Revista Matemática Iberoamericana
Vol. 6, N.os 1 y 2, 1990

On a Min-Max Procedure for the Existence of a Positive Solution for Certain Scalar Field Equations in $\mathbb{R}^N$

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Abstract

In this paper we mainly introduce a min-max procedure to prove the existence of positive solutions for certain semilinear elliptic equations in $\mathbb{R}^N$.

Introduction

In this paper, we investigate the existence of positive solutions for the following semilinear elliptic equation in $\mathbb{R}^N$:

$$\begin{cases} -\Delta u + u - q(x)|u|^{p-1}u = 0 \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $1 < p < \dfrac{N+2}{N-2}$, if $N \geqslant 3$; $1 < p < +\infty$, if $N = 1, 2$ and $q \in L^\infty(\mathbb{R}^N)$ satisfies the following hypotheses:

$$q(x) > 0 \quad \text{for every} \quad x \in \mathbb{R}^N. \tag{2}$$

There exists a positive constant, denoted as q_∞, such that

$$\lim_{|x| \to \infty} q(x) = q_\infty \tag{3}$$

and there exist some positive constants C and δ such that, for $|x|$ large,

$$q(x) \geqslant q_\infty - C \exp(-\delta|x|). \tag{4}$$

Under the hypotheses (2), (3) and (4), we are able to prove the existence of at least one positive solution to (1). However the main purpose of this paper is to derive the result through a min-max procedure, similar in spirit to [6], even though we have to replace (4) by a stronger hypothesis:

There exist some positive constants C and δ such that, for $|x|$ large,

$$q(x) \geqslant q_\infty - C \exp(-(2+\delta)|x|). \tag{5}$$

Without loss of generality we can assume that $\delta < p - 1$.

The existence of solutions of semilinear elliptic equations in $\mathbb{R}^N$ have been investigated, among others, in [3], [8], [13], [14], [5], [2], [18], [19], [11].

Weiyue Ding and Wei-Ming Ni established in [8] that (1) has at least one positive solution if $q(x) > 0$ is radially symmetric and bounded, for large $|x|$, by $|x|^l$ where l satisfies $0 < l < \frac{(N-1)(p-1)}{2}\cdot$ On the other hand Yi Li ([12]) has proved that for $N \geqslant 3$ (1) has no positive solution if $q(x) \geqslant 0$, $q \in C^{0,1}(\mathbb{R}^N)$ is radially symmetric and $q(x)|x|^{-(N-1)(p-1)/2}$ is nondecreasing in $|x|$. However, very little is known for the existence of nontrivial solution of (1) if $q(x)$ is not radially symmetric. In [8] [13], variational methods are used to prove the existence of a positive solution of (1) under various hypotheses, which ensure the existence of the global minimum of the functional associated with (1). Therefore when the minimum is not achieved the existence problem is left open. It is proved by A. Bahri and P. L. Lions in [2] that if we consider the problem on $\mathbb{R}^N \backslash \Omega$ instead of $\mathbb{R}^N$, where Ω is any smooth bounded open set, then the existence of positive solution can be established even when the minimum of the functional is not achieved. There the topology of $\mathbb{R}^N \backslash \Omega$ has been used. Our present situation is different since $\mathbb{R}^N$ is a contractable set. Therefore the technique developed in [2] cannot be applied directly. However, with the observation we have here, we can modify the argument of [2] to prove the existence of positive solution without using the topology of the domain.

In [5] and [11] the existence of multiple solutions has been studied under various hypotheses on $q(x)$.

1. Preliminaries

We first introduce some notations. Let

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + uv$$

denote the inner product of $H^1(\mathbb{R}^N)$ and

$$\|u\| = \langle u, u\rangle^{1/2}$$
$$\Sigma = \{u \in H^1(\mathbb{R}^N) \colon \|u\| = 1\}$$
$$\Sigma^+ = \{u \in \Sigma \colon u \geqslant 0 \text{ almost everywhere in } \mathbb{R}^N\}.$$

Let

$$J(u) = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2}{\left(\int_{\mathbb{R}^N} q(x)|u|^{p+1}\right)^{2/(p+1)}}$$
$$J_\infty(u) = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2}{\left(\int_{\mathbb{R}^N} q_\infty|u|^{p+1}\right)^{2/(p+1)}}$$
$$S_1 = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} J_\infty(u)$$
$$S_m = m^{(p-1)/(p+1)}S_1, \qquad m = 2, 3, 4, \ldots$$

It is well known that S_1 is actually achieved by some radially symmetric positive smooth function ω, which satisfies the following equation

$$\begin{cases} -\Delta\omega + \omega - q_\infty \omega^p = 0, \\ \omega \in H^1(\mathbb{R}^N). \end{cases} \tag{6}$$

It has also been proved by Kwong ([10]) recently that the positive solution of (6) is actually unique up to translations. Furthermore we know exactly how ω behaves at $+\infty$.

Theorem 1.1. *Let $q(x) \in L^\infty(\mathbb{R}^N)$ satisfy* (2), (3) *and* (5). *Then* (1) *has a positive solution in $H^1(\mathbb{R}^N)$ for $1 < p < (N+2)/(N-2)$,* if $N \geqslant 3$; $1 < p < +\infty$, *if $N = 1$,* 2.

Remark 1.1. The regularity of the solution found in Theorem 1.1 follows from standard elliptic theory.

Proposition 1.1. *Let ω be the positive solution of* (6), *then $\omega \in C^\infty(\mathbb{R}^N)$ and is radially symmetric after suitable translation, namely, $\omega = \omega(|x|)$. Further-*

more, there exists some positive constant $c > 0$, *such that,*

$$\omega(|x|)|x|^{(N-1)/2}\exp(|x|) \to c \quad as \quad |x| \to \infty \tag{7}$$

$$\omega'(|x|)|x|^{(N-1)/2}\exp(|x|) \to -c \quad as \quad |x| \to \infty \tag{8}$$

The proof of the above proposition can be found in [2] and the references there.

Proposition 1.2. *Let* $\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\psi \in C(\mathbb{R}^N)$ *be radially symmetric and satisfy for some* $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \in \mathbb{R}$

$$\varphi(x)\exp(\alpha|x|)|x|^\beta \to \gamma \quad as \quad |x| \to \infty, \tag{9}$$

$$\int_{\mathbb{R}^N} |\psi(x)| \exp(\alpha|x|)(1 + |x|^\beta) < \infty. \tag{10}$$

Then

$$\left(\int_{\mathbb{R}^N} \varphi(x+y)\psi(x)\,dx\right)\exp(\alpha|y|)|y|^\beta \to \gamma \int_{\mathbb{R}^N} \psi(x)\exp(-\alpha x_1)\,dx \quad as \quad |y| \to \infty.$$

PROOF. Since φ, ψ are radially symmetric, we only need to obtain the limit for $y = (|y|, 0, \dots, 0)$, $|y| \to +\infty$.

In the following we then assume that $y = (|y|, 0, \dots, 0)$. We first identify the pointwise limit of the integrand.

$$\begin{aligned}&\varphi(x+y)\psi(x)\exp(\alpha|y|)|y|^\beta \\ &= \psi(x)\varphi(x+y)\exp(\alpha|x+y|)|x+y|^\beta \exp(-\alpha|x+y|)\exp(\alpha|y|)|x+y|^{-\beta}|y|^\beta\end{aligned}$$

The limit (9) implies that

$$\varphi(x+y)\exp(\alpha|x+y|)|x+y|^\beta \to \gamma$$

pointwise as $|y| \to +\infty$. On the other hand,

$$\lim_{|y| \to +\infty} |x+y|^{-\beta}|y|^\beta = 1$$

is obvious.

Using the fact that $y = (|y|, 0, \dots, 0)$, we have

$$\begin{aligned}\exp(-\alpha|x+y|)\exp(\alpha|y|) &= \exp(-\alpha(|x|^2 + 2x \cdot y + |y|^2)^{1/2})\exp(\alpha|y|) \\ &= \exp\left(-\alpha|y|\left(1 + 2\frac{x \cdot y}{|y|} + \frac{|x|^2}{|y|^2}\right)^{1/2}\right)\exp(\alpha|y|) \\ &= \exp\left(-\alpha|y| - \alpha x_1 + O\left(\frac{1}{|y|}\right)\right)\exp(\alpha|y|).\end{aligned}$$

Therefore

$$\lim_{\substack{|y| \to +\infty \\ y = (|y|, 0, \dots, 0)}} \exp(-\alpha|x+y|)\exp(\alpha|y|) = \exp(-\alpha x_1).$$

Putting the above calculations together, we obtain that

$$\lim_{\substack{|y| \to +\infty \\ y = (|y|, 0, \dots, 0)}} \varphi(x+y)\psi(x)\exp(\alpha|y|)|y|^{\beta} = \gamma\psi(x)\exp(-\alpha x_1).$$

It follows from (9) that

$$\begin{aligned} &|\varphi(x+y)\psi(x)\exp(\alpha|y|)|y|^{\beta}| \\ &\qquad \leqslant C|\psi(x)|\exp(-\alpha|x+y|)(1+|x+y|)^{-\beta}\exp(\alpha|y|)|y|^{\beta} \\ &\qquad \leqslant C|\psi(x)|\exp(\alpha|x|)(1+|x+y|)^{-\beta}|y|^{\beta}. \end{aligned}$$

For $|y| \geqslant 2|x|$,

$$|\varphi(x+y)\psi(x)\exp(\alpha|y|)|y|^{\beta}| \leqslant C|\psi(x)|\exp(\alpha|x|).$$

For $|y| \leqslant 2|x|$,

$$|\varphi(x+y)\psi(x)\exp(\alpha|y|)|y|^{\beta}| \leqslant C|\psi(x)|\exp(\alpha|x|)|x|^{\beta}.$$

Now we can simply apply the Lebesgue dominated convergence theorem to conclude the proof of Proposition 1.2.

To apply a min-max method, we need to analyze where the functional J satisfies certain compactness condition. This has been well known by now. In the following, we are going to state some compactness lemmas. For the proof, see [13], [14] and [2].

Definition 1.1. *$J|_{\Sigma}$ is said to satisfy Palais-Smale condition at C', if for any Palais-Smale sequence $\{u_n\} \in \Sigma$, namely, $\{u_n\} \in \Sigma$, $J(u_n) \to C'$, $J|'_{\Sigma}(u_n) \to 0$ strongly in $H^1(\mathbb{R}^N)$, there exists a subsequence of $\{u_n\}$ which converges strongly in $H^1(\mathbb{R}^N)$.*

Lemma. *Let $\{u_n\} \in \Sigma$ be a Palais-Smale sequence, then there exists a subsequence (still denoted as $\{u_n\}$) for which the following holds: there exists an integer $m \geqslant 0$, sequences x_n^i for $1 \leqslant i \leqslant m$, functions $\bar{u}$, ω_i for $1 \leqslant i \leqslant m$ such that*

(11) $$-\Delta\bar{u} + \bar{u} = q(x)|\bar{u}|^{p-1}\bar{u} \quad in \quad \mathbb{R}^N, \qquad \bar{u} \in H^1(\mathbb{R}^N)$$

(12) $$-\Delta\omega_i + \omega_i = q_{\infty}|\omega_i|^{p-1}\omega_i \quad in \quad \mathbb{R}^N, \qquad \omega_i \in H^1(\mathbb{R}^N)$$

(13) $$|x_n^i| \to \infty, \qquad |x_n^i - x_n^j| \to \infty \quad as \quad n \to \infty \quad for \quad 1 \leqslant i \neq j \leqslant m$$

(14) $$u_n - \frac{\bar{u} + \sum_{i=1}^{m} \omega_i(\cdot - x_n^i)}{\left\| \bar{u} + \sum_{i=1}^{m} \omega_i(\cdot - x_n^i) \right\|} \to 0 \quad \textit{strongly in} \quad H^1(\mathbb{R}^N) \quad \textit{as} \quad n \to \infty$$

(15) $$J(u_n) \to \frac{\|\bar{u}\|^2 + \sum_{i=1}^{m} \|\omega_i\|^2}{\left(\int_{\mathbb{R}^N} q|\bar{u}|^{p+1} + \sum_{i=1}^{m} \int_{\mathbb{R}^N} q_\infty |\omega_i|^{p+1} \right)^{\frac{2}{p+1}}} \quad \textit{as} \quad n \to \infty$$

where we agree that in the case $m = 0$ *the above holds without* ω_i *and* x_n^i. *In addition, if* $u_n \geqslant 0$ *then* $\bar{u} \geqslant 0$, $\omega_i \geqslant 0$ *for all* $1 \leqslant i \leqslant m$. *Therefore* $\omega_i \equiv \omega$ *for all* $1 \leqslant i \leqslant m$ *due to the uniqueness upto translation of positive solution of* (6) (*see* [10]).

This result immediately implies that if $u_n \geqslant 0$ in the PS Lemma and (1) has no positive solution, then

(16) $$J(u_n) \to S_m$$

where $m \geqslant 1$ is some integer.

We are going to use a contradiction argument for the existence of positive solution of (1), namely, we start with the assumption that (1) has no positive solution, then we know that (PS) condition fails only at levels $\{S_m\}$. Furthermore if we are able to construct some min-max value which is different from $\{S_m\}$, the standard deformation lemma will give rise to a positive solution to (1), hence a contradiction. The purpose of the next two sections is to construct such a min-max value which is strictly between S_1 and S_2.

2. Energy Estimates

The energy estimates given in this section play an important role for the proof of the existence result. These estimates have essentially been given in [2], while we do need a result which is slightly different from that in [2]. This kind of phenomenon have been used by C. Taubes ([17]), A. Bahri and J. M. Coron ([1]), A. Bahri and P. L. Lions ([2]).

We first state a result which will be used later.

Lemma 2.1. *Let* $p > 1$ *be any real number, then there exists some constant* $C = C(p)$, *such that, for any nonnegative real numbers* a, b,

(17) $$(a + b)^{p+1} \geqslant a^{p+1} + b^{p+1} + (p + 1)(a^p b + ab^p) - Ca^{(p+1)/2} b^{(p+1)/2}$$

(In fact, if $p \geqslant 2$, *we may take* $C = C(p) = 0$.)

PROOF. If $a = 0$ or $b = 0$, (17) is obvious. Otherwise we assume, without loss of generality, that $a \leqslant b$, $ab \neq 0$. Let $x = a/b$, then $0 < x \leqslant 1$, (17) is equivalent to the following:

$$(1 + x)^{p+1} \geqslant 1 + x^{p+1} + (p + 1)(x + x^p) - Cx^{(p+1)/2}. \tag{18}$$

For x small, it follows from Taylor expansion that

$$(1 + x)^{p+1} = 1 + (p + 1)x + \frac{p(p + 1)}{2}x^2 + o(x^3),$$

therefore there exists some positive number $\delta_0 = \delta_0(p)$ between 0 and 1, such that, (18) holds for $0 < x \leqslant \delta_0$, $C \geqslant p + 1$. For $\delta_0 \leqslant x \leqslant 1$, we can choose $C = C(p)$ large enough to guarentee (18).

Lemma 2.2. *Under the hypotheses of Theorem* 1.1, *there exists a large number R_0, such that, for any $R \geqslant R_0$, $|x_1| \geqslant R$, $|x_2| \geqslant R$, $\sqrt{R} \leqslant |x_1 - x_2| \leqslant (2 + 1/2\sqrt{R}\,)\min\{|x_1|, |x_2|\}$, we have*

$$J(tu_1 + (1 - t)u_2) < S_2 \tag{19}$$

where $0 \leqslant t \leqslant 1$ and $u_i = \omega(\bullet - x_i)$.

PROOF. Let us first prove (19) for $t = 1/2$, namely,

$$J(u_1 + u_2) < S_2. \tag{20}$$

Let

$$A = \|\omega\|^2$$

then from (6) we know that

$$A = \int q_\infty \omega^{p+1}.$$

The following computation holds for large R.

$$\begin{aligned} J(u_1 + u_2) &= \frac{\|u_1 + u_2\|^2}{\left(\int q(u_1 + u_2)^{p+1}\right)^{2/(p+1)}} \\ &= \frac{\|u_1\|^2 + \|u_2\|^2 + 2\langle u_1, u_2\rangle}{\left(\int q_\infty(u_1 + u_2)^{p+1} + \int (q - q_\infty)(u_1 + u_2)^{p+1}\right)^{2/(p+1)}} \\ &\leqslant \frac{2A + 2\langle u_1, u_2\rangle}{\left(\int q_\infty(u_1 + u_2)^{p+1} - \int (q_\infty - q)^+(u_1 + u_2)^{p+1}\right)^{2/(p+1)}} \end{aligned}$$

We first estimate $\int u_1^p u_2$. In the following C will denote some positive constant independent of R.

$$\begin{aligned}\int u_1^p u_2 &= \int \omega(|x-x_1|)^p \omega(|x-x_2|)\,dx \\ &\geqslant \int_{|x-x_1|\le 1} \omega(|x-x_1|)^p \omega(|x-x_2|)\,dx \\ &\geqslant \frac{1}{C}\int_{|x-x_1|\le 1} \omega(|x-x_2|)\,dx \\ &\geqslant \frac{1}{C}\,\omega\,(|x_1-x_2|+1) \\ &\geqslant \frac{1}{C}\,|x_1-x_2|^{(1-N)/2}\exp(-|x_1-x_2|)\end{aligned}$$

The last inequality follows from (7).

Secondly we estimate $\int u_1^{(p+1)/2} u_2^{(p+1)/2}$.

$$\begin{aligned}\int u_1^{(p+1)/2} u_2^{(p+1)/2} &= \int \omega(|x-x_1|)^{(p+1)/2}\omega(|x-x_2|)^{(p+1)/2}\,dx \\ &\leqslant o(1)|x_1-x_2|^{(1-N)/2}\exp(-|x_1-x_2|) \\ &\leqslant o(1)\int u_1^p u_2\end{aligned}$$

where (17) and Proposition 1.2 have been used.

Here and in the sequel, $o(1)\to 0$ as R goes to ∞.

Thirdly we estimate $\int (q_\infty - q)^+(u_1+u_2)^{p+1}$. Use (5), (7) and Proposition 1.2, we have

$$\begin{aligned}\int (q_\infty-q)^+(u_1+u_2)^{p+1} &\leqslant C\sum_{i=1}^{2}\int (q_\infty-q)^+ u_i^{p+1} \\ &\leqslant C\sum_{i=1}^{2}\int \exp(-(2+\delta)|x|)\omega(|x-x_i|)^{p+1}\,dx \\ &\leqslant C\sum_{i=1}^{2}\exp\left(-\frac{2+\delta}{2}|x_i|\right) \\ &\leqslant C\sum_{i=1}^{2}\exp\left(-\frac{2+\delta}{2}\,\frac{|x_1-x_2|}{2+\dfrac{1}{2\sqrt{R}}}\right) \\ &= o(1)|x_1-x_2|^{(1-N)/2}\exp(-|x_1-x_2|) = o(1)\int u_1^p u_2\end{aligned}$$

where we have used the hypothesis

$$\sqrt{R} \leqslant |x_1 - x_2| \leqslant \left(2 + \frac{1}{2\sqrt{R}}\right) \min\{|x_1|, |x_2|\}.$$

According to Lemma 2.1 and the above estimates, we deduce that

$$\begin{aligned}
&\int q_\infty (u_1 + u_2)^{p+1} \\
&\quad \geqslant \int q_\infty (u_1^{p+1} + u_2^{p+1}) + (p+1) \int q_\infty (u_1^p u_2 + u_1 u_2^p) - C \int u_1^{(p+1)/2} u_2^{(p+1)/2} \\
&\quad = 2A + (2p+2)\langle u_1, u_2\rangle - C \int u_1^{(p+1)/2} u_2^{(p+1)/2} \\
&\quad \geqslant 2A + (2p + 2 - o(1))\langle u_1, u_2\rangle
\end{aligned}$$

where we have used the fact that

$$\int q_\infty u_1^p u_2 = \int q_\infty u_2^p u_1 = \langle u_1, u_2\rangle,$$

which follows from (6).

Therefore

$$\begin{aligned}
J(u_1 + u_2) &\leqslant \frac{2A + 2\langle u_1, u_2\rangle}{\left(\int q_\infty (u_1 + u_2)^{p+1} - \int (q_\infty - q)^+ (u_1 + u_2)^{p+1}\right)^{2/(p+1)}} \\
&\leqslant \frac{2A + 2\langle u_1, u_2\rangle}{(2A + (2p + 2 - o(1))\langle u_1, u_2\rangle)^{2/p+1)}} \\
&= S_2 \frac{1 + \frac{1}{A}\langle u_1, u_2\rangle}{\left(1 + \frac{p + 1 - o(1)}{A}\langle u_1, u_2\rangle\right)^{2/(p+1)}} \\
&= S_2 \frac{1 + \frac{1}{A}\langle u_1, u_2\rangle}{1 + \frac{2p + 2 - o(1)}{(p+1)A}\langle u_1, u_2\rangle}
\end{aligned}$$

Notice that $\frac{1}{A} < \frac{2p + 2 - o(1)}{(p+1)A}$ for R large, we obtain (20) from the above estimates. Next we are going to prove (19).

Let

$$\begin{cases} v_1 = t u_1 \\ v_2 = (1 - t) u_2 \end{cases}$$

where $0 \leqslant t \leqslant 1$.

$$\begin{aligned}\|v_1 + v_2\|^2 &= \|v_1\|^2 + \|v_2\|^2 + 2\langle v_1, v_2\rangle \\ &= t^2\|u_1\|^2 + (1-t)^2\|u_2\|^2 + 2t(1-t)\langle u_1, u_2\rangle \\ &= (t^2 + (1-t)^2)A + 2t(1-t)\langle u_1, u_2\rangle\end{aligned}$$

$$\int q|v_1 + v_2|^{p+1} \geqslant \int q_\infty |v_1 + v_2|^{p+1} - \int (q_\infty - q)^+ |v_1 + v_2|^{p+1}.$$

When t or $1 - t$ tends to zero, $v_1 + v_2$ tends to u_2 or u_1 and $J(v_1 + v_2)$ consequently converges to S_1. Therefore there exists some small constant $\delta' > 0$, such that, for any $\min\{t, 1-t\} \leqslant \delta'$, (20) holds. Notice that $\delta' > 0$ is independent of R large. In the following we always assume that $\min\{t, 1-t\} \geqslant \delta'$.

Arguing as before, we have,

$$\begin{aligned}&\int q_\infty |v_1 + v_2|^{p+1} \\ &\qquad \geqslant (t^{p+1} + (1-t)^{p+1})A + (p+1)(t^p(1-t) + t(1-t)^p)\langle u_1, u_2\rangle \\ &\qquad\quad - C\int u_1^{2/(p+1)} u_2^{2(p+1)} \\ &\qquad \geqslant (t^{p+1} + (1-t)^{p+1})A + (p+1)(t^p(1-t) + t(1-t)^p - o(1))\langle u_1, u_2\rangle.\end{aligned}$$

We have derived before that

$$\int (q_\infty - q)^+ (u_1 + u_2)^{p+1} = o(1)\langle u_1, u_2\rangle.$$

Notice that $\min\{t, 1-t\} \geqslant \delta'$, we have

$$\int (q_\infty - q)^+ |v_1 + v_2|^{p+1} \leqslant o(1)(t^p(1-t) + t(1-t)^p)\langle u_1, u_2\rangle.$$

Therefore

$$\begin{aligned}&J(v_1 + v_2) \\ &\leqslant \frac{(t^2 + (1-t)^2)A + 2t(1-t)\langle u_1, u_2\rangle}{\{(t^{p+1} + (1-t)^{p+1})A + (p+1)(t^p(1-t) + t(1-t)^p - o(1))\langle u_1, u_2\rangle\}^{2/(p+1)}} \\ &\leqslant \frac{t^2 + (1-t)^2}{(t^{p+1} + (1-t)^{p+1})^{2/(p+1)}} \frac{A}{A^{2/(p+1)}} \\ &\quad \times \frac{1 + \dfrac{2t(1-t)}{(t^2 + (1-t)^2)A}\langle u_1, u_2\rangle}{1 + \left(\dfrac{2(t^p(1-t) + t(1-t)^p)}{(t^{p+1} + (1-t)^{p+1})A} - o(1)\right)\langle u_1, u_2\rangle}.\end{aligned}$$

Since

$$\frac{A}{A^{2/(p+1)}} = S_1 = S_2\frac{2^{2/(p+1)}}{2},$$

we have

$$J(v_1 + v_2) \leqslant S_2 \frac{2^{2/(p+1)}}{2} \frac{t^2 + (1-t)^2}{(t^{p+1} + (1-t)^{p+1})^{2/(p+1)}}$$

$$\frac{1 + \dfrac{2t(1-t)}{(t^2 + (1-t)^2)A} \langle u_1, u_2 \rangle}{1 + \left(\dfrac{2(t^p(1-t) + t(1-t)^p)}{(t^{p+1} + (1-t)^{p+1})A} - o(1) \right) \langle u_1, u_2 \rangle}.$$

Notice that we have the following inequalities:

$$\frac{2^{2/(p+1)}}{2} \frac{t^2 + (1-t)^2}{(t^{p+1} + (1-t)^{p+1})^{2/(p+1)}} \leqslant 1 \quad \text{for} \quad 0 \leqslant t \leqslant 1 \tag{21}$$

$$\frac{t(1-t)}{t^2 + (1-t)^2} < \frac{t^p(1-t) + t(1-t)^p}{t^{p+1} + (1-t)^{p+1}} \quad \text{for} \quad 0 < t < 1 \tag{22}$$

(21) follows from the convexity of the function $x \mapsto x^{(p+1)/2}$, $x \geqslant 0$ and (22) is elementary.

Since t and $1 - t$ are bounded away from zero, we have

$$1 + \frac{2t(1-t)}{(t^2 + (1-t)^2)A} \langle u_1, u_2 \rangle < 1 + \left(\frac{2(t^p(1-t) + t(1-t)^p)}{(t^{p+1} + (1-t)^{p+1})A} - o(1) \right) \langle u_1, u_2 \rangle$$

for R sufficiently large.

Finally (19) follows from the above estimates.

3. The min-max Procedure

Let us define a map from Σ to the unit ball of $\mathbb{R}^N$

$$m(u) = \frac{1}{\|u\|_{p+1}^{p+1}} \int \frac{x}{|x|} |u|^{p+1} \, dx$$

where $\|u\|_{p+1}$ denotes the $L^{p+1}(\mathbb{R}^N)$ norm of u.

Clearly, m is continuous from Σ to $\mathbb{R}^N$ and $|m(u)| < 1$.

Let

$$I_a = \inf_{\substack{m(u) = a \\ u \in \Sigma}} \frac{\|u\|^2}{\left(\int q|u|^{p+1} \right)^{2/(p+1)}}$$

where $a \in \mathbb{R}^N$ and $|a| < 1$.

It is well known that (1) has a positive solution, which is actually a constant multiple of the global minimum of the functional J, if $\inf_{u\in\Sigma} J(u) < S_1$. (See [13].)

In the following we always consider the case when $\inf_{u\in\Sigma} J(u) \geqslant S_1$.

If for some $a \in \mathbb{R}^N$, $|a| < 1$, $I_a = S_1$, then there exists some $u \in \Sigma^+$, $m(u) = a$, such that $J(u) = S_1$. This follows from the concentration-compactness principle developed in [13]. In this case we will also obtain a solution of (1) which is a constant multiple of the global minimum of J. See [13] for details.

There is only one possibility left, namely,

$$I_a > S_1 \quad \text{for every} \quad a \in \mathbb{R}^N, \qquad |a| < 1.$$

Fix any $a \in \mathbb{R}^N$, $|a| < 1$, since $I_a > S_1$, there exists some positive constant R_1, such that,

$$J(\omega(\bullet - y)) < \frac{1}{2}(I_a + S_1) < I_a \quad \text{for every} \quad y \in \mathbb{R}^N, \qquad |y| \geqslant R_1. \tag{23}$$

Let $R > \max\{R_0, R_1\}$ be very large. (In the following R is always supposed to be very large), and

$$B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$$
$$\bar{x}_2 = \left(0, \dots, 0, R - \sqrt{R}\right)$$

We define a map h_0 from $\partial B_R(0)$ to Σ^+ by

$$h_0(x_1) = \frac{\omega(\bullet - x_1)}{\|\omega(\bullet - x_1)\|}$$

where $x_1 \in \partial B_R(0)$.

Since $R > R_1$, we have

$$J(h_0(x_1)) < \frac{1}{2}(I_a + S_1) < I_a \quad \text{for} \quad x_1 \in \partial B_R(0). \tag{24}$$

We define another map h^* from $B_R(0)$ to Σ^+ by

$$h^*(tx_1 + (1-t)\bar{x}_2) = \frac{t\omega(\bullet - x_1) + (1-t)\omega(\bullet - \bar{x}_2)}{\|t\omega(\bullet - x_1) + (1-t)\omega(\bullet - \bar{x}_2)\|}$$

where $0 \leqslant t \leqslant 1$, $x_1 \in \partial B_R(0)$. It is clear that $h^*|_{\partial B_R(0)} = h_0$.

It follows from Lemma 2.2 that

$$J(h^*(y)) < S_2 \quad \text{for all} \quad y \in B_R(0). \tag{25}$$

We next define some min-max value. Let

$$\Gamma = \{h: B_R(0) \to \Sigma^+ : h \text{ is continuous}, \ h|_{\partial B_R(0)} = h_0\} \tag{26}$$

and

$$c_0 = \inf_{h \in \Gamma} \max_{y \in B_R(0)} J(h(y)). \tag{27}$$

We will prove that

$$S_1 < I_a \leqslant c_0 < S_2 .$$

(25) implies that $c_0 < S_2$.
Consider the map

$$m \circ h: \quad B_R(0) \to R^N$$

for $h \in \Gamma$.
It is quite obvious that

$$\lim_{R \to +\infty} m \circ h_0(x_1) = \frac{x_1}{\|x_1\|} \quad \text{uniformly for} \quad x_1 \in \partial B_R(0). \tag{28}$$

By degree theory, for R large enough, we have

$$\deg(m \circ h, B_R(0), a) = 1.$$

In particular there exists some $y \in B_R(0)$, such that, $m \circ h(y) = a$, hence $c_0 \geqslant I_a > S_1$.

The PS Lemma in Section 1 guarentees that $J|_{\Sigma^+}$ satisfies PS condition at c_0 if (1) has no solution. Then by using some standard deformation argument and the maximum principle (see [16], [2]), c_0 is a critical value of $J|_\Sigma$ with some corresponding critical point $u > 0$, *i.e.*, $J|'_\Sigma(u) = 0$. By scaling u we obtain a positive solution of (1). Therefore in any case, (1) has a positive solution. The proof of Theorem 1.1 has been completed.

Remark 3.1. Under the hypotheses of Theorem 1.1, there exists a positive solution u with

$$J(u) < S_2 .$$

Remark 3.2. If we repace $\mathbb{R}^N$ by $\mathbb{R}^N \setminus \bar{\Omega}$ where Ω is any bounded smooth open set of $\mathbb{R}^N$, the proof still holds after simple modification.

With the observation in this paper, it is not difficult to see that we can modify the arguments in [2] to prove the following result.

Theorem 3.1. *Suppose that* $q(x) \in L^\infty(\mathbb{R}^N)$ *satisfy* (2), (3) *and* (4), *then* (1) *has a positive solution in* $H^1(\mathbb{R}^N)$ *for* $1 < p < (N+2)/(N-2)$, if $N \geqslant 3$; $1 < p < +\infty$, if $N = 1, 2$.

Acknowledgement. The authors want to express their sincere thanks to Professor Nirenberg for his useful comments and encouragement.

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Recibido: 31 de octubre de 1989.

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* Partially supported by NSF grant DMS-8907849.

Revista Matemática Iberoamericana
Vol. 6, n.^{os} 1 y 2, 1990

Para-Accretive Functions, the Weak Boundedness Property and the *Tb* Theorem

Y.-S. Han and E. T. Sawyer

Abstract

G. David, J.-L. Journé and S. Semmes have shown that if b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$, then the «*Tb* Theorem» holds: A linear operator T with Calderón-Zygmund kernel is bounded on L^2 if and only if $Tb_1 \in \mathrm{BMO}$, $T^*b_2 \in \mathrm{BMO}$ and $M_{b_2}TM_{b_1}$ has the weak boundedness property. Conversely they showed that when $b_1 = b_2 = b$, para-accretivity of b is necessary for the *Tb* Theorem to hold. In this paper we show that para-accretivity of both b_1 and b_2 is necessary for the *Tb* Theorem to hold in general. In addition, we give a characterization of para-accretivity in terms of the weak boundedness property and use this to give a sharp *Tb* Theorem for Besov and Triebel-Lizorkin spaces.

1. Introduction

We begin by recalling the definitions necessary for the statement of the *Tb* Theorem of G. David, J.-L. Journé and S. Semmes. For $0 < \eta < 1$, let $C_0^\eta(\mathbb{R}^n)$ denote the space of continuous functions f with compact support such that

$$\|f\|_{\mathrm{Lip}\,\eta} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\eta}$$

is finite. Suppose b_1 and b_2 are complex-valued bounded functions on $\mathbb{R}^n$, and that T is a linear operator such that $M_{b_2}TM_{b_1}$ is continuous from $C_0^\eta(\mathbb{R}^n)$ into its dual $C_0^\eta(\mathbb{R}^n)'$ for all $0 < \eta < 1$. Here M_b denotes the operation of multiplication by b. Suppose further that there is a continuous function $K(x, y)$ on $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$, called the kernel of T, that represents T in the sense that

$$(1.1) \qquad (M_{b_2}TM_{b_1}\varphi)(\psi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b_2(x)K(x, y)b_1(y)\varphi(y)\psi(x)\,dx\,dy$$

for all $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$, $0 < \eta < 1$, with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \varnothing$. We suppose that $K(x, y)$ satisfies the following size and smoothness estimates for some $\epsilon > 0$:

(1.2) (i) $|K(x, y)| \leqslant C|x - y|^{-n}$ for $x, y \in \mathbb{R}^n$,

(ii) $|K(x, y) - K(x', y)| \leqslant C\left(\dfrac{|x - x'|}{|x - y|}\right)^{\epsilon} |x - y|^{-n}$ for $x, x', y \in \mathbb{R}^n$ with $|x - x'| < \dfrac{1}{2}|x - y|$,

(iii) $|K(x, y) - K(x, y')| \leqslant C\left(\dfrac{|y - y'|}{|x - y|}\right)^{\epsilon} |x - y|^{-n}$ for $x, y, y' \in \mathbb{R}^n$ with $|y - y'| < \dfrac{1}{2}|x - y|$.

Kernels with the above properties are called Calderón-Zygmund kernels. See [DJS2] for details and examples.

A complex-valued bounded function is said to be para-accretive if ([DJS2])

(1.3) There is c positive such that for every cube Q in $\mathbb{R}^n$, there is a subcube I with

$$\left|\frac{1}{|Q|}\int_I b(x)\,dx\right| \geqslant c.$$

Note that the cube I in (1.3) satisfies

$$|I| \geqslant \frac{c}{\|b\|_{L^\infty}}|Q|.$$

Finally, a linear operator T from $C_0^\eta(\mathbb{R}^n)$ to $C_0^\eta(\mathbb{R}^n)'$, $0 < \eta < 1$, is said to satisfy the weak boundedness property if

$$(1.4) \qquad |(T\varphi)(\psi)| \leqslant C|Q|^{1 + 2\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{\mathrm{Lip}\,\eta}$$

for all cubes Q and $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ with support in Q. In [DJS2], this definition is shown to be independent of η. We can now state the Tb Theorem of G. David, J.-L. Journé and S. Semmes (see A. McIntosh and Y. Meyer [MM] for the first version of the Tb Theorem).

The Tb Theorem. ([DJS1], [DJS2]). *Suppose b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and that T is a linear operator such that $M_{b_2}TM_{b_1}$ is continuous from $C_0^\eta(\mathbb{R}^n)$ to $C_0^\eta(\mathbb{R}^n)'$ for some $0 < \eta < 1$, with a Calderón-Zygmund kernel $K(x, y)$, i.e.,* (1.1) *and* (1.2) (i), (ii), (iii) *hold. Then T is bounded on L^2 if and only if*

(1.5) (i) $Tb_1 \in \mathrm{BMO}$.
(ii) $T^*b_2 \in \mathrm{BMO}$ (*where T^* denotes the transpose of T*).
(iii) *$M_{b_2}TM_{b_1}$ satisfies the weak boundedness property.*

The reader is referred to Section 1 of [DJS2] for the definition of Tb_1 and T^*b_2- we only point out here that (1.1) and (1.2)(ii) are needed to define Tb_1 while (1.1) and (1.2)(iii) are needed for T^*b_2. We also mention in passing that the hypothesis (1.2)(i) on the size of the kernel $K(x, y)$ is not needed in the Tb Theorem since it is already implied by the other hypotheses. See the end of Section 3.

Conversely, it was shown ([DJS2; Proposition 1 in Section 9]) in the case $b_1 = b_2 = b$ is bounded, that if every linear operator T satifying (1.1), (1.2) and (1.5) is bounded on L^2, then b is para-accretive. The main result of this paper is that this converse result holds in general-namely, the para-accretivity of both b_1 and b_2 is necessary if the Tb Theorem is to hold. Two complex-valued bounded functions b_1 and b_2 are said to be jointly para-accretive if there is $c > 0$ such that for every cube Q in $\mathbb{R}^n$, there is a subcube I with

$$\frac{1}{|Q|} \max\left\{\left|\int_I b_1(x)\,dx\right|, \left|\int_I b_2(x)\,dx\right|\right\} \geqslant c.$$

Theorem 1. *Suppose b_1 and b_2 are complex-valued bounded functions. If b_2 is not para-accretive, then there exists a linear operator T with kernel K satisfying* (1.1), (1.2) (i), (ii), (iii) (*with* $\epsilon = 1$) *and such that*

(1.6) (i) $Tb_1 \in L^\infty$,
(ii) $T^*b_2 \in L^\infty$,
(iii) *$M_{b_2}TM_{b_1}$ has the weak boundedness property,*
(iv) *TM_{b_1} fails to have the weak boundedness property if b_1 and b_2 are jointly para-accretive, while T fails to have the weak boundedness property if b_1 and b_2 are not jointly para-accretive.*

Note that by (1.6)(iv), the operator T in Theorem 1 is not bounded on L^2 and thus the para-accretivity of b_2 is necessary for the Tb Theorem to hold. By duality, the para-accretivity of b_1 is also necessary.

A fairly straightforward consequence of Theorem 1 and a lemma of Y. Meyer ([M1]; Lemme 2) is the following characterization of para-accretivity in terms of the weak boundedness property. We thank Rodolfo Torres for discussions leading to this result. Let $\mathcal{C}$ denote the set of linear operators T with kernel $K(x,y)$ satisfying (1.1) (with $b_1 = b_2 = 1$) and (1.2)(i) and (ii) —but not necessarily (1.2)(iii)— and $T1 = 0$.

Theorem 2. *A complex-valued bounded function b is para-accretive if and only if for every T in $\mathcal{C}$, T has the weak boundedness property* (1.4) *whenever $M_b T$ does.*

Remark. Theorem 2 remains true if $\mathcal{C}$ is replaced by the larger class $\mathcal{C}'$ of operators T with kernel satisfying (1.1) (with $b_1 = b_2 = 1$) and (1.2)(i) and (ii) and $T1 \in \mathrm{BMO}$. See Section 3.

Note by contrast, that Lemme 2 of [M1] shows that for any bounded function b, $M_b T$ has the weak boundedness property whenever T in $\mathcal{C}$ does. We now recall a result of P. G. Lemarié [L].

The $T1$ Theorem for Besov Spaces. ([L]). *Suppose T in $\mathcal{C}$ satisfies the weak boundedness property* (1.4). *Then T is bounded on the homogeneous Besov space $\dot{B}_p^{\alpha,q}$ for $1 \leqslant p, q \leqslant \infty$ and $0 < \alpha < \epsilon$, where ϵ is the order of smoothness of K in the first variable in* (1.2)(ii).

As indicated in Section 14 of [DJS2], Lemarié's Theorem yields a Tb Theorem for Besov spaces —If T satisfies (1.1), (1.2)(i) and (ii) and $Tb_1 = 0$, and if $M_{b_2} T M_{b_1}$ has the weak boundedness property where b_2 is para-accretive, then TM_{b_1} is bounded on $\dot{B}_p^{\alpha,q}$ for $1 \leqslant p, q \leqslant \infty$ and $0 < \alpha < \epsilon$. Note that exactly half of the asymmetric hypotheses in the Tb Theorem (with BMO replaced by 0) are needed here. The other half imply by duality that $M_{b_2} T$ is bounded on $\dot{B}_p^{\alpha,q}$ for $1 \leqslant p, q \leqslant \infty$ and $-\epsilon < \alpha < 0$. See Section 14 of [DJS2] where these results are interpolated to yield another proof of the Tb Theorem for L^2.

In order to reduce this Tb Theorem to the $T1$ Theorem of Lemarié, simply observe that TM_{b_1} is in $\mathcal{C}$ and satisfies the weak boundedness property by the «only if» half of Theorem 2. The «if» half of Theorem 2 shows that the para-accretivity of b_2 cannot be removed.

The above considerations also apply to the homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ once we have shown that the conclusion of Lemarié's Theorem applies to $\dot{F}_p^{\alpha,q}$ in place of $\dot{B}_p^{\alpha,q}$ for $1 < p, q < \infty$ and $0 < \alpha < \epsilon$. The following result has been independently obtained by B. Jawerth, M. Taibleson and G. Weiss ([HJTW]).

Theorem 3. *Suppose T in $\mathcal{C}$ satisfies the weak boundedness property. Then T is bounded on $\dot{F}_p^{\alpha,q}$ for $1 < p, q < \infty$ and $0 < \alpha < \epsilon$, where ϵ is as in* (1.2)(ii).

Theorem 3 is easily obtained by adapting the proof of the $T1$ Theorem for L^2 outlined in Section 2 of [DJS2] and we will sketch the relevant details in Section 4 below. We remark that M. Frazier, Y.-S. Han, B. Jawerth and G. Weiss have shown ([FHJW]) that for T in $\mathcal{C}$ satisfying the weak boundedness property and the additional smoothness (1.2)(iii), T maps $\dot{F}_p^{\alpha,q}$-atoms to $\dot{F}_p^{\alpha,q}$-molecules (and so is bounded on $\dot{F}_p^{\alpha,q}$) for $1 < p, q < \infty$, $0 < \alpha < \epsilon$. Theorem m is proved in Section $(m + 1)$, $m = 1, 2, 3$.

2. Proof of Theorem 1

The proof of Theorem 1 splits into two cases.

Case 1. b_1 and b_2 are jointly para-accretive.

We modify the construction in Proposition 1 of Section 9 of [DJS2] (of an operator for which the *Tb* Theorem fails for a non-para-accretive function $b = b_1 = b_2$) in the spirit of a para-product. The basic idea evolves from the observation that if a Calderón-Zygmund kernel $K(x, y)$ equals $(1 + |x - y|)^{-n}$ for $|x - y| \leqslant N$, $-|x - y|^{-n}$ for $2N \leqslant |x - y| \leqslant N^2$ and zero for $|x - y| > 2N^2$ then $\| T1 \|_{L^\infty} \leqslant C$ and the weak boundedness constant C in (1.4) is at least $c \log N$. Suppose there is $c > 0$ such that for every cube Q in $\mathbb{R}^n$, there is a subcube I with

$$\frac{1}{|Q|} \max \left\{ \left| \int_I b_1(x)\, dx \right|, \left| \int_I b_2(x)\, dx \right| \right\} \geqslant c. \tag{2.1}$$

If b_1, b_2 are bounded in absolute value by M, then (2.1) forces

$$|I| \geqslant \frac{c}{M} |Q|$$

and so the ratio of the side lengths of I and Q is bounded below by

$$\delta = 1 \Bigg/ \left[\left(\frac{M}{c} \right)^{1/n} \right]$$

where $[x]$ denotes the greatest integer part of x. Since b_2 is not para-accretive, we can find a cube Q_k, for each $k > 0$, with the property

$$\sup_{\text{cubes} J \subset 3Q_k} \left| \frac{1}{|Q_k|} \int_J b_2(x)\, dx \right| \leqslant \frac{\delta^{kn}}{k}.$$

Thus

$$\text{(2.2)} \quad \left| \frac{1}{|J|} \int_J b_2(x)\, dx \right| \leqslant \frac{1}{k} \quad \text{for all cubes } J \subset 3Q_k \text{ with } |J|^{1/n} \geqslant \delta^k |Q_k|^{1/n}.$$

Momentarily fix k with $\delta^{kn}/k < c$. Then (2.1) and (2.2) imply that for every cube $J \subset 3Q_k$ with side length at least δ^k times that of Q_k, there is a cube $I \subset J$ of side length at least δ times that of J such that

$$\left| \frac{1}{|I|} \int_I b_1(x)\, dx \right| \geqslant c.$$

Let s_k denote the side length of Q_k. For $j = 0, 1, 2, \ldots, k-1$, let $\{J_i^j\}_{i=1}^{3^n \delta^{-jn}}$ denote the «dyadic» decomposition of $3Q_k$ into $3^n\, \delta^{-jn}$ congruent subcubes of side length $\delta^j s_k$ with pairwise disjoint interiors. For each cube J_i^j whose triple is contained in $3Q_k$, let $(J_i^j)'$ denote the translate of J_i^j by $\delta^j s_k(1, 1, \ldots, 1)$ and then set

$$J_i^{j*} = \frac{1}{3}(J_i^j)'.$$

By (2.1), there is a subcube I_i^j of J_i^{j*} with side length at least $\delta^{j+1} s_k/3$ and satisfying

$$\left| \frac{1}{|I_i^j|} \int_{I_i^j} b_1(x)\, dx \right| \geqslant c$$

(we may suppose $\delta \leqslant 1/3$).

We must now smooth out these averages. We claim that there are Lipschitz functions φ_i^j satisfying

(2.3) (i) $\operatorname{supp} \varphi_i^j \subset I_i^j$,

(ii) $|\varphi_i^j(y)| \leqslant |I_i^j|^{-1}$, for $y \in \mathbb{R}^n$,

(iii) $|\varphi_i^j(y) - \varphi_i^j(y')| \leqslant C \dfrac{|y - y'|}{|I_i^j|^{1/n}} |I_i^j|^{-1}$, for $y, y' \in \mathbb{R}^n$,

(iv) $\left| \int \varphi_i^j(y) b_1(y)\, dy \right| \geqslant c/2$,

where the constant C in (2.3)(iii) depends on M and c in (2.1). To construct the φ_i^j, simply choose φ_i^j to be supported in I_i^j with values between 0 and

$|I_i^j|^{-1}$, and to take the value $|I_i^j|^{-1}$ on γI_i^j where $\gamma < 1$ is so close to 1 that

$$\left|\int \varphi_i^j(y) b_1(y)\, dy\right| = \left|\frac{1}{|I_i^j|}\int_{I_i^j} b_1(y)\, dy + \int\left(\varphi_i^j - \frac{1}{|I_i^j|}\chi_{I_i^j}\right)(y) b_1(y)\, dy\right|$$
$$\geqslant c - M|I_i^j \setminus \gamma I_i^j|/|I_i^j| > c/2$$

Property (2.3)(iii) follows if the φ_i^j are taken to be translates and dilates of a fixed smooth φ.

Now we claim there exist Lipschitz functions ψ_i^j satisfying

(2.4) (i) $\operatorname{supp} \psi_i^j \subset \frac{3}{2} J_i^j$.

(ii) $0 \leqslant \psi_i^j \leqslant 1$.

(iii) $\sum_{i=1}^{3^n \delta^{-jn}} \psi_i^j(x) = 1, \qquad x \in 3Q_k, \quad 0 \leqslant j \leqslant k-1,$

(iv) $|\psi_i^j(x) - \psi_i^j(x')| \leqslant C \frac{|x - x'|}{|J_i^j|^{1/n}}$.

(v) $\left|\frac{1}{|J_i^j|}\int \psi_i^j(x) b_2(x)\, dx\right| \leqslant \frac{C}{k}$.

Define $\beta(x)$ on $\mathbb{R}$ to equal 1 for $|x| \leqslant 1/2$, 0 for $|x| \geqslant 3/2$ and to be linear on each of the intervals $[-3/2, -1/2]$ and $[1/2, 3/2]$. If the ψ_i^j are taken to be appropriate dilates and translates of

$$\psi(x) = \prod_{l=1}^{n} \beta(x_l),$$

then (2.4)(i)-(iv) hold immediately. Since ψ is a positive integral of characteristic functions of parallelepipeds whose sidelengths lie between 1 and 3, property (2.4)(v) would follow from (2.2) if only the cubes J in (2.2) were permitted to be parallelepipeds contained in $3Q_k$ with sidelengths at least $\delta^k |Q_k|^{1/n}$. However, it is an easy exercise to verify that one may replace the subcubes I in the definition of para-accretive in (1.3) by parallepipeds. Indeed, if

$$\left|\frac{1}{|Q|}\int_I b(x)\, dx\right| \geqslant c$$

for a parallelepiped I contained in Q, then there is N large, depending only on $\|b\|_\infty$ and c, such that if $\{J_i\}_{i=1}^{N^n}$ is the «dyadic» decomposition of Q into congruent subcubes of sidelength $|Q|^{1/n}/N$ and

$$I^* = \cup \{J_i : J_i \cap I \neq \varnothing\},$$

then

$$\left|\frac{1}{|Q|}\int_{I^*} b(x)\,dx\right| \geqslant \left|\frac{1}{|Q|}\int_I b(x)\,dx\right| - \frac{|I^*\backslash I|}{|Q|}\|b\|_\infty$$
$$\geqslant \frac{c}{2}.$$

It follows that

$$\left|\frac{1}{|Q|}\int_{J_i} b(x)\,dx\right| \geqslant \frac{c}{2N^n}$$

for at least one of the cubes J_i. This completes the proof of (2.4).

We wish to define an operator T_k with kernel of the form

$$K_k(x,y) = \sum_{j,i} \beta_i^j \psi_i^j(x)\varphi_i^j(y) \tag{2.5}$$

where the β_i^j are bounded constants so chosen that $\|T_k b_1\|_{L^\infty} \leqslant C$ and the weak boundedness constant for $T_k M_{b_1}$ (the best C in (1.4) with $T = T_k M_{b_1}$) is of the order of k. We will see that the size and smoothness estimates (1.2) for K_k, the boundedness of $|T_k^* b_2|$ by C and the weak boundedness of $M_{b_2} T_k M_{b_1}$ with constant of the order of 1, all follow independently of the particular choice of bounded β_i^j's.

In order to define the constants β_i^j, let

$$\Omega_0 = Q_k, \quad \Omega_1 = 2Q_k, \quad \Omega_2 = \frac{5}{2}Q_k, \ldots, \quad \Omega_j = (3 - 2^{1-j})Q_k$$

for $1 \leqslant j \leqslant \left[\frac{k-1}{2}\right]$. In the case $0 \leqslant j \leqslant \left[\frac{k-1}{2}\right]$, we define

$$\beta_i^j = \begin{cases} 1\Big/\int \varphi_i^j(y) b_1(y)\,dy & \text{if} \quad J_i^j \subset \Omega_j \\ 0 & \text{otherwise} \end{cases}$$

In the case $\left[\frac{k-1}{2}\right] + 1 \leqslant j \leqslant k-1$, we define

$$\beta_i^j = \begin{cases} -1\Big/\int \varphi_i^j(y) b_1(y)\,dy & \text{if} \quad J_i^j \subset \Omega_{k-1-j} \\ 0 & \text{otherwise} \end{cases}$$

With this choice of β_i^j we calim the following properties:

(2.6) (i) $\| T_k b_1 \|_{L^\infty} \leqslant C$.

(ii) $\| T_k^* b_2 \|_{L^\infty} \leqslant C$.

(iii) $|K_k(x,y)| \leqslant C|x-y|^{-n}$.

(iv) $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leqslant C \dfrac{|x-x'|}{|x-y|^{n+1}}$ whenever $|x-x'| < \dfrac{1}{2}|x-y|$,

(v) $M_{b_2} T_k M_{b_1}$ satisfies the weak boundedness property (1.4) with constant C independent of k,

(vi) $T_k M_{b_1}$ satisfies the weak boundedness property (1.4) only with constant $C \geqslant c'k$.

We begin by proving the key properties (i) and (vi) that rely on our particular choice of β_i^j. To see (i), fix x in $\Omega_l \backslash \Omega_{l-1}$ for some $0 \leqslant l \leqslant [(k-1)/2]$ (where $\Omega_{-1} = \varnothing$). We have

$$\begin{aligned} T_k b_1(x) &= \sum_{j=0}^{k-1} \sum_i \beta_i^j \psi_i^j(x) \int \varphi_i^j(y) b_1(y)\, dy \\ &= \sum_{j=0}^{k-1} A_j(x). \end{aligned}$$

For $0 \leqslant j \leqslant l-2$ and $k-l+1 \leqslant j \leqslant k-1$, $\beta_i^j \psi_i^j(x) = 0$ for all i since supp $\psi_i^j \cap \Omega_{l-2} = \varnothing$ if $\psi_i^j(x) \neq 0$ and so then $\beta_i^j = 0$ by definition. Thus $A_j(x) = 0$ for these ranges of j. For

$$l+1 \leqslant j \leqslant \left[\frac{k-1}{2}\right], \qquad \beta_i^j = 1 \Big/ \int \varphi_i^j(y) b_1(y)\, dy$$

whenever $\psi_i^j(x) \neq 0$ and so $A_j(x) = \sum_i \psi_i^j(x) = 1$ by (2.4)(iii). Similarly,

$$A_j(x) = -1 \quad \text{for} \quad \left[\frac{k-1}{2}\right] + 1 \leqslant j \leqslant k-l-2.$$

Finally, if j is one of the four remaining cases, $j = l-1, l, k-l-1$ or $k-l$, we simply use the crude estimate

$$|A_j(x)| \leqslant \sum_i \psi_i^j(x) = 1$$

which follows from

$$\left| \beta_i^j \int \varphi_i^j(y) b_1(y)\, dy \right| \leqslant 1.$$

Altogether we obtain

$$(2.7)\qquad \begin{aligned} |T_k b_1(x)| &= \left|\sum_{j=0}^{k-1} A_j(x)\right| \\ &\leqslant 4 + \left|\sum_{j=l+1}^{[(k-1)/2]} A_j(x) + \sum_{j=[(k-1)/2]+1}^{k-l-2} A_j(x)\right| \\ &= 4 + \left|\left[\frac{k-1}{2}\right] - l - \left[k - l - 2 - \left[\frac{k-1}{2}\right]\right]\right| \\ &= \begin{cases} 4 & \text{if } k \text{ even} \\ 5 & \text{if } k \text{ odd} \end{cases} \end{aligned}$$

Since, by the same argument, $|T_k b_1(x)| \leqslant 2$ for x outside $\Omega_{[(k-1)/2]}$, we have proved (2.6)(i).

To see (vi), let J denote one of the cubes J_i^j with $j = \left[\frac{k-1}{2}\right] + 1$ and such that the triple of J_i^j lies in Q_k. For any bounded functions φ and ψ we have

$$(2.8)\qquad \langle T_k \varphi, \psi \rangle = \sum_{j,i} \beta_i^j \iint \psi(x)\psi_i^j(x)\varphi_i^j(x)\varphi(y)\,dx\,dy.$$

If φ and ψ are both supported in $5J$, then all the integrals in the sum on the right side of (2.8) vanish for $0 \leqslant j \leqslant \left[\frac{k-1}{2}\right]$ since the cube $5J$ cannot simultaneously intersect the supports of ψ_i^j and φ_i^j if δ is small enough (*e.g.* $\delta < 1/60$) by (2.3)(i), (2.4)(i), the definition of J_i^{j*} and some elementary geometry. In particular, if $\psi = \chi_J$ and $\varphi = \chi_{5J} b_1$, then in addition, $\operatorname{supp} \varphi_i^j \subset 5J$ whenever $\operatorname{supp} \psi_i^j \cap J \neq \varnothing$ and so

$$(2.9)\qquad \begin{aligned} \langle T_k \chi_{5J} b_1, \chi_J \rangle &= \sum_{j=[(k-1)/2]+1}^{k-1} \sum_i \int_J \psi_i^j(x) \beta_i^j \int_{5J} \varphi_i^j(y) b_1(y)\,dy\,dx \\ &= -\sum_{j=[(k-1)/2]+1}^{k-1} \sum_i \int_J \psi_i^j(x)\,dx \\ &= -\left(k - 1 - \left[\frac{k-1}{2}\right]\right)|J|, \end{aligned}$$

by (2.4)(iii). We note in passing that (2.9) shows that the norm of T_k as an operator on L^2 is of the order at least k. Now choose ψ Lipschitz with support in γJ, taking the value 1 on J and choose φ Lipschitz with support in $5J$, taking the value 1 on $(5/\gamma)J$. Take γ so close to 1 that for $0 < \eta < 1$,

$$(2.9)' \qquad |\langle T_k M_{b_1} \varphi, \psi \rangle| \geqslant C_\eta \left[\frac{k-1}{4} \right] |J|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta}.$$

where C_η is independent of k. This proves (2.6)(vi).

The proofs of (ii) and (v), to which we now turn, are essentially the same as those given for Proposition 1 in Section 9 of [DJS2] to prove the analogous statements for their counterexample in the case $b_1 = b_2$. In fact,

$$|T_k^* b_2(y)| = \left| \sum_{j,i} \beta_i^j \varphi_i^j(y) \int \psi_i^j(x) b_2(x)\, dx \right|$$

$$\leqslant \sum_{j,i} \frac{2}{c} |I_i^j|^{-1} \chi_{I_i^j}(y) \frac{C}{k} |J_i^j|$$

by (2.3)(iv), (2.3)(i), (ii), and (2.4)(v). Since I_i^j has side length at least $\frac{1}{3}\delta^{j+1}s_k$ and J_i^j has side length $\delta^j s_k$, it follows that

$$|T_k^* b_2(y)| \leqslant \frac{2C}{ck} \sum_{j,i} 3\delta^{-1} \chi_{I_i^j}(y)$$

$$\leqslant \frac{6C}{c\delta},$$

which is (2.6)(ii).

To establish (v), we must show that

$$(2.10) \qquad |\langle M_{b_2} T_k M_{b_1} \varphi, \psi \rangle| \leqslant C |Q|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta}$$

for all $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ with support in the cube Q. We use the argument in Section 9 of [DJS2]. The point is that we only need small integrals for one of the b_i, in this case b_2. Fix a cube Q of side length s and Lip η functions φ, ψ with support in Q. Then

$$(2.11) \qquad \langle M_{b_2} T_k M_{b_1} \varphi, \psi \rangle = \sum_{j,i} \beta_i^j \iint \psi(x) b_2(x) \psi_i^j(x) \varphi_i^j(y) b_1(y) \varphi(y)\, dx\, dy$$

$$= \sum_j \mathrm{B}_j.$$

If $\delta^j s_k \geqslant s$, then we estimate B_j directly by

$$(2.12) \qquad |B_j| \leqslant \frac{2}{c} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty} M^2 s^{2n} \left(\frac{3\delta^{-j-1}}{s_k} \right)^n$$

$$\leqslant C \frac{s^{2n}}{(\delta^j s_k)^n} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty}$$

using (2.3)(iv) and (ii). If $\delta^j s_k < s$ and x_i^j denotes the centre of J_i^j, then

$$\begin{aligned}(2.13)\qquad B_J &= \sum_i \beta_i^j \iint [\psi(x) - \psi(x_i^j)] b_2(x)\psi_i^j(x)\varphi_i^j(y) b_1(y)\varphi(y)\,dx\,dy \\ &\quad + \sum_i \psi(x_i^j)\beta_i^j \iint b_2(x)\psi_i^j(x)\varphi_i^j(y) b_1(y)\varphi(y)\,dx\,dy \\ &= C_j + D_j.\end{aligned}$$

Now

$$\begin{aligned}(2.14)\qquad |C_j| &\leqslant \frac{2}{c}(\delta^j s_k)^\eta \|\psi\|_{\mathrm{Lip}\,\eta} M^2 \|\varphi\|_{L^\infty} \sum_i \int_{3Q}\int_Q \psi_i^j(x)\varphi_i^j(y)\,dy\,dx \\ &\leqslant C s^n (\delta^j s_k)^\eta \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}(2.15)\qquad |D_j| &\leqslant \|\psi\|_{L^\infty} \frac{2}{c} M \|\varphi\|_{L^\infty} \sum_i \int_Q \left| \int \psi_i^j(x) b_2(x)\,dx \right| \varphi_i^j(y)\,dy \\ &\leqslant C \frac{s^n}{k} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty}\end{aligned}$$

using (2.3)(iv), (ii) and (2.4)(v). Altogether then, (2.11)-(2.15) yield

$$\begin{aligned}|\langle M_{b_2} T_k M_{b_1}\varphi, \psi\rangle| &\leqslant \sum_{j:\,\delta^j s_k \geq s} C \frac{s^{2n}}{(\delta^j s_k)^n} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty} \\ &\quad + \sum_{j:\,\delta^j s_k < s} \left[C s^n (\delta^j s_k)^\eta \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{L^\infty} + C \frac{s^n}{k} \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty} \right] \\ &\leqslant C s^n \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty} + C s^{n+\eta} \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{L^\infty} \\ &\leqslant C s^{n+2\eta} \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{\mathrm{Lip}\,\eta}\end{aligned}$$

and this completes the proof of (2.10) and hence that of (2.6)(v).

Finally, the kernel $K_k(x, y)$ satisfies

$$\begin{aligned}|K_k(x,y)| &\leqslant \frac{2}{c} \sum_{j:\,|x-y| \leq 3\delta^j s_k} \frac{3}{(\delta^{j+1} s_k)^n} \\ &\leqslant C|x-y|^{-n}\end{aligned}$$

by (2.3)(i), (ii), (iv) and (2.4)(i) which proves (2.6)(iii). If $|x - x'| < \frac{1}{2}|x - x|$, then

$$
\begin{aligned}
|K_k(x,y) - K_k(x',y)| &\leqslant \frac{2}{c} \sum_{j:\,|x-y| \leq 3\delta^j s_k} \sum_i |\psi_i^j(x) - \psi_i^j(x')|\,|\varphi_i^j(y)| \\
&\leqslant C|x-x'| \sum_{j:\,|x-y| \leq 3\delta^j s_k} (\delta^j s_k)^{-1}(\delta^{j+1}s_k)^{-n} \\
&\leqslant C|x-x'|\,|x-y|^{-n-1}
\end{aligned}
$$

by (2.3)(i), (ii), (iv) and (2.4)(i)-(iv). If $|y-y'| < \frac{1}{2}|x-y|$, then

$$
\begin{aligned}
|K_k(x,y) - K_k(x,y')| &\leqslant \frac{2}{c} \sum_{j:\,|x-y| \leq 3\delta^j s_k} \sum_i \psi_i^j(x)|\varphi_i^j(y) - \varphi_i^j(y')| \\
&\leqslant C|y-y'| \sum_{j:\,|x-y| \leq 3\delta^j s_k} (\delta^{j+1}s_k)^{-n-1} \\
&\leqslant C|y-y'|\,|x-y|^{-n-1}
\end{aligned}
$$

by (2.3)(i)-(iii) and (2.4)(i)-(iii). This proves (2.6)(iv) and completes the proof of the properties (2.6).

Before assembling the operators T_k to form an operator T satisfying the conclusions (1.6)(i)-(iv) of Case 1, it is convenient to arrange for an additional property of the cubes Q_k:

(2.16) If $3Q_k \cap 3Q_l \neq \varnothing$, then either $s_k \leqslant \delta^l s_l$ or $s_l \leqslant \delta^k s_k$.

To achieve this by induction, suppose $Q_1, \ldots, Q_k$ are dyadic cubes satisfying (2.2) and (2.16). Let $\mathcal{S}_k$ consist of the (finitely many) dyadic cubes of side length at least $\delta^j s_j$ and at most $\delta^{-k-1}s_j$ whose triples intersect $3Q_j$, $j = 1, 2, \ldots, k$. Then choose Q_{k+1} to be a dyadic cube not in $\mathcal{S}_k$, and satisfying (2.2). Now define, as in [DJS2],

$$
T = \sum_{k=1}^{\infty} \frac{1}{k^2} T_{k^3}. \tag{2.17}
$$

The estimates (1.2)(i)-(iii) and (1.6)(i)-(iii) all follow easily from (2.6)(i)-(v) and (2.17) and it remains only to check that TM_{b_1} fails to have the weak boundedness property. For this, fix l^3 and $\varphi, \psi \in C_0^\eta(\mathbb{R}^n)$ supported in $5J$ (associated to Q_{l^3} as above) so that (2.9)′ holds with l^3 in place of k. Then

$$
\langle TM_{b_1}\varphi, \psi\rangle = l^{-2}\langle T_{l^3}M_{b_1}\varphi, \psi\rangle + \sum_{k \neq l} k^{-2}\langle T_{k^3}M_{b_1}\varphi, \psi\rangle. \tag{2.18}
$$

If $3Q_{k^3}$ intersects $3Q_{l^3}$ and $s_{l^3} \leqslant \delta^{k^3}s_{k^3}$, then the separation of the supports of

ψ_i^j and φ_i^j associated to Q_{k^3} (see (2.3)(i) and (2.4)(i)) shows that

$$\langle T_{k^3} M_{b_1} \varphi, \psi \rangle = 0.$$

Of course $\langle T_{k^3} M_{b_1} \varphi, \psi \rangle$ also vanishes if $3Q_{k^3} \cap 3Q_{l^3} = \varnothing$. Thus if

$$\langle T_{k^3} M_{b_1} \varphi, \psi \rangle \neq 0,$$

then by (2.16), $s_{k^3} \leqslant \delta^{l^3} s_{l^3}$. Suppose that $3Q_{k^3}$ intersects $5J$. From $s_{k^3} \leqslant \delta^{l^3} s_{l^3}$ and (2.6)(i) we have

$$\| T_{k^3} M_{b_1}(\chi_{6J}) \|_{L^\infty} = \| T_{k^3} b_1 \|_{L^\infty} \leqslant C.$$

Thus

$$\begin{aligned}
\langle T_{k^3} M_{b_1} \varphi, \psi \rangle &= \iint K_{k^3}(x, y)\psi(x) b_1(y)\varphi(y)\, dx\, dy \\
&= \iint K_{k^3}(x, y)\psi(x) b_1(y)[\varphi(y) - \chi_{6J}(y)\varphi(x)]\, dx\, dy \\
&\quad + \iint K_{k^3}(x, y)\psi(x) b_1(y)\chi_{6J}(y)\varphi(x)\, dx\, dy \\
&= A + B.
\end{aligned}$$

Now

$$\begin{aligned}
|A| &\leqslant C \int_{6J}\int_{5J} |x - y|^{-n} \|\psi\|_{L^\infty} M |x - y|^{\eta} \|\varphi\|_{\mathrm{Lip}\,\eta}\, dx\, dy \\
&\leqslant C|J|^{1+\eta/n} \|\psi\|_{L^\infty} \|\varphi\|_{\mathrm{Lip}\,\eta} \\
&\leqslant C|J|^{1+\eta/n} \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{\mathrm{Lip}\,\eta},
\end{aligned}$$

and

$$\begin{aligned}
|B| &= \left| \int T_{k^3} M_{b_1}(\chi_{6J})(x)\psi(x)\varphi(x)\, dx \right| \\
&\leqslant C|J|\ \|\psi\|_{L^\infty} \|\varphi\|_{L^\infty} \\
&\leqslant C|J|^{1+2\eta/n} \|\psi\|_{\mathrm{Lip}\,\eta} \|\varphi\|_{\mathrm{Lip}\,\eta}.
\end{aligned}$$

Summing in k yields

$$\Big| \sum_{k \neq l} k^{-2} \langle T_{k^3} M_{b_1} \varphi, \psi \rangle \Big| \leqslant C_\eta |J|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta},$$

$0 < \eta < 1$, and since (1.9)′ holds for l^3, *i.e.*

$$|l^{-2} \langle T_{l^3} M_{b_1} \varphi, \psi \rangle| \geqslant Cl|J|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta},$$

(2.18) shows that

$$|\langle T M_{b_1} \varphi, \psi \rangle| \geqslant Cl|J|^{1+2\eta/n} \|\varphi\|_{\mathrm{Lip}\,\eta} \|\psi\|_{\mathrm{Lip}\,\eta}.$$

Letting l tend to infinity shows that TM_{b_1} fails to have the weak boundedness property and this completes the proof of Theorem 1 in the case b_1 and b_2 are jointly para-accretive.

Case 2. b_1 and b_2 are not jointly para-accretive.

In this case, the proof is simply a discrete version of the proof of Proposition 1 in Section 9 of [DJS2]. If Case 1 fails, then we can find a cube Q_k, for each $k > 0$, with the property

$$\sup_{\text{cubes } J \subset 3Q_k} \frac{1}{|Q_k|} \max\left\{\left|\int_J b_1\right|, \left|\int_J b_2\right|\right\} \leqslant \frac{(1/2)^{kn}}{k}.$$

With k momentarily fixed, and J_i^j and ψ_i^j as in Case 1 (but with $\delta = 1/2$), define the kernel of T_k by

$$K_k(x,y) = \sum_{j,i:\, J_i^j \subset Q_k} (\delta^j s_k)^{-n} \psi_i^j(x)\psi_i^j(y).$$

Properties (2.6)(ii)-(v) hold for T_k just as in Case 1. Property (2.6)(i) now has the same proof as (2.6)(ii) and choosing nonnegative $\psi, \varphi \in C^\infty(\mathbb{R}^n)$ to be 1 on Q_k with support in $2Q_k$, we have

$$\langle T_k\varphi, \psi\rangle \geqslant \int_{Q_k}\int_{Q_k} K_k(x,y)\,dx\,dy \geqslant Ck|Q_k|,$$

i.e. T_k satisfies the weak boundedness property (1.4) only with a constant $C \geqslant c'k$. With

$$T = \sum_{k=1}^{\infty} \frac{1}{k^2} T_{k^3},$$

(1.6)(i)-(iii) hold and T fails to have the weak boundedness property. This completes the proof of Theorem 1.

3. Proof of Theorem 2

The «only if» half of Theorem 2 is a simple consequence of Theorem 1. If b is not para-accretive, then Theorem 1, with $b_1 = 1$ and $b_2 = b$ produces a linear operator T with kernel K satisfying (1.2)(i), (ii) and (iii) such that $T1 \in L^\infty$, and M_bT, but not T, satisfies the weak boundedness property. This operator T satisfies the requirements for membership in $\mathcal{C}$ except for $T1 = 0$. This however can be remedied by considering instead $\tilde{T} = T - \Pi_{T1}$ where Π_β denotes the para-product operator

$$\Pi_\beta(f)(x) = \int_0^\infty \psi_t * \{(\psi_t * \beta)(\varphi_t * f)\}(x)\frac{dt}{t}, \tag{3.1}$$

where $\psi, \varphi \in \mathfrak{D}$ with

$$\int \varphi = 1, \qquad \int \psi = 0, \qquad \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{for} \quad \xi \neq 0,$$

and

$$\eta_t(x) = t^{-n}\eta\left(\frac{x}{t}\right).$$

Since, for $\beta \in \mathrm{BMO}$, the kernel of Π_β satisfies (1.2)(i), (ii), (iii) and $\Pi_\beta(1) = \beta$ and Π_β is bounded on L^2(*cf.* [CM]), it follows that $\tilde{T} \in \mathcal{C}$ and $M_b\tilde{T}$, but not $\tilde{T}$, has the weak boundedness property. This proves the «only if» half of Theorem 2.

We now prove the «if» half of Theorem 2 for the larger class of operators $\mathcal{C}'$. Suppose b is para-accretive, $T \in \mathcal{C}'$, so that $T1 \in \mathrm{BMO}$, and $M_b T$ has the weak boundedness property. We follow the idea of the proof of Meyer's Lemme 2 in [M1]. Fix for the moment a cube Q with center x_0 and $\theta \in \mathfrak{D}$ with $\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n: |x_i| \leqslant 4, 1 \leqslant i \leqslant n\}$ and $\theta = 1$ on $\{x \in \mathbb{R}^n: |x_i| \leqslant 2, 1 \leqslant i \leqslant n\}$. Let

$$\chi_0(x) = \theta\left(\frac{x - x_0}{\frac{1}{2}|Q|^{1/n}}\right)$$

and $\chi_1 = 1 - \chi_0$. Then $\varphi = \varphi\chi_0$ for all φ in $C_0^\eta(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset Q$ and so we have

$$\begin{aligned} M_b T\varphi(x) &= b(x)\int K(x,y)[\varphi(y) - \varphi(x)]\chi_0(y)\,dy \\ &\quad + \varphi(x)b(x)\int K(x,y)\chi_0(y)\,dy \\ &= p(x) + q(x) \end{aligned} \tag{3.2}$$

where the equalities hold in the distribution sense.

Using the weak boundedness property of $M_b T$ and the size condition

$$|b(x)K(x,y)| \leqslant C|x-y|^{-n},$$

the proof of Lemme 3 in [M1] shows that

$$p(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} b(x)K(x,y)[\varphi(y) - \varphi(x)]\chi_0(y)\,dy$$

is actually a bounded function with

$$(3.3)\qquad |p(x)| \leqslant \overline{\lim_{\epsilon\to 0}} \int_{|x-y|>\epsilon} |b(x)|\,|K(x,y)|\,|\varphi(y)-\varphi(x)|\,|\chi_0(y)|\,dy$$

$$\leqslant C\|b\|_{L^\infty}\|\chi_0\|_{L^\infty}\int_{4Q}|x-y|^{\eta-n}\|\varphi\|_{\mathrm{Lip}\,\eta}\,dy$$

$$\leqslant C\|b\|_{L^\infty}|Q|^{\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}.$$

To estimate $q(x)$, let $\tilde{q}(x)$ denote the restriction of the distribution

$$T\chi_0(x) = \int K(x,y)\chi_0(y)\,dy$$

to the open cube

$$U = \frac{3}{2}Q.$$

In analogy with the argument on the bottom of page 246 of [M1], let $a(x)$ be a smooth H^1-atom with support in U. Then

$$\left|\int \tilde{q}(x)a(x)\,dx\right| = \left|\int T1(x)a(x)\,dx - \iint [K(x,y)-K(x_0,y)]a(x)\chi_1(y)\,dx\,dy\right|$$

since $T1 = T\chi_0 + T\chi_1$ and $\int a = 0$, and so

$$(3.4)\qquad \left|\int \tilde{q}(x)a(x)\,dx\right| \leqslant \|T1\|_{\mathrm{BMO}}\|a\|_{H^1} + C\|a\|_{L^1} \leqslant C\|a\|_{H^1},$$

by (1.2)(ii).

Inequality (3.4) shows that $\tilde{q} \in \mathrm{BMO}(U)$. We will now use the para-accretivity of b to estimate the average

$$w_Q = \frac{1}{|Q|}\int_Q \tilde{q}(x)\,dx.$$

For this we need

Lemma 3.5. *Suppose b is para-accretive and Q is a cube in $\mathbb{R}^n$. Then there is $\rho \in \mathfrak{D}$ with*

$$\operatorname{supp}\rho \subset Q,\quad \|\rho\|_{\mathrm{Lip}\,\eta} \leqslant C_1|Q|^{-1-\eta/n}\quad \textit{and}\quad \left|\int_Q b(x)\rho(x)\,dx\right| \geqslant C_2 > 0,$$

where C_1 and C_2 are constants independent of Q.

Assuming the lemma, we have for ρ as above,

$$\begin{aligned} C_2|w_Q| &\leqslant |\langle bw_Q, \rho\rangle| \\ &= |\langle M_b T\chi_0, \rho\rangle + \langle b(w_Q - \tilde{q}), \rho\rangle| \\ &\leqslant C|Q|^{1+2\eta/n}\|\chi_0\|_{\mathrm{Lip}\,\eta}\|\rho\|_{\mathrm{Lip}\,\eta} + C\|b\|_{L^\infty}\|\tilde{q}\|_{\mathrm{BMO}} \end{aligned}$$

since $M_b T$ satisfies (1.4) and $\|\rho\|_{L^\infty} \leqslant |Q|^{-1}$ with $\operatorname{supp}\rho \subset Q$, and so

$$C_2|w_Q| \leqslant C|Q|^{1+2\eta/n}|Q|^{-\eta/n}|Q|^{-1-\eta/n} + C \leqslant C.$$

Thus $|w_Q| \leqslant C$ where C is independent of Q and if $\psi \in C_0^\eta(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset Q$, then

$$\begin{aligned} |\langle q, M_{b^{-1}}\psi\rangle| &= |\langle \tilde{q}, \varphi\psi\rangle| \\ &\leqslant |\langle w_Q, \varphi\psi\rangle| + |\langle \tilde{q} - w_Q, \varphi\psi\rangle| \\ &\leqslant C|Q|\,\|\varphi\|_{L^\infty}\|\psi\|_{L^\infty}, \end{aligned}$$

since $\tilde{q} \in \mathrm{BMO}(U)$, and

$$|\langle p, M_{b^{-1}}\psi\rangle| \leqslant C\|b\|_{L^\infty}\|b^{-1}\|_{L^\infty}|Q|^{1+\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{L^\infty}$$

by (3.3). Using these inequalities and (3.2) we obtain

$$\begin{aligned} |\langle T\varphi, \psi\rangle| &= |\langle p, M_{b^{-1}}\psi\rangle + \langle q, M_{b^{-1}}\psi\rangle| \\ &\leqslant C\|b\|_{L^\infty}\|b^{-1}\|_{L^\infty}|Q|^{1+2\eta/n}\|\varphi\|_{\mathrm{Lip}\,\eta}\|\psi\|_{\mathrm{Lip}\,\eta} \end{aligned}$$

which is (1.4) since b^{-1} is bounded if b is para-accretive.

It remains to prove Lemma 3.5. Since b is para-accretive, there is a cube

$$I \subset \frac{1}{2}Q$$

such that

$$\left|\int_I b(x)\,dx\right| \geqslant \delta|Q|$$

where $\delta > 0$ depends only on b. Fix $\theta \in \mathfrak{D}$ with

$$\operatorname{supp}\theta \subset \{x \in \mathbb{R}^n : |x_i| \leqslant 1 + \epsilon, 1 \leqslant i \leqslant n\}, \qquad 0 \leqslant \theta \leqslant 1$$

and

$$\theta(x) = 1 \quad \text{if} \quad |x_i| \leqslant 1, \qquad 1 \leqslant i \leqslant n.$$

Let

$$\rho(x) = |Q|^{-1}\theta\left(\frac{x - x_I}{\frac{1}{2}|I|^{1/n}}\right)$$

where x_I is the centre of I and $\epsilon > 0$ is sufficiently small that $\operatorname{supp}\rho \subset 2I \subset Q$, and

$$\begin{aligned}\left|\int \rho(x)b(x)\,dx\right| &= \left|\int_I \rho(x)b(x)\,dx + \int_{(1+\epsilon)I\setminus I} \rho(x)b(x)\,dx\right| \\ &\geqslant \delta - \|b\|_{L^\infty}\frac{|(1+\epsilon)I\setminus I|}{|Q|} > \frac{\delta}{2} = C_2.\end{aligned}$$

For such ρ we have

$$\|\rho\|_{\mathrm{Lip}\,\eta} \leqslant C|Q|^{-1}|I|^{-\eta/n}\|\theta\|_{\mathrm{Lip}\,\eta} \leqslant C_1|Q|^{-1-\eta/n}$$

since

$$\delta|Q| \leqslant \left|\int_I b(x)\,dx\right| \leqslant \|b\|_{L^\infty}|I|$$

and this completes the proof of Lemma 3.5 and so also Theorem 2.

We close this section by proving the remark made in the introduction that the size condition (1.2)(i) on the kernel of T is not needed in the *Tb* Theorem for L^2. Suppose (1.1) and (1.2)(ii), (iii) hold and that $M_{b_2}TM_{b_1}$ satisfies the weak boundedness property with b_1 and b_2 para-accretive. Fix x and y in $\mathbb{R}^n$ and let $s = |x - y| > 0$. By Lemma 3.5, there are ρ_1 and ρ_2 in $\mathfrak{D}$ with

$$\operatorname{supp}\rho_1 \subset B\left(y, \frac{s}{3}\right), \qquad \operatorname{supp}\rho_2 \subset B\left(x, \frac{s}{3}\right), \quad \|\rho_i\|_{\mathrm{Lip}\,\eta} \leqslant Cs^{-n-\eta}$$

and

$$\left|\int b_i(u)\rho_i(u)\,du\right| \geqslant c > 0 \quad \text{for} \quad i = 1, 2.$$

Thus

$$\begin{aligned}c^2|K(x,y)| &\leqslant \left|\iint \rho_2(u)b_2(u)K(x,y)b_1(v)\rho_1(v)\,du\,dv\right| \\ &\leqslant \left|\iint \rho_2(u)b_2(u)[K(x,y) - K(u,v)]b_1(v)\rho_1(v)\,du\,dv\right| \\ &\quad + \left|\iint \rho_2(u)b_2(u)K(u,v)b_1(v)\rho_1(v)\,du\,dv\right| \\ &= A + B.\end{aligned} \tag{3.6}$$

The smoothness conditions (1.2)(ii), (iii) yield

$$\begin{aligned}|K(x,y)-K(u,v)| &\leqslant |K(x,y)-K(u,y)|+|K(u,y)-K(u,v)|\\ &\leqslant C|x-y|^{-n}\end{aligned}$$

for $u \in \operatorname{supp}\rho_2$ and $v \in \operatorname{supp}\rho_1$, and it follows that

$$\begin{aligned}A &\leqslant C|x-y|^{-n}\|b_1\|_{L^\infty}\|b_2\|_{L^\infty}\|\rho_1\|_{L^1}\|\rho_2\|_{L^1}\\ &\leqslant C|x-y|^{-n}.\end{aligned}$$

Since (1.1) holds and $M_{b_2}TM_{b_1}$ has the weak boundedness property,

$$\begin{aligned}B &= |\langle M_{b_2}TM_{b_1}\rho_1,\rho_2\rangle|\\ &\leqslant Cs^{n+2\eta/n}\|\rho_1\|_{\mathrm{Lip}\,\eta}\|\rho_2\|_{\mathrm{Lip}\,\eta}\\ &\leqslant Cs^{-n}\\ &= C|x-y|^{-n}.\end{aligned}$$

Combining the estimates for A and B with (3.6) yields (1.2)(ii) as required.

The above argument can easily be modified to show the same conclusion if one of the smoothness conditions (1.2)(ii), (iii) is replaced by a Hörmander condition. If both (1.2)(ii) and (iii) are replaced by Hörmander conditions, then the conclusion is that the integral size estimates

$$\int_{r<|x-v|<2r}|K(x,v)|\,dv \leqslant C$$

and

$$\int_{r<|u-y|<2r}|K(u,y)|\,du \leqslant C$$

hold for all x and y.

4. Proof of Theorem 3

As mentioned in the introduction, Theorem 3 is easily proved by adapting the proof of the $T1$ Theorem for L^2 that is outlined in Section 2 of [DJS2]-the main tool being the Calderón reproducing formula. The following sketch will highlight the main differences.

Choose $\phi \in C^\infty(\mathbb{R}^n)$ with support in the unit ball and mean value zero so that the identity operator is given by the Calderón reproducing formula

$$I=\int_0^\infty \phi_s\phi_s\frac{ds}{s} \tag{4.1}$$

where

$$\phi_s(x) = s^{-n}\phi(s^{-1}x)$$

and in the context of an operator, the symbol ϕ_s means convolution with $\phi_s(x)$. Formally (4.1) is

$$1 = \int_0^\infty |\hat{\phi}(s\xi)|^2 \frac{ds}{s}$$

and it is an easy matter to find ϕ with this property. For f and g test functions we have

$$\begin{aligned} \langle Tf, g\rangle &= \langle \mathrm{I}T\,\mathrm{I}f,\ g\rangle \\ &= \int_0^\infty \int_0^\infty \langle \phi_s\phi_s T\phi_t\phi_t f, g\rangle \frac{ds}{s}\frac{dt}{t} \\ &= \int_0^\infty \int_0^\infty \langle [\phi_s T\phi_t]\phi_t f, \phi_s g\rangle \frac{ds}{s}\frac{dt}{t} \end{aligned} \tag{4.2}$$

and thus we need to estimate the kernel of the operator $\phi_s T\phi_t$, which we denote by $\phi_s T\phi_t(x,y)$. We have

Lemma 4.3. *Suppose T has kernel $K(x,y)$ satisfying* (1.2)(i) *and* (ii) (*but not necessarily smoothness in the second variable,* (1.2)(iii)) *and that $T1 = 0$ and T has the weak boundedness property* (1.4). *Then*

$$|\phi_s T\phi_t(x,y)| \leqslant C\left[1 + \left|\log\frac{s}{t}\right|\right]\left[\left(\frac{s}{t}\right)^\epsilon \wedge 1\right]\frac{(s \vee t)^\epsilon}{[(s\wedge t) + |x-y|]^{n+\epsilon}}$$

where the symbols $\wedge$ and $\vee$ mean minimum and maximum respectively.

Assuming the lemma for the moment, we have

$$|(\phi_s T\phi_t)(\phi_t f)(x)| \leqslant C\omega\left(\frac{s}{t}\right)M(\phi_t f)(x)$$

where

$$\omega(r) = (1 + |\log r|)(r^\epsilon \wedge 1)$$

and M denotes the Hardy-Littlewood maximal operator. Setting

$$\theta(r) = r^{-\alpha}\omega(r),$$

we obtain

$$(4.4)\quad |\langle Tf, g\rangle| \leqslant \int_0^\infty\int_0^\infty \langle M(t^{-\alpha}\phi_t f), s^\alpha|\phi_s g|\rangle\theta\left(\frac{s}{t}\right)\frac{ds}{s}\frac{dt}{t}$$

$$\leqslant \left\langle \left\{\int_0^\infty\int_0^\infty M(t^{-\alpha}\phi_t f)(x)^q\theta\left(\frac{s}{t}\right)\frac{ds}{s}\frac{dt}{t}\right\}^{1/q}, \right.$$

$$\left.\left\{\int_0^\infty\int_0^\infty |s^\alpha\phi_s g(x)|^{q'}\theta\left(\frac{s}{t}\right)\frac{dt}{t}\frac{ds}{s}\right\}^{1/q'}\right\rangle$$

$$\leqslant \left\langle \left\{\int_0^\infty M(t^{-\alpha}\phi_t f)(x)^q\frac{dt}{t}\right\}^{1/q}, \left\{\int_0^\infty |s^\alpha\phi_s g(x)|^{q'}\frac{ds}{s}\right\}^{1/q'}\right\rangle$$

since $\displaystyle\int_0^\infty \theta\left(\frac{s}{t}\right)\frac{ds}{s} + \int_0^\infty \theta\left(\frac{s}{t}\right)\frac{dt}{t} < \infty$ for $0 < \alpha < \epsilon$,

$$\leqslant \left\|\left(\int_0^\infty [M(t^{-\alpha}\phi_t f)]^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p}\left\|\left(\int_0^\infty |s^\alpha\phi_s g|^{q'}\frac{ds}{s}\right)^{1/q'}\right\|_{L^{p'}}$$

$$\leqslant \left\|\left(\int_0^\infty |t^{-\alpha}\phi_t f|^q\frac{dt}{t}\right)^{1/q}\right\|_{L^p}\left\|\left(\int_0^\infty |s^\alpha\phi_s g|^{q'}\frac{ds}{s}\right)^{1/q'}\right\|_{L^{p'}}$$

by the Fefferman-Stein vector-valued inequality for $1 < p, q < \infty$ ([FS]),

$$= C\|f\|_{\dot F_p^{\alpha,q}}\|g\|_{\dot F_{p'}^{-\alpha,q'}}.$$

Since $(\dot F_p^{\alpha,q})' = \dot F_{p'}^{-\alpha,q'}$, it follows from (4.4) that T is bounded on $\dot F_p^{\alpha,q}$ for $1 < p, q < \infty$ and $0 < \alpha < \epsilon$.

Returning now to the lemma, we prove the estimate (4.3) in the crucial case where $s < t$ and $|x - y| \leqslant Ct$. The three remaining cases: $s < t$ and $|x - y| > Ct$, $t < s$ and $|x - y| \leqslant Cs$, $t < s$ and $|x - y| > Cs$, are similar but easier. Let $\eta_0 \in C^\infty(\mathbb{R}^n)$ be 1 on the unit ball and 0 outside its double. Set $\eta_1 = 1 - \eta_0$. Then following the proof of Lemma 7 in Section 6 of [DJS2], we have

$$(4.5)\qquad \phi_s T\phi_t(x, y) = \iint \phi_s(x - u)K(u, v)\phi_t(v - y)\,du\,dv$$

$$= \iint \phi_s(x - u)K(u, v)[\phi_t(v - y) - \phi_t(x - y)]\,du\,dv,$$

since $T1 = 0$, and so

$$\phi_s T\phi_t(x, y) = \iint \phi_s(x - u)K(u, v)[\phi_t(v - y) - \phi_t(x - y)]\eta_0\left(\frac{v - x}{s}\right)du\,dv$$

$$+ \iint \phi_s(x - u)[K(u, v) - K(x, v)][\phi_t(v - y) - \phi_t(x - y)]\eta_1\left(\frac{v - x}{s}\right)du\,dv = A + B$$

since $1 = \eta_0 + \eta_1$ and $\phi_s 1 = 0$. Now with $\psi(u) = \phi_s(x - u)$ and

$$\varphi(v) = [\phi_t(v - y) - \phi_t(x - y)]\eta_0\left[\frac{v - x}{s}\right],$$

$$|A| = |\langle T\varphi, \psi\rangle| \leqslant Cs^{n+2\eta}\|\varphi\|_{\mathrm{Lip}\eta}\|\psi\|_{\mathrm{Lip}\eta},$$

by the weak boundedness property (1.4),

$$\leqslant Cs^{n+2\eta}\left\{\left(\frac{s}{t}\right)t^{-n}s^{-\eta}\right\}\{s^{-n}s^{-\eta}\}$$

$$\leqslant C\left(\frac{s}{t}\right)t^{-n}$$

which is dominated by the right side of (4.3) when $s < t$, $|x - y| \leqslant Ct$ and $0 < \epsilon < 1$. Using the smoothness of $K(x, y)$ in x, (1.2)(ii), together with

$$\|\phi_t(v - y) - \phi_t(x - y)| \leqslant C\left(\frac{|v - x|}{t + |v - x|}\right)^{\epsilon} t^{-n},$$

$$|B| \leqslant C\iint_{|v-x|\geq s} |\phi_s(x - u)|\left(\frac{s}{|v - x|}\right)^{\epsilon}|v - x|^{-n}\left(\frac{|v - x|}{t + |v - x|}\right)^{\epsilon} t^{-n}\,du\,dv$$

$$\leqslant Cs^{\epsilon}t^{-n}\int_{|v-x|\geq t}|v - x|^{-n-\epsilon}\,dv + C\left(\frac{s}{t}\right)^{\epsilon}t^{-n}\int_{t\geq|v-x|\geq s}|v - x|^{-n}\,dv$$

$$\leqslant C\left(1 + \log\frac{t}{s}\right)\left(\frac{s}{t}\right)^{\epsilon}t^{-n}$$

which is again dominated by the right side of (4.3) when $s < t$ and $|x - y| \leqslant Ct$. This proves (4.3) for this case and completes our sketch of the proof of Theorem 3.

Remark. Theorem 3 remains true in the case $p = q = 2$ if the condition $T1 = 0$ is relaxed to the condition that $|s^{-\alpha}\phi_s(T1)(x)|^2\dfrac{dx\,ds}{s}$ is an α-Carleson measure for L^2 (*i.e.* (4.7) below holds). More precisely, if $T1 \neq 0$, then we must add to (4.5) the correction term

$$\chi_{\{s\leq Ct\}}\iint \phi_s(x - u)K(u, v)\phi_t(x - y)\,du\,dv = \phi_s(T1)(x)\phi_t(x - y)\chi_{\{s\leq Ct\}}$$

and estimate in (4.2) the new term

$$(4.6)\quad \int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_{\{s\le Ct\}}\phi_s(T1)(x)\phi_t(x-y)\phi_t f(y)\phi_s g(x)\,dx\,dy\frac{ds}{s}\frac{dt}{t}$$

$$=\int_0^\infty\int_{\mathbb{R}^n}\phi_s(T1)(x)\left\{\int_{C^{-1}s}^\infty\int_{\mathbb{R}^N}\phi_t(x-y)\phi_t f(y)\,dy\frac{dt}{t}\right\}\phi_s g(x)\,dx\frac{ds}{s}$$

$$=\int_0^\infty\int_{\mathbb{R}^n}\phi_s(T1)(x)P_s f(x)\phi_s g(x)\,dx\frac{ds}{s}$$

where

$$P_s=\int_{C^{-1}s}^\infty\phi_t\phi_t\frac{dt}{t}$$

satisfies $|P_s(x)|\leqslant Cs^{-n}$ and, if $\phi*\phi(x)=\theta(|x|)$ is radial, then

$$P_s(x)=\int_{C^{-1}s}^\infty t^{-n}\theta\left(\frac{|x|}{t}\right)\frac{dt}{t}=|x|^{-n}\int_0^{Cs^{-1}|x|}\theta(r)r^{n-1}\,dr=0$$

for $|x|>2C^{-1}s$, since $\phi*\phi$ is supported in double the unit ball and has mean value zero.

The integral in (4.6) is at most

$$\left(\iint_{\mathbb{R}^{n+1}_+}|P_s f(x)|^2|s^{-\alpha}\phi_s(T1)(x)|^2\,dx\frac{ds}{s}\right)^{1/2}\left(\iint_{\mathbb{R}^{n+1}_+}|s^{\alpha}\phi_s g(x)|^2\,dx\frac{ds}{s}\right)^{1/2}$$

and since the second factor is $\|g\|_{\dot F_2^{-\alpha,2}}$, duality shows that T will be bounded on $\dot F_2^{\alpha,2}$ provided (with $f=I_\alpha h$)

$$(4.7)\quad \iint_{\mathbb{R}^{n+1}_+}|P_s I_\alpha h(x)|^2|s^{-\alpha}\phi_s(T1)(x)|^2\,dx\frac{ds}{s}\leqslant C\int_{\mathbb{R}^n}|h(x)|^2\,dx$$

for all h in $L^2(\mathbb{R}^n)$. Characterizations of (4.7) can be found in [KS] and [NRS].

Finally, since the integral in (4.6) is

$$\int_{\mathbb{R}^n}\int_0^\infty\phi_s\{[\phi_s(T1)]P_s f\}(x)\frac{ds}{s}g(x)\,dx=\int_{\mathbb{R}^n}\Pi_{T1}f(x)g(x)\,dx$$

by (3.1), it follows that T is bounded on $\dot F_2^{\alpha,2}$ if and only if Π_{T1} is bounded on $\dot F_2^{\alpha,2}$. (If T is bounded on $\dot F_2^{\alpha,2}$, then T has the weak boundedness property and so $T1\in\dot B_\infty^{0,\infty}$ by [M2]. It then follows that Π_{T1} satisfies the standard estimates (1.2) (see [CM]) and the weak boundedness property is easily checked to hold. Thus $T-\Pi_{T1}$ is bounded on $\dot F_2^{\alpha,2}$ by Theorem 3.) While (4.7) is sufficient for this, we do not know if it is necessary.

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Recibido: 26 de octubre de 1989.

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* Research supported in part by NSERC Grant A5149.

Revista Matemática Iberoamericana
Vol. 6, n.os 1 y 2, 1990

Global Existence of Solutions for the Nonlinear Boltzmann Equation of Semiconductor Physics

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Abstract

In this paper we give a proof of the existence and uniqueness of smooth solutions for the nonlinear semiconductor Boltzmann equation. The method used allows to obtain global existence in time and uniqueness for dimensions 1 and 2. For dimension 3 we only can assure local existence in time and uniqueness. First, we define a sequence of solutions for a linearized equation and then, we prove the strong convergence of the sequence in a suitable space. The method relies in the use of interpolation estimates in order to control the decay of the solution when the wave vector goes to infinity.

1. Introduction

This paper is devoted to the Boltzmann equation of semiconductors, which is the basic equation of the kinetic model of semiconductors. In this model, each type of carriers is described by a distribution function $f(x, k, t)$, where $x \in \mathbb{R}^d$ is the spatial position, $k \in \mathbb{R}^d$ the wave vector of the carriers and $t \geqslant 0$, the time. We will denote by d the space dimension, which will be equal to 1, 2 or 3.

If all the physical constants are taken equal to unity, the Boltzmann equation, which gives the evolution of the distribution function, is written

$$\text{(1.1)} \qquad \begin{cases} \dfrac{\partial f}{\partial t} + v(k)\nabla_x f - E(x,t)\nabla_k f = Q(f) \\ f(x,k,0) = f_0(x,k) \end{cases}$$

where the electric field $E(x,t)$ is coupled to the distribution function by the Poisson equation

$$\text{(1.2)} \qquad E(x,t) = C(d)\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\rho(y,t)\,dy$$

being

$$\text{(1.3)} \qquad \rho(x,t) = \int_{\mathbb{R}^d} f(x,k,t)\,dk$$

the electric charge.

The velocity $v(k)$ is a known function deduced from the band diagram and the collision term $Q(f)$ will be given here according to

$$\text{(1.4)} \qquad Q(f)(x,k,t) = \int_{\mathbb{R}^d} [S(x,k',k)f(x,k',t)(1-f(x,k,t)) - S(x,k,k')f(x,k,t)(1-f(x,k',t))]\,dk'$$

where $S(x,k,k')\,dk'$ is the transition rate for an electron at the position x to be scattered from a state k, into a state belonging to a small volume dk' around the state k'.

We refer the reader to [1, 2, 3] for the physical background of (1.1)-(1.4). We have neglected, in our model, the electron-electron scattering as well as the generation-recombination processes.

In this paper, we show that the method used by P. Degond [5], to prove the existence of solutions for the Vlasov-Fokker-Planck equation, is also applicable to the Boltzmann equation of semiconductors, where nonlinear integral operators are included. So, we give a proof of the existence and uniqueness of global in time smooth solutions of (1.1)-(1.4) when the space dimension is 1 or 2. For dimension 3 the method used only allows to prove the existence and uniqueness of local in time smooth solutions. The global in time existence for dimension 3 is still an open problem, like usually occurs in Vlasov-Poisson type equations. The outline of the proof is essentially the same as in [5]. We define a sequence of solutions (f^n, E^n) for a suitable linearized equation and then we prove the strong convergence of this sequence.

We have used the linearization proposed by F. Poupaud in [4]. In this last reference, a proof of the existence and uniqueness of the semiconductor Boltzmann equation is given, but when all the integrals on the wave vector space are taken over a bounded domain. So, the control of the electric charge is not there a problem. However, in order to control the electric charge, when the integral which defines it is taken over $\mathbb{R}^d$, we need to estimate the decay of the solution when the wave vector goes to infinity. This will be obtained following the idea of [5] and using some estimates on the collision integral.

The outline of the paper is the following: in Section 2, we state the existence and uniqueness theorem. Section 3 is devoted to the definition of the iterative sequence on which the proof is based. We also give some a priori estimates. In Section 4, we obtain the basic estimates which allow to prove the convergence of the procedure in Section 5.

2. Existence and Uniqueness Theorem

We define the functional space

$$\chi = \{\varphi(x,y,z): \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}: \varphi(x,y,z) \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_y, L^1(\mathbb{R}^d_z)) \text{ and } (1+|z|^2)^{\gamma/2}\varphi(x,y,z) \in L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_z, L^1(\mathbb{R}^d_y))\}$$

with $\gamma > d$.

We assume the following considerations

$$S(x,k,k') \geqslant 0; \qquad S(x,k,k') \in \chi \tag{2.1}$$

$$|\nabla_{x,k,k'} S| \in \chi \tag{2.2}$$

$$\nabla_k v(k) \in L^\infty(\mathbb{R}^d). \tag{2.3}$$

Now, we can state our main result.

Theorem 2.1. *We suppose that the initial data $f_0(x,k)$ satisfies*

$$0 \leqslant f_0 \leqslant 1;\ f_0 \in W^{1,1}(\mathbb{R}^{2d});\ (1+|k|^2)^{\gamma/2}(|f_0| + |Df_0|) \in L^\infty(\mathbb{R}^{2d});\ \gamma > d \tag{2.4}$$

Then the semiconductor Boltzmann equation:

$$\frac{\partial f}{\partial t} + v(k)\nabla_x f - E(x,t)\nabla_k f = Q(f), \qquad f(x,k,0) = f_0(x,k) \tag{2.5}$$

$$Q(f)(x,k,t) = \int_{\mathbb{R}^d} [S(x,k',k)f(x,k',t)(1-f(x,k,t)) - S(x,k,k')f(x,k,t)(1-f(x,k',t))]\,dk' \tag{2.6}$$

$$E(x,t) = C(d) \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \rho(y,t)\, dy \tag{2.7}$$

$$\rho(x,t) = \int_{\mathbb{R}^d} f(x,k,t)\, dk \tag{2.8}$$

with the assumptions (2.1), (2.2) *and* (2.3), *admits a unique classical solution, in a time interval* $[0, T[$, *where*

$$T = \infty \quad \text{if} \quad d = 1 \quad \text{or} \quad 2,$$

and T is finite and depends on f_0, S and γ if $d = 3$.

This solution is such that

$$0 \leqslant f \leqslant 1 \tag{2.9}$$

$$f \in L^\infty_{\text{loc}}([0, T[, W^{1,1}(\mathbb{R}^{2d})) \tag{2.10}$$

$$(1 + |k|^2)^{\gamma/2}(|f| + |Df|) \in L^\infty_{\text{loc}}([0, T[, L^\infty(\mathbb{R}^{2d})) \tag{2.11}$$

$$E \in L^\infty_{\text{loc}}([0, T[, W^{1,\infty}(\mathbb{R}^d_x)). \tag{2.12}$$

3. The Iterative Sequence

As in [4], we define the following operators

$$A(f)(x,k,t) = \int S(x,k',k) f(x,k',t)\, dk' \tag{3.1}$$

$$B(f)(x,k,t) = \int S(x,k,k')(1 - f(x,k',t))\, dk'. \tag{3.2}$$

Then, the collision term $Q(f)$ can be written

$$Q(f)(x,k,t) = A(f)(x,k,t)(1 - f(x,k,t)) - B(f)(x,k,t) f(x,k,t)$$

and the Boltzmann equation (2.5) is given by

$$\begin{cases} \dfrac{\partial f}{\partial t} + v(k)\nabla_x f - E(x,t)\nabla_k f + \lambda(f) f = A(f) \\ \lambda(f) = A(f) + B(f) \end{cases} \tag{3.3}$$

The proof of the Theorem 2.1 will be based on the following iterative sequence (f^n, E^n):

We begin with

$$f^0(x,k,t) = f_0(x,k). \tag{3.4}$$

Then, if we consider that f^n is known, we can compute the charge ρ^n and the electric field E^n according to

$$\text{(3.5)} \qquad \begin{cases} \rho^n(x,t) = \displaystyle\int f^n(x,k,t)\,dk \\ E^n(x,t) = C(d)\displaystyle\int \frac{x-y}{|x-y|^d}\rho^n(y,t)\,dy \end{cases}$$

Finally f^{n+1} is defined as the solution of the following equation:

$$\text{(3.6)} \qquad \begin{cases} \dfrac{\partial f^{n+1}}{\partial t} + v(k)\nabla_x f^{n+1} - E^n(x,t)\nabla_k f^{n+1} + \lambda(f^n)f^{n+1} = A(f^n) \\ f^{n+1}(x,k,0) = f_0(x,k) \end{cases}$$

Now we can state

Proposition 3.1. *The functions f^n of the iterative sequence defined by* (3.4), (3.5) *and* (3.6) *satisfy*:

(i) $0 \leqslant f^n \leqslant 1$.
(ii) f^n *are uniformly bounded in* $L^\infty([0,T^*];\, L^1(\mathbb{R}^{2d}))$ *for any time* T^*.

PROOF. We assume that (i) holds for f^n. In view of (2.1), (3.1) and (3.2) we have:

$$\lambda(f^n) \geqslant 0; \qquad A(f^n) \geqslant 0.$$

So since the source and the initial data for the equation (3.6) are non-negative we get

$$f^{n+1} \geqslant 0.$$

In the other hand we can also write

$$\begin{cases} \left[\dfrac{\partial}{\partial t} + v(k)\nabla_x - E^n(x,t)\nabla_k + \lambda(f^n)\right](1-f^{n+1}) = B(f^n), \\ (1-f^{n+1})(x,k,0) = 1 - f_0(x,k) \geqslant 0. \end{cases}$$

The same argument leads to

$$1 - f^{n+1} \geqslant 0, \quad \textit{i.e.} \quad f^{n+1} \leqslant 1.$$

In order to prove (ii) we integrate (3.6) and we use the non negativity of $\lambda(f^n)$ to obtain

$$\|f^{n+1}(t)\|_1 \leqslant \|f_0\|_1 + \int_0^t \|A(f^n)(s)\|_1 \, ds.$$

Now, thanks to (2.1) and (2.4) we get

$$\begin{aligned} \|A(f^n)(s)\|_1 &= \iint f^n(x, k', s)\left(\int S(x, k', k)\, dk\right) dx\, dk' \\ &\leqslant C_1(S)\|f^n(s)\|_1 \\ \|f^{n+1}(t)\|_1 &\leqslant \|f_0\|_1 + C_1 \int_0^t \|f^n(s)\|_1 \, ds. \end{aligned}$$

Now if we note by $\delta(t)$ the solution of the linear equation

$$\dot{\delta}(t) = C_1 \delta(t), \qquad \delta(0) = \|f_0\|_1$$

it is a simple matter to check that

$$\|f^n(t)\|_1 \leqslant \delta(t) \quad \text{for every} \quad t \geqslant 0. \tag{3.7}$$

So, if we define $C = \underset{[0, T^*]}{\operatorname{Max}} \delta(t)$ we have

$$\|f^n(t)\|_1 \leqslant C(f_0, S, T^*) \quad \text{for every} \quad t \in [0, T^*], \qquad T^* < \infty$$

and (ii) is proved.

4. The Basic Estimates

In order to obtain the strong convergence of f^n we need to control the gradients of the functions. Furthermore, the use of interpolation inequalities (A.3) and (A.4) of Appendix requieres L^∞ estimates on ρ^n and $\nabla_x \rho^n$. There will be essentially obtained, as in [5], using some estimates on the decay at infinity, in the wave vector space, of f^n.

Lemma 4.1. *We assume that f_0 satisfies the hypotheses* (2.4). *Then we have for every n*

(i) *ρ^n is uniformly bounded in $L^\infty([0, T^*], L^\infty(\mathbb{R}^d))$,*
(ii) *E^n is uniformly bounded in $L^\infty([0, T^*], L^\infty(\mathbb{R}^d))$,*

where

$$T^* < \infty \quad \textit{if} \quad d = 1 \quad \textit{or} \quad 2,$$
$$T^* < T \quad \textit{finite if} \quad d = 3.$$

PROOF. Multiplying equation (3.6) by $(1 + |k|^2)^{\gamma/2}$ and defining

$$Y^n(x, k, t) = (1 + |k|^2)^{\gamma/2} f^n(x, k, t)$$

we get

$$\frac{\partial Y^{n+1}}{\partial t} + v(k)\,\nabla_x Y^{n+1} - E^n(x, t)\,\nabla_k Y^{n+1} + \lambda(f^n) Y^{n+1} = R_1^{n+1} + R_2^n \tag{4.1}$$

$$R_1^{n+1} = -\gamma(E^n, k)(1 + |k|^2)^{(\gamma - 2)/2} f^{n+1}$$
$$R_2^n = A(f^n)(1 + |k|^2)^{\gamma/2}$$

From (4.1) we can obtain

$$\| Y^{n+1}(t) \|_\infty \leqslant \| Y_0 \|_\infty + \int_0^t (\| R_1^{n+1}(s) \|_\infty + \| R_2^n(s) \|_\infty)\, ds. \tag{4.2}$$

But,

$$\begin{aligned} R_2^n(s) &= (1 + |k|^2)^{\gamma/2} \int S(x, k', k) f^n(x, k', s)\, dk' \\ &\leqslant \| Y^n(s) \|_\infty (1 + |k|^2)^{\gamma/2} \int S(x, k', k)\, dk' \end{aligned}$$

and thanks to (2.1)

$$\| R_2^n(s) \|_\infty \leqslant C(S, \gamma) \| Y^n(s) \|_\infty. \tag{4.3}$$

On the other hand

$$\| R_1^{n+1}(s) \|_\infty \leqslant \gamma \| E^n(s) \|_\infty \| (1 + |k|^2)^{(\gamma - 1)/2} f^{n+1}(s) \|_\infty. \tag{4.4}$$

Then, using the interpolation inequality (A.1) of Appendix and Proposition 3.1, we have

$$\begin{aligned} \| (1 + |k|^2)^{(\gamma - 1)/2} f^{n+1}(s) \|_\infty &\leqslant C(\gamma) \| f^{n+1}(s) \|_\infty^{1/\gamma} \| Y^{n+1}(s) \|_\infty^{1 - 1/\gamma} \\ &\leqslant C(\gamma) \| Y^{n+1}(s) \|_\infty^{1 - 1/\gamma}. \end{aligned} \tag{4.5}$$

Now, using (A.2), (A.3), (3.7) and again Proposition 3.1

$$\| E^n(s) \|_\infty \leqslant C(d) \| \rho^n(s) \|_1^{1/d} \| \rho^n(s) \|_\infty^{(d - 1)/d}$$
$$\| \rho^n(s) \|_1^{1/d} = \| f^n(s) \|_1^{1/d} \leqslant \delta(s)^{1/d} = \varphi(s)$$

$$\begin{aligned} \| \rho^n(s) \|_\infty &\leqslant C(\gamma, d) \| f^n(s) \|_\infty^{(\gamma - d)/\gamma} \| Y^n(s) \|_\infty^{d/\gamma} \\ &\leqslant C(\gamma, d) \| Y^n(s) \|_\infty^{d/\gamma}. \end{aligned} \tag{4.6}$$

Thus,

$$\| E^n(s) \|_\infty \leqslant C(\gamma, d) \varphi(s) \| Y^n(s) \|_\infty^{(d - 1)/\gamma}. \tag{4.7}$$

So, (4.7) becomes

$$\|Y^{n+1}(t)\|_\infty \leqslant C_1 + C_2 \int_0^t \|Y^n(s)\|_\infty \, ds + C_3 \int_0^t \varphi(s) \|Y^{n+1}(s)\|_\infty^{1-1/\gamma} \|Y^n\|_\infty^{(d-1)/\gamma} \, ds \tag{4.8}$$

where, from now on, C_i will denote constants depending on f_0, S, γ and d.

In order to estimate $\|Y^n(t)\|_\infty$ we define

$$y_n(t) = \operatorname{Max}\{1, \|Y^n(t)\|_\infty\}.$$

If $d < 2$, (4.8) simplifies into

$$y_{n+1}(t) \leqslant C_1 + C_2 \int_0^t y_n(s)\, ds + C_3 \int_0^t \varphi(s) y_{n+1}^{1-1/\gamma}(s) y_n^{1/\gamma}(s)\, ds.$$

Now if we consider the solution $\alpha(t)$ of the linear equation

$$\dot{\alpha}(t) = [C_2 + C_3 \varphi(t)]\alpha(t); \qquad \alpha(0) = C_1$$

it is easy to prove [5] by induction on n that

$$y_n(t) \leqslant \alpha(t) \quad \text{for every} \quad t \geqslant 0 \quad \text{and} \quad n \in \mathbb{N}$$

and thus

$$\|Y^n(t)\|_\infty \leqslant C(f_0, S, T^*, d, \gamma), \qquad t \in [0, T^*], \quad T^* < \infty; \quad n \in \mathbb{N}. \tag{4.9}$$

If $d = 3$, (4.8) leads to

$$y_{n+1}(t) \leqslant C_1 + C_2 \int_0^t y_n^{1+1/\gamma}(s)\, ds + C_3 \int_0^t \varphi(s) y_{n+1}^{1-1/\gamma}(s) y_n^{2/\gamma}(s)\, ds.$$

Now, considering the solution $\alpha(t)$ of the equation

$$\dot{\alpha}(t) = [C_2 + C_3 \varphi(t)]\alpha(t)^{1+1/\gamma}; \qquad \alpha(0) = C_1$$

which exists in a time interval $[0, T[$, where T depends on f_0, S and γ (through C_1, C_2 and C_3), the same reasoning can be applied to obtain

$$\|Y^n(t)\|_\infty \leqslant C(f_0, S, T^*, \gamma) \qquad t \in [0, T^*], \quad T^* < T; \qquad n \in \mathbb{N}. \tag{4.10}$$

Now, the propositions (i) and (ii) are obvious from (4.6) and (4.7).

Lemma 4.2. *We assume that f_0 satisfies the hypotheses* (2.4). *Then, considering Df as a vector ${}^t(\nabla_x f, \nabla_k f)$, we have*:

(i) *Df^n is uniformly bounded in $L^\infty([0, T^*], L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}))$,*
(ii) *$\nabla_x \rho^n$ is uniformly bounded in $L^\infty([0, T^*], L^\infty(\mathbb{R}^d))$,*
(iii) *$\nabla_x E^n$ is uniformly bounded in $L^\infty([0, T^*], L^\infty(\mathbb{R}^d))$,*

where

$$T^* < \infty \quad \text{if} \quad d = 1 \quad \text{or} \quad 2,$$
$$T^* < T \quad \text{finite if} \quad d = 3.$$

PROOF. If we differentiate equation (3.6) with respect to (x, k), we obtain

$$\frac{\partial}{\partial t}(Df^{n+1}) + v(k)\nabla_x(Df^{n+1}) - E^n(x,t)\nabla_k(Df^{n+1}) + \lambda(f^n)Df^{n+1} \tag{4.11}$$
$$= DA(f^n) - f^{n+1}D\lambda^n + \mathfrak{M}^n Df^{n+1}$$
$$= DA(f^n)(1 - f^{n+1}) - f^{n+1}DB(f^n) + \mathfrak{M}^n Df^{n+1}$$

where $\mathfrak{M}^n$ is the following matrix decomposed in 3×3 blocks

$$\mathfrak{M}^n = \begin{bmatrix} 0 & \nabla_x E^n \\ -\nabla_k v & 0 \end{bmatrix}.$$

Now, multiplying equation (4.11) by $(1 + |k|^2)^{\gamma/2}$ and defining

$$Z^n(x,k,t) = (1 + |k|^2)^{\gamma/2} Df^n(x,k,t)$$

we get

$$\frac{\partial Z^{n+1}}{\partial t} + v(k)\nabla_x Z^{n+1} - E^n(x,t)\nabla_k Z^{n+1} + \lambda(f^n) Z^{n+1} \tag{4.12}$$
$$= P_1^n + P_2^n + P_3^{n+1} + P_4^{n+1}$$

$$P_1^n = DA(f^n)(1 - f^{n+1})(1 + |k|^2)^{\gamma/2}$$
$$P_2^n = -f^{n+1}DB(f^n)(1 + |k|^2)^{\gamma/2}$$
$$P_3^{n+1} = -\gamma(E^n, k)(1 + |k|^2)^{(\gamma-2)/2} Df^{n+1}$$
$$P_4^{n+1} = \mathfrak{M}^n Z^{n+1}.$$

From (4.12) we obtain

$$\|Z^{n+1}(t)\|_\infty \tag{4.13}$$
$$\leqslant \|Z_0\|_\infty + \int_0^t [\|P_1^n(s)\|_\infty + \|P_2^n(s)\|_\infty + \|P_3^{n+1}(s)\|_\infty + \|P_4^{n+1}(s)\|_\infty]\, ds.$$

From now on, we will denote by $\psi_i(t)$ some known functions depending on f_0, S, γ and d; and obtained from the functions $\alpha(t)$ defined in Lemma 4.1. So the functions $\psi_i(t)$ will satisfy

$$\psi_i(t) \in L^\infty([0, T^*])$$

where $T^* < \infty$ if $d = 1$ or 2, $T^* < T$ finite if $d = 3$, and T depends on f_0, S and γ.

Estimate on P_1^n

$$\begin{aligned}\|P_1^n(t)\|_\infty &\leqslant \|1 - f^{n+1}(t)\|_\infty \|(1 + |k|^2)^{\gamma/2} DA(f^n)(t)\|_\infty \\ &\leqslant \|(1 + |k|^2)^{\gamma/2} DA(f^n)(t)\|_\infty .\end{aligned}$$

But

$$\begin{aligned}(1 + |k|^2)^{\gamma/2} DA(f^n) = (1 + |k|^2)^{\gamma/2} &\int \nabla_{x,k} S(x, k', k) f^n(x, k', t)\, dk' \\ &+ (1 + |k|^2)^{\gamma/2} \int S(x, k', k)\, \nabla_{x,k} f^n(x, k', t)\, dk'.\end{aligned}$$

So, we have

$$\begin{aligned}\|P_1^n(t)\|_\infty \leqslant \|Y^n(t)\|_\infty &\left\|(1 + |k|^2)^{\gamma/2} \int |\nabla_{x,k} S(x, k', k)|\, dk'\right\|_\infty + \\ &+ \|Z^n(t)\|_\infty \left\|(1 + |k|^2)^{\gamma/2} \int S(x, k', k)\, dk'\right\|_\infty\end{aligned}$$

and using hypotheses (2.1) and (2.2)

$$\begin{aligned}\|P_1^n(t)\|_\infty &\leqslant C(\gamma, S)(\alpha(t) + \|Z^n(t)\|_\infty) \\ &\leqslant \psi_1(t)(1 + \|Z^n(t)\|_\infty)\end{aligned} \tag{4.14}$$

Estimate on P_2^n

$$\|P_2^n(t)\|_\infty \leqslant \|Y^{n+1}(t)\|_\infty \|DB(f^n)(t)\|_\infty .$$

But

$$DB(f^n) = \int \nabla_{x,k} S(x, k, k')(1 - f^n(x, k', t))\, dk' - \int S(x, k, k') \nabla_{x,k} f^n(x, k', t)\, dk'.$$

Thus,

$$\|DB(f^n)(t)\|_\infty \leqslant \left\|\int |\nabla_{x,k} S(x, k, k')|\, dk'\right\|_\infty + \|Z^n(t)\|_\infty \left\|\int S(x, k, k')\, dk'\right\|_\infty$$

and again from (2.1) and (2.2)

$$\|DB(f^n)(t)\|_\infty \leqslant C(S)(1 + \|Z^n(t)\|_\infty). \tag{4.15}$$

So, we have

$$\begin{aligned}\|P_2^n(t)\|_\infty &\leqslant C(S)\alpha(t)(1 + \|Z^n(t)\|_\infty) \\ &\leqslant \psi_2(t)(1 + \|Z^n(t)\|_\infty).\end{aligned} \tag{4.16}$$

Estimate on P_3^{n+1}

$$\|P_3^{n+1}(t)\|_\infty \leqslant \gamma \|E^n(t)\|_\infty \|(1 + |k|^2)^{\gamma/2} Df^{n+1}(t)\|_\infty$$

and thanks to (4.7)

$$\|P_3^{n+1}(t)\|_\infty \leqslant C(\gamma, d)\varphi(t)\alpha(t)^{(d-1)/\gamma} \|Z^{n+1}(t)\|_\infty .$$

We recall that the function $\varphi(t) = \delta(t)^{1/d}$, where $\delta(t)$ was obtained in Proposition 3.1, belongs to $L^\infty([0, T^*])$, for any $T^* < \infty$.

So, we get

$$\|P_3^{n+1}(t)\|_\infty \leqslant \psi_3(t) \|Z^{n+1}(t)\|_\infty . \tag{4.17}$$

Estimate on P_4^{n+1}

$$\|P_4^{n+1}(t)\|_\infty \leqslant \|\mathfrak{M}^n(t)\|_\infty \|Z^{n+1}(t)\|_\infty .$$

We use (2.3) and the interpolation inequality (A.4) to obtain

$$\|\mathfrak{M}^n(t)\|_\infty \leqslant C(d)[1 + \|\rho^n(t)\|_\infty [1 + \operatorname{Log}(1 + \|\nabla_x \rho^n(t)\|_\infty)] + \|\rho^n(t)\|_1].$$

Now, from (3.7) and (4.6)

$$\begin{aligned} \|\mathfrak{M}^n(t)\|_\infty &\leqslant C(\gamma, d)[1 + \alpha(t)^{d/\gamma}[1 + \operatorname{Log}(1 + \|\nabla_x \rho^n(t)\|_\infty)] + \delta(t)] \\ &\leqslant \psi_4(t)[1 + \operatorname{Log}(1 + \|\nabla_x \rho^n(t)\|_\infty)]. \end{aligned} \tag{4.18}$$

But we have

$$\nabla_x \rho^n(t) = \int \nabla_x f^n(x, k, t)\, dk.$$

So, applying the interpolation inequality (A.2) we get

$$\|\nabla_x \rho^n(t)\|_\infty \leqslant \left\| \int |\nabla_x f^n(x, k, t)|\, dk \right\|_\infty \leqslant C(\gamma, d) \|Z^n(t)\|_\infty \tag{4.19}$$

and thus,

$$\|P_4^{n+1}(t)\|_\infty \leqslant \psi_5(t)[1 + \operatorname{Log}(1 + \|Z^n(t)\|_\infty)] \|Z^{n+1}(t)\|_\infty . \tag{4.20}$$

So, from (4.14), (4.16), (4.17) and (4.20), equation (4.13) becomes

$$\begin{aligned} \|Z^{n+1}(t)\|_\infty \leqslant \|Z_0\|_\infty &+ \int_0^t \psi_6(s)(1 + \|Z^n(s)\|_\infty + \|Z^{n+1}(s)\|_\infty)\, ds \\ &+ \int_0^t \psi_5(s) \|Z^{n+1}(s)\|_\infty \operatorname{Log}(1 + \|Z^n(s)\|_\infty)\, ds. \end{aligned} \tag{4.21}$$

In order to estimate $\|Z^n(t)\|_\infty$ we define the function

$$z_n(t) = \text{Max}\,\{e, \|Z^n(t)\|_\infty\}$$

and we obtain from (4.21)

$$z_{n+1}(t) \leqslant C_4 + \int_0^t \psi_7(s)(z_{n+1}(s) + z_n(s)) \operatorname{Log} z_n(s)\, ds.$$

Now, we consider the differential equation

$$\dot{\beta}(t) = 2\psi_7(t)\beta(t) \operatorname{Log} \beta(t); \qquad \beta(0) = C_4$$

whose solution

$$\beta(t) = \exp\left[(\operatorname{Log} C_4) \exp \int_0^t 2\psi_7(s)\, ds\right]$$

exists in a time interval $[0, T[$ with $T = \infty$ if $d = 1$ or 2, T finite and depending on f_0, S and γ if $d = 3$.

So, the same argument as for Lemma 4.1 proves that

$$\|Z^n(t)\|_\infty \leqslant \beta(t) \leqslant C(f_0, S, T^*, \gamma, d); \tag{4.22}$$

$$\text{for every} \quad t \in [0, T^*] \quad \begin{cases} T^* < \infty & \text{if} \quad d = 1 \quad \text{or} \quad 2 \\ T^* < T & \text{if} \quad d = 3 \end{cases}, \quad \text{and} \quad n \in \mathbb{N}.$$

Now, propositions (ii) and (iii) are obvious from (4.18), (4.19) and (4.22).

In order to finish the proof of lemma, we have to estimate $\|Df^n(t)\|_1$.

From (4.11) we can write

$$\|Df^{n+1}(t)\|_1 \tag{4.23}$$

$$\leqslant \|Df_0\|_1 + \int_0^t [\|DA(f^n)(s)\|_1 + \|f^{n+1}D\lambda(f^n)(s)\|_1 + \|\mathfrak{M}^n Df^{n+1}(s)\|_1]\, ds.$$

By considering the estimates (4.22) obtained above, we have

$$\|f^{n+1}D\lambda(f^n)(t)\|_1 \leqslant [\|DA(f^n)(t)\|_\infty + \|DB(f^n)(t)\|_\infty]\, \|f^{n+1}(t)\|_1$$

and thanks to (4.14), (4.15) and (3.7)

$$\|f^{n+1}D\lambda(f^n)(t)\|_1 \leqslant \psi_8(t). \tag{4.24}$$

Using (4.18) and (4.19),

$$\begin{aligned} \|\mathfrak{M}^n Df^{n+1}(t)\|_1 &\leqslant \|\mathfrak{M}^n(t)\|_\infty \|Df^{n+1}(t)\|_1 \\ &\leqslant \psi_2(t)\|Df^{n+1}(t)\|_1. \end{aligned} \tag{4.25}$$

$$\|DA(f^n)(t)\|_1 \leqslant \iint f^n(x, k', t)\, dx\, dk' \int |\nabla_{x,k} S(x, k', k)|\, dk$$
$$+ \iint |\nabla_{x,k} f^n(x, k', t)\, dx\, dk'| \int S(x, k', k)\, dk.$$

Applying hypotheses (2.1) and (2.2), and thanks to (3.7) we have

$$\begin{aligned} \|DA(f^n)(t)\|_1 &\leqslant C(S)[\|f^n(t)\|_1 + \|Df^n(t)\|_1] \\ &\leqslant \psi_{10}(t)[1 + \|Df^n(t)\|_1]. \end{aligned} \tag{4.26}$$

So, from (4.23), (4.24), (4.25) and (4.26) we obtain the following Gronwall inequality

$$\|Df^{n+1}(t)\|_1 \leqslant \|Df_0\|_1 + \int_0^t \psi_{11}(s)(1 + \|Df^{n+1}(s)\|_1 + \|Df^n(s)\|_1)\, ds.$$

Now if we use the same reasoning as for estimates on $Y^n(t)$ and $Z^n(t)$ we can write that

$$\|Df^{n+1}(t)\|_1 \leqslant C(f_0, S, T^*, \gamma, d); \tag{4.27}$$

$$\text{for every} \quad t \in [0, T^*] \quad \begin{cases} T^* < \infty & \text{if} \quad d = 1 \quad \text{or} \quad 2 \\ T^* < T & \text{if} \quad d = 3 \end{cases}; \quad \text{and} \quad n \in \mathbb{N}.$$

Thus, proposition (i) is proved from (4.22) and (4.27), and the proof of Lemma 4.2 is finished.

5. Convergence of the Iterative Sequence

We consider an arbitrary finite time T^* ($T^* < T(f_0, S, \gamma)$ if $d = 3$). Thanks to Proposition 3.1 and Lemmas 4.1 and 4.2, we have got the following convergences of subsequences

$$f^n \rightharpoonup f \quad \text{in} \quad L^\infty([0, T^*], L^\infty(\mathbb{R}^{2d})) \quad \text{weak-*} \tag{5.1}$$

$$(1 + |k|^2)^{\gamma/2} f^n \rightharpoonup (1 + |k|^2)^{\gamma/2} f \quad \text{in} \quad L^\infty([0, T^*], L^\infty(\mathbb{R}^{2d})) \quad \text{weak-*} \tag{5.2}$$

$$(1 + |k|^2)^{\gamma/2} Df^n \rightharpoonup (1 + |k|^2)^{\gamma/2} Df \text{ in } L^\infty([0, T^*], L^\infty(\mathbb{R}^{2d})) \quad \text{weak-*} \tag{5.3}$$

$$\rho^n \rightharpoonup \rho \quad \text{in} \quad L^\infty([0, T^*], W^{1,\infty}(\mathbb{R}^d)) \quad \text{weak-*} \tag{5.4}$$

$$E^n \rightharpoonup E \quad \text{in} \quad L^\infty([0, T^*], W^{1,\infty}(\mathbb{R}^d)) \quad \text{weak-*}. \tag{5.5}$$

To pass to the limit in the non-linear terms of equation (2.5) we need a strong convergence. So, we state

Lemma 5.1. *The functions* (f^n, E^n) *of the iterative sequence defined by* (3.4), (3.5) *and* (3.6) *satisfy*:

$$f^n \to f \quad in \quad L^\infty([0, T^*], L^1(\mathbb{R}^{2d})) \quad strong \tag{5.6}$$

$$E^n \to E \quad in \quad L^\infty([0, T^*], L^\infty(\mathbb{R}^d)) \quad strong \tag{5.7}$$

PROOF. The function $f^{n+1} - f^n$ satisfies the following equation

$$\begin{aligned} \frac{\partial}{\partial t}(f^{n+1} - f^n) + v(k)\nabla_x(f^{n+1} - f^n) - E^n(x, t)\nabla_k(f^{n+1} - f^n) \\ + \lambda(f^n)(f^{n+1} - f^n) = \\ A(f^n) - A(f^{n-1}) + (E^n(x, t) - E^{n-1}(x, t))\nabla_k f^n + (\lambda(f^{n-1}) - \lambda(f^n))f^n. \end{aligned} \tag{5.8}$$

Now, integrating (5.8) we can obtain

$$\|(f^{n+1} - f^n)(t)\|_1 \leqslant \int_0^t [\|Q_1^n(s)\|_1 + \|Q_2^n(s)\|_1 + \|Q_3^n(s)\|_1]\, ds \tag{5.9}$$

$$Q_1^n = (E^n - E^{n-1})\nabla_k f^n$$
$$Q_2^n = A(f^n) - A(f^{n-1})$$
$$Q_3^n = (\lambda(f^{n-1}) - \lambda(f^n))f^n$$

Estimate on Q_1^n

We can state

$$\|Q_1^n(s)\|_1 \leqslant C_5 \|(f^n - f^{n-1})(s)\|_1. \tag{5.10}$$

The proof of (5.10) can be found in [5] and is omitted here. The idea of the proof will be used in the proof of uniqueness.

Estimate on Q_2^n

Using (2.1) we get

$$\begin{aligned} \|Q_2^n(s)\|_1 &\leqslant \iint |f^n - f^{n-1}|(x, k', s)\, dx\, dk' \int S(x, k', k)\, dk \\ &\leqslant C(S)\|(f^n - f^{n-1})(s)\|_1. \end{aligned} \tag{5.11}$$

Estimate on Q_3^n

$$\|(\lambda(f^{n-1}) - \lambda(f^n))f^n(s)\|_1 \leqslant \|A(f^{n-1}) - A(f^n)\|_1 + \|(B(f^{n-1}) - B(f^n))(s)\|_1.$$

But

$$\|(B(f^{n-1}) - B(f^n))(s)\|_1 \leqslant \iint |f^n - f^{n-1}|(x, k', s)\, dx\, dk' \int S(x, k, k')\, dk.$$

So, applying (2.1) and (5.11) we have

$$\|Q_3^n(s)\|_1 \leqslant C(S)\|(f^n - f^{n-1})(s)\|_1. \tag{5.12}$$

Now, from (5.9), (5.10), (5.11) and (5.12) we have

$$\begin{aligned}\|(f^{n+1} - f^n)(t)\|_1 &\leqslant C_6 \int_0^t \|(f^n - f^{n-1})(s)\|_1\, ds \\ &\leqslant \frac{C_6^n t^n}{n!} \operatorname*{Max}_{t\in[0,T^*]} \|(f^1 - f^0)(t)\|_1\end{aligned}$$

which proves (5.6).

Now, it is clear that

$$\rho^n \to \rho \quad \text{in} \quad L^\infty([0,T^*], L^1(\mathbb{R}^d)) \quad \text{strong} \tag{5.13}$$

and using the interpolation inequality (A.3) and (5.4)

$$\begin{aligned}\|(E^n - E)(t)\|_\infty &\leqslant C(d)\|(\rho^n - \rho)(t)\|_\infty^{(d-1)/d}\|(\rho^n - \rho)(t)\|_1^{1/d} \\ &\leqslant C_7\|(\rho^n - \rho)(t)\|_1^{1/d}\end{aligned}$$

which proves (5.7) and that the electric field E found in (5.5) satisfies (2.7).

From the convergences (5.1)-(5.7), it is easy and classical to prove that the function f found in (5.1) satisfies equation (2.5) for almost every time, and that, since T^* is arbitrary, Propositions 2.10, 2.11 and 2.12 are satisfied. Proposition 2.9 is obvious from Proposition 3.1(i). In order to finish the proof of Theorem 2.1 it remains to prove the uniqueness.

Let (f_1, E_1) and (f_2, E_2) be two solutions of (2.4)-(2.8) satisfying (2.9)-(2.12). The function $f_1 - f_2$ satisfies the following equation

$$\begin{aligned}&\frac{\partial}{\partial t}(f_1 - f_2) + v(k)\nabla_x(f_1 - f_2) - E_1\nabla_k(f_1 - f_2) + \lambda(f_1)(f_1 - f_2) \\ &\qquad = A(f_1) - A(f_2) + (E_1 - E_2)\nabla_k f_2 + (\lambda(f_2) - \lambda(f_1))f_2\end{aligned} \tag{5.14}$$

and integrating (5.14) we get

$$\begin{aligned}\|(f_1 - f_2)(t)\|_1 \leqslant \int_0^t &[\|(A(f_1) - A(f_2))(s)\|_1 \\ &+ \|((E_1 - E_2)\nabla_k f_2)(s)\|_1 + \|(\lambda(f_2) - \lambda(f_1))f_2(s)\|_1]\, ds\end{aligned} \tag{5.15}$$

But, as in the proof of Lemma 5.1

$$\|(A(f_1) - A(f_2))(s)\|_1 + \|(\lambda(f_1) - \lambda(f_2))f_2(s)\|_1 \leqslant C_8\|(f_1 - f_2)(s)\|_1, \tag{5.16}$$

and

$$\|((E_1 - E_2)\nabla_k f_2)(s)\|_1 \leqslant \iint |\nabla_k f_2(x,k,s)|\,|(E_1 - E_2)(x,s)|\,dx\,dk$$

$$\leqslant \int |\rho_1(y,s) - \rho_2(y,s)|\,dy \iint \frac{1}{|x-y|^{d-1}} |\nabla_k f_2(x,k,s)|\,dx\,dk.$$

Now, using (A.3), (A.2),

$$\iint \frac{1}{|x-y|^{d-1}} |\nabla_k f_2(x,k,s)|\,dx\,dk$$

$$\leqslant \left(\iint |\nabla_k f_2(x,k,s)|\,dx\,dk\right)^{1/d} \left(\left\| \int |\nabla_k f_2(\cdot,k,s)|\,dk \right\|_\infty\right)^{(d-1)/d}$$

$$\leqslant C(\gamma,d)\|Df_2(s)\|_1^{1/d} \|(1+|k|^2)^{\gamma/2} Df_2(s)\|_\infty^{(d-1)/d}$$

and thanks to (2.10) and (2.11)

$$\|((E_1 - E_2)\nabla_k f_2)(s)\|_1 \leqslant C_9 \int |\rho_1(y,s) - \rho_2(y,s)|\,dy \tag{5.17}$$

$$\leqslant C_9 \|(f_1 - f_2)(s)\|_1.$$

So, from (5.15), (5.16) and (5.17) we get

$$\|(f_1 - f_2)(t)\|_1 \leqslant C_{10} \int_0^t \|(f_1 - f_2)(s)\|_1\,ds$$

which proves that

$$\|(f_1 - f_2)(t)\|_1 = 0 \quad \text{for every } t \text{ and so} \quad f_1 = f_2$$

and the proof of Theorem 2.1 is finished.

APPENDIX. Interpolation Inequalities

Lemma A.1. *For a function* $f(k)\colon \mathbb{R}^d \to \mathbb{R}$, *we have*

$$\|(1+|k|^2)^{(\gamma-1)/2} f\|_\infty \leqslant C(\gamma)\|f\|_\infty^{1/\gamma} \|(1+|k|^2)^{\gamma/2} f\|_\infty^{1-1/\gamma} \tag{A.1}$$

$$\int |f(k)|\,dk \leqslant C(\gamma,d)\|f\|_\infty^{(\gamma-d)/\gamma} \|(1+|k|^2)^{\gamma/2} f\|_\infty^{d/\gamma} \quad \textit{for} \quad \gamma > d. \tag{A.2}$$

PROOF. See [5].

Lemma A.2. *Let $\rho(x)$ be a function which belongs to $L^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ and let $E(x)$ be such that*

$$E(x) = \int \frac{x-y}{|x-y|^d} \rho(y)\, dy.$$

Then we have

$$\|E\|_\infty \leqslant C(d) \|\rho\|_1^{1/d} \|\rho\|_\infty^{(d-1)/d} \tag{A.3}$$

$$\|\nabla_x E\|_\infty \leqslant C(d)[1 + \|\rho\|_\infty[1 + \operatorname{Log}(1 + \|\nabla_x \rho\|_\infty)] + \|\rho\|_1]. \tag{A.4}$$

PROOF. See [5, 6, 7].

Acknowledgements. I would like express my gratitude to P. Degond for many helpful discussions and remarks.

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Recibido: 1 de febrero de 1990.

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REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 6, N.^{OS} 1 Y 2, 1990

Hermite Expansions on $\mathbb{R}^{2n}$ for Radial Functions

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1. Introduction

Let $\Phi_\nu(x)$ stand for the normalized Hermite functions on $\mathbb{R}^n$ which are eigen functions of the Hermite operator

$$H_n = \left(-\Delta + \frac{1}{4}|x|^2\right).$$

Let $P_N f$ denote the projection of $L^2(\mathbb{R}^n)$ onto the space spanned by $\{\Phi_\nu : |\nu| = N\}$. Then the Riesz means of order $\delta > 0$ are defined by

$$S_L^\delta f = \sum \left(1 - \frac{2N+n}{2L}\right)_+^\delta P_N f.$$

In [9] we proved the uniform estimates

$$\|S_L^\delta f\|_p \leqslant c\|f\|_p, \qquad 1 \leqslant p \leqslant \infty \tag{1.1}$$

whenever $\delta > (n-1)/2$. For a fixed p one is interested in finding the smallest value of δ so that the uniform estimates (1.1) will hold for that fixed p. If we define the critical index $\delta(p)$ by

$$\delta(p) = \max\left(n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right)$$

then $\delta > \delta(p)$ is known to be a necessary condition for the validity of (1.1).

This is the consequence of a transplantation theorem (see [4]). As proved in [9] for $p = 1$ the condition $\delta > \delta(1)$ is also sufficient to imply the uniform estimates. But for other values of p it is not known whether $\delta > \delta(p)$ is sufficient or not.

In [1] Chris Sogge studied the Riesz means of eigen function expansions associated to elliptic differential operators on a compact manifold. Adapting an argument of Fefferman-Stein [2] he showed that for

$$\left|\frac{1}{p} - \frac{1}{2}\right| \geqslant \frac{1}{n+1}$$

$$\|S_L^\delta f\|_p \leqslant C\|f\|_p, \qquad \delta > \delta(p) \tag{1.2}$$

provided that the operator is of order two. The main idea of the proof is that $L^p - L^2$ estimates of the projection operators associated with the expansions are sufficient to prove summability results if used together with the kernel estimates for large values of δ.

For the Hermite projection operators it is possible to prove the estimates

$$\|P_N f\|_2 \leqslant CN^{\delta(p)}\|f\|_p, \qquad \left|\frac{1}{p} - \frac{1}{2}\right| \geqslant \frac{1}{n+1}. \tag{1.3}$$

As it was proved in [9] we also have the pointwise estimate for the kernel of S_L^δ.

$$|S_L^\delta(x, y)| \leqslant C\{L^{n/2}(1 + L^{1/2}|x - y|)^{-\delta - 1} + L^{n/2}(1 + L^{1/2}|x + y|)^{-\delta - 1}\}.$$

So, one is tempted to use the same arguments as in [1] to study the L^p mapping properties of S_L^δ. Unfortunately the arguments break down owing to the fact that the eigenvalues of H_n are not squares of integers.

But things are not so bad, if we consider on $\mathbb{R}^{2n}$ only the radial functions then the above estimates for P_N can be improved. Indeed we will show that

$$\|P_N f\|_2 \leqslant CN^{\delta(p)/2 - 1/4}\|f\|_p, \qquad \left|\frac{1}{p} - \frac{1}{2}\right| \geqslant \frac{1}{4n}. \tag{1.4}$$

Also for radial functions $S_L^\delta f$ is given by a kernel which satisfies the estimate

$$|S_L^\delta(x, y)| \leqslant CL^n(1 + L^{1/2}|x - y|)^{-\delta - 1}. \tag{1.5}$$

By using the same arguments as in [1] we will prove that the estimates (1.4) and (1.5) imply the following result.

Theorem. *Let $f \in L^p(\mathbb{R}^{2n})$ be radial, $1 < p < 4n/(2n + 1)$ and let $\delta > \delta(p)$. Then there is a C independent of L such that $\|S_L^\delta f\| \leqslant C\|f\|_p$ holds.*

Corollary. *For f radial and $\delta > 0$ we have*

$$\|S_L^\delta f\|_p \leqslant C\|f\|_p, \qquad \frac{4n}{2n+1+\delta} < p < \frac{4n}{2n-1-2\delta}.$$

In the next section we obtain $L^p - L^2$ bounds for the projection operators and in the third section an estimate of the kernel is proved. Theorem and its corollary will be proved in the final section. The author wishes to thank Chris Sogge for clarifying certain points in the proof of the above theorem.

2. Bounds for the Projection Operators

Recall that the n-dimensional Hermite functions $\Phi_\nu(x)$ are defined by

$$\Phi_\nu(x) = (2^{|\nu|}\nu!\sqrt{2\pi})^{-n/2}e^{-|x|^2/2}\prod_{j=1}^{n} h_{\alpha_j}(x_j/\sqrt{2})$$

where

$$h_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}(e^{-x^2}).$$

Then $\{\Phi_\nu(x)\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$. On $L^2(\mathbb{R}^{2n})$ we have another orthonormal basis given by the special Hermite functions $\Phi_{\nu\mu}$. These functions are defined by

$$\Phi_{\nu\mu}(z) = \pi^{-n/2}\int_{\mathbb{R}^n} e^{i(xu)/\sqrt{2}}\Phi_\nu\left(u - \frac{y}{\sqrt{2}}\right)\Phi_\mu\left(u + \frac{y}{\sqrt{2}}\right)du,$$

where $z = x + iy \in \mathbb{C}^n$. If L is the operator $L = H_{2n} - iN$ where

$$N = \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

then it is not difficult to check that $\Phi_{\nu\mu}(z)$ are eigen functions of L:

$$L\Phi_{\nu\mu}(z) = (2|\nu| + n)\Phi_{\nu\mu}(z). \tag{2.1}$$

For these facts about the special Hermite functions we refer to the paper of Strichartz [7]. For the operator H_{2n} the eigen functions are $\Phi_\nu(z) = \Phi_\nu(x, y)$ and one has

$$H_{2n}\Phi_\nu(z) = (|\nu| + n)\Phi_\nu(z). \tag{2.2}$$

Let Q_N and P_N stand for the projections of $L^2(\mathbb{R}^{2n})$ onto the space spanned by $\{\Phi_{\nu\mu} : |\nu| = N\}$ and $\{\Phi_\nu : |\nu| = N\}$. We claim the following is true.

Lemma 2.1. *If f is a radial function on $\mathbb{R}^{2n}$ then one has $P_{2N}f = Q_N f$ and $P_{2N+1}f = 0$ for all N.*

PROOF. For radial functions f it is easily seen that $Nf = 0$ and hence $Lf = H_{2n}f$. Since $H_{2n}f$ is again radial we have $L^k f = H_{2n}^k f$ for all $k = 1, 2, \ldots$ Therefore, for $t > 0$, $e^{-tL}f = e^{-tH_{2n}}f$ which translates into

$$\sum_{N=0}^{\infty} \omega^{2N} Q_N f = \sum_{N=0}^{\infty} \omega^N P_N f$$

which is true even if ω is complex and $|\omega| < 1$. The last equality immediately implies that $P_{2N}f = Q_N f$ and $P_{2N+1}f = 0$.

The result of this Lemma is the key one to get improved estimates for the projection operators. Before proceeding further we need to recall several facts about the Weyl transform W. (A good reference for these is the paper of Mauceri [6].)

The Weyl transform W takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. It is defined by the equation

$$W(f) = \int_{\mathbb{C}^n} f(z) W(z)\, dz\, d\bar{z}$$

where $W(z)$ is the operator valued function

$$W(z)\varphi(\xi) = e^{ix(y/2 + \xi)}\varphi(\xi + y)$$

where $z = x + iy$ and $\xi \in \mathbb{R}^n$. When f is radial the Weyl transform reduce to the Laguerre transform

$$W(f) = \sum_{N=0}^{\infty} R_N(f)\tilde{P}_N$$

where $\tilde{P}_N$ is the projection of $L^2(\mathbb{R}^n)$ onto the N-th eigenspace of the operator $-\Delta + |x|^2$ and $R_N(f)$ are defined by

$$R_N(f) = \frac{N!}{(N+n-1)!} \int_{\mathbb{C}^n} f(z) L_N^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-|z|^2/4}\, dz\, d\bar{z}.$$

If we let

$$\varphi_N(z) = (2\pi)^{-n} L_N^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-|z|^2/4},$$

then it is clear that $W(\varphi_N) = \tilde{P}_N$ due to the orthogonality properties of the Laguerre functions.

Another important fact about the Weyl transform which we need is its action on Lf. If we define the operators Z_j and $\bar{Z}_j$ by

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} z_j$$

then a simple calculation shows that

$$-\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j) = \frac{1}{4}L.$$

Since

$$W\left(-\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j)f\right) = W(f)H,$$

where $H = -\Delta + |x|^2$ we have the equation

$$W(Lf) = 4W(f)H.$$

From this we conclude that $W(Q_N f) = W(f)\tilde{P}_N$ and when f is radial the above formula becomes $W(Q_N f) = R_N(f)\tilde{P}_N$. Since $W(\varphi_N) = \tilde{P}_N$ we have the result $Q_N f = R_N(f)\varphi_N$ for radial functions. Thus for radial functions $P_{2N} f = R_N(f)\varphi_N$ and we are ready to prove the following proposition.

Proposition 2.1. *When f is radial and $1 \leqslant p < 4n/(2n+1)$ we have*

$$\|P_N f\|_2 \leqslant C N^{n(1/p - 1/2) - 1/2}\|f\|_p.$$

PROOF. Since $\|\varphi_N\|_2 = CN^{(n-1)/2}$ we have

$$\begin{aligned}\|P_{2N} f\|_2 &= CN^{-(n-1)/2}\left|\int_{\mathbb{C}^n} f(z)\varphi_N(z)\,dz\,d\bar{z}\right| \\ &\leqslant CN^{-(n-1)/2}\|\varphi_N\|_q\|f\|_p\end{aligned}$$

where $1/p + 1/q = 1$. So we need to estimate the L^q norm of φ_N. A simple calculation shows that

$$\|\varphi_N\|_q = CN^{(n-1)/2}\|\mathcal{L}_N^{\alpha+\beta}(r)r^{-\beta/2}\|_q \tag{2.3}$$

where

$$\alpha + \beta = n - 1, \qquad \beta = 2(n-1)\left(\frac{1}{2} - \frac{1}{q}\right)$$

and $\mathcal{L}_N^{\delta}$ are the normalized Laguerre functions of type δ.

Now we will make use of the following estimates proved in [5]. Assume that $\alpha + \beta > -1$ and $\alpha > -2/q$. Then

$$\|\mathcal{L}_N^{\alpha+\beta}(r)r^{-\beta/2}\|_q \leqslant CN^{\beta/2-1/q} \tag{2.4}$$

if $1 \leqslant q \leqslant 4$ and $\beta > 2/q - 1/2$ or $q > 4$ and $\beta > 4/3q - 1/3$.

When $1 \leqslant p < 4/3$, $q > 4$ and $4/3q - 1/3 < 0$ so that $\beta > 4/3q - 1/3$.

When $4/3 \leqslant p < 4n/(2n+1)$, $4n/(2n-1) < q \leqslant 4$ and it is easily checked that $\beta > 2/q - 1/2$. Hence in view of the estimate (2.4) we immediately get

$$\|P_{2N}f\|_2 \leqslant CN^{(n-1)(1/2-1/q)-1/q}\|f\|_p.$$

This proves the proposition.

3. Estimating the Riesz Kernel

To get a good estimate for the kernel of the Riesz means we need to recall the definition of twisted convolution. The twisted convolution of two functions f and g both defined on $\mathbb{C}^n$ is defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-v)g(v)\bar{\omega}(z,v)\,dv\,d\bar{v} \tag{3.1}$$

where

$$\omega(z,v) = \exp\left(-\frac{i}{2}\operatorname{Im} z\bar{v}\right).$$

In terms of the real variables

$$f \times g(x) = \int_{\mathbb{R}^{2n}} f(x-y)g(y)e^{iP(x,y)}\,dy \tag{3.2}$$

where $P(x,y)$ is a real polynomial. An important result we need is the fact

$$W(f \times g) = W(f)W(g). \tag{3.3}$$

In the previous section we have observed that

$$W(Q_N f) = W(f)\tilde{P}_N = W(f)W(\varphi_N)$$

and hence in view of (3.3) we have $Q_N f = f \times \varphi_N$. Now for a radial function f

$$S_L^\delta f = \sum\left(1 - \frac{k+n}{L}\right)_+^\delta P_k f$$

$$= \sum\left(1 - \frac{2k+n}{L}\right)_+^\delta Q_k f = f \times s_L^\delta$$

where the kernel s_L^δ is defined by

$$s_L^\delta(z) = \sum \left(1 - \frac{2k+n}{L}\right)_+^\delta \varphi_k(z) \tag{3.4}$$

Thus we have

$$S_L^\delta f(x) = \int S_L^\delta(x, y) f(y)\, dy$$

where the kernel is given by

$$S_L^\delta(x, y) = s_L^\delta(x - y) e^{iP(x,y)}. \tag{3.5}$$

We can now prove the following estimate.

Proposition 3.1.

$$|S_L^\delta(x, y)| \leqslant CL^n(1 + L^{1/2}|x - y|)^{-\delta - n - 1/3} \tag{3.6}$$

PROOF. From (3.5) what we need is to prove

$$|s_L^\delta(z)| \leqslant CL^n(1 + L^{1/2}|z|)^{-\delta - n - 1/3}.$$

Consider the Cesaro means σ_L^δ defined by

$$\sigma_L^\delta f(z) = \frac{1}{A_L^\delta} \sum_{k=0}^{L} A_{L-k}^\delta Q_k f(z) \tag{3.7}$$

which is given by twisted convolution with

$$\sigma_L^\delta(z) = \frac{1}{A_L^\delta} \sum_{k=0}^{L} A_{L-k}^\delta \varphi_k(z). \tag{3.8}$$

There is a formula (see Gergen [3]) connecting $s_L^\delta(z)$ and $\sigma_L^\delta(z)$ viz.,

$$s_L^\delta(z) = \frac{1}{L^\delta} \sum_{k=0}^{L} V(L - k) A_k^\delta \sigma_k^\delta(z)$$

where the function V satisfies the estimate $|V(t)| \leqslant C\,(1 + t^2)^{-1}$. In view of this formula it is enough to prove

$$|\sigma_k^\delta(z)| \leqslant Ck^n(1 + k^{1/2}|z|)^{-\delta - n - 1/3}.$$

The following formula is true for Laguerre polynomials (see [5]):

$$\sum_{k=0}^{N} A_{N-k}^\alpha L_k^\beta(r) = L_N^{\alpha+\beta+1}(r).$$

Since

$$\sigma_k^\delta(z) = \frac{1}{A_k^\delta} \sum_{j=0}^{k} A_{k-j}^\delta L_j^{n-1}\left(\frac{1}{2}|z|^2\right) e^{-|z|^2/4}$$

we immediately get the formula

$$\sigma_k^\delta(z) = \frac{1}{A_k^\delta} L_k^{\delta+n}\left(\frac{1}{2}|z|^2\right) e^{-|z|^2/4}. \tag{3.9}$$

Now we can make use of the asymptotic estimates for the Laguerre Polynomials $L_k^{\delta+n}\left(\frac{1}{2}|z|^2\right)$ (see for example [5]). Indeed, if we let

$$\mathcal{L}_N^\alpha(r) = N^{-\alpha/2} L_N^\alpha(r) e^{-r/2} r^{\alpha/2}$$

then the following estimates are valid.

$$|\mathcal{L}_N^\alpha(r)| \leqslant C \begin{cases} (r\nu)^{\alpha/2} & \text{if} \quad 0 \leqslant r < \frac{1}{\nu} \\ (r\nu)^{-1/4} & \text{if} \quad \frac{1}{\nu} \leqslant r < \frac{\nu}{2} \\ [\nu(\nu^{1/3} + |\nu - r|)]^{-1/4} & \text{if} \quad \frac{\nu}{2} \leqslant r < \frac{3\nu}{2} \\ e^{-r\gamma} & \text{if} \quad r \geqslant \frac{3\nu}{2} \end{cases}$$

where $\nu = 4N + 2\alpha + 2$. In view of these estimates it is an easy matter to show that

$$|\sigma_k^\delta(z)| \leqslant Ck^n(1 + k^{1/2}|z|)^{-\delta-n-1/3}.$$

This completes the proof of the proposition

Remark. Incidentally, using the above estimate one can prove a pointwise convergence result for radial functions. When f is radial

$$S_L^\delta f(x) = \int s_L^\delta(x - y) e^{iP(x,y)} f(y)\, dy$$

shows that when $\delta > n - \frac{1}{3}$ we have

$$\operatorname{Sup} |S_L^\delta f(x)| \leqslant C\Lambda f(x)$$

where Λ is the Hardy-Littlewood maximal function. So when f is in $L^p(\mathbb{R}^{2n})$ and radial, $S_L^\delta f(x) \to f(x)$ a.e. as L tends to infinity.

We further remark that in [9] we have proved the pointwise convergence for any L^p function when $\delta > n - 1/3$. That required a considerable amount of work because there estimating the Riesz kernel was much difficult.

4. L^p Bounds for Riesz Means

In this section we will prove that when

$$1 < p < \frac{4n}{2n+1} \quad \text{and} \quad \delta > \delta(p)$$

the following uniform estimates are valid

$$\|S_L^\delta f\|_p \leqslant C\|f\|_p, \qquad f \text{ is radial.}$$

Following [1] we take a partition of unity

$$\sum_{-\infty}^{\infty} \varphi(2^\nu t) = 1$$

for $t > 0$ where φ is a C_0^∞ function supported in $(1/2, 2)$. For each ν set

$$\varphi_{L,\nu}^\delta(t) = \varphi\left(2^\nu\left(1 - \frac{t}{L}\right)\right)\left(1 - \frac{t}{L}\right)^\delta. \tag{4.1}$$

Furthermore, for $\nu = 1, 2, \ldots$ define

$$S_{L,\nu}^\delta f = \sum \varphi_{L,\nu}^\delta(\lambda_k) P_k f \tag{4.2}$$

$$S_{L,0}^\delta f = \sum \varphi_0\left(1 - \frac{\lambda_k}{L}\right)\left(1 - \frac{\lambda_k}{L}\right)^\delta P_k f \tag{4.3}$$

where $\lambda_k = k + n$ and

$$\varphi_0(t) = 1 - \sum_{\nu=1}^{\infty} \varphi(2^\nu t).$$

Then we have

$$S_L^\delta f = S_{L,0}^\delta f + \sum_{\nu=1}^{[\log\sqrt{L}\,]} S_{L,\nu}^\delta f + R_L^\delta f. \tag{4.4}$$

It is easily seen that $S_{L,0}^\delta$ satisfies the conditions of the Marcinkiewicz multiplier theorem (see [8]) and hence

$$\|S^\delta_{L,0}f\|_p \leqslant C\|f\|_p, \qquad 1 < p < \infty. \tag{4.5}$$

So what we need to prove is the existence of an $\epsilon > 0$ such that

$$\|S^\delta_{L,\nu}f\|_p \leqslant C2^{-\nu\epsilon}\|f\|_p, \tag{4.6}$$

$$\|R^\delta_L f\|_p \leqslant C2^{-\nu\epsilon}\|f\|_p. \tag{4.7}$$

As remarked in [1] it is enough to prove (4.6).

Proposition 4.1.

$$\|S^\delta_{L,\nu}f\|_2 \leqslant C\big(2^{-\nu}\sqrt{L}\big)^{1/2}2^{-\delta\nu}\big(\sqrt{L}\big)^{\delta(p)}\|f\|_p. \tag{4.8}$$

PROOF. Note that $\varphi^\delta_{L,\nu}(t) = 0$ unless t satisfies $2^{-\nu-1} \leqslant 1 - t/L \leqslant 2^{-\nu+1}$ and on the support $\varphi^\delta_{L,\nu}(t)$ has the bound $|\varphi^\delta_{L,\nu}(t)| \leqslant C2^{-\nu\delta}$. Consequently, by the orthogonality of the projections

$$\|S^\delta_{L,\nu}f\|_2^2 \leqslant C2^{-2\delta\nu}\sum \|P_kf\|_p^2$$

where the sum is extended over all k satisfying $2^{-\nu-1} \leqslant 1 - \dfrac{k+n}{L} \leqslant 2^{-\nu+1}$. Since we have

$$\|P_kf\|_2^2 \leqslant Ck^{2n(1/p-1/2)-1}\|f\|_p^2$$

a simple calculation shows that

$$\|S^\delta_{L,\nu}f\|_2^2 \leqslant C2^{-2\nu\delta}\big(2^{-\nu}\sqrt{L}\big)\big(\sqrt{L}\big)^{2\delta(p)}\|f\|_p^2.$$

This proves the proposition.

Once we have the estimate (4.8) and the kernel estimate (3.6) we can proceed as in [1] to prove the theorem mentioned in the introduction. For the sake of completeness and to clarify certain points we give a somewhat detailed proof of the theorem.

Proceeding as in [1] one can show that given a $\gamma > 0$, there is an $\epsilon > 0$ such that

$$\int_{|x-y|>2^{\nu(1+\gamma)}/\sqrt{L}} |S^\delta_L(x,y)|\,dy \leqslant C2^{-\nu\epsilon} \tag{4.9}$$

holds uniformly in x. The proof of this we will not elaborate since it is already given in full details in [1]. The proof makes use of the kernel estimate (3.6).

Let B be a ball of radius $2^{\nu(1+\gamma)}/\sqrt{L}$, then

$$\|S^\delta_{L,\nu}f\|_{L^p(B)} \leqslant |B|^{1/p-1/2}\|S^\delta_{L,\nu}f\|_{L^2(B)}.$$

If we use the bounds (4.8) we will get

$$\| S^{\delta}_{L,\nu} f \|_{L^p(B)} \leqslant C 2^{-\nu(\delta+1/2)} 2^{\nu(1+\gamma)(\delta(p)+1/2)} \| f \|_p .$$

Since $\delta > \delta(p)$ we can choose γ so that $\delta + 1/2 > (1+\gamma)\,(\delta(p) + 1/2)$ and with that choice of γ it is clear that there is an $\epsilon > 0$ such that the following is true

$$\| S^{\delta}_{L,\nu} f \|_{L^p(B)} \leqslant C 2^{-\nu\epsilon} \| f \|_p . \tag{4.10}$$

Split the kernel $S^{\delta}_{L,\nu}(x, y)$ into two parts by setting

$$\begin{aligned} K_1(x, y) &= S^{\delta}_{L,\nu}(x, y), \quad \text{if} \quad |x - y| \leqslant 2^{\nu(1+\gamma)}/\sqrt{L} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$K_2(x, y) = S^{\delta}_{L,\nu}(x, y) - K_1(x, y).$$

Here γ is the positive number already chosen. The estimate (4.9) immediately proves that

$$\| K_2 f \|_p \leqslant C 2^{-\nu\epsilon} \| f \|_p .$$

To prove a similar estimate for K_1 we proceed as follows.

Let $B(h)$ be the ball

$$|x - h| \leqslant \frac{\frac{1}{4} 2^{\nu(1+\gamma)}}{\sqrt{L}},$$

and let $B^*(h)$ be the ball

$$|x - h| \leqslant \frac{\frac{3}{4} 2^{\nu(1+\gamma)}}{\sqrt{L}},$$

and $B^{**}(h)$ the ball

$$|x - h| \leqslant \frac{\frac{5}{4} 2^{\nu(1+\gamma)}}{\sqrt{L}}.$$

We split the function f into three parts viz

$$f_1 = f\chi_{B^*}, \qquad f_2 = f\chi_{B^{**}\setminus B^*}, \qquad f_3 = f - f_1 - f_2 .$$

Since the kernel K_1 is supported in the region $|x-y| \leqslant \dfrac{2^{\nu(1+\gamma)}}{\sqrt{L}}$ we have

$$K_1 f = K_1 f_1 + K_1 f_2.$$

We will prove that

$$\int_{B(h)} |K_1 f(x)|^p \, dx \leqslant C 2^{-\nu\varepsilon p} \int_{B^{**}(h)} |f(y)|^p \, dy$$

Integration with respect to h will then prove the bound

$$\|K_1 f\|_p \leqslant C 2^{-\nu\varepsilon} \|f\|_p.$$

When $x \in B(h)$ and y is in the support of f_1 we have $|x-y| \leqslant 2^{\nu(1+\gamma)}/\sqrt{L}$ so that $K_1 f_1 = S^{\delta}_{L,\nu} f_1$. In view of (4.10) we get

$$\int_{B(h)} |K_1 f_1(x)|^p \, dx \leqslant C 2^{-\nu\varepsilon} \int_{B^{*}(h)} |f(y)|^p \, dy.$$

And when $x \in B(h)$ and y is in the support of f_2 we have

$$|x-y| \geqslant \frac{\frac{1}{2} 2^{\nu(1+\gamma)}}{\sqrt{L}},$$

and hence in view of (4.9) we get

$$\int_{B(h)} |K_1 f_2(x)|^p \, dx \leqslant C 2^{-\nu\varepsilon} \int_{B^{**}(h)} |f(y)|^p \, dy.$$

This completes the proof of the theorem.

We conclude this section with a proof of the corollary. Observe that when $p = \dfrac{4n}{2n+1}$, $\delta(p) = 0$. Given $\delta > 0$, a simple calculation shows that $\delta > \delta(p_0)$ for any

$$p_0 > \frac{4n}{2n+1+2\delta}$$

and hence

$$\|S^{\delta}_L f\|_{p_0} \leqslant C \|f\|_{p_0}.$$

By self adjointness and duality we also have

$$\|S^{\delta}_L f\|_{p'_0} \leqslant C \|f\|_{p'_0}.$$

An interpolation proves that $\|S^{\delta}_L f\|_p \leqslant C \|f\|_p$ for $p_0 < p < p'_0$. Hence the corollary.

Note added in the proof. Recently the author has proved the main theorem of this paper on $\mathbb{R}^{2n+1}$ also.

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Recibido: 28 de marzo de 1989.

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Revista Matemática Iberoamericana
Vol. 6, N.os 1 y 2, 1990

Littlewood-Paley-Stein Theory on $\mathbb{C}^n$ and Weyl Multipliers

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1. Introduction

On $\mathbb{C}^n$ consider the $2n$ linear differential operators

$$Z_j = \partial_j + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \bar{\partial}_j - \frac{1}{4}z_j, \qquad j = 1, 2, \ldots, n. \tag{1.1}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n + 1$ dimensional Heisenberg algebra. The only non trivial commutation relations are

$$[Z_j, \bar{Z}_j] = -\frac{1}{2}I, \qquad j = 1, 2, \ldots, n. \tag{1.2}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j)$$

is nonnegative, self-adjoint and elliptic. Hence it generates a diffusion semigroup T^t. Following Stein [8], we study g and g^* functions associated to this semigroup and apply the results to prove a multiplier theorem for the Weyl transform.

The above operators in (1.1) generate a family of 'twisted translations' and using them we can define 'twisted convolution' of functions on $\mathbb{C}^n$. It turns out that the operator g is defined by a twisted convolution operator whose kernel is of Calderón-Zygmund type but it takes values in a Hilbert space. Also twisted convolution operators with Calderón-Zygmund kernels can be thought of as ordinary convolutions with the kernel having an oscillatory factor. Such singular integral operators, called oscillatory singular integrals have been studied by Ricci and Stein [5] and Chanillo and Christ [1]. We study the g functions using oscillaroty singular integrals whose kernels are taking values in a Hilbert space.

As an application of the LPS theory we prove a multiplier theorem for the Weyl transform. The Weyl transform, which we denote by τ takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. It enjoys most of the properties of the ordinary Fourier transform. In analogy with the definition of Fourier multipliers, we can define Weyl multipliers. In [4] Mauceri has studied Weyl multipliers and has given sufficient conditions on an operator $M \in B(L^2(\mathbb{R}^n))$ so that it will be an L^p multiplier for the Weyl transform. In this paper we are concerned only with multipliers of the form $\phi(H)$ where H is the Hermite operator.

The multiplier theorem of Mauceri follows form a modified version of the Calderón-Zygmund theory in the general setting of homogeneous spaces developed by Coifman and Weiss [2]. On the other hand we follow the method used by Stein in his proof of Hormander-Mihlin multiplier theorem for the Fourier transform. The same method was successfully employed by Strichartz [9] to prove a multiplier theorem for the Spherical Harmonic expansions. Recently the author [11] used the LPS theory for the Hermite semigroup to prove a multiplier theorem for the Hermite expansions.

The plan of the paper is as follows. In the next section we briefly review the relevant facts about twisted convolution and the Weyl transform and state the main results of the paper. In Section 3 we apply Ricci-Stein theory of oscillatory singular integrals, after making necessary modifications, to study the functions g and g_k^*. Finally, in Section 4 we prove a version of the multiplier theorem for the Weyl transform.

2. Preliminaries and Main Results

Let

$$\omega(z, v) = \exp\left(-\frac{i}{2}\operatorname{Im}(z, \bar{v})\right)$$

and let $dv\, d\bar{v}$ stand for the Lebesgue measure on $\mathbb{C}^n$. Then the product

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - v) g(v) \bar{\omega}(z, v)\, dv\, d\bar{v} \tag{2.1}$$

is called the twisted convolution of the functions f and g on $\mathbb{C}^n$. It is well known [3] that ω satisfies the cocycle identities

(*a*) $\omega(z, 0) = \omega(z, z) = \omega(0, z) = 1$
(*b*) $\omega(z + v, u)\omega(z, u) = \omega(z, v + u)\omega(v, u)$

and that there exists an irreducible projective representation W of $\mathbb{C}^n$ into a separable Hilbert space H_W such that

$$W(z + v) = \omega(z, v) W(z) W(v).$$

Given a function f in $L^1(\mathbb{C}^n)$ its Weyl transform $\tau(f)$ is a bounded operator on H_W defined by

$$\tau(f) = \int_{\mathbb{C}^n} f(z) W(z)\, dz\, d\bar{z}. \tag{2.2}$$

The Weyl transform enjoys many of the properties of the ordinary Fourier transform. Indeed we have an analogue of the Fourier inversion formula:

$$f(z) = (2\pi)^{-n} \operatorname{tr} (W(z)^* \tau(f)) \tag{2.3}$$

and the Plancherel formula:

$$\| f \|_2^2 = (2\pi)^{-n} \| \tau(f) \|_{HS}^2 \tag{2.4}$$

where tr is the canonical semifinite trace on the algebra of bounded operators on H_W and $\| \cdot \|_{HS}$ is the Hilbert-Schmidt norm. Moreover, the Weyl transform of the twisted convolution is the product of the Weyl transforms,

$$\tau(f \times g) = \tau(f)\tau(g). \tag{2.5}$$

Let h^n be the Lie algebra generated by the following differential operators and the identity:

$$Z_j = \partial_j + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \bar{\partial}_j - \frac{1}{4} z_j, \qquad j = 1, \ldots, n. \tag{2.6}$$

Here $\partial_j = \partial/\partial z_j$ and $\bar{\partial}_j = \partial/\partial \bar{z}_j$. Let U^n stand for the universal enveloping algebra of h^n. Let us take $H_W = L^2(\mathbb{R}^n)$ and consider the Schrödinger representation defined by

$$W(z)\phi(\xi) = \exp \left\{ i \left(x, \frac{1}{2} y + \xi \right) \right\} \phi(\xi + y) \tag{2.6$'$}$$

where $z = x + iy \in \mathbb{C}^n$. The representation W extends to a representation of the enveloping algebra denoted by dW. From (2.6) it follows that for every

f in the Schwartz class

$$\tau(Z_j f) = \tau(f)\, dW(Z_j) = i\tau(f)A_j^* \tag{2.7}$$

$$\tau(\bar{Z}_j f) = \tau(f)\, dW(\bar{Z}_j) = i\tau(f)A_j \tag{2.8}$$

where A_j and A_j^* are the 'annihilation' and creation operators defined by

$$A_j = \partial/\partial\xi_j + \xi_j, \qquad j = 1, 2, \ldots, n \tag{2.9}$$

$$A_j^* = -\partial/\partial\xi_j + \xi_j, \qquad j = 1, 2, \ldots, n \tag{2.10}$$

Let

$$H = \frac{1}{2}\sum_{j=1}^{n} (A_j A_j^* + A_j^* A_j) = \sum_{j=1}^{n} (-\partial^2/\partial\xi_j^2 + \xi_j^2)$$

be the Hermite operator. Let $\{\Phi_\alpha\}$ be the family of n-dimensional Hermite functions which form an othonormal basis for $L^2(\mathbb{R}^n)$ and let P_N be the projection onto the eigenspace spanned by $\{\Phi_\alpha : |\alpha| = N\}$. Then H has the spectral resolution

$$H = \sum_{N=0}^{\infty} (2N + n)P_N. \tag{2.11}$$

If we let

$$L = -\frac{1}{2}\sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j)$$

then it follows that $\tau(Lf) = \tau(f)H$ or more generally

$$\tau(\phi(L)f) = \tau(f)\phi(H). \tag{2.12}$$

Let T^t be the semigroup generated by the operator L. Then following Stein [8] we define the following Littlewood-Paley g and g^* functions.

$$g(f, z)^2 = \int_0^\infty t|\partial_t T^t f(z)|^2\, dt \tag{2.13}$$

$$g_k(f, z)^2 = \int_0^\infty t^{2k-1}|\partial_t^k T^t f(z)|^2\, dt \tag{2.14}$$

$$g_k^*(f, z)^2 = \int_0^\infty \int_{\mathbb{C}^n} t^{1-n}(1 + t^{-1}|z - v|^2)^{-k}|\partial_t T^t f(v)|^2\, dv\, d\bar{v}\, dt \tag{2.15}$$

It is easy to verify that $g(f, z) \leqslant c g_k(f, z)$, $k \geqslant 1$. The following theorem is our main result on the boundedness propertities of g and g^* functions.

Theorem 2.1.

(i) $C_1\|f\|_p \leqslant \|g(f)\|_p \leqslant C_2\|f\|_p, \quad 1 < p < \infty.$

(ii) $\|g_k^*(f)\|_p \leqslant C\|f\|_p, \quad p > 2$ *provided* $k > n$.

The proof of this theorem will be given in the next section. We then want to apply this theorem to prove a multiplier theorem of the Weyl transform.

Recall that the Weyl transform takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. In analogy with the definition of Fourier multipliers, we say that a bounded operator M on $L^2(\mathbb{R}^n)$ is a Weyl multiplier on $L^p(\mathbb{C}^n)$ if the operator T_M initially defined on $L^1 \cap L^p$ by

$$\tau(T_M f) = \tau(f)M \tag{2.16}$$

extends to a bounded operator on $L^p(\mathbb{C}^n)$. Sufficient conditions on the operator M have been obtained by Mauceri [4] so that T_M is a bounded operator on $L^p(\mathbb{C}^n)$. In this paper we consider only multipliers M of the form $\phi(H)$ where H is the Hermite operator.

To state our result on the multiplier theorem for Weyl transform we introduce the following forward and backward difference operators

$$\Delta_+ \phi(N) = \phi(N+1) - \phi(N)$$
$$\Delta_- \phi(N) = \phi(N) - \phi(N-1)$$

Theorem 2.2. *Suppose that the function ϕ satisfies*

$$|\Delta_-^k \Delta_+^m \phi(N)| \leqslant CN^{-(k+m)} \tag{2.17}$$

with k, m positive integers such that $k + m = 0, 1, \ldots, \nu$, where $\nu = n + 1$ when n is odd and $\nu = n + 2$ when n is even. Then $\phi(H)$ is a Weyl multiplier on $L^p(\mathbb{C}^n)$, $1 < p < \infty$.

This theorem will be proved in Section 4. A good reference for the Weyl transform is [5].

3. Oscillatory Integral and LPS Theory

The aim of this section is to prove Theorem 2.1 on the boundedness of g and g_k^* functions. That will be done by first studying oscillatory singular integrals whose kernel takes values in a Hilbert space. To see how oscillatory singular integrals enter the picture let us analyze the operator $f \to \partial_t T^t f$ more closely.

In view of the equation (2.12) if we take the Weyl transform of $\partial_t T^t f$ we get

$$\tau(\partial_t T^t f) = \tau(f)\partial_t(e^{-tH}) \tag{3.1}$$

In other words, $\partial_t T^t f$ is given by a twisted convolution

$$\partial_t T^t f(z) = f \times k_t(z) \tag{3.2}$$

where $k_t(z) = \tau^{-1}(\partial_t e^{-tH})$ is the inverse Weyl transform of $\partial_t e^{-tH}$. Since $\partial_t e^{-tH}$ has the spectral resolution

$$\partial_t e^{-tH} = -\sum_{N=0}^{\infty} (2N+n)e^{-(2N+n)t}P_N \tag{3.3}$$

it is easy to calculate the kernel k_t explicitly. Indeed, Peetre [5] has shown that

$$\tau^{-1}(P_N) = (2\pi)^{-n}e^{-(1/4)|z|^2}L_N^{n-1}\left(\frac{1}{2}|z|^2\right) \tag{3.4}$$

where L_N^{n-1} are the Laguerre polynomials of degree N and type $n-1$.

Recall that for $\alpha > -1$, Laguerre polynomials $L_k^\alpha(x)$ are defined by the equation

$$e^{-x}x^\alpha L_k^\alpha(x) = \frac{1}{k!}\frac{d^k}{dx^k}(e^{-x}x^{k+\alpha}). \tag{3.5}$$

The Laguerre polynomials also satisfy the following generating function relation

$$\sum_{k=0}^{\infty} L_k^\alpha(x)r^k = (1-r)^{-\alpha-1}e^{-xr/(1-r)} \tag{3.6}$$

In view of the relations (3.3), (3.4) and (3.6) we see that

$$k_t(z) = (2\pi)^{-n}\partial_t\{(\sinh t)^{-n}e^{-1/4|z|^2\coth t}\}. \tag{3.7}$$

Writing out the twisted convolution $f \times k_t$ we have

$$\partial_t T^t f(z) = \int_{\mathbb{C}^n} f(v)e^{-i/2\,\mathrm{Im}(z\bar{v})}k_t(z-v)f(v)\,dv\,d\bar{v} \tag{3.8}$$

or equivalently

$$\partial_t T^t f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)}k_t(x-y)f(y)\,dy \tag{3.9}$$

where $P(x,y)$ is a real valued polynomial in x and y.

The kernel $k_t(x)$ can be considered as taking values in the Hilbert space $L^2(\mathbb{R}^+, t\,dt)$. The following lemma shows that it is a Calderón-Zygmund kernel.

Lemma 3.1. *The kernel k_t satisfies*

(i) $\|k_t(x)\| \leqslant C|x|^{-2n}$
(ii) $\|\nabla k_t(x)\| \leqslant C|x|^{-2n-1}$

where $\|\cdot\|$ stand for the norm of $L^2(\mathbb{R}^+, t\,dt)$.

PROOF. Since

$$k_t(x) = (2\pi)^{-n}\partial_t\{(\sinh t)^{-n}e^{-1/4|x|^2\coth t}\}$$

it is easy to see that the following estimate holds

$$|k_t(x)| \leqslant Ct^{-n-1}(1 + t^{-1}|x|^2)^{-n-1}.$$

From this it follows immediately that

$$\begin{aligned}\|k_t(x)\|^2 &\leqslant C\int_0^\infty t^{-2n-1}(1 + t^{-1}|x|^2)^{-2n-2}\,dt \\ &\leqslant C|x|^{-4n}\int_0^\infty t^{-2n-1}(1 + t^{-1})^{-2n-2}\,dt \\ &= C|x|^{-4n}\end{aligned}$$

as the t integral is convergent. This proves (i). The proof of the second estimate is similar. Any x_j derivative will in effect bring down a factor of $t^{-1/2}$ which accounts for the extra factor $|x|^{-1}$. The details are omitted.

Thus the operator $\partial_t T^t f$ can be considered as a oscillatory singular integral whose kernel takes values in the Hilbert space $L^2(\mathbb{R}^+, t\,dt)$. Having made this observation we proceed to prove Theorem 2.1. In doing so we closely follow Ricci-Stein [6]. As the first step we prove the following L^2 result.

Proposition 3.1. *For $f \in L^2(\mathbb{C}^n)$,*

$$\|g(f)\|_2 = \frac{1}{2}\|f\|_2.$$

PROOF. We have from the definition

$$\int_{\mathbb{C}^n} g(f, z)^2\,dz\,d\bar{z} = \int_0^\infty\int_{\mathbb{C}^n} t|\partial_t T^t f(x)|^2\,dz\,d\bar{z}\,dt.$$

To the inner integral we apply the Plancherel formula (2.4). The result is

$$\int_{\mathbb{C}^n} |\partial_t T^t f(z)|^2\,dz\,d\bar{z} = (2\pi)^{-n}\|\tau(\partial_t T^t f)\|_{HS}^2.$$

Since $\tau(\partial_t T^t f) = \tau(f)(\partial_t e^{-tH})$,

$$\|\tau(\partial_t T^t f)\|_{HS}^2 = \operatorname{tr}((\tau(f)(\partial_t e^{-tH}))^*(\tau(f)(\partial_t e^{-tH})))$$

As $\partial_t e^{-tH}$ is a self-adjoint operator and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ we have

$$\|\tau(\partial_t T^t f)\|_{HS}^2 = \operatorname{tr}(\tau(f)^*\tau(f)H^2e^{-2tH}).$$

Since $\{\Phi_\alpha\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$,

$$\operatorname{tr}(\tau(f)^*\tau(f)H^2e^{-2tH}) = \sum_{\alpha \geq 0}(\Phi_\alpha, \tau(f)^*\tau(f)H^2e^{-2tH}\phi_\alpha)$$

which equals $\sum_{\alpha\geq 0}(2|\alpha|+n)^2 e^{-2(2|\alpha|+n)t}(\Phi_\alpha, \tau(f)^*\tau(f)\Phi_\alpha)$. Integrating the last equation with respect to $t\,dt$ we get

$$\|g(f)\|_2^2 = (2\pi)^{-n}\sum_{\alpha\geq 0}\int_0^\infty (2|\alpha|+n)^2 t e^{-2(2|\alpha|+n)t}\,dt\,(\Phi_\alpha, \tau(f)^*\tau(f)\Phi_\alpha)$$

$$= (2\pi)^{-n}\frac{1}{4}\operatorname{tr}(\tau(f)^*\tau(f)) = \frac{1}{4}\|f\|_2^2.$$

Hence we have proved

$$\|g(f)\|_2 = \frac{1}{2}\|f\|_2.$$

To prove that g is bounded on $L^p(\mathbb{C}^n)$ we split the operator $\partial_t T^t f$ into two parts. Let $\alpha(x)$ be a C_0^∞ function supported in $|x| \leqslant 1$ such that $\alpha(x) = 1$ for $|x| \leqslant 3/4$. Let $\beta(x) = 1 - \alpha(x)$ and define

$$T_0 f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)} k_t(x-y)\alpha(x-y) f(y)\,dy \tag{3.10}$$

$$T_\infty f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)} k_t(x-y)\beta(x-y) f(y)\,dy \tag{3.11}$$

First we will take care of the local operator T_0. To do so, we first want to prove that the operator

$$\tilde{T}_0 f(x) = \int_{\mathbb{R}^{2n}} k_t(x-y) f(y)\alpha(x-y)\,dy \tag{3.12}$$

is bounded from $L^p(\mathbb{R}^{2n})$ to $L^p(\mathbb{R}^{2n}, L^2(\mathbb{R}^+, t\,dt))$. (Hereafter we simply say $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.) Observe that in view of Lemma 3.1 $\tilde{T}_0$ is a vector valued singular integral operator whose kernel satisfies the estimates

(i) $\|k_t(x)\alpha(x)\| \leqslant C|x|^{-2n}$,

(ii) $\|\nabla(k_t(x)\alpha(x))\| \leqslant C|x|^{-2n-1}$.

So if we know that $\tilde{T}_0$ is bounded on $L^2(\mathbb{R}^{2n})$ then we can apply the following theorem to conclude that $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.

Theorem 3.2. (Stein [7].) *Let $k(x)$ be a C^1 function away from the origin taking values in $B(H_1, H_2)$ where H_1 and H_2 are Hilbert spaces. Assume that*

(i) $\|k(x)\| \leqslant C|x|^{-n}$,

(ii) $\|\nabla k(x)\| \leqslant C|x|^{-n-1}$.

Let $f(x)$ takes values in H_1 and T be defined by

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y) f(y)\,dy.$$

If T is bounded on $L^2(R^n)$ it is also bounded on $L^p(R^n)$, $1 < p < \infty$.

To prove that $\tilde{T}_0$ is bounded on $L^2(\mathbb{R}^{2n})$ we need to know if T_0 is bounded on $L^2(\mathbb{R}^{2n})$ or not.

Lemma 3.2. *T_0 is bounded on $L^2(\mathbb{R}^{2n})$.*

PROOF. The operator $f \to \partial_t T^t f$ is bounded on $L^2(\mathbb{R}^{2n})$ and T_0 is a truncation of this operator. Hence T_0 is bounded on $L^2(\mathbb{R}^{2n})$. For a proof see the corresponding lemma in Ricci-Stein [6].

Lemma 3.3. *$\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.*

PROOF. As we have already mentioned we need only to prove that $\tilde{T}_0$ is bounded on L^2. We write

$$\begin{aligned}\tilde{T}_0 f(x) &= \int k_t(x-y)\alpha(x-y)f(y)\,dy \\ &= \int e^{iP(x,y)}k_t(x-y)\alpha(x-y)f(y)\,dy \\ &\quad - \int [e^{iP(x,y)}-1]k_t(x-y)\alpha(x-y)f(y)\,dy \\ &= T_0 f(x) + T_1 f(x).\end{aligned}$$

We will prove that

$$\int_{|x|\leq 1} \|\tilde{T}_0 f(x)\|^2\,dx \leqslant C\int_{|y|\leq 2} |f(y)|^2\,dy. \tag{3.13}$$

Since the kernel is supported in $|x-y| \leqslant 1$, when $|x| \leqslant 1$ only the points $|y| \leqslant 2$ matter. Since $P(y,y)=0$, $e^{iP(x,y)}-1 = O(|x-y|)$ and so the kernel of T_1 is integrable. Since T_0 is already known to be bounded on L^2 this proves (3.13). Since $\tilde{T}_0$ is translation invariant we also have

$$\int_{|x-h|\leq 1} \|\tilde{T}_0 f(x)\|^2\,dx \leqslant C\int_{|y-h|\leq 2} |f(y)|^2\,dy.$$

Integration with respect to h proves that $\tilde{T}_0$ is bounded on $L^2(\mathbb{R}^{2n})$. Hence by the previous Theorem 3.1, $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.

Now we use Lemma 3.3 to prove that T_0 is bounded on $L^p(\mathbb{R}^{2n})$.

Proposition 3.2. *T_0 is bounded on $L^p(\mathbb{R}^{2n})$, $1<p<\infty$.*

PROOF. The proof is similar to the corresponding theorem in Ricci-Stein [6]. We give the details for the sake of completeness. We use the same trick as in Lemma 3.3. We will first prove that

$$\int_{|x|\leq 1} \|T_0 f(x)\|^p\,dx \leqslant C\int_{|y|\leq 2} |f(y)|^p\,dy \tag{3.14}$$

$$T_0f(x) = \int [e^{iP(x,y)} - 1]k_t(x-y)\alpha(x-y)f(y)\,dy + \int k_t(x-y)\alpha(x-y)f(y)\,dy.$$

As before the kernel of the first integral is integrable and the second integral is $\tilde{T}_0 f$. Hence we have (3.14).

It is easily verified that

$$P(x+h, y+h) = P(x,y) + P(x,h) + P(h,y)$$

and therefore

$$\int_{|x-h|\leq 1} \| T_0 f(x)\|^p\,dx \leqslant C \int_{|y-h|\leq 2} |f(y)|^p\,dy$$

is also true. Integration with respect to h proves the Proposition.

Having taken care of T_0 we now turn our attention to the study of T_∞. Again we repeat the arguments of Ricci-Stein [6] but this time we will not give the details.

Let us take a partition of unity

$$1 = \sum_{j=-\infty}^{\infty} \psi_0(2^{-j}x)$$

where ψ_0 is supported in $1/2 \leqslant |x| \leqslant 1$ and write

$$k_t(x)\beta(x) = \sum_{j=0}^{\infty} k_t(x)\beta(x)\psi_0(2^{-j}x) = \sum_{j=0}^{\infty} k_t^j(x).$$

Let

$$T_j f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)} k_t^j(x-y) f(y)\,dy. \tag{3.15}$$

For these operators we wish to prove that their norm as operators on $L^2(\mathbb{R}^{2n})$ satisfy

$$\| T_j\| \leqslant C2^{-j\epsilon} \quad \text{for some} \quad \epsilon > 0. \tag{3.16}$$

To do this we consider $T_j^* T_j$ and estimate the kernel of $T_j^* T_j$. For that purpose we make the following observations regarding singular integral operators whose kernels take values in $B(H_1, H_2)$.

So, let $k(x,y)$ take values in $B(H_1, H_2)$ and $f(x)$ take values in H_1. Consider the operator

$$Tf(x) = \int k(x,y)f(x). \tag{3.17}$$

Assume that T is bounded from $L^2(\mathbb{R}^n, H_1)$ into $L^2(\mathbb{R}^n, H_2)$. This operator T has a formal adjoint T^* which maps $L^2(\mathbb{R}^n, H_2)$ into $L^2(\mathbb{R}^n, H_1)$. Let $k^*(x,y)$

denote the adjoint of the operator $k(x,y)$ which belongs to $B(H_2, H_1)$. Then it is easily verified that T^* is given by

$$T^*f(x) = \int k^*(y,x)f(y)\,dy. \tag{3.18}$$

From this it follows that

$$T^*Tf(x) = \iint k^*(y,x)k(y,z)f(z)\,dz\,dy.$$

Thus the kernel $G(x,z)$ of T^*T is

$$G(x,z) = \int k^*(y,x)k(y,z)\,dy$$

which is a bounded operator on H_1.

Let us specialize these observations to our operator T_j. The kernel of T_j is $e^{iP(x,y)}k_t^j(x-y)$ which takes values in the Hilbert space $L^2(\mathbb{R}^+, t\,dt)$. By taking $H_1 = \mathbb{C}$ and $H_2 = L^2(\mathbb{R}^+, t\,dt)$ we can assume that the kernel belongs to $B(H_1, H_2)$. The action of $e^{iP(x,y)} \times k_t^j(x-y)$ on $\mathbb{C}$ is simply given by $\lambda \to e^{iP(x,y)}k_t^j(x-y)\lambda$. Call this operator $s_j(x,y)$. The adjoint of $s_j(x,y)$ is then given by

$$s_j^*(x,y)g = \int_0^\infty e^{-iP(x,y)}k_t^j(x-y)g(t)t\,dt.$$

Therefore, the kernel $L_j(x,z)$ of $T_j^*T_j$ is

$$L_j(x,z) = \iint_0^\infty k_t^j(y-x)k_t^j(y-z)e^{i(P(y,z)-P(y,x))}\,t\,dt\,dy.$$

Thus the kernel of $T_j^*T_j$ is a scalar valued function which acts on H_1 by multiplication.

Having calculated the kernel of $T_j^*T_j$ now we can simply repeat the arguments of Ricci-Stein. Since k_t is a Calderón-Zygmund kernel the proof given in [6] goes through without any change. This completes the proof that T_∞ is bounded on L^p.

Now it is time to complete the proof of Theorem 3.1. Just now we showed that $g(f)$ is bounded on $L^p(\mathbb{C}^n)$. Since

$$\|g(f)\|_2 = \frac{1}{2}\|f\|_2,$$

the reverse inequality can be proved as in Stein [7]. Deduction of (ii) from (i) is routine and we refer to Stein [7] for details.

4. Multiplier Theorem for the Weyl Transform

In this section we will prove Theorem 3.2. In view of Theorem 3.1 it is enough to prove that

$$g_{k+1}(F, z) \leqslant C g_k^*(f, z) \tag{4.1}$$

where $F(z) = T_\phi f(z)$ for $k = n + 1$. Recall that the operator T_ϕ is defined by means of the Weyl transform as $\tau(T_\phi f) = \tau(f)\phi(H)$. Assume that the function ϕ satisfies the conditions stated in Theorem 3.2. Without loss of generality we further assume that $\phi(2N + n) = 0$ for $N \leqslant n + 1$.

Let $u(z, t) = T^t f(z)$ and $U(z, t) = T^t F(z)$. Then it is easily verified that

$$U(z, t + s) = u(z, s) \times M(t, z) \tag{4.2}$$

where $M(t, z)$ is the function defined by

$$M(t, z) = (2\pi)^{-n} \sum_{N=0}^{\infty} \phi(2N + n) e^{-(2N+n)t} e^{-1/4|z|^2} L_N^{n-1}\left(\frac{1}{2}|z|^2\right). \tag{4.3}$$

Taking one derivative with respect to s, k derivatives with respect to t and setting $t = s$ we get

$$\partial_t^{k+1} U(z, 2t) = (\partial_t T^t f) \times \partial_t^k M. \tag{4.4}$$

The following lemma translates the conditions on ϕ into properties of the function M.

Lemma 4.1. *Under the assumptions stated in Theorem* 3.2 *the following estimates are true.*

(i) $|\partial_t^k M(t, z)| \leqslant C t^{-n-k}$,

(ii) $\int_{\mathbb{C}^n} |z|^{2k} |\partial_t^k M(t, z)|^2 \, dz \, d\bar{z} \leqslant C t^{-n-k}$ *if k is even*,

(iii) $\int_{\mathbb{C}^n} |z|^{2k+2} |\partial_t^k M(t, z)|^2 \, dz \, d\bar{z} \leqslant C t^{-n-k+1}$ *if k is odd*.

Assuming the lemma for a moment we will complete the proof of Theorem 3.2. We want to prove (4.1) when $k = n + 1$. Recall that

$$g_{k+1}(F, z)^2 = \int_0^\infty t^{2k+1} |\partial_t^{k+1} T^t F(z)|^2 \, dt.$$

From equation (4.4) we have

$$\begin{aligned} |\partial_t^{k+1} T^{2t} F(z)| &\leqslant \int_{\mathbb{C}^n} |\partial_t T^t f(v)| \, |\partial_t^k M(t, z - v)| \, dv \, d\bar{v} \\ &= \int_{|z-v| \leq t^{1/2}} + \int_{|z-v| > t^{1/2}} = A_t(z) + B_t(z). \end{aligned}$$

Applying Schwarz inequality and using (i) of Lemma 4.1 we get

$$\begin{aligned} A_t(z)^2 &\leqslant \int_{|z-v|\leq t^{1/2}} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v} \int_{|z-v|\leq t^{1/2}} |\partial_t^k M(t, z-v)|^2 \,dv\,d\bar{v} \\ &\leqslant C \int_{|z-v|\leq t^{1/2}} t^{-n-2k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v} \\ &\leqslant C t^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v} \end{aligned}$$

When k is even another application of Schwarz inequality together with (ii) gives

$$\begin{aligned} B_t(z)^2 &\leqslant \int_{|z-v|>t^{1/2}} |z-v|^{-2k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v} \\ &\quad \int_{|z-v|>t^{1/2}} |z-v|^{2k} |\partial_t^k M(t, z-v)|^2 \,dv\,d\bar{v} \\ &\leqslant C t^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v}. \end{aligned}$$

When k is odd we use (iii):

$$\begin{aligned} B_t(z)^2 &\leqslant \int_{|z-v|>t^{1/2}} |z-v|^{-2k-2} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v} \\ &\quad \int_{|z-v|>t^{1/2}} |z-v|^{2k+2} |\partial_t^k M(t, z-v)|^2 \,d\bar{v}\,dv \\ &\leqslant C t^{-1} t^{-n-k+1} \int_{|z-v|>t^{1/2}} |z-v|^{-2k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v} \\ &\leqslant C t^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v}. \end{aligned}$$

Hence

$$|\partial_t^{k+1} T^t F(z)|^2 \leqslant C t^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \,dv\,d\bar{v}.$$

Multiplying by t^{2k+1} and integrating with respect to t we get

$$g_{k+1}(F, z)^2 \leqslant C g_k^*(f, z)^2.$$

Hence Theorem 3.2 is proved when $p > 2$. But it is easy to see that the adjoint of T_ϕ is also a multiplier of the same form and hence the Theorem 3.2 is completely proved.

Before going to the proof of Lemma 4.1 let us collect some facts about Laguerre polynomials which will be needed in the proof. If we set

$$\phi_n^\alpha(x) = r_n^{-1/2} e^{-x/2} L_n^\alpha(x) x^{\alpha/2}$$

where

$$r_n = \binom{n+\alpha}{n}$$

then $\{\phi_n^\alpha\}$ forms an orthonormal system for $L^2(0, \infty)$. We also observe that $r_n \sim n^\alpha$ as $n \to \infty$. The normalized Laguerre functions $\{\phi_k^\alpha\}$ satisfy the following generating function relation

$$(4.5)\quad \sum_{k=0}^{\infty} \phi_k^\alpha(x)\phi_k^\alpha(y)r^k$$
$$= \Gamma(\alpha+1)(1+r)^{-1}(-r)^{-\alpha/2}e^{-(x+y)/2}J_\alpha(2(-xyr)^{1/2}(1-r)^{-1})$$

where J_α is the Bessel function of order α. We also need the following recursion relation satisfied by the Laguerre polynomials

$$(4.6)\qquad kL_k^\alpha(x) = (-x+2k+\alpha-1)L_{k-1}^\alpha(x) - (k+\alpha-1)L_{k-2}^\alpha(x).$$

Finally the following asymptotic properties of the Bessel function are also needed

$$(4.7)\qquad |J_\alpha(z)| \leqslant C|z|^\alpha, \qquad |z| \leqslant 1$$
$$(4.8)\qquad |J_\alpha(iz)| \leqslant Cz^{-1/2}e^z, \qquad z \geqslant 1.$$

A good reference for all the above facts is Szego [10].

Let us start by proving (i). We write

$$\rho = \frac{1}{2}|z|^2.$$

Since ϕ is a bounded function and

$$\binom{N+n-1}{N} \sim N^{n-1}$$

we have

$$|\partial_t^k M(t,z)| \leqslant C \sum_{N=0}^{\infty} (2N+n)^{k+(n-1)/2}e^{-(2N+n)t}\left(\frac{N!}{(N+n-1)!}\right)^{1/2} L_N^{n-1}(\rho)e^{-\rho/2}.$$

Applying Schwarz inequality

$$|\partial_t^k M(t,z)|^2 \leqslant C\left\{\sum_{N=0}^{\infty}(2N+n)^{2k+n-1}e^{-(2N+n)t}\right\}$$
$$\times\left\{\sum_{N=0}^{\infty} e^{-(2N+n)t}\left(\frac{N!}{(N+n-1)!}\right)^{1/2} L_N^{n-1}(\rho)^2 e^{-\rho}\right\}.$$

The first term is $(-1)^{2k+n-1}$ times the $2k+n-1$ derivative of $e^{-nt}(1-e^{-2t})^{-1}$ and hence is bounded by constant times t^{-2k-n}. In view of (4.5) we have the formula

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho}$$

$$= \frac{1}{2}\Gamma(n)(-1)^{-(n-1)/2}(\sinh t)^{-1} e^{-\rho\coth t}\rho^{-(n-1)} J_{n-1}(i\rho\,\mathrm{cosech}\, t).$$

When $\rho\,\mathrm{cosech}\, t \leqslant 1$, in view of (4.7) we get

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho}$$

$$\leqslant C(\sinh t)^{-1}\rho^{-(n-1)}\rho^{n-1}(\mathrm{cosech}\, t)^{n-1} \leqslant Ct^{-n}$$

On the other hand when $\rho\,\mathrm{cosech}\, t > 1$ we use (4.8) to get the estimate

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho}$$

$$\leqslant C(\sinh t)^{-1} e^{-\rho\coth t}\rho^{-(n-1)}\rho^{-1/2}(\mathrm{cosech}\, t)^{-1/2} e^{\rho\,\mathrm{cosech}\, t}$$

$$= C(\sinh t)^{-1/2}\rho^{-n+1/2} e^{-\rho\tanh t/2}.$$

Since $\rho > \sinh t$ we get

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho} \leqslant C(\sinh t)^{-n} e^{-\rho\tanh t/2} \leqslant Ct^{-n}.$$

Hence $|\partial_t^k M(t,z)|^2 \leqslant Ct^{-2k-2n}$ and (i) is proved.

To prove (ii) we make use of the recursion relation (4.6). We write it in the following form

$$\rho L_N^{n-1}(\rho) = (2N+n)L_N^{n-1}(\rho) - (N+1)L_{N+1}^{n-1}(\rho) - (N+n-1)L_{N-1}^{n-1}(\rho).$$

Let us set

$$\psi(N) = (2N+n)^k e^{-(2N+n)t}\phi(2N+n).$$

In view of the recursion relation, as $\phi(2N+n) = 0$ for $N \leqslant n+1$,

$$\rho\partial_t^k M(t,z) = \sum [(2N+n)\psi(N) - N\psi(N-1) - (N+n)\psi(N+1)] L_N^{n-1}(\rho) e^{-\rho/2}$$

In terms of the operators Δ_+ and Δ_- we can write

$$\rho\partial_t^k M(t,z) = -\sum [N\Delta_-\Delta_+\psi(N) + n\Delta_-\psi(N)] L_N^{n-1}(\rho) e^{-\rho/2}.$$

Under the assumptions made on ϕ we observe that the effect of multiplying $\partial_t^k M$ by ρ is essentially to change $\psi(N)$ into $N^{-1}\psi(N)$. Now we can iterate this process. Since k is even in the present case after applying $\rho(k/2)$ times we get

$$\rho^{k/2}\partial_t^k M(t,z) = \sum \psi_k(N) L_N^{n-1}(\rho) e^{-\rho/2}$$

where we have an estimate of the form

$$|\psi_k(N)| \leqslant C(2N+n)^{k/2} e^{-(2N+n)t}.$$

Therefore,

$$\int_{\mathbb{C}^n} |z|^{2k} |\partial_t^k M(t,z)|^2 \, dz \, d\bar{z} = C \int_0^\infty |\rho^{k/2} \partial_t^k M(t,z)|^2 \rho^{n-1} \, d\rho$$
$$= C \int_0^\infty \Big| \sum \psi_k(N) e^{-\rho/2} \rho^{(n-1)/2} L_N^{n-1}(\rho) \Big|^2 d\rho$$

since $\{\phi_N^{n-1}(\rho)\}$ form an orthonormal system the last integral is equal to

$$\sum \psi_k(N)^2 \binom{N+n-1}{n} \leqslant C \sum (2N+n)^{k+n-1} e^{-(2N+n)t} \leqslant Ct^{-k-n}.$$

This proves (ii). The proof of (iii) is similar. Hence the lemma.

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Recibido: 9 de junio de 1988.

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Revista Matemática Iberoamericana
Vol. 6, n.os 3 y 4, 1990

Construction de Bases D'Ondelettes α-Hödériennes

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Résumé

Dans cet article, nous reprenons une méthode due à Ingrid Daubechies pour générer des bases orthonormales de fonctions dans $L^2(\mathbb{R})$ de la forme $\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ à partir de filtres miroir en quadrature (QMF) tels que l'ondelette ψ ait de bonnes propriétés de régularité. Une estimation de l'exposant de Hölder global optimal est obtenue en caractérisant précisément la décroissance de la fonction $\hat{\psi}$. Nous précisons finalement les liens exacts entre la régularité de l'ondelette et son ordre de cancellation (nombre de moments nuls).

Introduction

Depuis les travaux d'Yves Meyer ([1], [2]) et de J. O. Strönberg ([10]), on sait construire des bases orthonormées de l'espace $L^2(\mathbb{R})$ de la forme

$$\{2^{j/2}\psi(2^j x - k)\}_{j\in\mathbb{Z},\, k\in\mathbb{Z}}.$$

Le cas du système de Haar, inventé en 1911, s'identifie à une base de ce type où l'ondelette ψ est d'intégrale nulle, à support compact mais mal localisée en fréquence car discontinue. Dans certaines applications, telles que le codage, la compression des données, le calcul numérique performant ([4]) ou l'analyse mathématique des singularités ([5]), on souhaite que la fonction ψ présente plus de régularité et de moments nuls que dans l'exemple de la base de Haar.

D'autre part, les liens établis par Stéphane Mallat entre l'analyse multirésolution et les algorithmes pyramidaux montrent que l'implémentation numérique de la décomposition en ondelettes fait appel à la donnée unique d'une paire de QMF («quadrature mirror filter» au sens donné dans [6] par Smith et Barnwell). Réciproquement, s'ils satisfont certaines conditions, les QMF permettent de générer une analyse multirésolution et des ondelettes.

Il est donc intéressant de pouvoir estimer la régularité de la fonction ψ construite à partir d'un filtre donné. Les principaux travaux dans ce domaine ont été menés par Ingrid Daubechies ([7], [8]) qui a établi des conditions suffisantes portant sur les QMF pour obtenir de la régularité sur l'ondelette.

En poursuivant cette approche, nous allons chercher une estimation de l'exposant de Hölder global obtenu dans ces constructions. On verra notamment que, contrairement à ce qui a pu être conjecturé, la régularité de l'ondelette ne s'identifie pas au nombre de ses premiers moments qui s'annulent.

Nous commencerons par rappeler brièvement les liens entre les analyses multirésolutions et les QMF. Le lecteur désireux d'approfondir ses connaissances trouvera sur ce sujet des détails plus précis dans [3] et [9].

1. Construction des analyses multirésolutions

1.a. Analyse multirésolution et ondelettes

Par définition, une analyse multirésolution est une suite $\{V_j\}_{j\in\mathbb{Z}}$ de sous-espaces vectoriels fermés de $L^2(\mathbb{R})$ qui vérifie les propriétés suivantes

(1) $$V_j \subset V_{j+1}$$

(2) $$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$$

(3) $$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$$

(4) $$\bigcup_{j\in\mathbb{Z}} V_j \quad \text{est dense dans} \quad L^2(\mathbb{R})$$

(5) Il existe une fonction φ de V_0 telle que la famille $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ soit une base hilbertienne de V_0.

Remarquons qu'alors, $\{2^{j/2}\varphi(2^j - k)\}_{k\in\mathbb{Z}}$ est une base orthonormée de l'espace V_j qui est donc invariant par les translations de pas $k2^{-j}$.

La fonction φ vérifie deux propriétés importantes:

— $\int_{\mathbb{R}} \varphi(x)\,dx = \hat{\varphi}(0) = 1$ qui est liée à la propriété (4)

—φ satisfait l'identité

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + 2k\pi)|^2 = 1 \tag{6}$$

ou encore

$$\int_{\mathbb{R}} |\hat{\varphi}(\omega)|^2 e^{ik\omega} = 2\pi\delta_{0,k} \tag{7}$$

qui expriment l'orthonormalité de la suite $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$.

Si nous définissons à présent le sous-espace W_j comme étant le supplémentaire orthogonal de V_j dans V_{j+1}, il est clair que la somme orthogonale $\overset{\perp}{\bigoplus} W_j$ est dense dans $L^2(\mathbb{R})$ et que W_j s'obtient en dilatant W_{j+1} d'un rapport 2.

On peut alors trouver une fonction ψ telle que $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ soit une base orthonormée de W_0. La famille $\{2^{j/2}\psi(2^j x - k)\}_{j\in\mathbb{Z}, k\in\mathbb{Z}}$ est par conséquent une base orthonormée de $L^2(\mathbb{R})$ tout entier. La construction de l'ondelette ψ va être précisée grâce à l'introduction des QMF.

1.b. Analyse multirésolution et QMF

Les filtres QMF s'introduisent naturellement de la manière suivante: grâce à l'inclusion de V_{-1} dans V_0, on peut développer $\varphi(x/2)$ suivant les $\varphi(x-k)$.

On a donc

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} c_k \varphi(x-k). \tag{8}$$

Définissons à présent la fonction m_0 par

$$m_0(\omega) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\omega}.$$

C'est la fonction de transfert du filtre discret c_k et elle est 2π-périodique.

En prenant la transformée de Fourier de (8), il vient

$$\hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega). \tag{9}$$

Introduisons la relation (9) dans l'inégalité (6). L'expression

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(2\omega + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} (|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2)|\hat{\varphi}(\omega + 2k\pi)|^2$$

se simplifie en donnant la relation caractéristique des filtres QMF:

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1. \tag{10}$$

Par abus de langage nous confondrons désormais le filtre $\{c_k\}_{k\in\mathbb{Z}}$ avec sa fonction de transfert m_0. Nous appellerons «filtres QMF» les fonctions 2π-périodiques qui vérifient la propriété (10).

En traitement du signal, les «filtres QMF» désignent la paire constituée de la fonction m_0 et de son filtre conjugué défini par l'égalité

$$m_1(\omega) = e^{-i\omega}\overline{m_0(\omega + \pi)}$$

([6]). D'autre part, puisqu'on a $\hat{\varphi}(0) = 1$, il est clair d'après (9) que m_0 vaut 1 en 0.

Dans toute la suite on supposera que la fonction φ vérifie la propriété donnée par la définition suivante.

Définition 1. *Une analyse multirésolution est «localisée» si et seulement si la fonction $\hat{\varphi}(\omega)$ est dans l'espace de Sobolev $H^m(\mathbb{R})$, c'est-à-dire si*

$$\int_{\mathbb{R}} (1 + |x|)^{2m} |\varphi(x)|^2 \, dx < +\infty,$$

et ceci pour tout entier m.

La suite

$$\{c_k\}_{k\in\mathbb{Z}} = \left\{\int_{\mathbb{R}} \frac{1}{2}\varphi\left(\frac{x}{2}\right)\varphi(x-k)\,dx\right\}_{k\in\mathbb{Z}}$$

est alors rapidement décroissante et la fonction m_0 est donc régulière.

En itérant la relation (9), nous pouvons alors exprimer la fonction $\hat{\varphi}$ sous la forme d'un produit qui converge en tout point. On a

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_0\left(\frac{\omega}{2^k}\right). \tag{11}$$

Un choix possible pour l'ondelette ψ est alors le suivant

$$\hat{\psi}(2\omega) = m_1(\omega)\hat{\varphi}(\omega) = e^{-i\omega}\overline{m_0(\omega+\pi)} \prod_{k=1}^{+\infty} m_0\left(\frac{\omega}{2^k}\right). \tag{12}$$

Grâce au produit (11), on peut donc générer une analyse multirésolution et des bases d'ondelettes à partir de la donnée simple d'un QMF. Cependant, les filtres ayant la propriété (10) ne conviennent pas tous à la réalisation d'un tel programme. Nous allons maintenant préciser les caractéristiques des QMF associés à des analyses multirésolution.

1.c. QMF associés à des analyses multirésolution

Rappelons que nous nous plaçons dans le cadre des analyses localisées définies précédemment. Nous aurons besoin de la définition suivante.

Définition 2. *Un ensemble compact K de $\mathbb{R}$ est dit congru à $[-\pi, \pi]$ modulo 2π si et seulement si pour presque tout x dans $[-\pi, \pi]$, il existe un unique y dans K tel que $x - y$ soit un multiple entier de 2π. On exige de plus que cet ensemble soit formé d'une réunion finie d'intervalles.*

Nous pouvons alors caractériser précisément les QMF associés à des analyses multirésolution localisées.

Proposition 1. *Le produit* (11) *génère une analyse multirésolution localisée si et seulement si la fonction $m_0(\omega)$ est régulière et il existe un compact K, congru à $[-\pi; \pi]$ et contenant un voisinage de* 0, *sur lequel ce produit ne s'annule pas.*

La preuve de ce résultat est détaillée dans [9], ainsi que le sens de cette condition portant sur le QMF m_0.

Nous en présentons ici une esquisse:

—Dans un premier sens, partant d'une analyse multirésolution localisée, l'identité (6) perment alors de construire l'ensemble K souhaité. En effet, en utilisant la continuité de $\hat{\varphi}$, on peut écrire

$$\sum_{|x+2k\pi| \le A_\epsilon} |\hat{\varphi}(x+2k\pi)|^2 \geqslant 1 - \epsilon \quad \text{pour tout} \quad \epsilon > 0.$$

De cette localisation de l'identité (6), il découle que pour tout x dans $[-\pi; \pi]$, il existe un entier k_x tel que l'on ait $|\hat{\varphi}(x + 2k_x\pi)| \geqslant C > 0$ et $|k_x| \leqslant B$, les constantes B et C étant indépendantes de x. Ceci fournit un procédé pour la construction de notre compact par une union d'intervalles bien choisis. L'origine peut être choisie dans l'intérieur de K puisque $\hat{\varphi}(0) = 1$.

—Dans l'autre sens, on part d'un QMF ayant les propriétés requises. L'essentiel est alors de démontrer que si l'on construit $\hat{\varphi}(\omega)$ d'après (11), alors la suite $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ est orthonormée. On peut, pour cela, définir une suite de fonctions h_n par

$$\hat{h}_n(\omega) = \prod_{k=1}^{n} m_0\left(\frac{\omega}{2^k}\right) 1_{2^n K}(\omega) \tag{13}$$

$1_{2^n K}(\omega)$ est ici la fonction indicatrice du compact K dilaté d'un facteur 2^n.

Un raisonnement par récurrence sur n, utilisant la propriété (10) et la structure particulière de K permet de montrer que $\hat{h}_n$ vérifie la propriété (7) et donc que $\{h_n(x-k)\}_{k\in\mathbb{Z}}$ est une suite orthonormée.

Par ailleurs, puisque K contient un voisinage de l'origine, il est clair que $\hat{h}_n$ tend vers $\hat{\varphi}$ au sens de la convergence simple. On peut donc affirmer, grâce au lemme de Fatou, que $\hat{\varphi}$ est dans $L^2(\mathbb{R})$.

On remarque alors que l'on a

$$\hat{h}_n(\omega) = \frac{\hat{\varphi}(\omega)}{\varphi\left(\frac{\omega}{2^n}\right)}$$

si ω est dans $2^n K$ et $\hat{h}_n(\omega) = 0$ sinon.

Dans les deux cas, on a, d'après les hypothèses,

$$|\hat{h}_n(\omega)| \leqslant \frac{|\hat{\varphi}(\omega)|}{C}. \tag{14}$$

Ceci nous permet d'utiliser le théorème de la convergence dominée pour conclure que h_n tend vers φ dans $L^2(\mathbb{R})$ et que $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ est aussi une suite orthonormée.

L'appartenance de $\hat{\varphi}$ à tous les Sobolev $H^m(\mathbb{R})$ peut se démontrer en utilisant des produits finis du type (13) mais tronqués de manière plus douce que les $\hat{h}_n$. On raisonne là aussi par récurrence en exploitant essentiellement le caractère C^∞ de la fonction m_0.

Nous avons donc caractérisé les filtres QMF associés à une classe très large d'analyses multirésolution. On y trouve, par exemple, toutes les ondelettes à support compact dont le cas classique du système de Haar où les fonctions φ et ψ sont clairement discontinues.

Il est temps d'examiner la régularité exacte des fonctions φ et ψ obtenues dans ces constructions.

2. Les analyses multirésolutions α-régulières

Dans toute la suite, on s'intéresse uniquement aux propriétés de régularité de la fonction φ. Les mêmes propriétés se déduisent aisément pour l'ondelette ψ.

2.a. Les analyses r-régulières

Les analyses r-régulières ont été introduites par P. G. Lemarié et Yves Meyer ([2]).

r désigne ici un entier positif et on exige que la fonction φ vérifie les propriétés suivantes:

- φ est $r-1$ fois continûment dérivable.
- Pour tout m dans $\mathbb{N}$ et pour tout n tel que $0 \leqslant n \leqslant r$, on a

$$\int_{\mathbb{R}} (1+|x|)^m |\varphi^{(n)}(x)|\, dx < +\infty. \tag{15}$$

Les $r-1$ premières dérivées de φ sont alors à décroissance rapide.

Un exemple intéressant est fourni par les ondelettes de P. G. Lemarié et B. Battle ([11]) qui sont des fonctions splines d'ordre r.

Une propriété importante des analyses r-régulières remarquée par Yves Meyer ([2]) est l'annulation des r premiers moments de l'ondelette ψ. En effet, si l'on applique la formule de Taylor aux points $k2^{-j}$ dans l'intégrale nulle

$$\int_{\mathbb{R}} \psi(u)\bar{\psi}(2^{-j}u + 2^{-j}k)\, du,$$

en utilisant la propriété (15) pour majorer les restes, on voit que nécessairement

$$\int_{\mathbb{R}} x^n \psi(x)\, dx = 0 \quad \text{pour} \quad 0 \leqslant n \leqslant r.$$

On peut se demander si, réciproquement, cette dernière propriété entraîne la régularité de classe C^r ou C^{r-1}). Nous allons voir, par la suite, qu'il n'en est rien.

Dans un premier temps, revenons momentanément sur le procédé de construction des analyses multirésolution à partir des QMF.

2.b. L'algorithme en cascade

La démonstration de la Proposition 1 nous fournit en plus un procédé de construction numérique pour la fonction φ (qui permet aussi de calculer ψ).

La fonction φ est en effet approchée par la suite h_n. Par ailleurs, l'échantillonnage $\{h_n(2^{-n}k)\}_{k\in\mathbb{Z}}$ s'identifie aux coefficients de Fourier de la fonction $2^{n+1}\pi$-périodique

$$m_n(\omega) = \prod_{k=1}^{n} m_0\left(\frac{\omega}{2^k}\right).$$

Lorsque m_0 a une réponse impulsionnelle finie, le calcul de ces coefficients peut se faire de manière très simple, suivant un algorithme «en cascade» introduit par Ingrid Daubechies ([7]).

Dans cet algorithme, après avoir appliqué le filtre m_0 sur une suite de Dirac initiale, on l'itère, contracté à chaque fois d'un rapport 2.

Le résultat après n itération, présenté sur la figure 1, approche la fonction φ avec un pas d'échantillonnage de 2^{-n}.

Cependant la Proposition 1 ne nous fournit rien de plus que la convergence dans $L^2(\mathbb{R})$ de la suite h_n qui peut conduire à des résultats très chahutés comme le montre la figure 2. Le résultat suivant va nous donner des précisions sur cette convergence.

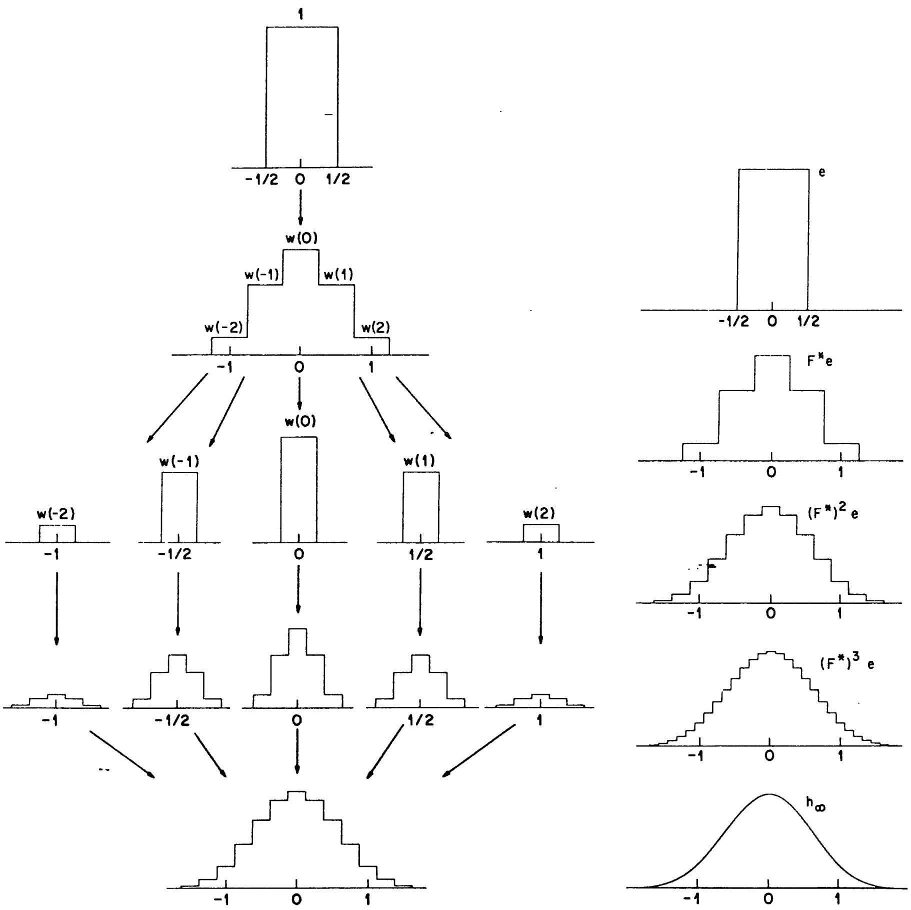

Fig. 1. L'Algorithme en cascade.

Proposition 2. *Supposons que les hypothèses de la Proposition* 1 *soient toutes remplies et que de plus, pour un entier r positif on ait*

$$\int_{\mathbb{R}} |\omega|^r |\hat{\varphi}(\omega)| \, d\omega < +\infty. \tag{16}$$

Alors $\omega^r \hat{h}_n(\omega)$ *tend vers* $\omega^r \hat{\varphi}(\omega)$ *dans* $L^1(\mathbb{R})$ *et la suite* h_n *tend donc vers* φ *au sens de la convergence uniforme pour toutes les dérivées jusqu'à l'ordre r.*

De même, l'approximation discrète $h_n(2^{-n}k)$ *tend uniformément vers* φ *pour tous les schémas de dérivation jusqu'à l'ordre r.*

Pour démontrer ce résultat on remarque simplement que l'on peut multiplier par $|\omega|^r$ les deux membres de l'inégalité (14) et appliquer de façon similaire un argument de convergence dominée.

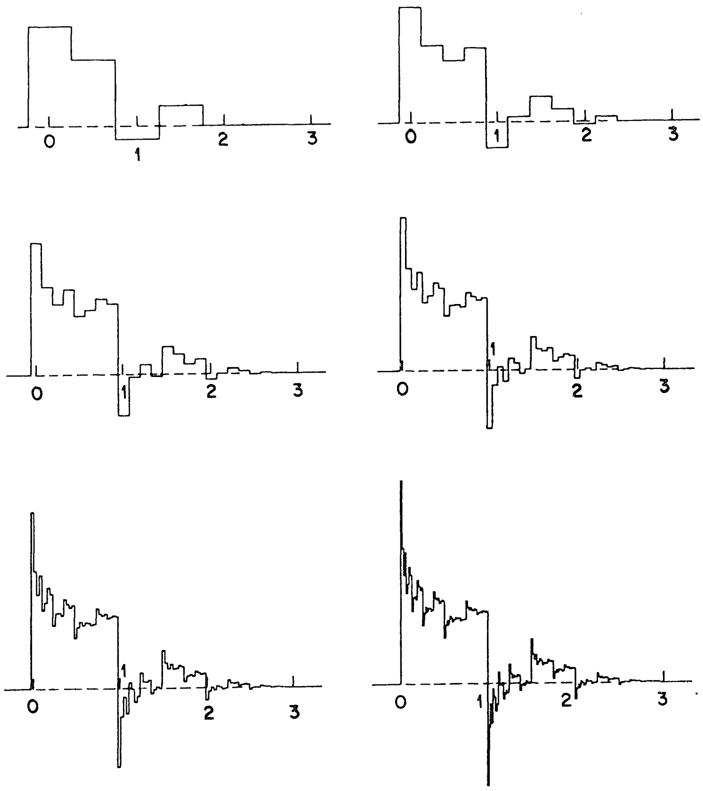

Fig. 2. Un cas où la convergence est mauvaise. †

Le fait que les approximations discrètes obtenues dans l'algorithme en cascade convergent uniformément ainsi que leurs schémas de dérivation provient simplement d'une généralisation du théorème des accroissements finis pour des dérivées d'ordre supérieur à 1.

Nous allons donc nous intéresser de plus près à la propriété (16), en l'étendant au cas où r n'est pas un entier.

2.c. Les analyses α-régulières

Dans toute la suite, α désigne un réel positif. Nous pouvons alors définir les analyses multirésolutions α-régulières et l'espace des fonctions de régularité C^α par

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† Ces figures sont extraites de [7].

$$\varphi \in C^\alpha \Leftrightarrow \int_{\mathbb{R}} |x|^\alpha |\hat{\varphi}| < +\infty.$$

Notons que, si l'on désigne par G^α l'espace classique des fonctions höldériennes d'exposant α on a alors

$$C^\alpha \subset G^\alpha.$$

En revanche G^α n'est pas inclu dans C^α. Si l'on se restreint aux fonctions à support compact on a $G^\alpha \subset C^{\alpha-(1/2)-\epsilon}$ pour $\epsilon > 0$. Il faut donc distinguer ces deux types d'espace.

Signalons enfin que la cascade décrite précédemment s'identifie à l'algorithme de reconstruction utilisé dans les travaux de Stéphane Mallat ([12]) sur les décompositions multiéchelles.

L'algorithme en cascade opère en effet cette reconstruction à partir d'un seul coefficient non nul à basse échelle et dans plusieurs applications, on peut exiger que la composante obtenue à l'issue d'une telle opération ait une régularité du type C^α.

Il est donc souhaitable de pouvoir caractériser ces classes de régularité à partir de la forme du QMF m_0.

3. Régularité et filtres QMF

3.a. Une condition suffisante de régularité

Un critère permettant d'obtenir de la régularité sur le fonction φ a été introduir par I. Daubechies et P. Tchamitchian ([7]). Il repose essentiellement sur le degré d'annulation de la fonction m_0 en π.

On peut en effet factoriser m_0 sous la forme

$$m_0(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^N f(\omega). \tag{17}$$

La fonction f est dans $C^\infty(\mathbb{R})$. Pour $j > 0$, on définit alors

$$B_j = \sup_{\omega \in \mathbb{R}} \left| \prod_{k=1}^{j} f\left(\frac{\omega}{2^k}\right) \right|$$

et $B_0 = 1$.

Notons

$$b_j = \frac{\log B_j}{j \log 2}.$$

Proposition 3. *Avec les définitions précédentes, il existe une constante C telle que pour tout ω dans ℝ on ait*

$$|\hat{\varphi}(\omega)| = \left| \prod_{k=1}^{+\infty} m_0\left(\frac{\omega}{2^k}\right) \right| \leqslant C(1 + |\omega|)^{-N+b_j}. \tag{18}$$

Pour obtenir cette inégalité, il suffit de remarquer que l'on a

$$\left| \frac{1 + e^{i\omega}}{2} \right| = \left| \cos\left(\frac{\omega}{2}\right) \right|.$$

On se sert alors de l'identité classique

$$\prod_{k=1}^{+\infty} \cos\left(\frac{\omega}{2^k}\right) = \frac{\sin \omega}{\omega}.$$

On observe ensuite que pour majorer le produit $\left| \prod_{k=1}^{+\infty} f\left(\frac{\omega}{2^k}\right) \right|$, il suffit de se restreindre aux premiers facteurs dans un nombre de l'ordre de $\log(\omega)/\log 2$. La majoration globale s'en déduit.

D'après cette estimation, il est clair que φ est dans C^α dès que α est strictement inférieur à $N - b_j - 1$.

Notons que N est aussi, d'après (12), le degré du premier terme dans le développement limité de $\hat{\psi}$ en 0. On retrouve donc ici des liens entre la régularité et l'annulation des premiers moments de l'ondelette.

Ce critère ne fournit cependant qu'une condition suffisante pour avoir de la régularité que l'on aimerait cerner de plus près. Nous allons tout d'abord affiner l'estimation (18) en exploitant le fait que cette inégalité vaut quel que soit l'entier j choisi.

3.b. L'exposant critique du QMF

On appellera exposant critique du QMF m_0, la quantité

$$b = \inf_{j>0} b_j.$$

Puisque $f(0) = 1$, il est clair que b est positif.

Le résultat suivant nous sera utile par la suite.

Lemme 1. *La suite b_j converge et admet b pour limite.*

DÉMONSTRATION. Soit $\epsilon > 0$. Il existe un entier j_ϵ tel que $|b_{j_\epsilon} - b| < \epsilon/2$ ou encore $b_{j_\epsilon} < b + \epsilon/2$. Posons alors pour j dans ℕ, la division euclidienne $j = n_j \cdot j_\epsilon + r_j$ avec $r_j < j_\epsilon$.

On a facilement

$$B_j \leqslant (B_{j_\epsilon})^{n_j}(B_1)^{r_j}$$

d'où

$$B_j \leqslant (B_{j_\epsilon})^{n_j}(B_1)^{j_\epsilon}$$

en passant aux logarithmes, il vient

$$b_j \leqslant b_{j_\epsilon} + \frac{b_1}{n_j}.$$

Si j est suffisament grand, on a alors $b_1/n_j < \epsilon/2$ et par conséquent $b_j < b + \epsilon$. Ceci montre que b est la limite des b_j.

Par ailleurs, nous pouvons reformuler le résultat de P. Tchamitchian et I. Daubechies, à l'aide de la quantité b. Il est clair, en effet, que la fonction φ est dans l'espace C^α pour tout α strictement inférieur à $N - b - 1$ (les ϵ différenciant b des b_j sont compris dans l'inégalité stricte).

Nous allons voir que cette nouvelle estimation est en fait optimale au sens suivant: si α est strictement supérieur à $N - b$ alors φ n'est pas dans C^α, ceci moyennant une légère hypothèse supplémentaire sur la fonction m_0.

4. Estimation de l'exposant de Hölder global

Enonçons tout de suite le résultat qui va être démontré.

Théorème. *Soit*

$$m_0(\omega) = \left(\frac{1 + e^{i\omega}}{2}\right)^N f(\omega)$$

un QMF satisfaisant les hypothèses de la Proposition 1 *et soit b son exposant critique. On suppose de plus que l'on a*

$$|f(\pi)| > |f(0)| = 1. \tag{19}$$

Alors la fonction φ *engendrée par le produit* (11) *n'est pas* C^α *si* α *est strictement supérieur à* $N - b$.

L'exposant optimal pour la régularité C^α *se situe donc dans l'intervalle* $[N - b - 1; N - b]$.

Avant de donner la preuve de cet énoncé, commentons l'hypothèse (19) faite sur la fonction m_0. Le lecteur se rendra compte qu'elle n'est pas strictement nécessaire dans ce qui va suivre. On peut en effet la remplacer par des hypothèses plus faibles ou plus générales. Cependant, (19) se trouve vérifiée dans tous les cas fréquemment utilisés et entre autre, celui des ondelettes à support compact introduites par Ingrid Daubechies ([7]). Dans ce cas, la fonction m_0 est un polynôme trigonométrique et l'on a

$$|f(\pi)|^2 = \sum_{j=0}^{N-1} \binom{n-1+j}{j} > 1.$$

Notons à présent

$$f_j(\omega) = \prod_{k=1}^{j} f\left(\frac{\omega}{2^j}\right)$$

et K le compact introduit dans les hypothèses de la Proposition 1.

La démonstration du théorème s'appuie sur deux lemmes concernant les fonctions $f_j(\omega)$.

Lemme 2. *Il existe une suite de réels ω_j tels que $2^{-j}\omega_j$ appartienne au compact K et $|f_j(\omega_j)| = B_j$. On a de plus*

$$0 < C_1 < |2^{-j}\omega_j| < \alpha_2 . \tag{20}$$

DÉMONSTRATION. Il est clair que f_j est $2^{j+1}\pi$-périodique. Puisque K est congru à $[-\pi; \pi]$ modulo 2π, B_j est donc aussi le maximum de $|f_j(\omega)|$ sur l'ensemble 2^jK.

Soit alors ω_j dans 2^jK tel que $|f_j(\omega_j)| = B_j$. Nous allons nous servir de la propriété (19) pour démontrer (20). On remarque que sous l'hypothèse (19) on a $|f(\omega + \pi)| > |f(\omega)|$ sur un voisinage de l'origine du type $|\omega| \leqslant C_1$.

Nous pouvons aussi l'écrire sous la forme

$$\left|f\left(\frac{\omega + 2^j\pi}{2^j}\right)\right| > \left|f\left(\frac{\omega}{2^j}\right)\right| \quad \text{lorsque} \quad \left|\frac{\omega}{2^j}\right| \leqslant C_1 .$$

On a par ailleurs

$$\left|f\left(\frac{\omega + 2^j\pi}{2^{j'}}\right)\right| = \left|f\left(\frac{\omega}{2^{j'}}\right)\right| \quad \text{pour} \quad j' < j.$$

Par conséquent

$$|f_j(\omega + 2^j\pi)| > |f_j(\omega)| \quad \text{si} \quad |2^j\omega| \leqslant C_1 .$$

Ceci nous montre que le point ω_j se trouve hors de l'intervalle $[-2^jC_1 ; 2^jC_1]$.

Finalement, puisque ω_j est dans $2^{-j}K$ et que K est un compact, on peut écrire

$$0 < C_1 < |2^{-j}\omega_j| < C_2. \tag{20}$$

L'hypothèse (19) nous fournit donc une information sur la croissance de la suite ω_j.

Lemme 3. *Soit ϵ dans $]0,1[$. On note $B = 2^b$. Il existe une constante C telle que $|f_j(\omega)|$ reste supérieure à $B^j/2$ sur un intervalle I_j inclus dans 2^jK, contenant ω_j et de taille $C(1-\epsilon)^j$.*

DÉMONSTRATION. Il est clair, tour d'abord que $|f_j(\omega_j)| = B_j \geqslant B^j$. Examinons à présent la dérivée de la fonction $f_j(\omega)$.

$$f_j'(\omega) = \sum_{l=1}^{j} 2^{-l} f'\left(\frac{\omega}{2^l}\right) \prod_{\substack{k=1 \\ k \neq l}}^{j} f\left(\frac{\omega}{2^k}\right). \tag{21}$$

Par conséquent,

$$\max_{\omega \in \mathbb{R}} |f_j'(\omega)| \leqslant C \sum_{l=0}^{j-1} 2^{-l} B_l B_{j-l-1}. \tag{22}$$

En se servant du Lemme 1, on a, pout tout $l \geqslant l_\epsilon$,

$$B_l \leqslant (B+\epsilon)^l. \tag{23}$$

À un changement de constante près dans (22), on peut alors écrire,

$$\max_{\omega \in \mathbb{R}} |f_j'(\omega)| \leqslant C(B+\epsilon)^j \tag{24}$$

Il en découle que $|f_j(\omega)|$ reste supérieur à $B^j/2$ sur un intervalle I_j, de taille $C(1-\epsilon)^j$ qui contient ω_j. Le compact K étant formé d'une réunion finie d'intervalles, il nous est possible, quitte à modifier encore une fois la constante C, de choisir I_j inclus dans 2^jK.

Notons que ces intervalles I_j peuvent se recouvrir, mais «pas trop» en ce sens: si nous considérons uniquement les intervalles I_{aj} où a est tel que $2^a C_1 > 2C_2$, il est clair que, pour J suffisamment grand, ces intervalles sont disjoints.

Pour prouver le théorème, nous allons étudier l'intégrale

$$\int_{\cup I_{aj}} |\omega|^\alpha |\hat{\varphi}(\omega)|\, d\omega,$$

ce qui revient, d'après la remarque précédente à préciser la nature de la série

(25) $$\sum_{j>0} \int_{I_{aj}} |\omega|^{\alpha} |\hat{\varphi}(\omega)| \, d\omega.$$

Pour cela, on décompose $\hat{\varphi}(\omega)$ de la manière suivante,

(26) $$|\hat{\varphi}(\omega)| = |f_j(\omega)| \left| \prod_{k=1}^{j} \left(\frac{1 + e^{i\omega/2^k}}{2} \right) \right|^N \left| \prod_{k=j+1}^{+\infty} m_0\left(\frac{\omega}{2^k} \right) \right| .$$

Nous allons traiter indépendamment ces trois facteurs.

- On a vu auparavant que sur I_j on a l'inégalité

$$|f_j(\omega)| \geqslant \frac{B^j}{2} .$$

- Puisque I_j est dans $2^j K$, en utilisant les hypothèses de la Proposition 1, on peut écrire pour tout ω de I_j.

$$\left| \prod_{k=j+1}^{+\infty} m_0\left(\frac{\omega}{2^k} \right) \right| \geqslant C > 0 \quad \text{indépendamment de } j.$$

- On a enfin

$$\left| \prod_{k=1}^{j} \left(\frac{1 + e^{i\omega/2^k}}{2} \right)^N \right| \geqslant \left| \frac{\sin \omega}{\omega} \right|^N .$$

En utilisant le Lemme 2 pour évaluer la distance de I_{aj} à l'origine, il vient,

(27) $$\int_{I_{aj}} |\omega|^{\alpha} |\hat{\varphi}(\omega)| \, d\omega \geqslant C B^{aj} 2^{\alpha aj} \int_{I_{aj}} \left| \frac{\sin \omega}{\omega} \right|^N d\omega.$$

L'intégrale contenue dans le membre de droite peut elle-même être minorée par

(28) $$\int_{I_{aj}} \left| \frac{\sin \omega}{\omega} \right|^N \geqslant C 2^{-aNj} (1 - \epsilon)^{aNj}.$$

Ceci nous conduit finalement à l'inégalité

(29) $$\int_{I_{aj}} |\omega|^{\alpha} |\hat{\varphi}(\omega)| \, d\omega \geqslant C 2^{aj(b + \alpha - N - N\epsilon)}.$$

(A chaque étape, on a modifié la constante C, ce qui n'a aucune importance.)

La série (25) explose donc lorsque α est supérieur à $N - b + N\epsilon$. ϵ étant arbitrairement petit, on en déduit que l'intégrale $\int_{\mathbb{R}} |\omega|^{\alpha} |\hat{\varphi}(\omega)|$ diverge dès que α est strictement supérieur à $N - b$. Ceci achève notre démonstration.

5. Applications

5.a. Estimation de l'exposant critique

Les propriétés de régularité dépendent donc essentiellement du degré N d'annulation des moments et de la quantité b. Il est par conséquent souhaitable de pouvoir estimer b par des techniques de calcul raisonnables.

Par définition, on a toujours $b \leqslant b_j$ pour tout j dans $\mathbb{N}^*$ et ceci nous permet de majorer l'exposant critique.

Pour le minorer, on peut faire la remarque suivante: $f_j(2^{j+1}\pi/3)$ est en fait un produit alterné des valeurs de f en $2\pi/3$ et en $-2\pi/3$. On a, par conséquent,

$$\text{(30)} \qquad \lim_{j \to +\infty} \frac{\log \left| f^j\left(\frac{2^j\pi}{3}\right)\right|}{j \log 2} = \frac{1}{2 \log 2} \log\left(\left| f\left(\frac{2\pi}{3}\right) f\left(\frac{-2\pi}{3}\right)\right|\right).$$

De cette limite, on déduit la minoration

$$\text{(31)} \qquad b \geqslant \frac{1}{\log 2} \log\left(\sqrt{\left| f\left(\frac{2\pi}{3}\right) f\left(-\frac{2\pi}{3}\right)\right|}\right).$$

Des estimations similaires peuvent être établies de manière générale, à l'aide des valeurs $f_j\left(\dfrac{2^{j+1}\pi}{2k+1}\right)$ pour k dans $\mathbb{N}^*$.

5.b. Cas des ondelettes à support compact

Rappelons dans ce cas la forme des filtres QMF à réponse impulsionnelle finie. On peut considérer pour cela la variable $y = \cos^2(\omega/2)$ et on a alors une famille de QMF m_0^N définis par

$$\text{(32)} \qquad |m_0^N(\omega)|^2 = y^N P_N(1-y)$$

avec

$$\text{(33)} \qquad P_N(y) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j.$$

Dans [8], Ingrid Daubechies utilise une technique différente pour estimer plus précisément les exposants de Hölder globaux et locaux de la fonction φ: on travaille dans ce cas directement sur φ (et non $\hat{\varphi}$) à l'aide des coefficients c_k de m_0 et de l'équation (8).

Cette méthode est plus précise (et de plus, locale) que celle que nous avons décrite (qui place l'exposant α dans un intervalle de taille 1). Elle devient cependant complexe lorsque N prend des valeurs élevées.

Nos estimations deviennent au contraire intéressantes lorsque N est grand et que l'on cherche un résultat de type asymptotique où un écart de 1 n'a pas d'importance.

Notons b_j^N, b^N et $f_N(\omega)$ les quantités b_j, b et le facteur $f(\omega)$ associés à la fonction m_0^N.

On sait d'après (32) que

$$\left|f_N\left(\frac{2\pi}{3}\right)\right| = \left|f_N\left(-\frac{2\pi}{3}\right)\right| = \left(P_N\left(\frac{3}{4}\right)\right)^{1/2}$$

et on a $P_N(3/4) \sim C3^N$ lorsque N tend vers $+\infty$. On a alors

$$\limsup_{N\to+\infty} \frac{1}{N}(N - b^N) \leqslant 1 - \frac{\log 3}{2\log 2} \approx 0{,}2075.$$

En utilisant b_4^N, Ingrid Daubechies obtient par ailleurs

$$\liminf_{N\to+\infty} \frac{1}{N}(N - b^N) > 0{,}1936.$$

L'exposant α est donc compris asymptotiquement entre $0{,}1936N$ et $0{,}2075N$ (figure 3). Ceci nous prouve notamment que la régularité ne s'identifie pas au nombre de premiers moments nuls de l'ondelette. Pour une régularité de classe C^r, il faut environ $N \approx 5r$ moments nuls. Notons, de plus, que le filtre m_0 est de taille $2N$ et aurait donc à peu près $10r$ coefficients.

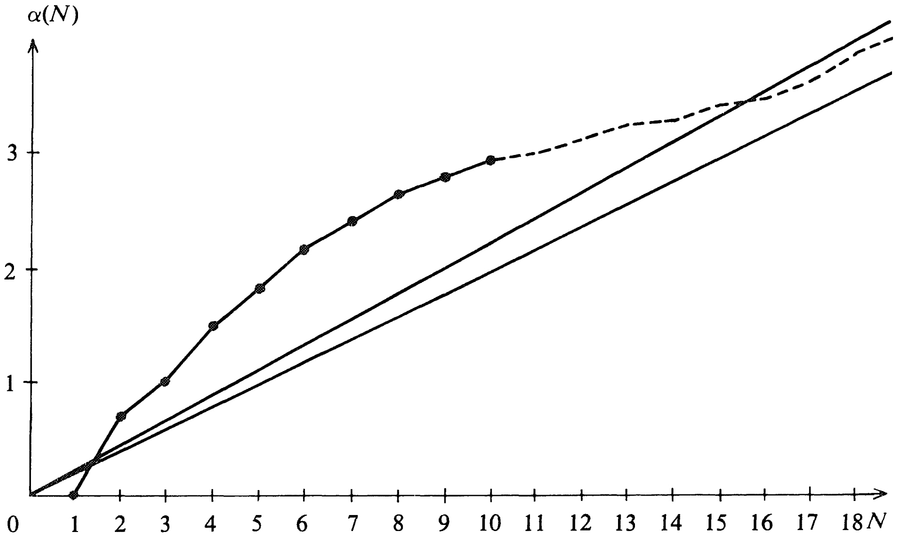

Fig. 3. Allure conjecturale de la courbe $\alpha(N)$ (les 10 premiers points sont estimés par Ingrid Daubechies dans [7] et [8]).

Il convient de tenir compte de cette estimation pour faire un compromis entre la régularité que l'on recherche et la taille des calculs mis en jeu par l'emploi des QMF.

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Recibido: 28 de marzo de 1990.

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Revista Matemática Iberoamericana
Vol. 6, N.^{os} 3 y 4, 1990

Sur les Variétés Riemanniennes a Flot Géodésique Topologiquement Transitif

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Introduction

Si M est une variété riemannienne C^∞ complète, nous pouvons considérer son flot géodésique $\{\varphi_t\}$ sur le fibré unitaire tangent T_1M. Soit X le champ engendré par $\{\varphi_t\}$.

D'autre part, si θ est un r-tenseur covariant sur M, on définit la fonction $F(\theta)\colon T_1M \to \mathbb{R}$ par $F(\theta)(v) = \theta(v, \dots, v)$. On peut donc calculer $XF(\theta)$ et exploiter les implications sur le tenseur du fait $XF(\theta) = 0$ lorsque le flot géodésique est topologiquement transitif. Remarquons que si $r = 1$ ou bien si $r = 2$ et θ est symétrique, la fonction $F(\theta)$ permet de «reconstruire» le tenseur.

Nous utilisons cette technique pour prouver les résultats suivants:

Soit (M, g) une variété riemannienne complète à flot géodésique topologiquement transitif. Alors

(*a*) Le groupe de Lie des isométries de (M, g) est discret.

(*b*) Si $f\colon (M, g) \to (M', g')$ est une correspondance géodésique, c'est-à-dire que f est un difféomorphisme qui envoie les géodésiques (non paramétrées) de (M, g) sur celles de (M', g'), alors f est une homothétie.

(*c*) Soit ρ le tenseur de Ricci de (M, g). Si $\nabla_v\rho(v, v) = 0$, pour tout champ de vecteurs v sur M, alors $\rho = cg$ où c est une constante.

Dans les dernières sections nous exploitons la même idée sur $O(M)$ (le fibré des repères orthonormés de M) et $\Omega_p(M)$ (le fibré des p-repères orthonormés) à la place de T_1M: les tenseurs sur M induisent des fonctions sur ces fibrés et nous étudions l'action des flots géodésiques généralisés (sur $O(M)$) et celle du flot de p-repères sur ces fonctions.

Rappelons que la condition la plus générale sur la variété riemannienne complète (M, g) impliquant la transitivité topologique de son flot géodésique sur T_1M est due à Eberlein (voir [2], Théorème 3.7). Enonçons ce résultat dans le cas compact:

(1) Si (M, g) est compacte et vérifie l'axiome de visibilité uniforme alors son flot géodésique est topologiquement transitif.

Il est intéressant de signaler aussi (voir [2], Théorème 5.1) que:

(2) Si (M, g) vérifie l'axiome de visibilité uniforme, alors toute métrique sur M sans points conjugués vérifie aussi cet axiome.

Si M est une variété à courbure sectionnelle non positive, des propriétés supplémentaires très faibles impliquent que M vérifie l'axiome de visibilité uniforme (voir [2], Théorème 4.1 et [3], Section 5). Il est en particulier ainsi si M est compacte à courbure sectionnelle négative.

D'autre part, Klingenberg ([8], remarque de la page 11) a montré que si M est compacte et son flot géodésique est d'Anosov, alors M vérifie l'axiome de visibilité uniforme; dans ce cas, la transitivité topologique du flot géodésique était bien connue mais (2) montre que cette propriété est héritée pour toute métrique sur M sans points conjugués.

Remarquons, enfin, que Gulliver [7] a construit des exemples de variétés (de dimension arbitraire) ayant des morceaux à courbure sectionnelle positive et dont le flot géodésique est d'Anosov.

1. Flot géodésique et 1-formes

Soit (M, g) une variété riemannienne complète, désignons par $\{\varphi_t\}$ son flot géodésique sur le fibré unitaire tangent T_1M. Soit X le champ engendré par $\{\varphi_t\}$.

Si φ est une 1-forme sur M on définit la fonction $F(\varphi)\colon T_1M \to \mathbb{R}$ par $F(\theta)(v) = \varphi(v)$. De même, si θ est un 2-tenseur covariant sur M, la fonction $F(\theta)\colon T_1M \to \mathbb{R}$ est donnée par $F(\theta)(v) = \theta(v, v)$.

Si Φ est un champ vectoriel sur M, $L_\Phi g$ désignera la dérivée de Lie de la métrique g par rapport à Φ; c'est-à-dire que $L_\Phi g$ est le 2-tenseur covariant symétrique

$$L_\Phi g(v, w) = g(\nabla_v \Phi, w) + g(\nabla_w \Phi, v).$$

Proposition 1. *Soit Φ un champ de vecteurs sur M et Φ^* sa 1-forme duale $\Phi^*(v) = g(\Phi, v)$. Alors*

$$XF(\Phi^*)(v) = \frac{1}{2} F(L_\Phi g)(v), \quad \textit{pour tout} \quad v \in T_1M.$$

Preuve. Désignons par π: $T_1M \to M$ la projection canonique. Remarquons que le champ $t \to \varphi_t v$ est parallèle sur la courbe $t \to \pi(\varphi_t v)$ car $\{\varphi_t\}$ est le flot géodésique; en conséquence:

$$X(F(\Phi^*))(v) = \frac{d}{dt}\bigg|_{t=0} F(\Phi^*)(\varphi_t v) = \frac{d}{dt}\bigg|_{t=0} g(\Phi(\pi\varphi_t v), \varphi_t v) = g(\nabla_v \Phi, v).$$

Soit $i(M)$ l'algèbre de Lie des champs de Killing (ou isométries infinitésimales) sur M et notons par $I(M)$ le groupe de Lie des isométries de M.

Rappelons (voir [9], Section 3 du Chapitre VI) que $\Phi \in i(M)$ si $L_\Phi g = 0$ (c'est-à-dire, si $F(\Phi^*)$ est invariante par le flot géodésique $\{\varphi_t\}$). D'autre part, comme la variété M est complète, les champs de $i(M)$ engendrent des groupes à un paramètre de difféomorphismes (isométries) de M. Il y a donc un isomorphisme naturel entre $i(M)$ et l'algèbre de Lie de $I(M)$.

Enfin, si le flot géodésique est topologiquement transitif et $\Phi \in i(M)$, alors $F(\Phi^*)$ est constante, donc $F(\Phi^*) = 0$ et $\Phi = 0$. En résumé:

Théorème 1. *Soit M une variété riemannienne complète à flot géodésique topologiquement transitif. Alors le groupe de Lie des isométries de M, $I(M)$, est discret (c'est-à-dire que son algèbre de Lie est nulle). En particulier, si M est compacte, $I(M)$ est fini.*

2. Flot géodésique et 2-tenseurs covariants

Comme dans la section précédente, si θ est un 2-tenseur covariant, on définit $F(\theta)$: $T_1M \to \mathbb{R}$ par $F(\theta)(v) = \theta(v, v)$.

Proposition 2. *Soit M une variété riemannienne complète; désignons par $\{\varphi_t\}$ le flot géodésique sur T_1M et par X le champ géodésique.*

Si θ est un 2-tenseur covariant, alors

$$XF(\theta)(v) = \nabla_v \theta(v, v), \quad \textit{pour tout} \quad v \in T_1M.$$

Preuve. Soit $v \in T_1M$ fixe et choisissons un repère $\{E_1, \ldots, E_n\}$ de $T_{\pi(v)}M$ où $E_1 = v$. Désignons par $E_i(t)$ le transporté par parallélisme de E_i sur la géodésique $s \to \pi(\varphi_s v)$ jusqu'à $\pi(\varphi_t v)$; en particulier $E_1(t) = \varphi_t v$.

Soit $\{E^1(t), \dots, E^n(t)\}$ le repère dual de $\{E_1(t), \dots, E_n(t)\}$ (c'est-à-dire que $E^i(t)(E_j(t)) = \delta^i_j$).

Écrivons le tenseur θ en $\pi(\varphi_t v)$ par

$$\theta(\pi\varphi_t v) = \theta_{ij}(t)E^i(t) \otimes E^j(t). \tag{1}$$

Alors le 2-tenseur covariant $\nabla_{E_1(t)}\theta$ s'écrira

$$(\nabla_{E_1(t)}\theta)(\pi\varphi_t v) = \left(\frac{d}{dt}\theta_{ij}(t)\right)E^i(t) \otimes E^j(t)$$
$$+ \theta_{ij}(t)(\nabla_{E_1(t)}E^i(t)) \otimes E^j(t)$$
$$+ \theta_{ij}(t)E^i(t) \otimes (\nabla_{E_1(t)}E^j(t)).$$

Il est aisé de montrer que $\nabla_{E_1(t)}E^\alpha(t) = 0$, pour tout $\alpha = 1, \dots, n$, en conséquence

$$\nabla_{E_1(t)}\theta = \left(\frac{d}{dt}\theta_{ij}(t)\right)E^i(t) \otimes E^j(t),$$

il en résulte en particulier que

$$[\nabla_{E_1(t)}\theta](E_1(t), E_1(t)) = \frac{d}{dt}\theta_{11}(t)$$

et la proposition en découle car, d'après (1):

$$\theta_{11}(t) = F(\theta)(E_1(t))$$

et

$$\frac{d}{dt}\theta_{11}(t) = [XF(\theta)](E_1(t)).$$

Corollaire 1. *Si le flot géodésique est topologiquement transitif et θ est un 2-tenseur covariant symétrique vérifiant $\nabla_v\theta(v, v) = 0$, pour tout $v \in TM$ alors, il existe une constante c telle que $\theta = cg$, où g est le tenseur métrique.*

PREUVE. D'après la proposition 2, $F(\theta)|_{T_1M} = c$.

2.1 Variétés dans la classe $\mathfrak{A}$

Soit M une variété riemannienne et ρ son tenseur de Ricci. On dit que M est dans la classe $\mathfrak{A}$ si $\nabla_v\rho(v, v) = 0$ pout tout $v \in TM$.

Gray [5] a montré que si M est compacte, à courbure sectionnelle négative et elle est dans la classe $\mathfrak{A}$ alors M est un espace d'Einstein, c'est-à-dire $\rho = cg$ où c est une constante et g est le tenseur métrique.

Le Corollarie 1 nous permet de généraliser ce résultat.

Théorème 2. *Si M est dans la classe $\mathfrak{A}$ et si son flot géodésique est tologiquement transitif, alors M est un espace d'Einstein.*

2.2. Correspondance géodésique

Soient M et M' des variétés différentielles de dimension $n > 1$ munies des connexions ∇ et ∇'. Un difféomorphisme $f: (M, \nabla) \to (M', \nabla')$ est dit application affine si $df(\nabla_Z Y) = \nabla'_{dfZ}\, df\, Y$, pour tous champs de vecteurs Z, Y sur M (voir [9], Section 1 du Chapitre VI). En particulier, f envoie les géodésiques paramétrées de (M, ∇) sur celles de (M', ∇'); c'est-à-dire, que $t \to f(\gamma(t))$ est géodésique en (M', ∇') si $t \to \gamma(t)$ l'est en (M, ∇).

Cette propriété caractérise les applications affines si les connexions ∇ et ∇' sont sans torsion (en effet, deux connexions ∇ et $\bar{\nabla}$ sur M ont les mêmes géodésiques si et seulement si $\Gamma^i_{jk} + \Gamma^i_{kj} = \bar{\Gamma}^i_{jk} + \bar{\Gamma}^i_{kj}$; voir [9], p. 146).

Considérons maintenant deux métriques g et g' sur les variétés M et M' respectivement, et soient ∇ et ∇' leurs connexions riemanniennes. D'après les remarques précédentes, les applications affines $f: (M, g) \to (M', g')$ sont les difféomorphismes qui préservent les géodésiques (paramétrées). On appelle correspondance géodésique entre (M, g) et (M', g') à tout difféomorphisme $f: M \to M'$ qui envoie les géodésiques (non paramétrées) de (M, g) sur celles de (M', g'); les applications affines sont donc des correspondances géodésiques.

Nous considérons le problème suivant: dans quelles conditions la correspondance géodésique $f: (M, g) \to (M', g')$ est-elle une homothétie? (c'est-à-dire, qu'il existe une constante c telle que $g'(dfv, dfw) = cg(v, w)$, pour tous $v, w \in TM$).

On peut donner la solution du problème pour les applications affines en termes du groupe d'holonomie: rappelons (voir [9], p. 71) que si $x \in (M, \nabla)$ le groupe d'holonomie en x, que sera désigné par $\psi(x)$, est le sous-groupe des transformations linéaires de $T_x M$ induites par le transport parallèle sur les courbes fermées en x. Si $\psi(x)$ ne laisse aucun sous-espace (non trivial) de $T_x M$ invariant, on dit que $\psi(x)$ est irreductible. Si M est connexe, tous les groupes $\psi(x)$, pour $x \in M$, sont isomorphes: on obtient ainsi le groupe d'holonomie de M, que sera désigné par ψ.

Proposition 3. *Soit (M, g) une variété riemannienne connexe et ψ son groupe d'holonomie. Alors: ψ est irréductible si et seulement si toute application affine $f: (M, g) \to (M', g')$ est une homothétie.*

PREUVE. Supposons que ψ soit irréductible. Prenons une application affine $f: (M, g) \to (M', g')$ et considérons la métrique $\bar{g}$ sur M définie par $\bar{g}(v, w) = g'(dfv, dfw)$. En particulier, si ∇ est la connexion de g, $\nabla\bar{g} = 0$ et $\bar{g}_x$ est invariant par $\psi(x)$. Or, ce groupe étant irréductible, on a $\bar{g}_x = c_x g_x$, où c_x est une constante (voir [9], p. 277); comme $\nabla\bar{g} = \nabla g = 0$, il en résulte que $c_x = c$ pour tout $x \in M$. Ainsi $\bar{g} = cg$ et f est une homothétie.

Supposons maintenant que ψ ne soit pas irréductible. Nous allons construire une métrique $\bar{g}$ sur M telle que $\bar{g} \neq cg$ et l'application identite Id: $(M, g) \to (M, \bar{g})$ soit une application affine.

Soit $x \in M$ fixé et T'_x un sous-espace non trivial de $T_x M$ invariant par $\psi(x)$. Etant donné $y \in M$, nous définissons T'_y comme le sous-espace de $T_y M$ obtenu par transport parallèle de T'_x sur une courbe joignant x et y (T'_y ne dépend pas de la courbe choisie). Soit T''_y le complément g-orthogonal de T'_y en $T_y M$. Les distributions $y \to T'_y$ et $y \to T''_y$ sont différentiables; en outre (voir [9], p. 180-183) elles sont involutives et vérifient aussi la propriété suivante:

Soit $y \in M$ et désignons par M'_y et M''_y les sous-variétés intégrales de T' et T'' par y. Alors ils existent: un voisinage ouvert V_y de y en M, des voisinages ouverts V'_y et V''_y de y en M'_y et M''_y respectivement, et un diféomorphisme $h: V_y \to V'_y \times V''_y$ vérifiant

(2) $$h: (V_y, g) \to (V'_y \times V''_y, g|_{V'_y} \times g|_{V''_y}) \quad \text{est une isométrie.}$$

Prenons deux constantes positives c' et c'' et définissons la métrique $\bar{g}$ sur M par les conditions

(3) $$\bar{g}|_{T'_y} = c' g|_{T'_y}, \qquad \bar{g}|_{T''_y} = c'' g|_{T''_y}$$

et les espaces T'_y et T''_y sont $\bar{g}$-orthogonaux, pour tout $y \in M$.

Il est aisé de montrer, utilisant (2) et (3), que

(4) $$h: (V_y, \bar{g}) \to (V'_y \times V''_y, \bar{g}|_{V'_y} \times \bar{g}|_{V''_y}) \quad \text{est une isométrie.}$$

D'après (3), il en résulte que $(V'_y, g|_{V'_y})$ et $(V'_y, \bar{g}|_{V'_y})$ ont les mêmes géodésiques (parametrées); il en est de même pour $(V''_y, g|_{V''_y})$ et $(V_y'', \bar{g}|_{V''_y})$. En conséquence, les métriques $\bar{g}|_{V'_y} \times \bar{g}|_{V''_y}$ et $g|_{V'_y} \times g|_{V''_y}$ ont les mêmes géodésiques or, (2) et (4) impliquent ce résultat pour les métriques g et $\bar{g}$ sur V_y. En résumé, l'identité Id: $(M, g) \to (M, \bar{g})$ est une application affine, mais $\bar{g} \neq cg$ si l'on prend $c' \neq c''$.

Néanmoins, dans le cas des correspondances géodésiques, l'irreductibilité de ψ ne suffit pas pour garantir qu'elles soient des homothéties comme le montre l'exemple suivant:

Considérons les coordonnées canoniques (x, y) de $\mathbb{R}^2$ et deux fonctions $\varphi_1(x) > \varphi_2(y) > 0$. Définissons les métriques g et $\bar{g}$ par

$$ds^2 = (\varphi_1 - \varphi_2)(dx^2 + dy^2)$$

$$d\bar{s}^2 = \frac{\varphi_1 - \varphi_2}{\varphi_1(\varphi_1 \varphi_2)} dx^2 + \frac{\varphi_1 - \varphi_2}{\varphi_2(\varphi_1 \varphi_2)} dy^2$$

Il en résulte que $\bar{g} \neq cg$ et on peut vérifier à l'aide de l'égalité (5) ci-dessous que l'identité Id: $(\mathbb{R}^2, g) \to (\mathbb{R}^2, \bar{g})$ est une correspondance géodésique (remarquons que l'exemple est naturel au vue des résultats de Levi-Civita; voir [10], p. 287 et [4], Section 41). La courbure de g est

$$k(x, y) = (\varphi_2 - \varphi_1)\left(\frac{\partial^2}{\partial x\, \partial x} \log \sqrt{\varphi_1 - \varphi_2} + \frac{\partial^2}{\partial y\, \partial y} \log \sqrt{\varphi_1 - \varphi_2} \right).$$

Alors, on peut choisir φ_1 et φ_2 de façon à avoir $k \neq 0$ ce qui implique que le groupe d'holonomie de $(\mathbb{R}^2, g)$ est irreductible.

Observons que si l'on prend $\varphi_i(x + n) = \varphi_i(x)$, pour tout $n \in \mathbb{Z}$ $(i = 1, 2)$, on a les mêmes résultats sur le tore T^2.

Théorème 3. *Soit (M, g) une variété riemannienne de dimension $n > 1$, complète, à flot géodésique topologiquement transitif.*

Alors, toute correspondance géodésique f: $(M, g) \to (M', g')$ est une homothétie.

PREUVE. Considérons la métrique $\bar{g}$, sur M, donnée par

$$\bar{g}(v, w) = g'(dfv, dfw).$$

Nous allons montrer que $\bar{g} = cg$, c'est-à-dire que f est une homothétie.

Prenons des coordonnées $(x^1, \ldots, x^n)$ sur un ouvert de M et soient g_{ij} et Γ^k_{ij} les composantes de la métrique g et de sa connexion; désignons par $\det g = \det (g_{ij})$. Pour $\bar{g}$ nous utilisons des notations analogues $\bar{g}_{ij}, \bar{\Gamma}^k_{ij}$ et $\det \bar{g} = \det (\bar{g}_{ij})$.

D'après l'hypothèse, g et $\bar{g}$ ont les mêmes géodésiques (non paramétrées) ce qui équivaut (voir [4], Section 40) à:

$$\bar{\Gamma}^k_{ij} - \Gamma^k_{ij} = \delta^k_j \frac{\partial}{\partial x^i} \psi + \delta^k_i \frac{\partial}{\partial x^j} \psi, \tag{5}$$

où

$$\psi = \frac{1}{2(n + 1)} \log \frac{\det \bar{g}}{\det g}.$$

En conséquence, si $\bar{\nabla}$ est la connexion de $\bar{g}$, on a:

$$\begin{aligned} 0 &= \bar{\nabla}_k \bar{g}_{ij} \\ &= \frac{\partial}{\partial x^k}\bar{g}_{ij} - \bar{\Gamma}^{\alpha}_{ki}\bar{g}_{\alpha j} - \bar{\Gamma}^{\beta}_{kj}\bar{g}_{i\beta} \\ &= \frac{\partial}{\partial x^k}\bar{g}_{ij} - \left(\Gamma^{\alpha}_{ki} + \delta^k_{\alpha}\frac{\partial}{\partial x^i}\psi + \delta^{\alpha}_i\frac{\partial}{\partial x^k}\psi\right)\bar{g}_{\alpha j} - \left(\Gamma^{\beta}_{kj} + \delta^{\beta}_j\frac{\partial}{\partial x^k}\psi + \delta^{\beta}_k\frac{\partial}{\partial x^j}\psi\right)\bar{g}_{i\beta} \end{aligned}$$

C'est-à-dire, que

$$\nabla_k \bar{g}_{ij} = 2\bar{g}_{ij}\frac{\partial}{\partial x^k}\psi + \bar{g}_{ik}\frac{\partial}{\partial x^j}\psi + \bar{g}_{jk}\frac{\partial}{\partial x^i}\psi. \tag{6}$$

Si l'on fait

$$\mu = \left(\frac{\det g}{\det \bar{g}}\right)^{1/(n+1)}$$

alors

$$\psi = -\frac{1}{2}\log \mu$$

et (6) s'écrit:

$$2\mu\nabla_k \bar{g}_{ij} = -2\bar{g}_{ij}\frac{\partial}{\partial x^k}\mu - \bar{g}_{ik}\frac{\partial}{\partial x^j}\mu - \bar{g}_{jk}\frac{\partial}{\partial x^i}\mu. \tag{7}$$

Comme μ ne dépend pas du système de coordonnées, nous pouvons définir le tenseur symmétrique

$$\theta_{ij} = \mu^2 \bar{g}_{ij}.$$

Il est aisé de montrer à l'aide de (7) que

$$\nabla_k \theta_{ij} + \nabla_i \theta_{jk} + \nabla_j \theta_{ki} = 0. \tag{8}$$

Or, θ étant symétrique, (8) revient à dire que $\nabla_v \theta(v, v) = 0$ pour tout $v \in TM$. En conséquence, d'après le Corollaire 1, il existe une constante λ telle que

$$\mu^2 \bar{g}_{ij} = \left(\frac{\det g}{\det g}\right)^{2/(n+1)} \bar{g}_{ij} = \lambda g_{ij}. \tag{9}$$

Alors $\mu^{2n} \det \bar{g} = \lambda^n \det g$ et on a

$$\mu^{n+1} = \frac{\det g}{\det \bar{g}} = \frac{\mu^{2n}}{\lambda^n},$$

or $n \neq 1$ implique μ est constante et (9) s'écrit $\bar{g}_{ij} = cg_{ij}$.

3. Les Flots Géodésiques Généralisés

Soit M une variété riemannienne complète de dimension n; désignons par $O(M)$ le fibré des repères orthonormés de M. Il s'agit d'une variété différentiable qui, en outre, est un fibré principal sur M dont la fibre est le groupe de Lie compact des matrices orthogonales $O(n)$ (voir [9], p. 60). L'application $\pi: O(M) \to M$ est la projection canonique.

Soit $C: (a, b) \to O(M)$ une courbe sur $O(M)$; désignons par c sa projection $\pi \circ C$. On peut écrire

$$C(t) = (E_1(t), \ldots, E_n(t)) \quad \text{où} \quad (E_1(t), \ldots, E_n(t)) \in \pi^{-1}(c(t)).$$

La courbe C est appelée courbe horizontale si $\nabla_{c'(t)} E_i(t) = 0$ pour tous $t \in (a, b)$, $i = 1, \ldots, n$. Etant donnée une courbe $c(t)$ sur M, et si l'on fixe $(E_1, \ldots, E_n) \in \pi^{-1}(c(0))$, il existe une unique courbe horizontale $C(t)$ telle que $C(0) = (E_1, \ldots, E_n)$ et $c(t) = \pi \circ C(t)$, pout tout t. On appelle $C(t)$ le relèvement horizontal de c par $(E_1, \ldots, E_n)$.

Soient $p \in M$ et $u \in \pi^{-1}(p)$ fixés. Un vecteur $W \in T_u(O(M))$ est horizontal (respectivement vertical) s'il est tangent à une courbe horizontale (respectivement si $d\pi W = 0$). Il en résulte que $T_u(O(M)) = \mathcal{V}(u) \otimes \mathfrak{H}(u)$, où $\mathcal{V}(u)$ et $\mathfrak{H}(h)$ sont les espaces des vecteurs verticaux et horizontaux de $T_u(O(M))$.

Définissons maintenant les flots géodésiques géneralisés sur $O(M)$: étant donné $(E_1, \ldots, E_n) \in \pi^{-1}(p)$, et si l'on fixe $i \in \{1, \ldots, n\}$, soit $\gamma_{E_i}(t)$ la géodésique sur M déterminée par les conditions $\gamma_{E_i}(0) = p$, $\gamma'_{E_i}(0) = E_i$. Soit $\varphi_t^i(E_1, \ldots, E_n)$ le repère en $\gamma_{E_i}(t)$ obtenu par transport parallèle de $(E_1, \ldots, E_n)$ sur γ_{E_i}. Le flot φ_t^i ainsi défini est appelé le i-ème flot géodésique généralisé. Le champ de φ_t^i est noté par X^i. Remarquons que $\mathfrak{H}(u)$ est engendré par $\{X^1(u), \ldots, X^n(u)\}$.

Désignons par $T_s^r(M)$ l'espace des tenseurs r-contravariants et s-covariants sur M. Nous pouvons associer à chaque tenseur $K \in T_s^r(M)$ une fonction $F(K): O(M) \to T_s^r(\mathbb{R}^n) \approx [\mathbb{R}^n]^{(s+r)}$ de la façon suivante:

Si $(E_1, \ldots, E_n) \in \pi^{-1}(p)$, soit $(E^1, \ldots, E^n)$ le repère dual $E^i(E_j) = \delta_j^i$. Alors $K(p) \in T_s^r(T_pM)$ s'écrira

$$K(p) = K_{j_1, \ldots, j_s}^{i_1, \ldots, i_r} E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes E^{j_1} \otimes \cdots \otimes E^{j_s}.$$

Nous définissons donc

$$F(K)(E_1, \ldots, E_n) = (K^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}) \in [\mathbb{R}^n]^{(s+r)} = T^r_s(\mathbb{R}^n).$$

Établissons maintenant les relations entre les flots X^i, les fonctions $F(K)$ et la dérivée covariante.

Proposition 4. *Soit $i \in \{1, \ldots, n\}$ fixé. Alors*

$$X^i(F(K))(E_1, \ldots, E_n) = F(\nabla_{E_i} K)(E_1, \ldots, E_n),$$

pour tous

$$(E_1, \ldots, E_n) \in O(M), \qquad K \in T^r_s(M).$$

En particulier, le tenseur K est parallèle (c'est-à-dire que $\nabla K = 0$) si et seulement si $F(K)$ est invariante par le i-ème flot géodésique généralisé.

PREUVE. Fixons $(E_1, \ldots, E_n)$ et K. Soit $(E_1(t), \ldots, E_n(t)) = \varphi^i_t(E_1, \ldots, E_n)$ et $(E^1(t), \ldots, E^n(t))$ son repère dual. Le tenseur K sur la géodésique $\gamma_{E_i}(t)$ s'écrit

$$K(\gamma_{E_i}(t)) = K^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(t) E_{i_1}(t) \otimes \cdots \otimes E_{i_r}(t) \otimes E^{j_1}(t) \otimes \cdots \otimes E^{j_s}(t).$$

En conséquence

$$F(K)(E_1(t), \ldots, E_n(t)) = (K^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(t))$$

et

$$X^i(F(K))(E_1(t), \ldots, E_n(t)) = \left(\frac{d}{dt} K^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(t)\right).$$

Étant donné que $\nabla_{E_i(t)} E_j(t) = 0$ pour tout $j = 1, \ldots, n$ il en résulte que $\nabla_{E_i(t)} E^j(t) = 0$. Alors on a

$$\nabla_{E_i(t)} K = \left[\frac{d}{dt} K^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(t)\right] E_{i_1}(t) \otimes \cdots \otimes E_{i_r}(t) \otimes E^{j_1}(t) \otimes \cdots \otimes E^{j_s}(t)$$

et la Proposition en découle.

Remarque. Comme les champs $X^1, \ldots, X^n$ engendrent les espaces horizontaux, nous pouvons écrire la proposition 4 de la façon suivante:

(10) Si $C(t)$ est une courbe horizontale et $c(t) = \pi \circ C(t)$, on a:

$$C'(t)(F(K)) = F(\nabla_{c'(t)} K).$$

Corollaire 2. *Soit $K \in T_s^r(M)$, désignons par $\nabla\nabla K \in T_{s+2}^r(M)$ le tenseur dérivée covariante seconde de K. Alors*

$$X^i X^i(F(K))(E_1, \ldots, E_n) = F(\nabla_{E_i}\nabla_{E_i}K)(E_1, \ldots, E_n).$$

PREUVE. Dans les notations ci-dessus, observons que le tenseur r-contravariant et s-covariant $\nabla_{E_i(t)}K$ est défini sur $\gamma_{E_i}(t)$; on peut considérer alors $\nabla_{E_i(t)}(\nabla_{E_i(t)}K)$.

D'après [9] (p. 125) on a

$$\nabla_{E_i(t)}\nabla_{E_i(t)}K = \nabla_{E_i(t)}(\nabla_{E_i(t)}K) - \nabla_{(\nabla_{E_i(t)}(E_i(t)))}K = \nabla_{E_i(t)}(\nabla_{E_i(t)}K), \tag{11}$$

car $\nabla_{E_i(t)}E_i(t) \equiv 0$.

La Proposition 4 implique

$$X^i(F(K))(E_1(t), \ldots, E_n(t)) = F(\nabla_{E_i(t)}K)(E_1(t), \ldots, E_n(t)).$$

En conséquence

$$\begin{aligned} X^i X^i(F(K))(E_1(t), \ldots, E_n(t)) &= X^i F(\nabla_{E_i(t)}K)(E_1(t), \ldots, E_n(t)) \\ &= F(\nabla_{E_i(t)}(\nabla_{E_i(t)}K))(E_1(t), \ldots, E_n(t)), \end{aligned}$$

où l'on a appliqué la Proposition 4 au tenseur $\nabla_{E_i(t)}K$. Le corollaire découle donc de (11).

La Proposition 4 permet aussi de retrouver le résultat de [11], p. 4:

Corollaire 3. *Si M est compacte, la nullité d'une dérivée covariante d'ordre quelconque d'un tenseur entraîne la nullité de la dérivée première de ce tenseur.*

PREUVE. D'après le corollaire antérieur, $\nabla\nabla K = 0$ équivant à $X^i X^i(F(K)) = 0$, ce qui s'écrit, sur l'orbite $(E_1(t), \ldots, E_n(t)) = \varphi_t^i(E_1, \ldots, E_n)$,

$$F(K)(E_1(t), \ldots, E_n(t)) = At + B \quad \text{où} \quad A, B \in [\mathbb{R}^n]^{(s+r)}.$$

Or, la compacité de M implique celle de $O(M)$, donc $A = 0$ et $F(K)$ est constante sur les trajectoires de X^i, la Proposition 4 implique alors que $\nabla K = 0$.

Un tenseur K est dit récurrent s'il existe une 1-forme α telle que $\nabla K = K \otimes \alpha$; c'est-à-dire que $\nabla_Z K = \alpha(Z)K$ cour tout champ Z sur M. Dans le cadre général des variétés munies de connexion lineáire, Wong [13] a caractérisé les tenseurs récurrents en termes des propriétés de la fonction $F(K)$ (voir aussi [9], p. 304). Nous nous intéressons aux relations entre les tenseurs récurrents et les tenseurs parallèles dans le cadre riemannien:

Proposition 5. *Soit K un tenseur récurrent sur la variété riemannienne M. Alors il existe un tenseur parallèle P et une fonction positive $\lambda: M \to \mathbb{R}$ vérifiant λP (réciproquement, tout tenseur de la forme λP est récurrent).*

PREUVE. Fixons une composante connexe de M, que nous désignerons aussi par M.

Si $K \equiv 0$, la proposition est triviale. Soit $p_0 \in M$ tel que $K(p_0) \neq 0$ et fixons $u_0 \in \pi^{-1}(p_0)$. Le fibré d'holonomie de u_0 est l'ensemble

$$O(M)(u_0) = \{u \in O(M)\text{: il existe } C\text{: } [0,1] \to O(M)\text{, horizontal, tel que } C(0) = u_0 \text{ et } C(1) = u\}.$$

Soit C une courbe horizontale avec $C(0) = u_0$ et désignons par c sa projection sur M, $\pi \circ C$. Comme K est récurrent, on a $\nabla_{c'(t)}K = \alpha(c'(t))K$. Alors $F(\nabla_{c'(t)}K) = \alpha(c'(t))F(K)$ et, d'après (10)

$$C'(t)[F(K)] = \alpha(c'(t))F(K).$$

En conséquence,

$$F(K)(C(t)) = F(K)(u_0)e^{\int_0^t \alpha(c'(s))\,ds} \quad \text{pour tout} \quad t. \tag{12}$$

Cela implique la propirété suivante

(13) Si K est un tenseur récurrent et il existe u_0 tel que

$$F(K)|_{O(M)(u_0)} = F(K)(u_0) \neq 0,$$

alors K est parallèle.

(En effet, il en résulte de (12) que $\alpha(c'(s)) = 0$, alors $\alpha \equiv 0$, car $c'(s)$ est arbitraire, et $\nabla K = K \otimes \alpha = 0$.)

La propriété (12) montre aussi qu'il existe une fonction positive $\varphi: O(M) \to \mathbb{R}$ telle que $F(K)(u) = F(K)(u_0)\varphi(u)$, pout tout $u \in O(M)(u_0)$.

Nous allons voir que φ est constante sur $O(M)(u_0) \cap \pi^{-1}(p)$, pour tout $p \in M$. En effet: soient u, $\bar{u} \in \pi^{-1}(p) \cap O(M)(u_0)$; si $u = (E_1, \ldots, E_n)$ et $\bar{u} = (\bar{E}_1, \ldots, \bar{E}_n)$ on a $\bar{E}_i = a_i^j E_j$ et $\bar{E}^i = b_j^i E^j$ où les matrices $A = (a_\alpha^\beta)$ et $B = (b_\alpha^\beta) = A^{-1}$ sont orthogonales, c'est-à-dire que

$$\sum_i a_i^\alpha a_i^\beta = \delta_{\alpha\beta} \quad \text{et} \quad \sum_i b_\alpha^i b_\beta^i = \delta_{\alpha\beta}. \tag{14}$$

Soient maintenant

$$F(K)(u) = (K_{j_1,\ldots,j_s}^{i_1,\ldots,i_r}) \quad \text{et} \quad F(K)(\bar{u}) = (\bar{K}_{j_1,\ldots,j_s}^{i_1,\ldots,i_r}).$$

Alors

$$\bar{K}^{i_1,\dots,i_r}_{j_1,\dots,j_s} = K^{l_1,\dots,l_r}_{m_1,\dots,m_s} b^{i_1}_{l_1}, \dots, b^{i_r}_{l_r} a^{m_1}_{j_1}, \dots, a^{m_s}_{j_s},$$

et (14) implique

$$(15) \qquad \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} (\bar{K}^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2 = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} (K^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2.$$

Étant donné que

$$F(K)(u) = F(K)(u_0)\varphi(u) \quad \text{et} \quad F(K)(\bar{u}) = F(K)(u_0)\varphi(\bar{u})$$

on a (pour tous $i_1, \dots, i_r, j_1, \dots, j_s$):

$$(K^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2 = \varphi(u)^2 (T^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2 \quad \text{et} \quad (\bar{K}^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2 = \varphi(\bar{u})^2 (T^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2,$$

où l'on a noté

$$F(K)(u_0) = (T^{i_1,\dots,i_r}_{j_1,\dots,j_s}).$$

En résumé, (15) s'écrit

$$\varphi(\bar{u})^2 \Big[\sum (T^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2 \Big] = \varphi(u)^2 \Big[\sum (T^{i_1,\dots,i_r}_{j_1,\dots,j_s})^2 \Big]$$

et comme $F(K)(u_0) \neq 0$ et φ est positive, on a $\varphi(\bar{u}) = \varphi(u)$.

Définissons alors la fonction $\lambda: M \to \mathbb{R}$ par $\lambda(p) = \varphi|_{O(M)(u_0) \cap \pi^{-1}(p)}$. Comme K est récurrent, il en est de même pour $(1/\lambda)K$ et, d'après la définition de λ, $F((1/\lambda)K)$ est constante ($\neq 0$) sur $O(M)(u_0)$. Alors la propriété (13) implique que $(1/\lambda)K$ est parallèle et la proposition est prouvée.

3.1. Implications de la transitivité topologique de X^i

Supposons maintenant que M est orientée et désignons par $O^+(M)$ le fibré des repères orthonormes positivement orientés de M; $O^+(M)$ est une fibre principal sur M dont la fibre est le groupe de Lie $SO(n) = \{a \in O(n): \det a = 1\}$.

Si $K \in T^r_s(M)$ nous avons, comme dans la section précédente, une fonction $F(K): O^+(M) \to T^r_s(\mathbb{R}^n)$.

Rappelons que, si $u = (E_1, \dots, E_n) \in O^+(M)$ et $a = (a^\beta_\alpha) \in SO(n)$, l'action à droite de $SO(n)$ sur $O^+(M)$ est définie par

$$\begin{array}{ccc} O^+(M) \times SO(n) & \longrightarrow & O^+(M) \\ (u, a) & \longrightarrow & ua \end{array}$$

où $ua = (\bar{E}_1, \dots, \bar{E}_n)$ et $\bar{E}_i = a^k_i E_k$.

De même, si $T = T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \times \cdots \times e_{i_r} \times e^{j_1} \times \cdots \times e^{j_s} \in T^r_s(\mathbb{R}^n)$ et $a = (a^\beta_\alpha) \in SO(n)$, l'action à gauche de $SO(n)$ sur $T^r_s(\mathbb{R}^n)$ est donnée par

$$\begin{array}{ccc} SO(n) \times T^r_s(\mathbb{R}^n) & \longrightarrow & T^r_s(\mathbb{R}^n) \\ (a, T) & \longrightarrow & aT \end{array}$$

où $aT = \bar{T}$ est le tenseur de composantes

$$\bar{T}^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = T^{l_1,\ldots,l_r}_{m_1,\ldots,m_s} a^{i_1}_{l_1}, \ldots, a^{i_r}_{l_r} b^{m_1}_{j_1}, \ldots, b^{m_s}_{j_s}$$

(ici $(b^\beta_\alpha) \equiv b = a^{-1}$).

Dans ces notations, il en résulte que

(16) $\qquad F(K)(ua) = a^{-1}[F(K)(u)] \quad$ pour tous $\quad u \in O^+(M), \quad a \in SO(n).$

Si le flot X^i est topologiquement transitif sur $O^+(M)$ et le tenseur K est parallèle, la Proposition 4 implique que

(17) $$F(K)|_{O^+(M)} \equiv T \in T^r_s(\mathbb{R}^n).$$

Et, d'après (16), le tenseur T vérifie

(18) $\qquad T = F(K)(ua) = a^{-1}F(K)(u) = a^{-1}T, \quad$ pour tout $\quad a \in SO(n)$

(on dit dans ce cas que T est invariant par l'action de $SO(n)$; voir [12] pour une classification de ces tenseurs).

Remarquons que les tenseurs $T \in T^r_s(\mathbb{R}^n)$ invariants par l'action de $SO(n)$ induisent d'une façon canonique des tenseurs parallèles sur la variété orientée M. Les résultats antérieurs impliquent que, si X^i est topologiquement transitif, alors ces tenseurs sont les seuls tenseurs parallèles sur M. La Proposition 5 donne alors la forme des tenseurs récurrents.

D'autre part, observons que si le tenseur de courbure de M vérifie (17) (donc (18)), alors M est à courbure sectionnelle constante.

Les résultats de L. W. Green [6] impliquent que X^i est topologiquement transitif sur $O^+(M)$ si M est compacte et à courbure sectionnelle négative 1/4-pincée (c'est-à-dire qu'il existe une constante $c > 0$ telle que les courbures sectionnelles de M sont bornées par $-c$ et $-1/4\,c$). Nous pouvons donc, énoncer le théorème suivant:

Théorème 4. *Soient M une variété riemannienne orientée et X^i le i-ème flot géodésique sur $O^+(M)$. Si X^i est topologiquement transitif (ce qui est vrai si M est compacte à courbure sectionnelle négative 1/4-pincée) alors un tenseur $K \in T^r_s(M)$ est parallèle si et seulement si $F(K)|_{O^+(M)} = T \in T^r_s(\mathbb{R}^n)$ et T est invariant par $SO(n)$.*

En particulier, si M est à tenseur de courbure parallèle, alors M est à courbure sectionnelle constante.

4. Le flot de p-repères

Considérons maintenant $\Omega_p(M)$ le fibré de p-repères orthonormés de la variété riemannienne M. Ici $1 \leqslant p \leqslant n - 1$ où $n = \dim M$. C'est-à-dire que pour chaque $x \in M$ nous considérons tous les p-repères orthonormés $(E_1, \ldots, E_p)$ de T_xM.

Si $\pi: \Omega_p(M) \to T_1M$ est la projection $\pi(E_1, \ldots, E_p) = E_1$, on peut voir $\Omega_p(M)$ comme un fibré sur T_1M dont la fibre $\pi^{-1}(E_1)$ est la variété de Stiefel $O(n-1)/O(n-1-(p-1))$.

Etant donné $(E_1, \ldots, E_p)$, soit γ_{E_1} la géodésique déterminée par E_1; désignons par $\phi_t^p(E_1, \ldots, E_p)$ le p-repère en $\gamma_{E_1}(t)$ obtenu par transport parallèle de $(E_1, \ldots, E_p)$ sur γ_{E_1}. Le flot ϕ_t^p ainsi défini sur $\Omega_p(M)$ est appelé le flot de p-repères et X_p notera son champ.

Si $(E_1, \ldots, E_p)$ est un p-repère en T_xM, soit $[E_1, \ldots, E_p]$ le sous-espace de T_xM qu'il engendre et désignons par E^i le covecteur dual de E_i pour la métrique g de M $(E^i = g(E_i, \bullet))$. Alors, étant donné $K \in T_s^r(M)$, la restriction de $K(x)$ à $[E_1, \ldots, E_p]$ s'écrira

$$K(x)|_{[E_1, \ldots, E_p]} = K_{j_1, \ldots, j_s}^{i_1, \ldots, i_r} E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes E^{j_1} \otimes \cdots \otimes E^{j_s}$$

où $1 \leqslant i_\alpha \leqslant p$, $1 \leqslant j_\beta \leqslant p$. Nous définissons la fonction $F(K): \Omega_p(M) \to T_s^r(\mathbb{R}^p)$ par $F(K)\,(E_1, \ldots, E_p) = (K_{j_1, \ldots, j_s}^{i_1, \ldots, i_r})$.

Il est aisé de montrer suivant la preuve de la Proposition 4 que

$$(19) \qquad X_p F(K)(E_1, \ldots, E_p) = F(\nabla_{E_1} K)(E_1, \ldots, E_p).$$

En particulier, si K est parallèle alors $F(K)$ est invariante par X_p. Comme $p < n$ nous ne pouvons pas affirmer la réciproque.

Néanmois, si le tenseur K a des propriétés supplémentaires, la fonction $F(K): \Omega_p(M) \to T_s^r(\mathbb{R}^p)$ peut garder toute l'information sur le tenseur (ce qui est toujours le cas pour $F(K): O(M) \to T_s^r(\mathbb{R}^n)$) même si p est petit.

Par exemple, si l'on prend le tenseur de courbure riemannienne $R \in T_4^0(M)$, il suffit de considérer $F(R): \Omega_2(M) \to T_4^0(\mathbb{R}^2)$. En particulier

(20) Si $F(R)|_{\Omega_2(M)} \equiv T \in T_4^0(\mathbb{R}^2)$, alors M est à courbure sectionnelle constante.

Il est bien connu (voir [1]) que si M est compacte, à courbure sectionnelle négative et de dimension impaire, alors le flot de 2-repères sur $\Omega_2(M)$ est ergodique. En conséquence (19) et (20) nous permettent d'énoncer le théorème suivant

Théorème 5. *Soit M une variété riemannienne compacte, de dimension impaire, à courbure sectionnelle négative.*

Si le tenseur de courbure est parallèle, alors M est à courbure sectionnelle constante.

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Recibido: 21 de junio de 1990.

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Revista Matemática Iberoamericana
Vol. 6, n.^{os} 3 y 4, 1990

Continuité-Sobolev de Certains Opérateurs Paradifférentiels

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Introduction

L'objet de ce travail est l'étude de la continuité des opérateurs d'intégrales singulières (au sens de Calderón-Zygmund) sur les espaces de Sobolev $\dot{H}^s$. Il complète le travail fondamental de David-Journé [6], concernant le cas $s = 0$, et ceux de P. G. Lemarié [10] et M. Meyer [11] concernant le cas $0 < s < 1$.

Dans ces travaux antérieurs le critère de continuité porte sur la fonction $T(1)$, image de la fonction identiquement égale à 1 par l'opérateur T. Lorsque $s \geqslant 1$ (et c'est le cas que nous étudions) il convient de généraliser l'objet $b = T(1)$; c'est ce que nous ferons dans les lignes qui suivent en définissant un ensemble fini de fonctions b_α.

Un autre progrès est une meilleure compréhension, grâce à l'usage des capacités de Riesz, de l'espace fonctionnel qui intervient déjà dans les travaux de M. Meyer et qui est noté $\mathrm{BMO}_2^{s,2}$ (cette notation est suggérée par David-Journé où le critère est $T(1) \in \mathrm{BMO}$, $T^*(1) \in \mathrm{BMO}$). Finalement nous arrivons à énoncer une condition nécessaire et suffisante maniable.

Il convient maintenant de préciser nos apports dans le sujet. Soit T un opérateur linéaire continu de $\mathfrak{D}(\mathbb{R}^n)$ dans $\mathfrak{D}'(\mathbb{R}^n)$; on dit que T est un opérateur d'intégrale singulière (à gauche) d'ordre $m \in \mathbb{N}^*$ (en abrégé $T \in \mathrm{SIO}(m)$) si son noyau-distribution est, hors de la diagonale $x = y$, une fonction de classe C^m par rapport à la première variable et vérifiant les estimations

$$|\partial_x^\alpha K(x,y)| \leqslant C|x-y|^{-n-|\alpha|} \quad \text{pour} \quad x \neq y \quad \text{et} \quad |\alpha| \leqslant m.$$

Désignons par R_α l'opérateur de multiplication par la fonction polynôme $x \to x^\alpha$, en particulier R_{e_i} est l'opérateur de multiplication par x_i; on note par $\Gamma^{e_i}(T)$ le commutateur $[T, R_{e_i}]$ et par récurrence $\Gamma^{\alpha+e_i}(T) = [\Gamma^\alpha(T), R_{e_i}]$. Si l'opérateur T est faiblement borné (au sens de David-Journé) nous avons montré [14] que, pour $|\alpha| \leqslant m-1$, $b_\alpha = \Gamma^\alpha(T)(1)$ est une distribution définie modulo les polynômes de degré $|\alpha|$ et appartenant à l'espace de Besov $\dot{B}_\infty^{|\alpha|,\infty}$. De plus, la continuité de T sur l'espace de Sobolev homogène $\dot{H}^s$ $(0 < s < m)$ est caractérisée par le contrôle des b_α pour $|\alpha| \leqslant [s]$ ($[s]$ = partie entière de s), de la façon suivante.

Soit $\psi \in \mathcal{D}(\mathbb{R}^n)$ une fonction portée par la couronne $1/2 \leqslant |\xi| \leqslant 2$, telle que

$$\sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1 \quad \text{pour} \quad \xi \neq 0.$$

La fonction φ définie par

$$\varphi(\xi) = 1 - \sum_{j \geq 1} \psi(2^{-j}\xi)$$

est de classe C^∞ portée par la boule $|\xi| \leqslant 2$ et $\varphi(\xi) = 1$ pour $|\xi| \leqslant 1$. Par la suite, on notera par Δ_j et S_j les opérateurs de convolution de symboles respectifs $\psi(2^{-j}\xi)$ et $\varphi(2^{-j}\xi)$.

Si $b \in \dot{B}_\infty^{|\alpha|,\infty}$ $(\alpha \in \mathbb{N}^n)$, l'opérateur de para-produit de J.M. Bony π_b^α (ou paradifférentiel) est défini par

$$\pi_b^\alpha(f) = \sum_{j \in \mathbb{Z}} \Delta_j(b) S_{j-3}(\partial^\alpha f);$$

c'est un opérateur faiblement borné appartenant à SIO (m) (pour tout $m \in \mathbb{N}$). De plus,

$$\Gamma^\alpha(\pi_b^\alpha)(1) = (\alpha!)b \quad \textit{et} \quad \Gamma^\beta(\pi_b^\alpha)(1) = 0 \quad \text{pour} \quad |\beta| < |\alpha|.$$

Avec ces notations, on obtient le théorème suivant [14].

Théorème 1. *Soient $m \in \mathbb{N}^*$, $T \in$ SIO (m) et $0 < s < m$. Alors T est continu sur $\dot{H}^s$ si et seulement si*

(i) *T est faiblement borné.*
(ii) *L'opérateur*

$$T_0 = \sum_{|\alpha| \leq [s]} \frac{1}{\alpha!} \pi_{b_\alpha}^\alpha$$

est continu sur $\dot{H}^s$ où

$$b_\alpha = \Gamma^\alpha(T)(1).$$

Le Théorème 1 permet le passage d'un opérateur d'intégrale singulière à un opérateur paradifférentiel; par contre son exploitation apparaît assez compliquée. Notre but dans ce qui suit, est donc d'expliciter ce critère; plus précisement, nous allons montrer que les opérateurs $\pi^{\alpha}_{b_\alpha}$ ($\alpha \in \mathbb{N}^n$ et $b_\alpha \in B^{|\alpha|,\infty}_{\infty}$) forment une famille indépendante pour la continuité-$\dot{H}^s$, autrement dit, la continuité de l'opérateur T_0 entraîne celle de chaque $\pi^{\alpha}_{b_\alpha}$.

En abordant cette étude, on montrera au passage les deux résultats suivants.

(*a*) Classiquement l'espace $\mathrm{BMO}^{s,2}_2$ est caractérisé à l'aide des capacités de Riesz [12], ou par l'opérateur de para-produit [11]; nous allons alors caractériser cet espace à l'aide d'une condition de Carleson généralisée, prolongeant d'une façon très naturelle celle de l'espace BMO.

(*b*) Pour tout $\alpha \in \mathbb{N}^n$ et pour tout $b \in \dot{B}^{|\alpha|,\infty}_{\infty}$ avec $|\alpha| \leqslant [s]$; l'opérateur π^{α}_{b} est continu sur $\dot{H}^s$ si et seulement si pour tout $\beta \in \mathbb{N}^n$ tel que $|\beta| = |\alpha|$, $\partial^\beta b \in \mathrm{BMO}^{s-|\alpha|,2}_2$. En particulier, lorsque $|\alpha| = s$ ($s \in \mathbb{N}$) et π^{α}_{b} est continu sur $\dot{H}^s$, alors b appartient à l'espace BMO-Sobolev $\dot{H}^s_\infty$($\dot{H}^s_\infty$ est une généralisation à «la Sobolev» de l'espace BMO, et sera donnée par la définition 2).

La rédaction de cet article a bénéficié de nombreuses suggestions de G. Bourdaud et de Y. Meyer, je souhaite vivement leur exprimer mes remerciements pour l'intérêt qu'ils ont apporté à ce travail.

1. La presque-orthogonalité

Pour $s \in \mathbb{R}$, l'espace de Sobolev homogène $\dot{H}^s$ est l'espace des distributions modulo les polynômes, défini par

$$\|f\|_{\dot{H}^s} = \left[\sum_{j\in\mathbb{Z}} 4^{sj}\|\Delta_j(f)\|_2^2\right]^{1/2} < +\infty.$$

Dans ce paragraphe, nous allons travailler dans un cadre un peu plus général qui est celui des espaces de Besov homogènes. Pour $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant +\infty$, l'espace de Besov $\dot{B}^{s,q}_p$ est défini par

$$\|f\|_{\dot{B}^{s,q}_p} = \left[\sum_{j\in\mathbb{Z}} 2^{sjq}\|\Delta_j(f)\|_p^q\right]^{1/q} < +\infty \quad \text{si} \quad q < +\infty,$$

et

$$\|f\|_{\dot{B}^{s,q}_p} = \operatorname{Sup}_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j(f)\|_p < +\infty \quad \text{si} \quad q = +\infty.$$

Soit $m \in \mathbb{N}$, pour $f \in \mathcal{S}(\mathbb{R}^n)$, on désigne par $F_m(f)$ la fonction en deux variables définie par

$$F_m(f)(x,y) = f(y) - \sum_{|\alpha| \le m} \frac{1}{\alpha!}(y-x)^\alpha (\partial^\alpha f)(x);$$

alors on a la proposition suivante.

Proposition 1. *Soient $h \in \mathcal{S}(\mathbb{R}^n)$, $m \in \mathbb{N}$; alors pour tout $s < m+1$ et pour tout $p < +\infty$, il existe une constante $C > 0$ telle que*

$$\left[\sum 2^{sjq}\left(\iint 2^{nj}|h(2^j(x-y))|\,|F_m(S_jf)(x,y)|^p\,dx\,dy\right)^{q/p}\right]^{1/q} \leqslant C\|f\|_{\dot{B}_p^{s,q}},$$

pour toute fonction $f \in \mathcal{S}(\mathbb{R}^n)$.

Considérons une fonction $g \in \mathcal{S}(\mathbb{R}^n)$; à l'aide de la formule de Taylor, on a

$$|F_m(g)(x,x+z)| \leqslant C|z|^{m+1}\left[\sum_{|\alpha|=m+1}\int_0^1 (1-t)^m |(\partial^\alpha g)(x+tz)|^p\,dt\right]^{1/p};$$

d'où

$$\int |F_m(g)(x,x+z)|^p\,dx \leqslant C|z|^{(m+1)p}\sum_{|\alpha|=m+1}\|(\partial^\alpha g)\|_p^p.$$

En posant $g = S_jf$, on obtient

$$\int |h(2^jz)|\,|F_m(S_jf)(x,x+z)|^p\,dx\,dz \leqslant C2^{-j(m+1)}\sum_{|\alpha|=m+1}\|\partial^\alpha(S_jf)\|_p^p.$$

Or,

$$\|\partial^\alpha(S_jf)\|_p^p \leqslant C\sum_{k\le j}\|\partial^\alpha(\Delta_kf)\|_p;$$

en utilisant l'inégalité de Bernstein, on a alors

$$\left[\int |h(2^jz)|\,|F_m(S_jf)(x,x+z)|^p\,dx\,dz\right]^{1/p}$$

est majorée par

$$C2^{-j(m+1)}\sum_{k\le j}2^{k(m+1)}\|\Delta_kf\|_p.$$

Comme $s < m+1$, l'inégalité de Young dans $l^q(\mathbb{Z})$ fait l'affaire. □

Proposition 2. *Soient $m \in \mathbb{N}$ et $b_\alpha \in \dot{B}_\infty^{|\alpha|,\infty}$ où $|\alpha| \leqslant m$. Si $s \in \mathbb{R}$ avec $m \leqslant s < m+1$ et si l'opérateur*

$$T = \sum_{|\alpha|\le m}\pi_{b_\alpha}^\alpha$$

est continu sur $\dot{B}_p^{s,q}$; *alors pour toute fonction* $h \in \mathcal{S}(\mathbb{R}^n)$ *à spectre dans la couronne* $1/2 \leqslant |\xi| \leqslant 2$, *il existe* $C > 0$ *tel que*

$$\left[\sum 2^{sjq} \left\| \sum_{|\alpha| \leq m} R_j(b_\alpha) S_{j-3}(\partial^\alpha f) \right\|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}_p^{s,q}}$$

pour toute fonction $f \in \mathcal{S}(\mathbb{R}^n)$ *et où* R_j *est l'opérateur de convolution avec* $2^{nj} h(2^j x)$.

La preuve de la proposition repose, entre autres, sur la Proposition 1 en utilisant une récurrence sur m.

Cas 1. $m = 0$.

Dans ce cas, on a $0 \leqslant s < 1$; on pose alors

$$X_j(f) = R_j(b_0) S_{j-3}(f) \quad \text{et} \quad Y_j(f) = R_j(Tf).$$

Puisque T est continu sur $\dot{B}_p^{s,q}$, il existe $C > 0$ tel que

$$\left[\sum 2^{sjq} \| Y_j(f) \|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}_p^{s,q}}.$$

Il suffit donc de montrer

$$\left[\sum 2^{sjq} \| Y_j(f) - X_j(f) \|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}_p^{s,q}}.$$

Or, $Y_j(f) - X_j(f) = A_j(f) + B_j(f)$ où

$$A_j(f) = \sum_{\nu=-2}^{2} R_j[\Delta_{j+\nu}(b_0) S_{j+\nu-3}(f)] - R_j[\Delta_{j+\nu}(b_0)] S_{j+\nu-3}(f)$$

et

$$B_j(f) = \sum_{\nu=-2}^{2} R_j[\Delta_{j+\nu}(b_0)][S_{j+\nu-3}(f) - S_{j-3}(f)].$$

Mais,

$$\sum_{\nu=-2}^{2} \| S_{j+\nu-3}(f) - S_{j-3}(f) \|_p \leqslant C \sum_{\nu=-4}^{1} \| \Delta_{k+\nu} f \|_p;$$

l'hypothèse $b_0 \in \dot{B}_\infty^{0,\infty}$ entraîne

$$\left[\sum 2^{sjq} \| B_j(f) \|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}_p^{s,q}}.$$

Pour estimer $A_j(f)$, on pose

$$b_j = \Delta_{j+\nu}(b_0), \qquad f_j = S_{j+\nu-3}(f)$$

et on néglige ν. On a

$$\begin{aligned} A_j(f) &= 2^{nj}\int h(2^j(x-y))b_j(y)(f_j(y)-f_j(x))\,dy \\ &= 2^{nj}\int h(2^j(x-y))b_j(y)F_0(f_j)(x,y)\,dy; \end{aligned}$$

alors

$$\|A_j(f)\|_p \leqslant C\Big[2^{nj}\int |h(2^j(x-y))|\,|F_0(f_j)(x,y)|^p\,dx\,dy\Big]^{1/p},$$

la Proposition 1 fait donc le nécessaire.

Cas 2. On suppose que $m \geqslant 1$ et que la proposition soit vraie pour tout entier $m' \leqslant m-1$. Posons

$$X_j(f) = \sum_{|\alpha|\leq m} R_j(b_\alpha)S_{j-3}(\partial^\alpha f) \quad \text{et} \quad Y_j(f) = R_j(Tf);$$

on a

$$Y_j(f) - X_j(f) = C_j(f) + D_j(f)$$

où

$$C_j(f) = \sum_{\nu=-2}^{2}\sum_{|\alpha|\leq m} R_j[(\Delta_{j+\nu}b_\alpha)S_{j+\nu-3}(\partial^\alpha f)] - R_j[(\Delta_{j+\nu}b_\alpha)]S_{j+\nu-3}(\partial^\alpha f)$$

et

$$D_j(f) = \sum_{\nu=-2}^{2}\sum_{|\alpha|\leq m} R_j(\Delta_{j+\nu}b_\alpha)[S_{j+\nu-3}(\partial^\alpha f) - S_{j-3}(\partial^\alpha f)].$$

De la même façon que le cas $m = 0$, on a

$$\Big[\sum 2^{sjq}\|D_j(f)\|_p^q\Big]^{1/q} \leqslant C\|f\|_{\dot{B}_p^{s,q}}.$$

D'autre part, en négligeant ν et en posant

$$b_{j,\alpha} = \Delta_{j+\nu}(b_\alpha) \quad \text{et} \quad f_j = S_{j+\nu-3}(\partial^\alpha f);$$

$C_j(f)$ s'écrit

$$C_j(f)(x) = E_j(f)(x) + E'_j(f)(x)$$

où

$$E_j(f)(x) = \sum_{|\alpha|\leq m} 2^{nj}\int h(2^j(x-y))b_{j,\alpha}(y)F_{m-|\alpha|}(\partial^\alpha f_j)(x,y)\,dy;$$

et

$$E'_j(f)(x) = \sum_{\substack{|\alpha| \leq m \\ 0 < |\beta| \leq m - |\alpha|}} \frac{1}{\beta!} \left[2^{nj} \int h(2^j(x-y)) b_{j,\alpha}(y)(y-x)^\beta \, dy \right] [\partial^{\alpha-\beta} f_j(x)].$$

Or, $|E_j(f)(x)|$ est majorée par

$$C \sum_{|\alpha| \leq m} 2^{-j|\alpha|} \left[\int 2^{nj} |h(2^j(x-y))| \, |F_{m-|\alpha|}(\partial^\alpha f_j)(x,y)|^p \, dy \right]^{1/p};$$

d'où

$$\|E_j(f)\|_p \leqslant C \sum_{|\alpha| \leq m} 2^{-j|\alpha|} \left[\iint 2^{nj} |h(2^j(x-y))| \, |F_{m-|\alpha|}(\partial^\alpha f_j)(x,y)|^p \, dx \, dy \right]^{1/p}.$$

Comme $s - |\alpha| < m - |\alpha| + 1$, la Proposition 1 nous montre que

$$\left[\sum 2^{sjq} \|E_j(f)\|_p^q \right] \leqslant C \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{\dot{B}_p^{s-|\alpha|,q}}$$
$$\leqslant C \|f\|_{\dot{B}_p^{s,q}}.$$

Pour estimer $E'_j(f)$, on pose

$$R_{j,\beta}(b)(x) = 2^{j(n+|\beta|)} \int h(2^j(x-y))(y-x)^\beta \Delta_j(b)(y) \, dy$$

et

$$T_\beta(f) = \sum_{|\alpha| \leq m - |\beta|} \pi^\alpha_{b_\alpha}(f).$$

Par interpolation, la continuité de T montre que T_β est continu sur $\dot{B}_p^{s-|\beta|,q}$, alors, l'hypothèse de récurrence prouve que, pour $|\beta| \neq 0$, on a

$$\left[\sum 2^{(s-|\beta|)jq} \left\| \sum_{|\alpha| \leq m - |\beta|} R_{j,\beta}(b_\alpha) S_{j-3}(\partial^\alpha g) \right\|_p^q \right]^{1/q} \leqslant C \|g\|_{\dot{B}_p^{s-|\beta|,q}}$$

pour toute fonction $g \in \mathcal{S}(\mathbb{R}^n)$.

Mais,

$$E'_j(f) = \sum_{0 < |\beta| \leq m} \frac{1}{\beta!} 2^{-j|\beta|} \sum_{|\alpha| \leq m - |\beta|} R_{j,\beta}(b_\alpha) S_{j-3}(\partial^{\alpha+\beta} f);$$

donc

$$\left[\sum 2^{sjq} \|E'_j(f)\|_p^q \right]^{1/q} \leqslant C \|f\|_{\dot{B}_p^{s,q}}. \quad \square$$

Pour conclure cette partie, rappelons un lemme de presque-orthogonalité classique (voir par exemple [3]).

Lemme. *Pour tout $t > 1$ et toute suite de fonctions $(f_j)_j$ à spectres respectifs dans les couronnes $t^{-1}2^j \leqslant |\xi| \leqslant t2^j$, on a*

$$\left\| \sum f_j \right\|_{\dot{B}^{s,q}_p} \leqslant C\left[\sum 2^{sjq} \| f_j \|_p^q \right]^{1/q}.$$

D'autre part, remarquons que pour $|\alpha| < s$ et $b_\alpha \in \dot{B}^{|\alpha|,\infty}_\infty$, on a

$$\left[\sum 2^{sjq} \| \Delta_j(b_\alpha)(\partial^\alpha f - S_{j-3}(\partial^\alpha f)) \|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}^{s,q}_p}. \tag{1}$$

En effet,

$$\| \Delta_j(b_\alpha)(\partial^\alpha f - S_{j-3}(\partial^\alpha f)) \|_p \leqslant C 2^{-j|\alpha|} \sum_{k \geq j-2} 2^{k|\alpha|} \| \Delta_k f \|_p;$$

comme $|\alpha| < s$, l'inégalité de Young montre (1).

De la Proposition 2 et du lemme de presque-orthogonalité, on tire le

Théorème 2. *Soient $s \in \mathbb{R}$, $m = [s]$ et $b_\alpha \in \dot{B}^{|\alpha|,\infty}_\infty$ pour $|\alpha| \leqslant m$; alors il y a équivalence entre les propriétés:*

1. *L'opérateur*

$$T = \sum_{|\alpha| \leq m} \pi^\alpha_{b_\alpha}$$

est continu sur $\dot{B}^{s,q}_p$.

2. *Il existe $C > 0$ tel que pour toute fonction $f \in \mathcal{S}(\mathbb{R}^n)$, on ait:*

(*a*) *si $s \notin \mathbb{N}$,*

$$\left[2^{sjq} \left\| \sum_{|\alpha| \leq m} \Delta_j(b_\alpha)(\partial^\alpha f) \right\|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}^{s,q}_p}; \tag{2}$$

(*b*) *si $s \in \mathbb{N}$,*

$$\left[\sum 2^{sjq} \left\| \sum_{|\alpha| \leq m-1} \Delta_j(b_\alpha)(\partial^\alpha f) + \sum_{|\alpha| = m} \Delta_j(b_\alpha) S_{j-3}(\partial^\alpha f) \right\|_p^q \right]^{1/q} \leqslant C \| f \|_{\dot{B}^{s,q}_p}. \tag{3}$$

2. Mesures de Carleson et continuité-Sobolev

Par la suite, les capacités de Riesz n'interviennent, peut-être, que comme artifice technique; cependant elles s'avèrent un outil assez fort pour montrer de nombreux théorèmes d'analyse; les travaux de D. R. Adams [2] et Coifman-Murai [5] en furent des exemples.

Soit Ω un ouvert de $\mathbb{R}^n$ et soit $s > 0$; la capacité de Riesz de Ω est définie par

$$\mathrm{Cap}_s(\Omega) = \inf \{ \| f \|^2_{\dot{H}^s} : f \in \mathcal{S}(\mathbb{R}^n), f \geqslant 1 \text{ sur } \Omega \}.$$

Les premières définitions des capacités ont été données à l'aide du potentiel de Riesz, ce qui oblige une restriction sur s; par contre, elles se confondent avec notre définition dès que $0 < s < n/2$. Nous utiliserons notamment la propriété suivante.

Proposition 3 ([2]). *Pour $0 < s < n/2$, il existe une constante $C > 0$ telle que pour toute fonction $f \in \mathcal{S}(\mathbb{R}^n)$, on ait*

$$\int_0^{+\infty} \lambda \operatorname{Cap}_s \{x : f^*(x) > \lambda\}\, d\lambda \leqslant C \|f\|_{\dot{H}^s}^2$$

où f^* désigne la fonction maximale de Hardy-Littlewood.

Passons maintenant aux mesures de Carleson; la définition que nous allons utiliser est celle de [9], dans sa version discrète.

Définition 1. *Soient $r \in \mathbb{R}$ et $(\mu_j)_{j \in \mathbb{Z}}$ une suite de mesures positives sur $\mathbb{R}^n$; on dit que $(\mu_j)_{j \in \mathbb{Z}}$ est une r-mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$ s'il existe $C > 0$ tel que pour tout $k \in \mathbb{Z}$ et toute boule B de rayon 2^{-k}, on ait*

$$\sum_{j \geq k} \mu_j(B) \leqslant C|B|^r$$

où $|B|$ désigne la mesure de B par rapport à la mesure de Lebesgue.

Lorsque $r = 1$, on dit que $(\mu_j)_{j \in \mathbb{Z}}$ est une mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$.

Définition 2. *Pour $s \in \mathbb{R}$, l'espace de* BMO*-Sobolev $\dot{H}^s_\infty$ est l'ensemble des distributions b telles que $(4^{sj}|\Delta_j(b)(x)|^2\, dx)_j$ soit une mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$.*

Proposition 4. *Soient $s \geqslant 0$, $\alpha \in \mathbb{N}^n$ avec $0 \leqslant s - |\alpha| < n/2$ et $b_\alpha \in \dot{B}_\infty^{|\alpha|,\infty}$; alors il y a équivalence entre les assertions:*

1. *L'opérateur $\pi^\alpha_{b_\alpha}$ est continu sur $\dot{H}^s$.*
2. *La suite $(4^{sj}|\Delta_j(b_\alpha)(x)|^2\, dx)_j$ est une r-mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$ où $r = 1 - 2(s - |\alpha|)/n$.*

Nous commençons par montrer le cas $|\alpha| \neq s$; pour $|\alpha| = s$, la preuve est de nature un peu différente et sera exposée à part.

Cas (a): $|\alpha| \neq s$. Supposons que $\pi^\alpha_{b_\alpha}$ soit continu sur $\dot{H}^s$ et soit $f \in \mathcal{D}(\mathbb{R}^n)$ une fonction telle que $f(x) = x^\alpha/\alpha!$ pour $|x| \leqslant 1$; pour $k \in \mathbb{Z}$ et B une boule de centre x_0 et de rayon 2^{-k}, on pose $g_k(x) = f(2^k(x - x_0))$.

Puisque $|\alpha| \leqslant [s]$, le Théorème 2 entraîne

$$\sum_j 4^{sj} \| \Delta_j(b_\alpha)\partial^\alpha g_k \|_2^2 \leqslant C \| g_k \|_{\dot{H}^s}^2 ;$$

l'invariance par translations-dilatations de $\dot{H}^s$ prouve que

$$\begin{aligned} \sum_j 4^{sj} \int_B |\Delta_j(b_\alpha)(x)|^2 \, dx &\leqslant C 4^{-k|\alpha|} \| g_k \|_{\dot{H}^s}^2 \\ &\leqslant C |B|^r. \end{aligned}$$

Inversement, supposons que $(4^{sj}|\Delta_j(b_\alpha)(x)|^2 \, dx)_j$ soit une r-mesure de Carleson et soit Ω un ouvert de $\mathbb{R}^n$, nous allons montrer que

$$\sum_j 4^{sj} \int_\Omega |\Delta_j(b_\alpha)(x)|^2 \, dx \leqslant C \operatorname{Cap}_{s-|\alpha|}(\Omega). \tag{4}$$

Pour cela, commençons par montrer (4) pour une boule $B = B(x_0, r)$. On remarque que $\operatorname{Cap}_{s-|\alpha|}(B) \approx |B|^r$; de plus si $k \in \mathbb{Z}$ tel que $2^{-k-1} \leqslant r < 2^{-k}$, alors

$$\begin{aligned} \sum_{j \geq k} 4^{sj} \int_B |\Delta_j(b_\alpha)(x)|^2 \, dx &\leqslant C \operatorname{Cap}_{s-|\alpha|}(B(x_0, 2^{-k})) \\ &\leqslant C \operatorname{Cap}_{s-|\alpha|}(B). \end{aligned}$$

Or, $\sum_{j \leq k} 4^{sj} \int_B |\Delta_j(b_\alpha)(x)|^2 \, dx$ est majorée par $C|B| \, \| b_\alpha \|_{\dot{B}_\infty^{|\alpha|,\infty}} \sum_{j \leq k} 4^{j(s-|\alpha|)}$, donc par $|B|^r$.

Considérons, maintenant un ouvert Ω borné de $\mathbb{R}^n$; d'après le lemme de recouvrement ([3] p. 109), il existe une suite de boules $B_j = B(x_j, r_j)$ deux à deux disjointes, contenues dans Ω et telles que $\Omega \subset \bigcup_{j \in \mathbb{Z}} (B(x_j, 5r_j))$.

En utilisant les résultats de [1], on peut vérifier facilement que

$$\sum_j \operatorname{Cap}_{s-|\alpha|}(B(x_j, r_j/2)) \leqslant C \operatorname{Cap}_{s-|\alpha|}(\Omega);$$

donc

$$\begin{aligned} \sum_j 4^{sj} \int_\Omega |\Delta_j(b_\alpha)(x)|^2 \, dx &\leqslant \sum_k \sum_j 4^{sj} \int_{B(x_k, 5r_k)} |\Delta_j(b_\alpha)(x)|^2 \, dx \\ &\leqslant C \sum_k \operatorname{Cap}_{s-|\alpha|}(B(x_k, 5r_k)) \\ &\leqslant C \operatorname{Cap}_{s-|\alpha|}(\Omega). \end{aligned}$$

Revenons à la continuité de $\pi^{\alpha}_{b_\alpha}$; pour $f \in \mathcal{S}(\mathbb{R}^n)$, on a

$$\sum_j 4^{sj} \| \Delta_j(b_\alpha) S_{j-3}(\partial^\alpha f) \|_2^2 \, dx \leqslant C \int_0^{+\infty} \lambda \left(\sum_j 4^{sj} \int_{\Omega_{\lambda,j}} |\Delta_j(b_\alpha)(x)|^2 \, dx \right) d\lambda$$

où

$$\Omega_{\lambda,j} = \{x \colon |S_{j-3}(\partial^\alpha f)| > \lambda\}.$$

Or, pour tout $j \in \mathbb{Z}$, $\Omega_{\lambda,j} \subset \Omega_\lambda = \{x \colon (\partial^\alpha f)^*(x) > \lambda\}$ (voir [3] p. 157-158); donc

$$\sum_j 4^{sj} \| \Delta_j(b_\alpha)(x) S_{j-3}(\partial^\alpha f) \|_p^2 \, dx \leqslant C \int_0^{+\infty} \lambda \, \mathrm{Cap}_{s-|\alpha|}(\Omega_\lambda) \, d\lambda;$$

la Proposition 3 nous donne alors la propriété souhaitée.

Cas (*b*): $|\alpha| = s$. Supposons que $(4^{sj}|\Delta_j(b_\alpha)(x)|^2 \, dx)_j$ soit une mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$; alors $b \in \dot{H}^s_\infty$ et

$$b' = \sum_j 4^{sj} \Delta_j(b_\alpha)$$

appartient à l'espace BMO (voir [7]), b' n'est définie qu'aux constantes près. Or, on sait classiquement (voir [6]) que $\pi_{b'_\alpha} = \pi^0_{b'_\alpha}$ est continu sur $L^2(\mathbb{R}^n)$; en appliquant la Proposition 2, on obtient

$$\sum_j 4^{sj} \| \Delta_j(b_\alpha)(x) S_{j-3}(g) \|_2^2 \leqslant C \| g \|_2^2;$$

d'où

$$\sum_j 4^{sj} \| \Delta_j(b_\alpha)(x) S_{j-3}(\partial^\alpha f) \|_2^2 \leqslant C \| \partial^\alpha f \|_2^2 \leqslant C \| f \|_{\dot{H}^s}^2.$$

Dans le cas où $\pi^{\alpha}_{b_\alpha}$ est continu sur $\dot{H}^s$, alors il existe $C > 0$ tel que pour toute $f \in \mathcal{S}(\mathbb{R}^n)$, on ait

$$\sum_j 4^{sj} \| \Delta_j(b_\alpha)(x) S_{j-3}(\partial^\alpha f) \|_2^2 \leqslant C \| f \|_{\dot{H}^s}^2.$$

Soit $f \in \mathcal{D}(\mathbb{R}^n)$ telle que $f(x) = x^\alpha/\alpha!$ pour $|x| \leqslant 2$, alors il existe $N \in \mathbb{N}$ tel que $|S_j(\partial^\alpha f)| \geqslant 1/2$ sur la boule $B(0, 1)$ pour $j \geqslant N$; ce qui montre

$$\sum_{j \geq N-3} 4^{sj} \int_{B(0,1)} |\Delta_j(b_\alpha)(x)|^2 \, dx \leqslant C \| f \|_{\dot{H}^s}^2.$$

Comme $b \in \dot{B}_{\infty}^{|\alpha|, \infty}$, on a alors

$$\sum_{j \geq 0} 4^{sj} \int_{B(0,1)} |\Delta_j(b_\alpha)(x)|^2 \, dx < +\infty.$$

Finalement, par translations et dilatations, on obtient la condition désirée. □

Proposition 5. *Soient $s > 0$, $b_\alpha \in \dot{B}_{\infty}^{|\alpha|, \infty}$ pour $|\alpha| \leqslant [s] = m$ et on suppose que l'opérateur*

$$T = \sum_{|\alpha| \leq m} \pi_{b_\alpha}^{\alpha}$$

soit continu sur $\dot{H}^s$. Alors pour tout α, $(4^{sj}|\Delta_j(b_\alpha)(x)|^2 \, dx)_{j \in \mathbb{Z}}$ est une r_α-mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$ où $r_\alpha = 1 - 2(s - |\alpha|)/n$.

Remarquons d'abord que pour tout $k \in \mathbb{Z}$ et tout $x_0 \in \mathbb{R}^n$, l'opérateur

$$T' = \sum_{|\alpha| \leq m} \pi_{b'_\alpha}^{\alpha}$$

est continu sur $\dot{H}^s$ où $b'_\alpha(x) = b_\alpha(2^{-k}(x - x_0))$; cette propriété vient du fait que $\dot{H}^s$, $\dot{B}_{\infty}^{|\alpha|, \infty}$ sont invariants par translations et dilatations.

Compte-tenu de cette propriété, il nous suffit de montrer la condition de Carleson pour la boule unité. Pour cela, notre preuve repose sur une récurrence sur α; de plus, via le Théorème 2, on utilise la propriété suivante:

$$\left[\sum 4^{sj} \left\| \sum_{|\alpha| \leq m-1} (\Delta_j b_\alpha)(\partial^\alpha f) + \sum_{|\alpha| = m} \Delta_j(b_\alpha) S_{j-3}(\partial^\alpha f) \right\|_2^2 \right]^{1/2} \leqslant C \|f\|_{\dot{H}^s}.$$

Cas (a): $\alpha = 0$. Soit $f \in \mathcal{D}(\mathbb{R}^n)$ telle que $f(x) = 1$ pour $|x| \leqslant 3/2$, alors

$$\sum_{j \geq 0} 4^{sj} \int_{B(0,1)} |\Delta_j(b_0)(x)|^2 \, dx$$

est majorée par $C(A_1 + A_2)^{1/2}$ où

$$A_1 = \left[\sum 4^{sj} \left\| \sum_{|\beta| \leq m-1} (\Delta_j b_\beta)(\partial^\beta f) + \sum_{|\alpha| = m} \Delta_j(b_\beta) S_{j-3}(\partial^\beta f) \right\|_2^2 \right]^{1/2},$$

et

$$A_2 = \sum_{|\beta| = m} \left[\sum 4^{sj} \left| \int_B |\Delta_j b_\beta(x) \partial^\beta f(x)|^2 \, dx \right. \right]^{1/2}.$$

Pour $m = 0$, on se trouve dans le cas de la Proposition 4; on suppose alors que $m \neq 0$. Dans ce cas pour tout $\beta \in \mathbb{N}^n$ tel que $|\beta| = m$, $\partial^\beta f$ s'annule sur la boule $B = B(0, 1)$; on peut vérifier facilement que $|S_j(\partial^\beta f)(x)| \leqslant C2^{-j}$ pour tout $x \in B$. De ce fait, on obtient

$$A_2 \leqslant C \sum_{|\beta| = m} \Big[\sum 4^{(s-1)j} \|\Delta_j b_\beta\|_\infty^2\Big]^{1/2}.$$

Comme $\|\Delta_j b_\beta\|_\infty \leqslant C2^{-jm}$ et $s - m - 1 < 0$, on trouve que A_2 est finie.

Pour terminer, remarquons que $A_1 \leqslant C\|f\|_{\dot H^s}$.

Cas (*b*). Supposons que la proposition soit vraie jusqu'a l'ordre $p \leqslant m - 1$ et soit $\alpha \in \mathbb{N}^n$ tel que $|\alpha| = p + 1$. Considérons une fonction $g \in \mathcal{D}(\mathbb{R}^n)$ telle que $g(x) = x^\alpha/\alpha!$ sur $B(0, 3/2)$; on a

$$\sum_{j \geq 0} 4^{sj} \int_B |\Delta_j b_\alpha(x) \partial^\alpha f(x)|^2\, dx$$

est majorée par $C(A_1' + A_2' + A_3')^2$ où

$$A_1' = \Big[\sum 4^{sj} \Big\| \sum_{|\beta| \leq m-1} \Delta_j(b_\beta)(\partial^\beta g) + \sum_{|\beta| = m} \Delta_j(b_\beta) S_{j-3}(\partial^\beta g)\Big\|_2^2\Big]^{1/2},$$

$$A_2' = \sum_{|\beta| = m} \Big[\sum_{j \geq 0} 4^{sj} \int_B |\Delta_j b_\beta(x) S_{j-3}(\partial^\beta g)(x)|^2\, dx\Big]^{1/2} \quad \text{et}$$

$$A_3' = \sum_{|\beta| < |\alpha|} \Big[\sum_{j \geq 0} 4^{sj} \int_B |\Delta_j b_\beta(x)(\partial^\beta g)(x)|^2\, dx\Big]^{1/2}.$$

Or, $A_1' \leqslant C\|g\|_{\dot H^s}$; de plus, le même raisonnement que (*a*) montre que A_2' est finie. Finalement l'hypothèse de récurrence nous montre que A_3' est aussi finie.

Cas (*c*): $|\alpha| = m$. On considère la même fonction $g(x) = x^\alpha/\alpha!$ sur la boule $B(0, 3/2)$; alors il existe $N \in \mathbb{N}$ tel que $|S_{j-3}(\partial^\alpha g)(x)| > 1/2$ sur la boule $B(0, 1)$; donc

$$\sum_{j \geq 0} 4^{sj} \int_B |\Delta_j b_\alpha(x)|^2\, dx$$

est majorée par $C(E_1 + E_2 + E_3)^{1/2}$ où

$$E_1 = \Big[\sum 4^{sj} \Big\| \sum_{|\beta| \leq m-1} \Delta_j(b_\beta)(\partial^\beta g) + \sum_{|\beta| = m} \Delta_j(b_\beta) S_{j-3}(\partial^\beta g)\Big\|_2^2\Big]^{1/2}$$

$$E_2 = \sum_{|\beta| < m} \Big[\sum_{j \geq 0} 4^{sj} \int_B |\Delta_j b_\beta(x)(\partial^\beta g)(x)|^2\, dx\Big]^{1/2}$$

$$E_3 = \sum_{|\beta| = m, \beta \neq \alpha} \Big[\sum_{j \geq 0} 4^{sj} \int_B |\Delta_j b_\beta(x) S_{j-3}(\partial^\beta g)(x)|^2\, dx\Big]^{1/2}.$$

En utilisant le même raisonnement que (b), on peut voir facilement que $E_1 + E_2 + E_3$ est finie. □

Proposition 6. *Sous les hypothèses de la Proposition 5, pour tout α tel que $|\alpha| \leqslant [s - n/2]$, on a $b_\alpha = 0$.*

Le cas où $|\alpha| < [s - n/2]$ est une conséquence immédiate de la Proposition 5; pour les autres cas, c'est à dire $|\alpha| = s - n/2$, on revient à la définition de $\Gamma^\alpha(T)(1)$. Comme $T(x^\beta) = 0$ pour $|\beta| < |\alpha|$, on a donc $\Gamma^\alpha(T)(1) = T(x^\alpha)$; nous allons expliquer le cas $s = n/2$, le cas général est de même nature.

L'opérateur T étant continu sur $\dot{H}^s = \dot{H}^{n/2}$, alors le transposé ${}^tT$ de T est continu sur $\dot{H}^{-n/2}$ et sur L^2, donc de l'espace de Hardy H^1 dans L^1. Soit $h \in \mathfrak{D}(\mathbb{R}^n)$ telle que

$$\int h(x)\,dx = 0,$$

alors $h \in H^{-n/2}$, donc ${}^tT(h) \in \dot{H}^{-n/2}$; de plus ${}^tT(h) \in L^1$; ce qui montre que

$$\int {}^tT(h)(x)\,dx = 0$$

(voir [4]).

Considérons une fonction $\theta \in \mathfrak{D}(\mathbb{R}^n)$ telle que $\theta = 1$ sur $B(0, 1)$; on sait que $\langle T(1), h\rangle = \lim_{j\to+\infty} \langle \theta_j, {}^tT(h)\rangle$ où $\theta_j(x) = \theta(2^{-j}x)$.

Comme $\lim_{j\to+\infty} \langle \theta_j, {}^tT(h)\rangle$ n'est rien d'autre que $\int {}^tT(h)(x)\,dx$, on obtient alors $\langle T(1), h\rangle = 0$; ce qui explique que $b_0 = 0$. □

Définition 3. *Pour $s \geqslant 0$, on désigne par $\mathrm{BMO}_2^{s,2}$ l'espace des distributions $b \in \dot{B}_\infty^{0,\infty}$ telles que l'opérateur $\pi_b = \pi_b^0$ soit continu sur $\dot{H}^s$.*

L'espace $\mathrm{BMO}_2^{s,2}$ a été introduit pour la première fois par Stegenga [12] et pour caractériser les multiplicateurs des espaces de Dirichlet; un peu plus tard M. Meyer [11] a montré la liaison entre ces espaces et les opérateurs intégraux singuliers, et on trouve une autre définition à l'aide du para-produit. Comme conséquence de la Proposition 6, on a $\mathrm{BMO}_2^{s,2} = \{0\}$ pour $s \geqslant n/2$; dans le cas $0 \leqslant s < n/2$, on a la caractérisation suivante:

Corollaire. *Pour $0 \leqslant s < n/2$, il y a équivalence entre les propriétés:*

1. $b \in \mathrm{BMO}_2^{s,2}$;
2. *La suite $(4^{sj}|\Delta_j(b_\alpha)(x)|^2\,dx)_j$ est une r-mesure de Carleson sur $\mathbb{R}^n \times \mathbb{Z}$ où $r = 1 - 2s/n$.*

Remarques.

1. Comme cas particulier du corollaire, on a $\mathrm{BMO}_2^{0,2} = \mathrm{BMO}$.
2. Désignons par $M(\dot{H}^s)$ l'espace des multiplicateurs ponctuels de $\dot{H}^s$; alors $b \in M(\dot{H}^s)$ si et seulement si $b \in L^\infty$ et $b \in \mathrm{BMO}_2^{s,2}$.
3. Dans le cas $s \geqslant n/2$, on a $M(\dot{H}^s) = \{0\}$; ce fait vient du choix de la réalisation de $\dot{H}^s$ qui est définie modulo les polynômes de degré $[s - n/2]$. Cette difficulté peut être contournée en utilisant la réalisation de $\dot{H}^s$ dans $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ (voir [4] ou [13]).

Revenons aux opérateurs paradifférentiels. En utilisant les propositions 4, 5 et 6, on obtient le théorème suivant.

Théorème 3. *Soient $s \geqslant 0$ et $b_\alpha \in \dot{B}_\infty^{|\alpha|,\infty}$ pour $|\alpha| \leqslant [s]$; alors il y a équivalence entre les propriétés:*

1. *L'opérateur*

$$T = \sum_{|\alpha| \leq [s]} \pi_{b_\alpha}^\alpha$$

est continu sur $\dot{H}^s$;

2. *Pour tout α, $\pi_{b_\alpha}^\alpha$ est continu sur $\dot{H}^s$;*
3. *Pour tout α et pour tout $\beta \in \mathbb{N}^n$ tel que $|\alpha| = |\beta|$, on a $\partial^\beta b_\alpha \in \mathrm{BMO}_2^{s-|\alpha|,2}$.*

4. Conclusions

(*a*) Nous devons généraliser les résultats exposés aux espaces de Sobolev $\dot{H}_p^s$ ou au moins l'équivalence entre (1) et (2) du Théorème 3. Les principales techniques peuvent se baser, d'une part, sur les travaux de Frazier-Torres-Weiss [8], et sur les capacités-$\dot{H}_p^s$ d'autre part.

(*b*) Le cas des espaces de Besov paraît plus délicat; à notre avis, il faut d'abord utiliser la localisation de $\dot{B}_p^{s,p}$ pour $0 \leqslant s < n/p$ (voir [13]), et ensuite procéder par les capacités-$B_p^{s,p}$ où $B_p^{s,p}$ désigne l'espace de Besov inhomogène.

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Recibido: 10 de octubre de 1989.

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Revista Matemática Iberoamericana
Vol. 6, n.os 3 y 4, 1990

Vector-Valued Multipliers on Stratified Groups

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Dedicated to the memory of J. L. Rubio de Francia

Introduction

Let $\mathcal{L}$ be a left invariant sublaplacian on a stratified Lie group G and denote by $\{E(\lambda): \lambda \geqslant 0\}$ its spectral resolution. The «multiplier operator» $m(\mathcal{L})$ can be defined for any Borel measurable function m on $[0, +\infty)$ by the spectral theorem according to the prescription

$$m(\mathcal{L}) = \int_0^\infty m(\lambda)\, dE(\lambda).$$

More generally, for every $t > 0$ consider the operator $m(t\mathcal{L})$ defined by the formula

$$m(t\mathcal{L}) = \int_0^\infty m(t\lambda)\, dE(\lambda)$$

and the maximal operator $m_*(\mathcal{L})$ associated to the family of operators $\{m(t\mathcal{L})\}_{t>0}$ defined by

$$m_*(\mathcal{L})f(x) = \sup_{t>0} |m(t\mathcal{L})f(x)|, \tag{0.1}$$

for all f in $\mathcal{S}$ and $x \in G$.

A natural problem is to find conditions on the function m which ensure the boundedness of the multiplier operator $m(\mathcal{L})$ or the maximal operator $m_*(\mathcal{L})$ on various spaces of distributions on G. When G is the Heisenberg group a criterion for the boundedness of the multiplier operator on the spaces L^p, $1 < p < \infty$, was given by De Michele and Mauceri [8] and Mauceri [15]. Hulanicki and Stein proved a Marcinkiewicz-type multiplier theorem for Hardy spaces on any stratified group G ([11], p. 208). Their result was improved by De Michele and Mauceri [9], who proved that if the function m satisfies a fractional order smoothness condition of order $s > Q(1/p - 1/2) + 1$, where Q is the homogeneous dimension of the group, then the multiplier operator is bounded on the Hardy space H^p, $0 < p \leqslant 1$. Later Christ ([5]) proved that smoothness of order $s > Q/2$ yields boundedness on L^p for $1 < p < \infty$. In [15] Mauceri gave also a general condition for the boundedness of the maximal operator $m_*(\mathcal{L})$ on L^p, $1 < p < \infty$, and from H^1 to L^1. The condition involves the behaviour at infinity of the Mellin transform of the function m. He also gave an application to the almost everywhere convergence of the Riesz means for the eigenfunction expansions of the sublaplacian.

In the first part of this paper we sharpen the multiplier theorem given in [9] proving that if the function m satisfies a smoothness condition of order $s > Q(1/p - 1/2)$, $0 < p \leqslant 1$, then the multiplier operator $m(\mathcal{L})$ is bounded on H^p. Christ's result follows from this by duality and interpolation. As an application of this result we study the boundedness of the strongly singular multipliers $\psi(\mathcal{L})\mathcal{L}^{-\beta}\exp(i\mathcal{L}^\alpha)$, $\alpha > 0$, $\operatorname{Re}(\beta) \geqslant 0$, where ψ is a smooth «cutoff» function which vanishes in a neighborhood of the origin and is identically 1 in a neighborhood of infinity. These multipliers were investigated in the context of $\mathbb{R}^n$ by Miyachi ([17]).

In the second part of the paper we extend the multiplier theorem to a vector-valued setting. Namely we consider a Banach space $\mathfrak{X}$ of functions on $\mathbb{R}_+$ endowed with a dilation invariant norm $\|\bullet\|_{\mathfrak{X}}$. We shall wiew the family of operators $\{m(t\mathcal{L})\}_{t>0}$ as a «vector-valued multiplier» mapping a scalar valued function f to the $\mathfrak{X}$-valued function $m(\bullet\mathcal{L})f$. Our aim is to give conditions on m which imply that for every test function f on G the function $(t,x) \mapsto m(t\mathcal{L})f(x)$ satisfies an a priori estimate

$$\left(\int_G \|m(\bullet\mathcal{L})f(x)\|_{\mathfrak{X}}^p\,dx\right)^{1/p} \leqslant C\|f\|_{H^p} \tag{0.2}$$

for some p in (0, 1] or a corresponding estimate with $\|f\|_{H^p}$ replaced by $\|f\|_p$ if $1 < p < \infty$. Estimate (0.2) implies that the operator $f \mapsto m(\bullet\mathcal{L})f$ extends to a bounded linear operator from H^p to the Bochner-Lebesgue space $L^p(\mathfrak{X})$ of all $\mathfrak{X}$-valued p-integrable functions on G. We study this problem when $\mathfrak{X}$ is either a Besov space defined in terms of the multiplicative structure of $\mathbb{R}_+$ or the space $C_0(\mathbb{R}_+)$ of all continuous functions vanishing at infinity on $\mathbb{R}_+$. In the first case estimate (0.2) can be viewed as a regularity result for the «means» $m(t\mathcal{L})f$, $t > 0$, of the function f. Mixed norm estimates analogous to (0.2) were studied in connection with the problem of regularity of spherical means in [4], [20], [23], [19], [18], in the Euclidean case, and in [6], [16] in the context of compact Lie groups. If $\mathfrak{X} = C_0(\mathbb{R}_+)$ estimate (0.2) implies the $H^p - L^p$ boundedness of the maximal operator $m_*(\mathcal{L})$ investigated in [15]. Our result (Theorem 2.6 below) improves the results given there. Maximal multiplier theorems in the context of $\mathbb{R}^n$ where given by J. L. Rubio de Francia in [21]. Our Corollary 2.7 is a version for stratified groups of a result of his.

1. The Multiplier Theorem

Some notation about stratified groups and Hardy spaces would be in order. However, since the notation used in the literature is quite standard, to avoid wasting of space, we refer the reader to the monograph by Folland and Stein [11] for all unexplained terminology and notation. For a more concise exposition, the reader may also consult Section 1 of [9]. Let G be a stratified group. Denote by $\{\delta_t : t > 0\}$ a family of dilations of the Lie algebra g of G; following a common abuse of notation we shall also denote by $\{\delta_t : t > 0\}$ the induced family of dilations of G. Let $\{E(\lambda) : \lambda \geqslant 0\}$ be the spectral resolution of a left invariant sublaplacian $\mathcal{L}$ on G. Since the spectral measure of $\{0\}$ is zero we shall regard our spectral multipliers as functions defined on $\mathbb{R}_+$ rather than on $[0, +\infty)$. If m is a bounded Borel function on $\mathbb{R}_+$ the operator $m(\mathcal{L})$ is bounded on L^2, the space of square integrable functions with respect to Haar measure, and commutes with left translations. Thus, by the Schwartz kernel theorem, there exists a tempered distribution k on G such that $m(\mathcal{L})f = f * k$ for all functions in the Schwartz space $\mathcal{S}$. Moreover, for every $t > 0$ the distribution kernel of the operator $m(t\mathcal{L})$ is $k_{\sqrt{t}}$, where $k_{\sqrt{t}}$ is the distribution obtained by «dilating and normalising by $\sqrt{t}$ the distribution k», *i.e.*

$$\langle k_{\sqrt{t}}, f \rangle = \langle k, f \circ \delta_{\sqrt{t}} \rangle$$

for all f in $\mathcal{S}$ ([11], Lemma 6.29). For $s \geqslant 0$ and $1 \leqslant p, q \leqslant \infty$ let $\Lambda^s_{p,q}(\mathbb{R}_+)$ be the space of all functions m on $\mathbb{R}_+$, whose pull-back $m \circ \exp$ via the exponential map is in the Besov space $\Lambda^s_{p,q}(\mathbb{R})$. The norm of a function m in $\Lambda^s_{p,q}(\mathbb{R}_+)$ is the norm of $m \circ \exp$ in $\Lambda^s_{p,q}(\mathbb{R})$. In particular we shall denote by $H^s_2(\mathbb{R}_+)$ the Sobolev space $\Lambda^s_{2,2}(\mathbb{R}_+)$. Throughout this paper we shall denote by ϕ a function in $C^\infty_c(\mathbb{R}_+)$ supported in $(1/2, 2)$, such that $\sum_{j \in \mathbb{Z}} \phi(2^j \lambda) = 1$ for every $\lambda > 0$. The main result of this section is

Theorem 1.1. *Suppose that m is a function on* $\mathbb{R}_+$ *which satisfies*

$$\sup_{k\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H_2^s} < \infty \tag{1.1}$$

for some $s > Q(1/p - 1/2)$ *and* $0 < p \leqslant 1$. *Then the multiplier operator* $m(\mathcal{L})$ *extends to a bounded operator on* H^q, *if* $p \leqslant q \leqslant 1$, *on* L^q *if* $1 < q < \infty$, *and on* BMO.

The proof of Theorem 1.1 is merely a refinement of the arguments of [9]. We shall begin by establishing some weighted norm inequalities for the distribution kernel of the operator $m(\mathcal{L})$, when m is a function with compact support in $\mathbb{R}_+$. The key step is the following lemma, which is an improved version of Lemma 3.1 in [9].

Lemma 1.2. *Suppose that* $\alpha \geqslant 0$, $1 \leqslant p \leqslant 2$. *Let m be a function in* $H_2^s(\mathbb{R}_+)$ *supported in* $(1/2, 2)$, *where* $s > \alpha/p + Q(1/p - 1/2)$. *Let k be the distribution kernel of* $m(\mathcal{L})$. *Then*

$$\int_G |x|^\alpha |Y^I k(x)|^p \, dx \leqslant C\|m\|_{H_2^s} \tag{1.2}$$

for every multiindex I.

PROOF. We first prove (1.2) in the case $p = 2$. A slight refinement of the argument of Lemma 3.1 in [9] shows that (1.2) holds for $s > (\alpha + 1)/2$ (we only need to recall that the Fourier coefficients of a function F in $H_2^s(\mathbb{T})$ satisfy the estimate $\sum_n (1 + |n|)^{s-\epsilon-1/2}|\hat{F}(n)| \leqslant C_\epsilon\|F\|_{H_2^s}$ for every $\epsilon > 0$). Let ψ be a function in $C_c^\infty(\mathbb{R}_+)$ supported in $(1/2, 2)$ such that $\psi(\lambda) = 1$ for every λ in the support of m. Given a bounded measurable function f on $\mathbb{R}_+$, denote by k_f the distribution kernel of the operator $\psi(\mathcal{L})f(\mathcal{L})$. The previous argument shows that $f \mapsto k_f$ is a bounded linear map from $H_2^s(\mathbb{R}_+)$ into $L^2(G, |x|^\alpha dx)$ for every s, α such that $\alpha \geqslant 0$ and $s > (\alpha + 1)/2$. On the other hand, by the Plancherel formula ([9], formula (3.7)) the map $f \mapsto k_f$, $f \in L^\infty \cap L^2(\mathbb{R}_+)$, extends to a bounded linear map from $L^2(\mathbb{R}_+)$ into L^2. Now fix β, $r > 0$ such that $r > \beta/2$. By interpolating between the $H_2^s(\mathbb{R}_+) - L^2(G, |x|^\alpha dx)$ estimate, which holds for $s > (\alpha + 1)/2$, and the $L^2(\mathbb{R}_+) - L^2$ estimate, and letting α tend to infinity, we get that $f \mapsto k_f$ maps $H_2^s(\mathbb{R}_+)$ into $L^2(G, |x|^\beta dx)$ continuously. This proves estimate (1.2) for $p = 2$ and $I = 0$. The estimate for the other values of p can be obtained by Hölder's inequality. The estimate for $I \neq 0$ follows from that for $I = 0$ as in the proof of Lemma 3.1 in [9]. This completes the proof.

Let k be a function in $C^\infty(G)$. For every positive integer N, denote by $P_x^{(N)}(k; \cdot)$ the right Taylor polynomial of k at x of homogeneous degree N ([11], p. 26). Set

$$\Delta^{(N)}k(x,y) = k(x,y) - P_x^{(N)}(k;y).$$

For $r,\ R > 0$ define

$$\omega_{N,r}(k;R) = R^{-Q-2r}\int_{|y|<R} dy \int_{|x|\geq 2R} dx\, |\Delta^{(N)}k(x,y)|^2 |x|^{Q+2r}.$$

By using Lemma 1.2 instead of Lemma 3.1 in [9], we obtain the following refinement of Lemma 3.2 in [9].

Lemma 1.3. *Let m be a function in $H_2^s(\mathbb{R}_+)$ supported in $(1/2, 2)$. Denote by k the distribution kernel of $m(\mathcal{L})$. Let r be a positive number, N a non-negative integer such that $N \leqslant r < N+1$. If $s > Q/2 + r$, then there exist positive constants δ, η and C such that*

$$\omega_{N,r}(k;R) \leqslant C\|m\|_{H_2^s}^2 \min\{R^\delta, R^{-\eta}\}. \tag{1.3}$$

PROOF OF THEOREM 1.1. Argue as in the proof of Theorem 1.1 in [9], using lemmata 1.2 and 1.3 instead of lemmata 3.1 and 3.2 therein to get the result for $0 < q \leqslant 1$. The case $q > 1$ follows by interpolation between the H^1 and the BMO estimates.

As an application of Theorem 1.1 we discuss the boundedness of the two-parameter family of operators $m_{\alpha,\beta}(\mathcal{L}) = \psi(\mathcal{L})\mathcal{L}^{-\beta}\exp(i\mathcal{L}^\alpha)$, $\alpha > 0$, $\mathrm{Re}(\beta) \geqslant 0$, where ψ is a smooth «cutoff» function which vanishes in a neighborhood of the origin and is identically 1 in a neighborhood of infinity. In the Euclidean context sharp H^p boundedness results have been given by Miyachi ([17]) in the case $\alpha \neq 1$ and by Peral ([20]) in the case $\alpha = 1$. See also [10].

Corollary 1.4. *Suppose that $\alpha > 0$, $\mathrm{Re}(\beta) \geqslant 0$, and $\mathrm{Re}(\beta)/\alpha > Q|1/p - 1/2|$. Then the operator $m_{\alpha,\beta}(\mathcal{L})$ is bounded on H^p if $0 < p \leqslant 1$, on L^p if $1 < p < \infty$ and on* BMO *if $p = \infty$.*

PROOF. A direct calculation shows that for every nonnegative integer n and large R

$$\|\phi(\cdot)m_{\alpha,\beta}(R\cdot)\|_{H_2^n} \leqslant C_{\alpha,\beta}R^{-\mathrm{Re}(\beta)+n\alpha}$$

(notice that the left hand side vanishes if R is small). By interpolation, a similar estimate holds with a nonnegative s in place of n; hence the result for $0 < p \leqslant 1$ and BMO follows directly from Theorem 1.1. The result for $1 < p < \infty$ follows by applying Stein's complex interpolation theorem to the analytic family of operators $\{m_{\alpha,\beta} : 0 \leqslant \mathrm{Re}(\beta) < Q/2 + \epsilon\}$, $\epsilon > 0$.

2. Vector-Valued Multipliers

If A is a Banach space we denote by $L^p(A)$, $0 < p \leqslant \infty$, the Bochner-Lebesgue space of all strongly measurable A-valued functions F on G, for which

$$\|F\|_{L^p(A)} = \left(\int_G \|F(x)\|_A^p \, dx \right)^{1/p}$$

(with the usual modification when $p = \infty$). If $0 < p \leqslant 1$ we denote by $H^p(A)$ the atomic Hardy space defined in terms of A-valued atoms (A-valued atoms are defined as in the scalar case except that absolute values are to be replaced by the norm in A). For a locally integrable A-valued function F we define the maximal function

$$F^{\#}(x) = \sup_{x \in B} \frac{1}{|B|} \int_B \|F(y) - F_B\|_A \, dy,$$

where B stands for an arbitrary ball in G and F_B is the average of F over B. Then $\mathrm{BMO}(A)$ is the space $\{F \in L^1_{\mathrm{loc}} : \|F\|_{\mathrm{BMO}(A)} = \|F^{\#}\|_\infty < \infty\}$. We recall that $H^1(B)^*$ imbeds isometrically into $\mathrm{BMO}(B^*)$ for every Banach space B. The imbedding is surjective if B is reflexive (more generally if B^* has the Radon-Nikodym property ([2], [3]).

Let $\mathcal{H}$ be a separable Hilbert space. By a result of Marcinkiewicz and Zygmund ([13]) if T is a bounded linear operator on L^2 the operator $T \otimes Id$ on $L^2 \otimes \mathcal{H}$ has a bounded extension $T_{\mathcal{H}}$ to $L^2(\mathcal{H})$. In particular we shall denote by $\{dE_{\mathcal{H}}(\lambda)\}$ the $L^2(\mathcal{H})$ projection-valued measure on $\mathbb{R}_+$ associated to the resolution of the identity $\{dE(\lambda)\}$ of the sublaplacian $\mathcal{L}$. If $\mathcal{K}$ is another Hilbert space we shall denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, endowed with the operator norm $\|\cdot\|_{\mathcal{H},\mathcal{K}}$. Notice that for every bounded continuous function M on $\mathbb{R}_+$, with values in $\mathcal{B}(\mathcal{H}, \mathcal{K})$, $ME_{\mathcal{H}}(\Omega) = E_{\mathcal{K}}(\Omega)M$ for each Borel subset Ω of $\mathbb{R}_+$. Thus for every function F in $L^2(\mathcal{H})$ the improper Riemann integral

$$M(\mathcal{L})F = \int_0^\infty M(\lambda) \, dE_{\mathcal{H}}(\lambda) F$$

converges in $L^2(\mathcal{K})$ and defines a bounded linear operator $M(\mathcal{L})$ from $L^2(\mathcal{H})$ into $L^2(\mathcal{K})$. Moreover $\|M(\mathcal{L})\|_{L^2(\mathcal{H}), L^2(\mathcal{K})} = \sup_{\lambda > 0} \|M(\lambda)\|_{\mathcal{H},\mathcal{K}}$.

Definition 2.1. *Let A and B be two Banach spaces such that $A \cap \mathcal{H}$ is dense in A. If the operator $M(\mathcal{L})$ extends to a bounded operator from the Hardy space $H^p(A)$ to $L^p(B)$ for some $p \in (0, 1]$ we say that M is a vector-valued multiplier of $H^p(A)$ into $L^p(B)$. In this definition the pair $(H^p(A), L^p(B))$ should be replaced by $(L^p(A), L^p(B))$ if $1 < p < \infty$ and by $(L^\infty(A), \mathrm{BMO}(B))$ if $p = \infty$.*

As in the scalar case, given $p \in (0, 1]$ and spaces A, B one wishes to find conditions on the operator-valued function M that guarantee that M is a multiplier of $H^p(A)$ into $L^p(B)$. In this paper we shall consider this problem in the following context. Let $\mathfrak{X}$ denote the Banach space $\Lambda^s_{2,q}(\mathbb{R}_+)$, $s > 0$, $1 \leqslant q \leqslant \infty$, of all measurable functions f on $\mathbb{R}_+$ whose pull back $f \circ \exp$ via the exponential map is in the usual Besov space $\Lambda^s_{2,q}(\mathbb{R})$ ([1]). Thus

$$\|f\|_{\mathfrak{X}} = \|f \circ \exp\|_{\Lambda^s_{2,q}(\mathbb{R})}$$

is a dilation invariant norm on $\mathfrak{X}$. If $m \in \mathfrak{X}$ and $\lambda \in \mathbb{R}_+$ we shall denote by $m(\cdot\lambda)$ the λ-dilate of the function m, namely the function $t \mapsto m(t\lambda)$. Thus the map $\lambda \mapsto m(\cdot\lambda)$ is a bounded, continuous $\mathfrak{X}$-valued function on $\mathbb{R}_+$, provided that $q < \infty$. We shall also view it as a $\mathcal{B}(\mathbb{C}, \mathfrak{X})$ and a $\mathcal{B}(\mathfrak{X}^*, \mathbb{C})$-valued function, via the natural isometric identifications of $\mathfrak{X}$ with $\mathcal{B}(\mathbb{C}, \mathfrak{X})$ and with a subspace of $\mathcal{B}(\mathfrak{X}^*, \mathbb{C})$. Thus if $\mathfrak{X}$ is the Hilbert space $H^s_2(\mathbb{R}_+) = \Lambda^s_{2,2}(\mathbb{R}_+)$ we shall denote by $m(\cdot\mathcal{L})$ both the operator of L^2 into $L^2(\mathfrak{X})$ defined by

$$m(\cdot\mathcal{L})f = \int_0^\infty m(\cdot\lambda)\, dE(\lambda) f, \qquad f \in L^2,$$

and the operator of $L^2(\mathfrak{X}^*)$ into L^2 defined by

$$\langle m(\cdot\mathcal{L}), F\rangle = \int_0^\infty \langle m(\cdot\mathcal{L}), dE_{\mathfrak{X}}(\lambda)F\rangle, \qquad F \in L^2(\mathfrak{X}^*).$$

Our multiplier theorem is then

Theorem 2.1. *Suppose that* $0 < p \leqslant \infty$, $\beta > 0$ *and* $s > Q\left|\frac{1}{p} - \frac{1}{2}\right| + \beta$. *If*

$$\sum_{k \in \mathbb{Z}} \|\phi(\cdot) m(2^k \cdot)\|_{H^s_2} < \infty$$

then $m(\cdot\mathcal{L})$ *extends to a bounded operator*

(i) *from* H^p *to* $L^p(\Lambda^\beta_{2,1}(\mathbb{R}_+))$ *if* $0 < p \leqslant 1$;
(ii) *from* L^p *to* $L^p(\Lambda^\beta_{2,1}(\mathbb{R}_+))$ *if* $1 < p < \infty$;
(iii) *from* L^∞ *to* $\mathrm{BMO}(\Lambda^\beta_{2,1}(\mathbb{R}_+))$ *if* $p = \infty$.

If $p = 2$ one can actually prove a sharper result. Indeed one has

Lemma 2.2. *If* $m \in \Lambda^s_{2,q}(\mathbb{R}_+)$, $s \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$, *then* $m(\cdot\mathcal{L})$ *extends to a bounded operator from* L^2 *to* $L^2(\Lambda^s_{2,q}(\mathbb{R}_+))$ *and from* $L^2(\Lambda^{-s}_{2,q}(\mathbb{R}_+))$ *to* L^2, *whose norm does not exceed* $C\|m\|_{\Lambda^s_{2,q}}$.

PROOF. Assume first that $q = 2$. Since $\lambda \to m(\cdot\lambda)$ is a bounded, continuous $\Lambda^s_{2,2}(\mathbb{R}_+)$-valued function, $m(\cdot\mathcal{L})$ is a bounded operator from L^2 to $L^2(\Lambda^s_{2,2}(\mathbb{R}_+))$ and from $L^2(\Lambda^{-s}_{2,2}(\mathbb{R}_+))$ to L^2, whose norm is $\sup_\lambda \|m(\cdot\lambda)\|_{\Lambda^s_{2,2}} = \|m\|_{\Lambda^s_{2,2}}$. The result for $q \neq 2$ follows by applying the $[\ ,\ ]_{\theta,q}$ interpolation method to the bilinear maps $(m,f) \mapsto m(\cdot\mathcal{L})f$ from $\Lambda^s_{2,2} \times L^2$ into $L^2(\Lambda^s_{2,2}(\mathbb{R}_+))$ and $(m,F) \mapsto \langle m(\cdot\mathcal{L}), F\rangle$ from $\Lambda^s_{2,2}(\mathbb{R}_+) \times L^2(\Lambda^{-s}_{2,2}(\mathbb{R}_+))$ into L^2 for different values of s ([1]).

We turn now to a description of the $H^p - L^p$ results. We begin by stating a result for vector-valued singular integrals. Let A and B be two Banach spaces. We consider kernels K which are strongly measurable functions defined on G and with values in the space $\mathcal{B}(A,B)$ of all bounded linear operators from A to B. We suppose that $\|K\|_{A,B}$ is locally integrable away from the origin. On $G \times G$ we shall consider the measure $d\mu_r(x,y) = |x|^{Q+2r}\,dx\,dy$, $r > 0$, and the sets $S_R = \{(x,y): |x| \geqslant 2R,\ |y| < R\}$, $R > 0$.

Definition 2.2. *If N is a nonnegative integer and $r > 0$ we say that K is a kernel of type $\mathfrak{N}_{N,r}(A,B)$ if there exists a polynomial $P_x(y) = \sum_{|I| \leq N} a_I(x) y^I$, $x, y \in G$, of homogeneous degree N, whose coefficients a_I are strongly measurable $\mathcal{B}(A,B)$-valued functions on G such that*

$$\mathfrak{N}_{N,r}(K) = \sup_{R>0}\left(R^{-Q-2r}\iint_{S_R} \|K(xy) - P_x(y)\|^2_{A,B}\, d\mu_r(x,y)\right)^{1/2} < \infty.$$

Definition 2.3. *A linear operator T mapping A-valued functions into B-valued functions is called a singular integral operator of type $\mathfrak{N}_{N,r}(A,B)$ if the following two conditions are satisfied:*

(i) *T is a bounded operator from $L^2(A)$ to $L^2(B)$;*
(ii) *there exists a kernel K of type $\mathfrak{N}_{N,r}(A,B)$ such that*

$$TF(x) = \int_G K(y^{-1})F(xy)\,dy$$

for every F in $L^2(A)$ with compact support and for almost every x in the complement of the support of F.

Theorem 2.3. *Suppose that $0 < p \leqslant 1$. If T is a singular integral operator of type $\mathfrak{N}_{N,r}(A,B)$ for some noninteger $r > Q(1/p - 1)$ and for $N = [r]$ then T can be extended to a bounded operator from $H^p(A)$ to $L^p(B)$. Moreover*

$$\|T\|_{H^p(A),L^p(B)} \leqslant C(\|T\|_{L^2(A),L^2(B)} + \mathfrak{N}_{N,r}(K)).$$

Remarks. The proof of Theorem 2.3 is a simple adaptation to the vector-valued case of proof of Theorem 2.1 in [9]. Related results on operator-valued singular integrals can be found in [22].

We shall apply Theorem 2.3 in the following context. Let $\mathfrak{X}$ denote the Sobolev space $H_2^s(\mathbb{R}_+)$, $s \geqslant 0$. If m is a function in $\mathfrak{X}$ by Lemma 2.2 the operator $m(\cdot\mathcal{L})$ is bounded from L^2 to $L^2(\mathfrak{X})$ and from $L^2(\mathfrak{X}^*)$ to L^2. Moreover it commutes with left translations. Thus, by the Schwartz kernel theorem, there exists a $\mathfrak{X}$-valued distribution K on G (*i.e.* a bounded linear map from $\mathcal{S}$ into $\mathfrak{X}$) such that $m(\cdot\mathcal{L})f = f * K$ for all f in $\mathcal{S}$. We shall show, essentially, that K is a locally integrable function away from the origin and satisfies conditions $\mathfrak{N}_{N,r}(\mathbb{C}, \mathcal{Y})$ and $\mathfrak{N}_{N,r}(\mathcal{Y}^*, \mathbb{C})$ for suitable N and r and for certain Besov spaces $\mathcal{Y}$ of functions on $\mathbb{R}_+$.

Remarks. If the function M is bounded on $\mathbb{R}_+$ and k is the distribution kernel of the operator $m(\mathcal{L})$ then the $\mathfrak{X}$-valued distribution kernel K of the operator $m(\cdot\mathcal{L})$ is the continuous linear map $f \mapsto \langle k_{\sqrt{t}}, f\rangle$ from $\mathcal{S}$ into $\mathfrak{X}$, because $k_{\sqrt{t}}$ is the kernel of the operator $m(t\mathcal{L})$ for every $t > 0$.

If $m \in C_c^\infty(\mathbb{R}_+)$ the kernel k is in $\mathcal{S}$ ([15], Proposition 2.7). Thus $K = k_{\sqrt{\cdot}}$ is a smooth $\mathfrak{X}$-valued function away from the origin. As in Section 1 we denote by $P_x^{(N)}$ the right Taylor polynomial of K at x of homogeneous degree N. Notice that the coefficients of $P_x^{(N)}$ are smooth $\mathfrak{X}$-valued functions of x on $G\setminus\{0\}$. We also denote by $\Delta^{(N)}K(x,y)$ the difference $K(xy) - P_x^{(N)}(y)$, for x, y in G, $x \neq 0$. For every $R > 0$ and x in $G\setminus\{0\}$ let $K_R(x)$ denote the R-dilate of $K(x)$ (as a distribution on $\mathbb{R}_+$). Then

$$\Delta^{(N)}K_R(x,y) = R^{-Q}(\Delta^{(N)}K)(R^{-1}x, R^{-1}y)$$

and, by using the invariance of the norm in $\mathfrak{X}$, it is an easy matter to show that

$$R^{-Q-2r}\iint_{S_R} \|\Delta^{(N)}K(x,y)\|_{\mathfrak{X}}^2\, d\mu_r(x,y)$$

is independent of R. Thus

$$\mathfrak{N}_{N,r}(K)^2 = \iint_{S_1} \|\Delta^{(N)}K(x,y)\|_{\mathfrak{X}}^2\, d\mu_r(x,y). \tag{2.2}$$

Lemma 2.4. *Suppose that $0 < p \leqslant 1$. Let $\mathcal{Y}$ denote the Besov space $\Lambda_{2,q}^\beta(\mathbb{R}_+)$, $\beta \geqslant 0$, $1 \leqslant q \leqslant \infty$. If $m \in H_2^s(\mathbb{R}_+)$ is supported in $(1/2, 2)$ and $s > Q(1/p - 1/2) + \beta$ then $m(\cdot\mathcal{L})$ is a bounded operator from H^p to $L^p(\mathcal{Y})$ and from $H^p(\mathcal{Y}^*)$ to L^p, whose norm does not exceed $C\,\|m\|_{H_2^s}$.*

PROOF. Assume first than $m \in C_c^\infty((1/2, 2))$ and $\mathcal{Y} = \Lambda_{2,2}^n(\mathbb{R}_+)$, where n is a nonnegative integer. We shall prove that $m(\cdot\mathcal{L})$ is a singular integral operator of types $\mathfrak{N}_{N,r}(\mathbb{C}, \mathcal{Y})$ and $\mathfrak{N}_{N,r}(\mathcal{Y}^*, \mathbb{C})$ for some $r > Q(1/p - 1)$ and for $N = [r]$.

Let k and K be the kernels of $m(\mathcal{L})$ and $m(\bullet\mathcal{L})$, respectively. By the previous remarks $k \in \mathcal{S}$, $K \in C^\infty(G\setminus\{0\})$ and, for every $x \neq 0$, $K(x)$ is the function $t \to k_{\sqrt{t}}(x)$. Therefore $K \in L^1_{\text{loc}}(\mathcal{Y})$ away from the origin. Since by Lemma 2.2 $m(\bullet\mathcal{L})$ is a bounded operator from L^2 to $L^2(\mathcal{Y})$ and from $L^2(\mathcal{Y}^*)$ to L^2 whose norm is $\|m\|_{\mathcal{Y}} \leqslant C\|m\|_{H^s_2}$, by Theorem 2.3 and (2.2) we only need to show that

$$\mathfrak{N}_{N,r}(K)^2 = \iint_{S_1} \|\Delta^{(N)}K(x,y)\|^2_{\mathcal{Y}}\, d\mu_r(x,y) \leqslant C\|m\|^2_{H^s_2}.$$

Let ρ denote the dilation invariant differential operator $t(d/dt)$ on $\mathbb{R}_+$. Then

$$\|\Delta^{(N)}K(x,y)\|^2_{\mathcal{Y}} = \sum_{j=0}^{n} \int_0^\infty |\rho^j \Delta^{(N)} k_{\sqrt{t}}(x,y)|^2 \frac{dt}{t}.$$

By the spectral theorem

$$\rho^j f * k_{\sqrt{t}} = \int_0^\infty \rho^j m(t\lambda)\, dE(\lambda) f.$$

Let $k^{(j)}$ be the kernel of the operator $\rho^j m(\mathcal{L})$. Then, by a straightforward, albeit tedious, computation

$$\rho^j \Delta^{(N)} k_{\sqrt{t}}(x,y) = \Delta^{(N)} k^{(j)}_{\sqrt{t}}(x,y) = t^{-Q/2}\Delta^{(N)}k^{(j)}(t^{-1/2}x, t^{-1/2}y).$$

Therefore, interchanging the order of integration, performing the change of variables $(t^{-1}x, t^{-1}y) = (\xi, \eta)$ and applying Lemma 1.3, we get that

$$\begin{aligned}\iint_{S_1} \|\Delta^{(N)}K(x,y)\|^2_{\mathcal{Y}}\, d\mu_r(x,y) &= \sum_{j=0}^{n} \iint_{S_1} \int_0^\infty |\rho^j \Delta^{(N)} k_{\sqrt{t}}(x,y)|^2\, d\mu_r(x,y) \frac{dt}{t} \\ &\leqslant C \sum_{j=0}^{n} \int_0^\infty \omega_{N,r}(k^{(j)}, t^{-1}) \frac{dt}{t} \\ &\leqslant C\|m\|^2_{\Lambda^\sigma_{2,2}}\end{aligned}$$

provided that $\sigma \geqslant Q/2 + r + n$. Choose $r > Q(1/p - 1)$ and σ such that $s > \sigma > Q/2 + r + n$. Since $H^s_2(\mathbb{R}_+)$ imbeds continuously in $\Lambda^\sigma_{2,2}(\mathbb{R}_+)$ we have proved that $\mathfrak{N}_{N,r}(K) \leqslant C\|m\|_{H^s_2}$. Thus, by Theorem 2.3, the norm of $m(\bullet\mathcal{L})$, qua operator from H^p to $L^p(\Lambda^n_{2,2}(\mathbb{R}_+))$ and from $H^p(\Lambda^{-n}_{2,2}(\mathbb{R}_+))$ to L^p, is bounded by $C\|m\|_{H^s_2}$ provided that $s > Q(1/p - 1/2) + n$. By interpolation, via the $[\ ,\ ]_{\theta,q}$, $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, method, we obtain that if $\sigma > Q(1/p - 1/2) + \beta$ and $m \in \Lambda^\sigma_{2,q}(\mathbb{R}_+)$ then the norm of $m(\bullet\mathcal{L})$, qua operator from H^p to $L^p(\Lambda^\beta_{2,q}(\mathbb{R}_+))$ and from $H^p(\Lambda^{-\beta}_{2,q}(\mathbb{R}_+))$ to L^p, is bounded by $C\|m\|_{\Lambda^\sigma_{2,q}}$. Since $H^s_2(\mathbb{R}_+)$ imbeds continuously in $\Lambda^\sigma_{2,q}(\mathbb{R}_+)$ for $s > \sigma$ the lemma is proved for m in $C^\infty_c((1/2, 2))$. In general, if m is a function in $H^s_2(\mathbb{R}_+)$ supported in $(1/2, 2)$, we pick a sequence $\{m_n\}$ of functions in $C^\infty_c((1/2, 2))$ which converges to m in $H^s_2(\mathbb{R}_+)$. Then $m_n(\bullet\mathcal{L})$ converges to $m(\bullet\mathcal{L})$ in the strong operator topology, qua operator from L^2 to $L^2(\mathcal{Y})$ and from $L^2(\mathcal{Y}^*)$ to L^2. Since $\{m_n(\cdot\mathcal{L})\}$ is also a Cauchy sequence in $\mathcal{B}(H^p, L^p(\mathcal{Y}))$ and in $\mathcal{B}(H^p(\mathcal{Y}^*), L^p)$, the lemma is proved.

The following corollary extends the result of Lemma 2.4 to the range $0 < p \leqslant \infty$.

Corollary 2.5. *Let $\mathcal{Y}$ denote the Besov space $\Lambda^{\beta}_{2,q}(\mathbb{R}_+)$, $\beta \geqslant 0$, $1 \leqslant q \leqslant \infty$. If m is a function in $H^s_2(\mathbb{R}_+)$ supported in $(1/2, 2)$ and $s > Q|1/p - 1/2| + \beta$, then the operator $m(\cdot\mathcal{L})$ is bounded from L^p to $L^p(\mathcal{Y})$, if $1 < p < \infty$, and from L^∞ to $BMO(\mathcal{Y})$, if $p = \infty$.*

PROOF. Assume first that $1 < q < \infty$, so that $\mathcal{Y}$ is reflexive. If $s > Q/2 + \beta$ the operator $m(\cdot\mathcal{L})$ is bounded from $H^1(\mathcal{Y}^*)$ to L^1 by Lemma 2.4. Thus its transpose is a bounded operator from L^∞ to BMO $(\mathcal{Y})$ with the same norm. Since the heat kernel ω satisfies $\omega(x) = \omega(x^{-1})$ for every x in G, the same property holds for the kernel of the operator $m(\cdot\mathcal{L})$. Thus the transpose of $m(\cdot\mathcal{L})$ is still $m(\cdot\mathcal{L})$. The result for $1 < p < \infty$ *follows by interpolation between the* $H^1 - L^1$ and the $L^2 - L^2$ estimates (Lemma 2.2) if $p < 2$ and the $L^2 - L^2$ and the L^∞ − BMO estimates if $p > 2$. To remove the restriction $q > 1$ choose $s_0,\ s_1 > Q|1/p - 1/2| + \beta$ and $\theta \in (0,1)$ such that $s = (1-\theta)s_0 + \theta s_1$ and apply the interpolation functor $[\ ,\ ]_{\theta,1}$ to the bilinear map $(m, f) \to m(\cdot\mathcal{L})f$ from $H^{s_i}_2(\mathbb{R}_+) \times L^p$ to $L^p(\Lambda^{\beta}_{2,\infty}(\mathbb{R}_+))$, $i = 0, 1$. Since $\Lambda^{\beta}_{2,1}(\mathbb{R}_+)$ imbeds continuously into $\Lambda^{\beta}_{2,\infty}(\mathbb{R}_+)$, the result holds for $q = \infty$.

PROOF OF THEOREM 2.1. We prove (i). The proofs of (ii) and (iii) are similar. Let m be a function on $\mathbb{R}_+$ satisfying (2.1) and ϕ be as in Section 1. Set $m_j(\lambda) = \phi(2^{-j}\lambda)m(\lambda)$ and $\mu_j(\lambda) = m_j(2^j\lambda)$. Notice that $\|m_j\|_{H^s_2} = \|\mu_j\|_{H^s_2}$ and that each μ_j is supported in $(1/2, 2)$. Hence the norm of $\mu_j(\cdot\mathcal{L})$ qua operator from H^p to $L^p(\Lambda^{\beta}_{2,1}(\mathbb{R}_+))$, $0 < p \leqslant 1$, is dominated by $C\|\mu_j\|_{H^s_2}$ (C independent of j), by Corollary 2.5. Also, the dilation invariance of the $\Lambda^{\beta}_{2,1}(\mathbb{R}_+)$ norm implies that $m_j(\cdot\mathcal{L})$ has the same norm as $\mu_j(\cdot\mathcal{L})$. Since m decomposes into the sum $\sum_{j\in\mathbb{Z}} m_j$, the norm of $m(\cdot\mathcal{L})$ can be estimated by $\sum_{j\in\mathbb{Z}} \|m_j\|_{H^s_2}$, which is convergent by (2.1). The proof of (i) is complete.

We now discuss some applications of the above results to maximal operators. Notice that the boundedness of the maximal operator $m_*(\mathcal{L})$ (see (0.1) for the definition) from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 \leqslant p \leqslant \infty$ is equivalent to the boundedness of the vector-valued multiplier $m(\cdot\mathcal{L})$ from H^p to $L^p(l^\infty(\mathbb{R}_+))$ if $0 < p \leqslant 1$ and from L^p to $L^p(l^\infty(\mathbb{R}_+))$ if $1 < p \leqslant \infty$. Our main result concerning maximal operators is the following.

Theorem 2.6. *Let m be a function on $\mathbb{R}_+$ such that*

$$\sum_{k\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H_2^s} = D < \infty. \tag{2.2}$$

Then

(i) *if $s > Q(1/p - 1/2) + 1/2$, $m(\cdot\mathcal{L})$ extends to a bounded operator from H^p to $L^p(C_0(\mathbb{R}_+))$, if $0 < p \leqslant 1$, and from L^p to $L^p(C_0(\mathbb{R}_+))$, if $1 < p \leqslant 2$, with norm not exceeding CD;*

(ii) *if $2 \leqslant p \leqslant \infty$ and $s > (Q-1)(1/2 - 1/p) + 1/2$, $m(\cdot\mathcal{L})$ extends to a bounded operator from L^p to $L^p(l^\infty(\mathbb{R}_+))$, with norm not exceeding CD.*

PROOF. Since $\Lambda^{\beta}_{2,1}(\mathbb{R}_+)$ imbeds continuously into $C_0(\mathbb{R}_+)$ by Bernstein's Theorem ([12], Theorem 1), (i) is an immediate consequence of Theorem 2.1.

To prove (ii), assume first that $s > Q/2$. Then the distribution kernel of the operator $m(\mathcal{L})$ is in L^1, by Lemma 1.2. Therefore the associated maximal operator $m_*(\mathcal{L})$ is bounded on L^∞, *i.e.* $m(\cdot\mathcal{L})$ is bounded from L^∞ to $L^\infty(l^\infty(\mathbb{R}_+))$. Assume now that $s = 1/2$. By Lemma 2.2 $m(\cdot\mathcal{L})$ extends to a bounded operator from L^2 to $L^2(\Lambda^{1/2}_{2,1}(\mathbb{R}_+))$, hence from L^2 to $L^2(C_0(\mathbb{R}_+))$. An easy interpolation argument concludes the proof.

Remark. In the Euclidean setting Dappa and Trebels ([7]) proved that the maximal operator $m_*(\mathcal{L})$ is bounded on L^p, $1 < p \leqslant 2$, and of weak type $1-1$, provided that

$$\sup_{j\in\mathbb{Z}} \|\phi(\cdot)m(2^k\cdot)\|_{H_2^s} < \infty$$

for some $s > (Q+1)/2$.

We now turn to some applications of Theorem 2.1 and Theorem 2.6.

Corollary 2.7. *Suppose that n is an integer larger than $Q/2$ and m is a function in $C^{(n)}(\mathbb{R}_+)$ which vanishes if $\lambda \leqslant 1$ and satisfies the estimate*

$$|m^{(j)}(\lambda)| \leqslant C\lambda^{-a}. \qquad j = 0, 1, \ldots, n$$

for some $a > 1/2$. Then the maximal operator $m_*(\mathcal{L})$ extends to a bounded operator from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 < p \leqslant \infty$ provided that

$$\frac{1}{r_a} = \frac{Q-2a}{2(Q-1)} < \frac{1}{p} < \frac{Q+2a-1}{2a} = \frac{1}{q_a}$$

(it must be understood that $r_a = \infty$ if $a \geqslant Q/2$).

PROOF. We retain the notation used in the proof of Theorem 2.1. An easy computation shows that

$$\|\mu_j\|_{H_2^0} \leqslant C2^{-ja} \quad \text{and} \quad \|\mu_j\|_{H_2^n} \leqslant C2^{j(n-a)}.$$

The desired result is an immediate consequence of an interpolation argument, Theorem 2.1 and Theorem 2.6.

Remark. Related results have been obtained in the Euclidean setting by J. L. Rubio de Francia ([21], Theorem B).

We now discuss the boundedness of the maximal operator associated to the Riesz means of $\mathcal{L}$. We improve some results obtained by the first author in [15].

For every $z \in \mathbb{C}$, with $\operatorname{Re}(z) > 0$, set

$$m_z(\lambda) = (1-\lambda)_+^z, \qquad \lambda > 0.$$

Corollary 2.8. *Let m_z be as above. Then*

(i) *if* $\operatorname{Re}(z) > Q(1/p - 1/2)$, $(m_z)_*(\mathcal{L})$ *extends to a bounded operator from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 < p \leqslant 2$;*
(ii) *if* $\operatorname{Re}(z) > (Q-1)(1/2 - 1/p)$ *and* $2 \leqslant p \leqslant \infty$, $(m_z)_*(\mathcal{L})$ *extends to a bounded operator on L^p.*

PROOF. Let ψ a smooth «cutoff» fuction on $\mathbb{R}_+$ which equals one if $\lambda < 1/2$ and vanishes if $\lambda > 3/4$. Write $m_z = m_z^1 + m_z^2$ where $m_z^1 = \psi m_z$. Since m_z^1 is the restriction to $\mathbb{R}_+$ of a function in the Schwartz class of $\mathbb{R}$, the distribution kernel of the operator $m_z^1(\mathcal{L})$ is in the Schwartz class on the group G ([15], Proposition 2.7); hence the associated maximal operator $(m_z^1)_*(\mathcal{L})$ is bounded from H^p to L^p if $0 < p \leqslant 1$ and on L^p if $1 < p \leqslant \infty$.

Notice that m_z^2 is a compactly supported function in $\Lambda_{2,\infty}^{\operatorname{Re}(z)+1/2}(\mathbb{R}_+)$ (hence in $H_2^{\operatorname{Re}(z)+1/2+\epsilon}(\mathbb{R}_+)$ for every $\epsilon > 0$). The desired result follows at once from Theorem 2.6.

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Recibido: 13 de julio de 1990.

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* Work partially supported by Ministero della Pubblica Istruzione, fondi 40 % e 60 %.

Revista Matemática Iberoamericana
Vol. 6, n.os 3 y 4, 1990

Unique Continuation for $|\Delta u| \leqslant V|\nabla u|$ and Related Problems

Thomas H. Wolff

Introduction

Much of this paper will be concerned with the proof of the following

Theorem 1. *Suppose $d \geqslant 3$, $r = \max\{d, (3d-4)/2\}$. If $V \in L^r_{\text{loc}}(\mathbb{R}^d)$, then the differential inequality $|\Delta u| \leqslant V|\nabla u|$ has the strong unique continuation property in the following sense: If u belongs to the Sobolev space $W^{2,p}_{\text{loc}}$ and if $|\Delta u| \leqslant V|\nabla u|$ and*

$$\lim_{R \to 0} R^{-N} \int_{|x|<R} |\nabla u|^{p'} = 0$$

for all N then u is constant.

Here we are using the notational convention

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{p} - \frac{1}{p'} = \frac{1}{r}.$$

In one sense Theorem 1 is just an ϵ improvement on a result of Y. M. Kim [10] stating the same with $r = (3d-2)/2$ ($r = d$ would be best possible; the result in [10] is itself a refinement of previous work [7], [8], [2]). However, we think it is of interest from the technical point of view. This is because of the counterexamples of Jerison and others [7], [8], [2] which show that no improvement on Kim's result can be obtained by the «Carleman method» as it is usually applied-*i.e.* as a direct consequence of a «Carleman» (weighted Sobolev) inequality.

The main point of this paper is in Section 4 where we give a variant of the Carleman method which in some circumstances lets one use the information in the Carleman inequalities more efficiently. We believe our method can be developed further, but seemingly difficult questions in real analysis come up and so far we have not been able to deal with them. See the remarks and conjecture at the end of Section 4.

In addition to this we give a partly new approach to proving the Carleman inequalities. Roughly speaking, the idea is that regardless of what weight one wants to use they just reflect properties of the Taylor remainder of the fundamental solution for Δ. So we make the main estimates with the weight $|x|^{-n}$ (the natural weight in the context of the Taylor remainder) and pass from these estimates to better estimates with respect to other weights using osculation by functions of the form $c|x-b|^{-n}$. See formula (3.1) and the proof of Proposition 3.2, and the proof of Lemma 5.1.

Here is an outline of the paper: In Section 1 we do asymptotics for the remainder term in the Taylor expansion of the fundamental solution and in Section 2 we apply this asymptotics to prove certain $L^p \to L^q$ estimates. Much of what we do in these two sections in probably not new —the methods certainly are not, although we could not find references for the actual results. In Section 3 we pass to the Carleman inequalities that we need for Theorem 1. In Section 4 we give a real variable lemma we need for our modified Carleman method, and then prove Theorem 1. The approach to Carleman inequalities in Sections 1-3 leads naturally to refinements of various known results. In Section 5 we make some observations of this type. In particular, we show how to lessen the differentiability assumptions in a result of Sogge [15] on unique continuation for variable coefficient operators.

We assume $d \geqslant 3$ throughout the paper. Theorem 1 (with $r = 2$) is known when $d = 2$. It may be derived by reading between the lines in [3] and is also a special case of [10]. We will use the notation $x \lesssim y$ to mean «x is less than or equal to a constant times y» and $x \approx y$ for «$x \lesssim y$ and $y \lesssim x$».

1. Taylor Expansion of the Fundamental Solution

Notation. $\Gamma_y(x) = \Gamma(x, y) = c_d|x-y|^{2-d}$: the solution vanishing at infinity of the equation $\Delta\Gamma_y = \delta_y$.

p_n^y: the degree $n-1$ Taylor polynomial at the origin of the function Γ_y.

$$I_n(x,y) = |y|^{d-2}\left(\frac{|y|}{|x|}\right)^n (\Gamma(x,y) - p_n^y(x)).$$

For $x, y \in \mathbb{R}^d$ we denote

$$r = r_{xy} = \frac{|x|}{|y|} \quad \text{and} \quad \theta = \theta_{xy} = \angle_{xOy} \in [0, \pi]$$

the unoriented angle subtended by x and y at the origin.

If $n \in \mathbb{Z}^+$ it is easily seen (*cf.* [13]) that

$$|x|^{-n} f(x) = \int I_n(x,y)|y|^{-(n+d-2)} \Delta f(y)\, dy \tag{1.1}$$

for all $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$. We claim the following estimates on I_n.

Proposition 1.1.

(i) $\left|\nabla_x^j\left(I_n - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^n \Gamma\right)\right| \lesssim n^{d-2+j}|y|^{-j}$ *when* $|x - y| < \frac{|y|}{2n}\cdot$

(ii) $|\nabla_x^j I_n| \lesssim n^{d-3+j} \min\{n, |1-r|^{-1}\}|x|^{-j}$ *when* $|x-y| \geqslant \frac{1}{2n}|y|$.

(iii) *We can write* $I_n(x,y) = \operatorname{Re}(a(r,\theta)e^{in\theta})$ *for a suitable complex valued function a satisfying*

$$\left|\frac{d^i}{dr^i}\frac{d^j}{d\theta^j} a(r,\theta)\right| \lesssim n^{d/2-2}|\sin\theta|^{1-d/2-j}(|\sin\theta| + |1-r|)^{-1-i}$$

when $|\sin\theta| \geqslant \frac{1}{2n}\cdot$

Here i and j run through $\mathbb{Z}^+ \cup \{0\}$, ∇^j means j^{th} gradient and the constants depend on d, i, j. As discussed in the introduction, it seems unlikely that Proposition 1.1 is new. We also want to note that C. Sogge's approach to unique continuation problems (*e.g.* [15], [16]) is based on related if less explicit asymptotics, and that E. Sawyer [13] had earlier used essentially (ii) of Proposition 1.1 to study unique continuation in $\mathbb{R}^3$ where the more delicate estimate (iii) is not needed.

PROOF OF (i) AND (ii). Homogeneity considerations reduce to the case where y is (say) the first standard basis element e. The Taylor expansion of Γ_e is $\sum Z_k$ where Z_k, a suitable normalization of the k^{th} zonal harmonic, satisfies $|Z_k(x)| \lesssim k^{d-3}|x|^k$ (*cf.* [21]) and therefore also $|\nabla^j Z_k| \lesssim k^{d-3+j}|x|^{k-j}$ (use that f harmonic implies $|\nabla f(x)| \lesssim r^{-1}\max\{|f(y)|: |y-a| \leqslant r\}$ with $r = |x|/k$).Thus

$$|\nabla_x^j(|x|^n(I_n - |x|^{-n}\Gamma))| = |\nabla_x^j p_n^y|$$
$$\lesssim \sum_{k=0}^{n-1} k^{d-3+j}|x|^{k-j}$$
$$\leqslant n^j|x|^{-j}\sum_{k=0}^{n-1} k^{d-3}|x|^k.$$

A calculation with the product rule gives

$$|\nabla_x^j(I_n - |x|^{-n}\Gamma)| \lesssim n^j|x|^{-j-n}\sum_{k=0}^{n-1} k^{d-3}|x|^k. \tag{1.2}$$

Statement (i) follows since $|x|^{k-j-n} \lesssim 1$ when $|x - e| \leqslant 1/2n$. We also obtain $|\nabla_x^j(I_n - |x|^{-n}\Gamma)| \lesssim n^{d-2+j}$ when $|x - e| > 1/2n$, $1 - 1/n < |x| < 1 + 1/n$. Since $|\nabla_x^j(|x|^{-n}\Gamma_e)| \lesssim n^{d-2+j}|x|^{-j}$ when $|x - e| > 1/2n$ and $|x| > 1 - 1/n$ we obtain (ii) in the region $1 - 1/n < |x| < 1 + 1/n$. When $|x| > 1 + 1/n$, (1.2) implies (ii) as follows: bound the right side of (1.2) by estimates $k^{d-3} \leqslant n^{d-3}$ and then summing a geometric series, use the triangle inequality and the bound $|\nabla_x^j(|x|^{-n}\Gamma_e)| \lesssim n^{d-2+j}|x|^{-j}$. When $|x| < 1 - 1/n$ we use instead $|x|^n I_n = \sum_{k \geq n} Z_k$ and obtain

$$|\nabla_x^j(|x|^n I_n)| \lesssim \sum_{k \geq n} k^{d-3+j}|x|^{k-j}$$
$$\lesssim n^{d-3+j}|x|^{n-j}(1 - |x|)^{-1}.$$

By the product rule $|\nabla_x^j I_n| \lesssim n^{d-3+j}|x|^{-j}(1 - |x|)^{-1}$, and (ii) is proved.

PROOF OF (iii). There are various methods for doing such asymptotics and we use a method based on contour integration. Similar arguments may be found in [19], p. 158, [17] and in numerous places in [20]. There are two cases depending on whether d is even or odd.

PROOF OF (iii) WHEN d IS EVEN. We write

$$c_d|y|^{d-2}|x - y|^{2-d} = c_d\left|\frac{x}{|y|} - \frac{y}{|y|}\right|^{2-d}$$
$$= c_d|(e^{-i\theta} - r)(e^{i\theta} - r)|^{1-d/2}$$
$$= f(r)$$

where $f(z) = c_d[(e^{-i\theta} - z)(e^{i\theta} - z)]^{1-d/2}$ is analytic except for poles at $e^{\pm i\theta}$. It follows that

$$I_n(x, y) = r^{-n}R_n(r)$$

where $R_n = f - p_n$ and p_n is the degree $n-1$ Taylor polynomial for f at zero. If $|z|$ is small then by elementary complex variables

$$\begin{aligned} z^{-n}R_n(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1/2} \zeta^{-n}f(\zeta)(\zeta-z)^{-1}\,d\zeta \\ &= -\Big(\operatorname*{Res}_{e^{i\theta}} + \operatorname*{Res}_{e^{-i\theta}}\Big)(\zeta^{-n}f(\zeta)(\zeta-z)^{-1}) \\ &\quad + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{|\zeta|=R}\zeta^{-n}f(\zeta)(\zeta-z)^{-1}\,d\zeta \end{aligned}$$

$$z^{-n}R_n(z) = -\Big(\operatorname*{Res}_{\zeta=e^{i\theta}} + \operatorname*{Res}_{\zeta=e^{-i\theta}}\Big)(\zeta^{-n}f(\zeta)(\zeta-z)^{-1}) \tag{1.3}$$

since the limit term is zero. By analytic continuation (1.3) is valid for all z. The residues may be evaluated by taking $d/2-2$ derivatives. Carrying this out with $z = r$,

$$I_n(x,y) = \operatorname{Re}(e^{in\theta}a(r,\theta))$$

where $a(r,\theta)$ is a finite sum of terms of the form

$$a_{klm}(\sin\theta)^{1-d/2-k}(e^{-i\theta}-r)^{-1-l}e^{\operatorname{Im}\theta},$$

a_{klm} being constants with $|a_{klm}| \approx n^m$, and $k+1+m = d/2-2$. If (as we are assuming) $n|\sin\theta|$ is bounded away from zero, the main term is the term with $m = d/2-2$ and we obtain

$$\begin{aligned} |a(r,\theta)| &\lesssim n^{d/2-2}|\sin\theta|^{1-d/2}|e^{-i\theta}-r|^{-1} \\ &\approx n^{d/2-2}|\sin\theta|^{1-d/2}(|\sin\theta| + |1-r|)^{-1}. \end{aligned}$$

Similarly we can estimate derivatives $\frac{d^i}{dr^i}\frac{d^j}{d\theta^j}a(r,\theta)$. Each r-derivative produces a factor of $(e^{-i\theta}-r)^{-1}$ while each θ derivative produces at worst a factor of $|\sin\theta|^{-1}$. Estimate (iii) follows.

PROOF OF (iii) WHEN d IS ODD. The reason things are a bit more complicated here is of course that $[(z-e^{i\theta})(z-e^{-i\theta})]^{1-d/2}$ is multivalued. We fix $\theta \in (0,\pi)$ and let θ_1 and θ_2 be sufficiently close to $-\theta$ and θ respectively. Let $\gamma = \{e^{it}: \theta_1 \leqslant t \leqslant \theta_2\}$ and let $f_{\theta_1\theta_2}$ be the branch of $[(z-e^{i\theta_1})(z-e^{i\theta_2})]^{-1/2}$ defined on $\mathbb{C}\setminus\gamma$ and with $f_{\theta_1\theta_2}(0)$ close to 1. We will write f instead of $f_{\theta_1\theta_2}$ when no confusion will result. With $q = (d-3)/2$, and C_1 a suitable constant,

$$C_1\frac{d^q}{d\theta_1^q}\frac{d^qf}{d\theta_2^q}\bigg|_{\theta_1=-\theta,\,\theta_2=\theta}$$

gives a branch of the function $[(z - e^{i\theta})(z - e^{-i\theta})]^{1-d/2}$. Any such branch changes sign across γ and therefore agrees with $|e^{i\theta} - r|^{2-d}$ for $r < 1$ and with $-|e^{i\theta} - r|^{2-d}$ for $r > 1$, or viceversa. That means $|y|^{d-2}(\Gamma_y(x) - p_n^y(x))$ has the form

$$C_2 \frac{d^q}{d\theta_1^q} \frac{d^q}{d\theta_2^q} R_n(r)$$

where C_2 is a suitable constant, $f = f_{\theta_1 \theta_2}$ is as above, p_n is the degree $n-1$ Taylor polynomial of f at $z = 0$, and

$$R_n(r) = \begin{cases} f(r) - p_n(r) & \text{when} \quad 0 < r < 1 \\ -f(r) - p_n(r) & \text{when} \quad r > 1. \end{cases}$$

Therefore

$$I_n(x, y) = C_2 \frac{d^q}{d\theta_1^q} \frac{d^q}{d\theta_2^q} (r^{-n} R_n(r)) \bigg|_{\theta_1 = -\theta, \theta_2 = \theta}.$$

We now rewrite $r^{-n} R_n(r)$ using contour integration.
Denote

$$\int_{\gamma -} f\,dz = \lim_{r \uparrow 1} \int_\gamma f\,dz$$

$$\int_{\gamma +} f\,dz = \lim_{r \downarrow 1} \int_\gamma f\,dz$$

Let

$$T_1 = \{re^{i\theta_1}: 1 < r < \infty\}$$

oriented with r decreasing and

$$T_2 = \{re^{i\theta_2}: 1 < r < \infty\}$$

oriented with r increasing. With γ oriented counterclockwise, and $|z|$ small,

$$z^{-n}(f(z) - p_n(z)) = \frac{1}{2\pi i} \int_{|\zeta| = 1/2} \zeta^{-n}(\zeta - z)^{-1} f(\zeta)\,d\zeta$$

$$= \frac{1}{2\pi i} \int_{\gamma - \gamma +} \zeta^{-n}(\zeta - z)^{-1} f(\zeta)\,d\zeta$$

(1.4) $$z^{-n}(f(z) - p_n(z)) = -\frac{1}{\pi i} \int_{\gamma +} \zeta^{-n}(\zeta - r)^{-1} f(\zeta)\,d\zeta$$

since f changes sign across γ. By analytic continuation (1.4) is valid for $z \in \mathbb{C}\backslash\gamma$. Now integration over the countour $T_1\gamma_+ T_2$ implies

$$r^{-n}(f(r) - p_n(r)) = \begin{cases} \dfrac{1}{\pi i}\displaystyle\int_{T_1 + T_2} \zeta^{-n}(\zeta - r)^{-1}f(\zeta)\,d\zeta & (0 < r < 1) \\ 2r^{-n}f(r) + \dfrac{1}{\pi i}\displaystyle\int_{T_1 + T_2} \zeta^{-n}(\zeta - r)^{-1}f(\zeta)\,d\zeta & (r > 1) \end{cases}$$

where the $2r^{-n}f(r)$ term comes from the residue at $\zeta = r$. In either case,

$$r^{-n}R_n(r) = \frac{1}{\pi i}\int_{T_1 + T_2} \zeta^{-n}(\zeta - r)^{-1}f(\zeta)\,d\zeta$$

i.e.

$$\text{(1.5)}\qquad r^{-n}R_n(r) = \frac{1}{\pi i}\left\{e^{-i(n-1)\theta_2}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1}f_{\theta_1\theta_2}(e^{i\theta_2}t)\,dt \right.$$
$$\left. - e^{-i(n-1)\theta_1}\int_1^\infty t^{-n}(e^{i\theta_1}t - r)^{-1}f_{\theta_1\theta_2}(e^{i\theta_1}t)\,dt\right\}.$$

Now define the quantities

$$I_{klm} = \frac{d^k}{dr^k}\frac{d^l}{d\theta_1^l}\frac{d^m}{d\theta_2^m}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1}f(e^{i\theta_2}t)\,dt$$

It is clear that I_{kl0} has the following form:

$$I_{kl0} = C_{kl}e^{il\theta_1}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1-k}(e^{i\theta_2}t - e^{i\theta_1})^{-l}f(e^{i\theta_2}t)\,dt.$$

An induction on m shows that I_{klm} has the following form:

$$I_{klm} = \sum_{i+j \le m}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1-k-i}(e^{i\theta_2}t - e^{i\theta_1})^{-l-j}f(e^{i\theta_2}t)p_{ijklm}(t, \theta_1, \theta_2)\,dt,$$

p_{ijklm} being polynomials in t of degree $\leqslant i + j$ with coefficients which are smooth functions of θ_1, θ_2 and independent of n. We claim that

$$\text{(1.6)}\qquad |I_{klm}(r, -\theta, \theta)| \lesssim n^{-1/2}|\sin\theta|^{-1/2-l-m}(|\sin\theta| + |1 - r|)^{-1-k}.$$

PROOF OF (1.6). Clearly

$$|I_{klm}(r, -\theta, \theta)|$$
$$\lesssim \sum_{i+j\le m}\int_1^\infty t^{-n}|e^{i\theta}t - r|^{-1-k-i}(|t - 1| + |\sin\theta|)^{-1/2-l-j}(t - 1)^{-1/2}t^{i+j}\,dt.$$

We can estimate $t \lesssim |\sin\theta|^{-1}|e^{i\theta}t - r|$ and $t \lesssim |\sin\theta|^{-1}((t-1) + |\sin\theta|)$, and therefore

$$|I_{klm}(r, -\theta, \theta)| \lesssim$$

$$\sum_{i+j\leq m} |\sin\theta|^{-i-j}\int_1^\infty t^{-n}|e^{i\theta}t - r|^{-1-k}(t-1+|\sin\theta|)^{-1/2-l}(t-1)^{-1/2}\,dt$$

$$\lesssim |\sin\theta|^{-1/2-l-m}\int_1^\infty t^{-n}(t-1)^{-1/2}|e^{i\theta}t - r|^{-1-k}\,dt. \tag{1.7}$$

If $r \leqslant 1 + |\sin\theta|$ we can estimate $|e^{i\theta}t - r| \gtrsim |\sin\theta| + |1-r|$, and (1.6) follows since

$$\int_1^\infty (t-1)^{-1/2}t^{-n}\,dt \lesssim n^{-1/2}.$$

If $r \geqslant 1 + |\sin\theta|$ we split the integral in (1.7) into $\int_1^{(1+r)/2}$ and $\int_{(1+r)/2}^\infty$. When $t < (1+r)/2$ we have $|e^{i\theta}t - r| \gtrsim |\sin\theta| + |1-r|$ and can argue as before. On the other hand

$$\begin{aligned}
\int_{(1+r)/2}^\infty & t^{-n}|e^{i\theta}t - r|^{-1-k}(t-1)^{-1/2}\,dt \\
&\lesssim (r|\sin\theta|)^{-1-k}\int_{(1+r)/2}^\infty t^{-n}(t-1)^{-1/2}\,dt \\
&\lesssim (r|\sin\theta|)^{-1-k}n^{-1/2}\left[\frac{1+r}{2}\right]^{1-n} \\
&= (r-1)^{-1-k}\left[\frac{r-1}{r|\sin\theta|}\right]^{1+k} n^{-1/2}\left[\frac{1+r}{2}\right]^{1-n} \\
&\lesssim n^{-1/2}(r-1)^{-1-k} \\
&\approx n^{-1/2}(|1-r| + |\sin\theta|)^{-1-k}
\end{aligned}$$

and (1.6) follows; we used $r - 1 \geqslant |\sin\theta| \geqslant 1/2n$ and to derive the next to last line, the fact that $(tx)^\alpha(1+x)^{-t}$ is uniformly bounded over $t \geqslant 0$ and $0 \leqslant x \leqslant 1$ for any fixed $\alpha > 0$.

A term

$$e^{i(n-1)\theta}\frac{d^q}{d\theta_1^q}\frac{d^q}{d\theta_2^q}\bigg|_{\substack{\theta_1=-\theta\\ \theta_2=\theta}} e^{-i(n-1)\theta_2}\int_1^\infty t^{-n}(e^{i\theta_2}t - r)^{-1}f(e^{i\theta_2}t)\,dt \tag{1.8}$$

is a sum at terms $(-i(n-1))^p I_{0qm}(r, -\theta, \theta)$ with $m+p=q$. The result of taking $ir-$ and $j\theta-$ derivatives of a term (1.8) is a sum of bounded constants times terms $(-in)^p I_{i,q+t,m+s}(-\theta,\theta)$ with $s+t=j$ and is therefore

$$\lesssim \sum_{m+p=q} \sum_{t+s=j} n^{p-1/2} |\sin\theta|^{-(m+q+t+s+1/2)} |e^{i\theta}-r|^{-1-i}$$

For $|\sin\theta| \gtrsim 1/n$ the worst terms here are the terms with $p=q$ and we conclude that $d^{i+j}/dr^i\,d\theta^j$ of a term (1.8) is

$$\lesssim n^{d/2-2} |\sin\theta|^{-(q+j+1/2)} |e^{i\theta}-r|^{-1-i}.$$

This and a similar estimate for the contribution from the second term in (1.5) prove (iii), so we are done with Proposition 1.1.

Now we let α be a multiindex. We derive an expression like (1.1) for $|x|^{-n}D^\alpha f(x)$. If α and β are multiindices then $\beta \leqslant \alpha$ means that $\beta_j \leqslant \alpha_j$ for each $j \in \{1, \ldots, d\}$.

Lemma 1.1. *If* $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ *then*

(1.9) $|x|^{-n}D^\alpha f(x)$

$$= \sum_{0 \leq \beta \leq \alpha} n^{|\beta|} u_{\alpha\beta n}(x) |x|^{|\alpha-\beta|} \left[\int D_x^{\alpha-\beta}\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|} \Gamma(x,y)\right) \right.$$

$$\left. \cdot |y|^{-(n+d-2+|\alpha|)} \Delta f(y)\,dy + D^{\alpha-\beta} \int \Gamma(x,y)|x|^{-(n+|\alpha|)} \Delta f(y)\,dy \right]$$

where $u_{\alpha\beta n}$ *are fixed (i.e. independent of* f*) functions homogeneous of degree zero and smooth on the unit sphere with bounds independent of* n*, and* $u_{\alpha 0 n} = 1$.

Remarks. (1) The convergence of the first set of integrals follows from Proposition 1.1(i).

(2) We are mainly interested in the case $|\alpha| \leqslant 2$, and in this case (1.9) may be written in simplified form. If $|\alpha - \beta| = 1$ one can differentiate under the integral sign in the second integral. If $|\alpha-\beta| = 2$ and $D^{\alpha-\beta}$ is a mixed partial one can still do this provided one interprets the resulting integral as a principal value. If $D^{\alpha-\beta} = \dfrac{d^2}{dx_j^2}$ then one obtains an extra δ-function term. See *e.g.* [5] p. 99. We obtain the following:

Corollary 1.1. *If* $|\alpha| \leqslant 1$, *or if* $|\alpha| = 2$ *and* $D^\alpha = \dfrac{d^2}{dx_i\,dx_j}$ *with* $i \neq j$, *then*

(1.10) $|x|^{-n}D^\alpha f(x)$

$$= \sum_{0\le\beta\le\alpha} n^\beta u_{\alpha\beta n}(x)|x|^{|\alpha-\beta|}\int D_x^{\alpha-\beta} I_{n+|\alpha|}(x,y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)\,dy$$

where we interpret the integral as a principal value if $|\alpha-\beta|=2$. *The formula remains valid when*

$$D^\alpha = \frac{d^2}{dx_j^2},$$

provided the left hand side is replaced by $|x|^{-n}\left(D^\alpha f(x) - \frac{1}{d}\Delta f(x)\right)$.

PROOF OF LEMMA 1.1. We use induction on $|\alpha|$, and when $\alpha=0$ it reduces to (1.1). Suppose it is proved for multiindices of length less than $|\alpha|$. Write down (1.1) with $n+|\alpha|$ instead of n and take the D^α derivative using the product rule:

$$|x|^{-(n+|\alpha|)}f(x) = \int I_{n+|\alpha|}(x,y)|y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy,$$

$$\sum_{0\le\beta\le\alpha} n^{|\beta|}v_{\alpha\beta n}(x)|x|^{-(n+|\alpha|+|\beta|)}D^{\alpha-\beta}f(x)$$

$$= \int D_x^\alpha\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right)|y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy$$

$$+ D^\alpha\int \Gamma(x,y)|x|^{-(n+|\alpha|)}\Delta f(y)\,dy$$

where the $v_{\alpha\beta n}$ satisfy the same conditions as the $u_{\alpha\beta n}$ and $v_{\alpha 0n}=1$. As mentioned above, the differentiation under the integral sign is justified because of Proposition 1.1(i). Isolate the $\beta=0$ term on the left hand side and multiply by $|x|^{|\alpha|}$:

$$(1.11)\quad \frac{D^\alpha f(x)}{|x|^n} = -\sum_{0<\beta\le\alpha} n^{|\beta|}v_{\alpha\beta n}(x)|x|^{-(n+|\beta|)}D^{\alpha-\beta}f(x)$$

$$+ |x|^{|\alpha|}\Bigg[\int D_x^\alpha\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right)$$

$$\cdot |y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy$$

$$+ D^\alpha\int\Gamma(x,y)|x|^{-(n+|\alpha|)}\Delta f(y)\,dy\Bigg].$$

The $|x|^{|\alpha|}[\ ]$ term has the form of the $\beta = 0$ term in (1.9). The quantity $|x|^{-(n+|\beta|)}D^{\alpha-\beta}f(x)$ may be evaluated using the inductive hypothesis with α replaced by $\alpha - \beta$ and n by $n + |\beta|$. The sum over β becomes

$$\sum_{0<\beta\le\alpha}\ \sum_{0\le\gamma\le\alpha-\beta} n^{|\beta|}v_{\alpha\beta n}(x)n^{|\gamma|}u_{\alpha-\beta,\gamma,n+|\beta|}(x)|x|^{|\alpha-\beta-\gamma|}$$

$$\left[\int D_x^{\alpha-\beta-\gamma}\left(I_{n+|\alpha|}(x,y) - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right)|y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy\right.$$

$$\left. + D^{\alpha-\beta-\gamma}\int \Gamma(x,y)|x|^{-(n+|\alpha|)}\Delta f(y)\,dy\right].$$

Let

$$\sigma = \beta + \gamma, \qquad u_{\alpha\sigma n} = -\sum_{\beta+\gamma=\sigma} v_{\alpha\beta n}u_{\alpha-\beta,\gamma,n+|\beta|},$$

then the above expression is

$$\sum_{0<\sigma\le\alpha} n^{|\sigma|}u_{\alpha\sigma n}(x)|x|^{|\alpha-\sigma|}\left[\int D_x^{\alpha-\sigma}\left(I_{n+|\alpha|} - |y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right)\right.$$

$$\cdot\, |y|^{-(n+|\alpha|+d-2)}\Delta f(y)\,dy$$

$$\left. + D^{\alpha-\sigma}\int \Gamma(x,y)|x|^{-(n+|\alpha|)}\Delta f(y)\,dy\right]$$

and we obtain (1.9).

Lemma 1.2. *Let $A(x,y)$ be any term $n^{|\beta|}u_{\alpha\beta n}(x)|x|^{|\alpha-\beta|}D_x^{\alpha-\beta}I_{n+|\alpha|}(x,y)$, $0 \leqslant \beta \leqslant \alpha$, $u_{\alpha\beta n}$ as above.*

(i) $$\left|A(x,y) - n^{|\beta|}u_{\alpha\beta n}(x)|x|^{|\alpha-\beta|}D_x^{\alpha-\beta}|y|^{d-2}\left(\frac{|y|}{|x|}\right)^{n+|\alpha|}\Gamma(x,y)\right| \lesssim n^{d-2+|\alpha|}$$

when $|x-y| < \dfrac{1}{2n}|y|$.

(ii) $|A(x,y)| \lesssim n^{d-3+|\alpha|}\min\{n, |1-r|^{-1}\}$ *when* $|x-y| > \dfrac{1}{2n}|y|$.

(iii) $A(x,y) = \mathrm{Re}\,(e^{in\theta}q(x,y))$ *where the amplitude q satisfies*

$$|D_y^\tau D_x^\sigma q| \lesssim n^{d/2-2+|\alpha|}|y|^{-|\tau|}|x|^{-|\sigma|}|\sin\theta|^{1-d/2-|\sigma|-|\tau|}(|\sin\theta| + |1-r|)^{-1}$$

when $|\sin\theta| > \dfrac{1}{2n}$.

PROOF. Parts (i) and (ii) follow immediately from the corresponding parts of Proposition 1.1. Part (iii) also follows this way but there is some calculation involved, which we sketch for the reader's convenience. First consider any function $b(r,\theta)$. Denote by δ^k any derivative of the form $\left(r\frac{d}{dr}\right)^i\left(\frac{d}{d\theta}\right)^{k-i}$. Then for any σ and τ, $D_y^\tau D_x^\sigma b(r,\theta)$ is a sum of terms of the form

$$|x|^{-|\sigma|}|y|^{-|\tau|}u(x,y)\delta^k b$$

where $k \leqslant |\sigma| + |\tau|$ and u denotes any fixed (*i.e.* independent of b) function homogeneous of degree zero in each variable ($u(\lambda x, \mu y) = u(x,y)$) and smooth on $S^{d-1}\times S^{d-1}$. This may be proved easily by induction on $|\sigma| + |\tau|$. Consider now the expression for $I_{n+|\alpha|}$ in (iii) of Proposition 1.1. Using the product rule to calculate $\delta^k(e^{i(n+|\alpha|)\theta}a)$, we may write $D_x^{\alpha-\beta}I_{n+|\alpha|}$ as a sum of terms

$$\operatorname{Re}\left(\frac{u(x,y)}{|x|^{|\alpha-\beta|}}e^{i(n+|\alpha|)\theta}n^l\delta^k a\right)$$

with $k+1 \leqslant |\alpha-\beta|$ and u as above. Absorbing $e^{i|\alpha|\theta}$ as well as relevant factors $u_{\alpha\beta n}$ into $u(x,y)$, we may therefore write $A(x,y)$ as a sum of terms

$$\operatorname{Re}(u(x,y)e^{in\theta}n^l\delta^k a)$$

where now $k + l \leqslant |\alpha|$.

I.e. $q(x,y)$ is a sum of terms $n^l u(x,y)\delta^k a$. If we take $D_y^\tau D_x^\sigma$ of such a term we obtain terms of the form

$$n^l|y|^{-|\tau|}|x|^{-|\sigma|}u(x,y)\delta^{k+m}a$$

where $k+l \leqslant |\alpha|$, $m \leqslant |\sigma| + |\tau|$. Proposition 1.1 implies (after a calculation) that such a term is

$$\lesssim n^{l+d/2-2}|\sin\theta|^{1-d/2-(k+m)}(|\sin\theta| + |1-r|)^{-1}|y|^{-|\tau|}|x|^{-|\sigma|}.$$

The worst terms here are the terms with $k=0$, $l=|\alpha|$, $m=|\sigma|+|\tau|$, and we obtain (iii).

Proposition 1.2. *When* $|\alpha| \leqslant 1$, *or* $|\alpha| = 2$ *and* $D^\alpha \neq \frac{d^2}{dx_j^2}$ *we have*

$$|x|^{-n}D^\alpha f(x) = \int I_n^{(\alpha)}(x,y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)\,dy$$

where $I_n^{(\alpha)}$ *satisfies* (i)-(iii) *below. When* $D^\alpha = \frac{d^2}{dx_j^2}$ *we have*

$$|x|^{-n}\left(D^{\alpha}f(x) - \frac{1}{d}\Delta f(x)\right) = \int I_n^{(\alpha)}(x, y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)\,dy$$

where $I_n^{(\alpha)}$ *satisfies* (i)-(iii).

(i) *If* $|x - y| < \dfrac{|y|}{2n}$ *then*

$$|I_n^{(\alpha)}(x, y)| \lesssim |y|^{d-2+|\alpha|}|x - y|^{-(d-2+|\alpha|)}$$

when $|\alpha| \leqslant 1$, *and*

$$|I_n^{(\alpha)}(x, y) - |y|^d D_x^{\alpha}(\Gamma(x, y))| \lesssim n|y|^{d-1}|x - y|^{-(d-1)}$$

when $|\alpha| = 2$.

(ii) *If* $|x - y| \geqslant \dfrac{|y|}{2n}$ *then*

$$|I_n^{(\alpha)}(x, y)| \lesssim n^{d-3+|\alpha|}\min\left\{n, \left|1 - \frac{|x|}{|y|}\right|^{-1}\right\}.$$

(iii) *Choose a coordinate system on the unit sphere* S^{d-1}, *let* e *and* f *be variables on* S^{d-1}, D_e^{σ}, D_f^{τ} *denote differentiation in the given coordinate system. Then for* $|\sin\theta| > \dfrac{1}{2n}$,

$$I_n^{(\alpha)}(x, y) = \operatorname{Re}(e^{in\theta}q(x, y))$$

where

$$|D_e^{\sigma}D_f^{\tau}I_n^{(\alpha)}(se, tf)| \lesssim n^{|\alpha|+d/2-2}|\sin\theta|^{1-d/2-|\sigma|-|\tau|}\left(|\sin\theta| + \left|1 - \frac{s}{t}\right|\right)^{-1}.$$

Remarks. (1) The form of the estimates in (iii) (the fact that the right hand side increases with $|\sigma|$ and $|\tau|$) shows they are independent of the coordinate system.

(2) A similar expression of the form

$$|x|^{-n}(D^{\alpha}f(x) - p_{\alpha}(D)\Delta f(x)) = \int I_n^{(\alpha)}(x, y)|y|^{-(n+d-2+|\alpha|)}\Delta f(y)$$

could be given for higher derivatives.

(3) The kernel $I_n^{(\alpha)}$ is of course obtained from Corollary 1.1. Thus it is a sum of terms of the type in Lemma 1.2. Part (ii) of Proposition 1.2 then follows immediately from (ii) of Lemma 1.2. Parts (i) and (iii) follow the same way after some manipulations with the product rule and (for (iii)) to compare derivatives in $\mathbb{R}^d$ with derivatives on the sphere. We leave them to the reader.

2. Estimates from Spheres to Spheres

Notation. If K is a function on $\mathbb{R}^d \times \mathbb{R}^d$, $s, t > 0$ then K^{st} is the function on $S^{d-1} \times S^{d-1}$ defined by $K^{st}(e, f) = K(se, tf)$.

We fix a coordinate system S^{d-1} and sufficiently large constants $\{C_j\}$ and, for $\lambda \in (0, 1)$, denote by χ_λ any function from $S^{d-1} \times S^{d-1}$ to $[0, 1]$ satisfing the following: $\chi_\lambda(x, y) = 0$ if $|\sin\theta(x, y)| < \lambda/100$ or $|\sin\theta(x, y)| > 100\lambda$, and $|D^\alpha \chi_\lambda| \leqslant C_{|\alpha|}\lambda^{-|\alpha|}$. We denote by χ^λ any function from $S^{d-1} \times S^{d-1}$ to $[0, 1]$ satisfying: $\chi^\lambda(x, y) = 0$ if $|\sin\theta(x, y)| > 100\lambda$, and $|D^\alpha\chi^\lambda| \leqslant C_{|\alpha|}\lambda^{-|\alpha|}$. That is, χ_λ is smooth cutoff to $|\sin\theta| \approx \lambda$ and χ^λ is a smooth cutoff to $|\sin\theta| \lesssim \lambda$.

We will identify a kernel with the operator it induces, and denote the norm of an operator acting from $L^p(X, \mu)$ to $L^p(Y, \nu)$ by $\|T\|_{L^p(X,\mu)\to L^q(Y,\mu)}$, or $\|T\|_{p\to q}$ if there is no confusion.

Proposition 2.1. *Let ψ be a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$ with $\psi(x, y) = 1$ when $|x - y| > \frac{1}{n}|y|$, $\psi(x, y) = 0$ when $|x - y| < \frac{1}{2n}|y|$, and $|\nabla^k\psi| \lesssim \left(\frac{|y|}{n}\right)^{-k}$ Let $I_n^{(\alpha)}$ be as in Proposition 1.2 and $K_n^{(\alpha)} = \psi I_n^{(\alpha)}$. Then for $1 \leqslant p \leqslant 2$, $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$, $\lambda > \frac{1}{200n}$, $\chi_\lambda K_n^{(\alpha)s,t}$ maps $L^p(S^{d-1})$ to $L^{p'}(S^{d-1})$ with norm $\lesssim n^{d/2r - 1/r - 1 + |\alpha|}\lambda^{-d/2r+1}\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}$. If in addition $\lambda \leqslant \frac{200}{n}$ then χ_λ then χ_λ may be replaced by χ^λ here.*

Remarks. (1) This will be proved by applying oscillating integral lemmas to the asymptotics in Proposition 1.2 (iii). This type of argument is very standard by now and has been used in closely related contexts by C. Sogge [15, 16]. We want to point out that only the most simpleminded mapping properties of oscillating integrals are used in our version, namely the «variable coefficient Plancherel» of Hormander [6] (or see [18], p. 347).

(2) In proving Theorem 1, we use only the case $|\alpha| = 1$. The other cases are used in Section 5.

Lemma 2.1. *Suppose $1 < p, q < \infty$, (X, μ), (Y, ν), (Z, σ), (W, τ) are measure spaces, $\{T_{wy}\}_{y\in Y, w\in W}$ is a measurable family of operators from $L^p(X, \mu)$ to $L^q(Z, \nu)$ and the kernel*

$$n(w, y) = \|T_{wy}\|_{L^p(X,\mu)\to L^q(Z,\sigma)}$$

defines a bounded operator $f \to \int n(w, y) f(y)\, d\nu(y)$ from $L^p(Y, \nu)$ to $L^q(W, \tau)$ with norm N. For $f\colon X \times Y \to \mathbb{C}$ define $f_y(x) = f(x, y)$ etc. Then T defined by

$$(Tf)_w = \int T_{wy}(f_y)\, d\nu(y)$$

is a bounded operator from $L^p(X \times Y, \mu \times \nu) \to L^q(Z \times W, \sigma \times \tau)$ *with norm* $\leqslant N$.

PROOF.
$$\begin{aligned}\|Tf\|_q &= \left(\int \left\| \int T_{wy} f_y\, d\nu(y)\right\|^q_{L^q(Z)} d\tau(w)\right)^{1/q}\\ &\leqslant \left(\int \left(\int \|T_{wy} f_y\|_{L^q(Z)}\, d\nu(y)\right)^q d\tau(w)\right)^{1/q}\\ &\leqslant \left(\int \left(\int n(w,y)\|f_y\|_{L^p(X)}\, d\nu(y)\right)^q d\tau(w)\right)^{1/q}\\ &\leqslant N\left(\int \|f_y\|^p_{L^p(X)}\, d\nu(y)\right)^{1/p}\\ &= N\|f\|_p.\end{aligned}$$

Lemma 2.2. *Suppose* $1 \leqslant p \leqslant 2$, $\dfrac{1}{p} - \dfrac{1}{p'} = \dfrac{1}{r}$, (X, μ) *and* (Y, ν) *are measure spaces,* $K\colon X \times Y \to \mathbb{C}$ *and* $u\colon X \to \mathbb{R}^+$, $v\colon Y \to \mathbb{R}^+$. *Define*

$$A = \sup_x \|(u(x)v(y))^{-1/p'} K(x,y)\|_{L^{r'}(Y,\, v(y)\nu)}$$

$$B = \sup_y \|(u(x)v(y))^{-1/p'} K(x,y)\|_{L^{r'}(X,\, u(x)\mu)}$$

Then the norm of $f \to \int K(x,y) f(y)\, d\nu(y)$ *as an operator from* $L^p(Y,\nu)$ *to* $L^{p'}(X,\mu)$ *is* $\leqslant (AB)^{1/2}$.

PROOF. If $u = v = 1$ this follows by interpolation: the norm of K from L^1 to $L^{r'}$ is $\leqslant B$ and the norm from L^r to L^∞ is $\leqslant A$. The general case may be reduced to the case $u = v = 1$ by observing that the norm of the operator in question is the same as the norm from $L^p(Y, v\nu)$ to $L^{p'}(X, u\mu)$ of the operator

$$f \to \int u(x)^{-1/p'} K(x,y) v(y)^{1/p} f(y)\, d\nu(y)$$

$$\left(= f \to \int u(x)^{-1/p'} K(x,y) v(y)^{-1/p'} f(y)\, d(v\nu)(y)\right).$$

Lemma 2.3. *Suppose* A, B *are dics in* $\mathbb{R}^{d-1}$ *with radii* δ, ϵ *respectively,* $\delta \leqslant \epsilon$. *Suppose* $\theta\colon A \times B \to \mathbb{R}$ *is* C^∞, $a \in C_0^\infty(A \times B)$. *Assume that on* supp a *we have* $|D_x^\alpha D_y^\beta \theta| \leqslant C_{\alpha\beta}\epsilon^{1-|\beta|}\delta^{-|\alpha|}$ *and at each point the matrix* $\nabla_x \nabla_y \theta$ *has at least* m *eigenvalues with magnitude* $\geqslant C^{-1}\delta^{-1}$. *Furthermore suppose* $|D_x^\alpha D_y^\beta a| \leqslant C_{\alpha\beta}\epsilon^{-|\beta|}\delta^{-|\alpha|}$. *Then the kernel* $a(x,y)e^{in\theta(x,y)}$ *is* $L^p(dy) \to L^{p'}(dx)$ *bounded with norm* $\lesssim (\delta\epsilon)^{(d-1)/p'}(n\epsilon)^{-m/p'}$ *for* $1 \leqslant p \leqslant 2$, *where the implicit constant depends on* $d, C, \{C_{\alpha\beta}\}$.

PROOF. It is enough to do the $p = 2$ case since the $p = 1$ case is easy and the rest follows by interpolation. We can assume A, B centered at zero. Consider instead the scaled kernel $\tilde{K}(x, y) = a(\delta x, \epsilon y)e^{in\theta(\delta x, \epsilon y)}$ on $D(0, 1) \times D(0, 1)$ and write it as $\tilde{a}(x, y)e^{i\tilde{n}\tilde{\theta}(x,y)}$ where $\tilde{n} = n\epsilon$, $\tilde{a}(x, y) = a(\delta x, \epsilon y)$ and $\tilde{\theta}(x, y) = \epsilon^{-1}\theta(\delta x, \epsilon y)$. This reduces us to the case $\delta = \epsilon = 1$. If $m = d - 1$ we would now be done by [18], p. 347. In general, we can assume by a partition of unity and linear change of coordinates that all eigenvalues of $\nabla_{\bar{x}}\nabla_{\bar{y}}\tilde{\theta}$ are $\gtrsim 1$, where we let $\bar{x}$ (respectively, $\bar{y}$) denote the first m coordinates of $x(y)$ and $\bar{\bar{x}}(\bar{\bar{y}})$ be the last $d - 1 - m$ coordinates. If we fix $\bar{\bar{x}}, \bar{\bar{y}}$ and let $K_{\bar{\bar{x}}\bar{\bar{y}}}$ be the operator $f \to \int e^{i\tilde{n}\tilde{\theta}(x,y)}\tilde{a}(x, y)f(y)\,d\bar{y}$ acting from $L^2(\mathbb{R}^m, d\bar{y})$ to $L^2(\mathbb{R}^m, d\bar{x})$ we have a norm bound $\tilde{n}^{-m/2}$ by [18], p. 347. By (for example) Lemma 2.1 the norm of $\tilde{K}$ is also $\lesssim \tilde{n}^{-m/2}$ and the lemma follows.

Lemma 2.4. *Suppose* $\theta\colon \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$, $a \in C_0^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1})$. *Assume* $|x - y| \approx \rho$ *for all* $x, y \in \operatorname{supp} a$, *and on* supp a, *we have* $|D_x^\alpha D_y^\beta \theta| \lesssim C_{\alpha\beta}\rho^{1-|\alpha|-|\beta|}$, *and* $\nabla_x\nabla_y\theta$ *has at least m eigenvalues with magnitude* $\geqslant (C\rho)^{-1}$. *Assume moreover that* $|\nabla_x^\alpha\nabla_y^\beta a| \leqslant C_{\alpha\beta}\rho^{-|\alpha|-|\beta|}$. *Then* $a(x, y)e^{in\theta(x,y)}$ *is* $L^p \to L^{p'}$ *bounded with norm* $\lesssim \rho^{2(d-1)/p'}(n\rho)^{-m/p'}$, $1 \leqslant p \leqslant 2$.

PROOF. Again need only be done when $p = 2$. Let T be the operator in question. Let $\{q_j\}$ be a partition of unity subordinate to a covering by discs of radius ρ and with each point belonging to a bounded number of them. Let T_j be the operator with kernel $q_j(x)\ a(x, y)e^{in\theta(x,y)}$. For each j there are a bounded number of k such that $T_j^*T_k$ or $T_jT_k^*$ or $T_k^*T_j$ or $T_kT_j^*$ is nonzero and it follows (*e.g.* Cotlar's lemma) that $\|T\| \lesssim \sup_j \|T_j\|$. On the other hand T_j satisfies the hypothesis of Lemma 2.3 ($\delta = \epsilon = \text{const} \cdot \rho$)-the result follows.

PROOF OF PROPOSITION 2.1. We will assume $|\alpha| = 1$ to simplify the notation. This is no loss of generality because both estimates (ii) and (iii) in Proposition 1.2 (which are the basis of the proof) depend on α through the factor $n^{|\alpha|}$ and the same is true of the estimate in Proposition 2.1. We then fix α and drop the α superscripts, *e.g.* we write K_n^{st} for $K_n^{(\alpha)st}$. We also define

$$D(n, \lambda) = n^{d/2r - 1/r}\lambda^{-d/2r+1}. \tag{2.1}$$

There are two cases $\dfrac{1}{200n} \leqslant \lambda \leqslant \dfrac{200}{n}$ and $\lambda > \dfrac{200}{n}\cdot$ In the first case we use Proposition 1.2(ii) to conclude that

$$\|\chi^\lambda K_n^{st}\|_\infty \leqslant n^{d-2}\min\left\{\left|1 - \frac{s}{t}\right|^{-1}, n\right\} \approx n^{d-2}\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}.$$

For fixed e, the set $\{f \in S^{d-1}: \chi^\lambda K_n(e,f) \neq 0\}$ has measure $\approx n^{-(d-1)}$, so

$$\sup_e \|\chi^\lambda K_n^{st}\|_{L^{r'}(df)} \lesssim n^{d-2-(d-1)/r'}\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}.$$

Likewise

$$\sup_f \|\chi^\lambda K_n^{st}\|_{L^{r'}(de)} \lesssim n^{d-2-(d-1)/r'}\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}$$

so we get (*e.g.* by Lemma 2.2)

$$\begin{aligned}\|\chi^\lambda K_n^{st}\|_{p\to p'} &\lesssim n^{d-2-(d-1)/r'}\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}\\ &= D\left(n,\frac{1}{n}\right)\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}\\ &\approx D(n,\lambda)\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}\end{aligned}$$

as claimed. For the case $\lambda > 200/n$, we first observe following *e.g.* C. Sogge [14] that the phase function $\theta: S^{d-1}\times S^{d-1}$ in Proposition 1.2 has the property that $\nabla_x\nabla_y\theta$ (relative to a coordinate system) has $d-2$ eigenvalues with magnitude $\approx |\sin\theta|^{-1}$. If we let a be the amplitude function in Proposition 1.2 (iii) and

$$\tilde{a}(e,f) = n^{1-d/2}\lambda^{d/2-1}\left(\lambda+\left|1-\frac{s}{t}\right|\right)a(se,tf)$$

then we have $|D_e^\alpha D_f^\beta \tilde{a}| \lesssim \lambda^{-(|\alpha|+|\beta|)}$. It follows by the product rule that also $|D_e^\alpha D_f^\beta(\chi_\lambda\tilde{a})| \lesssim \lambda^{-(|\alpha|+|\beta|)}$. Thus in local coordinates on S^{d-1}, $\chi_\lambda \tilde{a} e^{in\theta}$ satisfies the hypothesis of Lemma 2.4 with $m = d-2$, $\rho = \text{const}\cdot\lambda$. We obtain

$$\|\chi_\lambda \tilde{a} e^{in\theta}\|_{p\to p'} \lesssim \lambda^{2(d-1)/p'}(n\lambda)^{-(d-2)/p'} \tag{2.2}$$

$$\|\chi_\lambda K_n^{st}\|_{p\to p'} \lesssim n^{d/2-1}\lambda^{-(d/2-1)}\left(\lambda+\left|1-\frac{s}{t}\right|\right)^{-1}\lambda^{2(d-1)/p'}(n\lambda)^{(d-2)/p'}$$

which works out to

$$\|\chi_\lambda K_n^{st}\|_{p\to p'} \lesssim D(n,\lambda)\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}.$$

Proposition 2.2. *Fix α with $|\alpha| \leqslant 2$. For sufficiently large $\nu \in \mathbb{R}^+$ there is a kernel $L_\nu^{(\alpha)}$ such that (with the same modification as in Proposition 1.2 when $D^\alpha = \frac{d^2}{dx_j^2}$)*

$$|x|^{-\nu} D^\alpha f(x) = \int_{\mathbb{R}^d} L_\nu^{(\alpha)}(x, y) |y|^{-\nu} \Delta f(y)\, dy$$

and $L_\nu^{(\alpha)} = M_\nu^{(\alpha)} + N_\nu^{(\alpha)}$ where for suitable C,

(i) *if $1 < p, q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{2 - |\alpha|}{d}$ then $M_\nu^{(\alpha)}$ maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ with norm bounded independently of ν.*

(ii) *$N_\nu^{(\alpha)}(x, y) = 0$ if $|x - y| < (200\nu)^{-1}|y|$, and for any $\lambda > (200\nu)^{-1}$,*

$$\|\chi_\lambda N_\nu^{(\alpha)st}\|_{p \to p'} \lesssim (st)^{-(d-2+|\alpha|)/2} \nu^{|\alpha|-1} D(\nu, \lambda)$$

$$\cdot \begin{cases} \left(\frac{s}{t}\right)^\rho, & \text{if} \quad s < \frac{t}{2}, \\ \left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1}, & \text{if} \quad \frac{t}{2} \leqslant s < 2t, \\ \left(\frac{t}{s}\right)^{1-\rho}, & \text{if} \quad s > 2t. \end{cases}$$

Here ρ is defined to be the number in $[0, 1)$ such that $\nu - \frac{d - 2 + |\alpha|}{2} + \rho$ is a integer. χ_λ may be replaced by χ^λ if $\lambda < \frac{200}{\nu}$.

PROOF. We fix α and drop the α superscripts. We also assume for notational purposes that $D^\alpha \neq \frac{d^2}{dx_j^2}$. Write $\nu = n + \frac{d - 2 + |\alpha|}{2} - \rho$, where $n \in \mathbb{Z}$, and ρ is as above. Define

$$L_\nu(x, y) = |x|^{-(d-2+|\alpha|)/2+\rho} |y|^{-(d-2+|\alpha|)/2-\rho} I_n(x, y).$$

Then

$$\begin{aligned} \int L_\nu(x, y) |y|^{-\nu} \Delta f(y)\, dy &= |x|^{-(d-2+|\alpha|)/2+\rho} \int I_n(x, y) |y|^{-(n+d-2+|\alpha|)} \Delta f(y) \\ &= |x|^{-(d-2+|\alpha|)/2+\rho} |x|^{-n} D^\alpha f(x) \\ &= |x|^{-\nu} D^\alpha f(x). \end{aligned}$$

Also define $M_\nu = (1-\psi)L_\nu$, $N_\nu = \psi L_\nu$, ψ as in Proposition 2.1. Then (i) follows from (i) of Proposition 1.2 which implies $|M_\nu| \lesssim |x-y|^{-(d-2+|\alpha|)}$ when $|\alpha| \leqslant 1$ and may be treated as a fractional integral, and that M_ν may be treated as a (truncated) singular integral when $|\alpha| = 2$. As for (ii), Proposition 2.1. implies

$$\|\chi_\lambda N_\nu^{st}\|_{p\to p'} \lesssim n^{|\alpha|-1} D(n,\lambda)\left(\frac{s}{t}\right)^\rho (st)^{-(d-2+|\alpha|)/2}\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}$$

and we have

$$\left(\frac{s}{t}\right)^\rho\left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1} \lesssim \begin{cases} \left(\frac{s}{t}\right)^\rho, & \text{if } \quad s<\frac{t}{2}, \\ \left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}, & \text{if } \frac{t}{2}<s<2t, \\ \left(\frac{t}{s}\right)^{1-\rho}, & \text{if } \quad s>2t. \end{cases}$$

In Section 5 we will also want a certain variant on Proposition 2.2. Let e_0 be a point of S^{d-1} and, for given ν, let $\phi\colon S^{d-1}\to\mathbb{R}$ be such that $\phi(e)=1$ if $|e-e_0|<C\nu^{-1/2}$, $\phi(e)=0$ if $|e-e_0|>2C\nu^{-1/2}$ and $|D^\alpha\phi| \lesssim (\nu^{-1/2})^{-|\alpha|}$ (here C is a suitable constant). For $\lambda > 100C\nu^{-1/2}$ consider the kernel

$$\phi(e)\chi_\lambda(e,f)N_\nu^{(\alpha)}(se,tf)$$

where $N_\nu^{(\alpha)}$ is as in Proposition 2.2. We claim

Proposition 2.3. *With notation as in Proposition* 2.2

$$\|\phi(e)\chi_\lambda(e,f)N_\nu^{(\alpha)}(se,tf)\| \lesssim \left(\lambda\sqrt{\nu}\right)^{-(d-1)/p'}(st)^{-(d-2+|\alpha|)/2}\nu^{|\alpha|-1}D(\nu,\lambda)$$
$$\cdot\begin{cases} \left(\frac{s}{t}\right)^\rho, & \textit{if} \quad s<\frac{t}{2}, \\ \left(\left|1-\frac{s}{t}\right|+\lambda\right)^{-1}, & \textit{if } \frac{t}{2}\leqslant s<2t, \\ \left(\frac{t}{s}\right)^{1-\rho}, & \textit{if} \quad s>2t. \end{cases}$$

PROOF. Exactly as for Proposition 2.2 except that we use Lemma 2.3 instead of 2.4, with $\epsilon \approx \lambda$, $\delta \approx n^{-1/2}$. The effect is that instead of (2.2) we have

$$\|\phi\chi_\lambda\tilde{a}e^{in\theta}\|_{p\to p'} \lesssim n^{-(d-1)/2p'}\lambda^{(d-1)/p'}(n\lambda)^{-(d-2)/p'}.$$

The extra factor of $(n^{1/2}\lambda)^{-(d-1)/p'}$ remains throughout the proof and we end up with Proposition 2.3.

3. Carleman Inequalities

Notation. Fix p and r with $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$ and a multiindex α with $|\alpha| = 1$, and let $L_\nu = L_\nu^{(\alpha)}$ be the kernels in Proposition 2.2.

If $x, y \in \mathbb{R}^d$ we let $s = |x|$, $t = |y|$, $\sigma = \log(1/s)$, $\tau = \log(1/t)$. If $\gamma \subset \mathbb{R}$ then $\gamma_* = \{s \in \mathbb{R}: \log(1/s) \in \gamma\}$, $A(\gamma) = \{x \in \mathbb{R}^d: |x| \in \gamma_*\}$.

The characteristic function of a set E will be denoted 1_E. We keep the notation from Section 2, *e.g.* the functions χ_λ.

The purpose of this section is to prove Carleman type inequalities needed for Theorem 1. We actually prove two inequalities-the first will be used when $d \leqslant 4$ and the second when $d \geqslant 5$.

Proposition 3.1. *Suppose $r = d$. If ν is large enough, $\nu - (d-1)/2$ is not an integer, and $\beta, \gamma \subset \mathbb{R}$ are intervals with* $\min\{|\beta|, |\gamma|\} \geqslant \nu^{-1}$ *then, with the notation* $|\gamma|' = \min\{|\gamma|, 1\}$, *we have*

$$\|1_{A(\gamma)} L_\nu I_{A(\beta)}\|_{p \to p'} \lesssim (\nu \min\{|\gamma|', |\beta|'\})^{1/2 - 1/d}.$$

The implicit constant depends on $\operatorname{dist}\left(\nu - \frac{d-1}{2}, \mathbb{Z}\right)$.

To state the other inequality we fix $\psi: \mathbb{R}^+ \to \mathbb{R}$ increasing and convex. We have then (for $f \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$)

$$e^{\nu\psi(\sigma)} D^\alpha f(x) = \int P_\nu(x, y) e^{\nu\psi(\tau)} \Delta f(y)\, dy \tag{3.1}$$

where $P_\nu(x, y) = e^{-\nu(\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma))} L_{\nu\psi'(\sigma)}(x, y)$.

This is because of the following calculation:

$$\begin{aligned}
e^{\nu\psi(\sigma)} D^\alpha f(x) &= e^{\nu(\psi(\sigma) - \psi'(\sigma)\sigma)} |x|^{-\nu\psi'(\sigma)} D^\alpha f(x) \\
&= \int e^{\nu(\psi(\sigma) - \psi'(\sigma)\sigma)} L_{\nu\psi'(\sigma)}(x, y) |y|^{\nu\psi'(\sigma)} \Delta f(y)\, dy \\
&= \int e^{\nu(\psi(\sigma) - \psi'(\sigma)\sigma)} L_{\nu\psi'(\sigma)}(x, y) e^{-\nu(\psi(\tau) - \tau\psi'(\sigma))} e^{\nu\psi(\tau)} \Delta f(y)\, dy \\
&= \int P_\nu(x, y) e^{\nu\psi(\tau)} \Delta f(y)\, dy.
\end{aligned}$$

Proposition 3.2. *Suppose $d < r < \infty$, and $\psi\colon \mathbb{R}^+ \to \mathbb{R}$ is C^2 and satisfies the following conditions: there is $C > 0$ such that $C^{-1} < \psi'(\sigma) < C$ for all σ, and for any $\delta > 0$ there is $C_\delta > 0$ such that $\psi''(\sigma) \geqslant C_\delta e^{-\delta\sigma}$. Then for ν large enough and β, γ intervals with $\min\{|\beta|, |\gamma|\} \geqslant \nu^{-1}$ and with the notation $|\gamma|'' = \min\{|\gamma|, \nu^{-1/2}\}$, and $\delta = \min\{|\beta|'', |\gamma|''\}$, $\epsilon = \max\{|\beta|'', |\gamma|''\}$, we have*

$$\| 1_{A(\psi^{-1}\gamma)} P_\nu 1_{A(\psi^{-1}\beta)} \|_{p \to p'} \lesssim \nu^{(d-2)/2r} \delta^{1/2r'} \begin{cases} \delta^{(1-(d+1)/r)/2} & (r < d+1) \\ 1 + \log(\epsilon/\delta) & (r = d+1) \\ \epsilon^{(1-(d+1)/r)/2} & (r > d+1) \end{cases}$$

Remarks. (1) Note that our intervals $\psi^{-1}\gamma$, $\psi^{-1}\beta$ are contained in $\mathbb{R}^+$, *i.e.*, Proposition 3.2 is actually an estimate on functions supported in $D(0, 1)$.

(2) The assumptions on ψ are not the most general possible. For example, instead of the upper bound on ψ' we could assume that for every $\delta > 0$ there exists $C_\delta > 0$ such that $\psi' \leqslant C_\delta e^{\delta\sigma}$. What is significant (as in previous work on similar problems) is to have some kind of «strict convexity» hypothesis, *i.e.* lower bound on ψ''/ψ'. The first example in [8] or [2] shows that Proposition 3.2 fails when $\psi(\sigma) = \sigma$, $r \geqslant (3d-2)/2$, $|\gamma| = |\beta| = \infty$.

(3) A version of Proposition 3.1 could be proved also when $\nu - (d-1)/2$ is an integer, but there would be a dependence on $|\gamma|$, $|\beta|$ which blows up as $|\gamma| \to \infty$ or $|\beta| \to \infty$. This of course is the same phenomenon as appeared in [9]. Actually Proposition 3.1 and 3.2 are just convenient ways of recording the information in Section 2 and (in Section 4) we probably could have worked instead with Proposition 2.2 directly.

(4) The crucial point for us will be the dependence on $|\gamma|$ and $|\beta|$ when $|\gamma|$ and $|\beta|$ are less than the critical numbers 1 in Proposition 3.1, $\nu^{-1/2}$ in Proposition 3.2. See Corollary 3.1 below where we state what we actually use.

(5) As far as why we need both Propositions 3.1 and 3.2: 3.2 is a much stronger inequality and we need that when $d \geqslant 5$. On the other hand, nothing like 3.2 can be true when $r = d$ (as is the case when $d \leqslant 4$) because the problem is the scale invariant, and a scale invariant «strict convexity» hypothesis would have to be of the form $\psi''/\psi' \geqslant$ const., which is incompatible with ψ' being bounded. (Of course one needs ψ' bounded for the application to the SUCP.)

PROOFS. Propositions 3.1 and 3.2 both follow by integrating out Proposition 2.2 with respect to the radial variable. The following fact will be useful.

Lemma 3.1. *Suppose $\lambda > 0$, $\rho \geqslant 0$. Then for intervals $\gamma \subset \mathbb{R}$,*

$$\| e^{-\rho x^2}(|x| + \lambda)^{-1} \|_{L^{r'}(\gamma)} \lesssim \lambda^{-1} \min\{\lambda, |\gamma|, \rho^{-1/2}\}^{1/r'}$$

where the implicit constant only depends on $r' \in (1, \infty)$.

This seems to be most easily proved by splitting into six cases according to the relative sizes of λ, $|\gamma|$, and $\rho^{-1/2}$.

PROOF OF PROPOSITION 3.1. Since the right hand side of the inequality is always larger than or equal to 1 it will suffice to prove it for N_ν in place of L_ν. Choose a partition of unity on $S^{d-1} \times S^{d-1}$ consisting of functions $\{\gamma_{2^j\nu^{-1}}\}$ and $\chi^{\nu^{-1}}$ where j runs from 1 to $\log_2 \nu$. Fix j and let $\lambda = 2^j\nu^{-1}$ and consider

$$\left\| 1_{A(\gamma)}(x)\chi_\lambda\left(\frac{x}{|x|}, \frac{y}{|y|}\right)N_\nu(x,y)1_{A(\beta)}(y)\right\|_{p \to p'} \tag{3.2}$$

Also let $n(s,t)$ be the $L^p(S^{d-1}) \to L^{p'}(S^{d-1})$ norm of the kernel $(\chi_\lambda N_\nu)^{st}$. Regard $\mathbb{R}^d$ as $S^{d-1} \times \mathbb{R}^+$ with measure $d\theta \times s^{d-1}\,ds$ and apply Lemma 2.1, then Lemma 2.2 with $u(s) = s^{-d}$, $v(t) = t^{-d}$. This gives

$$\begin{aligned}(3.2) &\leqslant L^p(\beta_*, t^{d-1}\,dt) \to L^{p'}(\gamma_*, s^{d-1}\,ds) \text{ norm of } n(s,t)\\ &\leqslant (AB)^{1/2},\end{aligned}$$

where

$$A = \sup_{s \in \gamma_*} \|s^{d/p'}t^{d/p'}n(s,t)\|_{L^{d'}(\beta_*, dt/t)}$$

$$B = \sup_{t \in \beta_*} \|s^{d/p'}t^{d/p'}n(s,t)\|_{L^{d'}(\gamma_*, ds/s)}$$

Using Proposition 2.2 to estimate n we have (with $\delta = \operatorname{dist}(n - (d-1)/2, \mathbb{Z})$)

$$A \lesssim D(\nu,\lambda) \sup_s \|K(s,t)\|_{L^{d'}(\beta_*, dt/t)},$$

where

$$K(s,t) = \begin{cases} (|\sigma - \tau| + \lambda)^{-1} & \text{if } |\sigma - \tau| < 1 \\ e^{-\delta|\sigma - \tau|} & \text{if } |\sigma - \tau| > 1 \end{cases}$$

and $D(\nu,\lambda) = \nu^{1/2 - 1/d}\lambda^{1/2}$. Using Lemma 3.1 with $\rho = 0$,

$$\begin{aligned}\|(|\sigma - \tau| + \lambda)^{-1}\|_{L^{d'}(\beta_*, dt/t)} &= \|(|\sigma - \tau| + \lambda)^{-1}\|_{L^{d'}(\beta, d\tau)}\\ &\lesssim \lambda^{-1}\min\{\lambda, |\beta|\}^{1/d'}\end{aligned}$$

and of course $\|e^{-\delta|\sigma-\tau|}\|_{L^{d'}(\beta_*, dt/t)} \lesssim \min\{1, |\beta|\}^{1/d'}$. We conclude

$$\|K(s,t)\|_{L^{d'}(\beta_*, dt/t)} \lesssim \begin{cases} \lambda^{-1}\min\{\lambda, |\beta|\}^{1/d'} & \text{if } |\beta| \leqslant 1 \\ \lambda^{-1/d} & \text{if } |\beta| > 1 \end{cases}$$

$$A \lesssim \begin{cases} \nu^{1/2 - 1/d} \lambda^{-1/2} \min\{\lambda, |\beta|\}^{1/d'} & \text{if} \quad |\beta| \leqslant 1 \\ (\nu\lambda)^{1/2 - 1/d} & \text{if} \quad |\beta| > 1 \end{cases}$$

so in fact

$$A \lesssim \nu^{1/2 - 1/d} \lambda^{-1/2} \min\{|\lambda|, |\beta|'\}^{1/d'} \tag{3.3}$$

for all β.

There is an analogous estimate for B where $|\gamma|'$ substitutes for $|\beta|'$ in (3.3), hence an estimate for (3.2). Also we have the same estimate for

$$\| 1_{A(\gamma)} \chi^\lambda N_\nu 1_{A(\beta)} \|_{p \to p'}$$

when $\lambda \approx \nu^{-1}$. Summing over λ we obtain

$$\begin{aligned} &\| 1_{A(\gamma)} N_\nu 1_{A(\beta)} \|_{p \to p'} \\ &\quad \leqslant \nu^{1/2 - 1/d} \sum_{\lambda = 2^j \nu^{-1}, 0 \leq j \leq \log_2 \nu} \lambda^{-1/2} \min\{\lambda, |\beta|'\}^{1/2d'} \min\{\lambda, |\gamma|'\}^{1/2d'} \\ &\quad \lesssim [\nu \min\{|\beta|', |\gamma|'\}]^{1/2 - 1/d} \end{aligned}$$

and Proposition 3.1 follows.

PROOF OF PROPOSITION 3.2. Convexity of ψ implies that $\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma)$ is positive and in fact $\geqslant k(\sigma) \min\{1, (\tau - \sigma)^2\}$ where $k(\sigma)$ denotes any lower bound for $\psi''/2$ on the interval $(\sigma - 1, \sigma + 1)$. We are assuming there is such a bound of the form $C_\delta e^{-\delta\sigma}$ for any given $\delta > 0$.

Write $P_\nu = Q_\nu + R_\nu$ where Q_ν(respectively, R_ν) comes from substituting $M_\nu(N_\nu)$ for L_ν in the definition of P_ν. Then $|Q_n| \lesssim |M_\nu| \lesssim |x - y|^{-(d-1)}$ and since Q_ν vanishes for $|x - y| > \nu^{-1}$ it follows (*e.g.* from Lemma 2.2) that Q_ν is $L^p \to L^{p'}$ bounded with norm $\lesssim \nu^{d/r - 1}$. So it suffices to prove Proposition 3.2 for R_ν instead of P_ν. Introduce the same partition of unity as in the proof of Proposition 3.1 and consider

$$\| 1_{A(\psi^{-1}\gamma)} \chi_\lambda R_\nu 1_{A(\psi^{-1}\beta)} \|_{p \to p'}. \tag{3.4}$$

We bound (3.4) as in the proof of Proposition 3.1, *i.e.* use Lemma 2.1, then 2.2 with $u(s) = s^{-d}$, $v(t) = t^{-d}$ to obtain (3.4) $\leqslant (AB)^{1/2}$

$$A = D(\nu, \lambda) \sup_s \| e^{-\nu(\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma))} n(s, t)(st)^{d/p'} \|_{L^{r'}(\psi^{-1}\beta, d\tau)},$$

$$B = D(\nu, \lambda) \sup_t \| e^{-\nu(\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma))} n(s, t)(st)^{d/p'} \|_{L^{r'}(\psi^{-1}\gamma, d\sigma)}.$$

To calculate A, fix s and consider separately the contributions to the $L^{r'}$ norm from $\psi^{-1}\beta \cap \{|\sigma - \tau| > 1\}$ and $\psi^{-1}\beta \cap \{|\sigma - \tau| < 1\}$, using the lower bounds

$\psi(\tau) - \psi(\sigma) - \psi'(\sigma)(\tau - \sigma) \gtrsim C_\delta s^\delta \min\{1, (\sigma - \tau)^2\}$. Estimating the exponential factor by its value when $|\sigma - \tau| = 1$ and using Proposition 2.2 we see that the contribution from $\psi^{-1}\beta \cap \{|\sigma - \tau| > 1\}$ is $\lesssim D(\nu, \lambda)e^{-C_\delta \nu s^\delta} s^{(1-d/r)/2}$ which may be made $\lesssim \nu^{-T}$ for any given T by choosing δ small. The other is

$$\lesssim s^{(1-d/r)/2}\left(\int_{\psi^{-1}\beta} e^{-C_\delta \nu s^\delta(\sigma-\tau)^2}(|\sigma - \tau| + \lambda)^{-r'}\,d\tau\right)^{1/r'}. \tag{3.5}$$

Here $|\psi^{-1}\beta| \approx |\beta|$ because of the boundedness assumptions on ψ' so by Lemma 3.1,

$$\begin{aligned}(3.5) &\lesssim s^{(1-d/r)/2}\lambda^{-1}\min\{(\nu s^\delta)^{-1/2}, \lambda, |\beta|\}^{1/r'} \\ &\lesssim \lambda^{-1}\min\{\nu^{-1/2}, \lambda, |\beta|\}^{1/r'}\end{aligned}$$

provided $\delta \leqslant 1 - d/r$. It follows that $A \lesssim \lambda^{-1}\min\{\nu^{-1/2}, \lambda, |\beta|\}^{1/r'}D(\nu, \lambda)$. Similar (not identical!) estimates can be made for B leading to the same bound except that $|\beta|$ is replaced with $|\gamma|$. Therefore

$$\begin{aligned}&\| 1_{A(\psi^{-1}\gamma)} R_\nu 1_{A(\psi^{-1}\beta)} \|_{p \to p'} \\ &\qquad \lesssim \nu^{(d-2)/2r} \sum_{\substack{\lambda = 2^j \nu^{-1} \\ 0 \le j \le \log_2 \nu}} \lambda^{-d/2r}[\min\{\nu^{-1/2}, \lambda, |\beta|\}\min\{\nu^{-1/2}, \lambda, |\gamma|\}]^{1/2r'}.\end{aligned}$$

Proposition 3.2 follows by doing the sum separately over $\lambda < \min\{|\beta|'', |\gamma|''\}$, $\min\{|\beta|'', |\gamma|''\} < \lambda < \max\{|\beta|'', |\gamma|''\}$, $\lambda > \max\{|\beta|'', |\gamma|''\}$.

We record the following formal consequence which is what is actually used in the proof of Theorem 1.

Corollary 3.1. *Let* $r = \max\{d, (3d-4)/2\}$. *If* $d \geqslant 5$ *then* $\| 1_{A(\psi^{-1}\gamma)} P_\nu \|_{p \to p'} \lesssim (\nu|\gamma|'')^{1/r}$. *If* $d = 3$ *or* 4 *then* $\| 1_{A(\psi^{-1}\gamma)} L_\nu \|_{p \to p'} \lesssim (\nu|\gamma|')^{(d-2)/2r}$ *provided* $\nu - (d-1)/2$ *is kept bounded away from the integers.*

PROOF. This is just index juggling. The $d \leqslant 4$ case is the easiest. The $d \geqslant 5$ case splits into subcases $d = 5$, 6, or $\geqslant 7$ corresponding to the three alternatives in Proposition 3.2. We explain only the $d \geqslant 7$ case. We have (since here $r = (3d-4)/2 > d + 1$, $|\beta| = \infty$, $|\beta|'' = \nu^{-1/2}$, $\delta = |\gamma|''$, $\epsilon = \nu^{-1/2}$)

$$\begin{aligned}\| 1_{A(\psi^{-1}\gamma)} P_\nu \|_{p \to p'}^r &\lesssim \nu^{(3d-2)/8}(|\gamma|'')^{(3d-6)/4} \\ &= \nu|\gamma\|''(\nu^{1/2}|\gamma|'')^{(3d-10)/4} \\ &\leqslant \nu|\gamma|''.\end{aligned}$$

Remark. The proof of Proposition 3.1 also proves the Jerison-Kenig result: If we take $\alpha = 0$, $r = d/2$, and keep $\nu - (d-2)/2$ bounded away from the integers, then we obtain (instead of (3.3))

$$A \lesssim \nu^{1-2/d}\lambda^{-1}\min\{\lambda, |\beta|'\}^{1-2/d}.$$

Using this estimate and continuing as before, we eventually obtain

$$\| 1_{A(\gamma)} L_\nu 1_{A(\beta)} \|_{p\to p'} \leqslant C,$$

which is Jerison-Kenig if $\gamma = \beta = \mathbb{R}$ (and gives nothing better for other $|\gamma|$, $|\beta|$, as one must expect anyway since the Jerison-Kenig bound is the same, as the bound for fractional integrals).

Conversely, it seems likely that other known proofs of the Jerison-Kenig result could be modified to give Proposition 3.1.

4. Proof of Theorem 1

The main point is the following lemma.

Lemma 4.1. *Suppose μ is a positive measure on $\mathbb{R}$ without atoms and such that*

$$\lim_{T\to\infty} \frac{1}{T}\log \mu(\{x: |x| > T\}) = -\infty.$$

Define μ_k for $k \in \mathbb{R}$ by $d\mu_k(x) = e^{kx}\, d\mu(x)$. Suppose $N \in \mathbb{R}^+$. Then there are disjoint intervals $I_j \subset \mathbb{R}$ and numbers $k_j \in [N, 2N]$ such that (with C a positive universal constant)

(i) $\mu_{k_j}(I_j) \geqslant \frac{1}{2}\|\mu_{k_j}\|$

(ii) $\sum |I_j|^{-1} \geqslant CN.$

PROOF. Suppose μ satisfies the hypotheses and fix $k \in \mathbb{R}$. Let a_k be a number such that $\mu_k((-\infty, a_k)) = \|\mu_k\|/4$, and let b_k be a number such that $\mu_k((b_k, \infty)) = \|\mu_k\|/4$. These a_k and b_k exist since μ_k is finite and continuous. Define $\gamma_k = [a_k, b_k]$.

Claim.
$$|\gamma_k \cap \gamma_j| \lesssim \frac{1}{|k-j|}.$$

In the proof we assume $j > k$. We may suppose $b_k > a_j$; else there is nothing to prove. If $x > b_k$ and $y < a_j$ then $x > |\gamma_k \cap \gamma_j| + y$, so that

$$\begin{aligned}\frac{\mu_j((b_k,\infty))}{\mu_k((b_k,\infty))} &= \frac{1}{\mu_k((b_k,\infty))}\int_{b_k}^{\infty} e^{(j-k)x}\,d\mu_k(x)\\ &\geqslant e^{(j-k)|\gamma_k\cap\gamma_j|}\frac{1}{\mu_k((-\infty,a_j))}\int_{-\infty}^{a_j} e^{(j-k)y}\,d\mu_k(y)\\ &= e^{(j-k)|\gamma_k\cap\gamma_j|}\frac{\mu_j((-\infty,a_j))}{\mu_k((-\infty,a_j))}\end{aligned}$$

$$\begin{aligned}\|\mu_k\|\,\|\mu_j\| &\geqslant \mu_j((b_k,\infty))\mu_k((-\infty,a_j))\\ &> e^{(j-k)|\gamma_k\cap\gamma_j|}\mu_k((b_k,\infty))\mu_j((-\infty,a_j))\\ &= \frac{1}{16}e^{(j-k)|\gamma_k\cap\gamma_j|}\|\mu_k\|\,\|\mu_j\|\end{aligned}$$

and $|\gamma_k\cap\gamma_j| < 4\log 2/(j-k)$, proving the claim.

Now we restrict k, j, etc. to lie in $[N, 2N]$. If there is k with $|\gamma_k| \leqslant 1/N$ there is nothing to prove. Otherwise define γ_k to be minimal if $|\gamma_j\cap\gamma_k| > |\gamma_j|/2$ implies $|\gamma_j| > |\gamma_k|/2$. Consider the collection of all minimal intervals and take a subcover with the Besicovitch property (the subcover is the union of two families of pairwise disjoint intervals, and every minimal interval is contained in the union of two subcover intervals). Such a subcover exists because the faster-than-exponential decay implies an upper bound on the lengths of the γ_k. We will be done if we show the subcover intervals have property (ii).

For any γ_k we can find a chain $k = k_1, k_2, \ldots$ with $|\gamma_{k_{j+1}}| \leqslant |\gamma_{k_j}|/2$ and $|\gamma_{k_{j+1}}\cap\gamma_{k_j}| \geqslant |\gamma_{k_j}|/2$. Such a chain must terminte at a minimal interval γ_{k_m} since we are assuming a lower bound $|\gamma_k| \geqslant 1/N$. The claim shows $|k_{j+1}-k_j| \lesssim |\gamma_{k_j}\cap\gamma_{k_{j+1}}|^{-1} \leqslant 2|\gamma_{k_{j+1}}|^{-1}$. The geometric decrease of the $|\gamma_{k_j}|$ then shows $|k-k_m| \lesssim |\gamma_{k_m}|^{-1}$. There must be a subcover interval γ_j with $|\gamma_{k_m}\cap\gamma_j| \geqslant |\gamma_{k_m}|/2$, and minimality of γ_{k_m} implies $|\gamma_{k_m}| \geqslant |\gamma_j|/2$ so that $|k_m-j| \lesssim |\gamma_j|^{-1}$, $|k-j| \leqslant |k-k_m| + |k_m-j| \lesssim |\gamma_{k_m}|^{-1} + |\gamma_j|^{-1} \lesssim |\gamma_j|^{-1}$.

So we associated a subcover interval γ_j to each interval γ_k, in such a way that

$$|\{k\colon \gamma_j \text{ associated to } \gamma_k\}| \lesssim |\gamma_j|^{-1}.$$

Then

$$\begin{aligned}N = |[N,2N]| &\leqslant \sum|\{k\colon \gamma_j \text{ associted to } \gamma_k\}|\\ &\lesssim \sum|\gamma_j|^{-1}.\end{aligned}$$

In proving Theorem 1 in the $d \leqslant 4$ case we will need to restrict the possible values of k_j. Hence the following

Corollary 4.1. *Suppose $m \in \mathbb{R}$, $b \in \mathbb{R}$. Then Lemma* 4.1 *remains true if the following modifications are made: the numbers k_j are required to belong to the arithmetic progression $\{mn + b: n \in \mathbb{Z}\}$; and* (ii) *is replaced by*

$$\text{(ii)}' \qquad \sum_j \max\{|I_j|^{-1}, m\} \gtrsim N.$$

PROOF. This is actually a corollary of the *proof* of Lemma 4.1. We carry out the same argument requiring all the values k, j, etc., to belong to the given arithmetic progression. The only change is in the last paragraph of the proof where we now have associated a subcover interval γ_j to each γ_k in such a way that

$$\text{card}\{k: \gamma_j \text{ associated to } \gamma_k\} \lesssim \max\{1, (m|\gamma_j|)^{-1}\}$$

since the right-hand side is the cardinality of the set of arithmetic progression elements lying within $|\gamma_j|^{-1}$ of j.

In finishing the proof we may assume $m < N$; otherwise there is nothing to prove as we may take $\{I_j\}$ to be a singleton. When $m < N$, we have

$$\begin{aligned} N &\lesssim m \,\text{card}\{mn + b: N \leqslant mn + b \leqslant 2N\} \\ &\lesssim m \sum_{\text{subcover intervals } \gamma_j} \text{card}\{k: \gamma_k \text{ associated to } \gamma_j\} \\ &\lesssim m \sum \max\{1, (m|\gamma_j|)^{-1}\} \\ &= \sum \max\{m, |\gamma_j|^{-1}\}. \end{aligned}$$

We now finish Theorem 1.

First of all, if $\psi: (0, \infty) \to \mathbb{R}$ is increasing and convex and ψ' is bounded then the formula

$$e^{\nu\psi(\sigma)} D^\alpha f(x) = \int L_\nu(x, y) e^{\nu\psi(\tau)} \Delta f(y)\, dy \tag{4.1}$$

derived for $C_0^\infty(D(0,1)\setminus\{0\})$ functions in Section 3 extends to functions in $W^{2,p}$ with support in $D(0,1)$ and such that $\|\nabla f\|_{L^{p'}(D(0,r))}$ and $\|\nabla f\|_{L^p(D(0,r))}$ vanish faster than any power of r as $r \to 0$. This is standard. First, if supp f does not contain the origin then it follows using a mollifier. The general case follows using a cutoff function near 0 and controlling the error terms by the infinite order vanishing (and the fact that $\psi(\tau) \leqslant C\tau$).

In particular (4.1) is valid for ϕu if u is as in Theorem 1 and $\phi \in C_0^\infty$ with $\phi = 1$ in a neighborhood of 0 since the infinite order vanishing of Δu follows from that of ∇u using Hölder's inequality.

Let $r = \max\{d, (3d - 4)/2\}$. As in, *e.g.* [9], Theorem 1 will follow if we show there is $\epsilon_0 > 0$ such that $0 \leqslant S_0 \leqslant 1$ and $\|V\|_{L^r(D(0, S_0))} < \epsilon_0$ imply u vanishes identically on $D(0, S_0)$. So let ϵ_0 be small enough and suppose $\| V \|_{L^r(D(0,S_0))} < \epsilon_0$ but ∇u does not vanish identically on $D(0, S_0)$. Let $S_1 < S_0$ be such that ∇u does not vanish identically on $D(0, S_1)$ and choose $\phi \in C_0^\infty$ with $\phi = 1$ on $D(0, S_1)$ and $\operatorname{supp} \phi \subset D(0, S_0)$. Let $f = \phi u$.

We first consider the $d \geqslant 5$ case.

Let $\psi(\sigma) = \sigma - (\sigma + 1)^{1/2}$ (any other function satisfying the hypotheses of Proposition 3.2 would do as well). Define a measure μ on $\mathbb{R}$ by

$$\mu(\gamma) = \int_{A(\psi^{-1}\gamma)} (V|\nabla f|)^p.$$

The infinite order vanishing implies μ has the faster-than-exponential decay property assumed in Lemma 4.1. With notation as in Lemma 4.1 we have

$$\mu_k(\gamma) = \int_{A(\psi^{-1}\gamma)} (e^{\nu\psi} V|\nabla f|)^p$$

where $\nu = k/p$. For N sufficiently large, we let $\{I_j\}$ be the intervals from Lemma 4.1. We may assume they have length $\geqslant 1/N$ (else drop all but one of them and expand that one to length $1/N$). Fix j and denote I_j by I, k_j by k, and $\nu = k/p$. We have

$$\begin{aligned} \|\mu_k\| &\leqslant 2 \int_{A(\psi^{-1}I)} (e^{\nu\psi} V|\nabla f|)^p \\ &\leqslant 2\Big(\int_{A(\psi^{-1}I)} V^r\Big)^{p/r} \Big(\int_{A(\psi^{-1}I)} (e^{\nu\psi}|\nabla f|^{p'}\Big)^{p/p'} \\ &\lesssim \Big(\int_{A(\psi^{-1}I)} V^r\Big)^{p/r} (N|I|)^{p/r} \| e^{\nu\psi} \Delta f \|_p^p. \end{aligned}$$

The last line follows from Corollary 3.1 since $|\psi^{-1}I| \approx |I|$ (as ψ' is bounded away from 0 and ∞) and $\nu \approx N$. We calculate $\Delta f = \Delta(\phi u)$ by the product rule:

$$\begin{aligned} |\Delta f| &= |\phi\, \Delta u + 2\, \nabla\phi\, \nabla u + u\, \Delta\phi| \\ &\leqslant \phi V |\nabla u| + |2\, \nabla\phi\, \nabla u + u\, \Delta\phi| \\ &\leqslant V|\nabla f| + |u\, \nabla\phi| + |2\, \nabla\phi\, \nabla u + u\, \Delta\phi| \\ &= V|\nabla f| + E, \end{aligned}$$

where $E \in L^p$ is supported in $\{x : S_1 < |x| < S_0\}$. Thus

$$\|\mu_k\| \lesssim \Big(\int_{A(\psi^{-1}I)} V^r\Big)^{p/r} (N|I|)^{p/r} \Big(\|\mu_k\| + \int |e^{\nu\psi} E|^p \Big). \tag{4.2}$$

Now we use the usual trick: $\|\mu_k\|$ grows faster than $e^{k\psi(\sigma_1)}$ as $k \to \infty$ while the last term in (4.2) is $O(e^{k\psi(\sigma_1)})$. So for large enough N the last term may be absorbed leading to

$$\|\mu_k\| \lesssim \left(\int_{A(\psi^{-1}I)} V^r\right)^{p/r} (N|I|)^{p/r} \|\mu_k\|$$

$$\int_{A(\psi^{-1}I)} V^r \geqslant (N|I|)^{-1}.$$

We now sum over j. Using (ii) of Lemma 4.1,

$$\int_{\cup A(\psi^{-1}I_j)} V^r \gtrsim \sum_j (N|I_j|)^{-1} \gtrsim 1$$

which is a contradiction if ϵ_0 is small. This finishes the high dimensional case.

If $d = 3$ or 4 the argument is similar. Now we take $\psi(\sigma) = \sigma$. We define μ as above, but now instead of choosing the I_j by Lemma 4.1 we use Corollary 4.1. Thus there are disjoint intervals I_j such that for each j there is k_j with

$$p^{-1}k_j - \frac{d-1}{2} = \frac{1}{2} \bmod 1$$

and

$$\mu_{k_j}(I_j) \geqslant \frac{1}{2} \|\mu_{k_j}\|, \qquad \sum_j \max\{1, |I_j|^{-1}\} \geqslant N.$$

We now fix a value of j and argue as above. The only difference is that in applying Corollary 3.1 we obtain a factor $(N \min\{|I|, 1\})^{(d-2)/2r}$ instead of $(N|I|)^{1/r}$. We end up with

$$\int_{A(I)} V^r \gtrsim (N \min\{1, |I|\})^{-(d-2)/2}$$

and therefore

$$\begin{aligned} \int_{\cup_j A(\psi^{-1}I_j)} V^r &\gtrsim \sum_j (N \min\{1, |I_j|\})^{-(d-2)/2} \\ &\geqslant \left\{\sum_j (N \min\{1, |I_j|\})^{-1}\right\}^{(d-2)/2} \\ &\gtrsim 1 \end{aligned}$$

by Corollary 4.1. This is a contradiction as before.

We now have a question: does Lemma 4.1 extend to $\mathbb{R}^d$ in the following form? Suppose μ is a measure on $\mathbb{R}^d$ with faster —than— exponential decay (say, absolutely continuous) and define $d\mu_k(x) = e^{kx}\, d\mu(x)$. Then there should be rectangles $\{R_j\}$ such that

(i) the R_j are disjoint,
(ii) for each j there is $k \in [-N, N] \times \cdots \times [-N, N]$ such that $\mu_k(R_j) \gtrsim \|\mu_k\|/2$ for all $T > 0$,
(iii) $\sum |R_j|^{-1} \geqslant C^{-1}N^d$ where C only depends on d.

Remarks. (1) We believe that an affirmative answer should lead to a proof of the WUCP for $|\Delta u| \leqslant V|\nabla u|$ with $V \in L^d$, although we do not have a reduction of one problem to the other.

(2) Natural examples are a Gaussian and surface measure on the unit sphere. For the Gaussian, the R_j are cubes with equal side length and in the surface measure case they are the covering of the sphere by rectangles with dimensions $N^{-1} \times (N^{-1/2} \times \cdots \times N^{-1/2})$ familiar in connection with Stein's restriction problem and so forth. In both examples, the order N^d is attained. The second example shows that the R_j cannot in general be taken with sides parallel to the axes.

(3) Weaker disjointness conditions than (i) would also be of interest, *e.g.*

(i)′ $$\|\Sigma\chi_{R_j}\|_p^p \approx \|\Sigma\chi_{R_j}\| \quad \text{for all} \quad p < \infty,$$

or

(i)″ the number of R_j containing any given point is bounded by a power of $\log N$.

We can answer the question affirmatively with (i)″ replacing (i) if $d = 2$. Of course, the answer is also affirmative when $d = 1$ (Lemma 4.1). The two dimensional result and partial results in higher dimensions will appear in a subsequent paper. The author can now prove the WUCP for $|\Delta u| \leqslant A|u| + B|\nabla u|$, $A \in L^{\alpha/2}$, $B \in L^r$, $r > d$, along the lines described above.

5. Further Results

The effect of the «osculation by $|x|^{-n}$» argument in the proof of (3.1) was that by using weights with a sufficient amount of «convexity» one could localize in the radial variable to intervals of (logarithmic) length $n^{-1/2}$. One can ask whether it is possible to localize also in the other variables (*i.e.*, to discs of radius $n^{-1/2}$) by the same kind of argument. It turns out that this is possible in some cases and we will present consequences for the weak unique continuation problem. Our main goal is the following refinement of a result of C. Sogge [15].

Theorem 5.1. *Suppose*

$$L = \sum a_{ij} \frac{d^2}{dx_i dx_j}$$

is an elliptic operator with $C^{1+\eta}$ coefficients ($0 < \eta \leqslant 1 - 2/d$), and $V \in L^r$ where $r = (d-2)/2\eta$. Suppose $u \in W^{2,p}$ satisfies $|Lu| \leqslant V|u| + C|\nabla u|$ and vanishes on an open set. Then $u = 0$.

Remarks. (1) This result with $\eta = 1 - 2/d$ implies Sogge's (he proved the same assuming L has C^∞ coefficients). As far as the minimal regularity is concerned, nothing better than Lip 1 is possible even if $r = \infty$, as was shown by Plis [11]. Lip 1 is known to be sufficient when $r = \infty$ through work of Aronszajn, Cordes and Hormander in the 1950's. The argument below can be adapted to give this (only a linear change of variables is used in place of Lemma 5.6) but so can many other arguments. The new point is that there are L^r results with less than C^∞ coefficients. It appears likely that the optimal result for the WUCP will be that it holds provided L has Lip 1 coefficients and $V \in L^{d/2}_{\text{loc}}$. Sogge pointed out that if the main conjecture on $L^p \to L^p$ behavior of oscillating integrals could be established (*i.e.*, an affirmative answer given to the first question at the end of [6] when $r = q$) then the argument below could be used to prove the WUCP with Lip 1 coefficients and $V \in L^r_{\text{loc}}$ for any given $r > d/2$. This is because the $L^p \to L^p$ conjecture would improve the estimate in (iii) of Lemma 5.2 below. The general conjecture in [6] has been disproved by Bairgain.

(2) The argument will be based on freezing of coefficients —this is made possible by the localization effect in Lemma 5.1 below. Sogge used ψDO calculus which is naturally less efficient if one cares about the minimal regularity of the coefficients. On the other hand it must be pointed out that Sogge also treated the SUCP. One would expect this to be possible by our method also, but not without significant changes.

(3) The same kind of localization effect may be used to refine some results of Chanillo-Sawyer [4]. We discuss this at the end of the section.

We let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$ be the upper half space.

Lemma 5.1. *Suppose $d/2 \leqslant r \leqslant \infty$. Let ν be sufficiently large. For $a \in \mathbb{R}^d_+$ denote $D_a = D(a, \nu^{-1/2} a_d)$ and $\Gamma_a(x) = 1 + (|x-a|^2/4a_d x_d)$. Then for $f \in C_0^\infty(\mathbb{R}^d_+)$ we have*

(i) $\|x_d^{-\nu} f\|_{L^{p'}(D)} \lesssim \|\Gamma_a^{-\nu} x_d^{2-d/r-\nu} \Delta f\|_{L^p}$

(ii) $\|x_d^{-\nu} \nabla f\|_{L^p(D)} \lesssim \nu^{(d-2)/4r - 1/2} \|\Gamma_a^{-\nu} x_d^{1-\nu} \Delta f\|_{L^p}$

(iii) $\|x_d^{-\nu} H_f\|_{L^p(D)} \lesssim \nu^{(d-2)/4r + 1/2} \|\Gamma_a^{-\nu} x_d^{-\nu} \Delta f\|_{L^p}$.

We recall that $1/p - 1/p' = 1/r$. Also, we remark that the point of the functions $\Gamma_a^{-\nu}$ is that $\Gamma_a^{-\nu} \approx 1$ on D_a (since $|x-a|^2/4a_d x_d \lesssim \nu^{-1}$) and $\Gamma_a^{-\nu}$ dies off very fast outside D_a so that *e.g.* Lemma 5.5 below is valid. H_f is of course the Hessian matrix.

Lemma 5.1 will be a consequence of the following estimate with respect to the «natural» weights.

Lemma 5.2. *Suppose $d/2 \leqslant r \leqslant \infty$. Fix $e \in \mathbb{R}^n$ and denote $D = D(e, \nu^{-1/2}|e|)$, $\Omega = \{x \in \mathbb{R}^d : |x| > |e|/10,\ x \cdot e > 0\}$. Then for $f \in C_0^\infty(\Omega)$,*

(i) $\| |x|^{-\nu} f \|_{L^{p'}(D)} \lesssim \| |x|^{2-d/r-\nu} \Delta f \|_p$

(ii) $\| |x|^{-\nu} \nabla f \|_{L^p(D)} \lesssim \nu^{(d-2)/4r-1/2} \| |x|^{1-\nu} \Delta f \|_p$

(iii) $\| |x|^{-\nu} H_f \|_{L^p(D)} \lesssim \nu^{(d-2)/4r+1/2} \| |x|^{-\nu} \Delta f \|_p$.

PROOF OF LEMMA 5.1. There are three identical arguments for (i), (ii), (iii). We will do (iii).

Define $a^* = (a_1, \ldots, a_{d-1}, -a_d)$. Apply (iii) of Lemma 5.2 taking the origin to be at a^*, and with 2ν instead of ν. Observe that the assumption that $f \in C_0^\infty(\Omega)$ is in fact satisfied. We get

$$\| |x - a^*|^{-2\nu} H_f \|_{L^p(D)} \lesssim \nu^{(d-2)/2r+1/2} \| |x - a^*|^{-2\nu} \Delta f \|_p .$$

Now $|x - a^*|^{-2} = (4a_d)^{-1} x_d^{-1} \Gamma_a(x)^{-1}$. Accordingly

$$(4a_d)^{-\nu} \| \Gamma_a^{-\nu} x_d^{-\nu} H_f \|_{L^p(D)} \lesssim \nu^{(d-2)/2r+1/2} (4a_d)^{-\nu} \| \Gamma_a^{-\nu} x_d^{-\nu} \Delta f \|_p$$

and now we are done, since $\Gamma_a^{-\nu} \approx 1$ on D.

PROOF OF LEMMA 5.2. The reader will observe that part (i) can be derived from [9]. All three parts follow readily from our Propositions 2.2 and 2.3 by the same kind of arguments as we used in Section 3, only somewhat easier. Accordingly we will omit the details of the calculations.

The estimates in Lemma 5.2 scale correctly, so we may assume e is a unit vector. Recall the kernels L^α, M^α, N^α from Section 2. For any fixed m, the kernel $|x|^m L^\alpha_{t+m} |y|^{-m}$ maps $|y|^{-t} \Delta f$ to $|x|^{-t} D^\alpha f$ (or to $|x|^{-t}(D^\alpha f - (1/d)\Delta f)$ if $D^\alpha = d^2/dx_j^2$). So to prove Lemma 5.2 it will suffice to show that for some fixed m and sufficiently large ν we have bounds

$$\begin{aligned} &\| |x|^{m+d/r-2} L^\alpha_\nu |y|^{-m} \|_{L^p(\Omega) \to L^{p'}(D)} \leqslant 1 && (\alpha = 0) \\ &\| |x|^{m-1} L^\alpha_\nu |y|^{-m} \|_{L^p(\Omega) \to L^p(D)} \lesssim \nu^{(d-2)/4r-1/2} && (|\alpha| = 1) \\ &\| |x|^{m} L^\alpha_\nu |y|^{-m} \|_{L^p(\Omega) \to L^p(D)} \lesssim \nu^{(d-2)/4r+1/2} && (|\alpha| = 2) \end{aligned} \tag{5.1}$$

In fact it will suffice to prove the bounds (5.1) with the powers of x dropped from the left hand side (they are ≈ 1 on D) and with L^α_ν replaced by N^α_ν (the corresponding bounds for M_ν^α are easy estimates on fractional integrals when $|\alpha| < 2$ or singular integrals when $|\alpha| = 2$). Thus we will prove that

$$\begin{aligned}
&\|N_\nu^\alpha |y|^{-m}\|_{L^p(\Omega)\to L^{p'}(D)} \lesssim 1 && (\alpha = 0)\\
(5.2)\qquad &\|N_\nu^\alpha |y|^{-m}\|_{L^p(\Omega)\to L^p(D)} \lesssim \nu^{(d-2)/4r-1/2} && (|\alpha| = 1)\\
&\|N_\nu^\alpha |y|^{-m}\|_{L^p(\Omega)\to L^p(D)} \lesssim \nu^{(d-2)/4r+1/2} && (|\alpha| = 2)
\end{aligned}$$

provided the fixed positive number m is sufficiently large.

Let ψ be a smooth function which is 1 on the double of D, 0 outside the triple and has the natural bounds. Fix α and write

$$S(x, y) = N_\nu^\alpha(x, y)\psi(x)$$
$$T(x, y) = N_\nu^\alpha(x, y)(1 - \psi(x))$$

Consider first S. Let $\psi_\lambda(x, y)$ denote smooth cutoffs to $|x - y| \approx \lambda$, such that

$$1 = \sum_{\substack{\lambda = 2^j\nu^{-1}\\ 1 \le 2^j \le \nu^{1/2}}} \psi_\lambda$$

on supp S. Then with the χ_λ, K^{st} notation used in Sections 2 and 3, and with C a suitable constant, $(\psi_\lambda S)^{st}$ will be

$$\begin{cases}
0, & \text{if } |s - t| > C\lambda,\\
\chi_\lambda N_\nu^{\alpha st}, & \text{if } |s - t| > C\lambda,\\
\chi^{\nu^{-1}} N_\nu^{\alpha st} + \sum\limits_{\substack{\mu = 2^j\nu^{-1}\\ 1 \le 2^j \le \nu\lambda}} \chi_\mu S^{st}, & \text{if } C^{-1}\lambda < |s - t| < C\lambda.
\end{cases}$$

Using Proposition 2.2, and summing a geometric series in the case $C^{-1}\lambda < |s - t| < C\lambda$, we obtain

$$\|(\psi_\lambda S)^{st}\|_{p\to p'} \lesssim \nu^{|\alpha|-1} D(\nu, \lambda)\left(\left|1 - \frac{s}{t}\right| + \lambda\right)^{-1},$$

with $D(\nu, \lambda)$ as in (2.1). Lemmas 2.1 and 2.2 may thus be applied (take $u = v = 1$ in Lemma 2.2) to give (use Lemma 3.1 with $\rho = 0$, $|\gamma| \approx \lambda$ to calculate the relevant $L^{r'}$ norms)

$$\|\psi_\lambda S\|_{p\to p'} \lesssim \nu^{|\alpha|-1} D(\nu, \lambda)\lambda^{-1/r} \tag{5.2a}$$

and therefore also

$$\|\psi_\lambda S|y|^{-m}\|_{p\to p'} \lesssim \nu^{|\alpha|-1} D(\nu, \lambda)\lambda^{-1/r} \tag{5.3}$$

since $|y|^{-m} = 1$ on supp S.

Now we need the following observation.

Lemma 5.3. *Suppose* S: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is such that* $S(x,y) = 0$ *for* $|x - y| > \epsilon$. *Then* $(1 \leqslant p \leqslant 2, 1/r = 1/p - 1/p')$

$$\|S\|_{p\to p} \lesssim \epsilon^{d/r}\|S\|_{p\to p'}.$$

To prove this (easier than finding a reference!) let $\{B_j\}$ be a covering of $\mathbb{R}^d$ by balls with radius ϵ and finite overlap (*i.e.*, no point belongs to more than C of them where C is a fixed constant). Let $\tilde{B}_j$ be the double of B_j. The $\{\tilde{B}_j\}$ still have finite overlap, and $S(x,y) = 0$ if $x \in B_j$, $y \notin \tilde{B}_j$. So

$$\begin{aligned}
\|Sf\|_p^p &\leqslant \sum_j \int_{B_j} |Sf|^p \\
&\leqslant \sum_j \int_{B_j} |S(\chi_{\tilde{B}_j} f)|^p \\
&\leqslant \epsilon^{dp/r} \sum_j \|S(\chi_{\tilde{B}_j} f)\|_{p'}^p \quad \text{(Hölder)} \\
&\leqslant \epsilon^{dp/r} \sum_j \|\chi_{\tilde{B}_j} f\|_p^p \\
&\approx \epsilon^{dp/r}\|f\|_p^p.
\end{aligned}$$

Applying this in our situation when $|\alpha| = 1$ or 2 we obtain from (5.3)

$$\begin{aligned}
\alpha = 0&: \quad \|\psi_\lambda S|y|^{-m}\|_{p\to p'} \lesssim \nu^{-1}D(\nu,\lambda)\lambda^{-1/r}\ , \\
\alpha = 1&: \quad \|\psi_\lambda S|y|^{-m}\|_{p\to p} \lesssim D(\nu,\lambda)\lambda^{-1/r}\lambda^{d/r}\ , \\
\alpha = 2&: \quad \|\psi_\lambda S|y|^{-m}\|_{p\to p} \lesssim \nu D(\nu,\lambda)\lambda^{-1/r}\lambda^{d/r}.
\end{aligned}$$

Now we sum over $\lambda = 2^j\nu^{-1}$, $1 \leqslant 2^j \leqslant \nu^{1/2}$. For $\alpha = 1$ or 2 we have a convergent geometric series with the main term being the $\lambda = \nu^{-1/2}$ term, and we obtain

$$\|S|y|^{-m}\|_{p\to p} \lesssim \nu^{(d-2)/4r - 1/2 + |\alpha| - 1}.$$

For $\alpha = 0$ there are several cases according to the relative sizes of r and $(d+2)/2$, but we always get $\|S|y|^{-m}\|_{p\to p'} \leqslant 1$ (in fact, it is clear that only the case $r = d/2$ has to be considered).

Now we consider $T|y|^{-m}$. Choose a smooth cutoff $J(x,y) = J(\theta_{xy})$ with $J = 1$ when θ_{xy} is less than $C^{-1}\nu^{-1/2}$ (for suitable large C) and $J = 0$ when $\theta_{xy} > 2C^{-1}\nu^{-1/2}$. Write $T = JT + (1 - J)T$. $(JT)^{st}$ will vanish if $|s - t|$ is small compared with $\nu^{-1/2}$, and when $|s - t| \gtrsim \nu^{-1/2}$ we can write

$$(JT)^{st} = \chi^{\nu^{-1}} N_\nu^{\alpha st} + \sum_{\substack{\lambda = 2^j\nu^{-1} \\ 1 \leq 2^j \leq \nu^{1/2}}} \chi_\lambda N_\nu^{\alpha st}.$$

We may apply Proposition 2.2 to the summands and then sum a geometric series obtaining (remember we can assume $|s-t| \gtrsim \nu^{-1/2}$)

$$\|(JT)^{st}\|_{p\to p'} \lesssim t^{-(d-2+|\alpha|)/2}\nu^{|\alpha|-1}D(\nu, \nu^{-1/2})\left(\left|1-\frac{s}{t}\right| + \nu^{-1/2}\right)^{-1}.$$

Likewise $((1-J)T)^{st}$ may be written as

$$\sum_{\substack{\lambda = 2^j\nu^{-1/2} \\ 1 \le 2^j \le \nu^{1/2}}} \chi_\lambda N_\nu^{\alpha st}.$$

We then apply Proposition 2.3 (which was included for this purpose) to estimate the action of $\chi_\lambda N_\nu^{\alpha st}$ from L^p to $L^{p'}(D_s)$, where $D_s = \{e \in S^{d-1}: se \in D\}$, and sum a geometric series to obtain

$$\|((1-J)T)^{st}\|_{p\to p'} \lesssim t^{-(d-2+|\alpha|)/2}\nu^{|\alpha|-1}D(\nu, \nu^{-1/2})\left(\left|1-\frac{s}{t}\right| + \nu^{-1/2}\right)^{-1}$$

and therefore the same estimate for T^{st}. Thus

$$\|(T|y|^{-m})^{st}\|_{p\to p'} \lesssim t^{-(d-2+|\alpha|)/2-m}\nu^{|\alpha|-1}\nu^{(3d-4)/4r-1/2}\left(\left|1-\frac{s}{t}\right| + \nu^{-1/2}\right)^{-1}.$$

Also $(T|y|^{-m})^{st} = 0$ if $|s-1|$ is large compared with $\nu^{-1/2}$ or if $|t| < 1/10$. Denoting (for suitable C)

$$Q(s,t) = \begin{cases} t^{-(d-2+|\alpha|)/2-m}\left(\left|1-\frac{s}{t}\right| + \nu^{-1/2}\right)^{-1} & \left(\text{if } |t| > \frac{1}{10},\ |s-1| < C\nu^{-1/2}\right) \\ 0 & \text{(otherwise)}, \end{cases}$$

we have

$$\|Q(s,t)\|_{L^r(s^{d-1}ds)} \lesssim \nu^{1/2r} \quad \text{for any } t,$$
$$\|Q(s,t)\|_{L^r(t^{d-1}dt)} \lesssim \nu^{1/2r} \quad \text{for any } s,$$

provided m has been chosen large enough that there is no trouble at infinity. Lemmas 2.1, 2.2 therefore imply

$$\|T|y|^{-m}\|_{L^p(\Omega)\to L^{p'}(D)} \lesssim \nu^{|\alpha|-1}\nu^{(3d-2)/4r-1/2} \tag{5.4}$$

and therefore also (by Hölder's inequality)

$$\|T|y|^{-m}\|_{L^p(\Omega)\to L^p(D)} \lesssim \nu^{|\alpha|-1}\nu^{(d-2)/4r-1/2}.$$

Lemma 5.2 follows.

In Lemma 5.2 we needed the assumption $r \geqslant d/2$ because otherwise the «fractional integration» part M_ν^α was unbounded if $\alpha = 0$. On the other hand the estimates in the above argument work perfectly well if $r < d/2$. We record what we proved in case $\alpha = 0$ for use in connection with the Chanillo-Sawyer results at the end of the section.

Lemma 5.2. *Let m be a sufficiently large fixed positive number. If e is a unit vector,* $D = D(e, \nu^{-1/2})$, $\Omega = \{x : x \cdot e > 0,\ |x| > 1/10\}$, ψ *and* ψ_λ *are as above,* $S(x,y) = \psi(x,y)N_\nu^0(x,y)$, $T(x,y) = (1-\psi(x,y))N_\nu^0(x,y)$, *then for* $f \in L^p(\Omega)$, $1 \leqslant 2^j \leqslant \nu^{1/2}$

(i) $\|(\psi_\lambda S)f\|_{p'} \lesssim \nu^{-1}D(\nu,\lambda)\lambda^{-1/r}\|f\|_p$,

(ii) $\|Tf\|_{p'} \lesssim \nu^{(3d-2)/4r-3/2}\||y|^m f\|_p$.

PROOF. (i) Follows directly from (5.3) and (ii) from (5.4).

To prove Theorem 5.1 we need a version of Lemma 5.1 with the Laplacian replaced by a variable coefficient operator. This will work out because the disc D is small enough to permit a good approximation of the variable coefficient operator by constant coefficient one. Since the coefficients are $C^{1+\eta}$ and the linear term can be eliminated by changing coordinates (geodesic normal coordinates or something similar) we expect an approximation on D to within $O(\nu^{-1/2(1+\eta)})$, *i.e.*, the following lemma.

Lemma 5.4. *There is constant* $\beta > 0$ *so that if L is as in Theorem* 1, *B is a suitable constant depending on ellipticity bounds for L in* $\|x\| < 1$, *then for any* $\tau > 0$, *if* $a \in \mathbb{R}^d$ *is such that* $\|a\| < 1$ *and* a_d *is sufficiently small (depending on bounds for ellipticity and* $C^{1+\eta}$ *bounds for coefficients when* $|x| < 1$), *and if* ν *is sufficiently large and if we set* $\bar{D}_a = D\big(a, a_d/B\sqrt{\nu}\big)$ *then*

(i) $\|x_d^{-\nu}H_f\|_{L^p(\bar{D}_a)} \lesssim \nu^{(d-2)/4r+1/2}(\|\Gamma_a^{-\beta\nu}x_d^{-\nu}Lf\|_p + E_a)$

(ii) $\|x_d^{-\nu}\nabla f\|_{L^p(\bar{D}_a)} \lesssim \nu^{(d-2)/4r-1/2}(\|\Gamma_a^{-\beta\nu}x_d^{-\nu}Lf\|_p + E_a)$

(iii) $\|x_d^{-\nu}f\|_{L^{p'}(\bar{D}_a)} \lesssim \|\Gamma_a^{-\beta\nu}x_d^{-\nu}Lf\|_p + E_a$

where

$$E_a = (\nu^{-1/2}\tau)^{1+\eta}\|x_d^{-\nu}\Gamma_a^{-\beta\nu}H_f\|_p + \|x_d^{-\nu}\Gamma_a^{-\beta\nu}\nabla f\|_p.$$

The proof will be by a suitable change of variables and approximation by the Laplace operator as described above. Before getting into this, let us use Lemma 5.4 to prove Theorem 5.1. We need one other (elementary) lemma.

Lemma 5.5. *Let $\{a^j\}$ be a collection of points such that the discs*

$$\bar{D}_j = D\left(a^j, \frac{a_d^j}{B\sqrt{\nu}}\right)$$

cover $\mathbb{R}_+^d$ and have finite overlap. Then for any $\alpha > 0$, $\sum_j \Gamma_{a^j}(x)^{-\alpha\nu} \leqslant C_\alpha$ independently of $x \in \mathbb{R}_+^d$ and ν, provided ν is sufficiently large.

PROOF. We first observe that if $|a - b| < a_d/\sqrt{\nu}$ then $\Gamma_a(x)^{-\nu} \leqslant C\Gamma_b(x)^{-\alpha_0\nu}$ for suitable constants C and α_0. This may be seen as follows: If $|x - a| < 2a_d/\sqrt{\nu}$ then both sides are ≈ 1. If

$$|x - a| > \frac{2a_d}{\sqrt{\nu}} \quad \text{then} \quad \frac{|x-a|^2}{a_d x_d} \approx \frac{|x-b|^2}{b_d x_d}$$

so for suitable $\alpha_0 < 1$,

$$\begin{aligned} 1 + \frac{|x-a|^2}{4a_d x_d} &\geqslant 1 + \alpha_0 \frac{|x-b|^2}{4b_d x_d} \\ &\geqslant \left(1 + \frac{|x-b|^2}{4b_d x_d}\right)^{\alpha_0}, \end{aligned}$$

and

$$\left(1 + \frac{|x-a|^2}{4a_d x_d}\right)^{-\nu} \leqslant \left(1 + \frac{|x-b|^2}{4b_d x_d}\right)^{-\alpha_0\nu}.$$

It follows that the discrete sum in Lemma 5.5 may be replaced by an integral *i.e.*

$$\sum_j \Gamma_{a^j}(x)^{-\alpha\nu} \lesssim \int_{\mathbb{R}_+^d} \Gamma_a(x)^{-\alpha_0\alpha\nu} \frac{da}{(a_d/\sqrt{\nu})^d}.$$

The integral is scale invariant so we may assume $x_d = 1$. Then (let $\beta = \alpha_0\alpha$)

$$\begin{aligned} &\int_{\mathbb{R}^{d-1}} \left(1 + \frac{|x-a|^2}{4a_d x_d}\right)^{-\beta\nu} da_1 \cdots da_{d-1} \\ &\qquad = \left[\frac{4a_d}{(1+a_d)^2}\right]^{\beta\nu} \int_{\mathbb{R}^{d-1}} \left(1 + \left(\frac{t}{1+a_d}\right)^2\right)^{-\beta\nu} dt_1 \cdots dt_{d-1} \\ &\qquad \approx \left(\frac{4a_d}{(1+a_d)^2}\right)^{\beta\nu} \left(\frac{1+a_d}{\sqrt{\nu}}\right)^{d-1} \end{aligned}$$

$$\int_{\mathbb{R}^d_+} \Gamma_a^{-\beta\nu} \frac{da}{(a_d/\sqrt{\nu})^d} \approx \nu^{1/2} \int_0^\infty \left(\frac{4a_d}{(1+a_d)^2}\right)^{\beta\nu} (1+a_d)^{d-1} \frac{da_d}{a_d^d}$$
$$\lesssim \nu^{1/2} \int_0^\infty \left(\frac{4a_d}{(1+a_d)^2}\right)^{\beta\nu - d + 1} \frac{da_d}{a_d}$$
$$\lesssim 1$$

for large ν, since the integrand dies rapidly when $|a_d - 1| > \nu^{-1/2}$.

PROOF OF THEOREM 5.1. By the usual Carleman argument it will suffice to prove that there is $\rho > 0$ depending only on ellipticity and $C^{1+\eta}$ bounds for L inside the unit disc, such that

$$(5.5) \qquad \|x_d^{-\nu} f\|_{p'} + \nu^{\epsilon} \|x_d^{-\nu} \nabla f\|_p \lesssim \|x_d^{-\nu} Lf\|_p$$

for all $f \in C_0^\infty(D(0,\rho) \cap \mathbb{R}^d_+)$, where $\epsilon = 1/2 - (d-2)/4r > 0$. We remark that in addition to (5.5) we will also prove

$$(5.5)' \qquad \|x_d^{-\nu} H_f\|_p \lesssim \nu^{1-\epsilon} \|x_d^{-\nu} Lf\|_p$$

for $f \in C_0^\infty(D(0,\rho) \cap \mathbb{R}^d_+)$, with ϵ as above. To choose ρ, we first choose a small enough τ and then choose ρ according to the «sufficiently small» in Lemma 5.4.

To prove (5.5), (5.5)′, cover $D(0,\rho)$ with discs $\bar{D}_j = (a^j, a^j_d/B\sqrt{\nu})$ with finite overlap, write down the conclusion of Lemma 5.4(i) for each a^j, raise to the power p, and sum over j, obtaining

$$\sum_j \int_{\bar{D}_j} |x_d^{-\nu} H_f|^p \lesssim \nu^{p(d-2)/4r + p/2} \left(\sum_j \int |x_d^{-\nu} Lf|^p \Gamma_{a^j}^{-\beta\nu p} + \sum_j E^p_{a^j} \right).$$

Do the same for the conclusions of Lemma 5.4(ii) and (iii) after multiplying through by ν and $\nu^{(d-2)/4r + 1/2}$ respectively. Writing out E_{a^j} in longhand, we get

$$\sum_j \int_{\bar{D}_j} |x_d^{-\nu} H_f|^p + \nu^p \sum_j \int_{\bar{D}_j} |x_d^{-\nu} \nabla f|^p + \sum_j \nu^{p(d-2)/4r + p/2} \left(\int_{\bar{D}_j} |x_d^{-\nu} f|^{p'} \right)^{p/p'}$$
$$\lesssim \nu^{p(d-2)/4r + p/2} \sum_j \left\{ \int |x_d^{-\nu} Lf|^p \Gamma_{a^j}^{-\beta\nu p} + (\nu^{-1/2} t)^{p(1+\eta)} \int |x_d^{-\nu} H_f|^p \Gamma_{a^j}^{-\beta\nu p} \right.$$
$$\left. + \int |x_d^{-\nu} \nabla f|^p \Gamma^{-\beta\nu p} \right\}.$$

The $\bar{D}_j$ cover the support of f so the first sum on the left hand side dominates $\|x_d^{-\nu} H_f\|_p^p$. Likewise the second sum dominates $\nu^p \|x_d^{-\nu} \nabla f\|_p^p$, and (using that $\ell^1 \subset \ell^{p/p'}$) the third sum dominates $\nu^{p(d-2)/4r + p/2} \|x_d^{-\nu} f\|_{p'}^p$. On the right side we use Lemma 5.5 to bound $\sum \Gamma_{aj}^{-\beta\nu p}$ by a constant, and we also use $a_d \leqslant \rho$. The resulting inequality is

$$\|x_d^{-\nu} H_f\|_p^p + \nu^p \|x_d^{-\nu} \nabla f\|_p^p + \nu^{p(d-2)/4r+p/2} \|x_d^{-\nu} f\|_{p'}^p$$
$$\lesssim \nu^{p(d-2)/4r+p/2} (\|x_d^{-\nu} Lf\|_p^p + (\nu^{-1/2}\tau)^{(1+\nu)p} \|x_d^{-\nu} H_f\|_p^p + \|x_d^{-\nu} \nabla f\|_p^p).$$

Our assumption on r says that $\nu^{p(d-2)/4r+p/2}\nu^{-(1+\nu)p/2} = 1$. So for small τ we can bootstrap the second term on the right side. We can also bootstrap the third term obtaining

$$\|x_d^{-\nu} H_f\|_p^p + \nu^p \|x_d^{-\nu} \nabla f\|_p^p + \nu^{p(d-2)/4r+p/2} \|x_d^{-\nu} f\|_{p'}^p$$
$$\lesssim \nu^{p(d-2)/4r+p/2} \|x_d^{-\nu} Lf\|_p^p$$

which is equivalent to (5.5), (5.5)′.

It remains to prove Lemma 5.4. The following lemma provides the necessary change of coordinates.

Lemma 5.6. *Suppose L is as in Theorem* 5.1. *Then for* $p \in \mathbb{R}^d$ *and* $\epsilon > 0$ *small enough there is a* C^2 *(actually* C^∞*) diffeomorphism* $T: \mathbb{R}^d \to \mathbb{R}^d$ *with*

(i) $T(p) = p$, *and there is a linear map* $S: \mathbb{R}^d \to \mathbb{R}^d$ *with* $(Sx)_d = x_d$ *and such that* $Tx = p + S(x - p)$ *when* $|x - p| > \epsilon$.
(ii) $|DT(x) - S| \lesssim |x - p|$, $\quad |D^2T(x)| \lesssim 1$, $\quad |DT^{-1}(x) - S^{-1}| \lesssim |x - p|$, $|D^2T^{-1}(x)| \lesssim 1$.
(iii) $L(f \circ T) = (\Delta f) \circ T + O(|x - p|^{1+\eta}|H_f| \circ T) + O(|\nabla f| \circ T)$ *when* $|x - p| < \epsilon/2$.

All bounds depend only on ellipticity and $C^{1+\eta}$ *bounds for L near p (and not on* ϵ*).*

PROOF. Let us first do the case where $a_{ij}(p) = \delta_{ij}$ (Kronecker delta). We would like to start by choosing geodesic normal coordinates. We do not actually have geodesic normal coordinates unless a_{ij} is C^2, but we can use a second order approximation to them. We give this construction for the reader's convenience:

By assumption we have $a_{ij}(x) = \delta_{ij} = \ell_{ij}(x - p) + O(|x - p|^{1+\eta})$ where ℓ_{ij} are linear functions with $\ell_{ij} = \ell_{ji}$. We claim there are second order homogeneous polynomials Q_i with

$$\ell_{ij} + \frac{dQ_i}{dx_j} + \frac{dQ_j}{dx_i} = 0.$$

To see this let $a_{ijk} = \dfrac{1}{2}\left(\dfrac{d\ell_{jk}}{dx_i} - \dfrac{d\ell_{ik}}{dx_j} - \dfrac{d\ell_{ij}}{dx_k}\right)$. Then $a_{ijk} = a_{ikj}$ and

$$a_{ijk} + a_{jik} + \frac{d\ell_{ij}}{dx_k} = 0.$$

It follows that there are homogeneous quadratics Q_i with $\frac{d^2Q_i}{dx_j dx_k} = a_{ijk}$ and that they satisfy

$$\frac{d^2Q_i}{dx_j dx_k} + \frac{d^2Q_j}{dx_i dx_k} + \frac{d\ell_{ij}}{dx_k} = 0$$

for all i, j, k and $\frac{d^2Q_j}{dx_j} + \frac{d^2Q_j}{dx_i} + \ell_{ij} = 0$.

Now define $T(x) = x + \phi(x)Q(x-p)$ where $Q = (Q_1 \cdots Q_d)$ and $\phi(x) = 1$ when $|x-p| < \epsilon/2$, $\phi(x) = 0$ when $|x-p| > \epsilon$ and $|D^\alpha \phi| \lesssim \epsilon^{-|\alpha|}$.

Then $T(x) = x$ if $x = p$ or $|x-p| > \epsilon$, and a short calculation proves $|D^2T| \lesssim 1$, $|DT(x) - I| \lesssim |x-p|$. This implies if ϵ is small that T is a diffeomorphism of $\mathbb{R}^d$ and T^{-1} satisfies the same bounds. Thus it remains to prove (iii). By choice of Q we have

$$\sum_{i,j} (\delta_{ij} + \ell_{ij}(x-p))\left(\delta_{im} + \frac{dQ_m}{dx_i}(x-p)\right)\left(\delta_{jn} + \frac{dQ_n}{dx_j}(x-p)\right)$$
$$= \delta_{mn} + O(|x-p|^2).$$

On the other hand, if we denote $y = Tx$ then with the summation convention

$$L(f \circ T)(x) = a_{ij}\frac{dy_m}{dx_i}\frac{dy_n}{dx_j}\frac{d^2f}{dy_m dy_n} + a_{ij}\frac{d^2y_k}{dx_i dx_j}\frac{df}{dy_k}. \tag{5.6}$$

Up to terms of order $|x-p|^{1+\eta}$ we have $a_{ij} = \delta_{ij} + \ell_{ij}(x-p)$ and

$$\frac{dy_m}{dx_i} = \delta_{im} + \frac{dQ_m}{dx_i}(x-p)$$

when $|x-p| < \epsilon$. It follows that the first term on the right in (5.6) is $O(|x-p|^{1+\eta}|H_f \circ T|)$. The second term is clearly $O(|\nabla f| \circ T)$ so we are done with the case where $a_{ij}(p) = \delta_{ij}$. We can reduce the general case to this case by a preliminary affine change of variables: explictly, let $A = (a_{ij}(p))$ and choose S so that $S^*S = A^{-1}$ and $(Sx)_d = x_d$ (this is possible for any positive symmetric A^{-1}). The preceding argument then applies to the operator $L(f \circ S) \circ S^{-1}$ in place of L giving a change of variable T_0, and we let $T = S \circ T_0$.

Now, in the situation of Lemma 5.4, we can apply Lemma 5.6 with $p = a$ and $\epsilon = a_d/2$ (provided $|a| < 1$ and a_d is small enough). We obtain a change of variables T_a. In order to proceed we now need some further elementary properties of our weights especially as to how they behave under T_a.

Lemma 5.7.

(i) $\Gamma_a^{-\nu} \circ T_a \lesssim \Gamma_a^{-\gamma\nu}$ *for a suitable fixed constant* γ.
(ii) *For any* $\epsilon > 0$ *we will have* $x_d^{-\nu} \circ T_a \lesssim x_d^{-\nu}\Gamma_a^{\epsilon\nu}$ *provided* a_d *is sufficiently small.*
Moreover (i) *and* (ii) *also hold for* T_a^{-1}.
(iii) *For any given* k, $|x-a|^k\Gamma_a^{-\nu} \lesssim (\nu^{-1/2}a_d)^k\,\Gamma_a^{-\nu/2}$ *provided* ν *is large enough.*

PROOF. Part (iii) follows from the rapid decay of Γ_a outside D_a. To prove (i) and (ii) we consider the regions $|x-a| < a_d/2$ and $|x-a| > a_d/2$ separately. We abbreviate T_a by T.

In the first region, we have $(Tx)_d = x_d + O(|x-a|^2)$ by (ii) of Lemma 5.6 (and the fact that $(Sx) = x_d$) and therefore

$$(Tx)_d \geqslant x_d\left(1 - C\frac{|x-a|^2}{4x_d}\right).$$

If ϵ is given then by making a_d small we make this larger than or equal to $x_d(1 - C\epsilon\,|x-a|^2/4a_dx_d) \geqslant x_d \cdot \Gamma_a(x)^{-C'\epsilon}$ which gives (ii). As for (i), we have Lipschitz bounds on T and therefore $\Gamma_a(Tx) \geqslant (1 + C\,|x-a|^2/4a_dx_d)$ for suitable C hence $\Gamma_a(Tx) \geqslant \Gamma_a(x)^{C'}$. In the region $|x-a| > a_d/2$ (ii) is a tautology and (i) again follows from the Lipschitz property of T. Since all of this followed from (i) and (ii) of Lemma 5.6 it also holds for T^{-1}.

PROOF OF LEMMA 5.4. We will prove (i) and then indicate the modifications necessary to get (ii) and (iii). Where B was unspecified, we now specify it so be an upper bound for the Lipschitz norms of the T_a and T_a^{-1}. σ below is another fixed constant. $D = D\big(a, a_d/\sqrt{\nu}\,\big)$ and $\bar{D} = D\big(a, a_d/B\sqrt{\nu}\,\big)$.

We justify the following string of inequalities below.

$$
\begin{aligned}
\|x_d^{-\nu}H_f\|_{L^p(\bar{D})} &\lesssim \|(x_d^{-\nu}H_f)\circ T^{-1}\|_{L^p(D)} \\
&\lesssim \|(T^{-1}x)_d^{-\nu}H_{f\circ T^{-1}}\|_{L^p(D)} + \|(T^{-1}x)_d^{-\nu}\nabla(f\circ T^{-1})\|_{L^p(D)} \\
&\lesssim \|x_d^{-\nu}H_{f\circ T^{-1}}\|_{L^p(D)} + \|x_d^{-\nu}\nabla(f\circ T^{-1})\|_{L^p(D)} \\
&\lesssim \nu^{(d-2)/4r+1/2}\|\Delta(f\circ T^{-1})x_d^{-\nu}\Gamma_s^{-\nu}\|_p \qquad (5.7)\\
&\lesssim \nu^{(d-2)/4r+1/2}\big(\|(Lf)\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \qquad (5.8)\\
&\qquad + \|\,|x-a|^{1+\eta}H_f\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \\
&\qquad + \|(\nabla f)\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \\
&\qquad + \|x_d^{-\nu}\Gamma_a^{-\nu}H_f\circ T^{-1}\|_{L^p(|x-a|>(a_d/4B^2))}\big)
\end{aligned}
$$

$$\begin{aligned}\|x_d^{-\nu}H_f\|_{L^p(\bar D)} &\lesssim \nu^{(d-2)/4r+1/2}(\|(Lf)\circ T^{-1}x_d^{-\nu}\Gamma_a^{\nu}\|_p \\ &\quad + (\nu^{-1/2}a_d)^{1+\eta}\|H_f\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu/2}\|_p \\ &\quad + \|(\nabla f)\circ T^{-1}x_d^{-\nu}\Gamma_a^{-\nu}\|_p \\ &\quad + \nu^{-100}\|x_d^{-\nu}\Gamma_\alpha^{-\nu/2}H_f\circ T^{-1}\|_p) \\ &\lesssim \nu^{(d-2)/4r+1/2}(\|x_d^{-\nu}\Gamma_a^{-\sigma\nu}Lf\|_p \\ &\quad + [(\nu^{-1/2}a_d)^{1+\eta} + \nu^{-100}]\|x_d^{-\nu}\Gamma_a^{-\sigma\nu}H_f\|_p \\ &\quad + \|x_d^{-\nu}\Gamma_a^{-\sigma\nu}\nabla f\|_p).\end{aligned}$$

Justification. First inequality: T is bilipshitz and $T^{-1}D \subseteq \bar D$. Second inequality: $|H_f\circ T^{-1}| \lesssim |H_{f\circ T^{-1}}| + |\nabla(f\circ T^{-1})|$ if T has bounded first and second derivatives. Third inequality: Lemma 5.7 (ii) and the fact that $\Gamma_a^{-\nu}\approx 1$ on D. Forth inequality: Lemma 5.1. Fifth inequality (5.7) $\lesssim$ (5.8): split $\mathbb{R}_+^d$ in the two regions $|x-a| < a_d/4B$ and $|x-a| > a_d/4B$. The last two terms in (5.8) bound the L^p norm in (5.7) over the region $|x-a| > a_d/4B$ by the same calculation that was used to justify the second inequality. When $|x-a| < a_d/4B$ we know that $|T_a x - a| < a_d/4$, so we can write down (iii) of Lemma 5.6 for the function $f\circ T^{-1}$ and then compose with T^{-1} obtaining

$$(Lf)\circ T^{-1} = \Delta(f\circ T^{-1}) + O(|T^{-1}x - a|^{1+\eta}|H_{f\circ T^{-1}}|) + O(|\nabla(f\circ T^{-1})|).$$

Also $|T^{-1}x - a|$ is comparable with $|x-a|$ and $|H_{f\circ T^{-1}}|$ may be replaced by $|H_f\circ T^{-1}| + |(\nabla f)\circ T^{-1}|$ and $|\nabla(f\circ T^{-1})|$ by $|\nabla f\|\circ T^{-1}$ as in the second inequality. This shows (5.7) $\leqslant$ (5.8). To pass from (5.8) to the next line we estimate the second term in the parenthesis using Lemma 5.7 (iii) and the last term using that $\Gamma_a^{-\nu/2}$ is $\lesssim \nu^{-k}$ for any fixed k when $|x-a| > a_d/4B$. Finally the last inequality follows by reversing the change of variables and using Lemma 5.7(i) and (ii). One can take $\sigma = \gamma/5$ with γ as in Lemma 5.7(i). Lemma 5.4 follows since $(\nu^{-1/2}a_d)^{1+\eta} + \nu^{-100}$ may be made less than $(\tau\nu^{-1/2})^{1+\eta}$ for any given τ by making ν large and a_d small.

To obtain (ii) of Lemma 5.4, we start with $\|x_d^{-\nu}\nabla f\|_{L^p(\bar D)}$, make the change of variables T^{-1} and then use (ii) of Lemma 5.1 to obtain an expression like (5.7) but with $\nu^{(d-2)/4r-1/2}$ in place of $\nu^{(d-2)/4r+1/2}$. To obtain (iii) of Lemma 5.4 we start with $\|x_d^{-\nu}f\|_{L^{p'}(\bar D)}$ and proceed the same.

Now we prove a refinement of a result of Chanillo-Sawyer [4]. Recall (see [4] and references there) that the «Fefferman-Phong class» F_r is defined by the condition $V\in F_r$ if and only if $\|V\|_{L^r(B)} \leqslant C_V t^{d/r-2}$ for all t and all balls B of radius t. The F_r norm is the smallest possible C_V.

Theorem 5.2. *Suppose $d\geqslant 4$, $r > d/2-1$ and V is a function with sufficiently small F_r norm. Then the inequality $|\Delta u| \leqslant V|u|$ has the WUCP in the sense that if $u\in W^{2,p}_{\rm loc}$, $|\Delta u|\leqslant V|u|$, u vanishes on an open set then $u=0$.*

Remark. Chanillo-Sawyer required $r > d/2 - 1/2$ but they also treated the SUCP. An identical result to Theorem 5.2 has been proved independently by Ruiz-Vega [12] by a different argument. It should be pointed out that their version was circulated several months before ours. We include the result here only because it follows very easily from what we have been doing.

PROOF. By an argument in [4], it suffices to prove the following:

Lemma 5.8. *If $V \in F_r$ then we have $\|x_d^{-\nu} Vu\|_2 \lesssim \|x_d^{-\nu} V^{-1} \Delta u\|_2$ for all $u \in C_0^\infty(\mathbb{R}_+^d)$ and sufficiently large ν.*

Covering $\mathbb{R}_+^d$ by a family of discs $D(a^j, \nu^{-1/2} a_d^j)$ and using Lemma 5.5 as in the proof of Theorem 5.1, we see it will suffice to prove

Lemma 5.8′. *If $V \in F_r$ then $\|x_d^{-\nu} Vu\|_{L^2(D_a)} \lesssim \|\Gamma_a^{-\nu/2} x_d^{-\nu} V^{-1} \Delta u\|_{L^2}$ uniformly in $a \in \mathbb{R}_+^d$ and ν sufficiently large, and $u \in C_0^\infty(\mathbb{R}_+^d)$.*

This is a variant on Lemma 5.1 and again we reduce to a lemma in terms of the weights $|x|^{-\nu}$.

Lemma 5.8″. *If k is a large enough fixed positive number then, letting e be a unit vector, ν large enough, and defining*

$$D = D(e, \nu^{-1/2}), \qquad \Omega = \{x \colon |x| > 1/10, x \cdot e > 0\}$$

we have

$$\| |x|^{-\nu} Vu\|_{L^2(D)} \lesssim \|(1 + \nu^{1/2}|x - e|)^k |x|^{-\nu} V^{-1} \Delta u\|_{L^2(\Omega)}$$

for all $u \in C_0^\infty(\Omega)$.

PROOF OF LEMMA 5.8′. (assuming 5.8″). We may take a to be such that $a_d = 1/2$.

Using Lemma 5.8″ and the argument in the proof of Lemma 5.1, we obtain

$$\|x_d^{-\nu} Vu\|_{L^2(D_a)} \lesssim \|\Gamma_s^{-\nu}(1 + \nu^{1/2}|x - e|)^k x_d^{-\nu} V^{-1} \Delta u\|_{L^2}.$$

However, $(1 + \nu^{1/2}|x - e|)^k \Gamma_a^{-\nu} \lesssim \Gamma_a^{-\nu/2}$ by part (iii) of Lemma 5.6.

PROOF OF LEMMA 5.8″. It will suffice to show

$$\|V^{1/2} L_\nu V^{1/2} f\|_{L^2(D)} \lesssim \|(1 + \nu^{1/2}|x - e|)^k f\|_{L^2}. \tag{5.9}$$

If $f \in L^2(\Omega)$, where $L_\nu = L_\nu^0$. It is known (see references in [4]) that this kind of inequality is true if L_ν is replaced by $|x - y|^{2-d}$ (and therefore also if L_ν is replaced by M_ν) provided $V \in F_r$ for some $r > 1$. So it will suffice to prove (5.9) with L_ν replaced by N_ν. We write $N_\nu = S + T$ as in the proof of Lemma 5.2 and will prove (5.9) with L_ν replaced by N_ν. We write $N_\nu = S + T$ as in the proof of Lemma 5.2 and will prove (5.9) separately for S and for T. To deal with S, write $S = \sum_{\substack{\nu^{-1} \leq \lambda \leq \nu^{-1/2} \\ \lambda = 2^j \nu}} \psi_\lambda S$ as in the proof of Lemma 5.2. To estimate

$$\| V(x)^{1/2} \psi_\lambda (x, y) S(x, y) V(y)^{1/2} \|_{L^2(\Omega) \to L^2(D)} . \tag{5.10}$$

It will suffice (by a partition of unity and Cotlar's Lemma as in the proof of Lemma 2.4) to estimate the action on fuctions supported in a ball of radius λ. If B is such a ball, f vanishes off B and $\tilde{B}$ is a suitable fixed multiple of B then, using Hölder's inequality and Lemma 5.2′,

$$\begin{aligned} \| V^{1/2} (\psi_\lambda S) V^{1/2} f \|_2 &\leqslant \| V \|_{L^r(\tilde{B})}^{1/2} \| (\psi_\lambda S) V^{1/2} f \|_{p'} \\ &\lesssim \| V \|_{L^r(\tilde{B})}^{1/2} \| V^{1/2} f \|_{p'} \nu^{-1} D(\nu, \lambda) \lambda^{-1/r} \\ &\lesssim \| V \|_{L^r(\tilde{B})} \| f \|_2 \nu^{-1} D(\nu, \lambda) \lambda^{-1/r} . \end{aligned}$$

Now use the definition of F_r to estimate $\| V \|_{L^r(\tilde{B})} \lesssim \lambda^{d/r - 2}$. Substituting this in we obtain an estimate (5.10) $\lesssim (\nu\lambda)^{(d-2)/2r - 1}$. By our assumption on r the power of $\nu\lambda$ here is negative and we may sum over λ to obtain

$$\| V^{1/2} S V^{1/2} \|_{L^2(\Omega) \to L^2(D)} \lesssim 1 .$$

Now we have to consider T. Write

$$T(x, y) = \sum_{j=0}^{\infty} T(x, y) \phi_j(y) = \sum_{j=0}^{\infty} T_j(x, y),$$

where

$$\phi_j = \begin{cases} 1, & \text{if } \; 2^j \nu^{-1/2} < |y - e| < 2^{j+1} \nu^{-1/2}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider a given T_j. We know from Lemma 5.2′ that T is bounded from $L^p(\Omega, |y|^m \, dy)$ to $L^{p'}(D, dx)$ with norm $\lesssim \nu^{(3d-2)/4r - 3/2}$. Therefore T_j is bounded from $L^p(\Omega, |y|^m \, dy)$ to $L^{p'}(D, dx)$ with norm $\lesssim \nu^{(3d-2)/4r - 3/2}$. Therefore T_j is bounded from $L^p(\Omega, dy)$ to $L^{p'}(D, dx)$ with norm

$$\lesssim (1 + 2^j \nu^{-1/2})^m \nu^{(3d-2)/4r - 3/2} .$$

Using Hölder's inequality as in the estimation for S, $V^{1/2} T V^{1/2}$ is bounded from $L^2(\Omega, dy)$ to $L^2(D, dx)$ with norm

$$\begin{aligned}
&\lesssim (1 + 2^j \nu^{-1/2})^m \nu^{(3d-2)/4r-3/2} \| V \|^{1/2}_{L^r(D(e, 2^{j+1}\nu^{-1/2}))} \| V \|^{1/2}_{L^r(D(e, \nu^{-1/2}))} \\
&\lesssim (1 + 2^j \nu^{-1/2})^m \nu^{(3d-2)/4r-3/2} (2^j \nu^{-1/2})^{d/2r-1} \nu^{-(d/2r-1)/2} \\
&= 2^{j(d/2r-1)} (1 + 2^j \nu^{-1/2})^m \nu^{(d-2)/4r-1/2} \\
&\leqslant 2^{j(d/2r-1)} (1 + 2^j \nu^{-1/2})^m
\end{aligned}$$

by choice of r. The last expression may be bounded by $2^{-j}(1 + \nu^{1/2}|y - e|)^k$ for appropriate k if y is such that $T_j(x, y)$ is nonzero for some x, and therefore we obtain

$$\| V^{1/2} T_j V^{1/2} f \|_2 \lesssim 2^{-j} \| (1 + \nu^{1/2}|y - e|)^k f \|_2 .$$

We may now sum over j to obtain

$$\| V^{1/2} T V^{1/2} f \|_2 \lesssim \| (1 + \nu^{1/2}|y - e|)^k f \|_2$$

and we are done.

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Recibido: 3 de Octubre de 1990.

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* Partially supported by the NSF.

CORRIGENDA:

CONTINUITE-SOBOLEV DE CERTAINS OPERATEURS PARADIFFERENTIELS

Revista Matemática Iberoamericana, Vol. 6, N.° 3 y 4, p. 125-140 (1990).

Abdellah YOUSSFI

— *Pages 134-135:*

Dans la preuve de la proposition 4 et pour $0 < s - |\alpha| < n/2$, l'implication (2) entraîne (1) n'est pas correcte. Par contre elle reste valable por $s = |\alpha|$.

— *Page 139:*

Une partie du théoreme 3 est liée à la proposition 4; cette liaison concerne le cas $0 < s - |\alpha| < n/2$ pour montrer que (1) entraîne (2) et que (2) entraîne (3). L'énnoncé de ces deux implications est, pour l'instant, mis en cause. Une autre correction suivra celle-ci desqu'une démonstration correcte serait mise au point.

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