| < 1/2\} \subset G$ and so $\delta(0) \geqslant 1/2$ and

$$\delta(0)|\tilde{f}'(0)| \geqslant \frac{1}{24}(1 - |z_0|^2)|f'(z_0)|.$$

Applying Lemma 4 we get

$$\operatorname{Cap}(\partial G \cap T) \geqslant \exp\left(-\frac{k}{(1 - |z_0|^2)^2 |f'(z_0)|^2}\right),$$

and

$$\operatorname{Cap}(\tau(\partial G)\cap T) \geqslant (1-|z_0|)\exp\left(-\frac{k}{(1-|z_0|^2)^2|f'(z_0)|^2}\right).$$

If we use the fact that f is a Bloch function then we get the theorem with $A_f = \tau(\partial G)$. □

Theorem 6. *Let f be an inner function with $\|f\|_* < 1/2e$ and $f(z_0) = 0$. Then*

$$\operatorname{Cap} S(f) \geqslant (1-|z_0|^2)\exp\left(-\frac{k}{(1-|z_0|^2)^2|f'(z_0)|^2}\right),$$

k being an absolute constant.

PROOF. Assume that $f(0) = 0$ and let G be a component of $f^{-1}(\{|w| < 1/2\})$ containg 0, so that $\varnothing \neq \partial G \cap T \subset S(f)$.

Since $|z| < 1/2$ implies $|f(z)| \leqslant 2\|f\|_* < 1/2$ we see that $\{|z| < 1/2\} \subset G$ and so $\delta(0) \geqslant 1/2$. Now by Lemma 4 we get

$$\operatorname{Cap} S(f) \geqslant \operatorname{Cap}(\partial G \cap T) \geqslant \exp\left(-\frac{k}{|f'(0)|^2}\right).$$

If $z_0 \neq 0$ we deal with the function

$$\tilde{f}(z) = f\left(\frac{z+z_0}{1+z_0 z}\right). \quad \square$$

Remark. Makarov (written communication) has proved the conjecture that $\dim B_f = 1$ for every Bloch function. His method is more direct and completely different.

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Bases D'Ondelettes sur les Courbes Corde-Arc, Noyau de Cauchy et Espaces de Hardy Associés

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Résumé

On construit deux bases inconditionnelles de $L^2(\mathbb{R})$ adaptées à l'étude de l'intégrale de Cauchy sur une courbe corde-arc, et on étend la construction à $L^2(\mathbb{R}^d)$. Cela permet de donner une preuve simple du «Théorème $T(b)$» de G. David, J. L. Journé et S. Semmes. Un espace de Hardy à poids $H^1_b(\mathbb{R}^d)$ est défini et caractérisé par les bases précédentes. Enfin, on applique ces méthodes à l'étude du potentiel de double couche sur une surface lipschitzienne.

Introduction

Soit T un opérateur linéaire continu de $D(\mathbb{R}^d)$ dans $D'(\mathbb{R}^d)$.

Un des moyens que nous donne l'analyse fonctionnelle pour tester la continuité sur $L^2(\mathbb{R}^d)$ de cet opérateur est de disposer d'une famille dénombrable de fonctions f_λ, $\lambda \in \Lambda$, formant une base inconditionnelle de $L^2(\mathbb{R}^d)$ (on dit aussi base de Riesz). T admet alors une extension continue à $L^2(\mathbb{R}^d)$ si, et seulement si, la matrice M de l'opérateur T sur la base précédente est bornée sur $l^2(\Lambda)$.

Notre programme de travail est l'étude de certains opérateurs linéaires dits de Calderón-Zygmund à l'aide de cette remarque. (Nous rappelons leur définition ci-après.)

L'essentiel de notre tâche consiste alors à construire une base inconditionnelle de $L^2(\mathbb{R}^d)$ qui soit adaptée à ces opérateurs (Théorème 1): ce sera une base d'ondelettes possédant une propriété de compensation particulière.

Ainsi, nous pouvons redémontrer simplement le Théorème $T(b)$ de David, Journé et Semmes [DJS] dans le cas pseudo-accrétif (Théorème 4), et étudier des algèbres d'opérateurs de Calderón-Zygmund, isomorphes à celle étudiée par Lemarié [L] et contenant la valeur principale de Cauchy sur les courbes corde-arc (Théorème 5).

Dans [T], l'un des auteurs a réalisé ce programme pour démontrer la continuité L^2 de l'opérateur de Cauchy sur les graphes lipschitziens (théorème de Coifman, McIntosh et Meyer [CMM]). Notre travail en est le prolongement au cas «corde-arc».

Il n'en est pas pour autant une extension immédiate, et sans doute quelques mots d'explication, quoique nécessairement vagues, sont opportuns. Il y a deux grandes étapes dans l'élaboration des bases d'ondelettes dont nous avons besoin: d'abord leur construction, ensuite la preuve qu'il s'agit bien de bases inconditionnelles de $L^2(\mathbb{R}^d)$, et non pas seulement de familles complètes, par exemple. Cette dernière étape, dans [T], contient tous les arguments d'analyse réelle (en particulier l'emploi des mesures de Carleson) réputés indispensables à la preuve du théorème de Coifman, McIntosh et Meyer. Elle est inchangée dans le présent article.

En revanche, c'est dans la première étape, due en fait à Meyer et Lemarié dans le cas lipschitzien, et qui ne repose que sur des arguments de nature hilbertienne, que nous avons dû introduire une idée supplémentaire, qui revient à tenir compte de façon beaucoup plus forte de la géométrie du problème. Des explications plus détaillées sont bien sûr données dans le corps de l'article.

Nous concluons ce travail avec deux applications du résultat fondamental:

1. Nous définissons dans le cas corde-arc un espace $H^1_b(\mathbb{R}^d)$, substitut à l'espace atomique $H^1(\mathbb{R}^d)$ de Stein et Weiss utilisé classiquement (Théorème 6);
2. Grâce au calcul dans les algèbres de Clifford, nous montrons comment étudier le potentiel de double couche sur une surface lipschitzienne avec les idées précédentes.

Le contenu de ce travail se trouve dans la thèse de doctorat de l'un des auteurs [A1]. Nous tenons à remercier S. Jaffard et Y. Meyer de nous avoir communiqué certaines de leurs idées.

1. Définitions et enoncé du théorème fondamental

(1) *Opérateurs d'intégrales singulières*

Nous précisons ici la classe d'opérateurs que nous allons étudier. Soit $\Omega \subset \mathbb{R}^d \times \mathbb{R}^d$ l'ensemble des couples $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ tels que $x \neq y$.

Définition 1. *Soit $\delta \in]0,1]$. Un noyau δ-standard est une fonction continue $K(x, y)$ sur Ω verifiant les propriétés suivantes: il existe une constante $C \geqslant 0$ telle que*

(i) *Pour tout* $(x, y) \in \Omega \quad |K(x, y)| \leqslant C|x-y|^{-d}$.

(ii) *Pour* $(x, y) \in \Omega$ *et* $h \in \mathbb{R}^d$ *avec* $|h| \leqslant \dfrac{1}{2}|x-y|$

$$|K(x+h, y) - K(x, y)| \leqslant C|x-y|^{-d}\left(\frac{|h|}{|x-y|}\right)^{\delta},$$

(iii) *Pour* $(x, y) \in \Omega$, *et* $h \in \mathbb{R}^d$ *avec* $|h| \leqslant \dfrac{1}{2}|x-y|$,

$$|K(x, y+h) - K(x, y)| \leqslant C|x-y|^{-d}\left(\frac{|h|}{|x-y|}\right)^{\delta}.$$

Définition 2. *Soient b_1 et b_2 deux fonctions bornées sur $\mathbb{R}^d$ et $\delta \in]0, 1]$. Nous désignerons par $F(\delta, b_2, b_1)$ l'espace vectoriel des opérateur linéaires continus T de $D(\mathbb{R}^d)$ dans $D'(\mathbb{R}^d)$, dont le noyau distribution $S(x, y)$, restreint à Ω, s'écrit $b_2(x)K(x, y)b_1(y)$, K étant δ-standard, et tel que*

$$\langle Tg, f \rangle = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) b_2(x) K(x, y) b_1(y) g(y)\, dy\, dx$$

pour tout couple de fonctions $f, g \in D(\mathbb{R}^d)$ à supports disjoints.

David-Journé-Semmes [DJS] ont donné des conditions nécessaires et suffisantes pour que de tels opérateurs admettent une extension continue à $L^2(\mathbb{R}^d)$ et des conditions presque optimales sur les fonctions b_1 et b_2 pour que ce résultat s'applique. Nous utiliserons une notion un peu plus forte dite de pseudo-accrétivité.

Soit Q l'ensemble des cubes dyadiques

$$Q_{j,k} = \{x : 2^j x - k \in [0, 1]^d\}, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^d.$$

Définition 3. *Soit b une fonction bornée sur $\mathbb{R}^d$, ainsi que son inverse. On* *dit que b est pseudoaccrétive (de constante* $\delta_0 > 0$) *s'il existe* $\delta_0 > 0$ *telle que*

$$\text{(4)} \qquad \textit{pour tout} \quad Q \in \mathcal{Q}, \qquad |m_Q b| \geqslant \delta_0,$$

$m_Q(b)$ *désignant*

$$\frac{1}{|Q|}\int_Q b(x)\,dx,$$

dx étant la mesure de Lebesgue sur $\mathbb{R}^d$.

Voici l'exemple historique qui motive les définitions précédentes.

Soit Γ une courbe simple du plan complexe orientée et passant par l'infini. On suppose Γ rectifiable et l'on note $x \to z(x)$ une paramétrisation par longueur d'arc de Γ. On dit que Γ est corde-arc s'il existe un $\delta_0 > 0$ tel que

$$\text{(5)} \qquad \text{pour tout} \quad (x, y) \in \mathbb{R}^2, \qquad \delta_0|x - y| \leqslant |z(x) - z(y)| \leqslant |x - y|.$$

La fonction $z(x)$ admet une dérivée presque partout et $z'(x)$ vérifie alors la condition (4) pour tout intervalle réel I: z' est pseudoaccrétive.

Comme exemple de telles courbes, on peut citer une courbe lipschitzienne ou une spirale logarithmique.

Maintenant, écrivons l'opérateur de Cauchy associé à la courbe Γ:

$$\text{si} \quad f \in D(\mathbb{R}), \qquad T_\Gamma f(x) = \text{v.p.}\,\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{z'(y)}{z(y) - z(x)} f(y)\,dy,$$

cette valeur principale ayant un sens grâce à (5).

Il est immédiat de constater que $T_\Gamma \in F(1, 1, z')$.

Définition 4. $T \in F(\delta, 1, 1)$ *est dit de Calderón-Zygmund quand T est continu sur* $L^2(\mathbb{R}^d)$.

(2) *b-vaguelettes*

On désigne par b une fonction bornée. On pose $w_\eta(x) = (1 + |x|)^{-d-\eta}$ pour $x \in \mathbb{R}^d$ et $\eta > 0$.

Définition 5. *Soit* $r \in]0, 1]$. *On appelle famille de b-vaguelettes r-régulières toute famille de fonctions* (θ_Q) *indexée par les cubes dyadiques et vérifiant*

$$\text{(6)} \qquad \forall \eta > 0 \quad \exists C_\eta \geqslant 0,$$
$$|\theta_Q(x)| \leqslant C_\eta 2^{dj/2} w_\eta(2^j x - k)$$

pour $Q = Q_{j,k}$ *et* $x \in \mathbb{R}^d$,

(7) $$\forall \eta > 0 \quad \exists C_\eta \geqslant 0 \quad \forall (x, x') \in \mathbb{R}^d, \quad \forall Q \in \mathcal{Q}, \quad Q = Q_{j,k},$$
$$|\theta_Q(x) - \theta_Q(x')| \leqslant C_\eta 2^{(r + d/2)j} |x - x'|^r (w_\eta(2^j x - k) + w_\eta(2^j x' - k))$$

(8) $$\forall Q \in \mathcal{Q}, \qquad \int_{\mathbb{R}^d} b(x)\theta_Q(x)\, dx = 0.$$

On dira que la famille (θ_Q) *est une famille de b-molécules de régularité r si,* (8) *étant vérifiée,* (6) *et* (7) *sont seulement satisfaites avec* $\eta = r$.

Remarque. Dans le cas des b-molécules r-régulières, (7) peut être remplacée par (9):

(9) $\forall \delta \in]0, r]$,
$$|\theta_Q(x) - \theta_Q(x')| \leqslant C_r 2^{(\delta + d/2)j} |x - x'|^\delta (w_r(2^j x - k) + w_r(2^j x' - k)).$$

La raison de ce double formalisme est que pour la construction qui va suivre, nous aurons besoin d'estimations «à décroissance rapide» donc de b-vaguelettes, tandis que les propriétés des b-molécules nous suffiront pour étudier les opérateurs de la classe introduite ci-dessus.

On pose maintenant

$$B(f, g) = \int_{\mathbb{R}^d} f(x) b(x) \overline{g(x)}\, dx \quad \text{pour} \quad f, g \in L^2(\mathbb{R}^d).$$

Théorème 1. *Soit b une fonction pseudoaccrétive. Il existe un* $r \in]0, 1[$, *un ensemble fini* Λ *de cardinal* $2^d - 1$ *et, pour chaque* $\lambda \in \Lambda$, *deux familles* $(\theta_{\lambda, Q})$ *et* $(\tilde{\theta}_{\lambda, Q})$ *de b-vaguelettes r-régulières telles que*

(10) $$\forall f \in L^2(\mathbb{R}^d) \quad \int_{\mathbb{R}^d} |f(x)|^2\, dx \sim \sum_\lambda \sum_Q |B(f, \theta_{\lambda, Q})|^2 \sim \sum_\lambda \sum_Q |B(\tilde{\theta}_{\lambda, Q} f)|^2,$$

(11) $$\forall f \in L^2(\mathbb{R}^d) \quad f = \sum_\lambda \sum_Q B(f, \theta_{\lambda, Q}) \tilde{\theta}_{\lambda, Q} = \sum_\lambda \sum_Q B(\tilde{\theta}_{\lambda, Q} f) \theta_{\lambda, Q},$$

(12) $$\forall (\lambda, \lambda') \in \Lambda^2, \quad \forall (Q, Q') \in \mathcal{Q}^2, \quad B(\tilde{\theta}_{\lambda, Q}, \theta_{\lambda', Q'}) = \delta_{\lambda, \lambda'} \delta_{Q, Q'},$$

où δ *désigne le symbole de Kronecker.*

En d'autres termes, les collections de fonctions $(\theta_{\lambda, Q})$ et $(\tilde{\theta}_{\lambda, Q})$ forment deux bases inconditionnelles de $L^2(\mathbb{R}^d)$ bi-orthogonales pour la forme bilinéaire B.

La démonstration de ce théorème s'organisera comme suit. Nous nous placerons constamment en dimension 1, le cas de la dimension d étant traité dans la partie 6. Dans la partie 2, nous démontrerons deux séries de Lemmes abstraits. Nous élaborerons dans la partie 3 une analyse multirésolution de $L^2(\mathbb{R}^d)$ adéquate. Les lemmes abstraits seront appliqués dans la partie suivante pour construire les deux familles de b-vaguelettes vérifiant (12). Nous en terminerons dans la partie 5 en montrant les propriétés (10) et (11).

2. Lemmes abstraits

(1) *Matrices à décroissance rapide*

L'emploi de bases inconditionnelles conduit à remplacer les opérateurs par leurs matrices infinies. Celles-ci appartiendront à une algèbre de matrices que nous décrivons.

Soit (T, d) un espace métrique vérifiant les deux propriétés suivantes:

(13) $$\forall (t, t') \in T^2, \quad t \neq t' \qquad d(t, t') \geqslant 1.$$

(14) $$\forall R \geqslant 1, \quad \forall \tau \in T \qquad \text{Card}\,\{t : d(t, \tau) \leqslant R\} \leqslant CR^{\nu}$$

où C et ν ne dépendent que de T (T est donc nécessairement dénombrable).

Soit $\mathfrak{M}$ l'espace vectoriel des matrices complexes infinies $\alpha(t, t')$, $t \in T$, $t' \in T$ telles que $\sup_{t,t'} |\alpha(t, t')| < \infty$.

Soit $\mathfrak{M}_1$ le sous-espace vectoriel de $\mathfrak{M}$ constitué des matrices $\mathfrak{M} = (\alpha(t, t'))$ telles que

(15) $$\forall N \in \mathbb{N}^* \quad \exists C_N \geqslant 0 \qquad |\alpha(t, t')| \leqslant C_N(1 + d(t, t'))^{-N}.$$

Le résultat suivant est dû à S. Jaffard et Y. Meyer (non publié).

Lemme 1.

(i) $\mathfrak{M}_1$ *est une sous-algèbre de* $\mathcal{L}(l^2(T))$.
(ii) *soit* $M \in \mathfrak{M}_1$ *un isomorphisme de* $l^2(T)$ *sur* $l^2(T)$.

Alors $M^{-1} \in \mathfrak{M}_1$.

PREUVE.

(i) $\mathfrak{M}_1 \subset \mathcal{L}(l^2(T))$ s'obtient par le lemme de Schur. Il s'agit de vérifier

$$\sup_t \sum_{t'} |\alpha(t, t')| < \infty$$

et

$$\sup_{t'} \sum_t |\alpha(t, t')| < \infty.$$

Les deux inégalités se traitant de la même façon, nous examinons la première.

Soit t arbitraire et $N \geqslant 1 + \nu$. D'après (13) et (14) il vient

$$\begin{aligned}\sum_{t'} |\alpha(t, t')| &\leqslant \sum_{j \geq 0} \sum_{2^j \leq d(t, t') \leq 2^{j+1}} |\alpha(t, t')| \\ &\leqslant \sum_{j \geq 0} C_N (1 + 2^j)^{-N} 2^{j\nu} \\ &\leqslant C_N \sum_{j \geq 0} 2^{j(\nu - N)} < \infty.\end{aligned}$$

Soient M_1 et $M_2 \in \mathfrak{M}_1$. Montrons que $M_1 M_2 \in \mathfrak{M}_1$. Avec des notations évidentes, les coefficients de $M_1 M_2$ s'écrivent

$$\alpha(t, t'') = \sum_{t'} \alpha_1(t, t') \alpha_2(t', t'').$$

Posons $d(t, t'') = 2R$, alors, soit $d(t, t') \geqslant R$, soit $d(t', t'') \geqslant R$. Soit alors $N \geqslant 1 + \nu$. En utilisant (15) et (14) il vient

$$\begin{aligned}|\alpha(t, t'')| &\leqslant C_N \sum_{t'} (1 + d(t, t'))^{-N} (1 + d(t', t''))^{-N} \\ &\leqslant C_N \Big(\sum_{d(t, t') \geq R} (1 + d(t, t'))^{-N} + \sum_{d(t, t') \geq R} (1 + d(t, t'))^{-N} \Big) \\ &\leqslant C_{N, \nu} (1 + R)^{\nu - N}.\end{aligned}$$

(ii) Soit $\tau \in T$. Appelons X_τ l'opérateur non borné défini sur $l^2(T)$ par $(c_t) \to (d(t, \tau) c_t)$. Si $M \in \mathfrak{M}$, nous désignons par $[M, X_\tau]$ le commutateur $M X_\tau - X_\tau M$ lorsqu'il a un sens. Ensuite, si $M_0 \in \mathfrak{M}$, nous définissons une suite de matrices par $M_{n+1, \tau} = [M_{\eta, \tau}, X_\tau]$ et $M_{0, \tau} = M_0$. On voit facilement que

$$M_0 \in \mathfrak{M}_1 \Leftrightarrow \forall n \in \mathbb{N}^*, \qquad \sup |||M_{n, \tau}||| < \infty$$

où $|||M_{n, \tau}|||$ est la norme d'opérateurs de $M_{n, \tau}$ agissant sur $l^2(T)$. (Nous adopterons cette notation par la suite.)

En effet, les coefficients de M_0 étant $\alpha(t, t')$, ceux de $M_{n, \tau}$ s'écrivent $(d(t', \tau) - d(\tau, t))^n \alpha(t, t')$ et il suffit d'employer à nouveau le lemme de Schur.

Supposons maintenant $M_0 : l^2(T) \to l^2(T)$ inversible et posons $A_0 = M_0^{-1}$. On a alors

$$A_{1, \tau} = [A_0, X_\tau] = -A_0 [M_0, X_\tau] A_0 = -A_0 M_{1, \tau} A_0.$$

En utilisant les règles de calcul sur les commutateurs, il s'ensuit que $A_{n+1, \tau}$ s'écrit comme une fonction polynômiale de $A_0, A_{1, \tau}, \ldots, A_{n, \tau}, M_{1, \tau}, \ldots, M_{n, \tau}$ dont les coefficients ne dépendent pas du choix de τ. Une simple récurrence montre que, si $n \geqslant 1$, $A_{n,\tau}$ est opérateur borné sur $l^2(T)$ dont la norme est majorée uniformément en τ. Par conséquent $A_0 \in \mathfrak{M}_1$.

(2) *Formes pseudoaccrétives.*

Dans cette section, V et H désigneront deux espaces de Hilbert séparables sur $\mathbb{C}$.

Définition 6. *Soit β une forme sesquilinaire continue sur $V \times V$. Nous dirons que β est pseudoaccrétive sur $V \times V$ s'il existe une constante $\delta_0 > 0$ telle que l'on ait*

$$\sup_{\|v'\| \leq 1} |\beta(v, v')| \geqslant \delta_0 \|v\| \quad \textit{pour tout} \quad v \in V, \tag{16}$$

et

$$\sup_{\|v\| \leq 1} |\beta(v, v')| \geqslant \delta_0 \|v'\| \quad \textit{pour tout} \quad v' \in V. \tag{17}$$

Adoptons dès à présent la notation suivante: si β est une forme sesquilinéaire sur $V \times V$, on appellera β^* la forme adjointe donnée par:

$$\beta^*(v, v') = \overline{\beta(v', v)}, \qquad v', v' \in V.$$

Enfin les produits scalaires canoniques sur V et sur H sont notés $(\bullet|\bullet)_V$, $(\bullet|\bullet)_H$ ou plus simplement $(\bullet|\bullet)$.

Démontrons maintenant plusieurs résultats sur les formes pseudoaccrétives. Ceux-ci seront utilisés au cours de la partie 4.

Lemme 2. *Soit (v_t) une base inconditionnelle de V (t parcourant l'ensemble T). Soit β une forme sesquilinéaire continue sur $V \times V$ et M la matrice infinie de coefficients $\beta(v_s, v_t)$, $s, t \in T$. Alors β est pseudoaccrétive sur $V \times V$ si, et seulement si M est inversible sur $l^2(T)$.*

Supposons maintenant β pseudoaccrétive sur $V \times V$. Alors

(i) $$\|M^{-1}\| \leqslant C\delta_0^{-1}$$

où C ne dépend que de la base (v_t) et δ_0 est la constante figurant dans (16) et (17).

(ii) *Appelons $\alpha(s, t)$ les coefficients de M^{-1} et $\tilde{v}_s$ les vecteurs de V définis par*

$$\tilde{v}_s = \sum_t \alpha(s, t) v_t, \qquad s \in T. \tag{18}$$

La famille $(\tilde{v}_s)$ forme une base inconditionnelle de V telle que

$$\beta(\tilde{v}_s, v_t) = \delta_{s,t} \quad \textit{pour tout} \quad s, t \in T. \tag{19}$$

PREUVE. L'opérateur $A: l^2(T) \to V$ défini par

$$A\lambda = \sum_t \lambda_t v_t = v, \qquad \lambda = (\lambda_t) \in l^2(T),$$

est un isomorphisme de $l^2(T)$ sur V. Si $(\cdot|\cdot)$ désigne le produit scalaire sur $l^2(T)$, on a

$$\beta(v, v') = \beta(A\lambda, A\lambda') = (M\lambda|\lambda') = (\lambda|M^*\lambda').$$

La pseudoaccrétivité de β signifie alors que M et M^* sont injectifs et d'image fermée dans $l^2(T)$; c'est-à-dire que M est inversible sur $l^2(T)$. Dans ce cas, on obtient

$$|||M^{-1}||| \leqslant \|A\| \, \|A^{-1}\| \delta_0^{-1},$$

ce qui démontre (i).

Pour démontrer (ii), on commence par remarquer que (18) peut se réécrire

$$\tilde{v}_s = AM^{-1}A^{-1}v_s.$$

$AM^{-1}A^{-1}$ étant un isomorphisme de V sur lui-même, les vecteurs $(\tilde{v}_s)$ forment une base inconditionnelle de V. On a ensuite

$$\beta(\tilde{v}_s, v_t) = \beta(AM^{-1}A^{-1}v_s, v_t) = (A^{-1}v_s \,|\, A^{-1}v_t) = \delta_{s,t}$$

par définition de A.

Lemme 3. *Soient $l: V \to C$ une forme linéaire continue et β une forme pseudoaccrétive sur $V \times V$. Alors il existe un unique vecteur $v_0 \in V$ tel que $l(v) = \beta(v, v_0)$ pour tout $v \in V$.*

PREUVE. Soit R l'unique opérateur linéaire continu sur V défini par

$$(Rv \,|\, v') = \beta(v, v') \qquad v, v' \in V.$$

Les conditions (16) et (17) impliquent l'inversibilité de R sur V. Le théorème de représentation de Riesz nous donne ensuite un unique vecteur $w_0 \in V$ tel que $l(v) = (v \,|\, w_0)$ pout tout $v \in V$. Posons alors $v_0 = R^{*-1}w_0$. Il est immédiat de vérifier que v_0 convient.

Proposition 1.

(i) *Supposons $V \subset H$. Soit β une forme pseudoaccrétive sur $H \times H$ dont la restriction à $V \times V$ est encore pseudoaccrétive.*

Si nous introduisons

$$X = \{\theta \in H: \forall v \in V, \beta(v, \theta) = 0\},$$

alors H est la somme directe de V et X.

Soient ensuite $\pi_x: H \to X$ l'opérateur de projection sur X parallèlement à V, et W l'espace orthogonal à V dans H. Alors la restriction de π_x à W induit un isomorphisme de W sur X.

(ii) *Supposons V muni d'une base inconditionnelle (v_t) et appelons $(\tilde{v}_t)$ la base construite dans le Lemme* 2. *Alors*

$$\forall h \in H, \qquad \pi_x(h) = h - \sum_t \beta^*(h, \tilde{v}_t) v_t. \tag{20}$$

(iii) *Posons*

$$X^* = \{\theta^* \in H: \forall v \in V, \beta'(v, \theta^*) = 0\}.$$

En remplaçant β par β on obtient les mêmes résultats. De plus, la restriction de β à $X^* \times X$ satisfait des relations analogues à* (16) *et* (17).

PREUVE.

(i) Le fait que H soit la somme directe de V et X est une conséquence facile de la pseudoaccrétivité de β sur $V \times V$ et du Lemme 3.

Montrons maintenant que π_x est un isomorphisme de W sur X. Soit $\theta \in X$, cherchons $w \in W$ tel que $\pi_x(w) = \theta$. Ecrivons pour cela $\theta = v + w$ où $v \in V$ et $w \in W$. Alors $\theta = \pi_x(\theta) = \pi_x(v) + \pi_x(w) = \pi_x(w)$ par définition de π_x. Ensuite, pour montrer que π_x est injectif, on choisit $w \in W$ et l'on écrit

$$\|w\|^2 = (w \mid w) = (w \mid \pi_x(w)) \leqslant \|w\| \, \|\pi_x(w)\|,$$

la deuxième égalité provenant du fait que V et W sont orthogonaux. Donc $\|\pi_x w\| \geqslant \|w\|$ pour tout $w \in W$.

(ii) Commençons par remarquer que comme corollaire du Lemme 2, on a

$$\forall v \in V \qquad v = \sum_t \overline{\beta(v, \tilde{v}_t)} v_t = \sum_t \beta'(v, \tilde{v}_t) v_t.$$

Soit $\pi_v = I - \pi_x$ la projection sur V parallèlement à X. Alors, si $h \in H$,

$$\pi_v(h) = \sum_t \beta^*(\pi_v(h), \tilde{v}_t) v_t = \sum_t \beta^*(h, \tilde{v}_t) v_t,$$

ce qui démontre (20).

(iii) Regardons la restriction de β à $X^* \times X$. Fixons $\theta^* \in X^*$. D'après (16), il existe $h \in H$, $\|h\| \leqslant 1$, tel que

$$\beta(\theta^*, h) \geqslant \delta_0 \|\theta^*\|.$$

On écrit ensuite $h = v + \theta$ où $v \in V$ et $\theta \in X$ et il vient $\beta(\theta', h) = \beta(\theta^*, \theta)$ et $\|\theta\| \leqslant \|\pi_x h\| \leqslant \|\pi_x\| \, \|h\| \leqslant C$. On a donc

$$\sup_{\theta \in X, \|\theta\| \leq 1} |\beta(\theta^*, \theta)| \geqslant C^{-1}\delta_0 \|\theta^*\|.$$

L'autre inégalité s'obtient de façon symétrique.

2. Construction d'une Analyse Multirésolution Adéquate

Il est temps pour nous d'abandonner ces notions abstraites et de rentrer dans le vif de la démonstration du Théorème 1. Le cadre fonctionnel sera l'espace $L^2(\mathbb{R})$. Il conviendra de prendre $d = 1$ et de remplacer les cubes dyadiques Q par les intervalles dyadiques $I = I_{j,k} = [k2^{-j}, (k+1)2^{-j}[$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}$, dans les définitions de la partie 1. La collection de tous les intervalles dyadiques sera notée $\mathfrak{I}$.

Dans toute cette partie, on désignera par $b(x)$ une fonction pseudoaccrétive de constante δ_0. Nous lui associerons une forme sesquilinéaire continue sur $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, notée β, par

$$(21) \qquad \beta(f, f') = \int_{\mathbb{R}} f(x) b(x) \overline{f'(x)}\, dx \qquad f \in L^2(\mathbb{R}), \qquad f' \in L^2(\mathbb{R}).$$

Notons que β^* est alors associée à $\bar{b}\,(x)$.

Théorème 2. *Il existe un $r \in]0, 1[$ et une analyse multirésolution r-régulière de $L^2(\mathbb{R})$, notée $V_j, j \in \mathbb{Z}$, telle que, pour tout $j \in \mathbb{Z}$, la forme β restreinte à $V_j \times V_j$ soit pseudoaccrétive de constante $\delta_0/2$.*

Pour le confort du lecteur, nous allons rappeler ici la définition d'une analyse multirésolution r-régulière de $L^2(\mathbb{R})$ et les résultats que nous utiliserons. De plus amples développements sur ces analyses peuvent être trouvés dans [Ma] ou [M1].

Définition 7. *Une analyse multirésolution de $L^2(\mathbb{R})$ est une suite de sous-espaces fermés de $L^2(\mathbb{R})$, notée $V_j, j \in \mathbb{Z}$, obéissant aux conditions suivantes:*

(22) $V_j \subset V_{j+1}$ *pour tout* $j \in \mathbb{Z}$.

(23) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ *et* $\bigcup_{j\in\mathbb{Z}} V_j$ *est dense dans* $L^2(\mathbb{R})$.

(24) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$ *pour tout* $j \in \mathbb{Z}$.

(25) $f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0$ *pour tout* $k \in \mathbb{Z}$.

(26) *Il existe une fonction* $g \in V_0$ *telle que la famille de vecteurs* $g(x-k)$, $k \in \mathbb{Z}$, *forme une base inconditionnelle de* V_0.

On dira, sous ces hypothèses, que g engendre l'analyse multirésolution $V_j, j \in \mathbb{Z}$.

Celle-ci sera r-régulière si l'on peut choisir g r-régulière, c'est-à-dire si pour tout $N \in \mathbb{N}^*$ *la fonction* $(1+|x|)^N g(x)$ *appartient à* $C^r(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Rappelons que $C^r(\mathbb{R})$ est l'espace de Hölder homogène d'exposant r: $f \in C^r(\mathbb{R})$ si

$$\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^r} < \infty.$$

(Pour nous, r sera toujours pris dans $]0, 1[$.)

On peut exprimer de façon équivalente la r-régularité de g à l'aide des inégalités (6) et (7) où $d = 1$ et $Q = [0, 1[$ (*i.e.* $j = 0$, $k = 0$).

Prenons un exemple qui, pour nous sera fondamental. Soit $j \in \mathbb{Z}$. Soit V_j le sous-espace fermé de $L^2(\mathbb{R})$ constitué des fonctions en escalier sur les intervalles dyadiques $I_{j,k}$, $k \in \mathbb{Z}$. On vérifie facilement que $V_j, j \in \mathbb{Z}$, est une analyse multirésolution de $L^2(\mathbb{R})$ engendrée, par exemple, par $g(x) = \chi_{[0,1[}(x)$ (fonction indicatrice de $[0, 1[$). Cette analyse n'est par r-régulière quel que soit $r > 0$ puisque, excepté la fonction nulle, les fonctions de V_0 ne sont pas continues.

Voyons pourquoi elle est fondamentale en regard du Théorème 2. Fixons $j \in \mathbb{Z}$, examinons la restriction de β à $V_j \times V_j$. Posons pour cela $g_{j,k}(x) = 2^{j/2} g(2^j x - k)$, $k \in \mathbb{Z}$. D'après les règles (24) et (26), cette famille est une base inconditionnelle de V_j (et même orthonormée). Formons alors la matrice M_j de coefficients $\beta(g_{j,k}, g_{j,l})$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$. On voit facilement que

$$\text{si} \quad k \neq 1, \qquad \beta(g_{j,k}, g_{j,l}) = 0$$
$$\text{et si} \quad k = 1 \qquad \beta(g_{j,k}, g_{j,k}) = m_I b \quad \text{où} \quad I = I_{j,k}.$$

M_j est donc diagonale et, d'après la relation (4), ses coefficients diagonaux sont en module minorés par δ_0. M_j est donc inversible et, d'après le Lemme 2, β restreinte à $V_j \times V_j$ est pseudoaccrétive, la constante de pseudoaccrétivité étant même δ_0.

Nous aurions conclu le Théorème 2 si g avait été r-régulière. Pour démontrer ce théorème, notre tâche sera alors de modifier la situation précédente en lissant convenablement la fonction g par un argument de perturbation.

Tout d'abord, rappelons un résultat de S. Mallat [Ma], que nous avons légèrement amélioré pour la circonstance. On désignera par $\hat{f}(\xi)$ la transformée de Fourier de $f(x)$ donnée par

$$\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

Théorème. [Ma] *Soit $m(\xi)$ une fonction $C^\infty(\mathbb{R})$ 2π-périodique telle que $m(0) = 1$. Posons*

$$G(\xi) = \prod_{j \geq 1} m(\xi 2^{-j}).$$

(i) *On suppose que $m(\xi)$ ne s'annule pas sur $[-\pi/2, \pi/2]$ et que $|m(\xi)|^2 + |m(\xi + \pi)|^2 \leqslant 1$. Alors g définie par $\hat{g}(\xi) = G(\xi)$ appartient à $L^2(\mathbb{R})$ et la collection des vecteurs $g(x - k)$, $k \in \mathbb{Z}$ est une famille inconditionnelle de $L^2(\mathbb{R})$.*

(ii) *On suppose de plus, qu'il existe $G_1(\xi)$, $r \in]0, 1[$ et $C \geqslant 0$ tels que*

$$G(\xi) = \frac{\sin \xi/2}{\xi/2} G_1(\xi)$$

avec

$$|G_1(\xi)| \leqslant C(1 + |\xi|)^{-r}; \tag{27}$$

alors g est r-régulière et engendre une analyse multirésolution r-régulière de $L^2(\mathbb{R})$.

Pour obtenir l'estimation (27), on utilisera un résultat dû à Y. Meyer [M2].

Lemme. [M2] *Soit $q(\xi)$ une fonction C^∞ 2π-périodique et à valeurs dans $[0, 1]$. On suppose qu'il existe deux constantes $a \in]0, 1[$ et $b \geqslant 1$ telles que l'on ait*

$$0 \leqslant q(\xi) \leqslant a \quad si \quad \frac{2\pi}{3} \leqslant \xi \leqslant \frac{4\pi}{3},$$

$$|q(\xi)| \leqslant \frac{b}{2\pi} |\xi - \pi|$$

et

$$q(0) = 1.$$

Alors la fonction $\omega(\xi) = \prod_{j \geq 1} q(2^{-j}\xi)$ *est de classe* C^∞ *et il existe* $a \in]0, 1[$, $\alpha = \alpha(a, b)$, *et* $C \geqslant 0$ *tels que*

$$|\omega(\xi)| \leqslant C(1 + |\xi|)^{-\alpha}.$$

Ces résultats étant rappelés passons à la démonstration du Théorème 2. Pour commencer, nous fixons l'indice j égal à 0; les estimations que nous obtiendrons seront uniformes en j. La fonction $\chi_{[0,1[}(x)$ sera notée $G^{(0)}(x)$, l'espace V_0 associé, $V^{(0)}$, et les fonctions $g^{(0)}(x-k)$, $g_k^{(0)}(x)$.

Posons $q(\xi) = \exp[-\log^2 \cos^2 \xi/2]$ et $\omega(\xi) = \prod_{j \geq 1} q(\xi 2^{-j})$. Il est immédiat de vérifier les hypothèses du lemme précédent.

Ensuite, soit $\epsilon > 0$, posons $m_\epsilon(\xi) = e^{-i\xi/2} q(\xi)^\epsilon \cos \xi/2$. On définit une fonction $g^{(\epsilon)}(x)$ dans $L^2(\mathbb{R})$ par sa transformée de Fourier en posant

$$\hat{g}^{(\epsilon)}(\xi) = \prod_{j \geq 1} m_\epsilon(2^{-j}\xi) = e^{-i\xi/2} \frac{\sin \xi/2}{\xi/2} \omega(\xi)^\epsilon = \hat{g}^{(0)}(\xi)\omega(\xi)^\epsilon.$$

Remarquons que $q(\xi)$ est à valeurs dans $[0, 1]$, s'annule au point π (modulo 2π), est plate en ce point, donc $m_\epsilon(\xi) \in C^\infty(\mathbb{R})$, et l'on vérifie aisément les conditions du théorème de S. Mallat.

On appellera $V_j(\epsilon)$, $j \in \mathbb{Z}$, l'annalyse multirésolution ainsi construite. Elle est $\alpha\epsilon$-régulière, α étant donné par le lemme d'Y. Meyer.

Revenons à $j = 0$ et posons $V_0^{(\epsilon)} = V^{(\epsilon)}$. La famille de vecteurs $g_k^{(\epsilon)}(x)$, $k \in \mathbb{Z}$, est une base inconditionnelle de $V^{(\epsilon)}$. Formons alors la matrice $M^{(\epsilon)}$ de coefficients $\beta(g_k^{(\epsilon)}, g_l^{(\epsilon)})$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$. Suivant le Lemme 2, pour montrer que β restreinte à $V^{(\epsilon)} \times V^{(\epsilon)}$ est pseudoaccrétive, il suffit de montrer que $M^{(\epsilon)}$ est inversible sur $l^2(\mathbb{Z})$. Ceci découlera du lemme suivant.

Lemme 4. *Il existe une constante* $C \geqslant 0$ *telle que pour tout* $\epsilon \geqslant 0$

$$|||M^{(0)} - M^{(\epsilon)}||| \leqslant C\epsilon.$$

Admettons un instant ce lemme et terminons la démonstration du Théorème 2. Puisque $M^{(0)}$ est inversible, $M^{(\epsilon)}$ est inversible pour ϵ choisi assez petit et, par suite, β restreinte à $V^{(\epsilon)} \times V^{(\epsilon)}$ est pseudoaccrétive. Il est alors facile de voir que la constante de pseudoaccrétivité, que nous notons $\delta^{(\epsilon)}$, de β restreinte à $V^{(\epsilon)} \times V^{(\epsilon)}$ dépend continuement de ϵ. Valant δ_0 pour $\epsilon = 0$, on choisit définitivement ϵ tel que $\delta^{(\epsilon)} \geqslant \delta_0/2$. Ceci achève la preuve du Théorème 2.

PREUVE DU LEMME 4. On définit une fonction $h^{(\epsilon)}(x)$ par sa transformée de Fourier en posant

$$\hat{h}^{(\epsilon)}(\xi) = \frac{\partial}{\partial \epsilon} \hat{g}^{(\epsilon)}(\xi) = \log \omega(\xi) \cdot \hat{g}^{(\epsilon)}(\xi).$$

Une estimation grossière de $\log \omega(\xi)$ nous donne pour tout ξ

$$|\log \omega(\xi)| \leqslant C(1 + |\log^3 |\xi|| + |\log |\xi| \log^2 (\sin \xi/2)|.$$

La preuve en est laissée au lecteur. On obtient alors les estimations

$$|\hat{h}^{(\epsilon)}(\xi)|^2 \leqslant C(\alpha) \left| \frac{\sin \xi/2}{\xi/2} \right|^{1+\alpha} \tag{28}$$

et

$$|\hat{h}^{(\epsilon)}(\xi) - \hat{h}^{(\zeta)}(\xi)|^2 \leqslant C(\alpha)|\epsilon - \zeta|^2 \left| \frac{\sin \xi/2}{\xi/2} \right|^{1+\alpha},$$

pour tout $\xi \in \mathbb{R}$, $\epsilon \geqslant 0$, $\zeta \geqslant 0$ et $\alpha \in]0, 1[$. La fonction $\zeta \to \hat{h}^{(\zeta)}$ définit alors un chemin continu dans $L^2(\mathbb{R})$ et l'intégrale $\int_0^\epsilon \hat{h}^{(\zeta)}\, d\zeta$ est convergente dans $L^2(\mathbb{R})$. Sa valeur est donnée par

$$\int_0^\epsilon \hat{h}^{(\zeta)}\, d\zeta = \hat{g}^{(\epsilon)} - \hat{g}^{(0)}, \qquad \epsilon \geqslant 0.$$

La transformée de Fourier inverse et les translations étant continues sur $L^2(\mathbb{R})$, on obtient ensuite:

$$\hat{g}_k(\epsilon) - \hat{g}_k(0) = \int_0^\epsilon \hat{h}_k^{(\zeta)}\, d\zeta$$

pour tout $k \in \mathbb{Z}$, $\epsilon \geqslant 0$.

Admettons un instant les inégalités suivantes: il existe $C \geqslant 0$ telle que

$$\forall \epsilon \geqslant 0 \qquad \Big\| \sum_k \lambda_k h_k^{(\epsilon)} \Big\|^2 \leqslant C \sum_k |\lambda_k|^2 \tag{29}$$

$$\forall \epsilon \geqslant 0 \qquad \Big\| \sum_k \lambda_k g_k^{(\epsilon)} \Big\|^2 \leqslant C \sum_k |\lambda_k|^2, \tag{30}$$

la norme étant ici celle de $L^2(\mathbb{R})$.

Soient (λ_k) et (μ_k) deux suites finies de nombres complexes. On calcule alors $([M^{(0)} - M^{(\epsilon)}](\lambda_k) \mid (\mu_k)) = A(\epsilon)$:

$$\begin{aligned} |A(\epsilon)| &= \Big| \sum_k \sum_l \lambda_k \overline{\mu_l} [\beta(g_k^{(\epsilon)}, g_l^{(\epsilon)})] \Big| \\ &\leqslant \int_0^\epsilon \Big| \beta\Big(\sum_k \lambda_k h_k^{(\xi)}, \sum_l \mu_l g_l^{(\epsilon)} \Big) \Big| + \Big| \beta\Big(\sum_k \lambda_k g_k^{(0)}, \sum_l \mu_l h_l^{(\zeta)} \Big) \Big|\, d\zeta. \end{aligned}$$

On a utilisé pour cela la sesquilinéarité et la continuité de β. Donc

$$A(\epsilon) \leqslant C\epsilon \Big(\sum_k |\lambda_k|^2 \Big)^{1/2} \Big(\sum_l |\mu_l|^2 \Big)^{1/2}$$

d'après (29) et 30). Cette inegalité nous donne donc celle désirée.
Passons à la démonstration de (29).

$$\Big\|\sum_k \lambda_k h_k^{(\epsilon)}(x)\Big\|^2 = \frac{1}{2\pi}\Big\|\sum_k \lambda_k \hat{h}_k^{(\epsilon)}\Big\|^2 = \frac{1}{2\pi}\Big\|\Big(\sum_k \lambda_k e^{-ik\xi}\Big)\hat{h}^{(\epsilon)}(\xi)\Big\|^2.$$

Posons

$$m(\xi) = \sum_k \lambda_k e^{-ik\xi}.$$

On a alors

$$\Big\|\sum_k \lambda_k h_k^{(\epsilon)}(x)\Big\|^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |m(\xi)|^2 |\hat{h}^{(\epsilon)}(\xi)|^2 \, d\xi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} |m(\xi)|^2 \sum_k |\hat{h}^{(\epsilon)}(\xi + 2k\pi)|^2 \, d\xi.$$

Comme

$$\frac{1}{2\pi}\int_0^{2\pi} |m(\xi)|^2 \, d\xi = \sum_k |\lambda_k|^2,$$

il suffit de montrer

$$\sum_k |\hat{h}^{(\epsilon)}(\xi + 2k\pi)|^2 \leqslant C$$

uniformément en ϵ et cela découle aisément de l'inégalité (28).

Pour démontrer (30), on applique le même raisonnement et l'on remarque que, pour tout $\epsilon \geqslant 0$, $|\hat{g}^{(\epsilon)}(\xi)| \leqslant |\hat{g}^{(0)}(\xi)|$.

Nous avons donc terminé la preuve du Théorème 2. Nous abandonnons désormais la notation de ϵ en exposant. Celui-ci étant fixé, on désignera par V_j l'espace $V^{(\epsilon)}$ et $g^{(\epsilon)}(x)$ devient $g(x)$. Enfin, on posera $r = \alpha\epsilon$ qui est la régularité de $g(x)$.

4. La contruction des *b*-vaguelettes

Dans cette partie j est fixé égal à 0, sauf mention du contraire. Nous noterons V pour V_0 et H pour V_1. Suivant S. Mallat [Ma], si nous appelons W l'espace orthogonal à V dans H, il existe dans W une fonction $\psi(x)$ r-régulière telle que la collection des vecteurs $\psi(x-k)$, $k \in \mathbb{Z}$, forme une base orthonormée de W. De plus, $\psi(x)$ vérifie la relation

$$\int_{\mathbb{R}} \psi(x)\, dx = 0. \tag{31}$$

Remarquons que, puisque $g(x)$ est à valeurs réelles, l'algorithme constructif de S. Mallat permet de choisir $\psi(x)$ à valeurs réelles. Notons enfin que si W_j est l'espace orthogonal à V_j dans V_{j+1}, alors une base orthonormée de W_j est donnée par la famille $2^{j/2}\psi(2^jx - k)$, $k \in \mathbb{Z}$. De sorte que la collection $2^{j/2}\psi(2^jx - k), j, k \in \mathbb{Z}$ est une base orthonormée de $L^2(\mathbb{R})$ appelée base d'ondelettes de $L^2(\mathbb{R})$.

Construisons maintenant les b-vaguelettes. On note $\psi_k(x)$ la fonction $\psi(x - k)$. Soient X et X^* les deux sous-espaces de H introduits dans la proposition 1 et Π_X et Π_{X^*} les deux projecteurs associés. On définit deux suites de fonctions $(\theta_k^\#)$ et (θ_k^*) par

$$\theta_k^\#(x) = \Pi_X(\psi_k)(x) \in X, \qquad k \in \mathbb{Z} \tag{32}$$

et

$$\theta_k^*(x) = \Pi_{X^*}(\psi_k)(x) \in X^*, \qquad k \in \mathbb{Z}. \tag{33}$$

Image d'une base othornormée par un isomorphisme, $(\theta_k^\#)$ (resp. (θ_k^*)) est une base inconditionnelle de X (resp. X^*). Cherchons une famille, que nous noterons $(\theta_k^\sim)$, de vecteurs de X^* vérifiant

$$\beta(\theta_k^\sim, \theta_l^\#) = \delta_{k,l} \quad \text{pour tout} \quad (k, l) \in \mathbb{Z}^2. \tag{34}$$

On procède de la façon suivante. D'après le (iii) de la Proposition 1, β restreinte à $X^* \times X$ est (en étendant la définition 6) pseudoaccrétive. Une imitation du Lemme 2 (i) montre que la matrice N de coefficients $\beta(\theta_k^*, \theta_l^\#)$, $k, l \in \mathbb{Z}$, est inversible sur $l^2(\mathbb{Z})$. Suivant le Lemme 2 (ii), on appelle $\gamma(k, l)$ les coefficients de N^{-1} et l'on pose

$$\theta_k^\sim = \sum_l \gamma(k, l)\theta_k^*, \qquad k \in \mathbb{Z} \tag{35}$$

$(\theta_k^\sim)$ est alors la famille recherchée.

Nous allons maintenant nous intéresser aux estimations vérifiées par ces deux familles. Si $k \in \mathbb{Z}$, on a par exemple, d'après la formule (20) de la Proposition 1

$$\theta_k^\# = \psi_k - \sum_l \beta^*(\psi_k, g_l^\sim)g_l \tag{36}$$

où la fonction $g_l^\sim$ est définite par la relation (18) du Lemme 2 appliqué dans l'espace V aux vecteurs (g_l). La fonction $g(x)$ étant r-régulière, on voit facilement que la matrice $(\beta(g_k, g_l))$ appartient à l'algèbre $\mathfrak{M}_1$ introduite dans le Lemme 1: T est ici $\mathbb{Z}$ muni de la distance $d(k, l) = |k - l|$. Par suite, $M^{-1} \in \mathfrak{M}_1$ et la fonction $g_l^\sim(x)$ s'écrit $h_l(x - l)$ où h_l est à décroissance rapide en x à l'infini. La fonction h_l est en fait r-régulière. Ensuite, grâce à (36), on voit que $\theta_k^{\#}(x)$ s'écrit $u_k(x-k)$ où $u_k(x)$ est r-régulière. Le même raisonnement s'applique aux fonctions θ_k^*, puis on utilise à nouveau le Lemme 1 et (35) pour voir que $\theta_k^{\sim}(x+k)$ est r-régulière. Notons pour finir que les constantes obtenues dans les majorations sont uniformes par rapport à $k \in \mathbb{Z}$.

Montrons maintenant que

$$\int_{\mathbb{R}} b(x)\overline{\theta_k^{\#}(x)}\,dx = 0,$$

c'est-à-dire $\beta(1, \theta_k^{\#}) = 0$ où 1 désigne la fonction identiquement égale à 1.

On a $\beta(g_l, \theta_k^{\#}) = 0$ pour tout $l \in \mathbb{Z}$ par définition de X. On peut alors remarquer que $\sum_l g_l(x) = \sum_l g(x-l) = 1$. En effet, il suffit d'appliquer la formule sommatoire de Poisson et d'observer que $\hat{g}(2k\pi) = 0$ pour $k \neq 0$ tandis que $\hat{g}(0) = 1$. Comme $b\overline{\theta_k^{\#}} \in L^1(\mathbb{R})$, on somme sur tous les $l \in \mathbb{Z}$ et par convergence dominée on obtient $\beta(1, \theta_k^{\#}) = 0$.

De façon symétrique, on obtient $\beta^*(1, \theta_k^*) = 0$, puis en appliquant (35) et la convergence dominée il vient $\beta^*(1, \theta_k^{\sim}) = 0$, c'est-à-dire

$$\int_{\mathbb{R}} \theta_k^{\sim}(x) b(x)\,dx = 0.$$

Pour terminer cette partie, laissons varier l'indice j dans $\mathbb{Z}$. On obtient alors deux familles de fonctions $(\theta_{j,k}^{\#})$ et $(\theta_{j,k}^{\sim})$ de $L^2(\mathbb{R})$. Identifiant l'intervalle dyadique $I = I_{j,k}$ avec le couple (j, k), on notera $\theta_I^{\sim} = \theta_{j,k}^{\sim}$ et $\theta_I = \overline{\theta_{j,k}^{\#}}$. Il suit de la construction précédente que (θ_I) et $(\theta_I^{\sim})$ sont deux familles de b-vaguelettes r-régulières. Pour montrer (12), on prend $X_j = \{\theta \in V_{j+1} : \beta(v, \theta) = 0\ \forall v \in V_j\}$ et X_j^* en remplaçant β par β^*. On observe facilement que si $j \neq j'$ alors $\beta(X_j^*, X_j) = 0$, c'est-à-dire que les deux espaces sont orthogonaux par rapport à la forme β. Avec la relation (34), on obtient la relation (12).

5. Estimation Quadratique

Nous allons montrer que (θ_I) est une base inconditionnelle de $L^2(\mathbb{R})$. Pour ce faire, rappelons que (ψ_I) est une base orthonormée de $L^2(\mathbb{R})$. Nous démontrons le théorème suivant:

Théorème 3. *L'opérateur défini par linéarité par*

$$T(\psi_I) = \theta_I, \qquad I \in \mathcal{I}$$

est un isomorphisme de $L^2(\mathbb{R})$ sur lui-même.

Une fois ce théorème démontré, la preuve du Théorème 1 sera complètement achevée.

Commençons la preuve du Théorème 3 par le résultat suivant:

Proposition 2. *La collection des vecteurs θ_I est totale dans $L^2(\mathbb{R})$.*

PREUVE. Il suffit de montrer que la somme algébrique des espaces $X_j, X_{j-1}, X_{j-2}, \ldots$ est dense dans V_{j+1} pour tout $j \in \mathbb{Z}$. Appelons alors P_j le projecteur de $L^2(\mathbb{R})$ sur V_j parallèlement à $\{f \in L^2(\mathbb{R}): \beta(v, f) = 0\ \forall v \in V_j\} = Y_j$. On peut alors appliquer la Proposition 1 et le Lemme 2 en posant $V_j = V$, $L^2(\mathbb{R}) = H$ (β est pseudoaccrétive sur $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ car $b(x)^{-1}$ est bornée) et $Y_j = X$. On a donc la formule

$$P_j(f) = f - \sum_l \beta^*(f, g_{j,l}^{\sim}) g_{j,l} \quad \text{pour toute} \quad f \in L^2(\mathbb{R}).$$

Avec (37) et les propriétés de décroissance des fonctions $g_{j,l}^{\sim}(x)$ et $g_{j,l}(x)$, on montre que $\lim_{N \to +\infty} \|P_{-N}(f)\| = 0$.

Prouvons maintenant la continuité de l'opérateur T sur $L^2(\mathbb{R})$. Pour ce faire, nous utiliserons des résultats classiques de la théorie des opérateurs de Calderón-Zygmund. Grâce à l'orthogonalité des fonctions ψ_I entre elles par rapport au produit scalaire (et parce qu'elles sont à valeurs réelles), on peut écrire le noyau distribution de T comme

$$K(x, y) = \sum_{I \in \mathcal{J}} \theta_I(x) \psi_I(y) \quad \text{pour tout} \quad x \neq y.$$

Un simple calcul montre que $K(x, y)$ est δ-standard pour tout $\delta \in]0, r[$ (r étant la régularité des ondelettes et des b-vaguelettes). Pour montrer que T est continu, nous appliquons le Théorème $T(1)$ de David-Journé. Nous renvoyons le lecteur aux références suivantes pour les définitions et résultats concernant ce théorème: [DJ], [L], [M3].

Il s'agit de vérifier que T est faiblement borné. Ceci peut se voir de la façon suivante: soit $f \in D(\mathbb{R})$ et $g \in D(\mathbb{R})$ telle que $\int_{\mathbb{R}} g(x)\, dx = 0$. On calcule $\langle f, Tg \rangle$ (le crochet désigne maintenant la dualité distributions-fonctions formellement notée par $\int_{\mathbb{R}} f(x) \overline{Tg(x)}\, dx$).

On a

$$|\langle f, Tg \rangle| = \left| \sum_{I \in \mathcal{J}} \langle f, \theta_I \rangle \overline{\langle g, \psi_I \rangle} \right|.$$

Si $I = I_{j,k}$ on a $|\langle f, \theta_I \rangle| \leqslant C \|f\|_\infty 2^{-j/2}$, donc

$$|\langle f, Tg \rangle| \leqslant C \|f\|_\infty \sum_{I \in \mathcal{J}} 2^{-j/2} |\langle g, \psi_I \rangle|.$$

Or cette série définit une norme équivalente à celle de $B_1^{0,1}(\mathbb{R})$ (voir [LM]) donc

$$|\langle f, Tg\rangle| \leqslant C\|f\|_\infty \|g\|_{B_1^{0,1}}.$$

Ceci prouve la continuité faible de T (voir [L]).

On vérifie ensuite que $T(1)$ et ${}^tT(1)$ sont dans BMO($\mathbb{R}$). Comme

$$\int_{\mathbb{R}} \psi_I(y)\,dy = 0,$$

et grâce à la décroissance des θ_I, on obtient $T(1) = 0$. Ensuite, calculons ${}^tT(1)$. On a

$${}^tT(1)(y) = \sum_{I\in\mathcal{I}} c_I \psi_I(y) \quad \text{où} \quad c_I = \int_{\mathbb{R}} \theta_I(y)\,dy.$$

Utilisant un résultat de [LM], on sait que ${}^tT(1) \in \mathrm{BMO}(\mathbb{R})$ si, et seulement si, il existe $C \geqslant 0$ telle que pout tout $J \in \mathcal{I}$

$$\sum_{I\subset J, I\in\mathcal{I}} |c_I|^2 \leqslant C|J|,$$

qui est la condition de Carleson.

Calculons $\overline{c_I}$. Pour cela, on revient à la notation $I = I_{j,k}$ et $\overline{\theta_I}(x) = \theta^{\#}_{j,k}(x)$.

D'après la formule (36), on a

$$\theta^{\#}_{j,k}(x) = \psi_{j,k}(x) - \sum_l \beta^*(\psi_{j,k} \tilde{g_{j,l}}) g_{j,l}(x).$$

Posons

$$m_j(x) = \sum_l \left[\int_{\mathbb{R}} g_{j,l}\right] \tilde{g_{j,l}}(x).$$

Comme

$$\int_{\mathbb{R}} \psi_{j,k}(x)\,dx = 0,$$

on a

$$\overline{c_I} = \int_{\mathbb{R}} \theta^{\#}_{j,k}(x)\,dx = \beta^*(\psi_{j,k}, \overline{m_j}) = \langle \bar{b} \mid \overline{\psi_{j,k} m_j}\rangle.$$

Posons $\omega_I(x) = \overline{\psi_{j,k}(x) m_j(x)}$. La collection (ω_I) est une famille de 1-vaguelettes r-régulières. La régularité s'obtient immédiatement. Pour montrer que

$$\int_{\mathbb{R}} \omega_I(x)\,dx = 0,$$

il suffit d'observer que $m_j(x)$ appartient à l'adhérence de V_j dans $L^\infty(\mathbb{R})$ et que $\psi_{j,k}(x) \in W_j \cap L^1(\mathbb{R})$. Grâce à l'orthogonalité entre V_j et W_j, on voit que l'intégrale de $\omega_I(x)$ est nulle par convergence dominée.

Enfin $\bar{b}(x) \in L^\infty(\mathbb{R}) \subset \mathrm{BMO}(\mathbb{R})$. On a alors classiquement

$$\sum_{I \subset J, I \in \mathcal{I}} |\langle \bar{b} \mid \omega_I \rangle|^2 \leqslant C|J|$$

qui est la condition de Carleson recherchée.

D'après le Théorème $T(1)$, T admet une extension linéaire bornée sur $L^2(\mathbb{R})$.

Passons à l'invertibilité.

Soit T' l'opérateur de la classe $\mathfrak{F}(\delta, 1, b)$, où $\delta < r$, dont le noyau-distribution vérifie

$$S(x, y) = K'(x, y) b(y) = \Big(\sum_{I \in \mathcal{I}} \psi_I(x) \theta_I^{\sim}(y)\Big) b(y) \quad \text{pour} \quad x \neq y.$$

On montre de manière analogue à ce qui précède que l'opérateur de Calderón-Zygmund de la classe $\mathfrak{F}$ $(\delta, 1, 1)$ de noyau $K'(x, y)$ est borné sur $L^2(\mathbb{R})$. Donc T' est borné sur $L^2(\mathbb{R})$ et l'on a:

$$T'T = I \quad \text{sur} \quad L^2(\mathbb{R}) \quad \text{et} \quad TT' = I \quad \text{sur} \quad F$$

où $F = \mathrm{Span}\,\{\theta_I : I \in \mathcal{I}\}$. F étant dense dans $L^2(\mathbb{R})$ et T' étant borné sur $L^2(\mathbb{R})$, la seconde égalité se prolonge par densité à tout $L^2(\mathbb{R})$. Ceci montre que T' inverse T sur $L^2(\mathbb{R})$.

Notation. L'opérateur qui envoie (ψ_I) sur (θ_I) sera noté désormais T_b et celui qui envoie (ψ_I) sur $(\theta_I^{\sim})$, $T_b^{\sim}$. On a donc

$$T_b^{-1} = {}^tT_b^{\sim} M_b \tag{37}$$

où M_b est l'opérateur de multiplication par $b(x)$. L'importance de cette formule dépasse largement, comme nous allons le voir, le cadre du Théorème 3.4.

Remarque. La démonstration aurait pu se traiter sans introduire à la fois les espaces X^* et X. Nous avons cependant choisi cette approche pour pouvoir étendre les résultats qui suivent au cas où le corps des complexes est remplacé par une algèbre de Clifford (Partie 10).

6. Le cas de la dimension *d*

Il nous faut rappeler comment on construit une analyse multirésolution de $L^2(\mathbb{R}^d)$ à l'aide de celles de $L^2(\mathbb{R})$. La définition d'une analyse multirésolution de $L^2(\mathbb{R}^d)$ est analogue à la définition 7 où les translations entières par $\mathbb{Z}$ sont changées en celles indexées par $\mathbb{Z}^d$.

Soit Λ l'ensemble des indices $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ où les λ_i sont non tous nuls et appartiennent à $\{0, 1\}$. On posera $\bar{\Lambda} = \Lambda \cup \{(0, 0, \ldots, 0)\}$.

Soit $V_j, j \in \mathbb{Z}$ une analyse multirésolution de $L^2(\mathbb{R})$. Soit $\lambda \in \Lambda$, fixons $j \in \mathbb{Z}$. On appelle E_j^λ le complété dans $L^2(\mathbb{R})$ du produit tensoriel algébrique $E_j^{\lambda_1} \otimes E_j^{\lambda_2} \otimes \cdots \otimes E_j^{\lambda_d}$ si $\lambda = (\lambda_1, \ldots, \lambda_d)$ et si $E_j^0 = V_j$ et $E_j^1 = W_j$. Appelons alors $V_j, j \in \mathbb{Z}$ l'espace $E_j^{(0,\ldots,0)}$. Il est immédiat de vérifier que $V_j, j \in \mathbb{Z}$, est une analyse multirésolution de $L^2(\mathbb{R}^d)$. Les espaces E_j^λ étant 2 à 2 orthogonaux, on apelle W_j la somme orthogonale des $E_j^\lambda, \lambda \in \Lambda$. Il est clair que W_j est l'espace orthogonal à V_j dans V_{j+1}.

Partant de la fonction $g^{(0)}(x) = g^{(0)}(x_1)g^{(0)}(x_2) \cdots g^{(0)}(x_d)$ on construit

$$g^{(\epsilon)}(x) = g^{(\epsilon)}(x_1)g^{(\epsilon)}(x_2) \cdots g^{(\epsilon)}(x_d)$$

et la démonstration se déroule de la même façon.

7. Le théorème de continuité $T(b)$ de David-Journé-Semmes

Le premier Théorème $T(b)$ a été prouvé par McIntosh et Meyer dans [MM]: il affirmait essentiellement que, si $T(b) = {}^tT(b) = 0$, avec b fonction bornée accrétive (c'est-à-dire telle que $\operatorname{Re} b \geqslant \delta_0$), alors T est borné sur $L^2(\mathbb{R})$.

L'énoncé le plus général a été ensuite donné par David, Journé et Semmes ([DJS]). En voici une version légèrement plus faible:

Théorème 4. *Soit $b_1(x)$ et $b_2(x)$ deux fonctions pseudoaccrétives sur $\mathbb{R}^d$. Soit T un opérateur de la classe $F(\delta, b_2, b_1)$. Sont équivalents*:

(i) *T admet une extension linéraire continue sur $L^2(\mathbb{R}^d)$*

(ii) *T faiblement borné (WBP), $T(1) \in M_{b_2}\mathrm{BMO}(\mathbb{R}^d)$, ${}^tT(1) \in M_{b_1}\mathrm{BMO}(\mathbb{R}^d)$.*

La condition de bornitude faible est la même que celle adoptée dans [DJS]. Ici $T(1)$ est définie comme forme linéaire continue sur l'espace noté $\{D(\mathbb{R}^d)b_2(x)\}_0$ des fonctions de $D(\mathbb{R}^d)$ telles que

$$\int_{\mathbb{R}} f(x)b_2(x)\,dx = 0$$

tandis que ${}^tT(1)$ est définie sur $\{D(\mathbb{R}^d)b_1(x)\}_0$. Pour expliquer cela, on peut écrire $T = M_{b_2}SM_{b_1}$ où S peut être défini dans la classe $\mathfrak{F}(\delta, 1, 1)$. Suivant un théorème de Peetre, si S est continu sur $L^2(\mathbb{R}^d)$ alors SM_{b_1} envoie $L^\infty(\mathbb{R}^d)$ dans $\mathrm{BMO}(\mathbb{R}^d)$.

Nous nous proposons de démontrer le Théorème $T(b)$ à l'aide de la construction précédente.

Nous allons présenter la structure de la preuve sans rentrer dans les détails techniques. Cette structure est analogue à celle adoptée originellement dans [DJS] et nous référons le lecteur à cet article pour les justifications.

Nous nous plaçons pour ce faire en dimension 1 en prenant $b_1(x) = b_2(x) = b(x)$ pour simplifier les notations.

Comme nous venons de le voir, l'implication (i) $\Rightarrow$ (ii) est connue. Nous organisons la réciproque en deux étapes.

Première étape. Réduction au cas $T(1) = 0$ et ${}^tT(1) = 0$.

Il nous faut reprendre tout d'abord l'argument de perturbation en ϵ (Partie 3). Quitte à supposer ϵ plus petit, on impose la condition supplémentaire:

$$|B(g_I, g_I)| \geqslant \delta_0/2 \quad \text{pour tout} \quad I \in \mathcal{I}, \tag{38}$$

où $g(x)$ est la fonction «génératrice» de l'analyse multirésolution $V_j, j \in \mathbb{Z}$ et où $g_I(x) = 2^{j/2}g(2^jx - k)$ lorsque $I = I_{j,k}$.

Rappelons que (ψ_I) désigne la base d'ondelettes sur $L^2(\mathbb{R})$ associée et que r est la régularité des fonctions g et ψ et des systèmes de b-vaguelettes (θ_I) et $(\tilde{\theta}_I)$. Enfin δ désignera n'importe quel nombre pris dans $[0, r]$.

Reprenons l'opérateur T_b (Partie 5). Utilisant (37), on voit que T_b^{-1} envoie BMO $(\mathbb{R})$ dans BMO $(\mathbb{R})$ (T_b est même un isomorphisme de BMO$(\mathbb{R})$ sur lui-même). Soit alors $\beta \in$ BMO$(\mathbb{R})$ tel que $T(1) = b\beta$. On pose

$$c_I = \langle T_b^{-1}(\beta), \psi_I \rangle B(g_I, g_I)^{-1} \quad \text{pout tout} \quad I \in \mathcal{I}$$

et on définit un opérateur de paraproduit Π_1 dans la classe $F(\delta, b, b)$ dont le noyau-distribution est donné par $b(x)K_1(x, y)b(y)$ lorsque $(x, y) \in \Omega$ où

$$K_1(x, y) = \sum_{I \in \mathcal{I}} c_I \theta_I(x) g_I(y)^2.$$

On vérifie classiquement que Π_1 est borné sur $L^2(\mathbb{R})$, que $\Pi_1(1) = b\beta$ dans M_bBMO $(\mathbb{R})$, et que ${}^t\Pi_1(1) = 0$ dans M_bBMO $(\mathbb{R})$.

On définit ensuite un deuxième opérateur de paraproduit Π_2 dans $F(\delta, b, b)$. On prendra pour cela, si $\gamma \in$ BMO $(\mathbb{R})$ est tel que ${}^tT(1) = b\gamma$,

$$d_I = B(g_I, g_I)^{-1} \langle \tilde{T}_b^{-1}(\gamma), \psi_I \rangle$$

et

$$K_2(x, y) = \sum_{I \in \mathcal{I}} d_I (g_I(x))^2 \tilde{\theta}_I(y)$$

Π_2 envoie $L^2(\mathbb{R})$ dans $L^2(\mathbb{R})$, $\Pi_2(1) = 0$ et ${}^t\Pi_2(\gamma) = b\gamma$ dans M_bBMO $(\mathbb{R})$.

On est donc réduit à l'étude de $R = T - \Pi_1 - \Pi_2$ qui vérifie $R(1) = 0 = {}^tR(1)$ dans M_bBMO $(\mathbb{R})$ et $R \in F(\delta, b, b)$.

Deuxième étape. $T(1) = 0 = {}^tT(1)$ dans $M_b\mathrm{BMO}(\mathbb{R})$.

On écrit $T = M_b S M_b$ où S est faiblement borné et peut être défini dans la classe $F(\delta, 1, 1)$. Si $\nu < \delta$, d'après un théorème de Lemarié adapté dans [DJS], les opérateurs SM_b et ${}^tSM_b$ envoient continuement $C^\nu(\mathbb{R})$ dans $C^\nu(\mathbb{R})$. Ceci permet de calculer $SM_b(\theta_J)$ pour tout $J \in \mathcal{G}$. Des calculs classiques (*cf.* [L] ou [M3]) montrent que la collection des fonctions $\omega_J = SM_b(\theta_J)$ est une famille de b-molécules ν-régulières. On écrit ensuite grâce au Théorème 1:

$$\omega_J(x) = SM_b(\theta_J)(x) = \sum_{I\in\mathcal{G}} B(\tilde{\theta}_I, SM_b\theta_J)\theta_I(x)$$

$$= \sum_{I\in\mathcal{G}} \gamma_{I,J}\theta_I(x).$$

SM_b, et par conséquent T, est borné sur $L^2(\mathbb{R})$ si, et seulement si, la matrice $\mathfrak{M}$ de coefficients $\gamma_{I,J}, I, J \in \mathcal{G}$ est bornée sur $l^2(\mathcal{G})$. On peut estimer $\gamma_{I,J}$ de la façon suivante: supposons $|I| \leqslant |J|$, $\theta_I(x)b(x)$ est une fonction localisée dans l'intervalle I et d'intégrale nulle tandis que $\omega_J(x)$ est régulière et plate à l'échelle de θ_I. On écrit alors

$$\gamma_{I,J} = \int_{\mathbb{R}} \theta_I(x)b(x)(\omega_J(x) - \omega_J(x_I))\,dx$$

(x_I étant $\inf I$), ce qui conduit à la majoration (voir [M3] pour les détails):

$$|\gamma_{I,J}| \leqslant C \inf(|I|, |J|)^\nu \frac{|I|^{1/2}|J|^{1/2}}{(|I| + |J| + \operatorname{dist}(I, J))^{1+\nu}}$$

où ν est n'importe quel nombe tel que $0 < \nu < \delta < r$ et où $\operatorname{dist}(I, J) = |x_I - x_J|$ si $x_I = \inf I$ et $x_J = \inf J$.

Pour montrer enfin la continuité de M sur $l^2(\mathcal{G})$, on utilise un lemme de Schur à poids:

$$\Big|\sum_I \sum_J \lambda_I \gamma_{I,J} \overline{\mu_J}\Big| \leqslant \left[\sum_I |\lambda_I|^2 \Big(\sum_J |\gamma_{I,J}| \frac{|J|^\alpha}{|I|^\alpha}\Big)\right]^{1/2} \left[\sum_J |\mu_J|^2 \Big(\sum_I |\gamma_{I,J}| \frac{|I|^\alpha}{|J|^\alpha}\Big)\right]^{1/2}$$

$$\leqslant C\Big(\sum_I |\lambda_I|^2\Big)^{1/2}\Big(\sum_I |\mu_I|^2\Big)^{1/2}$$

dès que

$$\operatorname*{Sup}_I \left[\sum_J |\gamma_{I,J}| \Big(\frac{|J|}{|I|}\Big)^\alpha + \sum_J |\gamma_{J,I}| \Big(\frac{|J|}{|I|}\Big)^\alpha\right] < \infty.$$

Et cette relation se vérifie aisément pour tout $\alpha \in \left]\frac{1}{2} - \nu, \frac{1}{2} + \nu\right[$. Le Théorème 4 est donc démontré.

Par exemple, T_Γ s'étudie par la méthode de la deuxième étape, ce qui prouve une nouvelle fois la continuité sur $L^2(\mathbb{R})$ de l'opérateur de Cauchy sur une courbe corde-arc.

8. Un isomorphisme remarquable entre algèbres d'opérateurs

On appelle $\mathcal{A}(\delta, 1, b)$ le sous-espace vectoriel de $F(\delta, 1, b)$ constitué des opérateurs T continus sur $L^2(\mathbb{R})$ tels que $T(1) = 0$ dans BMO $(\mathbb{R}^d)$ et ${}^tT(1) = 0$ dans M_bBMO $(\mathbb{R}^d)$.

L'algèbre $\mathcal{A}_b^r$ que nous définissons ci-dessous, est une sous algèbre de l'algèbre A_b étudiée dans [MM]. Le point nouveau est la relation entre $\mathcal{A}_b^r$ et $\mathcal{A}_1^r$ démontré dans le théorème suivant.

Théorème 5. *Soit $b(x)$ une fonction pseudoaccrétive sur $L^2(\mathbb{R}^d)$. Il existe un $r \in]0, 1]$ tel que*

$$\mathcal{A}_b^r = \bigcup_{0<\delta<r} \mathcal{A}(\delta, 1, b)$$

constitue une sous-algèbre d'opérateurs continus sur $L^2(\mathbb{R}^d)$. Cette algèbre est isomorphe à l'algèbre

$$\mathcal{A}_1^r = \bigcup_{0<\delta<r} \mathcal{A}(\delta, 1, 1),$$

sous-algèbre de l'algèbre de Lemarié, par conjugaison intérieure dans $\mathcal{L}(L^2(\mathbb{R}^d))$.

Enfin, si $T \in \mathcal{A}(\delta, 1, b)$ alors $M_b^{-1}\, {}^tTM_b \in \mathcal{A}(\delta, 1, b)$.

PREUVE. Nous l'effectuons en dimension 1 pour simplifier l'écriture. Reprenons alors les fonctions (ψ_I), (θ_I) et $(\tilde{\theta}_I)$ que nous avons précédemment construites. Le nombre r désigne la régularité de ces fonctions. Alors l'application

$$C: \quad \begin{aligned} \mathcal{A}_b^r &\to \mathcal{A}_1^r \\ S &\to T_b^{-1} S T_b \end{aligned}$$

est un isomorphisme d'espaces vectoriels. Si nous admettons cela un instant, C étant un opérateur de conjugaison, et $\mathcal{A}_1^r$ une sous-algèbre de $\mathcal{L}(L^2(\mathbb{R}))$ (*cf.* [L]), il suit facilement que $\mathcal{A}_b^r$ est une sous-algèbre de $\mathcal{L}(L^2(\mathbb{R}))$.

Montrons maintenant que C est un isomorphisme de $\mathcal{A}_b^r$ sur $\mathcal{A}_1^r$. Par symétrie, il suffit de montrer que si $S \in \mathcal{A}_b^r$ alors $C(S) \in \mathcal{A}_1^r$.

Soient donc α, ν, δ tels que $0 < \alpha < \nu < \delta < r$.

Soit $S \in \mathcal{A}(\delta, 1, b)$, posons $A = T_b^{-1} S T_b$. Montrons que $A \in \mathcal{A}(\alpha, 1, 1)$. Tout d'abord A est continu sur $L^2(\mathbb{R})$. Désignons par $\gamma_{I,J}$ le coefficient $B(\tilde{\theta}_I, S\theta_J)$

où B désigne la forme bilinéaire usuelle. Il s'ensuit que

$$A\psi_I(x) = \sum_{I \in \mathcal{I}} \gamma_{I,J}\psi_J(x). \tag{39}$$

L'orthogonalité des fonctions $\psi_I(x)$ prises deux à deux nous permet de voir que le noyau-distribution de A s'écrit

$$K(x,y) = \sum_{I \in \mathcal{I}} A\psi_I(x)\psi_I(y) \quad \text{pour} \quad (x,y) \in \Omega.$$

Des calculs classiques montrent que les fonctions $A\psi_I(x)$ définies par (39) forment une famille de 1-molécules ν-régulières. Par suite $K(x,y)$ est un noyau α-standard et $A \in F(\alpha, 1, 1)$. Comme $A\psi_I(x)$ et $\psi_I(x)$ sont d'intégrale nulle, on vérifie aisément que $A(1) = {}^tA(1) = 0$ dans BMO ($\mathbb{R}$). Donc $A \in \mathcal{A}(\alpha, 1, 1)$.

Enfin, le dernier point du théorème ne présente aucune difficulté.

9. Une base inconditionnelle de l'espace $H^1_b(\mathbb{R}^d)$

Définition 8. *On appelle $H^1_b(\mathbb{R}^d)$ l'espace $M_b^{-1}H^1(\mathbb{R}^d)$ muni de la norme $\|f\|_{H^1_b(\mathbb{R}^d)} = \|g\|_{H^1(\mathbb{R}^d)}$ si $f = b^{-1}g$. $H^1(\mathbb{R}^d)$ est ici l'espace atomique de Stein et Weiss.*

On a le résultat suivant:

Théorème 6. *Soit $b(x) \in L^\infty(\mathbb{R}^d)$ une fonction pseudoaccrétive.*

(i) *T_b se prolonge en un isomorphisme de $H^1(\mathbb{R}^d)$ sur $H^1_b(\mathbb{R}^d)$.*

(ii) *$(\theta_{\lambda,Q})$ est une famille inconditionnelle de $H^1_b(\mathbb{R}^d)$ et*

$$f \in H^1_b(\mathbb{R}^d) \Leftrightarrow f = \sum_\lambda \sum_Q B(\tilde\theta_{\lambda,Q}, f)\theta_{\lambda,Q}$$

avec

$$\left(\sum_\lambda \sum_Q |B(\tilde\theta_{\lambda,Q}, f)|^2 \frac{1}{|Q|}\chi_Q(x)\right)^{1/2} \in L^1(\mathbb{R}^d).$$

(iii) *L'application* BMO ($\mathbb{R}^d$) $\to (H^1_b(\mathbb{R}^d))'$

$$\beta \to (f \to B(f, \beta))$$

est un isomorphisme.

(iv) *Pour tout $\beta \in$* BMO ($\mathbb{R}^d$), $\sum_\lambda \sum_Q B(\theta_{\lambda,Q}, \beta)\tilde\theta_{\lambda,Q}$ *converge vers β dans* BMO ($\mathbb{R}^d$) *pour la topologie de dual définie dans* (iii), *que nous noterons* σ^*_B(BMO ($\mathbb{R}^d$), $H^1_b(\mathbb{R}^d)$).

Remarque. On peut interchanger la famille $(\theta_{\lambda,Q})$ avec $(\tilde{\theta}_{\lambda,Q})$ et T_b avec $\tilde{T}_b$. Les énoncés restent identiques.

PREUVE. Nous allons encore une fois nous placer en dimension 1. On a vu que $\tilde{T}_b$ était un isomophisme de BMO ($\mathbb{R}$) sur BMO ($\mathbb{R}$). Par conséquent, ${}^t\tilde{T}_b$ est un isomorphisme de $H^1(\mathbb{R})$ sur $H^1(\mathbb{R})$. Or $T_b^{-1} = {}^t\tilde{T}_b M_b$. Il s'ensuit que T_b^{-1} est un isomorphisme de $H_b^1(\mathbb{R})$ sur $H^1(\mathbb{R})$ ce qui établit (i). Pour montrer (ii) nous allons utiliser un résultat de [LM] ou de [AEPT]:

$$h \in H^1(\mathbb{R}) \Leftrightarrow h = \sum_I \langle h \mid \psi_I \rangle \psi_I$$

avec

$$\left(\sum_I |\langle h \mid \psi_I \rangle|^2 \frac{1}{|I|} \chi_I(x) \right)^{1/2} \in L^1(\mathbb{R}).$$

Soit alors $f \in H_b^1(\mathbb{R})$. On écrit, au sens des distributions,

$$f = \sum_I \alpha_I \theta_I$$

et il vient $\alpha_I = B(\tilde{\theta}_\lambda, f)$ (qui a un sens puisque $\tilde{\theta}_I(x) b(x) f(x) \in L^1(\mathbb{R})$). La fonction $h = T_b^{-1}(f)$ appartient à $H^1(\mathbb{R})$ et il vient

$$h = \sum_I \alpha_I \psi_I \quad \text{avec} \quad \left(\sum_I |\alpha_I|^2 |I|^{-1} \chi_I(x) \right)^{1/2} \in L'(\mathbb{R}).$$

(iii) est une évidence car $B(\theta, \beta) = \langle M_b \theta, \beta \rangle$ si $\theta \in H_b^1(\mathbb{R})$ et $\beta \in$ BMO ($\mathbb{R}$). Or $M_b \theta \in H^1(R)$ et l'isomorphisme s'établit à l'aide de la dualité entre $H^1(\mathbb{R})$ et BMO ($\mathbb{R}$).

(iv) Soit $\beta \in$ BMO ($\mathbb{R}$). On sait que $\tilde{T}_b^{-1}(\beta) \in$ BMO ($\mathbb{R}$). Alors, d'après [LM],

$$\tilde{T}_b^{-1}(\beta) = \sum_I \langle \tilde{T}_b^{-1}(\beta), \psi_I \rangle \psi_I$$

et la convergence a lieu dans BMO ($\mathbb{R}$) pour la topologie σ^* (BMO ($\mathbb{R}$), $H^1(\mathbb{R})$). Soit alors $f \in H^1(\mathbb{R})$; on a

$$\left| \sum_I \langle \tilde{T}_b^{-1}(\beta), \psi_I \rangle \langle \psi_I, f \rangle \right| \leqslant C \|f\|_{H^1(\mathbb{R})} \|\tilde{T}_b^{-1}(\beta)\|_{\mathrm{BMO}(\mathbb{R})}.$$

Or $\langle \tilde{T}_b^{-1}(\beta), \psi_I \rangle = B(\beta, \theta_I)$ et si $\theta = T_b(f) \in H_b^1(\mathbb{R})$ on a

$$\left| B\left(\sum_I B(\beta, \theta) \tilde{\theta}_I, \theta \right) \right| \leqslant C \|\theta\|_{H_b^1(\mathbb{R})} \|\beta\|_{\mathrm{BMO}(\mathbb{R})}.$$

Donc $\sum_I B(\beta, \theta) \tilde{\theta}_I$ converge dans BMO pour la topologie faible σ_B^*(BMO ($\mathbb{R}$), $H_b^1(\mathbb{R})$) et il est facile de montrer que la limite est β.

10. Une excursion dans les algèbres de Clifford

David-Journé-Semmes ont montré que le $T(b)$ s'étendait au cas où le calcul dans les corps des complexes était remplacé par celui dans une algèbre de Clifford. Notre démonstration du $T(b)$ s'appuyant sur la théorie des espaces de Hilbert complexes, il n'est pas clair que nous puissions l'adopter. Nous allons donner les éléments nécessaires à cette extension.

Pourquoi une algèbre de Clifford? Parce que le potentiel de double couche sur une surface lipschitzienne de $\mathbb{R}^{d+1}$ se réécrit simplement par le biais des algèbres de Clifford. Cette observation nous a été communiquée par R. Coifman. Soit $X_0 = \varphi(x) = \varphi(x_1, x_2, \ldots, x_d)$ le graphe de $\varphi(x)$ où φ est telle que $\|\partial\varphi/\partial x_i\|_\infty < M$ pout tour $i \in \{1, \ldots, d\}$. Le potentiel de double couche est l'opérateur

$$T_\varphi f(x) = \text{v.p.} \int_{\mathbb{R}^d} K_\varphi(x,y) f(y)\, dy \quad \text{pour} \quad f \in D(\mathbb{R}^d)$$

où

$$K_\varphi(x,y) = \frac{\varphi(x) - \varphi(y) - \langle x - y, \nabla\varphi(y)\rangle_{\mathbb{R}^d}}{(|x-y|^2 + |\varphi(x) - \varphi(y)|^2)^{(d+1)/2}}.$$

El dans l'algèbre de Clifford $\mathfrak{A}_d$, $K_\varphi(x,y)$ est la partie réelle du noyau

$$\mathcal{K}(x,y)b(y) = \frac{\varphi(x) - \varphi(y)e_0 - \sum_{d}^{i=1} (x_i - y_i)e_i}{(|x-y|^2 + |\varphi(x) - \varphi(y)|^2)^{(d+1)/2}} \left(e_0 - \sum_{i=1}^{d} \frac{\partial\varphi}{\partial x_i}(y) e_i\right)$$

$\mathcal{K}(x,y)b(y)$ est donc un noyau du type de ceux que nous avons vu précédemment.

Nous allons rappeler brièvement la définition d'une algèbre de Clifford et voir comment adapter les méthodes précédentes.

Soient $n \in \mathbb{N}^*$ et $P = P(\{1, 2, \ldots, n\})$. L'algèbre de Clifford $\mathfrak{A}_n$ est engendrée sur $\mathbb{R}$ par 2^n vecteurs e_A, $A \in P$, construits de la façon suivante: on part de $n+1$ vecteurs $e_0, e_1, \ldots, e_n$ vérifiant $e_0^2 = e_0$, $e_i^2 = e_i$, $e_i e_0 = e_0 e_i$ et $e_i e_j = -e_j e_i$ pour tout $(i,j) \in \{1, \ldots, n\}^2$, $i \neq j$. On posera par la suite $e_0 = 1$ et si $A \in P$, $e_A = e_{i_1} e_{i_2} \cdots e_{i_r}$ lorsque A est la famille ordonnée $(i_1, i_2, \ldots, i_r)$.

$\mathfrak{A}_n$ est une algèbre non commutative pour la multiplication définie cidessus. $\mathfrak{A}_n$ est munie d'une involution $\alpha \to \bar{\alpha}$ définie par: si $\alpha = \sum_A \alpha_A e_A$, alors $\bar{\alpha} = \sum \alpha_A \bar{e}_A$ où $\bar{e}_A = (-1)^{r(A)} e_A$ et $r(A) = (\#A)(\#A+1)/2$. On vérifie que $\overline{\alpha\alpha'} = \bar{\alpha}' \cdot \bar{\alpha}$ pour tout $\alpha, \alpha' \in \mathfrak{A}_n$. Enfin, $\mathfrak{A}_n$ peut être munie du produit scalaire $(\alpha, \alpha') \to \operatorname{Re}(\alpha\bar{\alpha}')$ où pout tout $\mu \in \mathfrak{A}_n$, $\operatorname{Re}\mu$ désigne la composante de μ sur le vecteur $e_0 = 1$. On notera $|\ \ |$ la norme euclidienne induite sur $\mathfrak{A}_n$ par ce produit scalaire. $(\mathcal{A}_n, |\ \ |)$ est alors un espace euclidien et une algèbre normée.

Pour terminer, on distingue dans $\mathcal{A}_n$ le sous-espace de dimension $n+1$, noté $\mathcal{C}_n$ et appelé espace des vecteurs de Clifford, composé des vecteurs $x = \alpha_0 e_0 + \alpha_1 e_1 + \cdots + \alpha_n e_n$. $\mathcal{C}_n$ possède la propriété remarquable suivante: si $x = \alpha_0 e_0 + \alpha_1 e_1 + \cdots + \alpha_n e_n$ alors $\bar{x} = \alpha_0 e_0 - \alpha_1 e_1 - \cdots - \alpha_n e_n$ et $x\bar{x} = |x|^2 e_0 = |x|^2$. On définit alors $x^{-1} \in \mathcal{C}_n$ par $x^{-1} = \bar{x}/|x|^2$ pour $x \in \mathcal{C}_n - \{0\}$.

Nous renvoyons le lecteur à [BDS] pour toutes les justifications de ce que nous allons employer.

Passons aux définitions de l'espace $L^2(\mathbb{R}^d, \mathcal{A}_n)$. On dit que $f(x) \in L^2(\mathbb{R}^d, \mathcal{A}_n)$ si

$$f(x) = \sum_A f_A(x) e_A \quad \text{où} \quad f_A(x) \in L^2(\mathbb{R}^d, \mathbb{R}).$$

On munit $L^2(\mathbb{R}^d, \mathcal{A}_n)$ du produit scalaire réel

$$\langle f, g \rangle = \operatorname{Re} \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, dx \quad \text{et} \quad \|f\|^2 = \langle f, f \rangle$$

et l'on a également

$$\|f\|^2 = \sum_A \|f_A\|^2.$$

Soit $b(x) \in L^\infty(\mathbb{R}^d, \mathcal{A}_n)$ à valeurs dans $\mathcal{C}_n$ et soient β et β^* les formes associées à $b(x)$ et à $\bar{b}(x)$ par (21). On dira que β est pseudoaccrétive sur un sous-espace fermé $\mathcal{V}$ de $L^2(\mathbb{R}^d, \mathcal{A}_n)$ si $\mathcal{V}$ est un $\mathcal{A}_n$-module à gauche et si

$$\forall f \in \mathcal{V}, \qquad \sup_{\|f'\| \leq 1} \operatorname{Re} \beta(f, f') \geqslant \delta_0 \|f\|,$$

et symétriquement en échangeant les rôles de f et f'.

On généralise les lemmes abstraits de la façon suivante: si $\mathcal{V}$ est un sous-espace fermé de $L^2(\mathbb{R}^d, \mathcal{A}_n)$ et un $\mathcal{A}_n$-module à gauche, une forme $\mathcal{A}_n$-linéaire à gauche L sur $\mathcal{V}$ vérifie $L(\alpha v + v') = \alpha L(v) + L(v')$ pour tout $\alpha \in \mathcal{A}_n$, $v, v' \in \mathcal{V}$. Alors, à chaque forme $\mathcal{A}_n$-linéaire à gauche L continue sur $\mathcal{V}$ correspond une unique forme $\mathbb{R}$-linéaire, notée $\operatorname{Re} L$, telle que $\operatorname{Re} L(v) = \operatorname{Re}(L(v))$ pour tout $\mathcal{V}$. La réciproque est également vraie. Ce résultat permet de nous ramener au cadre classique des formes $\mathbb{R}$-linéaires.

Nous utiliserons églement le test d'inversibilité suivant.

Proposition 3. *Soit M une matrice à coefficients dans $\mathcal{A}_n$, notés $\alpha_{s,t}$, $s, t \in T$. On posera*

$$(M\lambda \mid \mu) = \sum_s \sum_t \lambda_s \alpha_{s,t} \overline{\mu_t}$$

qui fait de M un opérateur $\mathfrak{A}_n$-linéaire à gauche sur $l^2(T, \mathfrak{A}_n)$. On suppose que M est continu sur $l^2(\mathbb{C}, \mathfrak{A}_n)$ et vérifie

$$\forall \lambda \in l^2(T, \mathfrak{A}_n) \qquad \sup_{\|\mu\| \leq 1} \operatorname{Re}(M\lambda \,|\, \mu) \geqslant \delta_0 \|\lambda\|$$

et

$$\forall \mu \in l^2(T, \mathfrak{A}_n) \qquad \sup_{\|\lambda\| \leq 1} \operatorname{Re}(M\lambda \,|\, \mu) \geqslant \delta_0 \|\mu\|$$

où $\delta_0 > 0$. Alors M est inversible sur $l^2(T, \mathfrak{A}_n)$.

La preuve est laissée au lecteur. Nous lui laissons aussi le soin d'adapter les lemmes abstraits de la partie 2.

Une analyse multirésolution de $L^2(\mathbb{R}^d, \mathfrak{A}_n)$ est définie de la façon suivante: on dit que $f \in \mathbb{V}_j$ si $f = \sum f_A e_A$ avec pour tout A, $f_A \in V_j$ (*cf.* partie 6) et l'on définit de manière analogue $\mathbb{W}_j$. Il convient de remarquer l'importance du fait que la fonction g et l'ondelette ψ sont à valeurs réelles car $\mathbb{R}$ identifié ici à $\mathbb{R}e_0$ est le centre de l'algèbre $\mathfrak{A}_n$. Par suite $\mathbb{V}_j$ et $\mathbb{W}_j$ sont des $\mathfrak{A}_n$-module à gauche.

On applique alors l'argument de perturbation. Là aussi, la fonction $g^{(\epsilon)}$ et l'ondelette $\psi^{(\epsilon)}$ sont à valeurs réelles et les espaces $\mathbb{V}_j^{(\epsilon)}$ sont des $\mathfrak{A}_n$-modules à gauche. Les espaces $\mathbb{X}_j^{(\epsilon)}$ et $\mathbb{X}_j^{(\epsilon)*}$ que l'on est amené à définir seront également des $\mathfrak{A}_n$-modules à gauche, de sorte que nous pouvons utiliser les remarques précédentes.

Pour terminer, notons que la contruction des b-vaguelettes n'utilise pas un calcul commutatif. Le lecteur intéressé pourra en vérifier les détails. On obtient finalement le théorème suivant.

Théorème 7. *Soit $b(x) \in L^\infty(\mathbb{R}^d, \mathfrak{A}_n)$ à valeurs dans $\mathbb{C}_n$ et pseudoaccrétive. Il existe un $r \in]0, 1[$, un ensemble Λ de cardinal $2^d - 1$ et, pour chaque $\lambda \in \Lambda$, deux familles de b-vaguelettes r-régulières $(\theta_{\lambda,Q})$ et $(\tilde{\theta}_{\lambda,Q})$ à valeurs dans $\mathfrak{A}_n$ telles que*

(i) $\forall f \in L^2(\mathbb{R}^d, \mathfrak{A}_n)$,

$$f = \sum_\lambda \sum_Q B(f, \theta_{\lambda,Q})\tilde{\theta}_{\lambda,Q} = \sum_\lambda \sum_Q \theta_{\lambda,Q} B(f, \tilde{\theta}_{\lambda,Q}).$$

(ii) $$\|f\|^2 \approx \sum_\lambda \sum_Q |B(\tilde{\theta}_{\lambda,Q}, f)|^2 \approx \sum_\lambda \sum_Q |B(\theta_{\lambda,Q}, f)|^2.$$

(iii) $$B(\tilde{\theta}_{\lambda',Q'}, \theta_{\lambda,Q}) = \delta_{\lambda,\lambda'}\delta_{Q,Q'} \textit{ pour tous } \lambda, \lambda', Q, Q'.$$

Précisons que

$$\int_{\mathbb{R}^d} \tilde{\theta}_{\lambda,Q}(x) b(x)\, dx = 0 = \int_{\mathbb{R}^d} b(x)\theta_{\lambda,Q}(x)\, dx,$$

$\tilde{\theta}_{\lambda, Q}$ est une b-vaguelette «à gauche» tandis que $\theta_{\lambda, Q}$ en est une «à droite». B est la forme donnée par

$$B(f, g) = \int f b g.$$

Remarque. L'hypothèse $b(x)$ à valeurs dans $\mathcal{C}_n$ n'est pas nécessaire dans cet énoncé. Toutefois, elle apparait de façon cruciale dans le $T(b)$ où l'on a besoin de calculer $B(g_I, g_I)^{-1}$ comme l'inverse d'un vecteur de Clifford.

L'adaptation des théorèmes 4, 5, 6 est laissée au lecteur.

11. Commentaires

La théorie 6(ii) peut s'étendre, si $\frac{d}{d+r} < p < 1$, en définissant de façon similaire un espace $H_b^p(\mathbb{R}^d)$. Une démonstration directe de la caractérisation de $H^p(\mathbb{R}^d)$ par la fonction maximale $\left(\sum_\lambda \sum_Q |\alpha_{\lambda, Q}|^2 |Q|^{-1} \chi_Q(x)\right)^{1/2}$ peut se trouver dans [AEPT].

En dimension 1, l'espace $H_b^1(\mathbb{R})$ est lié de façon très étroite à l'analyse complexe sur les courbes corde-arc ($b(x)$ étant ici la fonction $z'(x)$). Voir [Z] par exemple. Lorsque $p < 1$, $H_b^p(\mathbb{R})$ est seulement un espace de distributions et on ne sait pas relier cet espace avec des espaces de fonctions holomorphes.

On peut également montrer que les familles $(\tilde{\theta}_{\lambda, Q})$ et $(\theta_{\lambda, Q})$ forment des bases inconditionnelles des espaces $L^p(\mathbb{R}^d)$, $1 < p < \infty$ et donner des critères d'appartenance à l'un de ces espaces d'une série $\sum_\lambda \sum_Q \alpha_{\lambda, Q} \theta_{\lambda, Q}$. On s'appuie pour cela sur l'article [LM] et l'on utilise les propriétés de l'opérateur T_b (ou $\tilde{T}_b$).

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Variational Characterization of Equations of Motion in Bundles of Embeddings

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1. Introduction

In this paper we study variational principles for a general situation which includes free boundary problems with surface tension. Following [2], our main result concerns a variational principle in a infinite dimensional principal bundle of embeddings of a compact region D in a manifold M having the same dimension as D. By considering arbitrary variations, free boundary problems are included, while variations parallel to the boundary permit to consider fluid motion or flow of Hamitonian vector fields in non compact regions, generalizing [3], [4].

In Section 2 the main result is stated and proved. The Lagrangian includes a boundary term allowing us to include surface tension [5], or to remove it. Section 3 applies our result to Hamiltonian vector fields, while Section 4 concerns free boundary problems.

2. Variational Principle in Bundles of Embeddings

Let M be an n-manifold and Ω a given volume on M. Let $D \subseteq M$ a submanifold having dimension n with boundary $\partial D \subseteq M$.

The set

$$P = \{\eta: D \to M: \eta \text{ is an embedding}\}$$

is a principal bundle having structure group

$$G = \{g: D \to D: g \text{ is a diffeomorphism}\}$$

acting on P on the right, by composition of maps.

Similarly, we define P_{vol}, G_{vol}, by adding the further incompressibility condition, namely

$$\eta * \Omega = \Omega, \qquad g * \Omega = \Omega,$$

where the star means the pull-back operation.

A typical example of this situation to be considered afterwards with some more detail, is the liquid drop D moving freely in $M = \mathbb{R}^3$ with

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3.$$

Thus, at each instant of time t, the element $\eta_t: D \to \mathbb{R}^3$ of P represents the position of the fluid particles at that instant namely, if $X = (X^1, X^2, X^3)$ are the coordinates of the position of a given particle at time $t = 0$, and $x = (x^1, x^2, x^3)$ the position of the same particle after the interval of time $[0, t]$ has passed, then $x = \eta_t(X)$. If the fluid is incompressible, then for each $X \in D$ and each t, we have

$$J_{\eta_t}(X) = 1$$

where J_{η_t} is the Jacobian of η_t. An equivalent condition is that $\eta_t * \Omega = \Omega$.

Now, back to the general situation, let $L: TM \to \mathbb{R}$ be a given lagrangian. This induces a Lagrangian $\mathcal{L}: TP \to \mathbb{R}$ defined by

$$\mathcal{L}(\eta, \dot{\eta}) = \int_D L(\eta(X), \dot{\eta}(X))\Omega(X).$$

It is sometimes useful to think of $(\eta, \dot{\eta})$ as the derivative with respect to t of a curve $x = \eta_t(X)$, $X \in M$. Thus

$$\dot{\eta}_t[\eta_t^{-1}(x)] = \frac{\partial \eta_t(\eta_t^{-1}(x))}{\partial t}$$

represents a vector field on $D_t = \eta_t(D) \subseteq M$.

Of course, $\mathcal{L}$ has a restriction

$$\mathcal{L}: TP_{\text{vol}} \to \mathbb{R}.$$

Let η_t; $t \in [t_0, t_1]$ be a curve in $P_{\text{vol}} \subseteq P$. Thus, for each $t \in [t_0, t_1]$, $\eta_t: D \to M$ is a volume preserving diffeomorphism. Now consider the following functional with $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed, defined on the curves (η_t, λ_t) where λ_t is a curve on the set $\mathfrak{F}(D)$ of real valued C^∞ functions on D

$$\begin{aligned}\mathcal{A}(\eta, \lambda) &= \int_{t_0}^{t} \left[\mathcal{L}(\eta, \dot{\eta}) + \lambda_t \frac{dJ_{\eta_t}}{dt} \right] dt \\ &= \int_{t_0}^{t_1} dt \int_D \left[L(\eta_t(X), \dot{\eta}_t(X)) + \lambda_t(X) \frac{d}{dt} J_{\eta_t}(X) \right] \Omega(X).\end{aligned}$$

The constraint J_{η_t} = constant, or equivalently $\dfrac{dJ_{\eta_t}}{dt} = 0$ with the Lagrange multiplier $\lambda_t \in \mathfrak{F}(D)$ together with the condition $J_{\eta_0} = 1$ gives the end this does not imply any loss of generality. Likewise, we can assume that

$$\Omega(X) = dX^1 \wedge \cdots \wedge dX^n.$$

This is because variational principles are essentially local in nature.

Sometimes we will write $\Omega(X) = d^3X$, whenever computations are simpler in the case $M = \mathbb{R}^3$.

Now think of a variation

$$\delta\eta_t = \frac{d}{d\epsilon} \eta_{t\epsilon} \bigg|_{\epsilon = 0}$$

such that

$$\delta\eta_{t_0} = 0, \qquad \delta\eta_{t_1} = 0,$$

and $\eta_{t\epsilon}$ is a curve on P for each $\epsilon \in (-\epsilon_1, \epsilon_2)$. On the other hand assume that $\eta_t \equiv \eta_{t_0}$ is a curve on P_{vol}. If (η_t, λ_t) is a critical point of $\mathcal{A}$, then we have

$$\delta\mathcal{A} = \frac{d}{d\epsilon} \mathcal{A} \bigg|_{\epsilon = 0} = 0.$$

This means that

$$\begin{aligned} 0 &= \delta \int_0^{t_1} dt \int_D \left\{ L[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] + \frac{d}{dt} J_{t\epsilon}(X) \lambda_t(X) \right\} d^3X \\ &= \int_{t_0}^{t_1} dt \int_D \left\{ \frac{\partial L}{\partial x} [\eta_t(X), \dot{\eta}_t(X)] \delta\eta_t + \frac{\partial L}{\partial \dot{x}} [\eta_t(X), \dot{\eta}_t(X)] \, \delta\dot{\eta}_t \right. \\ &\qquad \left. + \frac{d}{dt} [\nabla \cdot (\partial\eta_t \circ \eta_t^{-1}) \circ \eta_t(X)] \lambda_t(X) \right\} d^3X. \end{aligned} \tag{δ}$$

Since

$$\frac{d}{d\epsilon} J_{\eta_{t\epsilon}}(X)\Big|_{\epsilon=0} = \nabla \cdot (\delta\eta_t \circ \eta_t^{-1}) \circ \eta_t(X).$$

(To check this, let

$$v = \frac{d\eta_\epsilon}{d\epsilon}\Big|_{\epsilon=0} \circ \eta^{-1}(x).$$

Then

$$(\nabla \cdot v) \circ \eta = \frac{dJ_{\eta_\epsilon}}{d\epsilon}\Big|_{\epsilon=0}.$$

In fact we can assume without loss of generality that $\eta = \eta_0 = identity$. Thus

$$\left.\frac{d}{d\epsilon} \eta_\epsilon * (d^3x)\right|_{\epsilon=0} = \frac{dJ_{\eta_\epsilon}}{d\epsilon} d^3X = L_v(d^3x) = \nabla \cdot v\, d^3x.\Big)$$

Thus by applying integration by parts to (δ), we get

$$0 = \int_{t_0}^{t_1} dt \int_D \left\{ \left[\frac{\partial L}{\partial x}(\eta_t, \dot\eta_t) - \frac{d}{dt}\frac{\partial L}{\partial \dot x}(\eta_t, \dot\eta_t) \right] \delta\eta_t - \right.$$
$$\left. - [\nabla \cdot (\delta\eta_t \circ \eta_t^{-1})] \circ \eta_t \frac{d}{dt}\lambda_t \right\}(X)\, d^3X.$$

Since η_t is volume preserving we have $J_{\eta_t} = 1$, and then, by the change of variables formula for a multiple integral, we get

$$0 = \int_{t_0}^{t_1} dt \int_{\eta_t(D)} \left[\left(\frac{\partial L}{\partial x}(x, \dot x) - \frac{d}{dt}\frac{\partial L}{\partial \dot x}(x, \dot x) \right) \delta x_t - (\nabla \cdot \delta x_t)\eta_t(x) \right] d^3x.$$

where

$$\mu_t(x) = \frac{d}{dt}(\lambda_t \circ\ ) \circ \eta_t^{-1}(x).$$

But from Gauss' divergence theorem we have

$$\int_\Omega (\nabla \cdot Y)(x)\mu(x)\, d^3x = \int_\Omega [\nabla \cdot (\mu Y) - \nabla\mu \cdot Y]\, d^3x =$$
$$= \int_{\partial\Omega} \mu(Y, \bar n) - \int_\Omega \nabla\mu \cdot Y.$$

Applying this formula we finally get

$$0 = \int_{t_0}^{t_1} dt \int_{\eta_t(D)} \left[\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x}) + \nabla \mu_t(x) \right] \delta x_t \, d^3x - \int_{t_0}^{t_1} dt \int_{\partial \eta_t(D)} \mu(\delta x_t, \bar{n}).$$

At any event, we need the two separate integrals to be zero. From the first integral we have that

$$\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x}) = -\nabla \mu_t(x).$$

We consider two posibilities now.

(*a*) If the variations ∂x_t are arbitrary on the boundary, we need

$$\mu_t|_{\partial D} = 0.$$

(*b*) If the δx_t are parallel to the boundary, the second integral is automatically zero and there is no additional condition on μ_t.

Before we state our results, let us introduce some notation. The Euler-Lagrange operator will be denoted by $[L]_x$. In local coordinates

$$[L]_x = \left[\frac{\partial L}{\partial x}(x,\dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x,\dot{x}) \right] dx.$$

This is a well defined 1-form on M, once a curve $\eta_t(X) = x$ on P has been chosen. Here

$$\dot{x} = \dot{\eta}_t(X).$$

Summarizing the previous calculations we have

Lemma 1. *Let η_t be a curve on P_{vol} with $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed, and λ_t a curve on $\mathfrak{F}(D)$, and let*

$$\mu_t = \frac{d\lambda_t}{dt} \circ \eta_t^{-1}.$$

Then the following statements are equivalent.

(i) (η_t, λ_t) *is a critical point of* $\mathfrak{A}(\eta, \lambda)$ *in the set of curves* (η_t, λ_t) *such that* $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$, *and* ∂D_t *fixed (i.e.* $\delta\eta_t \,\|\, \partial D_t$*).*

(ii) $[L]_x = d\mu_t(x)$, $x \in \eta_t(D)$.

We will need now the following lemma, which gives a particular version of the Lagrange Multipliers Theorem.

Lemma 2. *Let η_t be a curve on P_{vol}. The following statements are equivalent*

(i) *η_t is a critical point of $\int_{t_0}^{t_1} \mathcal{L}(\eta_t, \dot{\eta}_t)\,dt$ on curves η_t belonging to P_{vol} with fixed end points $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ and ∂D_t fixed (i.e. $\partial\eta_t \parallel \partial D_t$).*

(ii) *There exists a curve $\lambda_t \in \mathfrak{F}(D)$ such that (η_t, λ_t) is a critical point of the functional $\mathfrak{A}$ on curves $\eta_t \in P$, $\lambda_t \in \mathfrak{F}(D)$ with the conditions $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$.*

Proof. That (ii) implies (i) is easy to check.

To prove that (i) implies (ii), we must show the global existence of λ_t.

Let η_t be a curve on P_{vol} satisfying (i). A variation $\eta_{t\epsilon}$ of η_t on P_{vol} can be constructed as follows.

Let Z be a vector field on D which is divergence-free ($\operatorname{div} Z = 0$) and parallel to the boundary ($Z \parallel \partial D$). Then for each t, $\eta_t * Z = Z_t$ is a vector field on $D_t = \eta_t(D)$ such that $\operatorname{div} Z_t = 0$ and $Z_t \parallel \partial D_t$. Let $\varphi(t, \epsilon)$ be any real valued function defined for $t \in [t_0, t_1]$ and $\epsilon > 0$. For our particular purposes, φ will be taken to be a bump function in the variable t for each ϵ, approximating the Dirac Delta function at $T \in [t_0, t_1]$ and satisfying $\varphi(t_0, \epsilon) = \varphi(t_1, \epsilon) = 0$.

Define $Z_{t\epsilon} = \varphi(t, \epsilon) Z_t$. Thus for each t, $Z_{t\epsilon}$ satisfies $\operatorname{div} Z_{t\epsilon} = 0$, $Z_{t\epsilon} \parallel \partial D_t$. For each t, let $F_{t\epsilon}$ be the flow of $Z_{t\epsilon}$ for $\epsilon > 0$. So for each t and ϵ, $F_{t\epsilon}: D_t \to D_t$ is a diffeomorphism. Define

$$\eta_{t\epsilon} = F_{t\epsilon} \circ \eta_t,$$

then $\eta_{t\epsilon}$ is a variation of η_t on P_{vol} satisfying

$$\eta_{t_0\epsilon} = \eta_0, \qquad \eta_{t_1\epsilon} = \eta_1$$

and

$$\left(\frac{d\eta_{t\epsilon}}{d\epsilon}(X)\right) \Big\| \, \partial D_t \quad \text{for all} \quad \epsilon > 0.$$

In general, if we are given a vector field $Z_{t\epsilon}$ for each t, ϵ such that $\operatorname{div} Z_{t\epsilon} = 0$, $Z_{t\epsilon} \parallel \partial D_t$ depending smoothly on the parameters, then we can construct in a similar way, a variation $\eta_{t\epsilon}$ as before.

Now we must compute

$$\frac{d}{d\epsilon} \int_{t_0}^{t_1} \mathcal{L}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)]\,dt = \frac{d}{d\epsilon} \int_{t_0}^{t_1} dt \int_D L[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)]\,d^3X$$

$$= \int_{t_0}^{t_1} dt \int_D \left\{ \frac{\partial L}{\partial x}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] \right\} \frac{d}{d\epsilon} \eta_{t\epsilon}(X)\,d^3X.$$

Since $\eta_{t\epsilon}$ is volume preserving we can change variables $x = \eta_{t\epsilon}(X)$, so that the right hand side equals

$$\int_{t_0}^{t_1} dt \int_{D_t} [L]_x Z_{t\epsilon}(x)\, d^3x.$$

Now choose $\varphi(t, \epsilon)$ such that $\varphi(t, \epsilon) \to \delta(t - T)$ for $\epsilon \to 0$. Then we finally get the condition

$$\int_{D_T} [L]_x Z_T(x)\, d^3x = 0.$$

At this point, we should remark that $Z_T = \eta_T * Z$ can be chosen to be an arbitrary vector field on D_T except for the conditions $\operatorname{div} Z_T = 0$, $Z_T \parallel \partial D_T$. By Hodge theorem, we can conclude that there exists μ_T globally defined on D_T and such that

$$[L]_x = d\mu_T(x), \qquad x \in D_T.$$

We leave to the reader to check that even though $\mu_T(x)$ is determined up to a constant, however we can choose $\mu_T(x)$ to be a C^∞ function of x, T and satisfying the previous condition.

To finish the proof, define λ_t by

$$\lambda_t(X) = \int^t \mu_t \circ \eta_t(X)\, dt$$

and apply Lemma 1. □

In order to state our main result, we need some notation. Let $\eta_t : M \to M$ be a curve on Diff (M) such that $J_{\eta_t} = 1$, *i.e.* η_t is volume preserving. For each $D \subseteq M$, a compact submanifold of M with boundary ∂D, define

$$\eta_t^D = \eta_t|_D$$

and denote by P^D the principal bundle of embeddings of D into M and by $P^D_{\text{vol}} \subseteq P^D$ the principal bundle of volume preserving embeddings. Define $\mathcal{L}^D_{\text{vol}}: TP^D_{\text{vol}} \to \mathbb{R}$ by

$$\mathcal{L}^D_{\text{vol}}(\eta_t^D, \dot\eta_t^D) = \int_D L[\eta_t^D(X), \dot\eta_t^D(X)]\, d^3X.$$

We also define for a given C^∞ curve λ_t on $\mathfrak{F}(M)$, $\lambda_t^D = \lambda_t|_D$ and

$$\mathcal{L}^D[\eta_t^D, \dot\eta_t^D, \lambda_t^D, \dot\lambda_t^D] = \int_D \left[L(\eta_t^D(X), \dot\eta_t^D(X)] + \lambda_t(X) \frac{d}{dt} J_{\eta_t}(X) \right] d^3X.$$

Theorem. *The following conditions on a curve $\eta_t \in \mathrm{Diff}_{\mathrm{vol}}(M)$ are equivalent.*

(i) *There exists* $\mu_t : M \to \mathbb{R}$, *a* C^∞ *curve on* $\mathfrak{F}(M)$ *such that*

$$[L]_{\eta_t(X)} = d\mu_t(\eta_t(X)).$$

(ii) *For each* D, η_t^D *is a critical point of*

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D[\eta_t(X),\ \dot{\eta}_t(X)]\, dt$$

on the set of curves η_t *on* P_{vol}^D *with fixed end point conditions* $\eta_{t_0} = \eta_{t_0}^D$, $\eta_{t_1} = \eta_{t_1}^D$ *and* ∂D_t *fixed (i.e.* $\delta\eta_t \parallel \partial D_t$*).*

(iii) *There exists a* C^∞ *curve* $\lambda_t \in \mathfrak{F}(M)$ *such that for each* D, (η_t^D, λ_t^D) *is a critical point of*

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D(\eta_t,\ \dot{\eta}_t, \lambda_t,\ \dot{\lambda}_t)\, dt$$

on the set of curves $(\eta_t, \lambda_t) \in P^D \times \mathfrak{F}(D)$ *with conditions* $\eta_{t_0} = \eta_{t_0}^D$, $\eta_{t_1} = \eta_{t_1}^D$ *and* ∂D_t *fixed (i.e.* $\delta\eta_t \parallel \partial D_t$*).*

Notice that η_t *and* λ_t *are related by*

$$\mu_t \circ \eta_t = \frac{d\lambda_t}{dt}.$$

PROOF. We first prove that (i) implies (ii). Let $D \subseteq M$ as before,

$$\mu_t^D = \mu_t \big|_{\eta_t(D)}$$

and

$$\lambda_t^D = \int^t \mu_t^D \circ \eta_t^D.$$

Thus by Lemma 1, (η_t^D, λ_t^D) is a critical point of $\mathfrak{A}(\eta^D, \lambda^D)$. By Lemma 2, we conclude that (ii) holds.

We now prove that (ii) implies (i). Using Lemma 2 and Lemma 1 we get for each D a function $\mu_t^D : \eta_t^D(D) \to \mathbb{R}$ such that

$$[L]_{\eta_t^D(X)} = d\mu_t^D[\eta_t^D(X)].$$

This shows that the Euler-Lagrange operator $[L]_{\eta_t^D(X)}$ is exact on $\eta_t^D(D)$. Since $\eta_t^D(D)$ can be chosen to be any given compact submanifold with boundary of M (having the same dimension as M), this immediately implies that μ_t^D can be taken as being the restriction to D of a globally defined 0-form μ_t.

Similarly, we can easily prove the equivalence between (iii) and (i) or (ii) by using Lemmas 1 and 2 if we define λ_t by

$$\mu_t \circ \eta_t = \frac{d\lambda_t}{dt}.$$

3. Hamiltonian Vector Fields

Let us consider a symplectic manifold M with volume element $\Omega = \omega^n$ where $\omega = d\alpha$ is its canonical 2-form and ω^n is the exterior power of order n. This problem was studied by Lacomba and Losco [4] for the case where M is a compact manifold with boundary.

We construct the principal fiber bundles P with structure group G and P_{vol} with structure group G_{vol} as in the general theory.

In this case we define, for a given curve η_t in $P_{\text{vol}} \subseteq P$ and the corresponding curve $Z_t = \dot{\eta}_t \cdot \eta_t^{-1}$ of vector fields on M, the Lagrangian

$$L[\eta_t(X),\ \dot{\eta}_t(X)] = i_{Z_t(X)}\alpha = \alpha[Z_t(X)].$$

For any compact submanifold with boundary $D \subseteq M$ as in Section 2, this induces a Lagrangian $\mathcal{L}^D_{\text{vol}}: TP^D_{\text{vol}} \to \mathbb{R}$ by

$$\mathcal{L}^D_{\text{vol}}(\eta^D_t,\ \dot{\eta}^D_t) = \int_D L[\eta^D_t(X),\ \dot{\eta}^D_t(X)]d^{2n}x.$$

If η^D_t is a critical point of $\int_{t_0}^{t_1} \mathcal{L}^D_{\text{vol}}[\eta_t(X),\ \dot{\eta}_t(X)]\,dt$ on the set of curves η_t on P^D_{vol} with fixed endpoint conditions $\eta_{t_0} = \eta^D_{t_0}$, $\eta_{t_1} = \eta^D_{t_1}$ and ∂D_t fixed, the main result implies the existence of a C^∞ curve $\mu_t: M \to \mathbb{R}$. From [4] and considering each $D \subseteq M$, we conclude that $\mu_t = H_t$ is a Hamiltonian function and Z_t is the associated Hamiltonian vector field. This means that the critical curves preserve not only the volume Ω, but also the symplectic from ω.

Notice that the arbitrariness of H_t permits to get any given Hamiltonian vector field.

We remark that this construction is still valid if (M, ω) is a non exact symplectic manifold. Since ω is closed, consider two different local expressions $\omega = d\alpha$ and $\omega = d\tilde{\alpha}$. Hence, $\tilde{\alpha} = \alpha + \gamma$ where γ is a closed form.

They produce two different but equivalent Lagrangians L, $\tilde{L}$ such that $\tilde{L} = L + \gamma$. It can be proved that the corresponding integrals

$$\int_{t_0}^{t_1} \mathcal{L}^D_{\text{vol}}[\eta_t(X),\ \dot{\eta}_t(X)]\,dt$$

give the same variational principle.

Indeed, we can write in local coordinates

$$\begin{aligned}\delta \int_{t_0}^{t_1} \mathcal{L}^D_{\text{vol}}[\eta_t(X),\ \dot{\eta}_t(X)]\,dt &= \int_{t_0}^{t_1} dt \int_D \{\omega[\dot{\eta}_t(X), \delta\eta_t(X)] - d\mu[\dot{\eta}_t(X), \delta\eta_t(X)]\}\, d^{2n}x \\ &= \int_{t_0}^{t_1} dt \int_{D_t} [\omega(\dot{x}, \delta x) - dH(\dot{x}, \delta x)]\, d^{2n}x.\end{aligned}$$

A related result for non exact symplectic manifolds appears in [1].

4. Free Boundary Problems

Free boundary problems, like a liquid incompressible homogeneous drop with surface tension, or a free elastic body, can be studied by using methods like those described in the previous sections. In this paper we will concentrate on the example of the liquid drop. A setting for this, from the Hamiltonian point of view can be found in [5]. However we may use part of that framework for our purposes, within the variational approach.

Let us denote P_{vol} the principal bundle of embeddings of the unit closed ball $D \subseteq \mathbb{R}^3$ into $\mathbb{R}^3$. Thus a curve $\eta_t \in P_{\text{vol}}$ represents a motion of the liquid drop. Note that the base B of the bundle P_{vol} consists of the set of all $\Sigma \subseteq \mathbb{R}^3$ where Σ is a 2-submanifold of $\mathbb{R}^3$ diffeomorphic to ∂D. Obviously every $\Sigma \in B$ can be written $\Sigma = \eta(\partial D)$ for some $\eta \in P_{\text{vol}}$. The group acting on P_{vol} on the right, is $G_{\text{vol}} = \text{Diff}_{\text{vol}}(D)$.

The surface tension coefficient being τ and the density being 1 and assuming that gravitational forces are absent, we can write the Lagrangian for the liquid drop as follows

$$\mathcal{L}(\eta_t, \dot{\eta}_t) = \int_D \frac{1}{2}[\dot{\eta}_t(X)]^2 \, d^3X - \tau \int_{\Sigma_t} d\Sigma_t$$

where $d\Sigma_t$ represents the area element on $\Sigma_t = \eta_t(\partial D)$.

Now suppose that η_t is a curve on P_{vol} which is a critical point of the functional $\int_{t_0}^{t_1} \mathcal{L}(\eta_t, \dot{\eta}_t)\, dt$ on the set of curves η_t such that $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed (note that we are not imposing here the condition ∂D_t fixed; thus variations $\delta\nu_t$ are allowed such that they are not necessarily assumed to be parallel to ∂D_t).

As we did in Section 2 where we assumed the condition $\delta\eta_t \parallel \partial D_t$, we can show that the problem of finding a critical curve η_t as stated above is equivalent to the problem at finding a critical curve (η_t, λ_t) of the functional

$$\mathfrak{A}(\eta, \lambda) = \int_{t_0}^{t_1} \left[\mathcal{L}(\eta, \dot{\eta}) + \lambda_t \frac{dJ_{\eta_t}}{dt} \right] dt$$

on curves (η_t, λ_t) with $\lambda \in \mathfrak{F}(D)$, $\eta_t \in P$, $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed.

By a similar computation to the one performed before the statement of Lemma 1 in Section 2, we can find that for a variation $\delta\eta_t$ with $\delta\eta_{t_0} = 0$, $\delta\eta_{t_1} = 0$, we have

$$0 = \int_{t_0}^{t_1} \int \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x, \dot{x}) + \nabla\mu_t(x) \right] \delta x_t \, d^3x - \int_{t_0}^{t_1} dt \int_{\partial D_t} \mu_t(x)(\delta x_t, \vec{n})\, d\Sigma - \tau \int_{t_t}^{t_1} dt \int_{\partial D_t} K(x)(\delta x_t, \vec{n})\, d\Sigma$$

where K is the mean curvature of ∂D_t and integrals on ∂D_t are both surface integrals (here, we are implicitly assuming the standard Riemannian metric given on $\mathbb{R}^3$).

The last term comes out as follows (see [5] for more details). A given variation $\eta_{t\epsilon}$ induces a variation $\eta_{t\epsilon}(\partial D) = D_{t\epsilon}$. Thus

$$\frac{d}{d\epsilon}\int_{D_{t\epsilon}} d\Sigma\,\bigg|_{\epsilon=0} = \int_{D_t} K(\delta x_t, \bar{n})\, d\Sigma.$$

A simple argument shows that equality to 0 for arbitrary variations δx_t will imply

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\nabla\mu_t, \qquad \text{for} \quad x \in D_t,$$

$$\mu_t(x) = -\tau K(x), \quad \text{for} \quad x \in \partial D_t,$$

the incompressibility condition $J_{\eta_t} = 1$ comes out after variations $\delta\lambda_t$ are considered, as usual. Putting all this together and taking into account that

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2,$$

we finally get

$$\frac{\partial^2 x}{\partial t^2} = -\nabla\mu_t \circ \eta_t, \quad \text{on} \quad D$$

$$\mu_t = \tau K, \qquad \text{on} \quad \partial D_t$$

$$J_{\eta_t} = 1, \qquad \text{on} \quad D.$$

We can write these equations in Eulerian (rather than Lagrangian) variables. Namely let

$$v = \frac{\partial x}{\partial t} \circ \eta_t^{-1}$$

be the Eulerian velocity.

Then we get

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v$$

and $J_{\eta_t} = 1$ implies $\nabla \cdot v = 0$. Thus

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla\mu, \quad \text{on} \quad D_t,$$

$$\nabla \cdot v = 0, \quad \text{on} \quad D_t,$$
$$\mu_t = 2K, \quad \text{on} \quad \partial D_t.$$

These are precisely the equations of motion of the liquid drop with surface tension.

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Existence of Solutions for some Elliptic Problems with Critical Sobolev Exponents

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geqslant 3$. In this paper we are concerned with the problem of finding $u \in H_0^1(\Omega)$ satisfying the nonlinear elliptic problems

$$\Delta u + |u|^{\frac{n+2}{n-2}} + f(x) = 0 \tag{1.1}$$

in Ω and $u(x) = 0$ on $\partial\Omega$, and

$$\Delta u + u + |u|^{\frac{n+2}{n-2}} + f(x) = 0 \tag{1.2}$$

in Ω and $u(x) = 0$ on $\partial\Omega$, when of $f \in L^\infty(\Omega)$.

The exponent $q = (n+2)/(n-2)$ is critical for the Sobolev embedding of $H_0^1(\Omega)$ in $L^{q+1}(\Omega)$. This embedding is not compact and therefore the operator $F: H_0^1(\Omega) \to H_0^1(\Omega)$, for which its fixed points are solutions of (1.1) or (1.2), is not compact. For these reasons, standard fixed-points methods can not be applied to find solutions of (1.1) and (1.2). In this paper we study (1.1) and (1.2) by making use of a fixed point theorem as well as one from approximation methods. Problems of type (1.1) and (1.2) have been studied in [2] and [4]. In these papers the authors find positive solutions. Their methods are variational and their work is related to the Yamabe Problem. For a complete description of the Problem of Yamabe, we refer to [6].

If $f = 0$, it is well known that the equation (1.1) has no positive solutions. (See [2 p. 422].) If $f \neq 0$ we will see in Theorem 3.3 that the equation (1.1) has a nontrivial solution and in the case $f > 0$, by the maximum principle, we have that the equation (1.1) has a positive solution.

Also, if $f = 0$, the equation (1.2) has a positive solution in the case that $1 \in (0, \lambda_1)$, where λ_1 denote the first eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω. (See [2, p. 441].) If $f \neq 0$, we will see in Theorem 3.4 that equation (1.2) has nontrivial solution and that if $f > 0$ then the solution is positive.

2. Preliminaries

Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by $g(x, u) = |u|^s + f$ in the case of problem (1.1), or $g(x, u) = u + |u|^s + f$, in the case of problem (1.2), and $0 < s \leqslant (n + 2)/(n - 2)$. Then the operator of Nemytsky $G: L^{s+1}(\Omega) \to L^{\frac{s+1}{s}}(\Omega)$ defined by $G(u)(x) = g(x, u(x))$ is continuous and bounded, so that for every $\epsilon > 0$ there exists $r = r(\epsilon)$ such that if $\|u\|_{L^{s+1}(\Omega)} \leqslant r$, then $\|g(x, u) - g(x, 0)\|_{L^{(s+1)/s}(\Omega)} \leqslant \epsilon$ and we have the following inequality, (see [5, p. 26]),

$$\|g(x, u)\|_{L^{(s+1)/s}(\Omega)} \leqslant \left[\left(\frac{\|u\|_{L^{s+1}(\Omega)}}{r}\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|f\|_{L^{(s+1)/s}(\Omega)}. \tag{2.1}$$

For short, we will indicate $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$, for all $p > 0$.

It is well-known (see [1, p. 40]) that if $1 \leqslant s + 1 < 2n/(n - 2)$, $n \geqslant 3$, the inclusion $H_0^1(\Omega) \to L^{s+1}(\Omega)$ is completely continuous. If $1 \leqslant s + 1 \leqslant 2n/(n - 2)$ the inclusion is only continuous and we have

$$\|u\|_{s+1} \leqslant \hat{K}(s)\|u\|_{1,2}, \tag{2.2}$$

where $\|\cdot\|_{1,2}$ is the norm of the space $H_0^1(\Omega)$. In the case $s = 1$, $\hat{K}(1) = 1/\sqrt{\lambda_1}$, where λ_1 is the first eigenvalue of the operator $-\Delta$.

If

$$s = \frac{n+2}{n-2}$$

then

$$\hat{K}\left(\frac{n+2}{n-2}\right) = \frac{n-1}{n-2}\frac{1}{\sqrt{n}}. \tag{2.3}$$

(see [1, p. 41]).

Since Measure $(\Omega) = |\Omega|$ is finite then

$$\|u\|_{s+1} \leqslant |\Omega|^{\frac{1}{s+1} - \frac{n-2}{2n}} \cdot \|u\|_{\frac{2n}{n-2}}. \tag{2.4}$$

(2.2) – (2.4) yield, for all $u \in H_0^1(\Omega)$,

$$\|u\|_{s+1} \leqslant K(s)\|u\|_{1,2}, \tag{2.5}$$

where

$$K(s) = \frac{1}{\sqrt{n}} \frac{n-1}{n-2} |\Omega|^{\frac{1}{s+1} - \frac{n-2}{2n}}.$$

SOLUTIONS OF (1.1) AND (1.2). Let

$$g(x, u) = |u|^s + f \quad \text{or} \quad g(x, u) = |u|^s + u + f, \qquad s \leqslant \frac{n+2}{n-2}\cdot$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.1) and (1.2) respectively, if for all $v \in H_0^1(\Omega)$

$$\langle u, v\rangle_{1,2} = \int_\Omega g(x, u(x))v(x)\,dx. \tag{2.6}$$

For $u \in H_0^1(\Omega)$ fixed, the right side of (2.6) defines a linear, continuous functional. Then by Riesz's Theorem there exists $F\colon H_0^1(\Omega) \to H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\langle F(u), v\rangle_{1,2} = \int_\Omega g(x, u)v \tag{2.7}$$

Then, by virtue of (2.6) and (2.7), $u \in H_0^1(\Omega)$ is a weak solution of (1.1) or (1.2) if and only if u is a fixed point of F. It is well-known that only for $s < (n+2)/(n-2)$, F is completely continuous. Our main tool will be the following Theorem due to Krasnosel'skii.

Theorem 2.1. *Let $F\colon H \to H$ be a completely continuous operator defined on a Hilbert space H. Let $D \subset H$ be an open and bounded set such that $0 \notin \partial D$. Suppose that for all $u \in \partial D$, $\langle F(u), u\rangle \leqslant \|u\|^2$. Then F has a fixed point in $\bar{D}$.*

PROOF. See [1, p. 271].

3. The Main Results

First, we are concerned with the following general problem

$$\left.\begin{aligned} \Delta u + g(x, u) &= 0 \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where $g: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is measurable in x, continuous in u and satisfies

$$|g(x, u)| \leqslant a(x) + b|u|^s, \tag{3.2}$$

where $a(x) \in L^{\frac{s+1}{s}}(\Omega)$, $b > 0$, $1 < s + 1 < 2n/(n-2)$ and $n \geqslant 3$. Then we have

Theorem 3.1. *Suppose that* $g(x, u)$ *satisfies* (3.2) *and* $0 < s < 1$, *then* (3.1) *has a weak solution.*

PROOF. In the same way as (2.1) we have

$$\|g(x, u)\|_{\frac{s+1}{s}} \leqslant \left[\left(\frac{\|u\|_{s+1}}{r}\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|g(x, 0)\|_{\frac{s+1}{s}}. \tag{3.3}$$

(2.5), (2.7) and (3.3) give

$$\langle F(u), u\rangle_{1,2} \leqslant \left\{\left[\left(\frac{K}{r}\|u\|_{1,2}\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|g(x, 0)\|_{\frac{s+1}{s}}\right\} K\|u\|_{1,2}. \tag{3.4}$$

We claim that there exists $y > 0$ such that for $\|u\|_{1,2} = y$

$$\left\{\left[\left(\frac{K}{r}y\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|g(x, 0)\|_{\frac{s+1}{s}}\right\} Ky \leqslant y^2. \tag{3.5}$$

Since $0 < s < 1$ it is easy to see that, for $y > 0$ sufficiently large, we get (3.5). For (3.4) and (3.5) we have that $\langle F(u), u\rangle_{1,2} \leqslant \|u\|_{1,2}^2$, where $\|u\|_{1,2}$ is sufficiently large. Then Theorem 3.1 follows from Theorem 2.1.

Remark. Theorem 3.1 is a consequence of Theorem 2.5 in [3] (see Example 2.9, p. 122). There, Example 2.9 is done from a variational point of view.

Theorem 3.2. *Suppose that* $g(x, u)$ *satisfies* (3.2) *and* $s = 1$. *Then* (3.1) *has a weak solution if* $\lambda_1 > b$.

PROOF. If $\|u\|_2 \leqslant r$ then $\|g(x, u)\|_2 \leqslant \|a(x)\|_2 + br$. Let $\epsilon = \|a(x)\|_2 + br$. (3.5) yields

$$(K^2y^2 + r^2)^{1/2}\frac{\epsilon}{r} + \|g(x, 0)\|_2 \leqslant \frac{y}{K}. \tag{3.6}$$

It is clear that there exists $y > 0$ such that y satisfies (3.6) if

$$\frac{1}{K^2} > K^2\frac{\epsilon^2}{r^2}. \tag{3.7}$$

In this case, we know that $K^2 = \frac{1}{\lambda_1}$, therefore (3.7) is equivalent to

$$\lambda_1 > \frac{\|a(x)\|_2}{r} + b. \tag{3.8}$$

For r sufficiently large we get (3.8). Theorem 2.1 now implies our result.

Now we will return to our main problems (1.1) and (1.2). We have the following

Theorem 3.3. *Assume that $f \in L^\infty(\Omega)$. Suppose that at least one of the following inequalities holds*

$$\|f\|_\infty < B(n)|\Omega|^{-\frac{n+2}{2n}}, \quad \textit{if} \quad |\Omega| > 1, \tag{3.9}$$

or

$$\|f\|_\infty < B(n), \qquad \textit{if} \quad |\Omega| \leqslant 1, \tag{3.10}$$

where

$$B(n) = \frac{\dfrac{4}{n+2}\left(\dfrac{n-2}{n+2}\right)^{\frac{n-2}{4}}}{\left(\dfrac{n-1}{n-2}\dfrac{1}{\sqrt{n}}\right)^{\frac{n+2}{2}}}, \qquad n \geqslant 3.$$

Then problem (1.1) *has at least a weak solution if we assume that $\partial\Omega$ is sufficiently smooth.*

PROOF. First we will consider the following problem

$$\left.\begin{aligned} \Delta u + |u|^s + f = 0 \quad &\text{in} \quad \Omega, \\ u(x) = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned}\right\} \tag{3.11}$$

and $1 < s < \frac{n+2}{n-2}$.

Let $g(x, u) = |u|^s + f$. As in Theorems 3.1 and 3.2, the problem (3.11) has a weak solution $u_s \in H_0^1(\Omega)$ if there exists $y > 0$ such that

$$\left[\left(\frac{K}{r}y\right)^{s+1} + 1\right]^{\frac{s}{s+1}} \epsilon + \|f\|_{\frac{s+1}{s}} \leqslant \frac{y}{K}, \tag{3.12}$$

where K is the constant in (2.5). In this case, it is easy to see that $r = r(\epsilon) = \epsilon^{1/s}$.

Therefore (3.12) becomes

$$[(K^2x)^{s+1} + \epsilon^{\frac{s+1}{s}}]^{\frac{s}{s+1}} + \|f\|_{\frac{s+1}{s}} \leqslant x, \tag{3.13}$$

where $x = y/K$.

Since we can take ϵ sufficiently small, (3.13) has a solution $x > 0$ if

$$(K^2x)^s + \|f\|_{\frac{s+1}{s}} < x \tag{3.14}$$

has a solution $x > 0$.

Now, it is easy to see that (3.14) has the solution

$$x_0 = \left(\frac{1}{sK^{2s}}\right)^{\frac{1}{s-1}}$$

if

$$\|f\|_{\frac{s+1}{s}} < \frac{\left(\frac{1}{s}\right)^{\frac{1}{s-1}} - \left(\frac{1}{s}\right)^{\frac{s}{s-1}}}{K^{\frac{2s}{s-1}}} = \lambda(s). \tag{3.15}$$

Finally,

$$\lim_{s \to \frac{n-2}{n-2}} \lambda(s) = B(n). \tag{3.16}$$

Also, for $s \in R$ such that $1 < s < \dfrac{n+2}{n-2}$ we have

$$\|f\|_{\frac{s+1}{s}} \leqslant \|f\|_\infty |\Omega|^{\frac{s}{s+1}}.$$

Therefore, if $|\Omega| > 1$, (3.9) yields

$$\|f\|_{\frac{s+1}{s}} \leqslant \|f\|_\infty |\Omega|^{\frac{s}{s+1}} < \|f\|_\infty |\Omega|^{\frac{n+2}{2n}} < B(n). \tag{3.17}$$

Or, if $|\Omega| \leqslant 1$, by (3.10) we have

$$\|f\|_{\frac{s+1}{s}} \leqslant \|f\|_\infty < B(n). \tag{3.18}$$

Using (3.16), (3.17) and (3.18) we get that there exists s_0 such that $1 < s_0 < \dfrac{n+2}{n-2}$ and, if $s \in \left(s_0, \dfrac{n+2}{n-2}\right)$, then (3.14) has a solution $x > 0$ and therefore (3.11) has a weak solution.

Now, by Theorem 2.1 we have that for $s \in \left(s_0, \dfrac{n+2}{n-2}\right)$ the weak solution $u_s \in H_0^1(\Omega)$ of (3.11) satisfies,

$$\|u_s\|_{1,2} \leqslant y_0 = Kx_0 = \left(\frac{1}{s}\right)^{\frac{1}{s-1}} K^{\frac{1+s}{1-s}}. \tag{3.19}$$

By (3.19) we obtain that for $s \in \left(s_0, \dfrac{n+2}{n-2}\right)$ the set $\{u_s\}$, such that u_s is a weak solution of (3.11), is bounded. Then there exists $\{u_k\} \subset \{u_s\}$ such that

$$W \lim_{k \to \frac{n+2}{n-2}} u_k = u_{\frac{n+2}{n-2}}, \tag{3.20}$$

for some $u_{\frac{n+2}{n-2}} \in H_0^1(\Omega)$. ($W\lim$ indicates weak limit.)

For simplicity we will use $\dfrac{n+2}{n-2} = N$.

Our next step is to show that u_N is a weak solution of (1.1).

Since $f \in L^\infty(\Omega)$, by an iterative argument called a bootstrapping procedure, (see [1, p. 50], we can see that $u_s \in H_0^1(\Omega)$, the weak solution of (3.11), satisfies that $u_s \in C^{0,\alpha}(\bar{\Omega})$, and since $\partial\Omega$ is sufficiently smooth, u_s is, in particular, continuous on $\bar{\Omega}$.

As in (2.7) let $F_s: H_0^1(\Omega) \to H_0^1(\Omega)$ be defined as

$$\langle F_s(u), v\rangle_{1,2} = \langle |u|^s + f, v\rangle_2, \tag{3.21}$$

for all $v \in H_0^1(\Omega)$.

For $s < N$, F_s is completely continuous. As we saw, there exists $s_0 < N$ such that for $s \in (s_0, N)$, $F_s(u_s) = u_s$, where u_s is a weak solution of (3.11).

Let $\{u_k\}$ a subsequence of $\{u_s\}$, $s \in (s_0, N)$, such that $u_k \rightharpoonup u_N$ if $k \to N$. ($\rightharpoonup$ denotes weak convergence). By (3.21) we have

$$\langle F_k(u_k), v\rangle_{1,2} = \langle u_k, v\rangle_{1,2} = \langle |u_k|^k + f, v\rangle_2, \tag{3.22}$$

for all $v \in H_0^1(\Omega)$. And therefore, for all $v \in H_0^1(\Omega)$

$$\lim_{k \to N} \langle |u_k|^k + f, v\rangle_2 = \langle u_N, v\rangle_{1,2}. \tag{3.23}$$

On the other hand, u_k, $k \in (s_0, N)$, is continuous. Hence

$$\lim_{r \to N} \| |u_k|^r - |u_k|^N \|_{\frac{2n}{n+2}} = 0,$$

and by Lebesgue's dominated convergence Theorem we have that $|u_k|^r \to |u_k|^N$ in $L^{\frac{2n}{n+2}}(\Omega)$ if $r \to N$. Now, for each $v \in H_0^1(\Omega)$ fixed, $\langle v, \bullet\rangle_2$ defines a linear and bounded functional on $L^{\frac{2n}{n+2}}(\Omega)$. Hence, for all $v \in H_0^1(\Omega)$

$$\lim_{r \to N} \langle |u_k|^r + f, v\rangle_2 = \langle |u_k|^N + f, v\rangle_2. \tag{3.24}$$

Also, $\{u_k\} \subset H_0^1(\Omega)$ is bounded, and since $H_0^1(\Omega)$ is embedded in $L^{\frac{2n}{n+2}}(\Omega)$, $\{u_k\}$ is bounded in $L^{\frac{2n}{n+2}}(\Omega)$ as well. Furthermore the Nemytsky operator defined by $|u|^N$ is bounded, so that $\{|u_k|^N\}$ is bounded in $L^{\frac{2n}{n+2}}(\Omega)$. Therefore, there exists

$h \in_{L^{2n/(n+2)}}(\Omega)$ and a subsequence of $\{u_k\}$, labeled in the same form, such that for all $v \in H_0^1(\Omega)$

$$\lim_{k \to N} \langle |u_k|^N + f, v \rangle_2 = \langle h + f, v \rangle_2. \tag{3.25}$$

By (3.24) and (3.25) we get

$$\lim_{r, k \to N} \langle |u_k|^r + f, v \rangle_2 = \langle h + f, v \rangle_2, \tag{3.26}$$

for each $v \in H_0^1(\Omega)$. (3.23) and (3.26) yield

$$\langle h + f, v \rangle_2 = \langle u_N, v \rangle_{1,2}, \tag{3.27}$$

for each $v \in H_0^1(\Omega)$.

Also, since $u_k \rightharpoonup u_N$ in $H_0^1(\Omega)$, we have

$$\lim_{k \to N} \langle |u_k|^r + f, v \rangle_2 = \langle |u_N|^r + f, v \rangle_2, \tag{3.28}$$

for all $v \in H_0^1(\Omega)$, and $r < N$. (3.23), (3.26), (3.27) and (3.28) yield

$$\lim_{r \to N} \langle |u_N|^r + f, v \rangle_2 = \langle h + f, v \rangle_2, \tag{3.29}$$

for each $v \in H_0^1(\Omega)$. Since $|u_N|^r v \leqslant (|u_N|^N + 1)|v|$, by Lebesgue's dominated convergence Theorem we have

$$\lim_{r \to N} \langle |u_N|^r + f, v \rangle_2 = \langle |u_N|^N + f, v \rangle_2, \tag{3.30}$$

for each $v \in H_0^1(\Omega)$. (3.30) and (3.29) yield

$$\langle |u_N|^N + f, v \rangle_2 = \langle u_N, v \rangle_{1,2}, \tag{3.31}$$

for each $v \in H_0^1(\Omega)$. By (3.31) we have that $u_N \in H_0^1(\Omega)$ is a weak solution of (1.1).

Theorem 3.4. *For $f \in L^\infty(\Omega)$ and for $n \geqslant 5$, suppose that*

$$|\Omega|^{2/n} < L(n), \tag{3.32}$$

and that at least one of following inequalities holds

$$\|f\|_\infty < \frac{A(n)}{|\Omega|^{\frac{n+2}{2n}}}, \quad \textit{if} \quad |\Omega| > 1, \tag{3.33}$$

or

$$\|f\|_\infty < A(n), \quad \textit{if} \quad |\Omega| \leqslant 1, \tag{3.34}$$

where

$$(3.35)\qquad L(n)=\frac{\left[\frac{4}{n+2}\left(\frac{n-2}{n+2}\right)^{\frac{n-2}{4}}\right]^{\frac{2}{n}}-\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{2\frac{n+2}{n-2}}}{\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{2}}$$

and

$$(3.36)\qquad A(n)=\frac{\left(\frac{n-2}{n+2}\right)^{\frac{n^2-4}{8n}}-\left(\frac{n-2}{n+2}\right)^{\frac{(n+2)^2}{8n}}}{\left[\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{2}|\Omega|^{\frac{2}{n}}+\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{\frac{2(n+2)}{n-2}}\right]^{\frac{n-2}{4}}}.$$

Then problem (1.2) *has at least a weak solution if we suppose that* $\partial\Omega$ *is sufficiently smooth.*

PROOF. As in Theorem 3.3 we will consider here the problem

$$(3.37)\qquad \left.\begin{aligned}\Delta u+u+|u|^s+f&=0 \quad \text{in} \quad \Omega,\\ u(x)&=0 \quad \text{on} \quad \partial\Omega,\end{aligned}\right\}$$

with $1<s<\dfrac{n+2}{n-2}$.

Let $g(x,u)=u+|u|^s+f$. Then if $\|u\|_{s+1}\leqslant r$, $\|u+|u|^s\|_{\frac{s+1}{s}}\leqslant\epsilon$ and the relationship betwen r and ϵ can be taken to be

$$(3.38)\qquad r|\Omega|^{\frac{s-1}{s+1}}+r^s=\epsilon.$$

As in the Theorem 3.3, problem (3.37) has a weak solution whenever there exists $x>0$ such that

$$(3.39)\qquad [(K^2x)^{s+1}+r^{s+1}]^{\frac{s}{s+1}}\frac{\epsilon}{r^s}+\|f\|_{\frac{s+1}{s}}\leqslant x.$$

If we take

$$(3.40)\qquad r=K^2,$$

K as in (2.5), then (3.39) becomes

$$(3.41)\qquad (x^{s+1}+1)^{\frac{s}{s+1}}\epsilon+\|f\|_{\frac{s+1}{s}}\leqslant x.$$

Now, it is easy to see that

$$x_0 = \left(\frac{1}{s}\right)^{\frac{s}{s^2-1}} \epsilon^{-\left(\frac{1}{s-1}\right)}$$

is a solution of (3.41) if the following two inequalities hold:

$$\epsilon^{\frac{s+1}{s-1}} < \left(\frac{1}{s}\right)^{\frac{1}{s-1}} - \left(\frac{1}{s}\right)^{\frac{s}{s-1}} \tag{3.42}$$

and

$$\|f\|_{\frac{s+1}{s}} < \frac{\left(\frac{1}{s}\right)^{\frac{s}{s^2-1}} - \left(\frac{1}{s}\right)^{\frac{s^2}{s^2-1}}}{\epsilon^{\frac{1}{s-1}}}. \tag{3.43}$$

Now, by (3.38), $\epsilon = |\Omega|^{\frac{s-1}{s+1}} K^2 + K^{2s}$, K as in (2.5). If we take the limit when $s \to N$, in both sides of (3.42) we obtain (3.32). Then (3.32) implies that there exists $s_1 \in R$, $1 < s_1 < N$, such that for all $s \in (s_1, N)$ the inequality (3.42) holds. Also, by (3.33) or (3.34) there exists $s_2 < N$ such that, for all $s \in (s_2, N)$ the inequality holds. Let $s_0 = \text{Max}\{s_1, s_2\}$; then for all $s \in (s_0, N)$ problem (3.37) has a weak solution. We may argue as in Theorem 3.3 and repeat all the formulas in (3.23) to (3.30) and obtain that there exists $u_N \in H_0^1(\Omega)$ a weak solution of the problem (1.2).

Remarks. If $|\Omega| < 1$, Theorem 3.4 holds for $n \geqslant 4$ and in that case, for $n \geqslant 5$, (3.32) is superfluous.

I am grateful to Dr. Yu Takeuchi who showed me that

$$\lim_{n\to\infty} A(n) = \lim_{n\to\infty} B(n) = \lim_{n\to\infty} L(n) = \infty.$$

Following the same argument of Theorem 3.4 we have the following

Theorem 3.5. *For $f \in L^\infty(\Omega)$ and $n \geqslant 4$ suppose that*

$$|\lambda|\,|\Omega|^{2/n} < L(n), \tag{3.44}$$

and that at least one of following inequalities holds

$$\|f\|_\infty < \frac{A(|\lambda|, n)}{|\Omega|^{\frac{n+2}{2n}}}, \quad \text{if} \quad |\Omega| > 1 \tag{3.45}$$

or

$$\|f\|_\infty < A(|\lambda|, n), \quad \text{if} \quad |\Omega| \leqslant 1, \tag{3.46}$$

where

$$(3.47)\quad A(|\lambda|,n)=\frac{\left(\frac{n-2}{n+2}\right)^{\frac{n^2-4}{8n}}-\left(\frac{n-2}{n+2}\right)^{\frac{(n+2)^2}{8n}}}{\left[\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^2|\lambda|\,|\Omega|^{\frac{2}{n}}+\left(\frac{n-1}{n-2}\frac{1}{\sqrt{n}}\right)^{\frac{2(n+2)}{n-2}}\right]^{\frac{n-2}{4}}}.$$

Then the problem

$$(3.48)\quad \left.\begin{aligned}\Delta u+\lambda u+|u|^N+f&=0 \quad \text{in} \quad \Omega,\\ u(x)&=0 \quad \text{on} \quad \partial\Omega,\end{aligned}\right\}$$

has at least a weak solution.

If we suppose that $f>0$ on Ω then, by the Maximum principle for the operator Δ, we have that problems (1.1), (1.2) and (3.48) for $\lambda\geqslant 0$, have positive solutions.

Finally, in Theorem 3.4, if $|\Omega|$ is sufficiently small then $\epsilon=|\Omega|^{\frac{s-1}{s+1}}+r^s$ is small too ($r=K^2$). Then inequality (3.41) holds for some $x>\|f\|_\infty$. We conclude by saying that Problem (1.2) has a weak solution if $|\Omega|$ is sufficiently small.

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