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Pointwise and Spectral Control of Plate Vibrations

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Résumé

On considère le problème du contrôle ponctuel (c'est-à dire au moyen d'une masse de Dirac située en un point fixé) des vibrations d'une plaque Ω. Sous des conditions aux limites générales, incluant les plaques posées ou encastrées, mais excluant (et pour cause) le cas où existent des vibrations propres multiples, nous montrons la contrôlabilité des combinaisons linéaires finies des fonctions propres en tout point de Ω qui n'est zéro *d'aucune fonction propre* et en tout temps strictement supérieur à la moitié de la surface de la plaque. Ce résultat est optimal car *aucune* combinaison linéaire finie non nulle de fonctions propres n'est ponctuellement contrôlable en un temps strictement inférieur à la moitié de la surface de la plaque. Sous la même condition sur le temps, mais pour un domaine Ω *quelconque* de $\mathbb{R}^2$, on résout le problème du contrôle spectral *interne*, c'est-à-dire que pour tout disque ouvert $\omega \subset \Omega$, une combinaison linéaire finie quelconque des fonctions propres peut être ramenée à l'équilibre au moyen d'un contrôle $h \in \mathcal{D}((0, T) \times \Omega)$ tel que $\operatorname{supp}(h) \subset (0, T) \times \omega$.

Abstract

We consider the problem of controlling pointwise (by means of a time dependent Dirac measure supported by a given point) the motion of a vibrating plate Ω. Under general boundary conditions, including the special cases of simply

supported or clamped plates, but of course excluding the cases where some multiple eigenvalues exist for the biharmonic operator, we show the controllability of finite linear combinations of the eigenfunctions at any point of Ω *where no eigenfunction vanishes* at any time greater than half of the plate's area. This result is optimal since *no* finite linear combination of the eigenfunctions other than 0 is pointwise controllable at a time smaller than half of the plate's area. Under the same condition on the time, but for an *arbitrary* domain Ω in $\mathbb{R}^2$, we solve the problem of *internal* spectral control, which means that for any open disk $\omega \subset \Omega$, any finite linear combination of the eigenfunctions can be set to equilibrium by means of a control function $h \in \mathfrak{D}((0, T) \times \Omega)$ supported in $(0, T) \times \omega$.

1. Introduction and Functional Setting

In order to make the theory more transparent, we shall consider the general case of a second order conservative evolution equation and apply only at the end our abstract results to the specific case of a 2-dimensional vibrating plate. Let Ω be a bounded domain of $\mathbb{R}^N$ (or a compact N-dimensional manifold without boundary) and A a positive self-adjoint operator in $H = L^2(\Omega)$. We assume that A satisfies the following properties

(1.1) A is *coercive* on H.
(1.2) $D(A^{1/2}) \subset C(\bar{\Omega})$ with continuous imbedding.

Given $T > 0$, $\xi \in \Omega$ and $[y^0, y^1] \in D(A^{1/2}) \times L^2(\Omega)$, we are interested in the existence of a *control function* $h \in L^2(]0, T[)$ such that $\operatorname{supp}(h) \subset [0, T]$ and for which the unique generalized solution y of

$$(1.3) \qquad \begin{cases} y'' + Ay = h(t)\delta(x - \xi) & \text{in} \quad]0, T[, \\ y(0, x) = y^0(x), & \\ y'(0, x) = y^1(x) & \text{in} \quad \Omega, \end{cases}$$

satisfies $y(T, x) = y'(T, x) = 0$ in Ω. If such a control h exists, we shall say that the state $[y^0, y^1]$ is «pointwise exactly L^2-controllable in ξ at time T».

The possibility of solving this «pointwise exact controllability problem» is related to the amount of information revealed by the restriction to $[0, T]$ of $t \mapsto \phi(t, \xi)$ where ϕ is an arbitrary solution of the homogeneous equation

$$(1.4) \qquad \phi'' + A\phi = 0 \quad \text{in} \quad]0, T[, \qquad \phi \in C(0, T; V) \cap C^1(0, T; H)$$

with $V = D(A^{1/2})$, $H = L^2(\Omega)$. (Note that as a consequence of (1.2) we have $\phi(t, \xi) \in C([0, T]$ for any such solution ϕ). In fact if any $[y^0, y^1]$ from a *dense* subset of $V \times H$ is exactly L^2-controllable in ξ at time T, then any solution

ϕ of the homogeneous equation (1.4) such that $\phi(t, \xi)$ vanishes identically on $]0, T[$ is the trivial solution $\phi \equiv 0$. *Conversely*, if any solution ϕ of the homogeneous equation (1.4) such that $\phi(t, \xi)$ vanishes identically on $]0, T[$ is the trivial solution $\phi \equiv 0$, then for each $(\phi^0, \phi^1) \in V \times H$, we consider the (clearly well-defined) *norm*

$$p(\phi^0, \phi^1) = \left\{ \int_0^T \phi^2(t, \xi)\, dt \right\}^{1/2}, \tag{1.5}$$

where ϕ is the solution of equation (1.4) with initial data (ϕ^0, ϕ^1). The following result then follows from the general HUM method of J. L. Lions ([17, 18, 19]).

Proposition 1.1. *A given state $[y^0, y^1] \in V \times H$ is exactly L^2-controllable at ξ in time T if and only if there exists a constant $C \geqslant 0$ such that for every $(\phi^0, \phi^1) \in V \times H$,*

$$\left| \int_\Omega (\phi^0 y^1 - \phi^1 y^0)\, dx \right| \leqslant C p(\phi^0, \phi^1). \tag{1.6}$$

As was clearly established in [6], the set of pointwise exactly L^2-controllable states (always a *dense* subset of $V \times H$ when p is a norm) is usually complicated and more precisely depends on the observation point ξ in a very complicated and unstable way, even in the simplest case of the standard vibrating string with fixed end! The only reasonable thing to be expected in general is that (1.6) might hold true when both y^0 and y^1 are finite linear combinations of the eigenfunctions of A, assuming that no eigenfunction vanishes at ξ. This implies in particular that all eigenvalues of A are simple, a condition that we shall assume in most of this text (Sections 2, 3 and 5). The controllability of all states for which y^0 and y^1 are finite linear combinations of the eigenfunctions of A is what we shall call «pointwise spectral controllability». Taking account of the form of the general solutions to the homogeneous equation, it is natural to apply the methods of harmonic analysis to solve this problem. Indeed, any solution of (1.4) can be written as a series

$$\phi(t, x) = \sum \left\{ \phi_n(x) \cos \sqrt{\lambda_n t} + \psi_n(x) \sin \sqrt{\lambda_n t} \right\}$$

where the functions ϕ_n, ψ_n are eigenfunctions of A associated to the eigenvalues λ_n, or in complex form

$$\phi(t, x) = \sum \varphi_j(x) e^{i\mu_j t}$$

where the μ_j stand for the (positive or negative) square roots of the eigenvalues λ_n. Thus for fixed x, it is a linear combination of some complex exponentials, the properties of which will be the key point of this work. Our main result will, maybe surprinsingly, turn out to be a consequence of one among the deepest classical results on harmonic analysis from the «sixties», namely the *Beurling Malliavin criterion* for computing the *completeness radius of a family of complex exponentials.* The application of this powerful machinery to our problem is the object of Sections 2 and 3 of this paper. The case of a vibrating plate with constant Lamé coefficients is a special case of our abstract result obtained for $N = 2$ and $A = \Delta^2$ with relevant boundary conditions. In Section 4, we shall combine the result of Section 2 with, essentially, a biorthogonality technique in the spirit of [2, 16, 22] to solve the easier problem of «internal spectral controllability» under slightly relaxed conditions on the domain. However we feel that much more should be done in this last direction, as already strongly suggested by the special cases considered in [6, 11, 15]. Finally in Section 5, we consider some additional examples and we discuss the relationship between spectral controllability and some uniqueness questions.

2. Some Properties of the Completeness Radius of a Family of Complex Exponentials

The main tool from harmonic analysis that we shall use in this paper is the notion of completeness radius and its characterization by some estimates.

2.1. Definition and some properties of the Completeness Radius

Definition 2.1.1. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of distinct real numbers. Consider all the functions of the form*

$$f(t) = \sum_{k\in J} f_k e^{i\lambda_k t}, \tag{2.1}$$

J being any finite subset of $\mathbb{Z}$. The completeness radius of Λ is defined as

$R(\Lambda) = \sup\{A > 0$: the functions f of the form (2.1) are dense in $C([-A, A])\}$.

In particular, if the functions f of the form (2.1) *are dense in* $C([-A, A])$ *for all $A > 0$, we set $R(\Lambda) = \infty$. On the other hand, if the density fails for all $A > 0$, we set $R(\Lambda) = 0$.*

Remark 2.1.2. A classical result from the theory of nonharmonic Fourier series (*cf. e.g.* [24, Theorem 8 p. 129]) asserts that either the functions of the form (2.1) are dense in $C([a, b])$, or no complex exponential of the form $e^{i\nu t}$ with ν different from all λ_n can be obtained as a limit of functions of the form (2.1) in $C([a, b])$. This interesting alternative is the main idea for the proof of Proposition 2.2.1 below.

Remark 2.1.3. For all $p \in [1, +\infty)$ we also have

$$R(\Lambda) = \sup\{A > 0: \text{ the functions } f \text{ of the form (2.1) are dense in } L^p([-A, A])\},$$

with the same conventions in the limiting cases $R(\Lambda) = 0, \infty$ (see [24]). In the sequel we shall be especially concerned by the case $p = 2$.

2.2. A «Density-Controllability» Alternative

The main result of this section is the following

Proposition 2.2.1. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. Then we have the following properties*

(1) *For each $T > 2R(\Lambda)$ and for each $n \in \mathbb{Z}$ there exists a constant C_n such that*

$$|f_n| \leqslant C_n\left\{\int_0^T |f(t)|^2\,dt\right\}^{1/2}, \tag{2.2}$$

for each function f of the form (2.1) *with $n \in J$.*

(2) *On the other hand for each $T < 2R(\Lambda)$ and for each finite sequence $\{\alpha_n\}_{n\in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that*

$$\left|\sum_{n\in F}\alpha_n f_n\right| \leqslant C\left\{\int_0^T |f(t)|^2\,dt\right\}^{1/2} \tag{2.2'}$$

for each f of the form (2.1) *with $F \subset J$.*

As a first step we will establish the following lemma.

Lemma 2.2.2. *Let $I = (0, T)$ and let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. Assume that the set of functions of the form* (2.1) *is not dense in $L^2(I)$. Then, for each $n \in \mathbb{Z}$ there exist a constant C_n such that* (2.2) *holds for each function f of the form* (2.1) *with $n \in J$.*

PROOF. If (2.2) is not satisfied for some n, we can find a sequence of functions $\{f^p\}$ of the form (2.1) such that $|f_n^p| = 1$ and $\int_I |f^p(x)|^2\,dx \to 0$ as $p \to +\infty$. It follows that the constant 1 is the limit in $L^2(I)$ of some functions g of the form

$$g(t) = \sum_{k\in J} g_k e^{i\mu_k t}, \tag{2.3}$$

where J is a finite subset of $\mathbb{Z} - \{n\}$, and: $\mu_k = \lambda_k - \lambda_n$. By repeated integration in t, we deduce that all polynomials of t with complex coefficients are also limits in $L^2(I)$ of some functions g of the form (2.3). Indeed, let $p \in \mathbb{N}$, $\epsilon > 0$ and g of the form (2.3) be such that

$$\left\| t^p - \sum_{k \in J} g_k e^{i\mu_k t} \right\|_2 \leqslant \epsilon$$

where $\| \ \|_2$ stands for the norm in $L^2(I)$. By integrating in t, we deduce easily the estimate

$$\left\| t^{p+1} - (p+1) \sum_{k \in J} g_k \frac{e^{i\mu_k t}}{\mu_k} + (p+1) \sum_{k \in J} \frac{g_k}{\mu_k} \right\|_\infty \leqslant (p+1)\epsilon T^{1/2}$$

where $\| \ \|_\infty$ stands for the norm in $L^\infty(I)$. Hence, in particular

$$\left\| t^{p+1} - (p+1) \sum_{k \in J} g_k \frac{e^{i\mu_k t}}{\mu_k} + (p+1) \sum_{k \in J} \frac{g_k}{\mu_k} \right\|_2 \leqslant (p+1)\epsilon T.$$

Then by approximating the constant $(p+1) \sum_{k \in J} (g_k/\mu_k)$ in $L^2(I)$ by functions of the form (2.3), we find a sequence of coefficients $\{g'_k\}_{k \in J'}$ for which

$$\left\| t^{p+1} - \sum_{k \in J} g'_k e^{i\mu_k t} \right\|_2 \leqslant 2(p+1)\epsilon T.$$

This proves the claim by induction on p since it has been proved already for $p = 0$. Finally by the Stone-Weierstrass density theorem, the functions g of the form (2.3) are dense in $L^2(I)$, and the same property follows at once for functions f of the form (2.1).

PROOF OF PROPOSITION 2.2.1. It follows clearly from the definition of $R(\Lambda)$ that for each $T > 2R(\Lambda)$ the functions f of the form (2.1) are not dense in $L^2(0, T)$, and therefore assertion 1) is an immediate consequence of Lemma 2.2.2. On the other hand for each *finite set* $F \subset \mathbb{Z}$, if we denote by Λ' the set $\{\lambda_n\}_{n \in \mathbb{Z} - F}$, then classically $R(\Lambda') = R(\Lambda)$. As a consequence for each $T < 2R(\Lambda)$, the set of functions f of the form (2.1) with $J \cap F = \varnothing$ is dense in $L^2(0, T)$, and therefore for each non trivial sequence $\{\alpha_n\}_{n \in F}$ of complex numbers, the function

$$a(t) = \sum_{k \in F} \alpha_k e^{i\lambda_k t}$$

can be approached in $L^2(0, T)$ by functions f of the form (2.1) with $J \cap F = \varnothing$. By taking the difference we find a sequence of functions of the form (2.1) tending to 0 in $L^2(0, T)$ and for which the left-hand side in (2.2)′ is *constant and positive*. This clearly establishes assertion 2).

2.3. Computation of a Beurling-Malliavin Density

The main result of this section is the following

Theorem 2.3.1. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d^+, d^- \geqslant 0$ and $0 \leqslant \alpha < 1$*

$$\# \{\lambda \in \Lambda : 0 \leqslant \lambda \leqslant t\} = d^+ t + O(t^\alpha) \tag{2.4}$$

and

$$\# \{\lambda \in \Lambda : -t \leqslant \lambda \leqslant 0\} = d^- t + O(t^\alpha).$$

Then we have

$$R(\Lambda) = \pi d, \qquad d = \max \{d^+, d^-\}. \tag{2.5}$$

Theorem 2.3.1. will be a consequence of the famous Beurling-Malliavin Theorem. In the important special case where $d^+ = d^-$, it will be sufficient to verify the following lemma.

Lemma 2.3.2. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d \geqslant 0$ and $0 \leqslant \alpha < 1$*

$$\# \{\lambda \in \Lambda : 0 \leqslant \lambda \leqslant t\} = dt + O(t^\alpha) \text{ and } \# \{\lambda \in \Lambda : -t \leqslant \lambda \leqslant 0\} = dt + O(t^\alpha). \tag{2.6}$$

Let us represent the generic compact interval of $\mathbb{R}$ by $\omega = [\omega_1, \omega_2]$ and define for each $\epsilon > 0$ the set

$$\Omega_\epsilon = \{\omega : \big| |\omega|^{-1} \# (\Lambda \cap \omega) - d \big| \geqslant \epsilon\}. \tag{2.7}$$

Then if we represent each interval ω by a point in the upper half-plane through the formulas

$$T(\omega) = (x, y) \quad \text{with} \quad x = (\omega_1 + \omega_2)/2 \quad \text{and} \quad y = |\omega| = \omega_2 - \omega_1, \tag{2.8}$$

we have

$$\forall \epsilon > 0, \qquad \iint_{T(\Omega_\epsilon)} \frac{dx\,dy}{1 + x^2 + y^2} < \infty, \tag{2.9}$$

for every $\epsilon > 0$.

PROOF. As a consequence of hypothesis (2.6) we have immediately

$$\# (\Lambda \cap \omega) - d|\omega| = O\{ |\omega_1|^\alpha + |\omega_2|^\alpha \}.$$

therefore we only need to check that for each $K \geqslant 0$, the set

$$A(K) = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant K(|x + y|^\alpha + |x - y|^\alpha)\}$$

satisfies

$$\iint_{A(K)} \frac{dx\, dy}{1 + x^2 + y^2} < \infty.$$

But obviously

$$A(K) \subset B = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant M(1 + x^2)^{\alpha/2}\}$$

for some constant M related to K. Finally we have

$$\int_B \frac{dx\, dy}{1 + x^2 + y^2} \leqslant \int_{-\infty}^{+\infty} dx \int_0^{M(1+x^2)^{\alpha/2}} \frac{dy}{1 + x^2 + y^2}$$

$$\leqslant M \int_{-\infty}^{+\infty} (1 + x^2)^{\alpha/2 - 1}\, dx < \infty.$$

The result follows immediately.

In order to complete the proof of Theorem 2.3.1, it will be useful to recall the main concepts required to formulate the general Beurling-Malliavin Theorem.

Definition 2.3.3. *A sequence of distinct real numbers $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is said to be regular with Beurling-Malliavin density equal to $d \geqslant 0$ if for each $\epsilon > 0$, the set Ω_ϵ given by* (2.7) *satisfies* (2.9) *with T given by* (2.8).

We now recall the main result of [1].

Theorem 2.3.4. (Beurling-Malliavin.) *Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of distinct real numbers. Then*

(*a*) *If Λ is regular with Beurling-Malliavin density equal to $d \geqslant 0$, we have*

$$R(\Lambda) = \pi d.$$

(*b*) *If Λ is not regular, then*

$$R(\Lambda) = \pi d,$$

where d is the infimum of all Beurling-Malliavin densities of regular sequences of distinct real numbers containing Λ.

PROOF OF THEOREM 2.3.1. (*a*) If $d^+ = d^-$, the result of Lemma 2.3.2. precisely means that Λ is regular with Beurling-Malliavin density equal to d, and (*a*) from the statement of Theorem 2.3.4. gives exactly (2.5).

(*b*) Otherwise, one easily finds that any regular sequence of distinct real numbers containing Λ has a density at least equal to d. On the other hand by «completing» Λ it is rather straightforward to build a sequence of distinct real numbers containing Λ and satisfying (2.6). As a consequence of Lemma 2.3.2., such a sequence must be regular with Beurling-Malliavin density equal to d. Then (2.5) follows at once from (*b*) in the statement of Theorem 2.3.4.

By combining the results of Proposition 2.2.1 and Theorem 2.3.1, we obtain

Corollary 2.3.5. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d^+ \geqslant 0$, $d^- \geqslant 0$ and $0 \leqslant \alpha < 1$*

(2.4) $$\#\{\lambda\in\Lambda: 0\leqslant\lambda\leqslant t\} = d^+t + O(t^\alpha) \quad \text{and}$$
$$\#\{\lambda\in\Lambda: -t\leqslant\lambda\leqslant 0\} = d^-t + O(t^\alpha).$$

Then letting $d = \max\{d^+, d^-\}$, we have the following properties

(1) *For each $T > 2\pi d$ and for each $n\in\mathbb{Z}$ there exists a constant C_n such that*

$$|f_n| \leqslant C_n\left\{\int_0^T |f(t)|^2\,dt\right\}^{1/2},$$

for each function f of the form (2.1) *with $n\in J$.*

(2) *On the other hand for each $T < 2\pi d$ and for each finite sequence $\{\alpha_n\}_{n\in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that*

$$\left|\sum_{n\in F}\alpha_n f_n\right| \leqslant C\left\{\int_0^T |f(t)^2\,dt\right\}^{1/2}$$

for each f of the form (2.1) *with $F\subset J$.*

The special case where $d^+ = d^-$ is especially important for the sequel (Sections 3 and 4) and therefore we state it separately for the reader's convenience

Corollary 2.3.6. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d\geqslant 0$ and $0\leqslant\alpha<1$*

(2.4) $\#\{\lambda\in\Lambda: 0\leqslant\lambda\leqslant t\} = dt + O(t^\alpha)$ *and* $\#\{\lambda\in\Lambda: -t\leqslant\lambda\leqslant 0\} = dt + O(t^\alpha)$.

Then we have the following properties

(1) *For each $T > 2\pi d$ and for each $n \in \mathbb{Z}$ there exists a constant C_n such that*

$$|f_n| \leqslant C_n \left\{ \int_0^T |f(t)|^2 \, dt \right\}^{1/2},$$

for each function f of the form (2.1) *with $n \in J$.*

(2) *On the other hand for each $T < 2\pi d$ and for each finite sequence $\{\alpha_n\}_{n \in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that*

$$\left| \sum_{n \in F} \alpha_n f_n \right| \leqslant C \left\{ \int_0^T |f(t)|^2 \, dt \right\}^{1/2}$$

for each f of the form (2.1) *with $F \subset J$.*

3. Application to Spectral Pointwise Control of some Plate Models

3.1. An Abstract Controllability Result

The main result of the section is the following.

Theorem 3.1.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ (or a compact N-dimensional manifold without boundary) and A a positive self-adjoint operator in $H = L^2(\Omega)$ satisfying conditions* (1.1) *and* (1.2) *with A^{-1} compact. We denote by $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ the increasing sequence of eigenvalues of $A^{1/2}$. We assume that all the eigenvalues λ_j are simple and that we have for some $d \geqslant 0$ and $0 \leqslant \alpha < 1$*

$$\#\{\lambda \in \Lambda^+ : \lambda \leqslant t\} = dt + O(t^\alpha). \tag{3.1}$$

Let $\xi \in \Omega$ be any point at which no eigenfunction of A vanishes, and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of A. Then

(1) *or every $T > 2\pi d$, and $(y^n, y^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with* $\operatorname{supp}(h) \subset [0, T]$ *and such that the unique solution y of* (1.3) *satisfies $y(T, x) = y'(T, x) = 0$ in Ω.*

(2) *This result is optimal: as soon as $T < 2\pi d$, there is no $(y^0, y^1) \in D \times D$ except the trivial state $(0, 0)$ for which such a control h exists.*

PROOF. This is a straightforward consequence of Proposition 1.1 and Corollary 2.3.6 applied with $\Lambda = \Lambda^+ \cup (-\Lambda^+)$. Indeed any solution of (1.4) with initial data in $D \times D$ has the form

$$u(t,x) = \sum_{j \in J} \{ u_j \cos \lambda_j t + v_j \sin \lambda_j t \} \varphi_j(x),$$

where the functions φ_j denote an orthonormal sequence of eigenfunctions of A and the coefficients u_j and v_j and given by the formulas

$$u_j = \int_\Omega u(0,x)\varphi_j(x)\,dx, \qquad v_j = \frac{1}{\lambda_j}\int_\Omega u'(0,x)\varphi_j(x)\,dx.$$

It is then clear that a direct application of Corollary 2.3.6 to the function $f(t) = u(t, \xi)$ with u as above provides exactly the result by taking into account Proposition 1.1.

3.2. Application to Simply Supported Plates

Let Ω be a bounded domain in $\mathbb{R}^2$. We denote by $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ the increasing sequence of eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$: it is known that under very general assumptions on Ω, for instance if $\partial\Omega$ is smooth, the counting function $n(t) = \#\{\lambda \in \Lambda^+ : \lambda \leqslant t\}$ where each $\lambda \in \Lambda^+$ is repeated according to its multiplicity satisfies the so called Weyl formula:

$$n(t) = dt + O(t^{1/2}) \quad \text{with} \quad d = (1/4\pi)\,\mathrm{vol}\,(\Omega). \tag{3.2}$$

As a special consequence of Theorem 3.1.1 we find

Theorem 3.2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ satisfying the «Weyl formula», assume that all eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ are simple and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta)$ in $H_0^1(\Omega)$. Let finally $\xi = (\xi_1, \xi_2) \in \Omega$ be any point at which no eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$ vanishes. Then*

For every $T > (1/2)\,\mathrm{vol}\,(\Omega)$, and $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\mathrm{supp}\,(h) \subset [0, T]$ such that the unique solution Ψ of

$$\begin{cases} \Psi_{tt} + \Delta^2\Psi = h(t)\delta_\xi(x,y) & in \quad]0, T[\times \Omega, \\ \Psi = \Delta\Psi = 0 & on \quad [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & in \quad \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & in \quad \Omega, \end{cases}$$

satisfies $\Psi(T, \bullet) = \Psi_t(T, \bullet) = 0$.

This result is optimal: more precisely if $T < (1/2)\,\mathrm{vol}\,(\Omega)$, no non-zero finite linear combination of the eigenfunctions of $(-\Delta)$ in $H_0^1(\Omega)$ is pointwise L^2-controllable.

When Ω is a rectangle of the form $(0, \pi) \times (0, L)$, with $(L/\pi)^2 \notin \mathbb{Q}$, let D denote the vector space of finite linear combinations of the basic eigenfunctions $\sin mx \sin(n\pi y/L)$, $m \in \mathbb{N}$, $n \in \mathbb{N}$. We have the following result:

Proposition 3.2.2. *Let $\xi = (\xi_1, \xi_2) \in \Omega$ be fixed with $\xi_1/\pi \notin \mathbb{Q}$, $\xi_2/L \notin \mathbb{Q}$. For each $T > (1/2)\pi L$ and each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with* supp$(h) \subset [0, T]$ *and such that the unique solution Ψ of*

$$\begin{cases} \Psi_{tt} + \Delta^2\Psi = h(t)\delta_\xi(x, y) & in \quad]0, T[\times \Omega, \\ \Psi = \Delta\Psi = 0 & on \quad [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & in \quad \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & in \quad \Omega, \end{cases}$$

satisfies $\Psi(T, \cdot) = \Psi_t(T, \cdot) = 0$.

3.3. Application to the Case of Clamped Plates and Other Boundary Conditions

Let Ω be a rectangle or a bounded domain in $\mathbb{R}^2$ with a smooth boundary. We denote by $\Lambda^+ = \{\lambda_j\}_{1 \le j \le +\infty}$ the increasing sequence of the square roots of the eigenvalues of Δ^2 with relevant homogeneous boundary conditions: under very general assumptions on these boundary conditions, the counting function

$$n'(t) = \#\{\lambda \in \Lambda^+ : \lambda \leqslant t\}$$

where each $\lambda \in \Lambda^+$ is repeated according to its multiplicity still satisfies the Weyl formula (3.1) with the same value of d. As a consequence of Theorem 3.1.1 we find for instance

Theorem 3.3.1. *Let Ω be a bounded smooth domain or a rectangle in $\mathbb{R}^2$ for which all the eigenvalues of Δ^2 in $H_0^2(\Omega)$ are simple, and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of Δ^2 in $H_0^2(\Omega)$. Let finally $\xi = (\xi_1, \xi_2) \in \Omega$ be any point at which no eigenfunction of Δ^2 in $H_0^2(\Omega)$ vanishes. Then*

For every $T > (1/2)\,\mathrm{vol}(\Omega)$, and $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with supp$(h) \subset [0, T]$ *such that the unique solution Ψ of*

$$\begin{cases} \Psi_{tt} + \Delta^2\Psi = h(t)\delta_\xi(x, y) & in \quad]0, T[\times \Omega, \\ \Psi = |\nabla\Psi| = 0 & on \quad [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & in \quad \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & in \quad \Omega, \end{cases}$$

satisfies $\Psi(T, \cdot) = \Psi_t(T, \cdot) = 0$.

This result is optimal: more precisely if $T < (1/2)\,\mathrm{vol}\,(\Omega)$, *no non zero finite linear combination of the eigenfunctions of* Δ^2 *in* $H_0^2(\Omega)$ *is pointwise* L^2-*controllable.*

PROOF. Let Ω be a bounded smooth domain or a rectangle in $\mathbb{R}^2$ for which all the eigenvalues of Δ^2 in $H_0^2(\Omega)$ are simple. Then Ivrii [10] asserts that under general positivity conditions, the fact that Δ^2 with the given boundary conditions is elliptic in the sense of Shapiro-Lopatinskii implies that the counting function $n'(t)$ satisfies (3.1). It is rather easy to check (*cf. e.g.* Wloka [23]) that the operator Δ^2 in $H_0^2(\Omega)$ satisfies the Shapiro-Lopatinskii condition, therefore Theorem 3.1.1. is applicable. (For a related weaker property *cf.* also Plejel [21].)

Remark 3.3.2. Of course the difficulty in general will be to determine the «strategic points» $\xi = (\xi_1, \xi_2)$ at which no eigenfunction of Δ^2 in $H_0^2(\Omega)$ vanishes. Even when Ω is a rectangle of the form $(0, \pi) \times (0, L)$, the eigenfunctions of Δ^2 in $H_0^2(\Omega)$ become more complicated than in the case of simply supported plates, and it is probably not so easy to find the strategic points. We know, however, that in the absence of multiple eigenvalues, almost every point is strategic.

Remark 3.3.3. In Section 5, the case of variable Lamé coefficients will be treated.

4. Some Applications to Spectral Internal Control

Let Ω be a bounded domain of $\mathbb{R}^N$ (or a compact N-dimensional manifold without boundary) and A a positive self-adjoint operator with compact resolvant in $H = L^2(\Omega)$. Let $\{\lambda_j\}_{1 \le j \le +\infty}$ be the increasing (without taking care of multiplicity) sequence of eigenvalues of A and for each j, let

$$F_j = \{u \in L^2(\Omega) : Au = \lambda_j u\}.$$

Then we have the following

Theorem 4.1. *Assume that A has the following properties*

(1) *For every j, the conditions* $u \in F_j$ *and* $u \equiv 0$ *on some non-empty open set imply* $u \equiv 0$.
(2) *There is a finite* $T_0 > 0$ *for which the functions*

$$\sum_{j \in J} \left\{ u_j e^{i\sqrt{\lambda_j}t} + v_j e^{-i\sqrt{\lambda_j}t} \right\}, \quad J \textit{ finite subset of } \mathbb{N} - \{0\}$$

where the u_j *and* v_j *are complex coefficients that are not dense in* $L^2(0, T_0; \mathbb{C})$.

Then for each $T > T_0$, *there are functions* $\{f_j\}_{1 \le j \le +\infty}$ *of class* C^∞ *with compact support in* $(0, T)$ *such that we have the following properties*

(*a*) *For every* $\Psi \in L^2(\Omega)$, *the unique solution* u *of*

$$(4.1) \quad u'' + Au = f_j(t)\Psi(x) \text{ in } \mathbb{R} \times \Omega, \qquad u(0,x) = u'(0,x) = 0 \text{ in } \Omega$$

fulfills $u(T, \bullet) \in F_j$ *and* $u'(T, \bullet) = 0$.

(*b*) *For every* ω *non-empty open subset of* Ω, *and every* $\varphi \in F_j$, *there exists* $\Psi \to D(\Omega)$ *with compact support in* ω *such that the unique solution* u *of* (3.1) *satisfies* $u(T) = \varphi$ *and* $u'(T) = 0$. *Moreover the solution* v *of*

$$(4.2) \quad v'' + Av = f_j'(t)\Psi(x) \text{ in } \mathbb{R} \times \Omega, \qquad u(0,x) = u'(0,x) = 0 \text{ in } \Omega$$

fulfills $v(T) = 0$ *and* $v'(T) = -\lambda_j \varphi$.

PROOF. Let $\mu_j = \lambda_j^{1/2}$ for all j. We shall prove the result in five steps.

(1) First of all it follows from (2) that for each j fixed, the function $\sin(\mu_j t)$ is not a limit in $L^2(0, T_0; \mathbb{C})$ of finite linear combinations of the functions $\exp(\pm i\mu_k t)$ for $k \neq j$. Indeed in such a case, the function $\exp(i\mu_j t)$ would be a limit in $L^2(0, T_0; \mathbb{C})$ of finite linear combinations of the functions $\exp(\pm i\mu_k t)$ for $k \neq j$ and of the function $\exp(-i\mu_j t)$, which by an argument similar to the proof of Lemma 2.2.2 would contradict property (2).

(2) In particular, there exists $h_j \in L^2(0, T_0; \mathbb{C})$ for which

$$\int_0^{T_0} h_j(t) e^{\pm i\mu_k t}\, dt = 0 \quad \text{for} \quad k \neq j; \qquad \int_0^{T_0} h_j(t) \sin(\mu_j t)\, dt \neq 0.$$

Replacing h_j by either its real or its imaginary part, we may assume $h_j \in L^2(0, T_0; \mathbb{R})$.

(3) Let now $T > T_0$, $0 < \eta \leqslant (T - T_0)/2$ and

$$\begin{aligned} h_j(t, \eta) &= 0 && \text{on} \quad (-\infty, \eta), \\ h_j(t, \eta) &= h_j(t - \eta) && \text{on} \quad (\eta, T_0 + \eta), \\ h_j(t, \eta) &= 0 && \text{on} \quad (T_0 + \eta, +\infty). \end{aligned}$$

For every $k \neq j$, we clearly have

$$\int_0^T h_j(t, \eta) e^{\pm i\mu_k t}\, dt = 0.$$

On the other hand, if η is small enough we have

$$\int_0^T h_j(t, \eta) \sin(\mu_j t)\, dt \neq 0.$$

For such a fixed η, let $h_j(t, \eta) = h_j(t)$. Define also $\rho_\epsilon \in D(0, \epsilon)$ with $\rho_\epsilon \geqslant 0$ and $\int \rho_\epsilon(x)\, dx = 1$, and let us introduce $h_j * \rho_\epsilon = h_{j,\epsilon}$. Then for each $\epsilon \in (0, \eta)$ we have $h_{j,\epsilon} \in D(0, T)$. In addition,

$$\int_0^T h_{j,\epsilon}(t) e^{\pm i\mu_k t}\, dt = 0 \quad \text{for} \quad k \neq j,$$

$$\lim_{\epsilon \to 0} \int_0^T h_{j,\epsilon}(t) \sin(\lambda_j t)\, dt = \int_0^T h_j(t) \sin(\lambda_j t)\, dt \neq 0.$$

By selecting $\epsilon > 0$ small enough and replacing $h_{j,\epsilon}$ by some proportional function, we obtain $g_j \in D(0, T)$ such that

$$\text{(4.3)} \qquad \int_0^T g_j(t) e^{\pm i\mu_k t}\, dt = 0 \quad \text{for } k \neq j \quad \text{and} \quad \int_0^T g_j(t) \sin(\mu_j t)\, dt = 1.$$

We can, in fact, also assume

$$\text{(4.4)} \qquad \int_0^T g_j(t) \cos \mu_j t\, dt = 0.$$

As a matter of fact, if

$$\int_0^T g_j(t) \cos \mu_j t\, dt = I \neq 0,$$

let $g_j^*(t) = g_j(t + \alpha) + c g_j(t)$, $\alpha \neq 0$ being taken small enough. Then

$$\begin{aligned}\int_0^T g_j^*(t) \cos \mu_j t\, dt &= cI + \int_0^T g_j(t) \cos \mu_j (t - \alpha)\, dt \\ &= (c + \cos(\alpha\mu_j))I + \sin(\alpha\mu_j),\end{aligned}$$

vanishes for

$$c = -\cos(\alpha\mu_j) - \sin(\alpha\mu_j)/I.$$

Taking c as above we have

$$\begin{aligned}\int_0^T g_j^*(t) \sin \mu_j t\, dt &= c + \int_0^T g_j(t) \sin \mu_j (t - \alpha)\, dt \\ &= c + \cos(\alpha\mu_j) - I \sin(\alpha\mu_j) \\ &= -(I + 1/I) \sin(\alpha\mu_j) \neq 0\end{aligned}$$

for α small.

We can then replace g_j by λg_j^* with $\lambda \neq 0$ properly chosen.

(4) Let $f_j(t) = g_j(T - t)$ for $t \in [0, T]$. The solution of (4.1) is given by

$$u(t, x) = \int_0^t f_j(t - s)\left\{\sum_m \sin(\mu_m s)\Psi_m(x)\right\} ds$$

with

$$\Psi_m = \mathrm{Proj}_{F_m}(\Psi) = P_m\Psi, \quad \text{for} \quad m \in \{1, 2, \ldots\}.$$

In particular we have

$$\begin{aligned} u(T, x) &= \int_0^T g_j(s)\left\{\sum_m \sin(\mu_m s)\Psi_m(x)\right\} ds \\ &= \sum_m \left\{\int_0^T g_j(s) \sin(\mu_m s)\right\}\Psi_m(x) \\ &= \Psi_j(x). \end{aligned}$$

On the other hand for t close to T we have

$$u'(t, x) = \int_0^t f_j'(t - s)\left\{\sum_m \sin(\mu_m s)\Psi_m(x)\right\} ds.$$

Therefore

$$\begin{aligned} u'(T, x) &= \int_0^T f_j'(T - s)\left\{\sum_m \sin(\mu_m s)\Psi_m(x)\right\} ds \\ &= \sum_m \left\{\int_0^T g_j'(s) \sin(\mu_m s)\, ds\right\}\Psi_m(x) = 0, \end{aligned}$$

since integration by parts gives

$$\int_0^T g_j'(s) \sin(\mu_m s)\, ds = -\lambda_m \int_0^T g_j(s) \cos(\mu_m s)\, ds = 0.$$

for every $m \in \mathbb{N} - \{0\}$. This establishes (a) with $u(T, \bullet) = \Psi_j$. Moreover we notice that $u' = v$ is the solution of (4.2) with initial data $(0, 0)$ and satisfies $v(T, \bullet) = 0$; $v'(T, \bullet) = u''(T, \bullet) = -Au(T, \bullet) = -\lambda_j\Psi_j$.

(5) To establish (b), we now use hypothesis (1). Indeed, for a fixed integer j, we consider an orthonormal basis $\{\varphi_1, \ldots, \varphi_r\}$ of F_j. To finish the proof we just need to show that we can find $\Psi \in \mathfrak{D}(\Omega)$ with support in ω such that $P_j\Psi = \varphi_1$ (say). In the opposite case, the linear form defined by

$$\Psi \in \mathfrak{D}(\omega) \mapsto \int_\omega \Psi(x)\varphi_1(x)\, dx$$

would vanish on the intersection of the kernels of the linear forms defined by

$$\Psi \in \mathfrak{D}(\omega) \mapsto \int_\omega \Psi(x)\varphi_k(x)\,dx, \qquad k \geqslant 2.$$

By a standard result of linear algebra we would deduce the existence of real coefficients $\{\alpha_k\}_{k\geq 2}$ for which

$$\int_\omega \Psi(x)\varphi_1(x)\,dx = \sum_{k\geq 2} \alpha_k \int_\omega \Psi(x)\varphi_k(x)\,dx,$$

for all $\Psi \in \mathfrak{D}(\omega)$. This immediately implies that for every $x \in \omega$,

$$\varphi_1(x) = \sum_{k\geq 2} \alpha_k \varphi_k(x).$$

This is in contradiction with hypothesis (1) and the linear independence of $\{\varphi_k\}_{k\geq 1}$.

Let us now denote by D the vector space of all (finite) linear combinations of the eigenfunctions of A and assume that all hypotheses of Theorem 4.1 are satisfied. Then by an immediate calculation we obtain the following result.

Corollary 4.2. *For each $(y^0, y^1) \in D \times D$, there exists $h \in \mathfrak{D}((0, T) \times \Omega)$ with* $\operatorname{supp}(h) \subset (0, T) \times \omega$ *and such that the unique solution y of*

$$\text{(4.5)} \qquad \begin{cases} y'' + Ay = h(t, x) & \text{in} \quad (0, T) \times \Omega, \\ y(0, x) = y^0(x) & \text{in} \quad \Omega, \\ y'(0, x) = y^1(x) & \text{in} \quad \Omega \end{cases}$$

satisfies $y(T, \bullet) = y'(T, \bullet) = 0$.

In particular, in the case of vibrating plates we obtain

Corollary 4.3. *Let Ω be a bounded smooth domain or a rectangle in $\mathbb{R}^2$ and let us denote by D the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta)$ in $H_0^1(\Omega)$. Then for any $T > (1/2)\operatorname{vol}(\Omega)$ and each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h \in \mathfrak{D}((0, T) \times \Omega)$ with* $\operatorname{supp}(h) \subset (0, T) \times \omega$ *and such that the unique solution Ψ of*

$$\text{(4.6)} \qquad \begin{cases} \Psi_{tt} + \Delta^2\Psi = h(t, x, y) & \text{in} \quad (0, T) \times \Omega, \\ \Psi = \Delta\Psi = 0 & \text{on} \quad [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & \text{in} \quad \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & \text{in} \quad \Omega, \end{cases}$$

satisfies $\Psi(T, \bullet) = \Psi_t(T, \bullet) = 0$. *In addition the same result is valid for the equation*

$$(4.6)' \quad \begin{cases} \Psi_{tt} + \Delta^2\Psi = h(t, x, y) & in \quad (0, T) \times \Omega, \\ \Psi = |\nabla\Psi| = 0 & on \quad [0, T] \times \partial\Omega, \\ \Psi(0; x, y) = \Psi^0(x, y) & in \quad \Omega, \\ \Psi_t(0; x, y) = \Psi^1(x) & in \quad \Omega. \end{cases}$$

PROOF. Property (1) is clearly satisfied in both cases. It is therefore sufficient to check (2) for all $T_0 > (1/2)\,\mathrm{vol}\,(\Omega)$. In the case of (4.6), when all the eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ are simple, this follows at once from the Weyl formula and Corollary 2.3.6. In fact, by considering for instance some artificial additional frequencies or by using a generalization of the Beurling-Malliavin theory for exponential-polynomial series, it is possible to extend Corollary 2.3.6 in the more general situation of a counting function allowing arbitrary finite repetitions of the frequencies. Then the Weyl formula implies (2) without requiring the eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ to be simple. The rest is clear. The same proof works for (4.6)′.

Remark 4.4. In the case of (4.6) in a rectangle, by using some results of J. P. Kahane [13], S. Jaffard [11] established the (in a sense stronger) result of exact internal controllability of any state with a finite energy, for any $T > 0$. However this result does not seem to imply immediately the existence of a C^∞ control for states $(\Psi^0, \Psi^1) \in D \times D$. The result of [11] has been recently generalized in arbitrarily many dimensions (for a product of intervals) by V. Komornik [15]. On the other hand, no such internal controllability result seems to be known for the clamped plate equation.

This theory is also applicable to cases where the open set Ω is replaced by a compact manifold without boundary. We obtain for instance the following result, valid in any dimension $N \geqslant 1$.

Corollary 4.5. *Let* Σ *be the unit sphere of* $\mathbb{R}^N$, *and let us denote by* $(-\Delta_\Sigma)$ *the Laplace-Beltrami operator on* Σ *and by* D *the vector space of all (finite) linear combinations of the eigenfunctions of* $(-\Delta_\Sigma)$. *For all* $T > 0$, *and all* ω *non-empty open subset of* Σ, *and for each* $(\Psi^0, \Psi^1) \in D \times D$, *there exists* $h \in \mathfrak{D}((0, T) \times \Sigma)$ *with* $\mathrm{supp}\,(h) \subset (0, T) \times \omega$ *and such that the unique solution* Ψ *of*

$$(4.7) \quad \begin{cases} \Psi_{tt} + \Delta_{\Sigma^2}\Psi = h(t, \sigma) & in \quad (0, T) \times \Sigma, \\ \Psi(0; \sigma) = \Psi^0(\sigma) & on \quad \Sigma \\ \Psi_t(0; \sigma) = \Psi^1(\sigma) & on \quad \Sigma \end{cases}$$

satisfies $\Psi(T, \bullet) = \Psi_t(T, \bullet) = 0$.

PROOF. Property (1) is clearly satisfied. Property (2) is a rather immediate consequence of the fact that the inverses of the positive eigenvalues of $(-\Delta_\Sigma)$ are summable. (*Cf. e.g.* [24], Theorem 15 p. 139.)

Remark 4.6. The control functions constructed here are of a special type. Their construction ultimately relies on the existence of a sequence of functions «biorthogonal» to some complex exponentials, a technique already widely used (*cf. e.g.*, H. O. Fattorini [2], J. Lagnese [16], D. L. Russel [22]) in control theory.

Remark 4.7. It seems rather reasonable to conjecture that the internal spectral controllability for the above plate equation is valid for general 2-dimensional domains and for every $T > 0$. The study of this conjecture and some related problems will be the object of further research.

5. Possible Extensions and Additional Remarks

5.1. One-Dimensional Vibrating Systems

We consider first the equation of vibrating strings

$$\begin{cases} u_{tt} - (a(x)u_x)_x = 0 & \text{on} \quad \mathbb{R} \times (0, L) \\ u(t, 0) = u(t, L) = 0 & \text{on} \quad \mathbb{R}, \end{cases}$$

where a is smooth and bounded from below by a positive constant. Let A be the (strongly elliptic) unbounded operator on $L^2(0, L)$ defined by

$$D(A) = H^2 \cap H_0^1(0, L); \qquad Av = -(a(x)v_x)_x \quad \text{for} \quad v \in D(A).$$

The solutions are of the form

$$u(t, x) = \sum_{n \in \mathbb{Z} - \{0\}} u_n e^{i\lambda_n t} w_n(x)$$

where $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is given by $\Lambda^+ \cup (-\Lambda^+)$ and $\Lambda^+ = \{\lambda_j\}_{1 \le j \le +\infty}$ is the increasing sequence of eigenvalues of $A^{1/2}$ in $H_0^1(0, L)$. Here Weyl's formula implies (*cf. e.g.* Hörmander [8], p. 273)

$$n(t) = ct + O(t^{1/2})$$

for some $c > 0$. Hence, by Theorem 3.1.1 we obtain that, appart from the nodal points (zeroes of the eigenfunctions of A) pointwise spectral controllability holds true for all times $T > 2\pi c$. This result extends to pointwise control some previous result of J. Lagnese [16] concerning internal exact controllability of strings.

Similarly we can consider vibrating beams given by

$$u_{tt} + (a(x)u_{xx})_{xx} = 0 \quad \text{on} \quad \mathbb{R} \times (0, L)$$

with either of the following boundary conditions

$$u(t,0) = u(t,L) = u_x(t,0) = u_x(t,L) = 0 \quad \text{on} \quad \mathbb{R},$$

or

$$u(t,0) = u(t,L) = u_{xx}(t,0) = u_{xx}(t,L) = 0 \quad \text{on} \quad \mathbb{R}.$$

Here of course $a(x)$ is assumed smooth and bounded from below by some positive constant. Here Theorem 3.1.1 provides pointwise spectral controllability for all times $T > 0$ since the completeness radius of the corresponding complex exponentials is obviously 0. Actually in such cases the result can also be deduced by means of a variant of Ingham's Lemma (*cf.* [9.5]).

5.2. The Case of Plates with Nonconstant Lamé Coefficients

We deal with a similar case as in Section 3.3 except that the bilaplacian is now replaced by

$$A = \sum_{i,j=1,2} \partial_{i,j} m_{i,j} \quad \text{with} \quad m_{i,j} = \frac{4\mu}{3}\left(\partial_{i,j} + \frac{\lambda}{\lambda + 2\mu}\delta_{i,j}\Delta\right).$$

The functions λ and μ are the Lamé coefficients which we suppose to be nonconstant in the plate, but C^∞. In order to obtain a «Weyl's formula» for this operator, we have to check that the assumptions given in [10] are fulfilled. The operator is symmetric since

$$\langle Au, v\rangle = \sum_{i,j} \int \frac{4\mu}{3} \partial_{i,j} u \overline{\partial_{i,j} v} + \frac{4\lambda\mu}{3(\lambda + 2\mu)} \int \Delta u \, \overline{\Delta v}.$$

The principal symbol of A is $(4\mu/3)(1 + \lambda/(\lambda + 2\mu))|\xi|^4$ which is positive definite. Thus we only have to check the Shapiro-Lopatinskii condition on the boundary. It is a condition on the principal part of the operator which must hold at each point x_0 of the boundary. Here, the principal part of A is a biplacian multiplied by the smooth function $a = (4\mu/3)(1 + \lambda/(\lambda + 2\mu))$. Thus, up to the multiplicative factor $a(x_0)$, the condition to check is exactly the same as if we had $A = \Delta^2$ with the corresponding boundary conditions, and the conclusion will be the same as for the bilaplacian; namely, for simply supported or clamped plates, formula (3.1) will hold with $d = (1/4\pi)\operatorname{vol}(\Omega)$, and thus, also the conclusion of the analog of Theorem 3.2.1.

5.3 Uniqueness and the Schröndinger Equation

In Section 3, we have given precisely the minimal time for pointwise spectral controllability. Is is clear that a time T of pointwise spectral controllability is also a uniqueness time in the sense that the trace of a solution at the observation point on $(0, T)$ determines the solution. In the case of second order problems (1.3)-(1.4), it is conjectured (*cf.* Kahane [14]) that the minimal uniqueness time is equal to the minimal time for pointwise spectral controllability.

Now let Ω be a bounded domain of $\mathbb{R}^2$ and A a positive self-adjoint operator in $H = L^2(\Omega)$. We assume that A satisfies the properties (1.1) and (1.2). Given $T > 0$, $\xi \in \Omega$ and y^0 a finite linear combination of the eigenfunctions of A, we are interested in the existence of a control function $h \in L^2(]0, T[)$ such that $\operatorname{supp}(h) \subset [0, T]$ and for which the unique generalized solution y of the Schrödinger type equation

$$y - iAy = h(t)\delta(x - \xi) \quad \text{in} \quad]0, T[, \qquad y(0, x) = y^0(x) \quad \text{in} \quad \Omega,$$

satisfies

$$y(T, x) = y'(T, x) = 0 \quad \text{in} \quad \Omega.$$

The solutions of the homogeneous equation $\phi' - iA\phi = 0$ are here given by

$$\phi(t, x) = \sum_{n \geq 1} \phi_n e^{i\lambda_n t} w_n(x),$$

where the numbers λ_n are the eigenvalues of A and the functions w_n are the associated eigenfunctions. For a given $\xi = (\xi_1, \xi_2) \in \Omega$, let

$$f(t) = \phi(t, \xi) = \sum_{n \geq 1} \alpha_n e^{i\lambda_n t},$$

Assume that the Weyl formula (3.1) holds for A with $d > 0$: then by Corollary 2.3.5 the minimal time for spectral controllability is easily seen to be also positive (more precisely equal to $2\pi d$).

On the other hand, let us show that *any positive time* T is in fact a «uniqueness time». From the Weyl formula (3.1), the properties of the initial data and the standard estimates on $\| w_n \|_\infty$ we deduce that the coefficients α_n have at most polynomial growth. Suppose that f vanishes identically on $[0, T]$ and let φ be a C^∞ nonnegative function supported inside $[-T/2, 0]$ with integral 1. Then the convolution product $\varphi * f$ vanishes identically on $[0, T/2]$ and we have

$$(\varphi * f)(t) = \sum_{n \geq 1} \alpha_n \hat{\varphi}(\lambda_n) e^{i\lambda_n t} = \sum_{n \geq 1} c_n e^{i\lambda_n t},$$

where the sequence (c_n) is quickly decreasing, hence $\varphi * f$ is C^∞. Let

$$\psi(t) = (\varphi * f)(t)\omega(t)$$

where $\omega(t)$ is in the Schwartz class with a compactly supported Fourier transform. Then ψ is in the Schwartz class with

$$\hat{\psi}(\xi) = [(\varphi \hat{*} f) * \hat{\omega}](\xi) = \left[\sum_{n \geq 1} c_n \delta_{\lambda_n}\right] * \hat{\omega}(\xi).$$

Since the numbers λ_n are all positive, and ω has a compactly supported Fourier transform, the Fourier transform of ψ vanishes on $(-\infty, \beta]$ for some β. Hence $g(t) = e^{i\beta t}\psi(t)$ is a C^∞ function which vanishes identically on $[0, T/2]$ and whose Fourier transform is supported by $[0, +\infty)$. A well known theorem of Helson and Szegö (*cf.* [7]) asserts that if $g \in L^2(\mathbb{R})$ has its Fourier transform supported in $[0, +\infty)$, then either $g \equiv 0$ or $\operatorname{Log}|g(t)|/(1 + t^2) \in L^1(\mathbb{R})$: in particular if g vanishes on an interval we must conclude that $g \equiv 0$. In our case we conclude that $\psi \equiv 0$ for any choice of functions φ, ω as above. It then follows immediately that $f \equiv 0$, hence uniqueness is established for any $T > 0$.

As a conclusion, in the case of the above Schrödinger equation, for all times between 0 and the minimal spectral controllability time $T_0 = 2\pi d$, there exists a dense family of pointwise controllable states, but none of them is a finite linear combination of the eigenfunctions of A. It would be, of course, of interest to decide what happens for our plate models for small positive times, and in particular to settle Kahane's conjecture.

5.4. The Plate Equation in Higher Dimensions

In dimensions higher than or equal to 3, the calculations are very similar to those in dimension 2 and we shall not repeat them. The results, on the other hand, are quite different: for instance, for a 3-dimensional «plate» the Weyl formula now gives

$$N(\lambda) = c\lambda^{3/2} + O(\lambda),$$

hence the Beurling-Malliavin density of the λ_n is infinity and for no finite $T > 0$ we have pointwise spectral controllability. The uniqueness problem is also open in this case. However, if we consider the associated Schrödinger equation, then the eigenvalues are positive and the proof of Section 5.2 is still applicable. Thus here, any positive time is a uniqueness time, while there is no finite time for pointwise spectral controllability!

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On Pseudospheres

John L. Lewis and Andrew Vogel
Dedicated to the memory of Allen Shields

1. Introduction

Denote points in Euclidean space, $\mathbb{R}^n$, by $x = (x_1, \ldots, x_n)$ and let $\bar{E}$, ∂E, denote the closure and boundary of $E \subset \mathbb{R}^n$, respectively. Put $B(x, r) = \{y: |y - x| < r\}$ when $r > 0$. Define k dimensional Hausdorff measure, $1 \leqslant k \leqslant n$, in $\mathbb{R}^n$ as follows: for fixed $\delta > 0$ and $E \subset \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subset \bigcup B(x_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \ldots$ Set

$$\phi_\delta^k(E) = \inf_{L(\delta)} \sum \alpha(k) r_i^k,$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi_\delta^k(E), \qquad 1 \leqslant k \leqslant n.$$

Let D be a bounded domain in $\mathbb{R}^n$ with $0 \in D$ and $H^{n-1}(\partial D) < +\infty$. We shall say D is a pseudo sphere if

(*a*) ∂D is homeomorphic to the unit sphere, S, in $\mathbb{R}^n$
(*b*) $g(0) = a \int_{\partial D} g \, dH^{n-1}$, whenever g is harmonic in D and continuous on $\bar{D}$.

In (*b*), a denotes a constant. The construction of pseudo spheres in $\mathbb{R}^2$, which are not circles, was first done by Keldysh and Lavrentiev to show the existence of domains not of Smirnov type (see [11, Ch. 3]). Also a completely different proof of existence has been given by Duren, Shapiro, and Shields in [3] (see also [2, Ch. 10]). Both proofs are heavily reliant on conformal mapping and $\mathbb{R}^2$ facts, such as: the logarithm of the gradient of a harmonic function is subharmonic.

In [12, p. 347], Shapiro asked whether there exists a pseudo sphere in $\mathbb{R}^n$ which is not a sphere. In this paper we answer Shapiro's question in the affirmative and even prove a little more:

Theorem 1. *There exists a pseudo sphere D in $\mathbb{R}^n$, $n \geqslant 3$, which is not a sphere. In fact D can be chosen so that there is a homeomorphism f from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f(S) = \partial D$ and*

$$c(\beta)^{-1}|x-y|^{1/\beta} \leqslant |f(x)-f(y)| \leqslant c(\beta)|x-y|^{\beta},$$

whenever $\beta \in (0,1)$ and $|x-y| \leqslant 1/2$.

In Theorem 1, as in the sequel, $c(\beta)$ denotes a positive constant depending only on β and n. Also, c will denote a positive constant depending only on n, not necessarily the same at each occurrence. Our method of proof is inspired by the proof of Keldysh and Lavrentiev in [9]. Here though conformal mapping techniques are not available. We outline our proof with $a = 1$ in (b). Let Ω be a bounded domain with $0 \in \Omega$ and let G be Green's function for Ω with pole at 0. That is,

$$G(x) - \frac{1}{n(n-2)\alpha(n)}|x|^{2-n}, \qquad x \in \mathbb{R}^n,$$

is harmonic in Ω and G has boundary value 0 in the sense of Perron-Wiener-Brelot. It is known that if $\partial\Omega$ is sufficiently smooth, then

$$\nabla G(x) = \left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}\right)$$

extends continuously to $\bar{\Omega} - \{0\}$. Under this assumption suppose that $|\nabla G| \geqslant 1$ on $\partial\Omega$. In Section 2, given ϵ, $0 < \epsilon \leqslant \epsilon_0$, we add smooth bumps to $\partial\Omega$ by «pushing out» $\partial\Omega$ along certain small surface elements in $\{x \in \partial\Omega\colon |\nabla G(x)| > 1 + \epsilon\}$ of approximate side length r, $0 < r \leqslant r_0$. Let Ω', G' be the smooth domain, and Green's function with pole at 0, obtained from this process. Then $\Omega \subset \Omega'$ and we shall choose the bumps so that for $\epsilon \leqslant t \leqslant 1$,

$$H^{n-1}(\partial\Omega') \geqslant H^{n-1}(\partial\Omega) + \eta(t)H^{n-1}\{x\colon |\nabla G(x)| > 1 + t\}, \tag{1.1}$$

where η is a positive function on $(0, \infty)$. It turns out that η can be chosen independent of Ω, Ω'. We note from the Hopf boundary maximum principle (see [6, Lemma 3.4]) and $|\nabla G| \geqslant 1$ on $\partial\Omega$, that $|\nabla G'| \geqslant 1$ on $\partial\Omega \cap \partial\Omega'$. Also from Schauder type estimates, it will follow that $|\nabla G'| \geqslant 1$ on the bumps. Hence,

$$|\nabla G'(x)| \geqslant 1, \qquad x \in \partial\Omega'. \tag{1.2}$$

Next we modify the identity mapping slightly in a neighborhood of each bump, to get h, a homeomorphism from $\mathbb{R}^n$ into $\mathbb{R}^n$, with $h(\partial\Omega) = \partial\Omega'$. In Section 3 using a lemma of Wolff ([14, Lemma 2.7]) we will show the bumps can be chosen so that

$$\int_{\partial\Omega'} |\nabla G'| \log |\nabla G'| \, dH^{n-1} \leqslant \int_{\partial\Omega} |\nabla G| \log |\nabla G| \, dH^{n-1}. \tag{1.3}$$

The proof of (1.3) is somewhat involved, but luckily much of the hardwork has been done for us by Wolff.

In Section 4 we use (1.1)-(1.3) and induction to construct D. More specifically put $D_0 = B(0, \rho)$ and let

$$G_0(x) = \frac{1}{n(n-2)\alpha(n)} (|x|^{2-n} - \rho^{2-n}), \qquad x \in B(0, \rho),$$

be Green's function for $B(0, \rho)$, where ρ is chosen so that if $x \in \partial B(0, \rho)$, then

$$|\nabla G(x)| = \frac{1}{n\alpha(n)} \rho^{1-n} = 2. \tag{1.4}$$

We put $\Omega = D_0$ and modify Ω as above to obtain $\Omega' = D_1$, $G' = G_1$, with ϵ replaced by ϵ_1 and h by h_1. Suppose D_k has been constructed for $0 \leqslant k \leqslant m$. Again we put $\Omega = D_m$ and modify Ω as above to obtain $\Omega' = D_{m+1}$, $G' = G_{m+1}$, with ϵ replaced by $\epsilon_{m+1} = 2^{-(m+1)}\epsilon_0$, and h by h_{m+1}. By induction we get $(D_k)_0^\infty, (h_k)_1^\infty, (G_k)_0^\infty$, satisfying (1.1), (1.2), with Ω', Ω, replaced by D_{k+1}, D_k, respectively. Let $h_0(x) = \rho x$, and let $f_k = h_k \circ h_{k-1} \circ \cdots \circ h_0$, where $\circ$ denotes composition. Then it will follow from our construction for $k = 1, 2, \ldots,$ that

$$c(\beta)^{-1}|x - y|^{1/\beta} \leqslant |f_k(x) - f_k(y)| \leqslant c(\beta)|x - y|^\beta, \tag{1.5}$$

when $x, y \in \mathbb{R}^n$ and $|x - y| \leqslant 1/4$. Moreover, each f_k is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f_k(S) = \partial D_k$. Set $D = \bigcup_0^\infty D_k$, and note from (1.5) that there exists a subsequence (f_{n_k}) of (f_k) which converges to a homeomorphism f of $\mathbb{R}^n$, satisfying the conclusions of Theorem 1. Thus (*a*) in the definition of a pseudo sphere is valid. To prove (*b*) we first note from Green's Theorem and (1.2) that

$$1 = \int_{\partial D_k} |\nabla G_k| \, dH^{n-1} \geqslant H^{n-1}(\partial D_k), \tag{1.6}$$

for $k = 0, 1, \ldots$ Second, observe for each $\delta > 0$ that

$$\lim_{k\to\infty} H^{n-1}\{x \in \partial D_k : |\nabla G_k(x)| > 1 + \delta\} = 0, \tag{1.7}$$

since otherwise we could use (1.1) and iteration to get a contradiction to (1.6) for large k. Next from (1.2), (1.3), and iteration we deduce that for $\alpha > 1$, $k = 0, 1, \ldots$

$$(1.8)\quad \log \alpha \int_{\{|\nabla G_k| > \alpha\}} |\nabla G_k| \, dH^{n-1} \leqslant \int_{\partial D_k} |\nabla G_k| \log |\nabla G_k| \, dH^{n-1} \leqslant c < +\infty.$$

Also in Section 4 we show that as $k \to \infty$,

$$(1.9)\qquad H^{n-1}|_{\partial D_{n_k}} \to H^{n-1}|_{\partial D},$$

weakly as measures on $\mathbb{R}^n$. Let $g \geqslant 0$ be a harmonic function in D which is continuous on $\bar{D}$. Then from (1.2), (1.9), and Green's Theorem we get

$$(1.10)\qquad g(0) = \int_{\partial D_{n_k}} g |\nabla G_{n_k}| \, dH^{n-1} \geqslant \int_{\partial D_{n_k}} g \, dH^{n-1} \to \int_{\partial D} g \, dH^{n-1},$$

as $k \to \infty$. To obtain the reverse inequality for fixed $\delta < 10^{-3}$ and $\alpha > 10^3$, put

$$\begin{aligned} E_k &= \{x \in \partial D_{n_k} : 1 \leqslant |\nabla G_{n_k}(x)| \leqslant 1 + \delta\} \\ F_k &= \{x \in \partial D_{n_k} : 1 + \delta < |\nabla G_{n_k}(x)| \leqslant \alpha\} \\ L_k &= \{x \in \partial D_{n_k} : |\nabla G_{n_k}(x)| > \alpha\}, \end{aligned}$$

for $k = 0, 1, 2, \ldots$ Then

$$g(0) = \int_{\partial D_{n_k}} g |\nabla G_{n_k}| \, dH^{n-1} = \int_{E_k} \cdots + \int_{F_k} \cdots + \int_{L_k} \cdots = I_1 + I_2 + I_3.$$

Clearly,

$$|I_1| \leqslant (1 + \delta) \int_{\partial D_{n_k}} g \, dH^{n-1}.$$

Also from (1.7) we find that

$$|I_2| \leqslant \alpha \|g\|_\infty H^{n-1}\{x \in \partial D_{n_k} : 1 + \delta < |\nabla G_{n_k}|\} \to 0,$$

as $k \to \infty$. Here, $\|g\|_\infty$ denotes the maximum of g in $\bar{D}$. Using (1.8) we get

$$|I_3| \leqslant \|g\|_\infty \int_{\{|\nabla G_{n_k}| > \alpha\}} |\nabla G_{n_k}| \, dH^{n-1} \leqslant \frac{c}{\log \alpha} \|g\|_\infty.$$

Letting $k \to \infty$ we obtain from the above estimates and (1.9) that

$$g(0) \leqslant (1 + \delta) \int_{\partial D} g \, dH^{n-1} + \frac{c}{\log \alpha} \|g\|_\infty.$$

Finally letting $\delta \to 0$, $\alpha \to \infty$, we have

$$g(0) \leqslant \int_{\partial D} g \, dH^{n-1}.$$

In view of (1.10) we conclude that

$$g(0) = \int_{\partial D} g\, dH^{n-1} \tag{1.11}$$

when $g \geqslant 0$ is continuous on $\bar{D}$ and harmonic in D. From (1.11) with $g \equiv 1$ we note that, $H^{n-1}(\partial D) = 1$. If g_1 is continuous on $\bar{D}$, harmonic in D, and $g_1 - m \geqslant 0$ in $\bar{D}$, then from (1.11) and the above note we deduce

$$g_1(0) = (g_1 - m)(0) + m = \int_{\partial D} (g_1 - m)\, dH^{n-1} + m = \int_{\partial D} g_1\, dH^{n-1}.$$

Thus, D is a pseudo sphere. The initial bumps on D_1 will be chosen to have low peaks relative to those added to form D_k, $k \geqslant 2$, in order to guarantee that D is not a ball.

We remark that D will be regular for the Dirichlet problem, so each continuous function on ∂D will have a harmonic extension to D which is continuous on $\bar{D}$. From (1.11) it follows that harmonic measure and H^{n-1} measure on ∂D are equal (see [7, Ch. 8] for the Dirichlet problem). Moreover, since $H^{n-1}(\partial D) = 1$, it follows (see [4, Section 5.8]) that D is of finite perimeter. Thus several other measures are equal to H^{n-1} measure on ∂D (see [5, Thm. 4.5.19, (16)] and [5, Thm. 3.2.26]). Also D will be a nontangentially accessible (NTA) domain in the sense of Kenig and Jerison [8]. Using the corkscrew condition for NTA domains ((i) in Section 3) it is easily deduced that every point in ∂D lies in the measure theoretic boundary of D (see [4, Section 5.8]). Hence D satisfies the hypotheses of Theorem 1 in [10], from which we conclude

$$\sup \{ |\nabla G^*(x)| : x \in D - B(0, \rho/2) \} = +\infty,$$

where G^* is Green's function for D with pole at 0. Next we remark that this paper leaves open the very interesting question as to whether f in Theorem 1 can also be chosen for some $K > 1$ to be a K quasiconformal mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, $n \geqslant 3$. In $\mathbb{R}^2$ it follows from a criteria of Ahlfors (see [1, Ch. 4]) and the Keldysh-Lavrentiev construction that the answer to the above question is yes, and in fact K can be chosen arbitrary near 1.

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2. Preliminary reductions

If $x \in \mathbb{R}^n$, we let $x' = (x_1, \ldots, x_{n-1})$ and shall write, $x = (x', x_n)$. We assume throughout this section that Ω is a bounded domain of class C^4 with $0 \in \Omega$.

More specifically, for each $y \in \partial\Omega$ there exists $s > 0$ such that $B(y, s) \cap \partial\Omega$ is a part of the graph of a four-times continuously differentiable function, defined on a hyperplane in $\mathbb{R}^n$, and $B(y, s) \cap \Omega$ lies above the graph. From compactness and a standard convering argument it follows for each $r > 0$ that there exists, $y^1, y^2, \ldots, y^N \in \partial\Omega$, such that

$$\partial\Omega \subset \bigcup_{i=1}^{N} B(y^i, 100r) \quad \text{and} \quad B(y^i, 10r) \cap B(y^j, 10r) = \varnothing, \qquad i \neq j.$$

Moreover, if $0 < r < r_0$, r_0 sufficiently small, and $y = (y', y_n) \in \{y^i\}_1^N$, then from the implicit function theorem we see there exists $\theta = \theta(\cdot, y)$, four-times continuously differentiable on $\mathbb{R}^{n-1}(\theta \in C^4(\mathbb{R}^{n-1}))$, with $\theta(0) = 0$, $\nabla'\theta(0) = 0$, such that after a possible rotation of axes:

$$\partial\Omega \cap B(y, 1000r^{1/2}) \subseteq \{(x' + y', \theta(x') + y_n) : x' \in \mathbb{R}^{n-1}\},$$
$$\Omega \cap B(y, 1000r^{1/2}) \subseteq \{(x' + y', x_n) : x_n - y_n > \theta(x'), x' \in \mathbb{R}^{n-1}\}$$

Here ∇' denotes the $\mathbb{R}^{n-1}$ gradient. Put

$$M_1 = \max_{y \in \{y^i\}_1^N} \left\{ \max_{x \in \partial\Omega \cap B(y, 1000r^{1/2})} \sum |\partial'_\alpha \theta(x', y)| \right\}$$

where the sum is taken over all multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ with $|\alpha| = \sum_{j=1}^{n-1} \alpha_j$, and $0 \leqslant |\alpha| \leqslant 4$. Also, ∂'_α denotes the corresponding partial derivative with respect to $(x')^\alpha$, $x' \in \mathbb{R}^{n-1}$. Given ϵ, $0 < \epsilon < \sigma_0 \leqslant 10^{-3}$, choose $r_0 > 0$ so small that for $0 < r \leqslant r_0$

$$M_1 r^{1/2} \leqslant 10^{-3} r^{1/4} < 10^{-9} \epsilon^4. \tag{2.1}$$

Again this choice is possible by compactness of $\partial\Omega$. In this section and the next section we allow r_0 to vary. At the end of this section we will fix σ_0 at a number, satisfying several conditions, which depends only on n. r_0 will depend on ϵ, M_1, n, and M_2, defined below.

As in Section 1 let G be Green's function for Ω with pole at 0 and assume $|\nabla G| \geqslant 1$ on $\partial\Omega$. Let

$$M_2 = \max_{y \in \{y^i\}_1^N} \left\{ \max_{x \in \bar{\Omega} \cap B(y, 1000r^{1/2})} \sum |\partial_\alpha G(x)| \right\},$$

where now $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $0 \leqslant |\alpha| \leqslant 4$, and ∂_α denotes the corresponding partial derivative with respect to x^α, $x \in \bar{\Omega}$. From Schauder's Theorem (see [6, Ch. 6]), it is clear that $M_2 < +\infty$. We choose r_0 still smaller, if necessary, so that in addition to the above conditions, we have

$$M_2 r^{1/2} \leqslant 10^{-3} r^{1/4} < 10^{-9} \epsilon^4. \tag{2.2}$$

Let l be the largest nonnegative integer such that $2^{-l}\sigma_0 > \epsilon$ and put $\sigma_k = 2^{-k}\sigma_0$, for $k = 0, 1, \ldots$ Set

$$E_k = \{x \in \partial\Omega \colon 1 + \sigma_k < |\nabla G(x)| \leqslant 1 + \sigma_{k-1}\}, \qquad 1 \leqslant k \leqslant l + 1,$$
$$E_0 = \{x \in \partial\Omega \colon |\nabla G(x)| > 1 + \sigma_0\}.$$

Let ψ, $0 \leqslant \psi \leqslant 1$, be a fixed C^∞ function on $\mathbb{R}^{n-1}$ with $\max_{\mathbb{R}^{n-1}} \psi = 1$ and support in the unit ball of $\mathbb{R}^{n-1}$, to be specified in Section 3. We form a domain Ω' of class C^4 by adding smooth bumps to $\partial\Omega$. More specifically, let L be the set of all $y \in \{y^i\}_1^N$ for which

$$B(y, 100r) \cap \bigcup_{k=0}^{i+1} E_k \neq \varnothing.$$

For fixed $y = (y', y_n) \in L$, let j be the smallest nonnegative integer with

$$B(y, 100r) \cap E_j \neq \varnothing. \tag{2.3}$$

Put

$$\xi(x') = \theta(x') - \sigma_j^2 r \lambda_j^{-1} \psi(\lambda_j x'/r) + y_n, \qquad x' \in \mathbb{R}^{n-1},$$

where $(\lambda_j)_0^\infty$ is an increasing sequence of positive numbers with $\lambda_j \geqslant 1/\sigma_j$, $j = 0, 1, \ldots$, which will be defined explicitly in Section 3. Also $(\lambda_j)_0^\infty$ will depend only on σ_0. Define Ω' by

(i) $\Omega - \bigcup_{z \in L} B(z, 10r) = \Omega' - \bigcup_{z \in L} B(z, 10r)$,

(ii) $\partial\Omega' \cap B(y, 10r) = \{(x' + y', \xi(x')) \colon x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r)$,

(iii) $\Omega' \cap B(y, 10r) = \{(x' + y', x_n) \colon x_n > \xi(x')\} \cap B(y, 10r)$.

Thus for each $y \in L$ and smallest j, $0 \leqslant j \leqslant l + 1$, satisfying (2.3), we add a bump to Ω under y, as defined above, to get Ω'. Clearly Ω' is of class C^4. Moreover, if r_0 is small enough, we claim as in (1.2) that

$$|\nabla G'(x)| \geqslant 1, \qquad x \in \partial\Omega'. \tag{2.4}$$

Indeed, if $x \in \partial\Omega' \cap \partial\Omega$, then it follows from the Hopf boundary maximum principle that (2.4) is true. To prove (2.4) for $x \in \partial\Omega' - \partial\Omega$, we let, $\hat{B}(t) = \{x' \in \mathbb{R}^{n-1} \colon |x'| < t\}$. We shall need the following lemma of Schauder type. In Lemma 1, ϕ, γ, are C^k functions on $\hat{B}(2)$, $k \geqslant 3$. Moreover, $\phi < 1/4$, and $\|\cdot\|_k$ denotes the C^k norm on $\hat{B}(2)$. Also, $c' = c'(\cdot, k)$, is an increasing function on $(0, \infty)$ which depends only on k.

Lemma 1. *Let*

$$H = \{(x', x_n): |x'| < 1 \text{ and } \phi(x') < x_n < 1\}.$$

Let u be harmonic in H, with $|u| \leqslant M_3 < +\infty$, *and suppose that* $u = \gamma$ *continuously on* $\{(x', \phi(x'))\} \cap \partial H$. *Then for* $k \geqslant 3$

$$\sum_{0 \leq |\alpha| \leq k} |\partial_\alpha u(x)| \leqslant c'(\|\phi\|_k)(\|\gamma\|_k + M_3), \qquad x \in B(0, 1/2) \cap \bar{H}.$$

Lemma 1 is given in [6, Corollary 6.7] for $C^{2,\alpha}$ domains with a constant depending on H. However, the proof is essentially unchanged if $C^{2,\alpha}$ is replaced by C^k, and $c'(\cdot)$ can be used for the resulting constant (see the remark following Lemma 6.5 in [6]). To prove (2.4) on a bump, we first let

$$Z(y, t) = \{(x', x_n): |x_n - y_n| < t, |x' - y'| < t\}$$

and note that since ψ has support in $\hat{B}(1)$,

$$(\partial\Omega' - \partial\Omega) \cap B(y, 10r) \subseteq Z(y, r\lambda_j^{-1}), \tag{2.5}$$

whenever $y \in L$ and j is the smallest integer satisfying (2.3). Second, observe from the Hopf boundary maximum principle and (2.5) that to prove (2.4) on a bump it suffices to show

$$|\nabla G^*(x)| \geqslant 1, \qquad x \in \bar{Z}(y, r\lambda_j^{-1}) \cap \partial\bar{\Omega}^*, \tag{2.6}$$

where Ω^* is obtained from Ω by adding just one bump at y as above, and G^* is the Green's function for Ω^* with pole at 0. To prove (2.6) let

$$F = \bar{Z}(y, r\lambda_j^{-1}) \cap \bar{\Omega}^*$$

and

$$M_4 = \max_{x \in F} |\nabla G^*(x)|.$$

Then from the mean value theorem of calculus and the fact that $G = 0$ on $\partial\Omega$, we deduce

$$0 \leqslant G^* - G \leqslant c M_4 \sigma_j^2 \lambda_j^{-1} r \tag{2.7}$$

on $\partial\Omega$. Since $G^* - G$ is harmonic in Ω, we see from the maximum principle for harmonic functions that (2.7) also holds in Ω. From (2.1), (2.2), (2.7), and the fact that

$$\nabla G(y) = \left(0, \ldots, \frac{\partial G(y)}{\partial y_n}\right)$$

we get for x in $\bar{\Omega} \cap \bar{B}(y, 20 r\lambda_j^{-1})$, (2.8) $|G^*(x) - |\nabla G(y)|(x_n - y_n)|$

$$\begin{aligned}
&\leqslant cM_4\sigma_j^2\lambda_j^{-1}r + |G(x) - |\nabla G(y)|(x_n - y_n)| \\
&\leqslant cM_4\sigma_j^2\lambda_j^{-1}r + \int_{y_n+\theta(x'-y')}^{x_n} \left| \frac{\partial G}{\partial t_n}(x', t_n) - \frac{\partial G}{\partial t_n}(y', y_n) \right| dt_n \\
&\quad + |\nabla G(y)|\, |\theta(x' - y')| \\
&\leqslant cM_4\sigma_j^2\lambda_j^{-1}r + cM_2(\lambda_j^{-1}r)^2 + cM_2M_1(\lambda_j^{-1}r)^2 \\
&\leqslant c(M_4\sigma_j^2 + \epsilon^2)\lambda_j^{-1}r.
\end{aligned}$$

Put, $\beta = 10r/\lambda_j$,

$$\begin{aligned}
\phi(x') &= \beta^{-1}(\xi(\beta x') - y_n), && x' \in \hat{B}(2), \\
u(x) &= \beta^{-1}G^*(\beta x + y) - |\nabla G(y)|x_n, && x \in H,
\end{aligned}$$

where H is defined relative to ϕ as in Lemma 1. Using (2.1) it is easily checked that $\|\phi\|_4 \leqslant c\sigma_j^2\|\psi\|_4 + c\epsilon^2$. Since $u = -|\nabla G(y)|\phi$ on $\partial H \cap B(1)$, we find from this inequality, (2.8), and Lemma 1 with $k = 4$ that

$$|\nabla u(x)| \leqslant c'(\|\phi\|_4)(M_4\sigma_j^2 + c\sigma_j^2|\nabla G(y)| + c\epsilon^2|\nabla G(y)| + c\epsilon^2)$$

$x \in B(0, 1/2) \cap H$, where

$$c'(\|\phi\|_4) \leqslant c'(\|\psi\|_4 + 1) = c_0.$$

From this inequality and the fact that $\epsilon \leqslant 2\sigma_j$, $|\nabla G(y)| \geqslant 1$, we deduce

(2.9) $$||\nabla G^*(x)| - |\nabla G(y)|| \leqslant c_0M_4\sigma_j^2 + c_1\sigma_j^2|\nabla G(y)|,$$

for $x \in \bar{Z}(y, r\lambda_j^{-1}) \cap \bar{\Omega}^*$. Let σ_0, $0 < \sigma_0 \leqslant 10^{-3}$, be so small that

(2.10) $$c_0 + c_1 < 10^{-3}\sigma_0^{-1}.$$

Choosing x so that

$$|\nabla G^*(x)| = M_4,$$

we conclude from the triangle inequality and (2.9) that

$$M_4(1 - c_0\sigma_j^2) \leqslant (1 + c_1\sigma_j^2)|\nabla G(y)|.$$

Hence,

(2.11) $$M_4 \leqslant (1 + 2c_0\sigma_j^2)(1 + c_1\sigma_j^2)|\nabla G(y)|.$$

Now from (2.2), (2.3), we see that $|\nabla G(y)| \geqslant 1 + \sigma_j/2$. Using this fact, (2.10), and (2.11), in (2.9), we deduce

$$|\nabla G^*(x)| \geqslant (1 - 2(c_0 + c_1)\sigma_j^2)|\nabla G(y)| \geqslant 1 + \frac{1}{4}\sigma_j.$$

Hence (2.6) is valid. From our earlier remarks it now follows that (2.4) is valid.

If

$$c_2 = \int_{\mathbb{R}^{n-1}} |\nabla' \psi(x')|^2 \, dx',$$

and

$$\sigma_0 \leqslant c_2 \leqslant \alpha(n-1)\Big(\max_{\mathbb{R}^{n-1}} |\nabla' \psi|\Big)^2 \leqslant \sigma_0^{-1} 10^{-6}, \tag{2.12}$$

then from (2.1), it follows that

$$\begin{aligned} H^{n-1}(Z(y, r\lambda_j^{-1}) \cap \partial\Omega') &= \int_{\hat{B}(r\lambda_j^{-1})} \sqrt{1 + |\nabla' \xi|^2} \, dx' \\ &\geqslant \int_{\hat{B}(r\lambda_j^{-1})} \sqrt{1 + \sigma_j^4 |\nabla' \psi(\lambda_j r^{-1} x')|^2} \, dx' - \epsilon^8 \alpha(n-1)(r/\lambda_j)^{(n-1)} \\ &= \Big(\int_{\hat{B}(1)} \sqrt{1 + \sigma_j^4 |\nabla' \psi(x')|^2} \, dx'\Big)(r/\lambda_j)^{(n-1)} - \epsilon^8 \alpha(n-1)(r/\lambda_j)^{(n-1)} \\ &\geqslant \Big(1 + \frac{1}{4}\sigma_j^4 c_2 - \epsilon^8\Big)\alpha(n-1)(r/\lambda_j)^{(n-1)} \\ &\geqslant \frac{1}{8}\sigma_j^4 c_2 \alpha(n-1)(r\lambda_j^{-1})^{(n-1)} + H^{n-1}(Z(y, r\lambda_j^{-1}) \cap \partial\Omega). \end{aligned} \tag{2.13}$$

Given $t \geqslant \epsilon$, let k be the least nonnegative integer such that $t \geqslant \sigma_k$, $0 \leqslant k \leqslant l + 1$. Let $J = J(k)$, be the set of all i such that (2.3) holds with $y = y^i$ and $j \leqslant k$. From (2.1) it is clear that

$$\begin{aligned} H^{n-1}\{x \in \partial\Omega : |\nabla G(x)| \geqslant 1 + t\} &\leqslant H^{n-1}\Big(\bigcup_{i \in J} B(y^i, 100r) \cap \partial\Omega\Big) \\ &\leqslant 2 \sum_{i \in J} \alpha(n-1)(100r)^{n-1}. \end{aligned} \tag{2.14}$$

Using (2.13), (2.14), and (2.5) we deduce

$$H^{n-1}(\partial\Omega') \geqslant H^{n-1}(\partial\Omega) + \frac{c_3 \sigma_k^4}{\lambda_k^{n-1}} H^{n-1}\{x \in \partial\Omega : |\nabla G(x)| > 1 + t\}, \tag{2.15}$$

where $c_3 > 0$ depends only on n. Let

$$\eta(t) = \begin{cases} \dfrac{c_3\sigma_o^4}{\lambda_o^{n-1}}, & \sigma_0 \leqslant t \\ \dfrac{c_3\sigma_k^4}{\lambda_k^{n-1}}, & \sigma_k \leqslant t < \sigma_{k-1}, \qquad k = 1, 2, \ldots \end{cases}$$

Clearly η does not depend on Ω or Ω'. Rewriting (2.15) in terms of η we obtain (1.1).

Next we define the homeomorphism h mentioned in Section 1. If $y \in L$ and j is the smallest positive integer for which (2.3) holds, define h on $Z(y, r)$ by $h(x) = (x', h^*(x))$, where

$$h^*(x', x_n) = \begin{cases} \dfrac{(r + y_n - \xi(x' - y'))(x_n - r - y_n)}{r - \theta(x' - y')} + r + y_n, & x \in Z(y, r) \cap \Omega \\ \dfrac{(\xi(x' - y') + r - y_n)(x_n + r - y_n)}{r + \theta(x' - y')} - r + y_n, & x \in Z(y, r) \cap (\mathbb{R}^n - \Omega) \end{cases}$$

Define $h(x) = x$ in the complement of the union of all $Z(y, r)$ for which (2.3) holds. We note that h restricted to $Z(y, r) = Z$ is simply a projection by lines parallel to the x_n axis of $Z \cap (\mathbb{R}^n - \Omega)$, $Z \cap \Omega$, respectively onto $Z \cap (\mathbb{R}^n - \Omega')$, $Z \cap \Omega'$, which keeps $\partial Z(y, r)$ fixed. Thus, h is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $h(\bar{\Omega}) = \bar{\Omega}'$. Moreover, using (2.1) it is easily checked that

$$(2.16) \qquad (1 - c_4\sigma_0^2)|x - z| \leqslant |h(x) - h(z)| \leqslant (1 + c_4\sigma_0^2)|x - z|,$$

when $x, z \in \mathbb{R}^n$ and

$$(2.17) \qquad |x - z| - c_4\sigma_0^2 r \leqslant |h(x) - h(z)| \leqslant |x - z| + c_4\sigma_0^2 r,$$

when $|x - z| > r$. Also for use in proving (1.9) we shall show for $x, z \in \partial\Omega$, that

$$(2.18) \qquad |h(x) - h(z)| \geqslant (1 - c_5 r^{1/2})|x - z|.$$

Indeed, suppose $x, z \in \partial\Omega$, $5r \leqslant |x - z| < 100 r^{1/2}$, $x \in B(w, 100r)$, and $z \in B(y, 100r)$, where $w, y \in \{y^i\}_1^N$. Let θ be defined relative to y as previously and recall that $B(y, 1000 r^{1/2}) \cap \partial\Omega$ can be expressed in terms of θ. Let $\nu(p)$ denote the outer unit normal to p in $\partial\Omega$ and let $\bullet$ denote inner product. Then

$$|\nu(y) \bullet \nu(w)| = (1 + |\nabla\theta(w' - y')|^2)^{-1/2} > 1 - \frac{1}{4} M_1^2 |w' - y'|^2.$$

Thus if δ denotes the angle between $\nu(y)$ and $\nu(w)$, then

$$\delta < 4M_1|w' - y'| < 164 M_1 |x - z|.$$

Next suppose $h(x) = (v', v_n)$ and $v' \neq x'$. Then we can draw the right triangle with vertices x, $h(x)$, and $P = (v', x_n)$. Let l_1, l_2, and l_3 be the sides of this triangle connecting x to $h(x)$, $h(x)$ to P, and P to x, respectively. Then from the definition of h we see that $\nu(w)$ is parallel to l_1, and so l_1, l_2 form an angle δ at $h(x)$. Also, $|x - h(x)| < r$, so from trigonometry and the above inequality,

$$|v' - x'| < r \sin \delta < 164 M_1 r |x - z|.$$

From this inequality and (2.1) we deduce

$$\begin{aligned} |h(z) - h(x)| &> |v' - z'| \\ &\geqslant |x' - z'| - |v' - x'| \\ &> (1 - cM_1^2 r)|x - z| \\ &> (1 - c_5 r^{1/2})|x - z|. \end{aligned}$$

Hence (2.18) is valid when $5r \leqslant |x - z| < 100 r^{1/2}$. If $|x - z| < 5r$, then (2.18) remains true as follows easily from the fact that the bumps are greater than $6r$ apart. If $100 r^{1/2} \leqslant |x - z|$ then it follows from (2.17) that (2.18) is true.

Finally in this section we fix σ_0 to be the largest number for which (2.10), (2.12), hold and

$$c_4 \sigma_0^2 \leqslant \frac{1}{2}. \tag{2.19}$$

Note from (2.12) that $0 < \sigma_0 \leqslant 10^{-3}$.

3. Wolff's lemma

To prove (1.3) in Section 1 we shall need some definitions. Let Ω_1 be a bounded domain. If diam $\Omega_1 = 1$, then Ω_1 is an NTA domain with constant A if it has the following properties:

(i) (Corkscrew condition.) For each $x \in \partial\Omega_1$, $0 < r < A^{-1}$, there are points $P_r(x) \in \Omega_1$, $Q_r(x) \in \mathbb{R}^n - \Omega_1$, with $|P_r(x) - x| \leqslant Ar$, $|Q_r(x) - x| \leqslant Ar$, and $\operatorname{dist}(P_r(x), \partial\Omega_1) \geqslant A^{-1} r$, $\operatorname{dist}(Q_r(x), \partial\Omega_1) \geqslant A^{-1} r$,

(ii) (Harnack chain condition.) For each $x, y \in \Omega_1$ there is a path γ from x to y with length $(\gamma) \leqslant A|x - y|$ and $\operatorname{dist}(\gamma(t), \partial\Omega_1) \geqslant A^{-1} \min\{|\gamma(t) - x|, |\gamma(t) - y|\}$.

In general Ω_1 is an NTA domain with constant A, if a scaling of it with diameter 1 has constant A. Ω_1 is said to be Lipschitz on scale t with constant A, provided for each $z \in \partial\Omega_1$, there is a coordinate system such that $\partial\Omega_1 \cap B(z, t)$ is the graph of a Lipschitz function defined on $\mathbb{R}^{n-1}$ with Lipschitz norm less than or equal to A. Moreover, $\Omega_1 \cap B(x, t)$ lies above the graph of this function.

Now suppose for some $w \in \partial\Omega_1$ and $t > 0$ that after a possible rotation of coordinates,

$$\begin{aligned} \partial\Omega_1 \cap B(w, t) &= \{x: x_n = w_n\} \cap B(w, t) \\ \Omega_1 \cap B(w, t) &= \{x: x_n > w_n\} \cap B(w, t) \end{aligned} \tag{3.1}$$

Let $p \leqslant 0$ be a C^∞ function with support in $\hat{B}(1)$, suppose $\lambda > 2\max_{\mathbb{R}^{n-1}} |p| + 1$, and define $\Omega_2 \supset \Omega_1$ as follows:

(*a*) $\Omega_1 - B(w, t) = \Omega_2 - B(w, t)$,

(*b*) $\partial\Omega_2 \cap B(w, t) = \{(x' + w', w_n + t\lambda^{-1}p(t^{-1}\lambda x')): x' \in \mathbb{R}^{n-1}\} \cap B(w, t)$,

(*c*) $\Omega_2 \cap B(w, t) = \{(x' + w', x_n): x_n > w_n + t\lambda^{-1}p(t^{-1}\lambda x')\} \cap B(w, t)$.

Let $\hat{p}$ be the continuous harmonic extension of p to $(\mathbb{R}^n)^+ = \{(x', x_n): x_n > 0\}$ and put

$$\Lambda(p) = \int_{\mathbb{R}^{n-1}} \left(\left(\frac{\partial \hat{p}}{\partial x_n} \right)^3 - 3|\nabla' p|^2 \frac{\partial \hat{p}}{\partial x_n} \right)(x', 0)\, dx'$$

where $\nabla' p$, as in Section 2, is the $\mathbb{R}^{n-1}$ gradient. Next if $d = \operatorname{diam} \Omega_1$, we assume

$$B(0, d/A) \subseteq \Omega_1 \subseteq B(0, Ad). \tag{3.2}$$

Denote Green's functions for Ω_1, Ω_2, with pole at 0, by G_1, G_2, respectively, and let ω_1 be harmonic measure on Ω_1 with respect to 0. If $\partial\Omega_1$ is sufficiently smooth we observe that

$$\omega_1(E) = \int_{E \cap \partial\Omega_1} |\nabla G_1|\, dH^{n-1}, \qquad E \text{ Borel}.$$

Then Wolff proved [14, Lemma 2.7].

Lemma 2. *Let* Ω_1 *be* NTA *and Lipschitz on scale* t *with constant* A. *Suppose* Ω_1 *satisfies* (3.1), (3.2), *and* Ω_2 *is obtained by adding a bump to* Ω_1 *as in* (*a*)-(*c*). *If* $\Lambda(p) < 0$, *then there exists* $\lambda^* = \lambda^*(A, p)$, $c_6 = c_6(A, p)$, *such that for* $\lambda \geqslant \lambda^*$,

$$\int_{\partial\Omega_2} |\nabla G_2| \log|\nabla G_2|\, dH^{n-1} \leqslant \int_{\partial\Omega_1} |\nabla G_1| \log |\nabla G_1|\, dH^{n-1} - \frac{c_6}{\lambda^{n-1}}\, \omega_1(B(w, t)).$$

Actually Wolff proves this Lemma only in $\mathbb{R}^3$, but the proof for $\mathbb{R}^n$, $n \geqslant 3$, is essentially unchanged. To show the existence of $p \leqslant 0$ for which $\Lambda(p) < 0$, Wolff first shows that $\Lambda(q) < 0$ for $n = 3$ when $q(x') = -|x' + e_3|^{-1}$, $x' \in \mathbb{R}^3$,

$e_3 = (0, 0, 1)$. In view of this function, the natural function to consider for $n \geqslant 3$ is

$$q(x') = -|x' + e_n|^{2-n}, \qquad e_n = (0, \ldots, 0, 1), \qquad x' \in \mathbb{R}^{n-1},$$

for which $\hat{q}(x) = -|x + e_n|^{2-n}$, $x \in (\mathbb{R}^n)^+$. Then

$$\Lambda(q) = (n-1)(n-2)^3\alpha(n-1)\int_0^\infty (r^2+1)^{-3n/2}(1-3r^2)r^{n-2}\,dr$$

$$= -\frac{(n-1)(n-2)^4\alpha(n-1)\Gamma(n-1/2)\Gamma(n/2-1/2)}{4\Gamma(3n/2)} < 0,$$

where Γ denotes the Euler gamma function and the integral was evaluated using the substitution $r = \tan\theta$, as well as, the beta function. Let Φ, $0 \leqslant \Phi \leqslant 1$, be a C^∞ function on $\mathbb{R}^{n-1}$ with support in $\hat{B}(2)$, $|\nabla'\Phi| \leqslant 1000$, and $\Phi = 1$ on $\hat{B}(1)$. Now if

$$q_m(x') = \Phi(m^{-1}x')q(x'), \qquad x' \in \mathbb{R}^{n-1},$$

then it follows easily from properties of conjugate harmonic functions (see [13, Ch. 6]) that

$$\Lambda(q_m) \to \Lambda(q) \quad \text{as} \quad m \to \infty.$$

Taking a suitable dilation of q_m for large m, we get $p \leqslant 0$ in $C^\infty(\mathbb{R}^{n-1})$ with $\operatorname{supp} p \subseteq \hat{B}(1)$, and $\Lambda(p) < 0$.

We now define ψ and $(\lambda_k)_0^\infty$ introduced in Section 2. Let ψ, $0 \leqslant \psi \leqslant 1$, be a fixed $C^\infty(\mathbb{R}^{n-1})$ function with support in $\hat{B}(1)$, $\max_{\mathbb{R}^{n-1}} \psi = 1$, and $\Lambda(\psi) > 0$. Recall that $\sigma_k = 2^{-k}\sigma_0$, $k = 0, 1, \ldots$, and define λ_k as follows: let $A = 200$ in Lemma 2 and $p = -\sigma_k^2\psi$. Let $\lambda_k' = \max\{\sigma_k^{-1}, b_k^{-1}, \lambda_k^*\}$, $k = 0, 1, \ldots$, where $b_k = c_6(200, -\sigma_k^2\psi)$, $\lambda_k^* = \lambda^*(200, -\sigma_k^2\psi)$. Put $\lambda_m = \max_{0 \le k \le m} \lambda_k'$, $m = 0, 1, \ldots$ and note that $(\lambda_k)_0^\infty$ depends only on n since σ_0 and ψ are fixed.

Let $\Omega, \Omega', \epsilon, r, L$, and $(E_k)_0^{l+1}$, be as in Section 2 and suppose also that Ω is NTA with constant 100. Moreover, we assume $B(0, \rho) \subseteq \Omega \subseteq B(0, 2)$, where ρ is as in (1.4). From our choice of r we see that Ω is Lipschitz on scale $r^{1/2}$ with constant 2. In order to apply Lemma 2, we need to add flat bumps under each $y \in L$. For fixed $y \in L$ let j be the smallest nonnegative integer for which (2.3) holds, *i.e.*

$$B(y, 100r) \cap E_j \neq \varnothing.$$

Suppose that $L = \{z_1, z_2, \ldots, z_m\}$ and put $L_k = \{z_1, \ldots, z_k\}$, $1 \leqslant k \leqslant m$. For fixed $y \in L$ we assume that $B(y, 1000r^{1/2}) \cap \Omega$, $B(y, 1000r^{1/2}) \cap \partial\Omega$, can be expressed as in Section 2 relative to θ. Let

$$\hat{\xi}(x') = -100M_1 r^2 \Phi\left(\frac{x'}{r}\right) + \left(1 - \Phi\left(\frac{x'}{r}\right)\right)\theta(x') + y_n, \qquad x' \in \mathbb{R}^{n-1},$$

$$\tilde{\xi}(x') = \hat{\xi}(x') - \sigma_j^2 r \lambda_j^{-1} \psi(\lambda_j x'/r), \qquad x' \in \mathbb{R}^{n-1},$$

where Φ was defined earlier in Section 3 and M_1 is as in (2.1). Define $\hat{\Omega}_k$, $1 \leqslant k \leqslant m$, as follows:

$$(\hat{\mathrm{I}}) \qquad \hat{\Omega}_k - \bigcup_{z \in L_k} B(z, 10r) = \Omega - \bigcup_{z \in L_k} B(z, 10r),$$

$$(\hat{\mathrm{II}}) \qquad \partial\hat{\Omega}_k \cap B(y, 10r) = \{(x' + y', \hat{\xi}(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r),$$

$$(\hat{\mathrm{III}}) \qquad \hat{\Omega}_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \hat{\xi}(x')\} \cap B(y, 10r),$$

for each $y \in L_k$. $\tilde{\Omega}_k \supseteq \hat{\Omega}_m$, $1 \leqslant k \leqslant m$, is defined similarly by

$$(\tilde{\mathrm{I}}) \qquad \tilde{\Omega}_k - \bigcup_{z \in L_k} B(z, 10r) = \hat{\Omega}_m - \bigcup_{z \in L_k} B(z, 10r),$$

$$(\tilde{\mathrm{II}}) \qquad \partial\tilde{\Omega}_k \cap B(y, 10r) = \{(x' + y', \tilde{\xi}(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r),$$

$$(\tilde{\mathrm{III}}) \qquad \tilde{\Omega}_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \tilde{\xi}(x')\} \cap B(y, 10r),$$

for each $y \in L_k$. From (2.1) and the definition of Ω' we see that $\hat{\Omega}_m \supseteq \Omega$, $\tilde{\Omega}_m \supseteq \Omega'$. Using the fact that Ω is NTA with constant 100 and local smoothness of $\hat{\Omega}_k$, $\tilde{\Omega}_k$, it is easily checked that $\hat{\Omega}_k$, $\tilde{\Omega}_k$, $1 \leqslant k \leqslant m$, are NTA and Lipschitz on scale r with constant 200. Let $\hat{\Omega}_0 = \Omega$, $\tilde{\Omega}_0 = \hat{\Omega}_m$. We first apply Lemma 2 with $t = r$, $\Omega_1 = \tilde{\Omega}_0$, $\Omega_2 = \tilde{\Omega}_1$, after a possible rotation. We next apply Lemma 2 with $\Omega_1 = \tilde{\Omega}_1$ and $\Omega_2 = \tilde{\Omega}_2, \ldots$, etc. Let $\hat{G}_k$, $\tilde{G}_k$, $\hat{\omega}_k$, $\tilde{\omega}_k$, be the Green's functions and harmonic measures relative to 0 for $\hat{\Omega}_k$, $\tilde{\Omega}_k$. Applying the above argument m times we obtain an inequality for $\hat{G}_m = \tilde{G}_0$ and $\tilde{G}_m$. Using the definition of $(\lambda_k)_0^\infty$, we conclude

$$(3.3) \quad \int_{\partial\tilde{\Omega}_m} |\nabla\tilde{G}_m| \log |\nabla\tilde{G}_m| \, dH^{n-1}$$
$$\leqslant \int_{\partial\hat{\Omega}_m} |\nabla\hat{G}_m| \log |\nabla\hat{G}_m| \, dH^{n-1} - c(\lambda_{l+1})^{-(n-1)} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_{k+1}, 2r)).$$

Next we define a function τ on $[0, 1]$ by $\tau(s) = \min\{\lambda_k : \sigma_k \leqslant s\}$, $0 < s \leqslant 1$. Choosing r_0 still smaller, if necessary, we assume, as we may, that for $0 < r \leqslant r_0$,

$$(3.4) \qquad r^{1/16} \leqslant \tau(\epsilon)^{-(n-1)}.$$

Note that $\tau(\epsilon) = \lambda_{l+1}$.

To prove (1.3) we must show that $\hat{G}_m$, $\tilde{G}_m$, in (3.3) can be replaced by G, G', with an error term at most,

$$c\tau(\epsilon)^{-(n-1)} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_k, 2r)).$$

To do so we introduce Ω'_k, $0 \leqslant k \leqslant m$, defined by, $\Omega'_0 = \Omega'$, and for $1 \leqslant k \leqslant m$,

$$\text{(I}'\text{)} \qquad \Omega'_k - \bigcup_{z \in L_k} B(z, 10r) = \Omega' - \bigcup_{z \in L_k} B(z, 10r),$$

$$\text{(II}'\text{)} \qquad \partial\Omega'_k \cap B(y, 10r) = \{(x' + y', \tilde{\xi}(x')) : x' \in \mathbb{R}^{n-1}\} \cap B(y, 10r),$$

$$\text{(III}'\text{)} \qquad \Omega'_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \tilde{\xi}(x')\} \cap B(y, 10r),$$

for each $y \in L_k$. Denote the corresponding Green's functions and harmonic measures relative to 0, by G'_k, ω'_k, $0 \leqslant k \leqslant m$. We shall also need the following facts about the NTA domain Ω_1 with constant A satisfying (3.2). If $z \in \partial\Omega_1$, then

$$\text{(3.5)} \qquad \begin{aligned} c(A)^{-1}\omega_1(B(z,t)) &\leqslant t^{n-2} \max_{B(z,t)\cap\Omega_1} G_1 \\ &\leqslant c(A) t^{n-2} G_1(P_t) \\ &\leqslant c(A)\omega_1(B(z,t)), \end{aligned}$$

for $0 < t < A^{-1}$, where $P_t = P_t(z)$. Moreover,

$$\text{(3.6)} \qquad \omega_1(B(z, 2t)) \leqslant c(A)\omega_1(B(z,t)).$$

(3.6) is called the doubling inequality for harmonic measure. If $z \in \partial\Omega_1$ and u, v are two positive harmonic functions in Ω_1 which vanish continuously on $\partial\Omega_1 - B(z,t)$, and $P_t = P_t(z)$, then for $x \in \Omega_1 - B(z, 2t)$

$$\text{(3.7)} \qquad c(A)^{-1}u(P_t)/v(P_t) \leqslant u(x)/v(x) \leqslant c(A)u(P_t)/v(P_t).$$

Moreover, (3.7) is valid when u and v vanish on $\partial\Omega_1 \cap B(z, 2t)$, and $x \in B(z,t) \cap \Omega_1$. (3.7) is called the rate inequality. Finally there exists $\mu = \mu(A) > 0$ so that for z and P_t as above, and $x \in B(z,t) \cap \Omega_1$,

$$\text{(3.8)} \qquad G_1(x) \leqslant c(|x - z|/t)^{\mu} G_1(P_t).$$

For the proof of (3.5)-(3.8) see [8, Sections 4 and 5].

From (3.5), (3.6), (3.8) with $t = A^{-1}$, and the fact that $\omega_1(B(z, A^{-1})) \geqslant c(A)^{-1}$, when $z \in \partial\Omega_1$, we see there exists $\nu(A)$, $0 < \nu < 1$, with

$$\text{(3.9)} \qquad c(A)^{-1}t^{1/\nu} \leqslant \omega_1(B(z,t)) \leqslant c(A)t^{\mu+n-2}, \qquad 0 < t < A^{-1}.$$

We claim that

$$(3.10)\qquad \sum_{k=0}^{m-1} \omega_k^*(B(z_{k+1}, 6r)) \leqslant c \sum_{k=0}^{m-1} \omega_k^+(B(z_{k+1}, 6r)),$$

whenever $*$ and $+$ are elements of $\{\hat{}, \sim, '\}$. Indeed from our construction and the maximum principle for harmonic functions we have,

$$\begin{aligned}\hat{\omega}_0(B(z_{k+1}, 6r) - B(z_{k+1}, 2r)) &\leqslant \omega_j^*(B(z_{k+1}, 6r) - B(z_{k+1}, 2r))\\ &\leqslant \tilde{\omega}_m(B(z_{k+1}, 6r) - B(z_{k+1}, 2r)),\end{aligned}$$

when $0 \leqslant j \leqslant m$, $0 \leqslant k \leqslant m-1$, and $* \in \{\hat{}, \sim, '\}$. Summing and using the doubling inequality it follows that

$$c^{-1} \sum_{k=0}^{m-1} \hat{\omega}_0(B(z_{k+1}, 6r)) \leqslant \sum_{k=0}^{m-1} \omega_k^*(B(z_{k+1}, 6r)) \leqslant c \sum_{k=0}^{m-1} \tilde{\omega}_m(B(z_{k+1}, 6r)).$$

On the other hand, from the maximum principle we deduce

$$\sum_{k=0}^{m-1} \tilde{\omega}_m(B(z_{k+1}, 6r)) \leqslant \sum_{k=0}^{m-1} \hat{\omega}_0(B(z_{k+1}, 6r)).$$

Hence our claim is true. We shall show for $0 \leqslant k \leqslant m-1$ that

$$\begin{aligned}(3.11)\quad &\int_{\partial\Omega'_k} |\nabla G'_k| \log |\nabla G'_k| \, dH^{n-1}\\ &\qquad \leqslant \int_{\partial\Omega'_{k+1}} |\nabla G'_{k+1}| \log |\nabla G'_{k+1}| \, dH^{n-1} + cr^{1/2}\omega'_k(B(z_{k+1}, 6r)),\end{aligned}$$

$$\begin{aligned}(3.12)\quad &\int_{\partial\hat{\Omega}_{k+1}} |\nabla \hat{G}_{k+1}| \log |\nabla \hat{G}_{k+1}| \, dH^{n-1}\\ &\qquad \leqslant \int_{\partial\hat{\Omega}_k} |\nabla \hat{G}_k| \log |\nabla \hat{G}_k| \, dH^{n-1} + cr^{1/2}\hat{\omega}_k(B(z_{k+1}, 6r)).\end{aligned}$$

Summing (3.11) and using (3.10), it then follows that

$$\begin{aligned}(3.13)\quad &\int_{\partial\Omega'} |\nabla G'| \log |\nabla G'| \, dH^{n-1}\\ &\qquad \leqslant \int_{\partial\tilde{\Omega}_m} |\nabla \tilde{G}_m| \log |\nabla \tilde{G}_m| \, dH^{n-1} + cr^{1/2} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_{k+1}, 6r)),\end{aligned}$$

where we have used the fact that $\Omega'_0 = \Omega_0$, $\Omega'_m = \tilde{\Omega}_m$.

Summing (3.12) and using (3.10), we find

$$(3.14)\quad \int_{\partial\hat{\Omega}_m} |\nabla \hat{G}_m| \log |\nabla \hat{G}_m|\, dH^{n-1} \leqslant \int_{\partial\Omega} |\nabla G| \log |\nabla G|\, dH^{n-1} + cr^{1/2} \sum_{k=0}^{m-1} \hat{\omega}_k(B(z_{k+1}, 6r)),$$

since $\hat{\Omega}_0 = \Omega$. Putting (3.13), (3.14), into (3.3) and using (3.6) we get (1.3) provided r_0 is small enough, thanks to (3.4). Thus (1.3) is true once we prove (3.11)-(3.12).

We prove only (3.11), (3.12), for $k = 0$, since the proof of all the other inequalities is the same. To prove (3.12) for $k = 0$ we first observe from (3.5) that

$$(3.15)\qquad \max_{B(z_1, 6r)\cap\hat{\Omega}_1} \hat{G}_1 \leqslant cr^{2-n}\hat{\omega}_1(B(z_1, 6r)).$$

Using (3.15), (2.1), and applying Lemma 1 with $k = 4$ after scaling $B(x_1, 6r) \cap \hat{\Omega}_1$, we find for x, y in the closure of $B(z_1, 3r) \cap \hat{\Omega}_1$,

$$(3.16)\qquad |\nabla \hat{G}_1(x) - \nabla \hat{G}_1(y)| \leqslant c|x - y| r^{-n}\hat{\omega}_1(B(z_1, 6r)),$$

while from (3.15), a barrier argument, (3.5)-(3.6) and (ii), we have

$$(3.17)\qquad c^{-1}r^{1-n}\hat{\omega}_1(B(z_1, 6r)) \leqslant |\nabla \hat{G}_1(x)| \leqslant cr^{1-n}\hat{\omega}_1(B(z_1, 6r)).$$

Clearly (3.17) and (3.9) imply

$$(3.18)\qquad |\log |\nabla \hat{G}_1(x)|| \leqslant -c \log r,$$

when x is in the closure of $B(z_1, 3r) \cap \hat{\Omega}_1$. Using (3.16)-(3.18), (3.6), (2.1), and parametrizing $\partial\Omega$ and $\partial\hat{\Omega}_1$ in terms of θ and ξ, for $y = z_1$, we obtain with $z_1 = (y', y_n)$, $\hat{x} = (x' + y', \xi(x'))$, $x = (x' + y', \theta(x') + y_n)$,

(3.19)

$$\left| \int_{\partial\Omega\cap B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1|\, dH^{n-1} - \int_{\partial\hat{\Omega}_1\cap B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1|\, dH^{n-1} \right|$$

$$\leqslant \int_{\hat{B}(3r)} ||\nabla \hat{G}_1| \log |\nabla \hat{G}_1||(x) \left| \sqrt{1 + |\nabla'\theta(x')|^2} - \sqrt{1 + |\nabla'\xi(x')|^2} \right| dx'$$

$$+ \int_{\hat{B}(3r)} ||\nabla \hat{G}_1|(x) - |\nabla \hat{G}_1|(\hat{x})| |\log |\nabla \hat{G}_1(x)|| \sqrt{1 + |\nabla'\xi(x')|^2}\, dx'$$

$$+ \int_{\hat{B}(3r)} |\nabla \hat{G}_1|(\hat{x})|\log|\nabla \hat{G}_1(x)| - \log|\nabla \hat{G}_1(\hat{x})|| \sqrt{1 + |\nabla' \xi(x')|^2}\, dx'$$

$$\leqslant (-cM_1^2 r^2 \log r - cM_1 r \log r + \log(1 + M_1 r))\hat{\omega}_1(B(z_1, 6r))$$

$$\leqslant cr^{1/2}\hat{\omega}_1(B(z_1, 6r)).$$

Next from (3.17), (2.1) and the fact that each point of $B(z_1, 6r) \cap \partial\hat{\Omega}_1$ lies within 200 $M_1 r^2$ of a point of $B(z_1, 6r) \cap \partial\Omega$, we get

$$(\hat{G}_1 - G)(x) \leqslant cM_1 r^{3-n}\hat{\omega}_1(B(z_1, 6r)) \tag{3.20}$$

for $x \in \partial\Omega$. From the maximum principle for harmonic functions and the fact that $\Omega \subseteq \hat{\Omega}_1$, we conclude this inequality holds in Ω. Let $\phi(x') = \theta(6rx')/6r$, and define H relative to ϕ as in Lemma 1. Put

$$u(x) = \frac{1}{6r}(\hat{G}_1(6rx + z_1) - G(6rx + z_1)), \qquad x \in \bar{H},$$

$$\phi_1(x') = \frac{1}{6r}(\xi(6rx') - y_n),$$

$$H_1 = \{x: |x'| < 8, \phi_1(x') < x_n < 2\},$$

$$u_1(x) = \frac{1}{6r}\hat{G}_1(6rx + z_1), \qquad x \in \bar{H}_1.$$

We note from (2.1) that

$$\max\{\|\phi\|_4, \|\phi_1\|_4\} \leqslant cM_1 r. \tag{3.21}$$

Using (3.20), (3.21), we first apply Lemma 1 with u, H, replaced by u_1, H_1. As in (3.16) we get

$$\sum_{0 \le |\alpha| \le 4} |\partial_\alpha u_1(x)| \leqslant cr^{1-n}\hat{\omega}_1(B(z_1, 6r)), \qquad x \in H. \tag{3.22}$$

We note that $u_1 = 0$ on $\partial H_1 \cap \{(x', \phi_1(x'))\}$ and $u = u_1 = \gamma$ on $\partial H \cap \{(x', \phi(x'))\}$. Using these notes and (3.21)-(3.22) we deduce

$$\sum_{|\alpha|=0}^{3} |\partial'_\alpha \gamma(x', \phi(x'))| = \sum_{|\alpha|=0}^{3} |\partial'_\alpha(u_1(x', \phi(x')) - u_1(x', \phi_1(x')))| \tag{3.23}$$

$$\leqslant cM_1 r^{2-n}\hat{\omega}_1(B(z_1, 6r)).$$

Applying Lemma 1 to u and H, with $k = 3$ we find from (3.20)-(3.23)

$$\sum_{|\alpha|=0}^{3} |\partial_\alpha u(x)| \leqslant cM_1 r^{2-n}\hat{\omega}_1(B(z_1, 6r)),$$

for $x \in B(0, 1/2) \cap H$. Hence if $x \in B(z_1, 3r) \cap \bar{\Omega}$, then

$$(3.24) \qquad |\nabla \hat{G}_1 - \nabla G|(x) \leqslant cM_1 r^{2-n} \hat{\omega}_1(B(z_1, 6r)) \leqslant c|\nabla \hat{G}_1(x)| M_1 r,$$

where the last inequality is just (3.17). From (3.24) and (2.1) we obtain

$$(3.25) \quad \left| \int_{\partial\Omega \cap B(z_1, 3r)} |\nabla G| \log |\nabla G| \, dH^{n-1} - \int_{\partial\Omega \cap B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| \, dH^{n-1} \right|$$

$$\leqslant \int_{\partial\Omega \cap B(z_1, 3r)} ||\nabla G| - |\nabla \hat{G}_1|| \, |\log |\nabla G|| \, dH^{n-1} + \int_{\partial\Omega \cap B(z_1, 3r)} |\nabla \hat{G}_1|$$

$$\times \left| \log \left(\frac{|\nabla G|}{|\nabla \hat{G}_1|} \right) \right| dH^{n-1}$$

$$\leqslant -cM_1 r \log r \, \hat{\omega}_1(B(z_1, 6r)) + \hat{\omega}_1(B(z_1, 6r)) \log(1 + cM_1 r)$$

$$\leqslant c r^{1/2} \hat{\omega}_1(B(z_1, 6r)).$$

Let $P = P_{3r}(z_1)$ and let $G(\bullet, Y)$ denote Green's function with pole at $Y \in \Omega$. Following Wolff (see [14, (2.7)]) we first note fom (3.20) and the rate inequality (3.7) with $u = \hat{G}_1 - G$, $v = G(\bullet, P)$, $t = 2r$, that

$$G(x, P)^{-1}(\hat{G}_1 - G)(x) \leqslant cM_1 r \, \hat{\omega}_1(B(z_1, 6r)), \qquad x \in \Omega - B(z_1, 3r).$$

Second, given w in $\partial\Omega - B(z_1, 3r)$, we apply the rate inequality with $u = G(\bullet, P)$, $v = G(\bullet, P_t(w))$, $t = 2|w - z_1|$ in $\Omega - B(z_1, t)$, provided $0 \in \Omega - B(z_1, 2t)$. We get for $x = 0$,

$$t^{n-2} G(P_t(w), P) \leqslant cG(0, P)/G(0, P_t(w)).$$

If $0 \in B(z_1, 2t)$, then it follows easily from Harnack's inequality and $t \geqslant \rho/2$ (since $B(0, \rho) \subseteq \Omega$) that

$$G(P_t(w), P) \leqslant ct^{2-n} G(0, P).$$

From the above inequalities, (3.8) and Harnack's inequality, we find for $P_t = P_t(w)$,

$$G(P_t, P) \leqslant ct^{2-n}(r/t)^{\mu}.$$

Third, we use the rate inequality in $B(w, 10^{-3}t) \cap \Omega$ with $u = \hat{G}(\bullet, P)$, $v = \hat{G}_1(\bullet, 0)$; the above inequalities, (3.5) and (3.6), to obtain

$$r^{-1}(\hat{\omega}_1(B(z_1, 6r)))^{-1} M_1^{-1}(\hat{G}_1 - G)(x) \hat{G}_1(x, 0)^{-1} \leqslant cG(x, P) \hat{G}_1(x, 0)^{-1}$$
$$\leqslant c(r/t)^{\mu} (\hat{\omega}_1(B(z_1, t)))^{-1},$$

for $x \in B(w, 10^{-3}t) \cap \Omega$. Letting $x \to w$ and using (2.1) we conclude from this inequality that

$$(3.26)\quad (|\nabla \hat{G}_1|^{-1}|\nabla \hat{G}_1 - \nabla G|)(w) \leqslant c r^{3/4+\mu}\hat{\omega}_1(B(z_1, 6r))(\hat{\omega}_1(B(z_1, t)))^{-1}|z_1 - w|^{-\mu}.$$

Now

$$(3.27)\quad \left| \int_{\partial\Omega - B(z_1,3r)} |\nabla G| \log |\nabla G| \, dH^{n-1} - \int_{\partial\hat{\Omega}_1 - B(z_1,3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| \, dH^{n-1} \right|$$
$$\leqslant \int_{\partial\Omega - B(z_1,3r)} \big||\nabla G| - |\nabla \hat{G}_1|\big| \, |\log |\nabla G|| \, dH^{n-1}$$
$$+ \int_{\partial\Omega - B(z_1,3r)} |\nabla \hat{G}_1| \, |\log (|\nabla G|/|\nabla \hat{G}_1|)| \, dH^{n-1}$$
$$= I_1 + I_2.$$

If $F_k = B(z_1, 3^{k+1}r) - B(z_1, 3^k r)$, $k = 1, 2, \ldots$ then from (3.26) we have

$$I_1 \leqslant \sum_{k=1}^{\infty} \int_{F_k \cap \partial\Omega} \big||\nabla G| - |\nabla \hat{G}_1|\big| \, |\log |\nabla G|| \, dH^{n-1}$$
$$\leqslant -c r^{3/4+\mu} \log r \, \hat{\omega}_1(B(z_1, 6r)) \left(\sum_{k=1}^{\infty} k 3^{-k\mu} \right) r^{-\mu}$$
$$\leqslant c r^{1/2} \hat{\omega}_1(B(z_1, 6r)).$$

A similar estimate holds for I_2. Using these estimates in (3.27) we get

$$(3.28)\quad \left| \int_{\partial\Omega - B(z_1,3r)} |\nabla G| \log |\nabla G| \, dH^{n-1} - \int_{\partial\hat{\Omega}_1 - B(z_1,3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| \, dH^{n-1} \right|$$
$$\leqslant c r^{1/2} \hat{\omega}_1(B(z_1, 6r)).$$

Next, since

$$\hat{\omega}_1(B(z_1, 6r)) \leqslant \hat{\omega}_0(B(z_1, 6r)),$$

we can replace $\hat{\omega}_1$ by $\hat{\omega}_0$ in (3.28), (3.25), and (3.19). Doing this and combining (3.28), (3.25), (3.19), we conclude that (3.12) is true for $k = 0$.

To prove (3.11) for $k = 0$, let j be the smallest positive integer such that $E_j \cap B(z_1, 10r) \neq \varnothing$. Put $r' = 10r/\lambda_j$ and let $z \in B(z_1, 6r) \cap \partial\Omega_1'$. Then it is easily checked that (3.16)-(3.18) hold with $\hat{G}_1, \hat{\omega}_1, r, z_1$, replaced by G_1', ω_1', r', z, respectively, when $x, y \in B(z, 3r')$. Now from (3.4) we have

$$(3.29)\quad \frac{r}{10} \geqslant \frac{r}{\lambda_j} \geqslant \frac{r}{\lambda_{l+1}} = \frac{r}{\tau(\epsilon)} \geqslant r^{\gamma}$$

where

$$\gamma = 1 + \frac{1}{16(n-1)} \leqslant \frac{33}{32}.$$

Let z^* be the point in $\partial\Omega'$ obtained by projecting z in the rotated x_n direction onto $\partial\Omega'$. Then from the new version of (3.16)-(3.18), and the fact that

$$|z - z^*| < 200 M_1 r^2 < r',$$

thanks to (2.1), (3.29) we find

$$\begin{aligned}
&|(|\nabla G_1'| \log |\nabla G_1'|)(z) - (|\nabla G_1'| \log |\nabla G_1'|)(z^*)| \\
&\qquad \leqslant ||\nabla G_1'|(z) - |\nabla G_1'|(z^*)| \log r' + |\nabla G_1'(z)| \log (|\nabla G_1'|(z)/|\nabla G_1'|(z^*))| \\
&\qquad \leqslant -cM_1 r^2 \log (r')(|\nabla G_1'(z)|/r').
\end{aligned}$$

Using this inequality, (3.29), and parametrizing $\partial\Omega'$, $\partial\Omega_1'$, we get as in (3.19)

(3.30)
$$\left| \int_{\partial\Omega' \cap B(z_1, 3r)} |\nabla G_1'| \log |\nabla G_1'| \, dH^{n-1} - \int_{\partial\Omega_1' \cap B(z_1, 3r)} |\nabla G_1'| \log |\nabla G_1'| \, dH^{n-1} \right| \leqslant c r^{1/2} \omega_1'(B(z_1, 6r)).$$

Next suppose $z \in \partial\Omega'$ and observe as in (3.20) that

(3.31) $(G_1' - G')(z) \leqslant cM_1 r^2 (r')^{1-n} \omega_1'(B(z_1, 6r')) \leqslant cM_1 r^2 (r')^{1-n} \omega_1'(B(z_1, 6r))$.

It follows from the maximum principle for harmonic functions that (3.31) holds in Ω'. If $z = (\bar{z} + y', \xi(\bar{z})) \in \partial\Omega'$, put

$$\phi'(x') = \frac{1}{6r'}(\xi(6r'x' + \bar{z}) - \xi(\bar{z})),$$

$$H' = \{x: |x'| < 1, \phi'(x') < x_n < 1\},$$

$$u'(x) = \frac{1}{6r'}(G_1'(6r'x + z) - G(6r'x + z)), \qquad x \in \bar{H}',$$

$$\phi_1'(x') = \frac{1}{6r'}(\tilde{\xi}(6r'x' + \bar{z}) - \xi(\bar{z})),$$

$$H_1' = \{x: |x'| < 8, \phi_1'(x') < x_n < 2\},$$

$$u_1' = \frac{1}{6r'} G_1'(6r'x + z), \qquad x \in \bar{H}_1'.$$

We note that

$$\|\phi'\|_4 + \|\phi_1'\|_4 \leqslant c,$$
$$\|\phi' - \phi_1'\|_4 \leqslant cM_1 r.$$

Using these inequalities in place of (3.21) and Lemma 1 we get

$$\sum_{0 \le |\alpha| \le 4} |\partial_\alpha u_1'(x)| \leqslant c(r')^{1-n}\omega_1'(B(z, 6r')) \leqslant c(r')^{1-n}\omega_1'(B(z_1, 6r))$$

in H'. Also, as in (3.23), we see for $u' = \gamma'$ on $\partial H' \cap \{(x', \phi'(x'))\}$, that

$$\sum_{|\alpha|=0}^{3} |\partial_\alpha' \gamma'|(x') \leqslant cM_1 r(r')^{1-n}\omega_1'(B(z_1, 6r)).$$

From this inequality, (3.31) and Lemma 1 it follows as in (3.24) that

$$\begin{aligned} (3.32) \quad |\nabla G_1' - \nabla G'|(x) &\leqslant cM_1 r^2 (r')^{-n}\omega_1'(B(z_1, 6r)) \\ &\leqslant cM_1(r^2/r')\omega_1'(B(z_1, 6r))(\omega_1'(B(z_1, 6r')))^{-1}|\nabla G_1'(x)|, \end{aligned}$$

$x \in B(z, 3r') \cap \bar{\Omega}'$. We cover $\partial\Omega' \cap B(z_1, 3r)$ by at most $c(r/r')^{n-1}$ balls, $B(z, 3r')$, $z \in \partial\Omega' \cap B(z_1, 3r)$. Using (3.32) in each ball and arguing as in (3.25) we have

(3.33)

$$\begin{aligned} &\left| \int_{\partial\Omega' \cap B(z_1', 3r)} |\nabla G'| \log |\nabla G'| \, dH^{n-1} - \int_{\partial\Omega' \cap B(z_1, 3r)} |\nabla G_1'| \log |\nabla G_1'| \, dH^{n-1} \right| \\ &\qquad \leqslant -cM_1 r(r/r')^n \log r \, \omega_1'(B(z_1, 6r)) \\ &\qquad \leqslant cr^{1/2}\omega_1'(B(z_1, 6r)), \end{aligned}$$

thanks to (3.29) and (2.1).

At this point we can use (3.31) in place of (3.20) and repeat the argument following (3.25) in the proof of (3.12) (for $k = 0$), since only NTA estimates were used. From (3.28) with G, $\hat{G}_1$, $\hat{\omega}_1$, replaced by G', G_1', ω_0' and (3.30), (3.33), with ω_1' replaced by ω_0', we conclude that (3.11) holds when $k = 0$. From our earlier remarks we now deduce that (1.3) is true.

4. Proof of Theorem 1

Recall that ψ, $0 \leqslant \psi \leqslant 1$, is a fixed C^∞ function with support in $\hat{B}(1)$, $\max_{\mathbb{R}^{n-1}} \psi = 1$, and $\Lambda(\psi) > 0$. Also σ_0, $0 < \sigma_0 \leqslant 10^{-3}$, was chosen to be the largest number for which (2.10), (2.12), and (2.19) are true. Finally, given ϵ, $0 < \epsilon \leqslant \sigma_0$, we note that $r_0 = r_0(\epsilon, M_1, M_2)$, was chosen so small that the inequalities in Sections 2 and 3 are true for $0 < r \leqslant r_0$.

We elaborate on the induction argument for the construction of D which was outlined in Section 1. Let $D_0 = B(0, \rho)$, where ρ satisfies (1.4). Put $\epsilon_0 = \sigma_0$ and $\epsilon_k = 2^{-k}\epsilon_0$, $k = 0, 1, 2, \ldots$ Choose a covering, $L_1 = \{B(z_{0i}, t_{0i})\}$, $1 \leqslant i \leqslant k_0$ of ∂D_0 such that $t_{0i} \leqslant 1/2$, $i = 1, 2, \ldots, k_0$, and

$$\alpha(n-1)\sum_{i=1}^{k_0} t_{0i}^{n-1} \leqslant H^{n-1}(\partial D_0) - \frac{1}{2}.$$

By compactness of D_0 we may assume $k_0 < \infty$. Let $2r_1' > 0$ denote the distance from ∂D_0 to $\mathbb{R}^n - \bigcup_1^{k_0} B(z_{0i}, t_{0i})$. We set $\Omega = D_0$, $\epsilon = \epsilon_1$, and apply the results in Section 2 with $r = r_1$, where r_1 is the smaller of $10^{-9}\rho, r_1'$, and $r_0 = r_0(\epsilon_1, M_1, M_2)$. Here M_1, M_2, are defined relative to D_0, G_0. Let $D_1 = \Omega'$ be the domain obtained by adding smooth bumps to D_0 and $h_1 = h$ the homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, which satisfies (2.16)-(2.18) with $r = r_1$. Moreover, $h_1(\partial D_0) = \partial D_1$. By induction, suppose for some $m \geqslant 1$ we have defined sequences: $(D_k)_0^m, (L_k)_1^m, (r_k')_1^m, (r_k)_1^m, (h_k)_1^m$. Let $L_{m+1} = \{B(z_{mi}, t_{mi})\}_1^{k_m}$, be a covering of ∂D_m such that $t_{mi} \leqslant 2^{-(m+1)}$, $1 \leqslant i \leqslant k_m$, and

$$\alpha(n-1)\sum_{1}^{k_m} t_{mi}^{n-1} \leqslant H^{n-1}(\partial D_m) - 2^{-(m+1)} \tag{4.1}$$

Let $2r_{m+1}' > 0$ be the distance from ∂D_m to $\mathbb{R}^n - \bigcup_1^{k_m} B(z_{mi}, t_{mi})$. Let $\Omega = D_m$, $\epsilon = \epsilon_m$, and $r = r_{m+1}$, where r_{m+1} is the smaller of $10^{-4m}r_m\rho$, r_{m+1}', and $r_0(\epsilon_{m+1}, M_1, M_2)$. Here M_1, M_2, are defined relative to D_m, G_m. Adding smooth bumps to Ω as in Section 2 we obtain $D_{m+1} = \Omega' \supseteq D_m$ and h_{m+1} a homeomorhism from $\mathbb{R}^n$ to $\mathbb{R}^n$ which satisfies (2.16)-(2.18) with $r = r_{m+1}$. Moreover, $h_{m+1}(\partial D_m) = \partial D_{m+1}$. By induction we get, $(D_k)_0^\infty$, $(H_k)_0^\infty$, $(r_k')_1^\infty$, $(r_k)_1^\infty$, and $(h_k)_1^\infty$. From our work in Section 2 we see that (1.1), (1.2), are true with Ω, Ω', G, G', replaced by $D_k, D_{k+1}, G_k, G_{k+1}$, respectively, $k = 0, 1, \ldots$

We claim that D_k, $k = 1, 2, \ldots$ is NTA with constant 100. Indeed, since $0 \leqslant \psi \leqslant 1$ and $r_k \leqslant 10^{-4k}\rho$, $k = 1, 2, \ldots$, it follows from the definition of D_k, by way of the triangle inequality, that

$$B(0, \rho) \subseteq D_k \subseteq B(0, 2\rho), \qquad k = 1, 2, \ldots \tag{4.2}$$

To prove D_k satisfies the corkscrew condition (i) in the definition of an NTA domain, we proceed by induction. If $0 < s < \rho$, and $z \in \partial D_0$, note that $B(z, s) \cap D_0$, $B(z, s) \cap (\mathbb{R}^n - D_0)$, each contain a ball of radius $s/4$. From this note and the fact that ∂D_1 lies within r_1 distance of ∂D_0, we deduce for $4r_1^{1/2} \leqslant s < \rho$, and $z \in \partial D_1$ that $B(z, s) \cap D_0$, $B(z, s) \cap (\mathbb{R}^n - D_0)$, each contain a ball of radius,

$$(1 - r_1)\frac{s}{4} - r_1 \geqslant \frac{1}{4}s(1 - 2r_1^{1/2}) = s_1.$$

If $0 < s \leqslant 4r_1^{1/2}$, then from our choice of $r_1 = r$, we have $z \in B(y, 100r_1)$, for some $y \in \{y^i\}_1^N$. Moreover, $B(y, 1000r_1^{1/2}) \cap D_1$, $B(y, 1000r_1^{1/2}) \cap \partial D_1$, can be expressed as in Section 2 relative to ξ. From (2.12) and (2.1) we observe that $|\nabla\xi| \leqslant 10^{-3}$. Using these facts and a little geometry it is easily seen that the above inequality remains valid when $0 < s \leqslant 4r_1^{1/2}$. By induction, suppose we have shown for some $m \geqslant 1$, that if $z \in \partial D_m$ and $0 < s < \rho$, then $B(z, s) \cap D_m$, $B(z, s) \cap (\mathbb{R}^n - D_m)$, each contain a ball of radius

$$\frac{1}{4} s\left(1 - 2 \sum_{k=1}^{m} r_k^{1/2}\right) = s_m. \tag{4.3}$$

If $4r_{m+1}^{1/2} \leqslant s < \rho$, and $z \in \partial D_{m+1}$, then since ∂D_{m+1} lies within r_{m+1} of ∂D_m, we deduce from (4.3) that $B(z, s) \cap D_{m+1}$, $B(z, s) \cap (\mathbb{R}^n - D_{m+1})$, each contain a ball of radius

$$\frac{1}{4}(s - r_{m+1})\left(1 - \sum_{k=1}^{m} r_k^{1/2}\right) - r_{m+1} \geqslant \frac{1}{4} s\left(1 - 2 \sum_{k=1}^{m+1} r_k^{1/2}\right) = s_{m+1}.$$

If $0 < s < 4r_{m+1}^{1/2}$, it follows from local smoothness of D_{m+1} that $B(z, s) \cap D_{m+1}$, $B(z, s) \cap (\mathbb{R}^n - D_{m+1})$, each contain a ball of radius s_{m+1}. Thus by induction we have shown for $z \in \partial D_k$, $k = 0, 1, \ldots,$ that $B(z, s) \cap D_k$, $B(z, s) \cap (\mathbb{R}^n - D_k)$, both contain a ball of radius

$$s_k \geqslant \frac{1}{4} s\left(1 - 2 \sum_{m=1}^{\infty} r_m^{1/2}\right) \geqslant \frac{1}{8} s,$$

when $0 < s < \rho$. Scalling D_k to have diameter 1, we see that (i) in Section 3 holds with $A = 16$.

To prove (ii), we proceed similarly. Suppose by induction, we have shown for some nonnegative integer m that whenever $x, z \in D_m$, we can join x to z by a curve γ with parameter interval, $[0, 1]$, in such a way that $\gamma(0) = x$, $\gamma(1) = z$, and

$$\text{(4.4)} \quad (a) \ \operatorname{dist}(\gamma(t), \partial D_m) \geqslant \frac{1}{16}\left(1 - 2 \sum_{k=1}^{m} r_k^{1/4}\right) \min\{|\gamma(t) - x|, |\gamma(t) - z|\},$$

$$\text{(4.5)} \quad (b) \qquad \operatorname{length} \gamma \leqslant 3\left(1 + \sum_{k=1}^{m} r_k^{1/4}\right)|x - z|.$$

In case $m = 0$, replace the sums in (4.4), (4.5) by 0. From inspection we see that (4.4), (4.5) hold when $m = 0$, since $D_0 = B(0, \rho)$. Next suppose $x, z \in D_{m+1}$ and $4r_{m+1}^{1/2} \leqslant |x - z|$. Since $D_m \subseteq D_{m+1}$, we note that (4.4) and (4.5) hold trivially unless either $x \notin D_m$ or $z \notin D_m$. If $x \notin D_m$, then $x \in B(y, r_{m+1}) \cap D_{m+1}$

for some $y \in \{y_j\}_1^N$, $y \in \partial D_m$, and $x = (x', x_n)$ in the corresponding rotated coordinate system. Put $x^* = (x', x_n + r_{m+1})$ and observe that $x^* \in D_m$. If $x \in D_m$, we also let $x^* = x$. Applying the same argument to z we get $x^*, z^* \in D_m$. Let γ^* be the curve joining x^* to z^* which satisfies (4.4), (4.5). If $x \neq x^*$, we modify γ^* as follows. Let t_0, $0 < t_0 < 1$, be the largest t with $\gamma^*(t) \in \bar{B}(y, r_{m+1}^{3/4})$. If $\gamma^*(t_0) = w = (w', w_n)$, we join x, w, to $\bar{x} = (x', y_n + r_{m+1}^{3/4})$, $\bar{w} = (w', y_n + r_{m+1}^{3/4})$, respectively by line segments, l_1, l_2. We then join $\bar{x}$ to $\bar{w}$ by a line segment l_3. Let $l_1 + l_2 + l_3$ denote the resulting curve from x to w with parameter interval $[0, t_0]$. If $z \notin D_m$, we see there exists $\hat{y} \in \{y^i\}_1^N$ and largest t_1, $0 < t_0 < t_1 < 1$, such that $z \in B(\hat{y}, r_{m+1})$, and

$$\{\gamma^*(t): 0 \leqslant t < t_1\} \cap \bar{B}(\hat{y}, r_{m+1}^{3/4}) = \varnothing.$$

As above, we get line segments $\tilde{l}_1, \tilde{l}_2, \tilde{l}_3$, with $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3$ joining $\gamma^*(t_1)$ to z. Moreover, $\tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3$ has parameter interval $[t_1, 1]$. Let $\hat{\gamma} = \gamma^*$ on $[t_0, t_1]$ and if $x \notin D_m$, then $\hat{\gamma} = l_1 + l_2 + l_3$ on $[0, t_0]$. Otherwise, $\hat{\gamma} = \gamma^*$ on $[0, t_0]$. If $z \notin D_m$, then $\hat{\gamma} = \tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3$ on $[t_1, 1]$, while if $z \in D_m$, then $\hat{\gamma} = \gamma^*$ on $[t_1, 1]$. From (4.5) we deduce

$$\begin{aligned} \text{(4.6)} \qquad \operatorname{length} \hat{\gamma} &\leqslant \operatorname{length} \gamma^* + 10 r_{m+1}^{3/4} \\ &\leqslant 3\left(1 + \sum_{k=1}^{m} r_k^{1/4}\right)|x^* - z^*| + 10 r_{m+1}^{3/4} \\ &\leqslant 3\left(1 + \sum_{k=1}^{m} r_k^{1/4}\right)|x - z| + 12 r_{m+1}^{3/4} \\ &\leqslant 3\left(1 + \sum_{k=1}^{m+1} r_k^{1/4}\right)|x - z|. \end{aligned}$$

Moreover, from local smoothness of ∂D_{m+1} it is easily checked for $t \in [0, t_0] \cup [t_1, 1]$, that

$$\operatorname{dist}(\hat{\gamma}(t), \partial D_{m+1}) \geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m+1} r_k^{1/4}\right) \min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\}.$$

If $t \in [t_0, t_1]$, then by construction

$$\begin{aligned} \min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\} &\geqslant r_{m+1}^{3/4} - r_{m+1} \\ &\geqslant \frac{1}{2} r_{m+1}^{3/4}. \end{aligned}$$

Using this inequality, (4.4), and the fact that $\gamma^* = \hat{\gamma}$ on $[t_0, t_1]$ we get for $t \in [t_0, t_1]$,

$$\text{(4.7)}\quad \operatorname{dist}(\hat{\gamma}(t), \partial D_{m+1}) \geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m} r_k^{1/4}\right)\min\{|\hat{\gamma}(t) - x^*|, |\hat{\gamma}(t) - z^*|\}$$
$$\geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m} r_k^{1/4}\right)\min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\} - \frac{r_{m+1}}{16}$$
$$\geqslant \frac{1}{16}\left(1 - 2\sum_{k=1}^{m+1} r_k^{1/4}\right)\min\{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\}.$$

If $|x - z| < 4r_{m+1}^{1/2}$, then from local smoothness of ∂D_{m+1}, we see there exists $\hat{\gamma}$ for which (4.6) and (4.7) hold. Thus by induction, we obtain (4.4), (4.5), for $m = 0, 1, 2, \ldots$ Since $\sum_1^\infty r_k^{1/4} \leqslant 1/10$, we conclude that D_m, $m = 0, 1, \ldots,$ is NTA with constant 100. From this fact, (4.2), and our work in Section 3 we now find that (1.3) holds with $\Omega = D_k$, $\Omega' = D_{k+1}$, $k = 0, 1, \ldots$

Next let $h_0(x) = \rho x$, and $f_k = h_k \circ h_{k-1} \circ \cdots \circ h_0$. Then f_k is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f_k(S) = \partial D_k$. From (2.16), (2.19), and iteration, we find

$$\text{(4.8)}\quad \begin{aligned} 2^{-k}\rho|x - z| &\leqslant \rho(1 - c_4\sigma_0^2)^k|x - z| \\ &\leqslant |f_k(x) - f_k(z)| \\ &\leqslant \rho(1 + c_4\sigma_0^2)^k|x - z| \\ &\leqslant \rho 2^k|x - z|, \end{aligned}$$

for $x, z \in \mathbb{R}^n$. If $r_j < |x - z|$ for some $j \geqslant 1$, then from (4.8) and the fact that $r_{k+1} \leqslant 10^{-4k} r_k \rho$, we deduce for $l \geqslant j$,

$$r_{l+1} < 2^{-l}\rho|x - z| \leqslant |f_l(x) - f_l(z)|.$$

From this inequality, (2.17), (2.19) and iteration we find for $k > j$,

$$|f_j(x) - f_j(z)| - \frac{1}{2}\sum_{m=j+1}^{k} r_m \leqslant |f_k(x) - f_k(z)| \leqslant |f_j(x) - f_j(z)| + \frac{1}{2}\sum_{m=j+1}^{k} r_m.$$

Using the above inequality, (4.8) with $j = k$, and the fact that

$$\sum_{m=j+1}^{\infty} r_m \leqslant \rho 10^{-j} r_j \leqslant \rho 10^{-j}|x - z|,$$

we get

$$\text{(4.9)}\quad 2^{-(j+1)}\rho|x - z| \leqslant |f_k(x) - f_k(z)| \leqslant \rho 2^{j+1}|x - z|.$$

Given $\beta \in (0, 1)$, we have

$$2^{j+1} \leqslant c(\beta)|x - z|^{\beta - 1},$$

when $r_j \leqslant |x - z| \leqslant r_{j-1}$, $j = 2, 3, \ldots$ for some $c(\beta)$, independent of j. Here we have used, $r_m \leqslant c10^{-m^2}$, $m = 1, 2, \ldots$, which follows easily from our choice of $(r_m)_1^\infty$. Using the above inequality in (4.9), we obtain

$$c(\beta)^{-1}|x - z|^{1/\beta} \leqslant |f_k(x) - f_k(z)| \leqslant c(\beta)|x - z|^\beta,$$

for $|x - z| \leqslant 1/4$. Hence (1.5) is true. As in Section 1 we put $D = \bigcup_0^\infty D_k$ and choose a subsequence (f_{n_k}) of (f_k) such that (f_{n_k}) converges uniformly to f on compact subsets of $\mathbb{R}^n$. We claim that D is not a sphere. Indeed, since $\max_{\mathbb{R}^{n-1}} \psi = 1$, and (2.1), (3.4) hold for r_1, ϵ_0, D_1, we see that if $\rho_1 = \rho + (2\lambda_0)^{-1}\sigma_0^2 r_1$, then $D_1 \cap (\mathbb{R}^n - B(0, \rho_1)) \neq \varnothing$. Also, by construction, there exists $x_0 \in \partial D_1$ with $|x_0| = \rho$. Using the definition of $(r_m)_1^\infty$ and the triangle inequality we see that $f(x_0) \in \partial D$ and $|f(x_0)| < \rho_1$. Therefore, D is not a sphere.

It remains only to prove (1.9) in order to obtain Theorem 1 from the remarks in Section 1. To this end let

$$p_j(x) = f \circ f_j^{-1}(x) = \lim_{k \to \infty} h_{n_k} \circ \cdots \circ h_{j+1}(x),$$

when $x \in \partial D_j$ and $j = 1, 2, \ldots$ Iterating (2.18) we deduce that if

$$e_j = \prod_{m=j+1}^{\infty} (1 - c_5 r_m^{1/2}),$$

then

$$e_j|x - y| \leqslant |p_j(x) - p_j(y)|, \qquad x, y \in \partial D_j.$$

If q_j denotes the inverse of p_j, it follows that

$$|q_j(x) - q_j(y)| \leqslant e_j^{-1}|x - y|, \tag{4.10}$$

when $x, y \in \partial D$. Next we use Kirsbraun's Theorem ([5, 2.10.43]) to extend q_j to $\mathbb{R}^n$ (also denoted q_j) in such a way that (4.10) holds whenever $x, y \in \mathbb{R}^n$.

From (4.10) it is easily seen by comparing coverings of each set that

$$H^{n-1}(q_j(F)) \leqslant e_j^{1-n} H^{n-1}(F), \qquad F \subseteq \mathbb{R}^n. \tag{4.11}$$

$j = 1, 2, \ldots$ Let $g \geqslant 0$ be a continuous function on $\mathbb{R}^n$, and put $\nu(E) = H^{n-1}(q_j^{-1}(E) \cap \partial D)$. Then from (4.11) with $F = q_j^{-1}(E) \cap \partial D$, we have

$$H^{n-1}(E \cap \partial D_j) \leqslant e_j^{1-n}\nu(E).$$

Also from the usual change of variables formula [5, Thm. 2.4.18] and the above inequality we get

$$e_j^{n-1} \int_{\partial D_j} g\, dH^{n-1} \leqslant \int_{\mathbb{R}^n} g\, d\nu = \int_{\partial D} g \circ q_j\, dH^{n-1}. \tag{4.12}$$

Letting $j \to \infty$, $j \in (n_k)_1^\infty$, we obtain from the definition of $(r_k)_1^\infty$ that $e_j \to 1$, while

$$\int_{\partial D} g \circ q_j\, dH^{n-1} \to \int_{\partial D} g\, dH^{n-1},$$

since $q_{n_k}(x) \to x$, uniformly on compact subsets of $\mathbb{R}^n$. Hence from (4.12) we have

$$\limsup_{k\to\infty} \int_{\partial D_{n_k}} g\, dH^{n-1} \leqslant \int_{\partial D} g\, dH^{n-1}. \tag{4.13}$$

On the other hand from our choice of $(r_k)_1^\infty$ we see that L_m, $m = 1, 2, \dots,$ is a covering for D. Thus if ϕ_δ^{n-1} is as in Section 1, then

$$\phi_{2^{-m}}^{n-1}(\partial D) \leqslant H^{n-1}(\partial D_m) - 2^{-m}.$$

Letting $m \to \infty$, we find

$$H^{n-1}(\partial D) \leqslant \liminf_{m\to\infty} H^{n-1}(\partial D_m). \tag{4.14}$$

From (4.13), (4.14), it follows that if $0 \leqslant g \leqslant 1$ on $\bar{D}$, then

$$\begin{aligned} H^{n-1}(\partial D) &\leqslant \liminf_{k\to\infty} H^{n-1}(\partial D_{n_k}) \\ &\leqslant \liminf_{k\to\infty} \int_{\partial D_{n_k}} g\, dH^{n-1} + \limsup_{k\to\infty} \int_{\partial D_{n_k}} (1-g)\, dH^{n-1} \\ &\leqslant \limsup_{k\to\infty} \int_{\partial D_{n_k}} g\, dH^{n-1} + \int_{\partial D} (1-g)\, dH^{n-1} \\ &\leqslant \int_{\partial D} g\, dH^{n-1} + \int_{\partial D} (1-g)\, dH^{n-1} \\ &= H^{n-1}(\partial D). \end{aligned}$$

Thus equality holds everywhere and so

$$\lim_{k\to\infty} \int_{\partial D_{n_k}} g\, dH^{n-1} = \int_{\partial D} g\, dH^{n-1} \tag{4.15}$$

when $0 \leqslant g \leqslant 1$. In general we can write, $g = ag_1 + b$, where $0 \leqslant g_1 \leqslant 1$ on D, for properly chosen $a, b \in \mathbb{R}$. Applying (4.15) to g_1, 1 we find that (4.15) holds when g is continuous on $\mathbb{R}^n$. Hence, (1.9) is true.

The proof of Theorem 1 is now complete.

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Hardy Spaces and Oscillatory Singular Integrals

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1. Introduction

Consider the oscillatory singular integral operator T:

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i(Bx,y)} K(x-y) f(y)\, dy, \tag{1}$$

where (Bx,y) is a real bilinear form, and K is a Calderón-Zygmund kernel, *i.e.* K is C^1 away from the origin, has mean-value zero on each sphere centered at the origin and satisfies

$$|K(x)| \leqslant C|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \leqslant C|x|^{-n-1}.$$

It is proved by D. H. Phong and E. M. Stein in [PS], that T is a bounded operator on L^p spaces, with bound independent of B. They also introduced some variants of the H^1 and BMO spaces (denoted by H^1_E and BMO_E, to avoid the confusion with the standard H^1 and BMO). Analogous to the fact that the classical singular integral operators are bounded from H^1 to L^1, Phong and Stein showed that T extends as a bounded operator from H^1_E to L^1. This fact was then used to prove the L^p boundedness by interpolating between L^2 and L^∞, (see [PS]).

The object of our study is a more general class of oscillatory singular integral operators. An operator in this class is obtained when the bilinear form in (1) is replaced by some real-valued polynomial in x and y. These operators have arisen in the study of Hilbert transform along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms, etc. F. Ricci and E. M. Stein have proved in [RS] that an operator of this kind is bounded on L^p spaces, with bound depending only on the total degree, not on the coefficients of the polynomial. The fact that these operators are of weak-type (1,1) was subsequently proved by S. Chanillo and M. Christ ([CC]).

It is our goal in this paper to establish a Hardy space theory for the class of oscillatory singular integral operators with polynomial phase functions. Given such an operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy, \tag{2}$$

where $P(x,y)$ is a real-valued polynomial, we will define the space H^1_E as some variant of the standard H^1 space, and this space H^1_E is closely associated with the given polynomial $P(x,y)$. First let us give the definition of the "atoms":

Definition. *Let Q be a cube with center x_Q, an atom is a function $a(x)$ which is supported in Q, so that*

$$|a(x)| \leqslant \frac{1}{|Q|},$$

and

$$\int_Q e^{iP(x_Q,y)} a(y)\, dy = 0.$$

The space H^1_E consists of the subspace of L^1 of functions f which can be written as $f = \sum \lambda_j a_j$, where a_j are atoms, and $\lambda_j \in \mathbb{C}$, with $\sum |\lambda_j| < \infty$. Consequently, we define BMO_E as the dual space of H^1_E. Our main result is

Theorem 1. *Suppose H^1_E and T are defined as above. Then T is a bounded operator from H^1_E to L^1. The bound of this operator can be taken to depend only on the total degree of P, (not on the coefficients of P).*

We notice that in the paper of Phong and Stein, the fact that the phase function is a real bilinear form makes it possible to apply the Plancherel's theorem to the Fourier transform (or partial Fourier transform) associated with B. When (Bx, y) is replaced by the polynomial $P(x,y)$, we no longer have this advantage. So we have to take a different approach, using some L^2 estimates of certain oscillatory integrals. This will become clear in our proof.

For $p < 1$, the Calderón-Zygmund singular integral operators are still bounded from H^p to L^p. However, this is no longer the case for the oscillatory singular integral operators. At the end of this article, we will present a simple example which shows that this fails even in the bilinear phase function case.

The main result presented in this paper was included in the author's thesis. The autor would like to thank his advisor, Professor E. M. Stein, for his encouragement and many helpful suggestions. The author would like to thank the referee for his comments.

2. Proof of Theorem 1

PROOF. Let us assume that a is a function supported in the cube Q_0, which is centered at the origin, and has sidelength 1, and a satisfies

$$|a| \leqslant 1, \qquad \int_{Q_0} a(y)\,dy = 0.$$

First we shall prove that if $P(x,y)$ is a polynomial in x, y, and $P(0,y) \equiv 0$, then

$$\left\| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) a(y)\,dy \right\|_{L^1} \leqslant C, \tag{3}$$

where C depends only on the total degree of P, and is otherwise independent of the coefficients of P.

To prove (3), we shall use induction on the degree l of y in $P(x,y)$.

If $l = 0$, then $e^{iP(x,y)}$ is only a function of x, therefore can be taken out of the integral sign, and (3) follows from the classical result of the standard H^1 theory. (See, for example [CW].)

Next we assume $l > 0$, and (3) is true for $l-1$. By the Ricci-Stein theorem on the L^p boundedness of T, we have

$$\begin{aligned}\int_{|x|\leq 2} |T(a)(x)|\,dx &\leqslant C\Big(\int_{|x|\leq 2} |T(a)(x)|^2\,dx\Big)^{1/2} \\ &\leqslant C\Big(\int_{\mathbb{R}^n} |a|^2\,dx\Big)^{1/2} \leqslant C.\end{aligned}$$

Write

$$P(x,y) = \sum_{|\alpha|\geq 1,\, |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta + Q(x,y),$$

where $Q(x,y)$ is a polynomial with degree in y less than or equal to $l-1$, and still satisfies $Q(0,y) \equiv 0$. For any $r > 0$, we have

$$\begin{aligned}\int_{2<|x|\leq r} |T(a)(x)|\,dx \leqslant& \int_{2<|x|\leq r} |(e^{iP(x,y)} - e^{iQ(x,y)}) K(x-y) a(y)\,dy|\,dx \\ &+ \int_{2<|x|\leq r} \left| \int_{\mathbb{R}^n} e^{iQ(x,y)} K(x-y) a(y)\,dy \right| dx.\end{aligned}$$

(If $r \leqslant 2$, all the above integrals are 0.)

By our inductive hypothesis, the second term is bounded. Also $|x-y| \geqslant |x|/2$, if $|x| > 2$, $|y| \leqslant 1$. So we have

$$\int_{2<|x|\leq r} |T(a)(x)|\,dx \leqslant C + C\int_{|x|\leq r} dx \int_{\mathbb{R}^n} \left| \exp\left(\sum_{\substack{|\alpha|\geq 1\\ |\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta \right) - 1 \right| \times \frac{|a(y)|}{|x|^n}\,dy$$

$$\leqslant C + C \sum_{\substack{|\alpha|\geq 1\\ |\beta|=l}} |a_{\alpha\beta}| \int_{|x|\leq r} |x|^{|\alpha|-n}\,dx$$

$$\leqslant C + C \sum_{\substack{|\alpha|\geq 1\\ |\beta|=l}} |a_{\alpha\beta}| r^{|\alpha|}.$$

Now, there exists (α_0, β_0) such that $|\alpha_0| \geqslant 1$, $|\beta_0| = l$, and

$$|a_{\alpha_0\beta_0}|^{1/|\alpha_0|} = \max_{\substack{|\alpha|\geq 1\\ |\beta|=l}} |a_{\alpha\beta}|^{1/|\alpha|}.$$

Put $r = |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$, we have

$$\int_{2<|x|\leq r} |T(a)(x)|\,dx \leqslant C,$$

where C depends only on the total degree of $P(x, y)$. Now we turn to the estimate of the remaining part

$$\int_{|x|>2,\,|x|>r} |T(a)(x)|\,dx.$$

We shall need the following lemmas:

Lemma 1. *Suppose*

$$\phi(x) = \sum_{|\nu|\leq k} a_\nu x^\nu$$

is a real-valued polynomial in $\mathbb{R}^n$ *of degree* k, *and* $\psi \in C_0^\infty$. *Then for any* ν, $|\nu| = k$, $a_\nu \neq 0$, *we have*

$$\left| \int_{\mathbb{R}^n} e^{i\phi(x)} \psi(x)\,dx \right| \leqslant C|a_\nu|^{-1/k} (\|\psi\|_{L^\infty} + \|\nabla\psi\|_{L^1}) \tag{4}$$

To see this, simply let ξ be an unit vector, such that

$$|(\xi \cdot \nabla_x)^k \phi(x)| \geqslant c|a_\nu|.$$

This is possible because

$$\frac{\partial^\nu \phi(x)}{\partial x^\nu} = \nu! a_\nu.$$

(See [ST], page 317.) Without loss of generality, we may assume

$$\xi = (1, 0, \ldots, 0).$$

Hence

$$\left| \frac{\partial^k \phi(y)}{\partial y_1^k} \right| \geqslant c|a_\nu|.$$

Now apply the one-dimensional Van der Corput's lemma to obtain (4). See also [ST].

Lemma 2. *Let*

$$P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$$

denote a polynomial in $\mathbb{R}^n$ *of degree* d. *Suppose* $\epsilon < 1/d$, *then*

$$\int_{|x| \leq 1} |P(x)|^{-\epsilon}\, dx \leqslant A_\epsilon \Big(\sum_{|\alpha| \leq d} |a_\alpha| \Big)^{-\epsilon}.$$

The bound A_ϵ *depends on* ϵ *(and the dimension* n*), but not on the coefficients* $\{a_\alpha\}$.

This is a result of Ricci and Stein. See [RS], page 182.

Now we continue our proof of Theorem 1. Let

$$R_j = \{x \in \mathbb{R}^n : 2^j \leqslant |x| < 2^{j+1}\},$$

for $j \geqslant 0$, and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy

$$\varphi(x) \equiv 1 \quad \text{for} \quad |x| \leqslant 1, \qquad \varphi(x) \equiv 0 \quad \text{for} \quad |x| \geqslant 2.$$

Define T_j by

$$(T_j f)(x) = \chi_{R_j}(x) \int_{\mathbb{R}^n} e^{iP(x,y)} \varphi(y) f(y)\, dy,$$

and consider the operator $T_j T_j^*$:

$$T_j T_j^*(f)(x) = \int_{\mathbb{R}^n} L_j(x, z) f(z)\, dz,$$

where

$$L_j(x,z) = \chi_{R_j}(x)\chi_{R_j}(z)\int_{\mathbb{R}^n} e^{i(P(x,y)-P(z,y))}|\varphi(y)|^2\,dy.$$

Write

$$P(x,y) - P(z,y) = \sum_{\substack{|\alpha|\geq 1\\ |\beta|=l}} a_{\alpha\beta}y^\beta(x^\alpha - z^\alpha) + (Q(x,y) - Q(z,y)),$$

where the degree of y in $Q(x,y) - Q(z,y)$ is less than or equal to $l-1$.

Applying Lemma 1, with $\nu = \beta_0$, we obtain

$$|L_j(x,z)| \leqslant C\left|\sum_{|\alpha|\geq 1} a_{\alpha\beta_0}(x^\alpha - z^\alpha)\right|^{-1/l}\chi_{R_j}(x)\chi_{R_j}(z).$$

On the other hand, it is obvious that $|L_j(x,z)| \leqslant C$, so let $N > 0$ be a large number (to be chosen later), we have

$$|L_j(x,z)| \leqslant C\left|\sum_{|\alpha|\geq 1} a_{\alpha\beta_0}(x^\alpha - z^\alpha)\right|^{-1/Nl}\chi_{R_j}(z).$$

By rescaling we would obtain the same norm if we were to replace $L_j(x,z)$ by $L_j'(x,z) = 2^{nj}L_j(2^jx, 2^jz)$, so we have

$$|L_j'(x,z)| \leqslant C2^{nj}\left|\sum_{|\alpha|\geq 1}(a_{\alpha\beta_0}2^{j|\alpha|})x^\alpha - \sum_{|\alpha|\geq 1} a_{\alpha\beta_0}2^{j|\alpha|}z^\alpha\right|^{-1/Nl}\chi_{R_0}(x)\chi_{R_0}(z).$$

Choosing N sufficiently large and applying Lemma 2, we get

$$\begin{aligned}\sup_z\int_{\mathbb{R}^n}|L_j'(x,z)|\,dx &\leqslant C2^{nj}\sup_z\left(\sum_{|\alpha|\geq 1}|a_{\alpha\beta_0}|2^{j|\alpha|} + \left|\sum_{|\alpha|\geq 1}a_{\alpha\beta_0}2^{j|\alpha|}z^\alpha\right|\right)^{-1/Nl}\\ &\leqslant C2^{nj}|a_{\alpha_0\beta_0}|^{-1/Nl}2^{-j|\alpha_0|/Nl}.\end{aligned}$$

Similar estimate holds for $\sup_x\int_{\mathbb{R}^n}|L'(x,z)|\,dz$, therefore we obtain

$$\|T_jT_j^*\| \leqslant C2^{nj}|a_{\alpha_0\beta_0}|^{-1/Nl}2^{-j|\alpha_0|/Nl},$$

so

$$\|T_j\|_{L^2\to L^2} \leqslant C2^{nj/2}|a_{\alpha_0\beta_0}|^{-1/2Nl}2^{-j|\alpha_0|/2Nl}$$

Now we have

$$\begin{aligned}\int_{|x|>2,\,|x|>r}|T(a)(x)|\,dx \leqslant &\int_{|x|>2,\,|x|>r}dx\int_{\mathbb{R}^n}|K(x-y)-K(x)|\,|a(y)|\,dy\\ &+\int_{|x|>2,\,|x|>r}|K(x)|\,dx\left|\int_{\mathbb{R}^n}e^{iP(x,y)}a(y)\,dy\right| = I_1 + I_2.\end{aligned}$$

The estimate for I_1 is easy

$$I_1 \leqslant \int_{|x|>2,\,|x|>r} dx \int_{\mathbb{R}^n} \frac{|y|\,|a(y)|}{|x|^{n+1}}\,dy$$

$$\leqslant C \int_{|x|>2} \frac{dx}{|x|^{n+1}} < C.$$

As for I_2, using our estimate on T_j and assuming $2^{j_0} \leqslant r < 2^{j_0+1}$, for some j_0, we have

$$I_2 \leqslant C \int_{|x|>2,\,|x|>r} \frac{1}{|x|^n} \left| \int_{\mathbb{R}^n} e^{iP(x,y)} a(y)\,dy \right| dx$$

$$\leqslant C \sum_{j \geq j_0} \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^n} |T_j(a)(x)|\,dx$$

$$\leqslant C \sum_{j \geq j_0} \left(\int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^{2n}}\,dx \right)^{1/2} \|T_j(a)\|_{L^2}$$

$$\leqslant C \sum_{j \geq j_0} 2^{-nj/2} 2^{nj/2} |a_{\alpha_0\beta_0}|^{-1/2Nl} 2^{-j|\alpha_0|/2Nl} \leqslant C,$$

because $2^{j_0} \geqslant (1/2)|a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$, and (3) is proved.

To prove the theorem, we only need to prove that $\|T(a)\|_{L^1} \leqslant C$, for all atoms a, and C is a constant which depends only on the total degree of $P(x, y)$.

Let a be an atom associated to the cube Q, and the center and sidelength of Q are x_Q and δ respectively. We observe that

$$\delta^n(T(a))(\delta x + x_Q) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(\delta x + x_Q,\, \delta y + x_Q)} K(x-y) \delta^n a(\delta y + x_Q)\,dy.$$

Write

$$P(\delta x + x_Q, \delta y + x_Q) = R(x, y) + P(x_Q, \delta y + x_Q),$$

where $R(x, y)$ is a polynomial which satisfies $R(0, y) \equiv 0$, and the total degree of R is not greater than that of P. Let

$$b(y) = e^{iP(x_Q,\, \delta y + x_Q)} \delta^n a(\delta y + a_Q),$$

by the definition of the atom, we have

$$\operatorname{supp}(b) \subset Q_0 \quad \text{and} \quad |b(y)| \leqslant 1,$$

also

$$\int_{Q_0} b(y)\,dy = \int_Q e^{iP(x_Q, y)} a(y)\,dy = 0.$$

Now invoking (3), we have

$$\|T(a)\|_{L^1} = \left\| \text{p.v.} \int_{\mathbb{R}^n} e^{iR(x,y)} K(x-y) b(y)\, dy \right\|_{L^1} \leqslant C.$$

This completes the proof of Theorem 5.

3. An Extension

In [RS], Ricci and Stein pointed out that the L^p boundedness still holds, if the Calderón-Zygmund kernel in the operator is replaced by some more general distribution. For H^1_E, the same thing is true, *i.e.*

Theorem 2. *If $K(x, y)$ is a distribution and C^1 away from the diagonal $\{x = y\}$, and satisfies:*

(i) $|K(x, y)| \leqslant C|x-y|^{-n}$ *and* $|\nabla K(x, y)| \leqslant C|x-y|^{-n-1}$.
(ii) *The operator*

$$f \to \int K(x, y) f(y)\, dy$$

extends as a bounded operator on $L^2(\mathbb{R}^n)$.
Then the operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x, y) f(y)\, dy \tag{5}$$

is bounded from H^1_E to L^1.

The proof of Theorem 2 is essentially the same as Theorem 1.

4. The Dual Space BMO_E

We define the sharp function $f^{\#}_E$ to be

$$(f^{\#}_E)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x) - f^E_Q(x)|\, dx,$$

where

$$f^E_Q(x) = e^{-iP(x_Q, x)} \left(\frac{1}{|Q|} \int_Q e^{-iP(x_Q, y)} f(y)\, dy \right)$$

and as the dual space of H_E^1, BMO_E is given by

$$\mathrm{BMO}_E = \{f \in L^1_{\mathrm{loc}} : f_E^\# \in L^\infty\}$$

and

$$\|f\|_{\mathrm{BMO}_E} = \|f_E^\#\|_{L^\infty}.$$

The dual statement of Theorem 2 is

Theorem 3. *The operator T^* (T given by (5)) extends as a bounded operator from L^∞ to* BMO_E.

5. A Counterexample

In this section, we shall give a simple example to show that the H^1 theory on the oscillatory singular integral operators cannot be extended to the H^p case, if $p < 1$.

Let T be defined as

$$(Ta)(x) = \text{p.v.} \int_{\mathbb{R}^1} e^{ixy} \frac{1}{x-y} a(y)\, dy.$$

Take $\delta > 0$, δ is very small, and a is a function supported on $I_\delta = [-\delta, \delta]$, given by

$$a(y) = \begin{cases} (2\delta)^{-1/p} & \text{if } \ y \in [\delta/2, \delta], \\ -(2\delta)^{-1/p} & \text{if } \ y \in [-\delta, -\delta/2], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that a satisfies

$$|a| \leqslant |I_\delta|^{-1/p}, \qquad \int_{I_\delta} a(y)\, dy = 0.$$

Therefore, we have

$$\operatorname{Im}(Ta)(x) = (2\delta)^{-1/p}\left(\int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x-y}\, dy + \int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x+y}\, dy \right).$$

Let $x \in (\pi/4\delta,\ \pi/3\delta)$, then $x - y > 0$, $x + y > 0$ for $y \in [\delta/2, \delta]$. Also $\pi/8 < xy < \pi/3$.

Hence

$$\operatorname{Im}(Ta)(x) > c_0(2\delta)^{-1/p}\left(\int_{\delta/2}^{\delta}\frac{1}{x-y}\,dy + \int_{\delta/2}^{\delta}\frac{1}{x+y}\,dy\right)$$

$$= c_0(2\delta)^{-1/p}\log\left(1+\frac{\delta x}{(x^2-\delta x/2-\delta^2/2)}\right)$$

$$> c_0'\delta^{1-1/p}x^{-1},$$

for some constant $c_0' > 0$. Then, we have

$$\int_{\mathbb{R}^1}|Ta(x)|^p\,dx \geqslant c_0'^p\int_{\pi/4\delta}^{\pi/3\delta}(\delta^{1-1/p})^p x^{-p}\,dx = c\delta^{2(p-1)}.$$

This is unbounded as $\delta\to 0$ and $p<1$.

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Local Properties of Stationary Solutions of some Nonlinear Singular Schrödinger Equations

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Abstract

We study the local behaviour of solutions of the following type of equation $-\Delta u - V(x)u + g(u) = 0$ when V is singular at some points and g is a nondecreasing function. Emphasis is put on the case when $V(x) = c|x|^{-2}$ and g has a power-like growth.

Introduction

In this article we study the local behaviour of a solution u of the following time-independent, N-dimensional, nonlinear Schrödinger equation

$$-\Delta u - V(x)u + g(u) = 0 \tag{0.1}$$

near an isolated singularity of the potential V, g being some asymptotically nondecreasing real valued function. In many physical examples V is a Coulombian potential:

$$V(x) = \sum_{i=1}^{k} z_i |x - a_i|^{-1} \tag{0.2}$$

in the case of a nucleus in the Thomas-Fermi-Dirac-von Weizsäcker theory [3], [4]. However it is mathematically more exciting when V can be compared with $|x - a|^{-2}$ near the isolated singularity a. In that case the interference between the Laplacian, the potential and the nonlinearity is very strong. The model equation is the following

$$-\Delta u - \frac{c}{|x|^2} u + u|u|^{q-1} = 0 \tag{0.3}$$

where $q > 1$ and c is some real number. If we look for a specific solution of (0.3) under the form

$$u_s(r) = \alpha r^{\beta}, \tag{0.4}$$

then

$$\beta = -\frac{2}{q-1}$$

and

$$\alpha^{q-1} = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right).$$

Henceforth the solution u_s exists if and only if

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) > 0. \tag{0.5}$$

It is worth noticing that if (0.5) does not hold then

$$c \leqslant \left(\frac{N-2}{2}\right)^2, \tag{0.6}$$

and this condition plays a fundamental role in the description of the fundamental solutions of the equation

$$\Delta \phi + \frac{c}{|x|^2} \phi = 0. \tag{0.7}$$

If (0.6) is satisfied let β be $\sqrt{(N-2)^2 - 4c}$ and μ_i the two fundamental solutions of (0.7), that is

$$\mu_1(x) = \begin{cases} |x|^{-(N-2+\beta)/2} & \text{if } c < \left(\dfrac{N-2}{2}\right)^2, \\ |x|^{-(N-2)/2} \operatorname{Ln}(1/|x|) & \text{if } c = \left(\dfrac{N-2}{2}\right)^2, \end{cases} \tag{0.8}$$

$$\mu_2(x) = \begin{cases} |x|^{-(N-2-\beta)/2} & \text{if } c < \left(\dfrac{N-2}{2}\right)^2, \\ |x|^{-(N-2)/2} & \text{if } c = \left(\dfrac{N-2}{2}\right)^2. \end{cases} \tag{0.9}$$

It is important to notice that μ_2 is the regular solution of (0.7) in the sense that $c|\bullet|^{-2}\mu_2(\bullet)$ is locally integrable in $\mathbb{R}^N$ and

$$\Delta\mu_2 + \frac{c}{|x|^2}\mu_2 = 0 \tag{0.10}$$

holds in $D'(\mathbb{R}^N)$ (if $c \leqslant 0$, u is continuous), as the same holds for μ_1 if and only if $c > 0$; in any case $\mu_2 = o(\mu_1)$ near 0. Our first removability result deals with the meaning of the equation in the sense of distributions.

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, g a continuous real valued function satisfying*

$$\begin{cases} \liminf\limits_{r \to \infty} g(r)/r^q > 0 \\ \limsup\limits_{r \to -\infty} g(r)/(-r^q) < 0 \end{cases} \tag{0.11}$$

and $V \in C^0(\Omega^*)$ is such that

$$-\infty < |x|^2 V(x) \leqslant c \tag{0.12}$$

near 0 for some constants $q > 1$ and c. If we assume either $q > N/(N-2)$, or $1 < q \leqslant N/(N-2)$ and

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) \leqslant 0, \tag{0.13}$$

any $u \in C^1(\Omega^*)$ satisfying

$$-\Delta u - Vu + g(u) = 0 \tag{0.14}$$

in $D'(\Omega^*)$ can be extended as a solution of the same equation in $D'(\Omega)$.

We must remark that if (0.5) is satisfied with $q > N/(N-2)$ there exist singular solutions of the model problem (0.3) with a rather weak singularity; this must be compared with

$$\Delta u + u^q = 0 \tag{0.15}$$

for which the same holds when $q > N/(N-2)$. Our second removability result is to compare a solution of (0.14) in Ω^* with the regular solution of (0.10).

Theorem 1.2. *Let Ω and V be as in Theorem 1.1 and let g be a continuous real valued function satisfying (0.11) for some $q > 1$. Assume also that*

$$0 = g(0) = g^{-1}(0) \tag{0.16}$$

and

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) \leqslant 0 \tag{0.17}$$

hold. Then if u is any $C^1(\Omega^*)$ function satisfying (0.14) in $D'(\Omega^*)$, u/μ_2 remains locally bounded in Ω.

It is important to notice that, as (0.17) holds, (0.6) also holds which allows us to have a comparison principle.

Our second section is devoted to the extension of Vázquez-Veron's isotropy theorems [23], [24] to the potential case. Let us introduce some notations: let S^{N-1} be the unit sphere in $\mathbb{R}^N$, $(r, \sigma) \in \mathbb{R}_*^+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$ and $\bar{\rho}(r)$ the spherical average of a function $\rho(r, \sigma)$, that is

$$\bar{\rho}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \rho(r, \sigma)\, d\sigma. \tag{0.18}$$

Theorem 2.1. *Assume Ω is an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, g is a continuous nondecreasing real valued function and $u \in C^1(\Omega^*)$ is a solution of (0.14) in $D'(\Omega^*)$ where $V \in C^0(\Omega^*)$ is a radial function such that*

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2, \quad \text{for every} \quad x \in \Omega^*. \tag{0.19}$$

If u satisfies

$$\liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0, \tag{0.20}$$

then $u(x)/\mu_1(x)$ admits a limit in $\mathbb{R} \cup \{-\infty, \infty\}$ as x tends to 0.

A similar isotropy result holds for a solution u of (0.14) in an exterior domain of $\mathbb{R}^N$. An interesting class of solutions of (0.14) in Ω^* are those which present a singular linear behaviour near 0. As the equation in the sense of distributions in Ω is not very singnificative except when $c = 0$ where the Dirac mass plays a fundamental role [6], [10], the good criterion for the behaviour of linear singularities will be the existence of a finite, not always 0, limit of $u(x)/\mu_1(x)$ as x tends to 0, as in [29].

Theorem 3.1. *Assume g is a continuous nondecreasing real valued function and*

$$c < \left(\frac{N-2}{2}\right)^2. \tag{0.21}$$

Then the equation

$$-\Delta u - \frac{c}{|x|^2} u + g(u) = 0 \tag{0.22}$$

admits solutions u in

$$B_1(0) \backslash \{0\} = \{x \in \mathbb{R}^N : 0 < |x| < 1\}$$

such that

$$\lim_{x \to 0} u(x)/\mu_1(x) = \gamma, \tag{0.23}$$

where γ is any arbitrary real number if and only if

$$\int_1^\infty (g(t) + |g(-t)|) t^{2(1-\alpha_1)/\alpha_1} \, dt < \infty, \tag{0.24}$$

where

$$\alpha_1 = -\left(N - 2 + \sqrt{(N-2)^2 - 4c}\right)/2.$$

When $c = 0$, condition (0.24) is the one introduced by Brézis and Bénilan [6] for solving equations of type

$$-\Delta u + g(u) = m \tag{0.25}$$

where m is a bounded measure. When

$$c = \left(\frac{N-2}{2}\right)^2$$

the situation is quite more complicated (see Vázquez [22] for the case $N = 2$).

We define

$$(0.26)\qquad \begin{cases} b_g^+ = \inf\left\{b > 0 : \int_0^1 g(t^{-(N-2)/(N+2)}\,\mathrm{Ln}\,(1/t)/b)\,dt < \infty\right\}, \\ b_g^- = \inf\left\{b > 0 : \int_0^1 g(t^{-(N-2)/(N+2)}\,\mathrm{Ln}\,t/b)\,dt > -\infty\right\}, \end{cases}$$

and we prove

Theorem 3.2. *Assume g is a continuous nondecreasing real valued function. Then the equation*

$$(0.27)\qquad -\Delta u - \left(\frac{N-2}{2|x|}\right)^2 u + g(u) = 0$$

admits solutions u in $B_1(0)\setminus\{0\}$ *such that*

$$(0.28)\qquad \lim_{x\to 0} |x|^{(N-2)/2} u(x)/\mathrm{Ln}\,(1/|x|) = \gamma,$$

where γ *is a real number, if and only if*

$$(0.29)\qquad -(N+2)/(2b_g^-) \leqslant \gamma \leqslant (N+2)/(2b_g^+).$$

The Dirichlet problems corresponding to Theorems 3.1, 3.2 *are also solved.*

In the last section we study the limit properties of the solutions u of (0.3) (as $|x|$ tends to 0 or ∞ as well). If we perform the classical transformation

$$(0.30)\qquad u(r,\sigma) = r^{-2/(q-1)} v(t,\sigma), \qquad t = \mathrm{Ln}\, r$$

and denote by $\Delta_{S^{N-1}}$ the Laplace-Beltrami operator on S^{N-1}, then

$$(0.31)\qquad v_{tt} + \left(N - 2\frac{q+1}{q-1}\right)v_t + \Delta_{S^{N-1}} v + \lambda v - v|v|^{q-1} = 0$$

holds in $(-\infty, 0)$ or $(0, \infty)$ with

$$(0.32)\qquad \lambda = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right),$$

moreover u is bounded. When $q \neq (N+2)/(N-2)$ the study of this equation is an extension of previous results of Veron [26] [27], Chen-Matano-Veron [13] and Bidaut-Veron-Veron [7]. A typical result is the following

Theorem 4.1. *Assume* $q \in (1,\infty)\setminus\{(N+2)/(N-2)\}$ *and u is a solution of* (0.3) *in* $B_1(0)\setminus\{0\}$. *Then* $r^{2/(q-1)}u(r,\bullet)$ *converges in the* $C^3(S^{N-1})$*-topology* *to some compact connected subset ξ' of the set ξ of the $C^3(S^{N-1})$-functions ω satisfying*

$$-\Delta_{S^{N-1}}\omega + \omega|\omega|^{q-1} = \lambda\omega \tag{0.33}$$

on S^{N-1}. Moreover there exists precisely one $\omega \in \xi$ such that

$$\lim_{r\to 0} r^{2/(q-1)}u(r, \cdot) = \omega(\cdot), \tag{0.34}$$

at least in the following cases:

(i) *u is nonnegative,*
(ii) $\lambda \leqslant N - 1$,
(iii) *q is an odd integer,*
(iv) *ξ' is an hyperbolic limit manifold in the sense of Simon* [21],
(v) $N = 2$ *and* $c \leqslant 1$.

When $q = (N + 2)/(N - 2)$ the study is more complicated, in particular because of the conformal invariance of (0.3) and the existence of solitary waves satisfying (0.3) (see [7]). Convergence results hold at least for nonnegative solutions [17]. When $\lambda \leqslant 0$ there always holds

$$\lim_{x\to 0} |x|^{2/(q-1)}u(x) = 0 \tag{0.35}$$

for any solution u of (0.3) in $B_1(0)\setminus\{0\}$ and the exact behaviour is given by μ_2 from Theorem 1.2 except in the particular case $\lambda = 0$, $q > (N + 2)/(N - 2)$ and we prove

Theorem 4.2. *Assume $0 < c \leqslant ((N - 2)/2)^2$, $\lambda = 0$ and $q > (N + 2)/(N - 2)$. If u is any solution of (0.3) in $B_1(0)\setminus\{0\}$, then the following limit exists*

$$\lim_{x\to 0} u(x)/(\mu_2(x)\,\mathrm{Ln}\,(1/|x|)^{\alpha_2/2}) = l \tag{0.36}$$

with

$$\alpha_2 = \left(2 - N + \sqrt{(N - 2)^2 - 4c}\right)/2 = -2/(q - 1)$$

and

$$l \in \{0, \pm((N(q - 1) - 2(q + 1))/(q - 1)^2)^{1/(q-1)}\}. \tag{0.37}$$

When $\lambda > 0$ it may happen that (0.35) holds. In that case the behaviour of u near 0 is most often described by the solutions of

$$-\Delta\zeta - \frac{c}{|x|^2}\zeta = 0, \tag{0.38}$$

satisfying (0.35) except when $-2/(q-1)$ is a solution of the algebraic equation

$$X^2 + (N-2)X + c - k(k+N-2) = 0 \tag{0.39}$$

for some integer k. Only when $N = 2$ and $c \leqslant 0$ this spectral case is understood. In some cases, when the rate of blow-up of u near 0 is of order $|x|^{-(N-2)/2}$, u may behave as a finite superposition of travelling waves near 0 (up to the damping factor $|x|^{(N-2)/2}$).

We also study the asymptotics of the solution of (0.3) in an exterior domain and end this section with some orbit connecting questions where the structure of the set of the stationnary solutions of (0.33) plays a fundamental role.

Our paper is organized as follows

(1) Removable singularities.
(2) The isotropy theorems.
(3) Solutions with linear singularities.
(4) The power case.

1. Removable Singularities

In this section we assume that $\Omega \supset \bar{B}_1(0)$, $\Omega^* = \Omega\setminus\{0\}$ and we first prove the following a priori estimate of Osserman's type [19], [10], [31].

Lemma 1.1. *Assume $u \in L^\infty_{\mathrm{loc}}(\Omega^*)$ satisfies $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega^*)$ and*

$$-\Delta u - \frac{c}{|x|^2}u + au^q \leqslant b \tag{1.1}$$

a.e. on $\{x \in \Omega : u(x) \geqslant 0\}$, for some constants $a > 0$, b and $c \geqslant 0$ and $q > 1$. Then

$$u(x) \leqslant A|x|^{-2/(q-1)} + B \qquad (\text{for all} \quad x \in \bar{B}_{1/2}(0)\setminus\{0\}), \tag{1.2}$$

where

$$A = \sigma(N,q)\left(\frac{1+c}{a}\right)^{1/(q-1)}, \qquad B = \sigma(N,q)\left(\frac{b}{a}\right)^{1/q}, \tag{1.3}$$

with $\sigma(N,q) > 0$.

PROOF. Let x_0 be such that $0 < |x_0| < 1/2$. Set

$$G = \left\{x \in \Omega \colon |x - x_0| < \frac{1}{2}|x_0|\right\}$$

and $G^+ = \{x \in G \colon u(x) \geqslant 0\}$. The function u is essentially bounded in G and

$$-\Delta u + \frac{a}{2}u^q \leqslant \beta = \max_{r>0}\left\{b + \frac{4c}{|x_0|^2}r - \frac{a}{2}r^q\right\} \tag{1.4}$$

a.e. in G^+. If we compute β we find

$$\beta = b + \frac{4(q-1)c}{|x_0|^2}\left(\frac{8c}{aq|x_0|^2}\right)^{1/(q-1)} \tag{1.5}$$

As in [10], [31] we consider a function v under the following form

$$v(x) = \rho\left(\frac{1}{4}|x_0|^2 - |x - x_0|^2\right)^{-2/(q-1)} + \tau. \tag{1.6}$$

If

$$\eta = \max\left\{\frac{2N}{q-1}, \frac{4(q+1)}{(q-1)^2}\right\}, \qquad \rho = \left(\frac{2\eta}{a}\right)^{1/(q-1)}, \qquad \tau = \left(\frac{2\beta}{a}\right)^{1/q},$$

v satisfies

$$-\Delta v + \frac{a}{2}v^q \geqslant \beta \tag{1.7}$$

in G. Using Kato's inequality as in [10], [11], we deduce $v \geqslant u$ in G, which implies $v(x_0) \geqslant u(x_0)$ and gives (1.3).

Lemma 1.2. *Assume* $1 < q \leqslant N/(N-2)$,

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) \leqslant 0 \tag{1.8}$$

and $u \in L^\infty_{\mathrm{loc}}(\Omega^*)$ *satisfies* $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega^*)$ *and* (1.1) *a.e. on* $\{x \in \Omega \colon u(x) \geqslant 0\}$ *for some constants* $a > 0$ *and* $b \geqslant 0$. *Then* $\mu_2 u^+ \in L^q_{\mathrm{loc}}(\Omega)$.

PROOF. From Kato's inequality we have

$$-\Delta u^+ - \frac{c}{|x|^2}u^+ + a(u^+)^q \leqslant b \tag{1.9}$$

in $D'(\Omega^*)$, and from (1.2) and (1.8) we deduce that $c \leqslant 0$ and

$$u^+ \leqslant K_0 \mu_1 \tag{1.10}$$

near 0, for some $K_0 > 0$. Let ϕ be an element of $C_0^\infty(\Omega)$, $\phi \geqslant 0$, η_n be a C^∞-function in Ω such that $0 \leqslant \eta_n \leqslant 1$ and

$$\eta_n(x) = \begin{cases} 0 & \text{if} \quad 0 < |x| < 1/n \\ 1 & \text{if} \quad\quad |x| \geqslant 2/n \end{cases} \tag{1.11}$$

with $|\nabla \eta_n| \leqslant Kn$, $|\Delta \eta_n| \leqslant Kn^2$.

Then we claim that

$$\int (u^+)^q \mu_2 \phi \, dx < \infty.$$

As a test function we take $\phi \eta_n \mu_2$ and get

$$\int u^+ \left(-\Delta(\phi \eta_n \mu_2) - \frac{c}{|x|^2}(\phi \eta_n \mu_2) \right) + a \int (u^+)^q \phi \eta_n \mu_2 \leqslant K(\phi). \tag{1.12}$$

But

$$\Delta(\phi \eta_n \mu_2) = \phi \eta_n \, \Delta \mu_2 + \mu_2 \, \Delta(\phi \eta_n) + 2 \, \nabla \mu_2 \, \nabla(\phi \eta_n)$$

and (1.12) becomes

$$\int u^+ (-\mu_2 \, \Delta(\phi \eta_n) - 2 \, \nabla \mu_2 \, \nabla(\phi \eta_n)) + a \int (u^+)^q \phi \eta_n \mu_2 \leqslant K(\phi). \tag{1.13}$$

Let $\Gamma_n = \{x \in \Omega : 1/n < |x| < 2/n\}$ and let χ_{Γ_n} be the characteristic function of Γ_n. There exist K_1, K_2 such that $K_i > 0$ and

$$|\Delta(\phi \eta_n)| \leqslant K_1 + K_2 n^2 \chi_{\Gamma_n}, \qquad |\nabla(\phi \eta_n)| \leqslant K_1 + K_2 n \chi_{\Gamma_n}. \tag{1.14}$$

Plugging into (1.12) implies

$$\begin{aligned} a \int (u^+)^q \phi \eta_n \mu_2 < K(\phi) &+ K_0 \int \mu_1 \mu_2 (K_1 + K_2 n^2 \chi_{\Gamma_n}) \\ &+ 2K_0 \int \mu_1 |\nabla \mu_2| (K_1 + K_2 n \chi_{\Gamma_n}). \end{aligned} \tag{1.15}$$

As $\mu_1 \mu_2 = |x|^{2-N}$ and $\mu_1 |\nabla \mu_2| = K'|x|^{1-N}$, the right-hand side of (1.15) is bounded independently of n. Letting n tend to infinity implies the claim.

Lemma 1.3. *Under the hypotheses of Theorem* 1.1, *$g(u)$ and Vu are locally integrable in Ω.*

PROOF. We shall treat separately the cases $1 < q \leqslant N/(N-2)$ and $q > N/(N-2)$ but from (0.11) and Lemma 1.1 in any case $|x|^{2/(q-1)}u(x)$ is locally bounded in Ω.

Case 1. $1 < q \leqslant N/(N-2)$. From Kato's inequality we have

$$-\Delta u^{+} - Vu^{+} + \operatorname{sign}^{+}(u)g(u) \leqslant 0 \tag{1.16}$$

in $D'(\Omega^*)$. Let ζ_ϵ be $\mu_2/(\epsilon+\mu_2)$ ($\epsilon > 0$). As a test function we take $\phi\eta_n\zeta_\epsilon$ where ϕ and η_n are as in Lemma 1.2 with $\phi \equiv 1$ in $\bar{B}_{1/2}(0)$. We get

$$\int u^{+}((-\Delta(\phi\eta_n\zeta_\epsilon) - V\phi\eta_n\zeta_\epsilon) + \int \operatorname{sign}^{+}(u)g(u)\phi\eta_n\zeta_\epsilon \leqslant 0. \tag{1.17}$$

As

$$\Delta(\phi\eta_n\zeta_\epsilon) = \phi\eta_n\,\Delta\zeta_\epsilon + \zeta_\epsilon\,\Delta(\phi\eta_n) + 2\,\nabla\zeta_\epsilon\,\nabla(\phi\eta_n)$$

and

$$\nabla\zeta_\epsilon = \frac{\epsilon}{(\epsilon+\mu_\epsilon)^2}\nabla\mu_2,$$

$$\Delta\zeta_\epsilon = \frac{\epsilon}{(\epsilon+\mu_2)^2}\,\Delta\mu_2 - \frac{2\epsilon}{(\epsilon+\mu_2)^3}\,|\nabla\mu_2|^2.$$

As

$$\Delta\mu_2 = -\frac{c}{|x|^2}\mu_2,$$

we get

$$\begin{aligned} -\Delta(\phi\eta_n\zeta_\epsilon) - V\phi\eta_n\zeta_\epsilon = &-\left(-\frac{c\epsilon\mu_2}{|x|^2(\epsilon+\mu_2)^2} - \frac{2\epsilon}{(\epsilon+\mu_2)^3}\,|\nabla\mu_2|^2\right)\phi\eta_n \\ &- V\phi\eta_n\frac{\mu_2}{\epsilon+\mu_2} \\ &- \zeta_\epsilon\,\Delta(\phi\eta_n) \\ &- \frac{2\epsilon}{(\epsilon+\mu_2)^2}\,\nabla\mu_2\,\nabla(\phi\eta_n). \end{aligned}$$

Henceforth (1.17) implies

$$\begin{aligned} &\int \frac{\mu_2}{\epsilon+\mu_2}\phi\eta_n(-Vu^{+} + \operatorname{sign}^{+}(u)g(u)) \\ &\leqslant -\int \frac{c\epsilon\mu_2 u^{+}}{|x|^2(\epsilon+\mu_2)^2}\phi\eta_n + \int u^{+}\zeta_\epsilon\,\Delta(\phi\eta_n) + 2\epsilon\int \frac{u^{+}}{(\epsilon+\mu_2)^2}\nabla\mu_2\,\nabla(\phi\eta_n). \end{aligned} \tag{1.18}$$

We also have

$$|\Delta(\phi\eta_n)| \leqslant K_1\chi_{\Gamma_2} + n^2K_2\chi_{\Gamma_n}, \tag{1.19}$$

$$|\nabla(\phi\eta_n)| \leqslant K_1\chi_{\Gamma_2} + nK_2\chi_{\Gamma_n}, \tag{1.20}$$

and

$$\left|\int u^+\zeta_\epsilon\,\Delta(\phi\eta_n)\right| \leqslant K_1\int_{\Gamma_2} u^+ + n^2K_2\int_{\Gamma_n} u^+\zeta_\epsilon. \tag{1.21}$$

As

$$\int_{\Gamma_n} u^+\zeta_\epsilon \leqslant \left(\int_{\Gamma_n}(u^+)^q\mu_\epsilon\right)^{1/q}\left(\int_{\Gamma_n}\frac{\mu_2}{(\epsilon+\mu_2)^{q/(q-1)}}\right)^{(q-1)/q},$$

and

$$\left(\int_{\Gamma_n}\frac{\mu_2}{(\epsilon+\mu_2)^{q/(q-1)}}\right)^{(q-1)/q} \leqslant \frac{c(N)}{\epsilon}n^{-(\alpha_2+N)(q-1)/q}$$

where

$$\alpha_2 = \frac{2-N+\sqrt{(N-2)^2-4c}}{2}$$

is such that $\mu_2(x) = |x|^{\alpha_2}$; as (1.8) holds $\alpha_1 \leqslant -2/(q-1)$ and as $\alpha_1 + \alpha_2 = 2 - N$ we deduce that

$$-(\alpha_2+N)(q-1)/q + 2 \leqslant 0. \tag{1.22}$$

From Lemma 1.2 $\mu_2(u^+)^q$ is locally integrable in Ω; henceforth

$$\lim_{n\to\infty} n^2\int_{\Gamma_n} u^+\zeta_\epsilon = 0. \tag{1.23}$$

In the same way

$$\epsilon\int\frac{u^+}{(\epsilon+\mu_2)^2}\nabla\mu_2\,\nabla(\phi\eta_n) \leqslant 2\epsilon K_1\int_{\Gamma_2} u^+\frac{|\nabla\mu_2|}{\mu_2} + 2Kn\int_{\Gamma_n}\frac{\epsilon u^+}{(\epsilon+\mu_2)^2}|\nabla\mu_2|$$

and

$$\int_{\Gamma_n}\frac{\epsilon u^+}{(\epsilon+\mu_2)^2}|\nabla\mu_2| \leqslant \int_{\Gamma_n} u^+\zeta_\epsilon\frac{\epsilon\alpha_2}{|x|(\epsilon+\mu_2)} \leqslant n\alpha_2\int_{\Gamma_n} u^+\zeta_\epsilon,$$

which yields

$$\lim_{n\to\infty} n \int_{\Gamma_n} \frac{u^+}{(\epsilon+\mu_2)^2} \nabla\mu_2 = 0. \tag{1.24}$$

For the last right-hand side term of (1.18) we have

$$-c\epsilon \int \frac{u^+\mu_2}{|x|^2(\epsilon+\mu_2)^2}\phi\eta_n \leqslant -cn^2\int_{\Gamma_n} u^+\zeta_\epsilon. \tag{1.25}$$

Using (1.23), (1.24) and the facts that V is negative and $\text{sign}^+(u)g(u)$ is bounded below by some constant imply

$$\int \phi(-Vu^+ + \text{sign}^+(u)g(u)) \leqslant K \tag{1.26}$$

for some $K > 0$. In the same say

$$\int \phi(-Vu^- + \text{sign}^-(u)g(u)) \leqslant K. \tag{1.27}$$

Henceforth Vu and $g(u)$ are locally integrable in Ω.

Case 2. $q > N/(N-2)$. From Lemma 1.1 and (0.12) uV is locally integrable in Ω. Taking $\eta_n\phi$ as a test function in (1.16) implies

$$\int u^+(-\Delta(\phi\eta_n) - V\phi\eta_n) + \int \text{sign}^+(u)g(u)\phi\eta_n \leqslant 0. \tag{1.28}$$

Using (1.19) yields

$$\left|\int u^+\,\Delta(\phi\eta_n)\right| \leqslant K_1\int_{\Gamma_2} u^+ + n^2K_2\int_{\Gamma_n} u^+ \leqslant K_1\int_{\Gamma_2} u^+ + K_2K'n^{2-N+2/(q-1)} \leqslant K''.$$

Letting n tend to infinity implies

$$\int \text{sign}^+(u)g(u)\phi < \infty.$$

In the same way

$$\int \text{sign}^-(u)g(u)\phi < \infty$$

which ends the proof

PROOF OF THEOREM 1.1. *Case* 1. $1 < q \leqslant N/(N-2)$.

As a test function we take $\eta_n \zeta_\epsilon \phi$ where η_n and ζ_ϵ are as in Lemma 1.3 and $\phi \in C_0^\infty(\Omega)$. We have

$$\int u(-\Delta(\eta_n \phi \zeta_\epsilon)) - \int V u \eta_n \phi \zeta_\epsilon + \int g(u) \eta_n \phi \zeta_\epsilon = 0 \tag{1.29}$$

and

$$\Delta(\eta_n \phi \zeta_\epsilon) = \eta_n \phi \, \Delta\zeta_\epsilon + \eta_n \zeta_\epsilon \, \Delta\phi + \phi \zeta_\epsilon \, \Delta\eta_n + 2\eta_n \nabla\phi \, \nabla\zeta_\epsilon + 2\phi \, \nabla\eta_n \, \nabla\zeta_\epsilon + 2\zeta_\epsilon \nabla\phi \, \nabla\eta_n .$$

As in Lemma 1.3, Case 1 it is easy to let n tend to infinity and obtain, for $\epsilon > 0$ fixed,

$$\int u(-\phi \, \Delta\zeta_\epsilon - \zeta_\epsilon \, \Delta\phi + 2 \, \nabla\phi \, \nabla\zeta_\epsilon) + \int (g(u) - Vu)\phi\zeta_\epsilon = 0. \tag{1.30}$$

But

$$|u \, \nabla\phi \, \nabla\zeta_\epsilon| \leqslant |\nabla\phi| \, |u| \frac{\epsilon\alpha_2\mu_2}{|x|(\epsilon + \mu_2)^2} \leqslant \frac{\alpha_2}{2|x|} |\nabla\phi| \, |u|,$$

$$\begin{aligned} |u\phi \, \Delta\zeta_\epsilon| &\leqslant |c| \, |u| \, |\phi| \frac{\epsilon\mu_2}{|x|^2(\epsilon + \mu_2)^2} + 2\alpha_2^2 |u| \, |\phi| \frac{\epsilon\mu_2^2}{|x|^2(\epsilon + \mu_2)^3} \\ &\leqslant \frac{|c|}{2} |\phi| \frac{|u|}{|x|^2} + \frac{8\alpha_2^2}{27} |\phi| \frac{|u|}{|x|^2}. \end{aligned}$$

If $c = 0$ the two terms $|u \, \nabla\phi \, \nabla\zeta_\epsilon|$ and $|u\phi \, \Delta\zeta_\epsilon|$ vanish; if $c < 0$ we know from Lemma 1.3 that $u/|x|^2$ is locally integrable in Ω. Henceforth, from Lebesgue's theorem, we get

$$\lim_{\epsilon \to 0} \int u(-\phi \, \Delta\zeta_\epsilon - \zeta_\epsilon \, \Delta\phi + 2 \, \nabla\phi \, \nabla\zeta_\epsilon) = \int (-u \, \Delta\phi) \tag{1.31}$$

and

$$\int (-u \, \Delta\phi) - \int V u \phi + \int g(u)\phi = 0. \tag{1.32}$$

Case 2. $q > N/(N-2)$.

As a test function we just take $\phi\eta_n$ and we have from Lemma 1.3 and Hölder's inequality

$$\lim_{n \to \infty} \int -u \, \Delta(\phi\eta_n) - V u \phi \eta_n + g(u)\phi\eta_n = \int -u \, \Delta\phi - V u \phi + g(u)\phi, \tag{1.33}$$

which ends the proof.

Lemma 1.4. *Assume Ω is as above and V is continuous in Ω^* and satisfies*

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2 \tag{1.34}$$

near 0. *If $w \in C^0(\Omega^*)$ is a nonnegative function satisfying*

$$\Delta w + Vw \geqslant 0 \tag{1.35}$$

in $D'(\Omega^*)$ and $w = o(\mu_1)$ near 0, *then u/μ_2 remains locally bounded in Ω.*

PROOF. Let M be the supremum of w on $\{x: |x| = 1\}$ and, for $\epsilon > 0$, $\Phi_\epsilon = M\mu_2 + \epsilon\mu_1$. We write (1.35) in spherical coordinates and get

$$w_{rr} + \frac{N-1}{r^2} w_r + \frac{c}{r^2} w + \frac{1}{r^2} \Delta_{S^{N-1}} w \geqslant 0 \tag{1.36}$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} (we have used (1.35)). We shall distinguish two cases:

Case 1. $c < \left(\frac{N-2}{2}\right)^2$. We write

$$v(s, \sigma) = w(r, \sigma)/\mu_1(r), \qquad s = r^\beta, \qquad \beta = \sqrt{(N-2)^2 - 4c} \tag{1.37}$$

and get

$$s^2 v_{ss} + \frac{1}{\beta^2} \Delta_{S^{N-1}} v \geqslant 0. \tag{1.38}$$

We write $\phi_\epsilon(s, \sigma) = \Phi_\epsilon(r, \sigma)/\mu_1(r)$, $t = \mathrm{Ln}(1/s)$ and $\psi(t, \sigma) = (v - \phi_\epsilon)(s, \sigma)$. The following relation holds in $D'(\mathbb{R}_*^+ \times S^{N-1})$

$$\psi_{tt} + \psi_t + \frac{1}{\beta^2} \Delta_{S^{N-1}} \psi \geqslant 0. \tag{1.39}$$

By convolution on t we may assume that $\psi \in C^\infty(\mathbb{R}_*^+, C^0(S^{N-1}))$ and if we approximate ψ by the solution χ_η ($\eta > 0$) of

$$-\eta \Delta_{S^{N-1}} \chi_\eta + \chi_\eta = \psi, \tag{1.40}$$

which converges to ψ in $L^2(S^{N-1})$ as η tends to 0, we deduce that

$$\frac{d^2}{dt^2} \|\psi^+(t, \cdot)\|_{L^2(S^{N-1})} + \frac{d}{dt} \|\psi^+(t, \cdot)\|_{L^2} \geqslant 0, \tag{1.41}$$

which implies that the function $s \to \|(v - \phi_\epsilon)^+(s, \cdot)\|_{L^2(S^{N-1})}$ is convex on $(0, 1)$. As it vanishes at 0 and 1, it is always 0. Letting ϵ tend to 0 implies the claim.

Case 2. $c = \left(\dfrac{N-2}{2}\right)^2$. We just write

$$v(t, \sigma) = w(r, \sigma)/r^{-(N-2)/2}, \qquad t = \mathrm{Ln}\,(1/r) \tag{1.42}$$

and v satisfies

$$v_{tt} + \Delta_{S^{N-1}} v \geqslant 0 \tag{1.43}$$

in $D'(\mathbb{R}_*^+ \times S^{N-1})$. By the same approximation we see that the convexity and the fact that $v = o(t)$ at infinity imply the estimate $w \leqslant K\mu_2$.

PROOF OF THEOREM 1.2. *Case* 1. We assume that

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) < 0. \tag{1.44}$$

In that case $2/(q-1) < (N - 2 + \beta)/2$ and $u = o(\mu_1)$ near 0. As we have

$$\Delta u^+ + \frac{c}{|x|^2} u^+ \geqslant \mathrm{sign}^+(u) g(u) \geqslant 0 \tag{1.45}$$

in $D'(\Omega^*)$, we deduce $u^+ \leqslant k\mu_2$. We do the same with u^-.

Case 2. We assume that $c \leqslant 0$.
In that case u^+ satisfies

$$\Delta u^+ \geqslant \mathrm{sign}^+(u) g(u) \geqslant a(u^+)^q - b. \tag{1.46}$$

From Brézis-Veron's result [11] u^+ is locally bounded in Ω; henceforth $u^+ \leqslant k\mu_2$. The same with u^-.

Case 3. We assume

$$0 < c \leqslant \left(\frac{N-2}{2}\right)^2, \tag{1.47}$$

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) = 0. \tag{1.48}$$

For $n \in \mathbb{N}^*$ let ϕ_n be the solution of

$$(1.49) \qquad \begin{cases} \Delta\phi_n + \dfrac{c}{|x|^2}\phi_n - a\phi_n^q = 0 & \text{in} \quad B_1(0)\setminus B_{1/n}(0) \\ \phi_n = \max\{u^+(x): |x| = 1\} & \text{on} \quad \partial B_1(0) \\ \phi_n = \max\{u^+(x): |x| = 1/n\} & \text{on} \quad \partial B_{1/n}(0) \end{cases}$$

(ϕ_n exists by minimization techniques and it is positive and unique) where a is defined as in (1.46) or Lemma 1.1. Let σ be $b/(c + 2N)$. Then $\psi_n = \phi_n + \sigma|x|^2$ satisfies

$$(1.50) \qquad \Delta\phi_n + \frac{c}{|x|^2}\psi_n - a\psi_n^q \leqslant b.$$

We then deduce, as in Lemma 1.4, that $u^+ \leqslant \psi_n$ in $B_1(0)\setminus B_{1/n}(0)$. As ϕ_n remains locally bounded in $B_1(0)\setminus B_{1/n}(0)$, independently of n (Lemma 1.1), we deduce that (up to a subsequence) it converges in the $C^1_{\text{loc}}(\bar{B}_1(0)\setminus\{0\})$-topology to a function ϕ which is radial and satisfies

$$(1.51) \qquad \begin{cases} \Delta\phi + \dfrac{c}{|x|^2}\phi - a\phi^q = 0 & \text{in} \quad B_1(0)\setminus\{0\}, \\ \phi = \max\{u^+(x): |x| = 1\} & \text{on} \quad \partial B_1(0), \end{cases}$$

and

$$(1.52) \qquad u(x) \leqslant \phi(x) + \sigma|x|^2$$

in $\bar{B}_1(0)\setminus\{0\}$. Moreover, in the range (1.48), we have

$$(1.53) \qquad \phi(x) \leqslant c\mu_1(x) = c|x|^{\alpha_1} = c|x|^{-2/(q-1)}.$$

If we set $\phi(x) = \tilde{\phi}(r)$ and

$$(1.54) \qquad \eta(t) = r^{2/(q-1)}\tilde{\phi}(r), \qquad t = \text{Ln}\,(1/r)$$

then we get

$$(1.55) \qquad \eta_{tt} - \left(N - 2\frac{q+1}{q-1}\right)\eta_t - a\eta^q = 0$$

in $(0, +\infty)$.

(i) If $q = (N+2)/(N-2)$ the first order coefficient is 0 and

$$(1.56) \qquad W(\eta, \eta_t) = \frac{1}{2}\eta_t^2 - \frac{a}{q+1}\eta^{q+1}$$

is constant. As η is nonnegative and bounded, the only admissible constant is 0 and $\eta(t)$ tends to 0 as t tends to infinity.

(ii) If $q \neq (N+2)/(N-2)$ then

$$\frac{d}{dt} W(\eta, \eta_t) = \left(N - 2\frac{q+1}{q-1}\right)\eta_t^2. \tag{1.57}$$

From La Salle invariance principle $\lim_{t \to \infty} \eta(t) = 0$.

Henceforth

$$\lim_{x \to 0} u^+(x)/\mu_1(x) = 0. \tag{1.58}$$

As the same holds for u^- we deduce the claim from Lemma 1.4.

Remark 1.1. Using Theorems 2.1 and 3.1 it is possible to extend Theorem 1.2 to the case where g satisfies

$$\begin{cases} \liminf_{r \to \infty} \dfrac{g(r)\,\mathrm{Ln}\, r}{r^q} > 0 \\ \limsup_{r \to -\infty} \dfrac{g(r)\,\mathrm{Ln}\, r}{|r|^q} < 0 \end{cases} \tag{1.59}$$

and (0.17) (see [24] for the zero potential case and [16]).

2. The isotropy Theorems

In this section Ω is an open subset of $\mathbb{R}^N$ containing $\bar{B}_1(0)$, $\Omega^* = \Omega \setminus \{0\}$, g is a continuous nondecreasing real valued function and $V \in C(\Omega^*)$ is a radial potential such that

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2, \quad \text{for all} \quad x \in \Omega^*. \tag{2.1}$$

We are interested in solutions $u \in C^1(\Omega^*)$ of

$$-\Delta u - Vu + g(u) = 0. \tag{2.2}$$

Lemma 2.1. *Assume $u \in C^1(\Omega^*)$ satisfies* (2.2) *in $D'(\Omega^*)$ and*

$$\liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0. \tag{2.3}$$

Then there exists a constant $K \geqslant 0$ *such that*

$$\|u(r,\cdot) - \bar{u}(r)\|_{L^2(S^{N-1})} \leqslant K r^{(2-N+\sqrt{N^2-4c})/2} \tag{2.4}$$

for $0 < r \leqslant 1$.

PROOF. *Case* 1. $c < \left(\dfrac{N-2}{2}\right)^2$. In radial coordinates we have

$$u_{rr} + \frac{N-1}{r} u_r + V(r)u + \frac{1}{r^2}\Delta_{S^{N-1}} u = g(u). \tag{2.5}$$

We write u as in (1.37), that is

$$v(r,\sigma) = u(r,\sigma)/\mu_1(r), \qquad s = r^\beta, \qquad \beta = \sqrt{(N-2)^2 - 4c}, \tag{2.6}$$

and get

$$s^2 v_{ss} + \frac{1}{\beta^2}\Delta_{S^{N-1}} v + \frac{1}{\beta^2}(s^{2/\beta}V(s^{1/\beta}) - c)v = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} v) \tag{2.7}$$

with $\alpha_1 = (2 - N - \beta)/2$.

Let $\bar{\rho}(s)$ be the spherical average of a function $\rho(s,\sigma)$; then

$$s^2 \bar{v}_{ss} + \frac{1}{\beta^2}(V(s^{1/\beta}) - c)\bar{v} = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta}\overline{g(s^{\alpha_1/\beta} v)}.$$

As

$$\int_{S^{N-1}} (-\Delta v(v - \bar{v}))\, d\sigma \geqslant (N-1)\int_{S^{N-1}} (v - \bar{v})^2\, d\sigma,$$

$$\int_{S^{N-1}} (g(s^{\alpha_1/\beta} v) - \overline{g(s^{\bar{\alpha}_1/\beta} v)})(v - \bar{v})\, d\sigma \geqslant 0,$$

and (2.1) we get

$$s^2 \int_{S^{N-1}} (v_{ss} - \bar{v}_{ss})(v - \bar{v})\, d\sigma - \frac{N-1}{\beta^2}\int_{S^{N-1}} (v - \bar{v})^2\, d\sigma \geqslant 0. \tag{2.8}$$

Setting

$$X(s) = \left(\int_{S^{N-1}} (v - \bar{v})^2(s)\, d\sigma\right)^{1/2},$$

we obtain

$$s^2 X_{ss} - \frac{N-1}{\beta^2} X \geqslant 0 \tag{2.9}$$

in $D'(0, 1)$. As $X(s_n) = o(s_n^{(\beta - \sqrt{N^2-4c})/2\beta})$ for some sequence $\{s_n\}$ converging to 0 we deduce that

$$X(s) \leqslant K s^{(\beta + \sqrt{N^2-4c})/2\beta}, \tag{2.10}$$

which is (2.4).

Case 2. $c = \left(\frac{N-2}{2}\right)^2$. We write u as in (1.42):

$$w(t, \sigma) = r^{(N-2)/2} u(r, \sigma), \qquad t = \mathrm{Ln}\,(1/r); \tag{2.11}$$

then

$$w_{tt} + \Delta_{S^{N+1}} w + (\tilde{V}(t) - c) w = e^{(2-N)t/2} g(e^{(N-2)t/2} w) \tag{2.12}$$

where

$$\tilde{V}(t) = e^{(2-N)t} V(e^{(N-2)t/2}).$$

If we set

$$X(t) = \left(\int_{S^{N-1}} (w - \bar{w})^2(t)\, d\sigma \right)^{1/2},$$

then

$$X_{tt} - (N-1) X \geqslant 0 \tag{2.13}$$

in $D'(\mathbb{R}_*^+)$, and $X(t_n) = o(e^{\sqrt{N-1}\, t_n})$ for some sequence $\{t_n\}$ tending to ∞. The maximum principle implies that

$$X(t) \leqslant K e^{-\sqrt{N-1}\, t} \tag{2.14}$$

which is (2.4).

In order to have a L^∞ estimate we need the following result the proof of which is essentially contained in [24].

Lemma 2.2. *Assume γ, a and b are positive numbers such that $a < b$ and ϕ, $\psi \in L^2(S^{N-1})$. Then there exists a unique function $\Phi \in C([a, b]; L^2(S^{N-1}))$*

$\cap C^\infty((a, b) \times S^{N-1})$ *such that*

$$\text{(2.15)} \qquad \begin{cases} s^2\Phi_{ss} + \dfrac{1}{\gamma}\Delta_{S^{N-1}}\Phi = 0 & \textit{in} \quad (a, b) \times S^{N-1}, \\ \Phi(a, \cdot) = \phi(\cdot); \qquad \Phi(b, \cdot) = \psi(\cdot) & \textit{in} \quad S^{N-1}. \end{cases}$$

Moreover there exists a constant $C_1 > 0$ *such that*

$$\text{(2.16)} \qquad \|\Phi(s, \cdot)\|_{L^\infty(S^{N-1})} \leqslant C_1\left\{\left(1 + \frac{1}{\operatorname{Ln}(s/a)}\right)^{(N-1)/2} \|\Phi\|_{L^2(S^{N-1})} + \left(1 + \frac{1}{\operatorname{Ln}(b/s)}\right)^{(N-1)/2} \|\psi\|_{L^2(S^{N-1})}\right\}.$$

This result is, up to change of variable and unknown, essentially an estimate concerning harmonic functions in an annulus.

Lemma 2.3. *Assume the hypotheses of Lemma* 2.1 *hold with* $c < (N-2)^2/4$. *Then the function* v *introduced in* (2.6) *satisfies*

$$\text{(2.17)} \qquad \|v(s, \cdot) - \bar{v}(s)\|_{L^\infty(S^{N-1})} \leqslant \tilde{c}s^{(\beta + \sqrt{N^2 - 4c})/2\beta}$$

for some $\tilde{c} > 0$ *and any* $s \in (0, 1/2]$.

PROOF. Let y be the solution of

$$\text{(2.18)} \qquad \begin{cases} s^2 y_{ss} = \dfrac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} y) & \text{in} \quad (a, b), \\ y(a) = \rho, \qquad y(b) = \tau, \end{cases}$$

with $0 < a < b < 1$, ρ and τ real numbers. Let w be $v - y$, ϕ be $(v(a, \cdot) - \rho)^+$, ψ be $(v(b, \cdot) - \tau)^+$ and Φ be the solution of (2.15). Then $\Phi \geqslant 0$. If we define h as

$$h = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta}(g(s^{\alpha_1/\beta} v) - g(s^{\alpha_1/\beta} y))/w,$$

then $h \geqslant 0$ and

$$\text{(2.19)} \qquad s^2 w_{ss} + \frac{1}{\beta^2}\Delta_{S^{N-1}} w = hw.$$

Henceforth Φ is a super-solution for (2.19) and $\Phi \geqslant w$. Using (2.16) with $\rho = \bar{v}(a)$, $\tau = \bar{v}(b)$ we get

$$(2.20)\quad v(s,\sigma) - y(s) \leqslant C_1\left\{\left(1+\frac{1}{\operatorname{Ln}(s/a)}\right)^{(N-1)/2} \|(v(a,\bullet)-\bar{v}(a))^+\|_{L^2(S^{N-1})} + \left(1+\frac{1}{\operatorname{Ln}(b/s)}\right)^{(N-1)/2} \|(v(b,\bullet)-\bar{v}(b))^+\|_{L^2(S^{N-1})}\right\}$$

for any $\sigma \in S^{N-1}$ and any $a < s < b$.

If we take $\dfrac{s}{a} = \dfrac{b}{s} = 2$ and use estimate (2.4) in the s variable, we get

$$(2.21)\qquad v(s,\sigma) - y(s) \leqslant C_2 s^{(\beta + \sqrt{N^2-4c})/2\beta}.$$

In the same way we have

$$(2.22)\qquad y(s) - v(s,\sigma) \leqslant C_2 s^{(\beta + \sqrt{N^2-4c})/2\beta},$$

which implies the claim.

Lemma 2.4. *Assume the hypotheses of Lemma* 2.1. *hold and* $c = (N-2)^2/4$. *Then the function* w *introduced in* (2.12) *satisfies*

$$(2.23)\qquad \|w(t,\bullet) - \bar{w}(t)\|_{L^\infty(S^{N-1})} \leqslant \tilde{c}e^{-\sqrt{N-1}\,t}$$

for some $\tilde{c} > 0$ *and any* $t > 0$.

PROOF. It is essentially the same at the one of Lemma 2.3 except that Lemma 2.2 is replaced by the following estimate: for $a > 0$ and $\phi \in L^2(S^{N-1})$ the unique bounded solution Φ of

$$(2.24)\qquad \begin{cases} \Phi_{tt} + \Delta_{S^{N-1}}\Phi = 0 & \text{in} \quad (a, +\infty) \times S^{N-1}, \\ \Phi(a,\bullet) = \phi(\bullet) & \text{on} \quad S^{N-1}. \end{cases}$$

satisfies

$$(2.25)\qquad \|\Phi(t,\bullet)\|_{L^\infty(S^{N-1})} \leqslant \tilde{c}\left(1+\frac{1}{t-a}\right)^{(N-1)/2} \|\phi\|_{L^2(S^{N-1})}.$$

This is essentially Poisson's formula.

PROOF OF THEOREM 2.1. *Case* 1. $c < \left(\dfrac{N-2}{2}\right)^2.$

The proof follows the ideas of [24] and we have to distinguish according $\bar{v}$ is bounded or not near 0

Case 1.1. $\bar{v}$ is bounded near 0. There exist a sequence $\{s_n\}$ tending to 0 and some z such that $\bar{v}(s_n)$ converges to z as $n \to \infty$. Assuming $z > 0$, we write $\tilde{g}(r) = g(r) - g(0)$ and call $\tilde{v}$ the solution of

$$\text{(2.26)} \quad \begin{cases} s^2 \tilde{v}_{ss} + \dfrac{1}{\beta^2}(s^{2/\beta} V(s^{1/\beta}) - c)\tilde{v} = \dfrac{1}{\beta^2} s^{(2-\alpha_1)/\beta} \tilde{g}(s^{\alpha_1/\beta}\tilde{v}) \quad \text{on} \quad (s_n, s_{n_0}), \\ \tilde{v}(s_n) = \tilde{v}(s_{n_0}) = z/2, \end{cases}$$

where n_0 is such that $v(s_n, \sigma) > z/2$ for all $n \geqslant n_0$, and $\sigma \in S^{N-1}$. It is clear that $\tilde{v} \geqslant 0$. If Λ is the solution of

$$\text{(2.27)} \quad \begin{cases} s^2 \Lambda_{ss} + \dfrac{1}{\beta^2}(s^{2/\beta} V(s^{1/\beta}) - c)\Lambda + \dfrac{1}{\beta^2} s^{(2-\alpha_1)/\beta} |g(0)| = 0 \quad \text{on} \quad (s_n, s_{n_0}), \\ \Lambda(s_n) = \Lambda(s_{n_0}) = 0, \end{cases}$$

then $\Lambda \geqslant 0$ and $\Lambda(s) \leqslant Ks$ for some constant independent of n. If we set $v^* = \tilde{v} - \Lambda$, then v^* is a sub-solution for (2.7) which implies

$$\text{(2.28)} \qquad v(s, \sigma) \geqslant v^*(s) \geqslant -Ks \qquad (\text{for every} \quad (s, \sigma) \in [s_n, s_{n_0}] \times S^{N-1}),$$

and $v_k(s, \sigma) = v(s, \sigma) + Ks$ is nonnegative in $(0, s_{n_0}] \times S^{N-1}$. As the spherical average $\bar{v}_K$ of v_K satisfies

$$\text{(2.29)} \quad s^2(\bar{v}_K)_{ss} + \frac{1}{\beta^2}(s^{2/\beta} V(s^{1/\beta}) - c)\bar{v}_K \geqslant \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(-K) + \frac{K}{\beta^2}(s^{2/\beta} V(s^{1/\beta}) - c)s,$$

there exist two constants M and $N \geqslant 0$ such that the function $E(s) = \bar{v}_K(s) + Ms^{(2-\alpha_1)/\beta} + N(s \operatorname{Ln} s - s)$ is convex. As $E(s_n)$ tends to z we deduce $\lim_{s \to 0} E(s) = z$, which yields

$$\text{(2.30)} \qquad \lim_{s \to 0} \bar{v}(s) = z = \lim_{s \to 0} v(s, \cdot),$$

uniformly on S^{N-1}.

If $z < 0$ we proceed similarly. If $z = 0$, then it is clear by using the technique above that

$$\text{(2.31)} \qquad \lim_{s \to 0} \bar{v}(s) = 0 = \lim_{s \to 0} \| v(s, \cdot) \|_{L^\infty(S^{N-1})}.$$

Case 1.2. $\bar{v}$ is unbounded near 0. Then there exists a sequence $\{s_n\}$ tending to 0 such that $\lim_{s \to 0} \bar{v}(s_n) = \infty$ ($-\infty$ in the same way). We conclude by the same convexity argument as in Case 1.1 that

$$\lim_{s \to 0} v(s, \sigma) = \infty \tag{2.32}$$

uniformly on S^{N-1}.

Case 2. $c = \left(\frac{N-2}{2}\right)^2$. We essentially follow the ideas of Case 1 but use the t variable ($t > 0$) and Lemma 2.4. If $\bar{w}$ is bounded in $\mathbb{R}^+$ then

$$\lim_{|x| \to 0} u(x)/\mu_1(x) = 0. \tag{2.33}$$

If $\bar{w}$ is not bounded we deduce from convexity arguments that

$$\text{either} \quad \lim_{t \to \infty} w(t, \cdot) = +\infty, \quad \text{or} \quad \lim_{t \to \infty} w(t, \cdot) = -\infty, \tag{2.34}$$

uniformly on S^{N-1}. Assuming the first case we also have from convexity the fact that $\bar{v}(t)/t$ admits a limit in $\mathbb{R}^+ \cup \{+\infty\}$. This limit is the same as the one of $u(x)/\mu_1(x)$ as x tends to zero and this ends the proof.

Remark 2.1. It is interesting to notice that (2.3) is automatically satisfied as soon as g has a fast enough growth, that is

$$\begin{cases} \liminf_{r \to \infty} g(r)/r^q = \infty, \\ \limsup_{r \to -\infty} g(r)/(-r)^q = -\infty, \end{cases} \tag{2.35}$$

for some $q > 1$ such that

$$c + \frac{q+1}{q-1}\left(\frac{q+1}{q-1} - N\right) = 0. \tag{2.36}$$

In that case we have

$$\frac{2}{q-1} = \frac{N - 2 + \sqrt{N^2 - 4c}}{2}$$

and

$$u(x) = o(|x|^{-2/(q-1)})$$

from (2.35) and Lemma 1.1. In the zero potential case the limit exponent q is $(N+1)/(N-1)$.

As we have proved Theorem 2.1 we can prove a similar result for the solution of (2.2) in an exterior domain G

Theorem 2.2. *Assume $G \supset \{x \in \mathbb{R}^N : |x| \geqslant 1\}$ and $u \in C^1(G)$ satisfies (2.2) in G where V is a radial potential defined in G and satisfying*

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \left(\frac{N-2}{2}\right)^2, \quad \text{for every} \quad x \in G. \tag{2.37}$$

If $g(0) = 0$ and u satisfies

$$\liminf_{r \to \infty} r^{(N-2-\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0, \tag{2.38}$$

then $u(x)/\mu_2(x)$ admits a limit in $\mathbb{R} \cup \{\infty, -\infty\}$ as $|x|$ tends to infinity. Moreover, if $\lim_{|x| \to \infty} u(x)/\mu_2(x) = 0$, there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{|x| \to \infty} u(x)/\mu_1(x) = \gamma. \tag{2.39}$$

The zero potential case of this result can be found in [31]. We can apply this type of methods to symmetry problems as in [28].

Corollary 2.1. *Assume V is a radial potential defined in $\mathbb{R}^N \setminus \{0\}$ and satisfying*

$$-\infty < |x|^2 V(x) \leqslant c \leqslant \frac{N^2}{4}, \quad \text{for every} \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{2.40}$$

and g is a nondecreasing real valued function. If $u \in C^1(\mathbb{R}^N \setminus \{0\})$ satisfies

$$\liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0, \tag{2.41}$$

$$\liminf_{r \to \infty} r^{(N-2-\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0; \tag{2.42}$$

then u is a radial function.

It is important to notice that the hypothesis on V is weaker as the proof essentially deals with the study of the following differential inequality

$$X_{rr} + \frac{N-1}{r} X_r + \left(V - \frac{N-1}{r^2}\right) X \geqslant 0. \tag{2.42}$$

Other symmetry results for Schrödinger operator with singular radial potentials can be found in [28], [5].

3. Solution with Linear Singularities

We assume that Ω and Ω^* are as in Section 1 and g is a continuous nondecreasing real valued function.

PROOF OF THEOREM 3.1. We recall that we assume $c < (N-2)^2/4$.

Step 1. Suppose (0.24) is satisfied, that is

$$\int_1^\infty (g(t) + |g(-t)|)t^{2(1-\alpha_1)/\alpha_1}\,dt < \infty, \tag{3.1}$$

then we claim that for any $\gamma \in \mathbb{R}$ there exists $u \in C^1(\Omega^*)$ satisfying

$$-\Delta u - \frac{c}{|x|^2}u + g(u) = 0 \tag{3.2}$$

in Ω^* and

$$\lim_{x\to 0} u(x)/\mu_1(x) = \gamma. \tag{3.3}$$

We take $\gamma \geqslant 0$ and for $\epsilon > 0$ let y_ϵ be the solution of

$$\begin{cases} s^2(y_\epsilon)_{ss} = \dfrac{1}{\beta^2}(s+\epsilon)^{(2-\alpha_1)/\beta}g((s+\epsilon)^{\alpha_1/\beta}y_\epsilon) & \text{in} \quad (0,1), \\ y_\epsilon(0) = \gamma, \qquad y_\epsilon(1) = 0. \end{cases} \tag{3.4}$$

In order to avoid technical difficulties we suppose $g(0) = 0$. Henceforth y_ϵ is positive, convex, nondecreasing and

$$y_\epsilon(s) \leqslant \gamma + s(1-\gamma), \quad \text{for all} \quad 0 < s < 1. \tag{3.5}$$

From (3.4) we get

$$(y_\epsilon)_s(s) = (y_\epsilon)_s(1) - \frac{1}{\beta^2}\int_s^1 (\tau+\epsilon)^{(2-\alpha_1)/\beta-2}g((\tau+\epsilon)^{\alpha_1/\beta}y_\epsilon)\,d\tau \tag{3.6}$$

for $0 < s \leqslant 1$, and $(y_\epsilon)_s(1)$ is bounded from (3.4)-(3.5). From (3.6) we deduce

$$|y_\epsilon(s_1) - y_\epsilon(s_2)| \leqslant a(s_2 - s_1) + \frac{1}{\beta^2}\int_{s_1}^{s_2}\int_s^1 (\tau+\epsilon)^{(2-\alpha_1)/\beta-2}g((\tau+\epsilon)^{\alpha_1/\beta}y_\epsilon)\,d\tau \tag{3.7}$$

for some constant $a > 0$ and $0 < s_1 < s_2 \leqslant 1$. But

$$(3.8)\quad \int_{s_1}^{s_2}\int_s^1 (\tau+\epsilon)^{(2-\alpha_1)/\beta-2} g((\tau+\epsilon)^{\alpha_1/\beta} y_\epsilon)\,d\tau\,ds$$

$$\leqslant \int_{s_1}^{s_2+\epsilon}\int_s^2 t^{(2-\alpha_1)/\beta-2} g(\gamma t^{\alpha_1/\beta})\,dt\,ds$$

(we assume $\epsilon < 1$). Set

$$\phi(x) = \int_x^2\int_s^2 t^{(2-\alpha_1)/\beta-2} g(\gamma t^{\alpha_1/\beta})\,dt\,ds,$$

then

$$(3.9)\qquad \lim_{x\to 0}\phi(x) = \int_0^2\int_s^2 t^{(2-\alpha_1)/\beta-2} g(\gamma t^{\alpha_1/\beta})\,dt\,ds$$

$$= l\int_{\gamma 2^{\alpha_1/\beta}}^{\infty} t^{2(1-\alpha_1)/\alpha_1} g(t)\,dt < \infty$$

from hypothesis ($l = l(\alpha_1, \beta) > 0$). Henceforth ϕ is extendable to $[0, 2]$ as a uniformly continuous function $\tilde{\phi}$ and (3.7) reads as

$$(3.10)\qquad |y_\epsilon(s_1) - y_\epsilon(s_2)| \leqslant a|s_2 - s_1| + \frac{1}{\beta^2}(\tilde{\phi}(s_2+\epsilon) - \tilde{\phi}(s_1+\epsilon))$$

which implies the equicontinuity of $\{y_\epsilon\}_{0<\epsilon<1}$ in $C([0, 1])$ and the existence of a $y \in C([0, 1])$ satisfying

$$(3.11)\qquad \begin{cases} s^2 y_{ss} = \dfrac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} y) & \text{in} \quad (0, 1], \\ y(0) = \gamma, \qquad y(1) = 0. \end{cases}$$

The function $u_\gamma(x) = |x|^{\alpha_1} y(|x|^\beta)$ is a solution of (3.2) satisfying (3.3).

Step 2. We assume that there exists $\gamma > 0$ such that (3.3) holds for some $u \in C^1(\Omega^*)$ and that

$$(3.11)\qquad \int_1^\infty t^{2(1-\alpha_1)/\alpha_1} g(t)\,dt = \infty.$$

As $\lim_{r\to 0} \bar{u}(r)/\mu_1 = \gamma$ and

$$(3.12)\qquad \bar{u}_{rr} + \frac{N-1}{r}\bar{u}_r + \frac{c}{r^2}\bar{u} = \overline{g(u(r))} \quad \text{in} \quad (0, 1]$$

we deduce, from the monotonicity of g, that

$$\bar{u}_{rr} + \frac{N-1}{r}\bar{u}_r + \frac{c}{r^2}\bar{u} \geqslant g(\gamma r^{\alpha_1}/2). \tag{3.13}$$

Defining $\psi(s)$ by $\bar{u}(r)/r^{\alpha_1}$ with $s = r^{\beta}$, then

$$s^2\psi_{ss} \geqslant \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(\gamma s^{\alpha_1/\beta}/2). \tag{3.14}$$

Integrating (3.14) twice yields

$$\psi(s) \geqslant \psi(1) + \psi_s(1)(s-1) + \frac{1}{\beta^2}\int_s^1\int_t^1 \sigma^{(2-\alpha_1)/\beta - 2} g(\gamma\sigma^{\alpha_1/\beta})\, d\sigma\, dt. \tag{3.15}$$

As

$$\lim_{s\downarrow 0}\int_s^1\int_t^1 \sigma^{(2-\alpha_1)/\beta - 2} g(\gamma\sigma^{\alpha_1/\beta})\, d\sigma\, dt = \infty$$

we derive

$$\lim_{s\to 0}\psi(s) = \lim_{r\to 0}\bar{u}(r)/r^{\alpha_1} = \infty, \tag{3.16}$$

a contradiction.

Remark 3.1. With the above techniques it is easy to show that if Ω is bounded with a regular boundary $\partial\Omega$, for any $\phi \in C(\partial\Omega)$ and any $\gamma \in \mathbb{R}$ there exists a unique $u_\gamma \in C(\bar{\Omega}\setminus\{0\}) \cap C^1(\Omega^*)$ satisfying

$$\begin{cases} -\Delta u_\gamma - \dfrac{c}{|x|^2} u_\gamma + g(u_\gamma) = 0 & \text{in } \Omega^*, \\ u_\gamma = \phi \quad \text{on} \quad \partial\Omega, & \lim_{x\to 0} u_\gamma(x)/\mu_1(x) = \gamma. \end{cases} \tag{3.17}$$

PROOF OF THEOREM 3.2. Here we assume that $c = (N-2)^2/4$. We recall the definition of b_g^+, b_g^-:

$$\begin{cases} b_g^+ = \inf\left\{b > 0 \colon \displaystyle\int_0^1 g(t^{-(N-2)/(N+2)}\operatorname{Ln}(1/t)/b)\, dt < \infty\right\}, \\ b_g^- = \inf\left\{b > 0 \colon \displaystyle\int_0^1 g(t^{-(N-2)/(N+2)}\operatorname{Ln} t/b)\, dt > -\infty\right\}. \end{cases}$$

Step 1. *Existence result.* It is clear that if $[-(N+2)/2b_g^-, (N+2)/2b_g^+] = \{0\}$ there exists u satisfying the equation (0.27) with a zero limit in (0.28); so we shall assume $b_g^+ < \infty$, and consider any $\gamma \in (0, (N+2)/2b_g^+]$.

Case 1. $\gamma < (N+2)/2b_g^+$. For $\epsilon > 0$ let y_ϵ be the solution of

$$(3.19)\quad \begin{cases} (y_\epsilon)_{ss} = (s+\epsilon)^{-3}\exp\left(-\dfrac{N+2}{2(s+\epsilon)}\right)g\left(\dfrac{y_\epsilon}{s+\epsilon}\right)\exp\left(\dfrac{N-2}{2(s+\epsilon)}\right) \text{ on } [0,1], \\ y_\epsilon(0) = \gamma, \qquad y_\epsilon(1) = 0. \end{cases}$$

We assume again that $g(0) = 0$; y_ϵ is decreasing, positive and convex; therefore

$$(3.20)\quad |y_\epsilon(s_1) - y_\epsilon(s_2)| \leqslant a|s_1 - s_2| + \int_{s_1+\epsilon}^{s_2+\epsilon}\int_s^2 t^{-3}e^{-(N+2)/2t}g(\gamma e^{-(N-2)/2t}t^{-1})\,dt\,ds,$$

for $0 < s_1 < s_2 < 1$ ($\epsilon < 1$). Let ϕ be defined by

$$(3.21)\qquad \phi(x) = \int_x^2\int_s t^{-3}e^{-(N+2)/2t}g(\gamma e^{-(N-2)/2t}t^{-1})\,dt\,ds,$$

then

$$(3.22)\quad \lim_{x\to 0}\phi(x) = l\int_0^{e^{-(N+2)/4}} g((2\gamma/(N+2))t^{-(N-2)/(N+2)}\operatorname{Ln}(1/t))\,dt < \infty.$$

As in the proof of Theorem 3.1, $\{y_\epsilon\}$ is equicontinuous in [0, 1] and there exists $y \in C([0,1])$ such that

$$(3.23)\qquad \begin{cases} y_{ss} = s^{-3}\exp(-(N+2)/2s)g(ye^{-(N-2)/2s}s^{-1}) \quad \text{on} \quad (0,1), \\ y(0) = \gamma, \qquad y(1) = 0. \end{cases}$$

If we set

$$u_\gamma(x) = |x|^{-(N-2)/2}\operatorname{Ln}(1/|x|)y(-1/\operatorname{Ln}|x|),$$

then

$$(3.24)\qquad \begin{cases} -\Delta u_\gamma - \left(\dfrac{N-2}{2|x|}\right)^2 u_\gamma + g(u_\gamma) = 0 \quad \text{in} \quad B_{e^{-1}}(0)\setminus\{0\}, \\ u_\gamma(x) = 0 \quad \text{on} \quad \partial B_{e^{-1}}(0), \\ \lim\limits_{x\to 0} u_\gamma(x)/\mu_1(x) = \gamma. \end{cases}$$

Case 2. $\gamma = (N+2)/2b_g^+$. Let y_n be the solution of

$$(3.25)\quad \begin{cases} (y_n)_{ss} = s^{-3}\exp(-(N+2)/2s)g(y_n e^{-(N-2)/2s}s^{-1}) & \text{on} \quad (0,1), \\ y_n(0) = (N+2)/(2b_g^+) - \dfrac{1}{n}, \quad y_n(1) = 0. \end{cases}$$

The function y_n is again decreasing, positive and convex and the sequence $\{y_n\}$ is increasing and bounded. From Dini's Theorem it is uniformly convergent on (0, 1] and its limit y is continuous on [0, 1] and satisfies (3.23) with $\gamma = (N+2)/2b_g^+$.

Step 2. Assume there exists γ and a solution u of (0.27) such that

$$\lim_{x\to 0} \frac{u(x)}{\mu_1(x)} = \gamma$$

and we assume for example that

$$(3.26)\quad \gamma > (N+2)/2b_g^+.$$

(we proceed similarly if $\gamma < -(N+2)/2b_g^-$). We define

$$(3.27)\quad u(r,\sigma) = \frac{1}{t}\exp((N-2)/2t)v(t,\sigma), \qquad t = -1/\operatorname{Ln} r,$$

and v satisfies

$$(3.28)\quad \begin{cases} v_{ss} + \Delta_{S^{N-1}}v = s^{-3}\exp(-(N+2)/2s)g(ve^{-(N-2)/2s}s^{-1}) & \text{in} \quad \Omega^*, \\ \lim\limits_{s\to 0} v(s,\bullet) = \gamma \quad \text{uniformly on} \quad S^{N-1}. \end{cases}$$

We consider $\epsilon_0 \in (0, \gamma - (N+2)/2b_g^+)$ and set $\lambda = \gamma - \epsilon_0 > (N+2)/2b_g^+$. For s small enough we have $v(s,\sigma) \geqslant \lambda$ (for every $\sigma \in S^{N-1}$) and it is the same with the spherical average $\bar{v}(s)$. Therefore

$$(3.29)\quad \bar{v}_{ss} \geqslant s^{-3}\exp(-(N+2)/2s)g(\lambda s^{-1}\exp((N-2)/2s)).$$

Integrating (3.29) twice as in step 1 and using the definition of b_g^+ implies

$$\lim_{s\to 0} \bar{v}(s) = +\infty,$$

a contradiction.

Remark 3.2. The Dirichlet problem is also solvable in the case $c = (N-2)^2/4$.

Remark 3.3. If $g(r)$ behaves like r^q $(q > 1)$ at infinity (and $-|r|^q$ at $-\infty$), (3.1) means that

$$0 < -\alpha_1 < \frac{2}{q-1} \quad \text{or} \quad 1 < q < \frac{N+2+\beta}{N-2+\beta} \tag{3.30}$$

and

$$c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right) < 0.$$

In the critical case $c = \left(\frac{N-2}{2}\right)^2$, the role of the Sobolev exponent $\frac{N+2}{N-2}$ is enlightened:

$$b_g^+, b_g^- = \begin{cases} \infty & \text{if} \quad q \geqslant (N+2)/(N-2), \\ 0 & \text{if} \quad 1 < q < (N+2)/(N-2). \end{cases} \tag{3.31}$$

Remark 3.4. Let u_γ be the solution on (3.17), for any γ if $c < (N-2)^2/4$ or if $c = (N-2)^2/4$ or if $c = (N-2)^2/4$ and $b_g^+ = 0$. Then the mapping $\gamma \mapsto u_\gamma$ is increasing. If we assume that

$$\int_A^\infty \frac{ds}{\sqrt{sg(s)}} < \infty \tag{3.32}$$

for some $A > 0$, u_γ is bounded above in $\bar{\Omega}\backslash\{0\}$ by a continuous function in $\bar{\Omega}\backslash\{0\}$. Then $u_\infty = \lim_{\gamma\to\infty} u_\gamma$ exists. In the case $g(r) = |r|^{q-1}r$ we shall prove in Section 4 that

$$\lim_{x\to 0} |x|^{2/(q-1)} u_\infty(x) = \left(c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)^{1/(q-1)} \tag{3.34}$$

Moreover u_∞ is the unique solution of (3.17) with $\gamma = \infty$ (see [31] for example).

4. The Power Case

In this Section we study the solutions of (0.3), that is

$$-\Delta u - \frac{c}{|x|^2} u + |u|^{q-1} u = 0 \tag{4.1}$$

in $B_1(0)\backslash\{0\}$ or in $\complement B_1(0)$ or in $\mathbb{R}^N\backslash\{0\}$. As some of the results are direct extensions of [13] and [7], we shall abreviate their proof.

PROOF OF THEOREM 4.1. It is clear from the classical energy method that in the case $q \neq (N+2)/(N-2)$, $r^{2/(q-1)}u(r, \bullet)$ converges in the $C^3(S^{N-1})$-topology to a compact connected subset of the set ξ of solutions of the following equation on S^{N-1}:

$$-\Delta_{S^{N-1}}\omega + |\omega|^{q-1}\omega = \left(c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)\omega, \tag{4.2}$$

(see [2], [7], [13], [26], [27]). Set

$$\lambda = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right),$$

then

(i) if $u \geqslant 0$, $\xi \cap C^+(S^{N-1})$ is reduced to 0 and $\lambda^{1/(q-1)}$ $(\lambda > 0)$,
(ii) if $0 < \lambda \leqslant N-1$, ξ is reduced to 0, $\lambda^{1/(q-1)}$ and $-\lambda^{1/(q-1)}$,
(iii) if q is an odd integer $r \mapsto r^q$ is a real analytic function and we can apply Bidaut-Veron-Veron and Simon's theorem [7], [8], [21].
(v) if ξ' is an hyperbolic limit manifold, that is for any $\omega \in \xi'$ and any $\psi \in C^2(S^{N-1})$ satisfying

$$-\Delta\psi + q|\omega|^{q-1}\psi - \left(c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)\right)\psi = 0 \tag{4.3}$$

there exists a one-parameter family $\{\omega_s\}_{0 \leq s \leq 1}$ of elements of ξ' such that

$$\lim_{s \to 0} s^{-1}(\omega_s - \omega) = \psi \tag{4.4}$$

in $C^2(S^{N-1})$. We can use Simon's result [21, Theorem 6.6]. Henceforth we are left with (v): $N = 2$, $c \leqslant 1$.

Lemma 4.1. *Assume A is an open subset strictly included into*

$$\bar{B}_1^+(0)\backslash\{0\} = \{(r, \theta) \in \mathbb{R}^2 : 0 < r \leqslant 1, 0 \leqslant \theta \leqslant r\}$$

and assume $c \leqslant 1$. Then

$$\lambda_1(A) = \inf\left\{\frac{1}{2}\int_A \left(|\nabla\phi|^2 - \frac{c}{|x|^2}f^2\right)dx : \phi \in W_0^{1,2}(A)\right\} > 0.$$

PROOF. Let $A = \{(t, \theta) : (e^t, \theta) \in A\}$ and $\psi(t, \theta) = \phi(r, \theta)$ with $r = e^t$. Then

$$\int_A \left(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2 - \frac{c}{r^2}\phi^2\right) r\,dr\,d\theta = \int_A (\psi_t^2 + \psi_\theta^2 - c\psi^2)\,dt\,d\theta.$$

As $A \subset (\operatorname{Ln} a, \operatorname{Ln} b) \times (0, \pi)$ for some $0 < a < b < 1$ and as the first eigenvalue of $-\Delta$ in $W_0^{1,2}((\operatorname{Ln} a, \operatorname{Ln} b) \times (0, \pi))$ is $\pi^2\left(\frac{1}{\pi^2} + \frac{1}{(\operatorname{Ln} b/a)^2}\right)$ we deduce by the monotonicity property that

$$\lambda_1(A) \geqslant 1 - c + \frac{\pi^2}{(\operatorname{Ln} b/a)^2} > 0,$$

which is the claim.

We define v by (0.30); as $N = 2$, it satisfies

$$v_{tt} - \frac{4}{q-1} v_t + v_{\theta\theta} + \left(c + \left(\frac{2}{q-1}\right)^2\right) v - |v|^{q-1} v = 0 \tag{4.5}$$

in $(-\infty, 0) \times S^1$. As in [13] we are left with the situation where the α-limit set of the negative trajectory of $v(t, \cdot)$ defined by

$$\Gamma^- = \bigcap_{t<0} \overline{\bigcup_{\tau \leq t} v(t, \cdot)}^{C^3(S^1)} \tag{4.6}$$

is included into one of the non trivial-S^1-action connected component of the set of solutions of

$$\omega_{\theta\theta} + \left(c + \left(\frac{2}{q-1}\right)^2\right)\omega - |\omega|^{q-1}\omega = 0 \quad \text{on} \quad S^1, \tag{4.7}$$

that is

$$\Gamma^- \subset \{\omega(\cdot + \alpha) : \alpha \in S^1\}, \tag{4.8}$$

where ω is a solution of (4.7) with anti-period π/k ($k \in \mathbb{N}^*$). The following result is then an extension of [13 Lemma 1.6].

Lemma 4.2. *Let ω be an element of Γ^-. If $\omega_\theta(\theta_0) > 0$ (resp. < 0) at some $\theta_0 \in S^1$, then there exists $t^* \leqslant 0$ such that*

$$v_\theta(t, \theta_0) \geqslant 0 \qquad (\textit{resp.} \leqslant 0) \tag{4.9}$$

for any $t \leqslant t^*$.

Proof. For proving it we may assume $\theta_0 = 0$ and define

$$\tilde{u}(r, \theta) = u(r, \theta) - u(r, -\theta); \tag{4.10}$$

then $\tilde{u}$ satisfies

$$-\Delta \tilde{u} - \frac{c}{|x|^2} \tilde{u} + d(x)\tilde{u} = 0 \tag{4.11}$$

in $B_1^+(0)\backslash\{0\}$ where $d(x) \geqslant 0$. Let us set

$$\theta^+ = \{x \in \bar{B}_1^+ \backslash \{0\} : \tilde{u}(x) > 0\}; \qquad \theta^- = \{x \in \bar{B}_1^- \backslash \{0\} : \tilde{u}(x) < 0\}. \tag{4.12}$$

If C is a connected component of θ^+ or θ^-, we claim that

$$0 \in \partial C \quad \text{or} \quad C \cap \partial B_1(0) \neq \varnothing. \tag{4.13}$$

Assume the contrary; if C is such a component, there exists a, b such that $0 < a < b < 1$ and

$$\bar{C} \subset \{(r, \theta) : a < r < b, \quad 0 \leqslant \theta \leqslant \pi\} = \Gamma_{a,b}^+. \tag{4.14}$$

Extending $\tilde{u}$ by 0 in $\Gamma_{a,b}^+\backslash C$ then the new function $\tilde{u}^e$ belongs to $W_0^{1,2}(\Gamma_{a,b}^+)$ and

$$\int_{\Gamma_{a,b}^+} \left(|\nabla \tilde{u}^e|^2 - \frac{c}{|x|^2} (\tilde{u}^e)^2 + d(x)(\tilde{u}^e)^2 \right) = 0. \tag{4.15}$$

Then $\tilde{u} = 0$ in C from Lemma 4.1, contradiction. The remaining of the proof of Lemma 4.2 goes as in [10 Lemma 1.6 (i)].

Remark 4.1. Using Lemma 4.2 and comparison principles implies that if $\omega_\theta(\theta) > 0$ for $\theta \in [\theta_0, \theta_1] \subset S^1$, then there exists $t^* \leqslant 0$ such that

$$v_\theta(t, \theta) \geqslant 0, \qquad \int_{\theta_0}^{\theta_1} v_\theta(t, \theta)\, d\theta > 0 \tag{4.16}$$

for any $t \leqslant t^*$, $\theta \in [\theta_0, \theta_1]$. However it is interesting to notice that the other assertions of [13, Lemma 1.6] do not hold for $0 < x \leqslant 1$ as they involve Neuman boundary data.

The remaining of the proof of Theorem 4.1 goes exactly as in [13, Theorem 1.1].

Remark 4.2. The potential $c|x|^{-2}$ of Theorem 4.1 can be replaced by a more general potential V such that $V \in C^{1+\epsilon}(\bar{B}_1(0)\backslash\{0\})$ and $r^2 V(r, \cdot)$ converges to c as r tends to 0 in the $C^{1+\epsilon}(S^{N-1})$-topology. In the case (iv) we have also to assume: either $c < 1$ or $|x|^2 V(x) \leqslant 1$ in some punctured neighborhood of 0.

It is clear that if

$$\lambda = c + \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right)$$

is nonpositive, the set ξ of the solutions of (4.2) is reduced to $\{0\}$ and from Theorem 1.2 u is described by μ_2. However, if $q > (N+2)/(N-2)$ and if $\lambda = 0$ we have

$$\alpha_1 < \alpha_2 = -2/(q-1). \tag{4.17}$$

The superposition of the linear and the nonlinear effect gives rise to the phenomena described in Theorem 4.2.

PROOF OF THEOREM 4.2. *Step* 1. *A priori estimate.* We claim that for any $\epsilon \in (0, 1)$ there exists $K_\epsilon > 0$ such that

$$|u(x)| \leqslant L(N, q)(|x|^2 \operatorname{Ln}(1/|x|))^{-1/(q-1)}(1 + K_\epsilon(\operatorname{Ln}(1/|x|))^{\epsilon - 1}) \tag{4.18}$$

where

$$L(N, q) = \left(\left(\frac{1}{q-1}\right)\left(N - 2\frac{q+1}{q-1}\right)\right)^{1/(q-1)}. \tag{4.19}$$

We use the function $v(t, \sigma)$ defined in (0.30) and v satisfies

$$v_{tt} + \left(N - 2\frac{q+1}{q-1}\right)v_t + \Delta_{S^{N-1}}v - v|v|^{q-1} = 0 \tag{4.20}$$

in $(-\infty, 0) \times S^{N-1}$ and $\lim_{t \to -\infty} v(t, \sigma) = 0$ uniformly on S^{N-1}.

Let $\psi(t) = L(N, q)(-t)^{-1/(q-1)} + M(-t)^{-\rho}$, $M, \rho > 0$, then

$$\begin{aligned} \psi_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\psi_t - \psi^q &= L(N, q)\frac{q}{(q-1)^2}(-t)^{-2-1/(q-1)} \\ &\quad + \rho(\rho+1)M(-t)^{-\rho-2} + L^q(N, q)(-t)^{-q/(q-1)} \\ &\quad + M\rho\left(N - 2\frac{q+1}{q-1}\right)(-t)^{-\rho-1} \\ &\quad - L^{q-1}(N, q)(-t)^{-q/(q-1)} - qML^{q-1}(N, q)t^{-\rho-1} \\ &\quad + o(t^{-\rho-1}). \end{aligned} \tag{4.21}$$

If we choose $\dfrac{1}{q-1} < \rho < \dfrac{q}{q-1}$ then $-2 - \dfrac{1}{q-1} < -\rho - 1$ and

$$M_\rho\left(N - 2\frac{q+1}{q-1}\right) < qML^{q-1}(N, q). \tag{4.22}$$

Henceforth, there exists $T < 0$ such that

$$\psi_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\psi_t - \psi^q \leqslant 0 \tag{4.23}$$

in $(-\infty, T) \times S^{N-1}$. Choosing M large enough we conclude that $v \leqslant \psi$. Arguing similarly for the negative part of v yields (4.18), (4.19).

Step 2. *End of the proof.* Let us define

$$\zeta(t, \sigma) = (-t)^{1/(q-1)}v(t, \sigma). \tag{4.24}$$

ζ is bounded in $(-\infty, -1] \times S^{N-1}$ where it satisfies

$$\zeta_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\zeta_t + \Delta_{S^{N-1}}\zeta - \frac{2}{t(q-1)}\zeta_t + \frac{q}{t^2(q-1)}\zeta + \frac{1}{t}(|\zeta|^{q-1}\zeta - L^{q-1}(N, q)\zeta) = 0. \tag{4.25}$$

From Agmon-Douglis-Niremberg [15] and Schauder theory all the derivatives $(\partial^\alpha/\partial t^\alpha)\nabla_\beta\zeta$ up to the order 3 are uniformly bounded in $(-\infty, -1] \times S^{N-1}$. Henceforth the α-limit set Γ^- of the trajectory of $\zeta(t, \cdot)$, $t \leqslant -1$, is a non-empty compact subset of $C^2(S^{N-1})$. Multiplying (4.25) by ζ_t and integrating over $(-\infty, -1] \times S^{N-1}$ yields

$$\int_{-\infty}^{-1}\int_{S^{N-1}} \zeta_t^2 d\sigma\, dt < \infty, \tag{4.26}$$

after some easy integrations by parts. This immediately implies

$$\int_{-\infty}^{-1}\int_{S^{N-1}} \zeta_{tt}^2 d\sigma\, dt < \infty.$$

The uniform continuity of ζ_t and ζ_{tt} yields

$$\lim_{t \to -\infty} \|\zeta_t(t, \cdot)\|_{L^2(S^{N-1})} = \lim_{t \to -\infty} \|\zeta_{tt}(t, \cdot)\|_{L^2(S^{N-1})} = 0. \tag{4.27}$$

If we multiply (4.25) by $\phi \in C^\infty(S^{N-1})$ and take some element $l \in \Gamma^-$ we get

$$\int_{S^{N-1}} l \Delta\phi \, d\sigma = 0. \tag{4.28}$$

Henforth Γ^- is reduced to a constant l which satisfies

$$l \in [-L(N,q), L(N,q)], \tag{4.29}$$

from (4.18). If we integrate (4.25) on $(t, -1) \times S^{N-1}$ we get

$$\int_t^{-1} \int_{S^{N-1}} \frac{1}{T}(L^{q-1}(N,q) - \zeta^{q-1})\zeta \, d\sigma \, dt = \Phi(t) \tag{4.30}$$

where $\Phi(t)$ admits a limit as t tends to $-\infty$. Henceforth it is the same with the left-hand side of (4.30) and l must satisfy

$$(L^{q-1}(N,q) - l^{q-1})l = 0 \tag{4.31}$$

which ends the proof.

Remark 4.3. A similar argument can be found in [2] for the study of the isolated singularities of the solutions of

$$-\Delta u = u^{N/(N-2)} \qquad (u > 0) \tag{4.32}$$

or in [27] for the study of the long range behaviour of the solutions of

$$-\Delta u + |u|^{2/(N-2)}u = 0 \tag{4.33}$$

in an exterior domain.

When

$$\lim_{x \to 0} |x|^{2/(q-1)}u(x) = 0$$

we are usually in the situation where the behaviour of u near 0 is essentially of linear type. If we look for solutions of

$$\Delta\zeta + \frac{c}{|x|^2}\zeta = 0 \tag{4.34}$$

in $B_1(0)\backslash\{0\}$ under the form

$$\zeta(r,\sigma) = y(r)\phi(\sigma), \tag{4.35}$$

we find out that ϕ must be an eigenfunction of $-\Delta_{S^{N-1}}$ with corresponding eigenvalue $\lambda_k = k(k+N-2)$ and y must satisfy

$$r^2 y_{rr} + r(N-1)y_r + (c-\lambda_k)y = 0 \tag{4.36}$$

($k \in \mathbb{N}$); the corresponding characteristic equation is

$$X^2 + (N-2)X + c - \lambda_k = 0, \tag{4.37}$$

with discriminant $\delta_k = (N-2+2k)^2 - 4c$. If $\delta_k \geqslant 0$ (4.36) admits two fundamental solutions with constant sign

$$\mu_1^k(x) = \begin{cases} |x|^{-(N-2+\beta_k)/2} & \text{if } \delta_k > 0, \\ |x|^{-(N-2)/2}\operatorname{Ln}(1/|x|) & \text{if } \delta_k = 0, \end{cases} \tag{4.38}$$

$$\mu_2^k(x) = \begin{cases} |x|^{-(N-2-\beta_k)/2} & \text{if } \delta_k > 0, \\ |x|^{-(N-2)/2} & \text{if } \delta_k = 0, \end{cases} \tag{4.39}$$

with $\beta_k = \sqrt{\delta_k}$. If $\delta_k < 0$ the space of solutions of (4.36) is generated by

$$\begin{cases} \nu_1^k(x) = |x|^{(2-N)/2}\cos\left(\sqrt{-\delta_k}\operatorname{Ln}\sqrt{r}\right), \\ \nu_2^k(x) = |x|^{(2-N)/2}\sin\left(\sqrt{-\delta_k}\operatorname{Ln}\sqrt{r}\right). \end{cases} \tag{4.40}$$

Surprisingly the case $q > (N+2)/(N-2)$ is simpler than the case $1 < q < (N+2)/(N-2)$.

Theorem 4.3. *Assume $q > (N+2)/(N-2)$ and that $-2/(q-1)$ is not a solution of* (4.37) *for some $k \in \mathbb{N}$. If u is a solution of* (4.1) *in $B_1(0)\setminus\{0\}$ such that*

$$\lim_{x\to 0} |x|^{2/(q-1)}u(x) = 0; \tag{4.41}$$

then we have the following alternative.

(i) *either there exists $l \in \mathbb{N}$ satisfying $\delta_l > 0$ and $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_l I)$, $\psi \neq 0$, such that*

$$\lim_{r\to 0} r^{(N-2-\beta_l)/2}u(r,\cdot) = \psi(\cdot) \tag{4.42}$$

in the $C^2(S^{N-1})$-topology,

(ii) *or $u \equiv 0$.*

PROOF. As $-2/(q-1)$ is not a root of (4.37) we can apply [13, Lemma 2.1]. Henceforth there exists $\epsilon < 0$ such that

(4.43) $$|u(x)| \leqslant M|x|^{-2/(q-1)+\epsilon}$$

near 0. Let k_0 be the smallest integer such that $\delta_{k_0} > 0$; then, as in [13], we derive the estimate

(4.44) $$|u(x)| \leqslant M|x|^{-(N-2-\beta_{k_0})/2}$$

near 0 and this estimate yields easily that there exists $\psi_0 \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_{k_0} I)$ such that

(4.45) $$\lim_{x \to 0} r^{(N-2-\beta_{k_0})/2} u(r, \sigma) = \psi_0 .$$

If $\psi_0 = 0$, then, as in [13], we obtain

(4.46) $$|u(x)| \leqslant M|x|^{-(N-2-\beta_{k_0+1})/2},$$

etc., and we carry on as above. If we assume that

(4.47) $$\lim_{x \to 0} |x|^{(N-2-\beta_k)/2} u(x) = 0$$

for any $k \in \mathbb{N}$, we conclude that $u \equiv 0$ from Aronszajn's unique continuation theorem [1].

If $1 < q < (N+2)/(N-2)$ we have $(N-2)/2 < 2/(q-1)$ and the properties of u will depend on the sign of $(N-2)^2 - 4c$.

Theorem 4.4. *Assume* $1 < q < (N+2)/(N-2)$, *that* $-2/(q-1)$ *is not a root of* (4.37) *for some* $k \in \mathbb{N}$ *and* $(N-2)^2 \geqslant 4c$. *If* u *is a solution of* (4.1) *in* $B_1(0)\setminus\{0\}$ *satisfying* (4.41); *then let* k_0 *be the largest integer such that*

(4.48) $$(N-2+\beta_{k_0})/2 < 2/(q-1);$$

(i) *either there exist an integer* $k \in [0, k_0]$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that*

(4.49) $$\lim_{r \to 0} u(r, \cdot)/\mu_1^k(r) = \psi(\cdot)$$

in the $C^2(S^{N-1})$*-topology,*

(ii) *or there exist an integer* $k \geqslant 0$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that* (4.49) *holds with* μ_2^k *instead of* μ_1^k,

(iii) *or* $u \equiv 0$.

The proof is the same as the one of Theorem 4.3.

Theorem 4.5. *Assume* $1 < q < (N+2)/(N-2)$, *that* $-2/(q-1)$ *is not a root of* (4.37) *for some* $k \in \mathbb{N}$ *and* $(N-2)^2 < 4c$. *If* u *is a solution of* (4.1) *in* $B_1(0)\setminus\{0\}$ *satisfying* (4.41), *let* $k_0 \geqslant 1$ *be the smallest integer such that* $\delta_{k_0} \geqslant 0$.

Case I. $2/(q-1) \geqslant (N-2+\beta_{k_0})/2$. *Let* $k_1 \geqslant k_0$ *be the largest integer such that*

$$(N-2+\beta_{k_1})/2 < 2/(q-1). \tag{4.50}$$

Then

(i) *either there exist an integer* $k \in [k_0, k_1]$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that* (4.49) *holds.*
(ii) *or there exists* k_0 *couples of functions* (ϕ_k, ψ_k) *both belonging to* $\operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *for* $k \in \mathbb{N} \cap [0, k_0-1]$, *one of the above functions at least being nonzero, such that*

$$\lim_{r\to 0}\Big\{r^{(N-2)/2}u(r,\cdot) - \sum_{k=0}^{k_0-1}\Big(\cos\big(\sqrt{-\delta_k}\,\mathrm{Ln}\sqrt{r}\,\big)\phi_k + \sin\big(\sqrt{-\delta_k}\,\mathrm{Ln}\sqrt{r}\,\big)\psi_k\Big)\Big\} = 0 \tag{4.51}$$

in the $C^2(S^{N-1})$*-topology,*
(iii) *or there exist an integer* $k \geqslant k_0$ *and a nonzero* $\psi \in \operatorname{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ *such that* (4.49) *holds with* μ_2^k *instead of* μ_1^k,
(iv) *or* $u \equiv 0$.

Case II. $(N-2-\beta_{k_0})/2 > 2/(q-1)$. *Only the parts* (ii), (iii) *and* (iv) *of Case I hold.*

PROOF. As in the proof of Theorem 4.2 the (i) of Case I is clear as $-2/(q-1)$ is not a root of (4.37). Henceforth we may assume that

$$|u(x)| \leqslant M|x|^{(2-N)/2} \tag{4.52}$$

and define

$$w(t,\sigma) = r^{(N-2)/2}u(r,\sigma), \qquad t = \mathrm{Ln}\, r. \tag{4.53}$$

Therefore w satisfies

$$w_{tt} + \Delta_{S^{N-1}}w + \left(c - \frac{(N-2)^2}{4}\right)w + e^{(q-1)(N-2)t/2}w|w|^{q-1} = 0 \tag{4.54}$$

in $(-\infty, 0] \times S^{N-1}$ where it stays bounded. Let w^k be the projection of w onto $\mathrm{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$ for $0 \leqslant k \leqslant k_0 - 1$. Then w_k satisfies

$$w_{tt}^k + \left(c - \frac{(N-2-2k)^2}{4}\right)w^k + e^{(q-1)(N-2)t/2} f_k = 0 \tag{4.55}$$

where f_k is bounded, and it is easy to check that

$$\lim_{t\to\infty} \left(w^k(t) - \cos\left(\sqrt{-\delta_k}\, t/2\right)\phi_k - \sin\left(\sqrt{-\delta_k}\, t/2\right)\psi_k\right) = 0 \tag{4.56}$$

for some ϕ_k, ψ_k in $\mathrm{Ker}(\Delta_{S^{N-1}} + \lambda_k I)$. As the α-limit set of the trajectory of $(w(t, \bullet))_{t\leq 0}$ is included into the direct sum of the $\mathrm{Ker}(\Delta_{S^{n-1}} + \lambda_k I)$ for $k = 0, \ldots, k_0 - 1$, we get (ii). The remaining of the proof is as in Theorem 4.2.

Remark 4.4. If $N = 2$ and $c \leqslant 0$, Theorem 4.4 holds for any $q > 1$.

Similar types of results (with some times many cases to examine) hold for the exterior problem. We just give the basic ones corresponding to Theorem 4.1-4.2.

Theorem 4.6. *Assume* $q \in (1, \infty)\setminus\{(N+2)/(N-2)\}$ *and* u *is a solution of* (4.1) *in* $G \supset \{x: |x| \geqslant 1\}$. *Then* $r^{2/(q-1)}u(r, \bullet)$ *converges in the* $C^3(S^{N-1})$-*topology to some compact connected subset of the set* ξ *of the* $C^3(S^{N-1})$-*functions satisfying* (4.2). *Moreover there exists precisely one* $\omega \in \xi$ *such that*

$$\lim_{r\to 0} r^{2/(q-1)}u'r, \bullet) = \omega(\bullet), \tag{4.57}$$

at least if one of the conditions (i)-(v) *of Theorem* 4.1 *is fulfilled.*

Theorem 4.7. *Assume* $1 < q < (N+2)/(N-2)$, $0 < c \leqslant (N-2)/2)^2$ *and* $\lambda = 0$. *If* u *is any solution of* (4.1) *in* $G \supset \{x: |x| \geqslant 1\}$, *then the following limit exists*

$$\lim_{|x|\to\infty} u(x)/(\mu_1(x)(\mathrm{Ln}\,|x|)^{\alpha_1/2}) = \tilde{l} \tag{4.58}$$

with

$$\alpha_1 = \frac{2 - N + \sqrt{(N-2)^2 - 4c}}{2} = -\frac{2}{q-1}$$

and

$$\tilde{l} \in \{0, \pm((2(q+1) - N(q-1))/(q-1)^2)^{1/(q-1)}\}. \tag{4.59}$$

From Theorems 4.1 and 4.6 we know that a global solution of (4.1) in $\mathbb{R}^N\setminus\{0\}$ satisfies

$$\lim_{r\to 0} r^{2/(q-1)}u(r,\cdot)\in\xi_-,\qquad \lim_{r\to\infty} r^{2/(q-1)}u(r,\cdot)\in\xi_+ \tag{4.60}$$

where ξ_- and ξ_+ are two compact connected subsets of ξ. If we define

$$E(\eta)=\int_{S^{N-1}}\left(\frac{1}{2}|\nabla\eta|^2+\frac{1}{q+1}|\eta|^{q+1}-\frac{\lambda}{2}\eta^2\right)d\sigma, \tag{4.61}$$

then $E_{|\xi_-}=E_-$, $E_{|\xi_+}=E_+$ and

$$\left(N-2\frac{q+1}{q-1}\right)\int_{-\infty}^{\infty}\int_{S^{N-1}}v_t^2\,d\sigma\,\delta\tau=E_+-E_-, \tag{4.62}$$

where we have used the notations of (0.30). This relation tells us what are the set of elements of ξ for which a connecting orbit may exist. The way to constructing connecting orbits is to go through a semiflow as in [13] and to constructing such a semiflow we need an existence and uniqueness result for some initial boundary value problem.

Theorem 4.8. *Assume $1<q$. Then for any $\phi\in C(\partial B_1(o))$ there exists a unique $u\in C(\complement B_1(0))\cap C^3(\complement\bar B_1(0))$ satisfying*

$$-\Delta u-\frac{c}{|x|^2}u+u|u|^{q-1}=0 \tag{4.63}$$

in $\complement(\bar B_1(0))$ and $u=\phi$ on $\partial B_1(0)$ if one of the two following conditions is fulfilled

(I) *$\lambda>0$, u and ϕ are nonnegative, and either $1<q\leqslant(N+2)/(N-2)$, or $q>(N+2)/(N-2)$ and $c>(N-2)^2/4$,*
(II) *either $c\leqslant 0$, or $0<c\leqslant(N-2)^2/4$ and $1<q\leqslant(N+2)/(N-2)$.*

PROOF. *Case I-Step* 1. *Uniqueness.* If $\lambda>0$ and $u\geqslant 0$, we know, from Theorem 4.1 if $q\neq(N+2)/(N-2)$ or from [17] when $q=(N+2)/(N-2)$, that

$$\lim_{r\to\infty} r^{2/(q-1)}u(r,\cdot)=L\in\{0,\lambda^{1/(q-1)}\}. \tag{4.64}$$

If $\phi\neq 0$, $u>0$ in $\bar B_1(0)$ from the strong maximum principle. If $L=0$ we get

$$\bar v_{tt}+\left(N-2\frac{q+1}{q-1}\right)\bar v_t+(\lambda-\epsilon(t))\bar v=0, \tag{4.65}$$

where $\epsilon(t)$ is a positive function tending to 0 at infinity. Let $\tilde{\delta}$ be the discriminant of the equation with constant coefficients associated to (4.65)

$$\tilde{\delta} = \left(N - 2\frac{q+1}{q-1}\right)^2 - 4\lambda = (N-2)^2 - 4c. \tag{4.66}$$

If $1 < q < (N+2)/(N-2)$, 0 is a source and $\bar{v}$ cannot tend to 0 except if it is identically 0; if $q \geqslant (N+2)/(N-2)$, then $\tilde{\delta} < 0$ and any solution of (4.65) tending to 0 at infinity must oscillate around 0, a contradiction. Henceforth $L = \lambda^{1/(q-1)}$. If we linearize (0.31) at $\lambda^{1/(q-1)}$ we obtain the following equation

$$\phi_{tt} + \left(N - 2\frac{q+1}{q-1}\right)\phi_t + \Delta_{S^{N-1}}\phi - (q-1)\lambda\phi = 0. \tag{4.67}$$

As this equation satisfies the maximum principle we deduce that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|v(t, \cdot) - \lambda^{1/(q-1)}\|_{L^\infty(S^{N-1})} \leqslant C_\epsilon e^{-(\tau-\epsilon)t}, \tag{4.68}$$

where

$$\tau = \frac{1}{2}\left\{\left(N - 2\frac{q+1}{q-1}\right) + \sqrt{\left(N - 2\frac{q+1}{q-1}\right)^2 + 4\lambda(q-1)}\right\}, \tag{4.69}$$

which implies

$$\|u(r, \cdot) - \lambda^{1/(q-1)}r^{-2/(q-1)}\|_{L^\infty(S^{N-1})} \leqslant C_\epsilon r^{-2/(q-1)-\tau+\epsilon}. \tag{4.70}$$

Assume now u and $\hat{u}$ are two solutions of (4.63) with same initial data $\phi \geqslant 0$, u and $\hat{u} \geqslant 0$. Then for any $R > 1$ we have

$$-\int_{B_R(0)}\left(\frac{\Delta\hat{u}}{u} - \frac{\Delta\hat{u}}{\hat{u}}\right)(u^2 - \hat{u}^2) + \int_{B_R(0)}(|u|^{q-1} - |\hat{u}|^{q-1})(u^2 - v^2) = 0, \tag{4.71}$$

and

$$\int_{B_R(0)}\left(\frac{\Delta u}{u} - \frac{\Delta\hat{u}}{\hat{u}}\right)(u^2 - \hat{u}^2) = -\int_{\partial B_R(0)}\left(\frac{u_\nu}{u} - \frac{\hat{u}_\nu}{\hat{u}}\right)(u^2 - \hat{u}^2)$$
$$+ \int_{B_R(0)}\left(\left|\nabla u - \frac{u}{\hat{u}}\nabla\hat{u}\right|^2 + \left|\nabla\hat{u} - \frac{\hat{u}}{u}\nabla u\right|^2\right).$$

But from (4.70) we have

$$\int_{\partial B_R(0)}\left(\frac{u_\nu}{u} - \frac{\hat{u}_\nu}{\hat{u}}\right)(u^2 - \hat{u}^2) = O(R^{(N-2-4/(q-1)-\tau+\epsilon)}) \tag{4.72}$$

and

$$N-2-\frac{4}{q-1}-\tau=\frac{1}{2}\left\{N-2\frac{q+1}{q-1}-\sqrt{\left(N-2\frac{q+1}{q-1}\right)^2+4\lambda(q-1)}\right\}$$

which is negative. For ϵ small enough we deduce

$$(4.73)\quad \int_{B_R(0)}\left(\left|\nabla u-\frac{u}{\hat u}\nabla\hat u\right|^2+\left|\nabla\hat u-\frac{\hat u}{u}\nabla u\right|^2\right) + \int_{B_R(0)}(|u|^{q-1}-|\hat u|^{q-1})(u^2-v^2)=0,$$

which implies the uniqueness.

Step 2. *Existence.* We consider the following iterative scheme

$$(4.74)\quad \begin{cases} -\Delta u_n+u_n^q=\dfrac{c}{|x|^2}u_{n-1} & \text{in} \quad \complement\bar B_1(0) \qquad (n\geqslant 1),\\ u_n=\phi & \text{on} \quad \partial B_1(0)\\ u_0=0. \end{cases}$$

$\{u_n\}$ is increasing. For $\Lambda>\lambda^{1/(q-1)}$ the function $\psi_\Lambda(r)=\Lambda r^{-2/(q-1)}$ satisfies

$$(4.75)\quad -\Delta\psi_\Lambda+\psi_\Lambda^q\geqslant\frac{c}{|x|^2}\psi_\Lambda\,.$$

If we choose $\Lambda\geqslant\|\phi\|_{L^\infty(\partial B_1(0))}$ we deduce $\psi_\Lambda\geqslant u_1$ and finally $0<u_1<u_2<\cdots<u_n<\psi_\Lambda$. Clearly u_n converges to a solution u of (4.63) with initial value ϕ.

Case II. Step 1. *Uniqueness.* Let u and $\hat u$ be two solutions, $w=u-\hat u$, $v(s,\sigma)=w(r,\sigma)/\mu_1(r)$, $s=r^\beta$ (we assume $c<(N-2)^2/4$, the case $c=(N-2)^2/4$ is treated by the same technique, see Lemma 1.4); then

$$(4.76)\quad s^2(\|w(s,\bullet)\|_{L^2(S^{N-1})})_s\geqslant 0$$

in $D'(0,+\infty)$. As $\|w(s,\bullet)\|_{L^2(S^{N-1})}=o(s)$ at infinity, $w\equiv 0$.

Step 2. Existence. We approximate u by the solution of the following problem in $B_n(0)\backslash B_1(0)$

$$(4.77)\quad \begin{cases} -\Delta u_n-\dfrac{c}{|x|^2}u_n+|u_n|^{q-1}u_n=0 & \text{in} \quad B_n(0)\backslash B_1(0), \quad n\geqslant 2,\\ u_n=\phi & \text{on} \quad \partial B_1(0),\\ u_n=0 & \text{on} \quad \partial B_n(0). \end{cases}$$

u_n is unique (see Step 1), uniformly bounded, therefore it is convergent to the desired u.

Remark 4.5. Using a phase plane analysis for the radial solutions of (0.31) we can see that Theorem 4.8 is optimal. It is also of some interest to notice that if $\lambda \leqslant 0$ there exists no positive solution of (4.63) in $\complement B_1(0)$: if u were such a solution, then $\lim_{r\to\infty} r^{2/(q-1)}u(r, \cdot) = 0$ from Theorem 4.1 if $q < (N+2)/(N-2)$ and [17] if $q = (N+2)/(N-2)$ and then

$$(4.78) \qquad \left(N - 2\frac{q+1}{q-1}\right)\int_0^\infty \int_{S^{N-1}} v_t^2 d\sigma\, dt = -E(\phi) + \frac{1}{2}\int_{S^{N-1}} v_t^2(0, \cdot)\, d\sigma$$

where we use the notations of (0.30) and (4.61), and $u(x) = \phi$ on $\partial B_1(0)$. If $q = (N+2)/(N-2)$ we deduce $E(\phi) = 0 \Rightarrow \phi = 0$. But we can replace $\phi = v(0, \cdot)$ by $v(T, \cdot)$ for any $T \geqslant 0$.

Remark 4.6. Thanks to Theorem 4.8 we can define a semiflow Φ on $X = C^+(S^{N-1})$ in Case I or on $C(S^{N-1})$ in Case II by the formula

$$(4.79) \qquad \Phi_t(\phi)(\cdot) = u(t, \cdot) \qquad (t \geqslant 0)$$

if u satisfies (4.63) in $\complement \bar{B}_1(0)$ and $u = \phi$ on $\partial B_1(0)$. Clearly Φ satisfies

(i) $\Phi_0 = I$,
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$,
(iii) $(t, \phi) \mapsto \Phi_t(\phi)$ is continuous in (t, ϕ).

The proof of those assertions is the same as the one of [13, Proposition 3.2]. Moreover Φ is strongly order preserving in the sense that being given ϕ_1 and ϕ_2 on X, $\phi_1 \geqslant \phi_2$, $\phi_1 \neq \phi_2$, then for any $t > 0$ there exists $\delta > 0$ such that for any η_1, $\eta_2 \in X$, satisfying

$$\|\phi_1 - \eta_1\|_{C^0(S^{N-1})} \leqslant \delta,$$
$$\|\phi_2 - \eta_2\|_{C^0(S^{N-1})} \leqslant \delta$$

we have

$$(4.80) \qquad \begin{aligned} &\Phi_t(\eta_1) \geqslant \Phi_t(\eta_2), \\ &\Phi_t(\eta_1) \neq \Phi_t(\eta_2). \end{aligned}$$

Finally, if B is a bounded subset of X and $t > 0$, $\Phi_t(B)$ is relatively compact in X. Those results are what we need to apply Matano's Theorem concerning heteroclinic orbits of Φ connecting two equilibria ω_1 and ω_2 such that $[\omega_1, \omega_2]$ contains no other equilibria than ω_1 and ω_2 [18].

Remark 4.7. In order to apply Matano's method we need to know what is the structure of the set ξ of the solutions of

$$-\Delta_{S^{N-1}}\omega + \omega|\omega|^{q-1} = \lambda\omega \tag{4.81}$$

on S^{N-1}. The complete structure is far out of reach, but using the geometric technique we have introduced in [30] one can describe some of the solutions of (4.81) associted to a tesselation of S^{N-1}. We first recall that if G is a subgroup of $O(N)$ generated by reflections through hyperplanes containing 0 and if G is finite, then G contains a finite number of reflections through hyperplanes $(H_k)_{k\in K}$ containing 0 and those hyperplanes divide $\mathbb{R}^N$ into a finite number of angular polyhedra $(P_i)_{i\in I}$, each of them being limited by at most N faces [12], [9]. Moreover those polyedra are all equal and G acts transitively on them. The intersections of those angular polyedra with S^{N-1} are spherical simplexes $(S_i)_{i\in I}$ on which G also acts transitively. Henceforth we can consider only model simplex S as *a fundamental domain for G*. The complete description of those finite groups generated by reflections can be found in [9] but on $\mathbb{R}^3$ there exists only five types of subgroups: type I is generated by reflections through two hyperplanes with angle π/n; type II is generated by the reflections through two hyperplanes with angle π/n and a reflection through an hyperplane orthogonal to them; type III, IV and V are associated to Plato's polyedra [14] and have respectively 24, 48 and 120 elements. In order to construct a solution of (4.81) with a high degree of complexity we consider a finite subgroup of reflections G with fundamental simplicial domain S on S^{N-1} and we call $\lambda(S)$ the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(S)$. It is clear that $\lambda(S)$ is an eigenvalue of $\Delta_{S^{N-1}}$ on S^{N-1}. If $\lambda > \lambda(S)$ we call ω_S the unique positive solution of the following equation on S

$$-\Delta_{S^{N-1}}\omega_S + \omega_S^q = \lambda\omega_S, \tag{4.82}$$

ω_S vanishing on ∂S (ω_S is a minimizer). We then extend ω_S by reflection to whole S^{N-1} according the formula

$$\omega_{G|S_i} = \det(g_i)\,\omega_S \circ g_i^{-1} \tag{4.83}$$

if $S_i = g_i(S)$ for some $g_i \in G$. As the vertices have codimension 2 in S^{N-1} and ω_G is bounded, ω_G belongs to ξ [20]. For $\lambda > 0$ let ξ^* be the subset of ξ of solutions of (4.81) containing the three constants and the solutions which are of type ω_G for some finite subgroup of reflections G (as ω_G is constructed, $\omega_G \circ \tau$, for any $\tau \in O(N)$, is of the same type). If ω_G and $\omega_{G'}$ are two non-constant elements of ξ^* associated to G and G' with fundamental simplicial domains S and S' and if S is a disjoint union of a finite number k of $g_j'(S')$, $1 \leqslant j \leqslant k$, $g_j' \in G'$ we shall say that *the frequency of* $\omega_{G'}$ *is a multiple of the frequency of* ω_G. As $\omega_{G|S}$ is energy minimizing we clearly have

$$(4.84)\quad \int_S \left(\frac{1}{2}|\nabla\omega_G|^2 + \frac{1}{q+1}|\omega_G|^{q+1} - \frac{\lambda}{2}\omega_G^2\right) < \int_S \left(\frac{1}{2}|\nabla\omega_{G'}|^2 + \frac{1}{q-1}|\omega_{G'}|^{q+1} - \frac{\lambda}{2}\omega_{G'}^2\right)$$

which clearly implies

$$(4.85)\qquad E(\omega_G) < E(\omega_{G'}) < 0.$$

Using also the techniques of Theorem 4.8, for example the increasing iterative scheme

$$(4.86)\qquad \begin{cases} -\Delta_{S^{N-1}}\omega_n + \omega_n^q = \lambda\omega_{n-1} & \text{in } S \\ \omega_n = 0 & \text{on } \partial S \end{cases}$$

with $\omega_0 = \omega^+_{G'|S'}$ $(S' \subset S)$, we deduce

$$(4.87)\qquad \omega_G > \omega_{G'} \quad \text{in } S.$$

The following result the proof of which is an extension of Theorem 4.8 will be useful in the sequel for constructing a semiflow.

Theorem 4.9. *Assume Ω is an open subset of S^{N-1}, K_Ω is the piece of cone defined by*

$$(4.88)\qquad K_\Omega = \{\tau\sigma\colon \tau > 1, \sigma \in \Omega\},$$

$\lambda(\Omega)$ is the first eigenvalue of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(\Omega)$ and $\tilde{\partial}K_\Omega$ is the lateral boundary of K_Ω; let q be bigger than 1. Then for any $\phi \in C_0(\Omega)$ there exists a unique $u \in C(\bar{K}_\Omega) \cap C^3(K_\Omega)$ satisfying

$$(4.89)\qquad -\Delta u - \frac{c}{|x|^2}u + u|u|^{q-1} = 0$$

in K_Ω, $u = 0$ on $\tilde{\partial}K_\Omega$, $u = \phi$ on Ω if one of the following two conditions is fulfilled

(I) $\lambda > \lambda_1(\Omega)$, *u and ϕ are nonnegative, and either* $1 < q \leqslant (N+2)/(N-2)$ *or* $q > (N+2)/(N-2)$ *and* $c > \lambda_1(\Omega) + (N-2)^2/4$,

(II) *either* $c \leqslant \lambda_1(\Omega)$, *or* $\lambda_1(\Omega) < c \leqslant \lambda_1(\Omega) + (N-2)^2/4$ *and* $1 < q \leqslant (N+2)/(N-2)$.

With this result we can define a semiflow Φ^G on $X = C_0^+(\Omega)$ in Case I or on $C_0(\Omega)$ in Case II and, if $\partial\Omega$ is Lipschitz, Φ^G is a strongly order preserving semiflow on X mapping bounded subsets of X into relatively compact subsets of X for $t > 0$. We are now able to study the solutions u of (4.63) in $\mathbb{R}^N \setminus \{0\}$ such that

$$(4.90) \qquad \begin{cases} \lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) = \omega_1, & \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) = \omega_2 \quad \text{or} \\ \lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) = \omega_2, & \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) = \omega_1 \end{cases}$$

and by extension of the notations of Remark 4.7 we shall say that the two constants $\pm\lambda^{1/(q-1)}$ are of type ω_G with $G = \{I_d\}$ and $S = S^{N-1}$, $\lambda(S) = 0$.

Theorem 4.10. *Assume $q > 1$, ω_1 and $\omega_2 \in \xi^*$. Then there exists a solution u of* (4.63) *in* $\mathbb{R}^N \setminus \{0\}$ *satisfying* (4.90) *if*

(*A*) $\omega_1 = 0$, $\omega_2 = \omega_G$ *for some G and one of the following two conditions is fulfilled*

(i) $1 < q \leqslant (N+2)/(N-2)$,
(ii) $q > (N+2)/(N-2)$ *and* $c > \lambda(S) + (N-2)^2/4$;

(*B*) $\omega_1 = \lambda^{1/(q-1)}$, $\omega_2 = \omega_G$ *for some non trivial G and either $c \leqslant 0$, or $0 < c \leqslant (N-2)^2/4$ and $1 < q \leqslant (N+2)/(N-2)$*;

(*C*) $\omega_1 = \omega_G$, $\omega_2 = \omega_{G'}$ *the frequency of $\omega_{G'}$ is a multiple of the frequency of ω_G and either $c \leqslant \lambda(S)$ or $\lambda(S) < c \leqslant \lambda(S) + (N-2)^2/4$ and $1 < q \leqslant (N+2)/(N-2)$*.

The proof of the Theorem is essentially a consequence of the construction of Remark 4.7 and of Theorem 4.8 for B and C and Theorem 4.9 applied in K_S for A; in that last case the positive solution constructed in the cone with basis 0 and vertex 0 is extended by reflection to be a solution of (4.63) in $\mathbb{R}^N \setminus \{0\}$. It must also be noticed that the case A, B and C imply $\lambda > \lambda(S)$, $\lambda > \lambda(S)$ and $\lambda > \lambda(S') > \lambda(S)$ respectively.

Remark 4.8. The complete set of the critical values of E is not known, in particular is it true that all the connected components of ξ have different energy value (the energy is constant on each connected component from Sard's theorem)? Such an exclusion principle is valid on ξ^*.

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Ondelettes sur l'intervalle

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1. Introduction

En 1873 Dubois-Reymond construisit une fonction continue de la variable réelle x et 2π-périodique dont la série de Fourier diverge en un point donné. Ce contre-exemple amena A. Haar à se poser, puis à résoudre le problème de l'existence d'une base orthonormée $h_0(x), h_1(x), \ldots, h_m(x), \ldots$ de $L^2[0,1]$ ayant la propriété que, pour toute fonction continue $f(x)$, la série $\sum_0^\infty \langle f, h_m \rangle h_m(x)$ converge uniformément vers $f(x)$. A Haar choisit $h_0(x) = 1$, $h_1(x) = h(x)$ où $h(x) = 1$ sur $[0, 1/2[$, $h(x) = -1$ sur $[1/2, 1[$ et $h(x) = 0$ ailleurs. Enfin, pour $m = 2^j + k$, $0 \leqslant k < 2^j$, $j \geqslant 0$, il pose $h_m(x) = 2^{j/2} h(2^j x - k)$.

Les sommes partielles $\langle f, h_0 \rangle h_0(x) + \cdots + \langle f, h_m \rangle h_m(x) = S_m(f)(x)$ sont des approximations de $f(x)$ par des fonctions en escalier. Mais si, pour un certain exposant $s \in]0, 1[$, $f(x)$ vérifiait en outre,

$$f(x+h) - f(x) = o(h^s) \tag{1.1}$$

uniformément en $x \in \mathbb{R}$, les sommes partielles $S_m(f)$ ne pourraient converger vers $f(x)$ pour la norme de l'espace de Banach C_0^s défini par (1.1).

Peut-on modifier la construction du système de Haar et obtenir une base orthonomée de $L^2[0,1]$ convenant à l'analyse et à la synthèse des espaces de Hölder C^s définis par (1.1)?

Ce problème a été étudié depuis le travail de pionnier de Haar. G. Faber puis J. Schauder ont commencé par remplacer les fonctions $h_m(x)$ du système de Haar par leurs primitives $\Delta_m(x)$. En changeant la normalisation, il vient $\Delta_m(x) = \Delta(2^j x - k)$ où $m = 2^j + k$, $0 \leqslant k < 2^j$, et où $\Delta(x) = 2x$ si $0 \leqslant x \leqslant 1/2$, $2 - 2x$ si $1/2 \leqslant x \leqslant 1$ et 0 hors de l'intervalle $[0, 1]$.

L'approximation d'une fonction continue sur [0, 1] par les sommes partielles de la série $a + bx + \sum_1^\infty \alpha_m \Delta_m(x)$ revient à approcher le graphe de $f(x)$ par des lignes polygonales inscrites et constitue donc une amélioration de l'approximation à l'aide des fonctions en escalier. Si $f(x)$ appartient à $C_0^s[0, 1]$, on a $\alpha_m = o(m^{-s})$ et réciproquement, si cette condition est vérifiée, la série $a + bx + \sum_1^\infty \alpha_m \Delta_m(x)$ converge vers $f(x)$ en norme $C^s[0, 1]$. En ce sens, le système de Schauder $\Delta_m(x)$, $m \geqslant 1$ (complété par 1 et x) est une base *inconditionnelle* de l'espace $C_0^s[0, 1]$.

En revanche, le système de Schauder ne peut plus servir à l'analyse de l'espace $L^2[0, 1]$. Une façon de le voir est d'observer que les coefficients α_m se calculent par

$$\alpha_m = f\left(\left(\frac{k+1}{2}\right)2^{-j}\right) - \frac{1}{2}[f(k2^{-j}) + f((k+1)2^{-j})] \tag{1.2}$$

et que cette formule n'a plus aucun sens si $f(x)$ appartient à $L^2[0, 1]$.

Pour corriger ce défaut de la base de Schauder, Ph. Franklin a eu l'idée d'orthogonormaliser la suite $1, x, \Delta_1(x), \ldots, \Delta_m(x), \ldots$ en utilisant le procédé de Gram-Schmidt. On obtient alors une suite $f_m(x)$, $m \geqslant -1$, où $f_{-1}(x) = 1$, $f_0(x) = 2\sqrt{3}(x - 1/2)$, etc... Le système de Franklin est un peu tombé dans l'oubli parce que les fonctions $f_m(x)$ ne sont pas fournies par un algorithme aussi simple que celui des fonctions $h_m(x)$ du système de Haar. Cependant en 1963, Ciesielski démontra que tout se passe comme si l'on avait $f_m(x) = 2^{j/2}\psi(2^j x - k)$ lorsque $m = 2^j + k$. Il prouve en effet que $|f_m(x)| \leqslant C2^{j/2} \exp(-\gamma|2^j - k|)$ pour un certain exposant $\gamma > 0$. Cette estimation, jointe à

$$\int_0^1 f_m(x)\, dx = \int_0^1 x f_m(x)\, dx = 0,$$

fournit la caractérisation attendue des espaces $C^s[0, 1]$ par $\alpha_m = O(m^{-1/2-s})$. La différence avec la base de Schauder vient de ce que les normalisations sont différentes.

Nous nous proposons de corriger le défaut du système de Franklin. Plus précisément, pour tout entier $N \geqslant 1$, nous allons construire une base orthonormée $f_m^{(N)}$ de $L^2[0, 1]$ ayant une structure algorithmique aussi simple que celle du système de Haar. En outre, pour une constante absolue $\gamma > 0$ qui est estimée dans [1], les espaces $C^s[0, 1]$ seront caractérisés par $\alpha_m = O(m^{-1/2-s})$ lorsque $0 < s < \gamma(N - 1)$.

Pour l'essentiel notre nouvelle base a exactement la structure du système de Haar puisqu'elle contient toutes les fonctions $2^{j/2}\psi(2^j x - k)$, $j \geqslant 0$, $k \geqslant 0$, dont le support est inclus dans [0, 1]. Ici $\psi(x)$ désigne «l'ondelette de Daubechies» dont le support est l'intervalle $[0, 2N - 1]$.

Malheureusement les fonctions précédentes engendrent un sous-espace de $L^2[0, 1]$ de codimension infinie. Ceci est dû aux *effets de bord* produits par 0 et

1. Pour tenir compte de ces effets de bord, il nous faudra adjoindre aux fonctions précédentes (les $2^{j/2}\psi(2^jx-k)$) les fonctions $2^{j/2}\psi_1^{\#}(2^jx), \ldots, 2^{j/2}\psi_{N-1}^{\#}(2^jx)$, «affectées à 0» et les fonctions $2^{j/2}\psi_1^b(2^j(1-x)), \ldots, 2^{j/2}\psi_{N-1}^b(2^j(1-x))$ «affectées à 1». Nous apprendrons à construire les $2N-2$ fonctions $\psi_l^{\#}$ et ψ_l^b, $1 \leqslant l \leqslant N-1$. Il manquera alors un ensemble fini explicite pour constituer la base orthonormée cherchée de $L^2[0,1]$.

2. Rappels sur les ondelettes d'Ingrid Daubechies et sur les filtres associés

On part d'un entier $N \geqslant 1$ et de $2N$ coefficients $h_0, h_1, \ldots, h_{2N-1}$ tels que le polynôme trigonométrique

$$m_0(\xi) = h_0 + h_1 e^{i\xi} + \cdots + h_{2N-1}e^{i(2N-1)\xi}$$

vérifie les trois conditions suivantes

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1 \tag{2.1}$$

$$m_0(0) = 1 \tag{2.2}$$

$$m_0(\xi) \neq 0 \quad \text{si} \quad -\frac{\pi}{2} \leqslant \xi \leqslant \frac{\pi}{2}. \tag{2.3}$$

Ces trois conditions assurent l'existence d'une suite orthonormée $\varphi(x-k)$, $k \in \mathbb{Z}$, de fonctions de $L^2(\mathbb{R})$ telle que l'on ait

$$\frac{1}{2}\varphi(x) = h_0\varphi(2x) + \cdots + h_k\varphi(2x-k) + \cdots + h_{2N-1}\varphi(2x-(2N-1)) \tag{2.4}$$

et

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1.$$

On construit $\varphi(x)$ à l'aide de

$$\int_{-\infty}^{\infty} e^{ix\xi}\varphi(x)\,dx = m_0(\xi/2)m_0(\xi/4)\ldots \tag{2.5}$$

et le support de φ est inclus dans $[0, 2N-1]$.

Nous ferons dans tout ce qui suit le choix particulier suivant. On part de la fonction $c(\sin t)^{2N-1}$ où $c = c(N) > 0$ est la constante définie par

$$c\int_0^{\pi} (\sin t)^{2N-1}\,dt = 1.$$

On désigne par $g(t)$ la primitive de $c(\sin t)^{2N-1}$ qui est nulle en π. On a alors $g(-t) = g(t)$, $g(t) \geqslant 0$ pour tout t et $g(t) + g(t+\pi) = 1$. Le lemme de F. Riesz permet alors de choisir

$$m_0(t) = h_0 + h_1 e^{it} + \cdots + h_{2N-1} e^{i(2N-1)t}$$

de sorte que

$$|m_0(t)|^2 = g(t).$$

Notons que

$$|m(t)|^2 = h_0 \bar{h}_{2N-1} e^{-i(2N-1)t} + \cdots + \bar{h}_0 h_{2N-1} e^{i(2N-1)t}$$

et que $h_0 h_{2N-1} \neq 0$.

Ces choix de $m_0(\xi)$ conduisent à des ondelettes de régularité $C^{\gamma(N-1)}$ où $\gamma > 0$ est une constante universelle et où l'on suppose $N \geqslant 2$. Le cas $N = 1$ correspond au système de Haar.

On définit ensuite l'ondelette $\psi(x)$ en introduisant

$$\begin{aligned} m_1(\xi) &= e^{i(2N-1)\xi} \overline{m_0(\xi+\pi)} \\ &= g_0 + g_1 e^{i\xi} + \cdots + g_{2N-1} e^{i(2N-1)\xi} \end{aligned}$$

et en posant

$$\frac{1}{2}\psi(x) = g_0 \varphi(2x) + g_1 \varphi(2x-1) + \cdots + g_{2N-1} \varphi(2x - 2N + 1). \tag{2.6}$$

Les identités (2.4) et (2.6) conduisent à

$$\begin{aligned} \varphi(2x) = {} & \bar{h}_0 \varphi(x) + \bar{h}_2 \varphi(x+1) + \cdots + \bar{h}_{2N-2} \varphi(x+N-1) \\ & + \bar{g}_0 \psi(x) + \bar{g}_2 \psi(x+1) + \cdots + \bar{g}_{2N-2} \psi(x+N-1) \end{aligned} \tag{2.7}$$

et à

$$\begin{aligned} \varphi(2x-1) = {} & \bar{h}_1 \varphi(x) + \bar{h}_3 \varphi(x+1) + \cdots + \bar{h}_{2N-1} \varphi(x+N-1) \\ & + \bar{g}_1 \psi(x) + \bar{g}_3 \psi(x+1) + \cdots + \bar{g}_{2N-1} \psi(x+N-1). \end{aligned} \tag{2.8}$$

Voici l'interprétation géométrique de ces identités. Les fonctions $\varphi(x-k)$, $k \in \mathbb{Z}$, sont une base orthonormée d'un espace que l'on note V_0. On définit V_j, $j \in \mathbb{Z}$, par

$$\forall f \in L^2(\mathbb{R}), \qquad f(x) \in V_0 \Leftrightarrow f(2^j x) \in V_j \tag{2.9}$$

et les fonctions $\psi(x-k)$, $k \in \mathbb{Z}$, forment une base orthonormée du complément orthogonal W_0 de V_0 dans V_1. En particulier $\varphi(2x)$ appartient à V_1 et

(2.7) fournit sa décomposition $u + v$ où $u \in V_0$ et $v \in W_0$. Il en est de même pour $\varphi(2x - 1)$ qui est décomposé grâce à (2.8). Finalement toutes les fonctions $\varphi(x - k)$, $k \in \mathbb{Z}$, se décomposent, si k est pair, grâce à (2.7) et si k est impair, grâce à (2.8). Cela signifie que (2.7) et (2.8) sont les formules de changement de base permettant de passer de la base orthonormée $\sqrt{2}\,\varphi(2x - k)$, $k \in \mathbb{Z}$, de V_1 à la nouvelle base orthonormée que l'on obtient en réunissant les bases $\varphi(x - k)$, $k \in \mathbb{Z}$, de V_0 et $\psi(x - k)$, $k \in \mathbb{Z}$, de W_0.

3. L'analyse multirésolution $V_j^{[0,1]}$ de $L^2[0, 1]$

On part de la suite emboîtée V_j, $-\infty < j < \infty$, de sous-espaces fermés de $L^2(\mathbb{R})$, définis par

(3.1) $\varphi(x - k)$, $k \in \mathbb{Z}$, est une base orthonormée de V_0

(3.2) $f(x) \in V_0 \Leftrightarrow f(2^j x) \in V_j$, pour toute $f \in L^2(\mathbb{R})$.

et l'on désigne, pour tout $j \in \mathbb{N}$, par $V_j^{[0,1]}$ l'espace des restrictions à [0, 1] des fonctions de V_j. Puisque $\varphi(x)$ est une fonction à support compact, il est évident que $V_j^{[0,1]}$ est de dimension finie. En fait, on a un résultat beaucoup plus précis.

Désignons par $S(j)$ l'intervalle d'entiers k définis par $-2N + 2 \leqslant k \leqslant 2^j - 1$ ou, ce qui est équivalent, par la condition que le support de la fonction $\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$ rencontre l'intervalle $]0, 1[$. Alors on a

Théorème 1. *Pour tout entier $j \geqslant 0$, les fonctions $\varphi_{j,k}$, $k \in S(j)$ constituent une base de l'espace $V_j^{[0,1]}$ des restrictions à* [0, 1] *des fonctions de V_j.*

Cet énoncé peut se formuler de façon équivalente et cette seconde formulation nous sera utile, par la suite.

Corollaire. *Soit*

$$f(x) = \sum_{-\infty}^{\infty} c_k \varphi(2^j x - k)$$

une fonction arbitraire de V_j. Supposons $j \geqslant 0$ et $f(x) = 0$ si $0 \leqslant x \leqslant 1$. Alors $f(x)$ est formellement nulle sur [0, 1], *c'est-à-dire que $c_k = 0$ pour tout $k \in S(j)$.*

Nous établirons le théorème par une récurrence descendante sur j. Nous commencerons par traiter le cas facile où $2^j \geqslant 4N - 4$ en utilisant le lemme suivant

Lemme 1. *Si*

$$f(x) = \sum_{-\infty}^{\infty} c_k \varphi(x-k)$$

est nulle sur $]-\infty, 0]$, *alors* $f(x)$ *est formellement nulle sur* $]-\infty, 0]$, *c'est-à-dire* $f(x) = c_0\varphi(x) + c_1\varphi(x-1) + \cdots$

Pour le montrer, on commence par observer que

$$c_k = \int_{-\infty}^{\infty} f(x)\overline{\varphi(x-k)}\,dx = 0 \quad \text{si} \quad k \leqslant -2N+1.$$

On désigne alors par l le plus petit des entiers k tels que $c_k \neq 0$. Si $l \geqslant 0$, il n'y a rien à démontrer et si $l < 0$, il suffit d'observer que $f(x)$, restreinte à $[l, l+1]$ est nulle, par hypothèse, mais est aussi égale à $c_l\varphi(x-l)$. Puisque le support de φ est exactement $[0, 2N-1]$, on aboutit à une contradiction.

Pour démontrer le Théorème 1, nous définissons j_0 comme le plus petit entier j tel que $2^j \geqslant 4N-4$. Nous commençons par établir le Théorème 1 quand $j \geqslant j_0$. Dans ce cas l'ensemble $S(j)$ des entiers k tels que le support de $\varphi_{j,k}$ rencontre $]0, 1[$ se divise en trois ensembles disjoints $S_1(j)$, $S_2(j)$ et $S_3(j)$ selon que l'intérieur du support de $\varphi_{j,k}$ contient 0, que le support de $\varphi_{j,k}$ est inclus dans $[0, 1]$ ou que l'intérieur du support de $\varphi_{j,k}$ contient 1. Lorsque $j \geqslant j_0$ et que $k \in S_1(j)$, le support de $\varphi_{j,k}$ est inclus dans $]-\infty, 1/2]$ et lorsque $j \geqslant j_0$ et $k \in S_3(j)$, ce support est inclus dans $[1/2, \infty[$. Si $j \geqslant j_0$ et si

$$f(x) = \sum_{k \in S(j)} c_k \varphi_{j,k}(x)$$

est nulle sur $[0, 1]$, alors

$$c_k = \int f(x)\overline{\varphi_{j,k}(x)}\,dx = 0 \quad \text{si} \quad k \in S_2(j).$$

Ensuite

$$f_1(x) = \sum_{k \in S_1(j)} c_k \varphi_{j,k}(x)$$

est nulle sur $[1/2, \infty[$,

$$f_3(x) = \sum_{k \in S_3(j)} c_k \varphi_{j,k}(x)$$

l'est sur $]-\infty, 1/2]$ et $f_1(x) + f_2(x) + f_3(x) = f(x)$ l'est sur $[0, 1]$. Il en résulte que $f_1(x)$ est nulle sur $[0, \infty[$. Le Lemme 1 s'applique et $c_k = 0$ si $k \in S_1(j)$. On en déduit, de même, que $c_k = 0$ si $k \in S_3(j)$.

Nous désignerons par P_j la propriété: les fonctions $\varphi_{j,k}(x)$, $k \in S(j)$, forment une base de $V_j^{[0,1]}$. Nous nous proposons maintenant de démontrer que P_j implique P_{j-1}. Pour cela, on utilise le lemme suivant

Lemme 2. *Si les $h_0, \ldots, h_{2N-1}$ sont les coefficients utilisés pour construire $m_0(\xi)$, alors toute suite $u_0, \ldots, u_{2N-3}$ vérifiant, pour $0 \leqslant k \leqslant N-2$, les relations*

$$h_0 u_k + h_2 u_{k+1} + \cdots + h_{2N-2} u_{k+N-1} = 0$$

et

$$h_1 u_k + h_3 u_{k+1} + \cdots + h_{2N-1} u_{k+N-1} = 0$$

est nécessairement la suite nulle.

On pose, en effet,

$$U(z) = h_0 + h_2 z + \cdots + h_{2N-2} z^{N-1} \quad \text{et}$$
$$V(z) = h_1 + h_3 z + \cdots + h_{2N-1} z^{N-1}$$

et l'on a

$$m_0(\xi) = U(e^{2i\xi}) + e^{i\xi} V(e^{2i\xi}).$$

Puisque

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1,$$

il vient

$$|U(e^{i\xi})|^2 + |V(e^{i\xi})|^2 = \frac{1}{2}\cdot$$

Puisque les coefficients h_k sont réels, il en découle que

$$U(z)U(z^{-1}) + V(z)V(z^{-1}) = \frac{1}{2}$$

pour tout $z \neq 0$. Donc $U(z)$ et $V(z)$ n'ont aucun zéro commun $z \neq 0$ et, puisque $h_0 \neq 0$, $U(z)$ et $V(z)$ n'ont aucun zéro commun.

Désignons par $z_1, \ldots, z_{N-1}$ les zéros, supposés simples, de $U(z)$ et par $\zeta_1, \ldots, \zeta_{N-1}$ ceux de $V(z)$, que l'on supposera également simples dans un premier temps. On a alors

$$u_k = c_1 z_1^k + \cdots + c_{N-1} z_{N-1}^k \quad \text{pour} \quad 0 \leqslant k \leqslant 2N-3$$

et, de même,

$$u_k = \gamma_1 \zeta_1^k + \cdots + \gamma_{N-1} \zeta_{N-1}^k \quad \text{pour} \quad 0 \leqslant k \leqslant 2N-3.$$

Il en découle que

$$c_1 z_1^k + \cdots + c_{N-1} z_{N-1}^k - \gamma_1 \zeta_1^k - \cdots - \gamma_{N-1} \zeta_{N-1}^k = 0$$

pour $0 \leqslant k \leqslant 2N-3$. Or ces relations de liaison entraînent la nullité du déterminant de Van der Monde

$$\begin{vmatrix} 1 & \ldots & 1 \\ z_1 & \ldots & \zeta_{N-1} \\ z_1^{2N-3} & \ldots & \zeta_{N-1}^{2N-3} \end{vmatrix}$$

ce qui est absurde, puisque

$$\{z_1, \ldots, z_{N-1}\} \cap \{\zeta_1, \ldots, \zeta_{N-1}\} = \emptyset.$$

Ce raisonnement s'adapte immédiatement au cas de racines multiples de $U(z)$ ou de $V(z)$.

Revenons à la preuve du Théorème 1 et montrons que P_{j+1} implique P_j, pour tout $j \geqslant 0$.

Supposons donc que

$$\sum_{k \in S(j)} x_k \varphi(2^j x - k) = 0$$

sur $[0, 1]$. On écrit

$$\frac{1}{2} \varphi(2^j x - k) = h_0 \varphi(2^{j+1} - 2k) + h_1 \varphi(2^{j+1}x - 2k - 1) + \cdots$$
$$+ h_{2N-1} \varphi(2^{j+1}x - 2k - 2N + 1).$$

On a donc, si $0 \leqslant x \leqslant 1$,

$$0 = \sum_{k \in S(j)} x_k \varphi(2^j x - k)$$
$$= \sum_0^{2N-1} \sum_k x_k h_l \varphi(2^{j+1}x - 2k - l).$$

La propriété P_{j+1} nous apprend que toute somme $\sum_{-\infty}^{\infty} y_k \varphi(2^{j+1}x - k)$ qui est *identiquement* nulle sur $[0, 1]$ est, en fait, *formellement* nulle sur $[0, 1]$: chaque terme qui la compose est nul sur $[0, 1]$. On a donc les $2^{j+1} + 2N - 2$ relations

$$y_{-2N+2} = x_{-N+1}h_0 + x_{-N}h_2 + \cdots + x_{-2N+2}h_{2N-2} = 0$$
$$y_{-2N+3} = x_{-N+1}h_1 + x_{-N}h_3 + \cdots + x_{-2N+2}h_{2N-1} = 0$$
$$y_{-2N+4} = x_{-N+2}h_0 + x_{-N+1}h_2 + \cdots + x_{-2N+3}h_{2N-2} = 0$$
$$y_{-2N+5} = x_{-N+2}h_1 + x_{-N+1}h_3 + \cdots + x_{-2N+3}h_{2N-1} = 0$$
$$\cdots\cdots$$
$$y_{2^{j+1}-2} = x_{2^j-1}h_0 + \cdots + x_{2^j-N}h_{2N-2} = 0$$
$$y_{2^{j+1}-1} = x_{2^j-1}h_1 + \cdots + x_{2^j-N}h_{2N-1} = 0.$$

On utilise alors le Lemme 2 et il vient $x_{-2N+2} = \cdots = x_{2^j-1} = 0$, comme annoncé.

Nous pouvons compléter le Théorème 1 par l'énoncé quantitatif correspondant. A savoir l'existence de deux constantes $C_2 > C_1 > 0$, indépendantes de $j \geqslant 0$, telles que l'on ait pour toute suite $\alpha_{j,k}$, $k \in S(j)$, de coefficients

$$C_1\Big(\sum_{k\in S(j)} |\alpha_{j,k}|^2\Big)^{1/2} \leqslant \Big\| \sum_{k\in S(j)} \alpha_{j,k}\varphi_{j,k}(x)\Big\|_{L^2(0,1)}$$
$$\leqslant C_2\Big(\sum_{k\in S(j)} |\alpha_{j,k}|^2\Big)^{1/2}.$$

Si $0 \leqslant j < j_0$, il n'y a rien à démontrer et (3.3) paraphrase le Théorème 1. Si $j \geqslant j_0$, on pose

$$f(x) = \sum_{k\in S(j)} \alpha_{j,k}\varphi_{j,k}(x)$$

puis

$$f_1(x) = \sum_{k\in S_1(j)} (\cdots), \qquad f_2(x) = \sum_{k\in S_2(j)} (\cdots) \quad \text{et} \quad f_3(x) = \sum_{k\in S_3(j)} (\cdots),$$

L'inégalité de Bessel fournit, tout d'abord,

$$\sum_{k\in S_2(j)} |\alpha_{j,k}|^2 \leqslant \int_0^1 |f(x)|^2\,dx.$$

Il en découle que

$$\|f_1 + f_3\|_{L^2[0,1]} \leqslant 2\|f\|_{L^2[0,1]}.$$

Mais les supports de f_1 et de f_3 sont disjoints puisque $j \geqslant j_0$. Il en résulte que $\|f_1\|_2 + \|f_3\|_2 \leqslant 2\sqrt{2}\,\|f\|_2$. Les normes de $f_1(x)$ et de $f_3(x)$ s'évaluent enfin par simple changement d'échelle et les estimations ne dépendent pas de j.

L'inégalité de droite de (3.3) est encore plus facile, en utilisant la décomposition $f = f_1 + f_2 + f_3$.

Avant de passer à la construction des ondelettes, nous allons déduire du Théorème 1 la construction d'une base orthonormée de $V_j^{[0,1]}$. Nous nous limiterons au cas où $j \geqslant j_0$. Alors les fonctions $\varphi_{j,k}$, $k \in S_1(j)$, $k \in S_2(j)$ ou $k \in S_3(j)$ forment une base (non orthogonale) de $V_j^{[0,1]}$. Il importe de remarquer que, pour le produit scalaire de $L^2[0, 1]$, les fonctions $\varphi_{j,k}$, $k \in S_1(j)$ sont orthogonales aux fonctions $\varphi_{j,k}$, $k \in S_2(j)$. Cette orthogonalité tient à ce que l'on a

$$\int_{-\infty}^{\infty} \varphi(x)\bar{\varphi}(x-k)\,dx = 0 \quad \text{si} \quad k \in \mathbb{Z}, \qquad k \neq 0.$$

Ensuite les fonctions $\varphi_{j,k}$, $k \in S_1(j)$ sont orthogonales aux fonctions $\varphi_{j,k}$, $k \in S_3(j)$ parce que leurs supports sont disjoints.

Pour transformer en une base orthonormée la base des $\varphi_{j,k}$, $k \in S(j)$, il suffit donc de rendre orthogonales entre elles les fonctions du paquet $\varphi_{j,k}$, $k \in S_1(j)$ et, de même, de rendre orthogonales entre elles les fonctions du paquet $\varphi_{j,k}$, $k \in S_3(j)$. Dans chaque cas, il s'agit de $N-1$ fonctions et les calculs à faire sont invariants par dilatation. On obtient donc, à la place des $N-1$ fonctions $\varphi_{j,k}(x)$, $k \in S_1(j)$, $N-1$ nouvelles fonctions

$$2^{j/2}\varphi_1^{\#}(2^j x), \ldots, 2^{j/2}\varphi_{N-1}^{\#}(2^j x)$$

et de même, à la place des $N-1$ fonctions $\varphi_{j,k}$, $k \in S_3(J)$, $N-1$ nouvelles fonctions

$$2^{j/2}\varphi_1^{b}(2^j(1-x)), \ldots, 2^{j/2}\varphi_{N-1}^{b}(2^j(1-x)).$$

Nous pouvons conclure en énonçant le résultat suivant

Proposition 1. *Pour $j \geqslant j_0$, la collection des fonctions*

$$2^{j/2}\varphi_1^{\#}(2^j x), \ldots, 2^{j/2}\varphi_{N-1}^{\#}(2^j x), 2^{j/2}\varphi(2^j x - k), \qquad 0 \leqslant k \leqslant 2^j - 2N + 1$$

et, finalement

$$2^{j/2}\varphi_1^{b}(2^j(1-x)), \ldots, 2^{j/2}\varphi_{N-1}^{b}(2^j(1-x))$$

est une base orthonormée de $V_j^{[0,1]}$.

Une dernière remarque nous sera utile dans ce qui suit

Lemme 3. *L'espace vectoriel $V_0^{[0,1]}$ (de dimension $2N-1$) contient l'espace vectoriel (de dimension N) des restrictions à $[0, 1]$ des polynômes de degré $\leqslant N-1$.*

Pour le voir, on commence par observer que

$$\sum_{-\infty}^{\infty} (x-k)^m \varphi(x-k) = c_m \quad \text{pour} \quad 0 \leqslant m \leqslant N-1,$$

on le montre en appliquant la formule sommatoire de Poisson et en observant que

$$\left(\frac{d}{dx}\right)^m \varphi(x) = 0 \quad \text{si} \quad x = 2k\pi, \qquad k \neq 0, \quad \text{et} \quad 0 \leqslant m \leqslant N-1.$$

Il en découle que

$$\sum_{-\infty}^{\infty} k^m \varphi(x-k) = P_m(x)$$

où $P_m(x)$ est un polynôme de degré m.

En restreignant cette identité à $[0, 1]$, on obtient le Lemme 3.

La construction des ondelettes sur l'intervalle $[0, 1]$ débute par l'orthonormalisation de la base de $V_0^{[0,1]}$ constituée des fonctions $\varphi(x + 2N - 2)$, $\varphi(x + 2N - 3), \ldots, \varphi(x)$. Compte tenu du Lemme 3, nous commencerons par substituer à cette base celle composée des monômes $1, x, \ldots, x^{N-1}$ puis de $N - 1$ fonctions $g_1(x), \ldots, g_{N-1}(x)$ de $V_0^{[0,1]}$. L'orthonormalisation de cette nouvelle base fournit donc N polynômes orthogonaux suivis de $N - 1$ fonctions de $V_0^{[0,1]}$ dont les moments d'ordre $\leqslant N - 1$ sont tous nuls. Ces fonctions seront donc des ondelettes et il en sera de même de celles que nous allons maintenant construire.

4. La construction des ondelettes sur l'intervalle

Cette construction repose sur l'énoncé suivant qui permet de compléter la base $\varphi_{j,k}$, $k \in S(j)$, en une base de $V_{j+1}^{[0,1]}$.

Théorème 2. *Pour tout $j \geqslant 0$, une base de $V_{j+1}^{[0,1]}$ est constituée de la réunion de la base $\varphi_{j,k}$ de $V_j^{[0,1]}$ et des fonctions $\psi_{j,k}$ telles que $-N + 1 \leqslant k \leqslant 2^j - N$.*

Pour établir ce résultat, on commence par démontrer le résultat suivant

Lemme 4. *Les fonctions $\psi(2^j x - k)$, $-2N + 2 \leqslant k \leqslant -N$, une fois restreintes à $[0, 1]$, appartiennent à $V_j^{[0,1]}$.*

Nous poserons $2^j x = t$ et définirons $V_0^{[0,\infty[}$ comme l'espace des restrictions à $[0, \infty[$ des fonctions de $V_0 = V_0(\mathbb{R})$. Le Lemme 4 résultera du résultat plus précis suivant

Lemme 5. *Les fonctions* $\psi(x-k)$, $-2N+2 \leqslant k \leqslant -N$, *une fois restreintes à* $[0, \infty[$, *appartiennent à* $V_0^{[0,\infty[}$.

Cela signifie que si plus la moitié du support de $\psi(x-k)$ tombe à l'exterieur de $[0, \infty[$, la restriction à $[0, \infty[$ de $\psi(x-k)$ appartient à V_0 mais, en revanche, que les $N-1$ fonctions $\psi(x+N-1), \ldots, \psi(x+1)$, une fois restreintes à $[0, \infty[$, sont linéairement indépendantes modulo V_0.

Revenons au Lemme 5. Pour l'établir, on retourne à (2.7) où l'on remplace successivement x par $x+2N-2$, puis par $x+2N-3, \ldots$ et, enfin, x par $x+N$. Puisque $\varphi(2x+2N-1)=0$ si $x \geqslant 0$, il vient successivement, pour $x \geqslant 0$,

$$\bar{g}_0 \psi(x+2N-2) + \bar{h}_0 \varphi(x+2N-2) = 0$$

puis

$$\bar{g}_0 \psi(x+2N-3) + \bar{g}_2 \psi(x+2N-2) + \bar{h}_0 \psi(x+2N-3) + \bar{h}_2 \varphi(x+2N-2) = 0$$

et enfin

$$\bar{g}_0 \psi(x+N) + \cdots + \bar{g}_{2N-4} \psi(x+2N-2) + \bar{h}_0 \varphi(x+N) + \cdots + \bar{h}_{2N-4} \varphi(x+2N-2) = 0.$$

A l'aide de ces relations, on démontre successivement que $\psi(x+2N-2)$, restreint à $[0, \infty[$, appartient à $V_0^{[0,\infty[}$ puis qu'il en est de même pour $\psi(x+2N-3)$ et, de proche en proche, pour tous les $\psi(x+k)$, $2N-2 \geqslant k \geqslant N$.

Le Lemme 4 résulte du Lemme 5 par simple changement d'échelle.

Pour démontrer le Théorème 2, on observe d'abord que le nombre de fonctions proposées est exactement la dimension $(2^{j+1}+2N-2)$ de $V_{j+1}^{[0,1]}$. Pour établir le théorème, il suffit d'établir que ces fonctions constituent un système générateur.

On part donc d'une fonction arbitraire f de V_{j+1} et l'on veut montrer que la restriction de f à $[0, 1]$ s'écrit $g+h$ où $g \in V_j^{[0,1]}$ et

$$h(x) = \sum_{-N+1}^{2^j-N} \alpha(j,k)\psi_{j,k}(x). \tag{4.1}$$

En fait, $f = u + v$ où $u \in V_j$ et $v \in W_j$. À ce titre, $v(x) = \sum_k \beta(j,k)\, \psi_{j,k}(x)$. Dans cette série, on peut distinguer sept ensembles de valeurs de k. Si $k \leqslant -2N+1$, la restriction de $\psi_{j,k}$ à $[0, 1]$ est nulle et on n'a pas à considérer les termes correspondants. Si $-2N+2 \leqslant k \leqslant -N$, la restriction de $\psi_{j,k}$ à $[0, 1]$

appartient à $V_j^{[0,1]}$ (Lemme 4) et contribue à la fonction $g(x)$. Les termes tels que $-N+1 \leqslant k \leqslant 2^j - N$ sont ceux qui nous intéressent.

Si $2^j - N + 1 \leqslant k \leqslant 2^j - 1$, la restriction de $\psi_{j,k}$ à $[0,1]$ appartient à $V_j^{[0,1]}$. On le démontre par un raisonnement identique à celui qui a conduit au Lemme 4. Enfin si $k \geqslant 2^j$, la restriction de $\psi_{j,k}$ à $[0,1]$ est nulle.

Le Théorème 2 est donc démontré.

La construction de la base orthonormée d'ondelettes sur l'intervalle $[0,1]$ suit désormais le schéma classique des analyses multirésolutions.

On dispose d'une suite emboîtée de sous-espaces $V_j^{[0,1]}$ de $L^2[0,1]$, $j \geqslant 0$. La réunion des $V_j^{[0,1]}$ est dense dans $L^2[0,1]$ puisque la réunion des V_j est dense dans $L^2(\mathbb{R})$. On désigne alors par $W_j^{[0,1]}$ le complément orthogonal de $V_j^{[0,1]}$ dans $V_{j+1}^{[0,1]}$. On observera que $W_j^{[0,1]}$ n'est pas l'espace des restrictions à $[0,1]$ des fonctions de W_j. En effet le Lemme 4 nous apprend que la restriction à $[0,1]$ de $\psi(2^jx - k)$ appartient à $V_j^{[0,1]}$ si $-2N+2 \leqslant k \leqslant -N$ ou si $2^j - N + 1 \leqslant k \leqslant 2^j - 1$. Lorsque $-N+1 \leqslant k \leqslant -1$ ou $2^j - 2N + 2 \leqslant k \leqslant 2^j - N$, les restrictions des fonctions $\psi(2^jx - k)$ n'appartiennent certes pas à $V_j^{[0,1]}$ mais ne sont pas pour autant orthogonales à $V_j^{[0,1]}$.

En tout état de cause, on a

$$L^2[0,1] = V_0^{[0,1]} \oplus W_0^{[0,1]} \oplus W_1^{[0,1]} \oplus \cdots \oplus W_j^{[0,1]} \oplus \cdots$$

Nous disposons déjà d'une base orthonormée de $V_0^{[0,1]}$. Nous nous proposons de construire une base orthonormée de $W_j^{[0,1]}$ pour chaque $j \geqslant 0$. À cet effet, on utilise le Théorème 2, en distinguant les cas $0 \leqslant j < j_0$ et $j \geqslant j_0$ ($2^{j_0} \geqslant 4N - 4$).

Dans le premier cas, il suffit de projeter orthogonalement sur W_j les fonctions $\psi_{j,k}$ telles que $-N+1 \leqslant k \leqslant 2^j - N$. Puisque nous disposons déjà d'une base orthonormée de V_j, l'opérateur de projection orthogonale sur V_j est explicite. Une fois projetés sur W_j, les $\psi_{j,k}$ deviennent des fonctions $h_{j,k}$ qu'il convient ensuite d'orthonormaliser entre elles pour $-N+1 \leqslant k \leqslant 2^j - N$.

Lorsque $j \geqslant j_0$, tout se clarifie. En effet les $\psi_{j,k}$ telles que $0 \leqslant k \leqslant 2^j - 2N + 1$ appartiennent de «plein droit» à $W_j^{[0,1]}$. Pour obtenir la base orthonormée de $W_j^{[0,1]}$, il suffit d'adjoindre à ces $2^j - 2N + 2$ fonctions les $2N - 2$ ondelettes manquantes (la dimension de $W_j^{[0,1]}$ est 2^j). Ces $2N - 2$ ondelettes manquantes se composent de $N - 1$ ondelettes «affectées à 0» et de $N - 1$ ondelettes «affectées à 1».

Pour construire les premières, on part des $N - 1$ fonctions $\psi_{j,k}$ telles que $-N+1 \leqslant k \leqslant -1$. Elles sont orthogonales aux $\varphi_{j,k}$ lorsque $k \geqslant 0$ tout comme elles le sont aux $\psi_{j,k}$ pour $k \geqslant 0$. Ce qui manque à ces fonctions $\psi_{j,k}$ est l'orthogonalité aux $N - 1$ fonctions $2^{j/2}\varphi_1^{\#}(2^jx), \ldots, 2^{j/2}\varphi_{N-1}^{\#}(2^jx)$. Ces $N - 1$ fonctions formant une suite orthogonale, les corrections rendant les $\psi_{j,k}$ $(-N+1 \leqslant k \leqslant -1)$ orthogonales à $V_j^{[0,1]}$ sont évidentes. On obtient alors $N-1$ fonctions $2^{j/2}h_1(2^jx), \ldots, 2^{j/2}h_{N-1}(2^jx)$ où $h_1, \ldots, h_{N-1}$ ne dépendent pas de j. Il suffit enfin d'orthonormaliser ces $N-1$ fonctions pour obtenir les $N-1$ ondelettes «affectées à 0», à savoir $2^{j/2}\psi_1^{\#}(2^jx), \ldots, 2^{j/2}\psi_{N-1}^{\#}(2^jx)$.

La construction des $N-1$ ondelettes «affectées à 1», à savoir $2^{j/2}\psi_1^b(2^j(1-x))$, $\ldots, 2^{j/2}\psi_{N-1}^b(2^j(1-x))$ est semblable.

Il est plus commode d'indexer les fonctions que nous venons de construire par l'ensemble $\mathcal{I} \cup E$ où $\mathcal{I}$ est la collection de tous les intervalles dyadiques inclus dans $[0, 1]$ et E est un ensemble de cardinalité $2N-1$. On commencera par examiner les intervalles $I = [k2^{-j}, (k+1)2^{-j}[$ où $j \geqslant j_0$, $0 \leqslant k < 2^j$. On distingue trois cas. Désignons systématiquement par $(2N-1)I = \tilde{I}$ l'intervalle ayant le même centre que I et pour longueur $2N-1$ fois celle (notée $|I|$) de I. Le premier cas est celui où ni 0 ni 1 n'appartiennent à l'intérieur de $\tilde{I}$. Alors $\tilde{I}$ est inclus dans $[0, 1]$ et l'on pose

$$\psi_I(x) = 2^{j/2}\psi(2^jx - k + N - 1). \tag{4.5}$$

On observera que le support est exactement $\tilde{I}$. Si 0 appartient à l'intérieur de $\tilde{I}$, on a $0 \leqslant k \leqslant N-2$ et l'on pose

$$\psi_I(x) = 2^{j/2}\psi_{k+1}^{\#}(2^jx). \tag{4.6}$$

Si enfin 1 appartient à l'intérieur de $\tilde{I}$, on a $2^j - (N-1) \leqslant k \leqslant 2^j - 1$ et l'on pose $l = 2^j - k$ et

$$\psi_I(x) = 2^{j/2}\psi_l^b(2^j(1-x)). \tag{4.7}$$

Finalement il reste à indexer les $2^{j_0} + 2N - 2$ fonctions manquantes. Nous disposons pour cela de $1 + 2 + \cdots + 2^{j_0 - 1}$ intervalles dyadiques $I \subset [0, 1]$ qui n'ont pas encore servi. Désignons par $\underline{j}_0$ le plus petit entier j tel que $2^j \geqslant 2N-1$; si $\underline{j}_0 \leqslant j < j_0 - 1$, on incorpore dans la base orthonormée de W_j les fonctions $2^{j/2}\psi(2^jx - k)$, $0 \leqslant k \leqslant 2^j - 2N + 1$, dont le support est inclus dans $[0, 1]$. On complète ces fonctions (que l'on notera bien évidemment ψ_I en revenant à (4.5)) en une base orthonormée de W_j que l'on indexe arbitrairement à l'aide des intervalles I de longueur 2^{-j} non encore utilisés.

Nous venons, pour chaque $j \geqslant 0$, de former une base orthonormée de W_j qui se compose des ondelettes ψ_I, I intervalle dyadique de longueur 2^{-j} inclus dans $[0, 1]$; l'ensemble de ces intervalles dyadiques sera noté E_j. Enfin ψ_I est donné par (4.5) chaque fois que $(2N-1)I = \tilde{I}$ est inclus dans $[0, 1]$.

Pour obtenir une base orthonormée de $L^2[0, 1]$, il nous reste à former une base orthonormée $\varphi_0(x), \ldots, \varphi_{2N-2}(x)$ de V_0. Remarquons tout d'abord que $1, x, \ldots, x^{N-1}$ appartiennent à V_0 et $\varphi_0, \ldots, \varphi_{N-1}$ ne sont autres que ces monômes orthogonalisés par le procédé de Gram-Schmidt. On complète ces polynômes orthogonaux en une base orthonormée de V_0.

Nous venons d'établir le résultat suivant

Théorème 3. *L'ensemble des fonctions* $\psi_I(x)$, $I \in \mathcal{I}$, *et des* $2N-1$ *fonctions* $\varphi_0(x), \ldots, \varphi_{2N-2}(x)$ *constitue une base orthonormée de* $L^2[0,1]$.

5. Analyse des espaces de Hölder $C^s[0,1])$, $s > 0$, dans la base précédente

Rappelons que $C^s[0,1]$, $s > 0$, désigne l'espace des restrictions à l'intervalle $[0,1]$ des fonctions de $C^s(\mathbb{R})$. Nous désignerons par $C_0^s(\mathbb{R})$ la fermeture, pour la norme de l'espace de Banach C^s, des fonctions de la classe $\mathfrak{D}(\mathbb{R})$ de Schwartz. Si $s = 1$, convenons que C^1 ne désignera pas l'espace usuel mais la classe de Zygmund Λ_* définie par la condition

$$|f(x+h) + f(x-h) - 2f(x)| \leqslant C|h|. \tag{5.1}$$

Si $1 < s \leqslant 2$, on écrira $s = 1 + r$ et $f \in C^s(\mathbb{R})$ signifie que $f'(x)$, la dérivée de f, appartient à C^r etc...

Alors il vient, pour toute fonction $f(x) \in L^2[0,1]$,

Proposition 2. *Si* $0 < s < N$ *et si* $f(x)$ *appartient à* $C^s[0,1]$, *alors on a*

$$\int_0^1 f(x)\psi_I(x)\,dx = O(|I|^{1/2+s}). \tag{5.2}$$

Réciproquement si $0 < s < \gamma(N-1)$, *cette condition caractérise l'espace* $C^s[0,1]$.

Rappelons que $\gamma(N-1)$ mesure la régularité de l'ondelette $\psi(x)$ et de la fonction $\varphi(x)$ qui lui est associée. La preuve de ce résultat est semblable à celle que le lecteur pourra trouver dans [2].

On a des résultats analogues en ce qui concerne l'espace $H^s[0,1]$, $s \geqslant 0$, des restrictions à $[0,1]$ des fonctions $f(x) \in H^s(\mathbb{R})$.

Un cas remarquable est celui de l'espace BMO de John et Nirenberg dont nous rappelons la définition.

Définition 1. *Une fonction* $f(x)$, *appartenant à* $L^2[0,1]$, *appartient en outre, à* BMO $[0,1]$ *s'il existe une constante* $C \geqslant 0$ *telle que pour tout intervalle* $I \subset [0,1]$, *on puisse trouver une constante* $\gamma(I)$ *de sorte que l'on ait*

$$\left(\frac{1}{|I|}\int_I |f(x) - \gamma(I)|^2\,dx\right)^{1/2} \leqslant C. \tag{5.3}$$

La borne inférieure de ces constantes C est la norme BMO de f. Mais on peut souhaiter éviter (ce que nous ferons) que les constantes aient pour norme 0. Alors on ajoute $\left|\int_0^1 f(x)\,dx\right|$ à la norme précédente.

L'espace BMO $[0, 1]$ se compose des restrictions à $[0, 1]$ des fonctions de BMO $(\mathbb{R})$. Pour le montrer, il suffit de construire l'opérateur de prolongement d'une fonction $f \in$ BMO $[0, 1]$ en une fonction $g \in$ BMO $(\mathbb{R})$. Ce prolongement s'effectue en imposant à $g(x)$ d'être paire et périodique de période 2.

L'espace BMO $[0, 1]$ est le dual E^* d'un espace de Banach E; E est le sous-espace de l'espace de Hardy $H^1(\mathbb{R})$ composé de toutes les fonctions de $H^1(\mathbb{R})$ dont le support est inclus dans $[0, 1]$.

On suppose $N \geqslant 2$ et l'on a alors (avec les notations du Théorème 3)

Proposition 3. *Une fonction $f(x)$ de carré sommable sur* $[0, 1]$ *appartient à* BMO $[0, 1]$ *si et seulement si l'on a, pour une certaine constante C,*

$$\sum_{J \subset I} |\langle f, \psi_J \rangle|^2 \leqslant C|I| \tag{5.4}$$

où I est un intervalle dyadique arbitraire inclus dans $[0, 1]$ *et où la somme porte sur tous les sous-intervalles dyadiques $J \subset I$.*

Pour conclure cette section, observons que l'analyse en ondelettes des espaces $C^s[0, 1]$ ou BMO $[0, 1]$ fournit automatiquement des opérateurs de prolongement à $\mathbb{R}$ tout entier. Explicitons ce point dans le cas de BMO $[0, 1]$. Les ondelettes permettant de reconstituer f sont soit de la forme $2^{j/2}\psi(2^j x - k)$ et leur support est alors inclus dans $[0, 1]$, soit de la forme $2^{j/2}\psi_1^{\#}(2^j x), \ldots, 2^{j/2}\psi_{N-1}^{\#}(2^j x)$, soit de la forme $2^{j/2}\psi_1^{b}(2^j(1-x)), \ldots, 2^{j/2}\psi_{N-1}^{b}(2^j(1-x))$, soit enfin l'une des fonctions $\varphi_0, \ldots, \varphi_{2N-2}$. Si bien que $f \in$ BMO $[0, 1]$ s'écrit canoniquement $f = f_1 + f_2 + f_3 + f_4$ où f_1 est une fonction de BMO $(\mathbb{R})$ dont le support est inclus dans $[0, 1]$,

$$f_2(x) = \sum_{j \geq j_0} \sum_1^{N-1} \alpha(j, m)\psi_m^{\#}(2^j x) \qquad \text{et} \quad \sup_{j \geq j_0} |\alpha(j, m)| < \infty$$

$$f_3(x) = \sum_{j \geq j_0} \sum_1^{N-1} \beta(j, m)\psi_m^{\#}(2^j(1-x)) \quad \text{et} \quad \sup_{j \geq j_0} |\beta(j, m)| < \infty$$

et finalement

$$f_4(x) = \gamma_0\varphi_0(x) + \cdots + \gamma_{2N-2}\varphi_{2N-2}(x).$$

Le prolongement des fonctions $\psi_m^{\#}(x)$ et $\psi_m^{b}(x)$ à $\mathbb{R}$ tout entier est fourni par construction. Naturellement ces fonctions, une fois prolongées, perdent leur caractère oscillant. On a

$$\int_0^{\infty} x^k \psi_m^{\#}(x)\, dx = 0$$

si $0 \leqslant k \leqslant N-1$, $1 \leqslant m \leqslant N-1$ mais l'intégrale correspondante sur $]-\infty, \infty[$ n'est pas nulle. Néanmoins $f_2(x)$ et $f_3(x)$ appartiennent à BMO ($\mathbb{R}$) car seule la régularité et le support compact des fonctions $\psi_m^{\#}$ et ψ_m^b importent.

6. Opérateurs de Calderón-Zygmund sur l'intervalle

Commençons par étendre au carré unité $[0, 1] \times [0, 1] = Q_0$ la base orthonormée de $L^2[0, 1]$ que nous venons de construire. On procède comme dans [2] et l'on observe que la réunion des sous-espaces $V_j^{[0,1]} \otimes V_j^{[0,1]}$ de $L^2(Q_0)$ est dense dans $L^2(Q_0)$. On a donc

(6.1) $$L^2(Q_0) = (V_0 \otimes V_0) \oplus (W_0 \otimes V_0) \oplus (V_0 \otimes W_0) \oplus (W_0 \otimes W_0) \oplus \cdots$$

Une base orthonormée de $V_0 \otimes V_0$ se compose des fonctions $\varphi_l(x)\varphi_m(y)$, $0 \leqslant l \leqslant 2N-2$, $0 \leqslant m \leqslant 2N-2$, déjà utilisées. Nous utiliserons de même une base orthonormée φ_I de $V_j = V_j^{[0,1]}$, construite de la façon suivante. La base φ_I est indexée par l'ensemble F_j des intervalles dyadiques de longueur 2^{-j} inclus dans $[-(2N-2)2^{-j}, 1+(2N-2)2^{-j}]$. Si $\tilde{I} = (2N-1)I$ est inclus dans $[0, 1]$, alors $\varphi_I(x) = 2^{j/2}\varphi(2^j x - k + N - 1)$ et si 0 ou 1 appartiennent à l'intérieur de $\tilde{I}$, la construction de φ_I est calquée sur celle que nous avons donnée pour ψ_I.

Une base orthonormée de $W_j \otimes V_j$ est donc composée des fonctions $\psi_I(x)\varphi_J(y)$ où $I \in E_j$ (ensemble des intervalles dyadiques I inclus dans $[0, 1]$ et de longueur 2^{-j}) et $J \in F_j$. De même une base orthonormée de $V_j \otimes W_j$ est composée de fonctions $\varphi_I(x)\psi_J(y)$, $I \in F_j$, $J \in E_j$. Enfin une base orthonormée de $W_j \otimes W_j$ est composée des fonctions $\psi_I(x)\psi_J(y)$, $I \in E_j$, $J \in E_j$.

Tout ce que nous dit dans le cas de l'intervalle $[0, 1]$ convient au cas de $[0, 1] \times [0, 1]$. C'est-à-dire que la décomposition dans cette base fournit un prolongement canonique des objets que l'on analyse.

Nous allons vérifier cette assertion en analysant, dans la base que nous venons de construire, les noyaux distributions $S(x, y)$ des opérateurs de Calderón-Zygmund opérant sur $L^2[0, 1]$.

Commençons par une forme bilinéaire $J: C^1[0, 1] \times C^1[0, 1] \to \mathbb{C}$ définie par la distribution correspondante $S \in \mathfrak{D}'(\mathbb{R}^2)$ dont le support est inclus dans $[0, 1]^2$.

Nous supposerons que la restriction à $y \neq x$ de $S(x, y)$ est une fonction $K(x, y)$ vérifiant les conditions

(6.2) $$|K(x, y)| \leqslant C_0 |x-y|^{-1}, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad y \neq x,$$

(6.3) il existe un exposant $\gamma \in]0, 1[$ tel que l'on ait

$$|K(x', y) - K(x, y)| \leqslant C_0 |x' - x|^\gamma |x - y|^{-1-\gamma}$$

chaque fois que

$$|x' - x| \leqslant \frac{1}{2} |x - y|, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant x' \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad y \neq x,$$

on a de même,

(6.4) $$|K(x, y') - K(x, y)| \leqslant C_0 |y' - y|^\gamma |x - y|^{-1-\gamma}$$

chaque fois que

$$|y' - y| \leqslant \frac{1}{2} |x - y|, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y' \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad x \neq y.$$

Définissons maintenant la continuité faible. A cet effet, pour tout intervalle fermé $I \subset [0, 1]$ et toute fonction $f \in C^1[0, 1]$ à support dans I, nous posons

$$N_I(f) = |I|^{1/2} \sup_I |f(x)| + |I|^{3/2} \sup_I |f'(x)|;$$

si $I = [0, a]$, $a < 1$, nous ne demandons pas que f ou f' s'annulent en 0 et de même si $I = [b, 1]$.

Nous dirons que J a la propriété de continuité faible s'il existe une constante C telle que, pour tout intervalle fermé $I \subset [0, 1]$ et tout couple de deux fonctions f et g de $C^1[0, 1]$ à supports dans I, on ait

(6.5) $$|J(f, g)| \leqslant C N_I(f) N_I(g).$$

Désignons enfin par T l'opérateur défini par la forme J et par

$$\langle T(f), g \rangle = J(f, g).$$

Le problème que nous posons est de savoir si un opérateur T vérifiant (6.2), (6.3), (6.4) et (6.5) se prolonge en un opérateur continu sur $L^2[0, 1]$. A cet effet, analysons $S(x, y)$ dans la base d'ondelettes du carré. On obtient, grâce à la continuité faible et aux propriétés (6.2), (6.3) et (6.4)

(6.6) $$|J(\varphi_I, \psi_J)| \leqslant C_1 (1 + |k - l|)^{-1-\gamma}$$

(6.7) $$|J(\psi_I, \varphi_J)| \leqslant C_1 (1 + |k - l|)^{-1-\gamma}$$

(6.8) $$|J(\psi_I, \psi_J)| \leqslant C_1 (1 + |k - l|)^{-1-\gamma}$$

lorsque $I = [k 2^{-j}, (k + 1) 2^{-j}[$ et $J = [l 2^{-j}, (l + 1) 2^{-j}[$. Inversement ces propriétés caractérisent les distributions S vérifiant (6.2), (6.3), (6.4) et la propriété de continuité faible.

Désignons par G_j l'ensemble des intervalles dyadiques I de longueur 2^{-j} tels que $(2N-1)I = \tilde{I} \subset [0,1]$. On part de la décomposition de la distribution $S(x,y)$ dans la base d'ondelettes de $L^2(Q_0)$. On regroupe tous les termes $\alpha(I,J)\varphi_I(x)\psi_J(y) + \beta(I,J)\psi_I(x)\varphi_J(y) + \gamma(I,J)\psi_I(x)\psi_j(y)$ tels que $I \in G_j$ et $J \in G_j$ et leur somme est notée $\tilde{S}(x,y)$. Alors $\tilde{S}$ appartient automatiquement à $\mathfrak{D}(\mathbb{R}^2)$ et la restriction $\tilde{K}(x,y)$ de $\tilde{S}$ à $y \neq x$ vérifie (6.2), (6.3) et (6.4) dans $\mathbb{R}^2$ tout entier. En outre $\tilde{K}(x,y) = 0$ hors de Q_0.

La différence $R(x,y) = S(x,y) - \tilde{S}(x,y)$ vérifie

$$|R(x,y)| \leqslant \frac{C_2}{|x|+|y|} + \frac{C_2}{|x-1|+|y-1|} \tag{6.9}$$

et l'opérateur associé est automatiquement borné sur $L^2[0,1]$. En outre le prolongement canonique de $R(x,y)$ hors de Q_0 est un noyau à support dans $(2N-1)Q_0$ qui vérifie (6.2), (6.3) et (6.4) dans tout $\mathbb{R}^2$.

Il est alors immédiat de conclure. Si T se prolonge en un opérateur linéaire continu sur $L^2[0,1]$, alors l'opérateur $\tilde{T}$ est continu de $L^2(\mathbb{R})$ dans lui-même. Le noyau-distribution $\tilde{S}(x,y)$ de $\tilde{T}$, une fois restreint à $y \neq x$, vérifie (6.2), (6.3) et (6.4). Le théorème de David et Journé s'applique donc et la continuité en question équivaut à $\tilde{T}(1) \in \mathrm{BMO}$ et ${}^t\tilde{T}(1) \in \mathrm{BMO}$. Finalement il faut comparer $\tilde{T}(1)$ à $T(1)$ d'une part, ${}^t\tilde{T}(1)$ à ${}^tT(1)$ d'autre part. En revenant aux décompositions des noyaux $S(x,y)$ et $\tilde{S}(x,y)$, on voit que $\tilde{T}(1)$ et $T(1)$ ne diffèrent que par deux fonctions de BMO que nous avons appelées $f_2(x)$ et $f_3(x)$ dans la Section 5.

Ainsi la continuité de l'opérateur T équivaut à $T(1) \in \mathrm{BMO}\,[0,1]$ et ${}^tT(1) \in \mathrm{BMO}\,[0,1]$.

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Spherical Means on the Heisenberg Group and a Restriction Theorem for the Symplectic Fourier Transform

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Dedicated to Prof. E. M. Stein on his 60th birthday

1. Introduction

Spherical mean value operators on a compact Riemannian manifold M have been extensively studied by Sunada in a series of papers [14], [15] and [16]. He has studied the eigenvalue problem $L_r f = \alpha f$ associated with the spherical mean value operator L_r. The question about the eigenvalues $\alpha = 1$ and $\alpha = -1$ are related to the ergodicity and mixing properties of the geodesic random walk of step size r on the manifold M. In a recent article [7] Pati-Shahshahani-Sitaram have investigated the eigenvalue problem in the case when M is a compact symmetric space. In this case they are able to identify the eigenvalues completely in terms of the elementary spherical functions associated to M. They provide alternate proofs of some results of Sunada regarding the eigenvalues 1 and -1. Let us briefly recall their result.

Let $M = G/K$ be a compact symmetric space. Then the spherical mean value operator L_r can be identified with a convolution operator $L_r f = f * \nu_r$ where ν_r is a certain probability measure which can be viewed as a K-biinvariant measure on G. Let $\hat{G}_1$ denote the collection of all pairwise inequivalent, irreducible, unitary representations of class 1 of G. For each $\pi \in \hat{G}_1$ there is an elementary spherical function ϕ_π associated with it. The main result of [7] can now be stated as follows.

Theorem 0. *All the eigenvalues of the operator L_r are of the form*

$$\psi_\pi(r) = \int_G \phi_\pi(x)\, d\nu_r$$

where $\pi \in \hat{G}_1$.

The aim of this paper is to study spherical mean value operators on the reduced Heisenberg group H^n/Γ. Here H^n is the Heisenberg group and Γ is the subgroup $\{(0, 2\pi k): k \in \mathbb{Z}\}$ of H^n. The Heisenberg group H^n is a nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. The coordinates on H^n are (z, t) where $z = x + iy$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The group law is defined by

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\operatorname{Im} z\bar{w}\right).$$

The Haar measure in H^n is the Lebesgue measure $dz\,dt$. The group H^n/Γ is a nilpotent Lie group with compact centre. A function on H^n is said to be radial or rotation invariant if it is invariant under rotations in the variable z.

By a spherical mean value operator we mean an operator of the form $T_\mu f = f * \mu$ where μ is a rotation invariant compactly supported probability measure on H^n/Γ. We are able to identify all the eigenvalues of the operator T_μ. For each $k = 0, 1, 2, \ldots$ and $\lambda \neq 0$ there are certain radial functions e_k^λ on the Heisenberg group H^n which can be thought of as the elementary spherical functions for the Heisenberg group. As in the case of the compact symmetric space, the eigenvalues are then given by the averages of e_k^λ with respect to μ.

Theorem 1. *Assume that μ has no mass at the centre of H^n/Γ. Then all the eigenvalues of the operator T_μ are given by*

$$\alpha_k(j) = \int_{H^n/\Gamma} e_k^j(z,t)\, d\mu,$$

where j is an integer. Further, any function of the form $f * e_k^{-j}$ satisfies

$$T_\mu(f * e_k^{-j}) = \alpha_k(j)(f * e_k^{-j}).$$

We can make more precise statements regarding the eigenvalues if we take $\mu = \mu_{r,t}$ where $\mu_{r,t}$ is the normalized Lebesgue (surface) measure on the sphere $S_{r,t} = \{(z,t): |z| = r\}$ in H^n/Γ. Let $M_{r,t}$ stand for T_μ when $\mu = \mu_{r,t}$. The elementary spherical functions $e_k^j(z, s)$ are radial functions of z and slightly abusing the notation we write $e_k^j(r, s)$ in place of $e_k^j(z, s)$ when $|z| = r$.

Theorem 2.
(i) *All the eigenvalues of the operator $M_{r,t}$ are given by $\alpha_k(j) = e_k^j(r, t)$.*
(ii) $\alpha = 1$ *and* $\alpha = -1$ *are not eigenvalues of the operator $M_{r,t}$ for any $r > 0$.*

By writing down the Fourier series of $f * \mu_{r,t}$ we can see that it involves operators of the form $g \times \mu_r$ where $g \times \mu_r$ is the twisted convolution of g with the surface measure on the sphere $|z| = r$ in $\mathbb{C}^n$. The spectral properties of the operator $T_r g = (2\pi)^n g \times \mu_r$ are worth studying and we have the following theorem.

Theorem 3.

(i) *All the eigenvalues of the operator T_r are given by*

$$\alpha_k = \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)$$

where ϕ_k are the Laguerre functions of type $(n-1)$.
(ii) *For each k the eigenspace corresponding to the eigenvalue α_k is infinite dimensional; hence the operator T_r is not compact.*
(iii) $\alpha = 1$ *and* $\alpha = -1$ *are not eigenvalues of T_r and $\alpha = 0$ is an eigenvalue if and only if $\phi_k(r) = 0$ for some k.*

The operators T_r also arise naturally in connection with certain restriction operators R_r for the symplectic Fourier transform on $\mathbb{R}^{2n}$. In Section 5 we will show that we can write

$$f(z) = (4\pi)^{-2n}\omega_{2n}\int_0^\infty R_r f(z) r^{2n-1}\,dr$$

where R_r *are the restriction operators. These restriction operators* R_r are related to T_r by $R_r f = (2\pi)^{-n}T_r(\mathfrak{F}_s f)$ where $\mathfrak{F}_s f$ is the symplectic Fourier transform of f. Using the above relation and the spectral properties of T_r we are able to prove the following theorem regarding the mapping properties of the restriction operators R_r.

Theorem 4. *Assume that $n \geqslant 3$. Then the following are true.*

(i) $\|R_r f\|_{p'} \leqslant C_r \|f\|_p$, *for* $1 \leqslant p \leqslant 2$,

(ii) $\|R_r f\|_p \leqslant C_r \|f\|_p$, *for* $\dfrac{2n}{n+1} \leqslant p \leqslant 2$,

(iii) $\|R_r f\|_q \leqslant C_r \|f\|_p$, *for* $1 \leqslant p \leqslant \dfrac{2n}{n+1}$, $\quad q = \dfrac{n-1}{n+1} p'$.

To prove this theorem we need to use some mapping properties of the projection operators associated with special Hermite expansions. We will also show that the operators R_r are regularising in the sense that they take $L^2(\mathbb{C}^n)$ into $\mathcal{W}^s(\mathbb{C}^n)$ where $\mathcal{W}^s(\mathbb{C}^n)$ are the twisted Sobolev spaces to be defined in the sequel. The plan of the paper is as follows. In the next section we will define the functións e_k^λ and show that they have all the properties satisfied by the elementary spherical functions. In Section 3 we will prove Theorem 1. The spectral properties of T_r will be taken up in Section 4 and finally the restriction operators will be studied in Section 5.

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2. Elementary Spherical Functions on the Heisenberg Group

Let us briefly recall the definition and properties of elementary spherical functions. Let G be a semisimple, noncompact, connected Lie group with finite centre and K a maximal compact subgroup. Let $C_c(K\backslash G/K)$ denote the space of continuous functions with compact support on G which satisfy $f(k_1 g k_2) = f(g)$ for all k_1, k_2 in K. Such functions are called spherical or K-biinvariant. Then $C_c(K\backslash G/K)$ forms a commutative Banach algebra under convolution. An elementary spherical function ϕ is then defined to be a K-biinvariant continuous function with $\phi(e) = 1$ such that $f \to f * \phi(e)$ defines an algebra homomorphism of $C_c(K\backslash G/K)$.

The elementary spherical functions are characterised by the following properties (see [3]).

(i) They are eigenfunctions of the convolution operator:

$$f * \phi = \hat{\phi}(f)\phi,$$

where

$$\hat{\phi}(f) = \int_G f(x^{-1})\phi(x)\,dx.$$

(ii) They are eigenfunctions for a large class of left invariant differential operators on G.

(iii) They satisfy

$$\int_K \phi(xky)\,dk = \phi(x)\phi(y).$$

Let us now consider the case of the Heisenberg group H^n. The role of the K-biinvariant functions will be played by the radial functions on H^n. If $L^1_{\mathrm{rad}}(H^n)$ stand for the subspace of $L^1(H^n)$ containing all the radial functions then it is well known that L^1_{rad} is a commutative Banach algebra under convolution (see Hulanicki-Ricci [5]). This will play the role of $C_c(K\backslash G/K)$. On the Heisenberg group we have the following $(2n+1)$ left invariant vector fields X_j, Y_j, T:

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

The sublaplacian on the Heisenberg group is defined by

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

Let ϕ_k be the Laguerre functions of type $(n-1)$ defined by the generating function identity

$$\sum_{k=0}^{\infty} r^k\phi_k(z) = (1-r)^{-n}e^{-(1/4)(1+r)/(1-r)|z|^2}. \tag{2.1}$$

For any nonzero real number λ we set $\phi_k^\lambda(z) = \phi_k(|\lambda|^{1/2}z)$ and define e_k^λ by

$$e_k^\lambda(z,t) = \frac{k!\,(n-1)!}{(k+n-1)!}e^{i\lambda t}\phi_k^\lambda(z). \tag{2.2}$$

It follows from the properties of the Laguerre functions (see Szego [17]) that $e_k^\lambda(0,0) = 1$. We claim that these functions satisfy the following properties.

Theorem 2.1.

(i) *For any polynomial p with constant coefficients one has*

$$p(\mathcal{L})\,e_k^\lambda = p(2|\lambda|(2k+n))\,e_k^\lambda. \tag{2.3}$$

(ii) *For any radial function f on H^n one has*

$$f * e_k^\lambda = (2\pi)^n R_k(-\lambda,f)\,e_k^\lambda \tag{2.4}$$

where $R_k(\lambda,f)$ is defined by the formula

$$R_k(\lambda,f) = (2\pi)^{-n}\frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n}\tilde{f}(z,\lambda)\phi_k^\lambda(z)\,dz, \tag{2.5}$$

$\tilde{f}(z, \lambda)$ *being the inverse Fourier transform*

$$\tilde{f}(z, \lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z, t)\, dt.$$

(iii) *For any* (w, s) *in* H^n *with* $|w| = r$ *one has the identity*

$$\int_{|w'|=1} e_k^{\lambda}((z, t)\cdot(-rw', -s))\, d\sigma(w') = e_k^{\lambda}(z, t)\, e_k^{-\lambda}(w, s)$$

where $d\sigma$ *is the normalized surface measure on* $|w'| = 1$.

Thus we see that the functions e_k^{λ} have all the properties satisfied by the elementary spherical functions on a semisimple Lie group. So, they can be rightly called the elementary spherical functions for the Heisenberg group. The above properties of the function e_k^{λ} are fairly wellknown in the literature though not stated in the above form (see *e.g.* Stempak [12] and Strichartz [13]). Nevertheless, we will give a proof of the above theorem here.

To prove the theorem we need to recall several facts about the twisted convolution and the Weyl transform (see Folland [2], Mauceri [6] and Peetre [8]). The twisted convolution of two functions f and g defined on $\mathbb{C}^n$ is defined to be

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - w) g(w)\, e^{i/2\, \mathrm{Im}\, z\bar{w}}\, dw. \tag{2.7}$$

The Weyl transform of a function f is the bounded operator $W(f)$ acting on $L^2(\mathbb{R}^n)$ given by

$$W(f)\phi(\xi) = \int_{\mathbb{C}^n} f(z) W(z)\phi(\xi)\, dz \tag{2.8}$$

where $\phi \in L^2(\mathbb{R}^n)$ and $W(z)$ is the operator valued function

$$W(z)\phi(\xi) = e^{i(x\xi + xy/2)}\phi(\xi + y). \tag{2.9}$$

The relation between the Weyl transform and the twisted convolution is given by $W(f \times g) = W(f)W(g)$.

The Hermite functions $\Phi_\alpha(x)$ play an important role in the harmonic analysis on the Heisenberg group (see Folland [2]). These are eigenfunctions of the Hermite operator $H = (-\Delta + |x|^2)$, $H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha$. Let P_k be the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the k^{th} eigenspace spanned by $\{\Phi_\alpha : |\alpha| = k\}$. We also need certain properties of the special Hermite functions. Let us define

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\Phi_\alpha\left(\xi + \frac{y}{2}\right)\Phi_\beta\left(\xi - \frac{y}{2}\right) d\xi. \tag{2.10}$$

Then it is well known that they form a complete orthonormal system for $L^2(\mathbb{C}^n)$ (see Strichartz [13]). Let $\pi(\alpha, \beta)$ denote the operator defined by

$$\pi(\alpha, \beta)\phi = (\phi, \Phi_\alpha)\Phi_\beta.$$

Then one has the following proposition (see Folland [2] and Peetre [8]).

Proposition 2.1.

(i) $W(\bar{\Phi}_{\alpha\beta}) = (2\pi)^{n/2}\pi(\alpha, \beta)$ *and consequently* $\Phi_{\alpha\beta} \times \Phi_{\nu\delta} = 0$ *if* $\beta \neq \nu$ *and* $\Phi_{\alpha\beta} \times \Phi_{\beta\delta} = (2\pi)^{n/2}\Phi_{\alpha\delta}$.
(ii) $W(\phi_k) = (2\pi)^n P_k$ *and consequently* $\phi_j \times \phi_k = (2\pi)^n \delta_{jk}\phi_k$ *where* δ_{jk} *is the Kronecker* δ.

By abusing the notation slightly let us write $\phi_k(r)$ in place of $\phi_k(z)$ when $|z| = r$. Then the functions

$$\psi_k(r) = \left(\frac{2^{1-n}k!}{(k+n-1)!}\right)^{1/2} \phi_k(r)$$

form a complete orthonormal system in $L^2(\mathbb{R}_+, r^{2n-1}\,dr)$. If f is a radial function on $\mathbb{C}^n$ then we can expand f interms of $\psi_k(r)$ obtaining

$$f(z) = \sum_{k=0}^{\infty} R_k(f)\phi_k \tag{2.11}$$

where

$$R_k(f) = (2\pi)^{-n}\frac{k!\,(n-1)!}{(k+n-1)!}\int_{\mathbb{C}^n} f(z)\phi_k(z)\,dz.$$

This proves that when f is a radial function one has $f \times \phi_k = (2\pi)^n R_k(f)\phi_k$.

Now we are ready to prove Theorem 2.1. The assertion (i) is already proved in Strichartz [13] and so we will not prove it here. For (ii) an easy calculation reveals that

$$f * e_k^\lambda(z, t) = \frac{k!\,(n-1)!}{(k+n-1)!}e^{i\lambda t}\int_{\mathbb{C}^n} \tilde{f}(z - w, -\lambda)\phi_k^\lambda(w)\,e^{-i(\lambda/2)\operatorname{Im} z\bar{w}}\,dw.$$

It is therefore enough to show that the above integral is equal to $(2\pi)^n R_k(-\lambda, f)$ $\phi_k^\lambda(z)$. By rescaling we can assume that $\lambda = -1$. But then we need to show that

$$\int_{\mathbb{C}^n} \tilde{f}(z - w, 1)\phi_k(w)\,e^{i/2\operatorname{Im} z\bar{w}}\,dw = (2\pi)^n R_k(1, f)\phi_k(z)$$

which follows from the above remark as f is radial.

The proof of the assertion (iii) is similar. We have

$$\int_{|w'|=1} e_k^\lambda\left(z-w, t-s-\frac{1}{2}\operatorname{Im} z\bar{w}\right) d\sigma(w')$$

$$= \frac{k!\,(n-1)!}{(k+n-1)!} e^{i\lambda(t-s)} \int_{|w'|=1} \phi_k^\lambda(z-w)\, e^{-i(\lambda/2)\operatorname{Im} z\bar{w}}\, d\sigma(w').$$

Again we can assume that $\lambda = -1$. The function

$$F_k(z, w) = \int_{|w'|=1} \phi_k(z-w)\, e^{i/2 \operatorname{Im} z\bar{w}}\, d\sigma(w')$$

is a radial function of w and hence in view of (2.11)

$$F_k(z, w) = \sum_{j=0}^{\infty} R_j(F_k)\phi_j(w).$$

But

$$R_j(F_k) = (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} F_k(z, w)\phi_j(w)\, dw$$

$$= (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k \times \phi_j(z).$$

This proves that

$$F_k(z, w) = \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(z)\phi_k(w).$$

We have proved that

$$\int_{|w'|=1} e_k^\lambda\left(z-w, t-s-\frac{1}{2}\operatorname{Im} z\bar{w}\right) d\sigma(w') = e_k^\lambda(z, t)\, e_k^{-\lambda}(w, s).$$

Hence the theorem.

We would like to end this section with the following remark. Recently Benson-Jenkins-Ratcliff [1] has studied «spherical functions» on the Heisenberg group. Let K be a compact group of automorphisms of H^n such that the convolution algebra L_K^1 of K-invariant functions is commutative. A bounded, continuous K-invariant function φ such that $f \to \int f\varphi$ is an algebra homomorphism on L_K^1 is called a K-spherical function. In [1] the authors have studied the K-spherical functions for various different K. When $K = U(n)$, the K-spherical functions include our e_k^λ. (We are indebted to G. B. Folland and the referee for bringing the above work to our attention.)

3. Spherical Mean Value Operator on the Heisenberg Group

Let H^n be the n dimensional Heisenberg group defined in the previous section. Let Γ be the discrete subgroup $\{(0, 0, 2\pi k): k \in \mathbb{Z}\}$. Then the quotient group H^n/Γ is called the reduced Heisenberg group. For $g \in SO(2n, \mathbb{R})$ we define a rotation

$$\bar{g}: H^n/\Gamma \to H^n/\Gamma \quad \text{by} \quad \bar{g}(z, t) = (gz, t).$$

By a radial measure we mean a measure μ such that for every $g \in SO(2n, \mathbb{R})$ and every Borel set $S \subset H^n/\Gamma$ one has $\mu(S) = \mu(g^{-1}S)$. Let μ be such a rotation invariant probability measure with compact support. Then the operator $T_\mu f = (f * \mu)$ is called a spherical mean value operator. In the following theorem we identify all eigenvalues of T_μ as averages of the elementary spherical functions e_k^j as claimed in the introduction.

Theorem 3.1. *Assume that μ has no mass at the centre of H^n/Γ. Then all the eigenvalues of the operator T_μ are given by $\alpha_k(j)$ where*

$$\alpha_k(j) = \int_{H^n/\Gamma} e_k^j(z, t)\, d\mu.$$

Any function of the form $f * e_k^{-j}$ is an eigenfunction corresponding to $\alpha_k(j)$.

To prove this theorem we need to recall several results about the Fourier transform on the Heisenberg group (a good reference is Geller [4]). For each real $\lambda \neq 0$ we have an irreducible representation $\pi_\lambda(z, t)$ acting on $L^2(\mathbb{R}^n)$. It is defined by

$$\pi_\lambda(z, t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + 1/2 x\cdot y)}\phi(\xi + y). \tag{3.1}$$

The Fourier transform of a function f on H^n is the operator valued function $\hat{f}(\lambda)$ defined by

$$\hat{f}(\lambda) = \int_{H^n} f(z, t)\pi_\lambda(z, t)\, dz\, dt.$$

When f is a radial function $\hat{f}(\lambda)$ is given by the formula

$$\hat{f}(\lambda) = (2\pi)^n \sum_{k=0}^{\infty} R_k(\lambda, f) P_k(\lambda). \tag{3.2}$$

Here $R_k(\lambda, f)$ are as defined in (2.5) and $P_k(\lambda)$ are the projections of $L^2(\mathbb{R}^n)$ onto the space spanned by $\{|\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x): |\alpha| = k\}$. In the case of H^n/Γ, π_λ is a representation only if λ is an integer say $\lambda = j$.

Let χ_k be the function defined by

(3.3) $$\chi_k(z, t) = (2\pi)^{-n} \sum_{j=-\infty}^{\infty} e^{-ijt} e^{-j^2/2} \phi_k^j(z) |j|^n.$$

Then it is easy to see that $\chi_k \in L^2(H^n)$ and $\hat{\chi}_k(j) = e^{-j^2/2} P_k(j)$. We can now calculate the convolution $\mu * \chi_k$.

Lemma 3.1. *Let μ be a rotation invariant probability measure and let χ_k be as above. Then $(\mu * \chi_k)^\wedge(j) = \alpha_k(j)\hat{\chi}_k(j)$.*

PROOF. Since $\mu * \chi_k$ is a radial function we can calculate the Fourier transform using formula (3.2). A calculation shows that

$$\int e^{ijt} \mu * \chi_k(z, t)\, dt = (2\pi)^{-n} |j|^n e^{-j^2/2} \int_{\mathbb{C}^n} \phi_k^j(z - w) e^{-i(j/2)\operatorname{Im} z\bar{w}} d\mu^j(w)$$

where $\mu^j(w)$ is the j-th Fourier coefficient of μ in the t variable. It follows that

$$R_k(j, \mu * \chi_k) = (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} e^{-j^2/2} \int_{\mathbb{C}^n} \phi_k^j(w)\, d\mu^j(w)$$

and $R_i(j, \mu * \chi_k) = 0$ for $i \neq k$. This shows that

$$(\mu * \chi_k)^\wedge(j) = e^{-j^2/2} \left(\int_{H^n/\Gamma} e_k^j(z, t)\, d\mu \right) P_k(j).$$

This completes the proof of Lemma 3.1.

We can now prove the first part of Theorem 3.1. We claim that there exists k and $j \neq 0$ such that $(f * \chi_k)^\wedge(j) \neq 0$. Assuming the claim for a moment we will prove the theorem. Let $f * \mu = \alpha f$ for a non zero f in $L^2(H^n/\Gamma)$. Then in view of the lemma $\alpha_k(j)(f * \chi_k)^\wedge(j) = (f * \mu * \chi_k)^\wedge(j) = \alpha(f * \chi_k)^\wedge(j)$. This proves that $\alpha = \alpha_k(j)$ as $(f * \chi_k)^\wedge(j) \neq 0$.

We will now prove the claim. If $(f * \chi_k)^\wedge(j) = 0$ for all k and $j \neq 0$ then calculating the Fourier coefficients of $f * \chi_k$ one can see that all the Fourier coefficients of f except the zero-th one are zero and consequently

$$f(z, t) = Af(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z, t)\, dt.$$

As $f \neq 0$, $Af \neq 0$ and $f * \mu = \alpha f$ becomes $Af * \mu_0 = \alpha Af$ where now the convolution is on $\mathbb{R}^{2n}$ and μ_0 is the compactly supported measure

$$d\mu_0(z) = \int_0^{2\pi} d\mu(z, t).$$

But then Af has to be zero which is a contradiction.

To prove the second part of the theorem we need to recall Choquet's theorem. Let K^n stand for the set of all rotation invariant probability measures on H^n/Γ. Let ext (K^n) stand for the set of all extreme points of K^n. Then one has ext $(K^n) = E \cup \Delta_n$ where $E = \{\mu_{r,t}\}$ and $\Delta_n = \{\delta_t : t \in \mathbb{R}\}$. Here $\mu_{r,t}$, $r > 0$ is the normalized Lebesgue measure on the sphere $S_{r,t} = \{(z, t) : |z| = r\}$ in H^n/Γ and δ_t are the Dirac measures. Given $\mu \in K^n$, according to Choquet's theorem there is a measure M such that

$$\mu(B) = \int_{E \cup \Delta_n} \sigma(B)\, dM(\sigma). \tag{3.4}$$

If μ has no mass at the centre of H^n/Γ then $M(\Delta_n) = 0$ (see Stempak [11]) and we have

$$\mu(B) = \int_E \sigma(B)\, dM(\sigma). \tag{3.5}$$

Let us consider $f * e_k^{-j} * \mu$. In view of (3.5) we have

$$f * e_k^{-j} * \mu = \int_E (f * e_k^{-j} * \sigma)\, dM(\sigma). \tag{3.6}$$

When $\sigma = \mu_{r,t}$ we can easily calculate $e_k^{-j} * \sigma$. In fact,

$$e_k^{-j} * \mu_{r,t}(z, s) = \frac{k!\,(n-1)!}{(k+n-1)!} e^{ij(t-s)} \int_{\mathbb{C}^n} e^{i(j/2)\,\mathrm{Im}\, z\bar{w}} \phi_k^j(z - w)\, d\mu_r.$$

Recall that μ_r are the normalized surface measures on the sphere $|z| = r$. Since the functions $|j|^{n/2}\psi_k(|j|^{1/2}r)$ where $\psi_k(r)$ are defined in the previous section form an orthonormal basis for $L^2(\mathbb{R}_+, r^{2n-1}\, dr)$, we can expand the radial function

$$G_k(r) = \int_{\mathbb{C}^n} e^{i(j/2)\,\mathrm{Im}\, z\bar{w}} \phi_k^j(z - w)\, d\mu_r$$

in terms of them. In view of the relations $\phi_j \times \phi_k = (2\pi)^n \delta_{jk} \phi_k$ one calculates that $e_k^{-j} * \mu_{r,t} = e_k^j(w, t)\, e_k^{-j}(z, s)$ where $|w| = r$. This means that

$$f * e_k^{-j} * \mu_{r,t} = e_k^j(w, t)(f * e_k^{-j}) = \mu_{r,t}(e_k^j) f * e_k^{-j}$$

where

$$\mu_{r,t}(e_k^j) = \int e_k^j(z, s)\, d\mu_{r,t} = e_k^j(w, t), \qquad |w| = r.$$

Putting this back in (3.6) we get

$$f * e_k^{-j} * \mu = \left(\int_E \sigma(e_k^j)\, dM(\sigma)\right)(f * e_k^{-j}) = \mu(e_k^j) f * e_k^{-j}.$$

This completes the proof of Theorem 3.1.

When $\mu = \mu_{r,t}$ it is immediate that $\alpha_k(j) = e^j_k(r, t)$. This proves part (i) of Theorem 2. To prove that $\alpha = \pm 1$ are not eigenvalues of the operator $T_\mu = M_{r,t}$ we will prove in the next section (see Proposition 4.2) that for all $r > 0$ one has

$$\frac{k!\,(n-1)!}{(k+n-1)!}|\phi_k(r)| < 1.$$

This will then complete the proof of Theorem 2.

4. Spetral Properties of the Operator $T_r f = (2\pi)^n f \times \mu_r$

As we have seen in the introduction the study of the spherical mean value operator on the quotient group H^n/Γ involves operators of the form $f \to f \times \mu_r$. These operators are interesting in their own right and we will show in the next section that they are connected to the restriction operators for the symplectic Fourier transform. We study the spectral properties of the operators $T_r f = (2\pi)^n f \times \mu_r$ where μ_r is the normalized surface measure on the sphere $|z| = r$. For the operators T_r we have the following alternate description.

Theorem 4.1.

$$T_r f(z) = \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r) f \times \phi_k(z). \tag{4.1}$$

This Theorem is an immediate consequence of the following Proposition in view of the relation $W(f \times g) = W(f)W(g)$.

Proposition 4.1.

$$W(\mu_r) = \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r) P_k. \tag{4.2}$$

PROOF. Let $p_\epsilon(z)$ be the Poisson Kernel defined by

$$p_\epsilon(z) = \Gamma\left(\frac{2n+1}{2}\right)\pi^{-(2n+1)/2}\epsilon(\epsilon^2 + |z|^2)^{-(2n+1)/2}.$$

If F is a continuous function vanishing at infinity then we kno:/ that

$$\int_{\mathbb{C}^n} F(z)\, d\mu_r = \lim_{\epsilon \to 0} \int_{\mathbb{C}^n} p_\epsilon * \mu_r(z) F(z)\, dz.$$

Given functions ϕ and ψ in $L^2(\mathbb{R}^n)$ let us define $F(z)$ by $F(z) = (W(z)\phi, \psi)$. Since

$$(W(z)\phi, \psi) = e^{ixy/2} \int_{\mathbb{R}^n} e^{ix\xi}\phi(\xi + y)\bar{\psi}(\xi)\, d\xi$$

it is clear that $F(z)$ is a continuous function vanishing at infinity. Hence we have

$$\begin{aligned}(W(\mu_r)\phi, \psi) &= \int_{\mathbb{C}^n} (W(z)\phi, \psi)\, d\mu_r \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{C}^n} p_\epsilon * \mu_r(z) F(z)\, dz \\ &= \lim_{\epsilon \to 0} (W(p_\epsilon * \mu_r)\phi, \psi).\end{aligned}$$

Replacing ϕ by $P_k\phi$ we get

$$(W(\mu_r)P_k\phi, \psi) = \lim_{\epsilon \to 0} (W(p_\epsilon * \mu_r)P_k\phi, \psi).$$

Since $p_\epsilon * \mu_r$ is a radial function, its Weyl transform is given by

$$W(p_\epsilon * \mu_r) = (2\pi)^n \sum_{k=0}^{\infty} R_k(p_\epsilon * \mu_r)P_k$$

and consequently

$$(W(\mu_r)P_k\phi, \psi) = \lim_{\epsilon \to 0} (2\pi)^n R_k(p_\epsilon * \mu_r)(P_k\phi, \psi).$$

Let us now calculate $R_k(p_\epsilon * \mu_r)$. We have

$$\int_{\mathbb{C}^n} p_\epsilon * \mu_r(z)\phi_k(z)\, dz = \int_{\mathbb{C}^n} p_\epsilon * \phi_k(z)\, d\mu_r(z).$$

As $p_\epsilon * \phi_k(z) \to \phi_k(z)$ uniformly as $\epsilon \to 0$ one gets that

$$\lim_{\epsilon \to 0} R_k(p_\epsilon * \mu_r) = (2\pi)^{-n} \frac{k!\,(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} \phi_k(z)\, d\mu_r(z).$$

This proves that

$$(W(\mu_r)P_k\phi, \psi) = \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r)(P_k\phi, \psi).$$

Hence the proposition.

In view of Theorem 4.1 it is easy to see that T_r is a bounded self adjoint operator. In fact, the functions ϕ_k satisfy the estimate

$$|\phi_k(r)| < \frac{k!\,(n-1)!}{(k+n-1)!}$$

(see Proposition 4.2) and $f \times \phi_k$ are orthogonal projections associated to the special Hermite expansions (see [18]) and hence T_r is a bounded self adjoint operator on $L^2(\mathbb{C}^n)$. It is also clear that all the eigenvalues of T_r are given by

$$\alpha_k = \frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)$$

and any function of the form $f \times \phi_k$ is an eigenfunction corresponding to the eigenvalue α_k.

In view of the relations $W(\bar{\Phi}_{\alpha\beta}) = (2\pi)^{n/2}\pi(\alpha,\beta)$ and $W(\phi_k) = (2\pi)^n P_k$ one checks that $\bar{\Phi}_{\alpha\beta} \times \phi_k = (2\pi)^n \bar{\Phi}_{\alpha\beta}$ provided $|\alpha| = k$. This means that the functions $\bar{\Phi}_{\alpha\beta}$ are eigenfunctions of T_r with eigenvalue α_k. As $\Phi_{\alpha\beta}$ are lineary independent this shows that the eigenspace corresponding to each α_k is infinite dimensional. Thus parts (i) and (ii) of Theorem 3 are proved. Part (iii) follows from the next proposition. The result of the proposition is not new but the novelty lies in the proof.

Proposition 4.2. *For any k and $r > 0$,*

$$|\phi_k(r)| < \frac{(k+n-1)!}{k!\,(n-1)!} = \phi_k(0). \tag{4.3}$$

PROOF. The proof is based on the following fact. If the Fourier transform of a function f is positive then $|f(x)| < f(0)$ for $x \neq 0$. We will show that the Fourier transform of the function $L_k^{n-1}(1/2\,|x|^2)\,e^{-1/4|x|^2}$ on $\mathbb{R}^n$ is positive. This means that

$$\left| L_k^{n-1}\left(\frac{1}{2}|x|^2\right) e^{-1/4|x|^2} \right| = |\phi_k(r)| < \frac{(k+n-1)!}{k!\,(n-1)!}.$$

To do this we calculate the kernel $K(x,y)$ of the projection P_k in different ways.

From the very definition one has the formula

$$K(x,y) = \sum_{|\alpha|=k} \Phi_\alpha(x)\Phi_\alpha(y).$$

On the other hand, as $W(\phi_k) = (2\pi)^n P_k$, $K(x,y)$ is also given by

$$K(x,y) = \int_{\mathbb{R}^n} \phi_k(\xi, y-x)\, e^{1/2\, i\xi(x+y)}\, d\xi$$

where we have written

$$\phi_k(\xi, \eta) = \phi_k\left(\frac{1}{2}(|\xi|^2 + |\eta|^2)\right).$$

Therefore, setting $x = y$ we get

$$(2\pi)^n \sum_{|\alpha| = k} (\Phi_\alpha(x))^2 = \int_{\mathbb{R}^n} e^{ix\xi} L_k^{n-1}\left(\frac{1}{2}|\xi|^2\right) e^{-1/4|\xi|^2} d\xi.$$

This proves that the Fourier transform of the Laguerre function is non-negative.

We will conclude this section with a result analogous to a theorem of Ragozin on the convolution of rotation invariant measures on $\mathbb{R}^n$. In [9] Ragozin proved that it μ is the surface measure on the unit sphere in $\mathbb{R}^n$, $n \geqslant 2$ then $\mu * \mu$ is absolutely continuous with respect to the Lebesgue measure. Here we will prove a similar result for the twisted convolution. Moreover, we will identify the density explicitly.

Proposition 4.3. *Assume that $n \geqslant 2$. Then $\mu_r \times \mu_r$ is absolutely continuous with respect to the Lebesgue measure. The density is given by*

$$J(z) = (2\pi)^{-2n} \sum_{k=0}^{\infty} \left(\frac{k!\,(n-1)!}{(k+n-1)!}\right)^2 (\phi_k(r))^2 \phi_k(z) \tag{4.4}$$

where the series converges uniformly on every compact subset of the form $0 < a \leqslant |z| \leqslant b$.

PROOF. In view of Theorem 4.1 one has

$$f \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r) f \times \phi_k(z).$$

This in turn gives us

$$f \times \mu_r \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \left(\frac{k!\,(n-1)!}{(k+n-1)!}\right)^2 \phi_k(r)^2 f \times \phi_k(z).$$

This shows that $\mu_r \times \mu_r$ is given by $J(z)\,dz$. If $a \leqslant |z| \leqslant b$ then one has the asymptotic estimate (see Szego [17]),

$$L_k^{n-1}(r^2)\, e^{-r^2/2} r^{n-1} = K^{-(n-1)/2} \frac{(k+n-1)!}{k!} J_{n-1}\left(2r\sqrt{K}\right) + O(k^{(n-1)/2-3/4}) \tag{4.5}$$

where $K = k + n/2$ and the bound holds uniformly in $a \leqslant r \leqslant b$. As $n \geqslant 2$, this shows that the series defining $J(z)$ converges uniformly on compact sets of the form $0 < a \leqslant |z| \leqslant b$.

5. A Restriction Theorem for the Symplectic Fourier Transform on $\mathbb{R}^{2n}$

The operator $T_r f = (2\pi)^n f * \mu_r$ is related to a restriction operator R_r for the symplectic Fourier transform as we are going to see now. Before that let us briefly recall the usual restriction operators for the Fourier transform on $\mathbb{R}^n$. If we define

$$Q_r f(x) = (2\pi)^{-n} \int_{|w|=1} e^{irxw} \hat{f}(rw)\, d\sigma(w), \tag{5.1}$$

then the Fourier inversion formula can be written as

$$f(x) = \int_0^\infty Q_r f(x) r^{n-1}\, dr. \tag{5.2}$$

The operators f going to $Q_r f$ are called the restriction operators for the Fourier transform. It is well known that

$$\|Q_r f\|_{p'} \leqslant C_r \|f\|_p, \qquad 1 \leqslant p \leqslant \frac{2(n+1)}{n+3}. \tag{5.3}$$

It is also known that such an estimate is not possible when $p > \frac{2n}{n+1}$. As a consequence of (5.3) one can prove the Stein-Tomas [19] restriction theorem

$$\left(\int_{|w|=1} |\hat{f}(w)|^2\, d\sigma \right)^{1/2} \leqslant C \|f\|_p, \qquad 1 \leqslant p \leqslant \frac{2(n+1)}{n+3} \tag{5.4}$$

which justifies the name restriction operators.

Let us now consider the symplectic Fourier transform on $\mathbb{R}^{2n}$. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the symplectic Fourier transform is defined as

$$\mathfrak{F}_s f(z) = \int_{\mathbb{C}^n} f(w)\, e^{-i/2 \operatorname{Im} z\bar{w}}\, dw$$

and the inversion formula is given by

$$f(z) = (4\pi)^{-2n} \int_{\mathbb{C}^n} \mathfrak{F}_s f(w)\, e^{-i/2 \operatorname{Im} z\bar{w}}\, dw.$$

We can rewrite the inversion formula as

$$f(z) = (4\pi)^{-2n} \int_{\mathbb{C}^n} \mathfrak{F}_s f(z-w)\, e^{i/2 \operatorname{Im} z\bar{w}}\, dw.$$

Let ω_{2n} be the surface area of $|z| = 1$. We can now write

$$f(z) = (4\pi)^{-2n}\omega_{2n}\int_0^\infty r^{2n-1}\,dr\int_{|w|=r}\mathfrak{F}_s f(z-w)\,e^{i/2\,\mathrm{Im}\,z\bar{w}}\,d\mu_r$$

where μ_r is the normalized surface measure on $|w| = r$. If we define $R_r f$ by

$$R_r f(z) = \int_{|w|=r}\mathfrak{F}_s f(z-w)\,e^{i/2\,\mathrm{Im}\,z\bar{w}}\,d\mu_r = \mathfrak{F}_s f \times \mu_r \tag{5.5}$$

then we have obtained the inversion in the form

$$f(z) = (4\pi)^{-2n}\omega_{2n}\int_0^\infty R_r f(z) r^{2n-1}\,dr. \tag{5.6}$$

This is the analogue of (5.2) and that is the reason why we call them restriction operators. Unlike the operators $Q_r f$, these $R_r f$ are no longer eigenfunctions of the Laplacian.

The relation between T_r and R_r is now clear: $R_r f = (2\pi)^{-n} T_r(\mathfrak{F}_s f)$. In view of this we have the alternate formula

$$R_r f = (2\pi)^{-n}\sum_{k=0}^{\infty}\frac{k!\,(n-1)!}{(k+n-1)!}\phi_k(r)(\mathfrak{F}_s f \times \phi_k). \tag{5.7}$$

Using the bounds for the functions ϕ_k one immediately gets

$$\|R_r f\|_2 \leqslant C\|\mathfrak{F}_s f\|_2 \leqslant C\|f\|_2.$$

If we interpolate with the trivial estimate

$$\|R_r f\|_\infty = \|\mathfrak{F}_s f \times \mu_r\|_\infty \leqslant C\|\mathfrak{F}_s f\|_\infty \leqslant C\|f\|_1$$

we obtain the following boundedness result.

Proposition 5.1. *For $1 \leqslant p \leqslant 2$, one has*

$$\|R_r f\|_{p'} \leqslant C\|f\|_p. \tag{5.8}$$

Using the asymptotic properties of the Laguerre functions $\phi_k(r)$ we can prove the following regularity theorem. To state the result we introduce the twisted Sobolev spaces $\mathcal{W}^s(\mathbb{C}^n)$. On $\mathbb{C}^n$ consider the $2n$ vector fields

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4}z_j$$

and the operator

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j).$$

The Hermite operator H and L are related by $W(Lf) = W(f)H$. Using this formula we can define L^s by $W(L^s f) = W(f)H^s$. We then define the twisted Sobolev spaces by

$$\mathcal{W}^s(\mathbb{C}^n) = \{f \in L^2(\mathbb{C}^n) : L^s f \in L^2(\mathbb{C}^n)\}. \tag{5.9}$$

With this definition we can now prove the following theorem.

Theorem 5.1. *If* $s \leqslant (2n-1)/4$ *then* R_r *maps* $L^2(\mathbb{C}^n)$ *continuously into* $\mathcal{W}^s(\mathbb{C}^n)$.

PROOF. Since $L^s(\phi_k) = (2k+n)^s \phi_k$ we have

$$L^s(R_r f) = L^s(\mathfrak{F}_s f \times \mu_r) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r)(2k+n)^s(\mathfrak{F}_s f \times \phi_k)$$

(where we have used the relation $L^s(f \times g) = (f \times L^s g)$). In view of the estimate (4.5) it is clear that $(2k+n)^s \dfrac{k!\,(n-1)!}{(k+n-1)!} \phi_k(r)$ is bounded as long as $s \leqslant (2n-1)/4$ uniformly in k. This proves that $L^s(R_r f)$ belongs to $\mathcal{W}^s(\mathbb{C}^n)$ and $\|R_r f\|_{\mathcal{W}^s} \leqslant C\|f\|_2$.

We will now proceed to prove parts (ii) and (iii) of Theorem 4 of the introduction. We have already shown that $\|R_r f\|_{p'} \leqslant C\|f\|_p$ for $1 \leqslant p \leqslant 2$. The assertion (ii) that $\|R_r f\|_p \leqslant C\|f\|_p$ for $2n/(n+1) \leqslant p \leqslant 2$ will follow once we show that the following is true.

Proposition 5.2. *Assume that* $n \geqslant 3$. *Then*

$$\|f \times \mu_r\|_{2n/(n+1)} \leqslant C\|f\|_{2n/(n-1)}. \tag{5.10}$$

To see that the assertion (ii) *follows from* (5.10) *we observe that*

$$\begin{aligned}\|R_r f\|_{2n/(n+1)} &= \|\mathfrak{F}_s f \times \mu_r\|_{2n/(n+1)} \\ &\leqslant C\|\mathfrak{F}_s f\|_{2n/(n-1)} \leqslant C\|f\|_{2n/(n+1)}.\end{aligned}$$

An interpolation with $\|R_r f\|_2 \leqslant C\|f\|_2$ *proves the assertion.*

To prove Proposition 5.2 we need the following estimate for the projections $f \to f \times \phi_k$.

Proposition 5.3.

$$\|f \times \phi_k\|_{2n/(n-1)} \leqslant C\|f\|_{2n/(n+1)}. \tag{5.11}$$

As the projections f going to $f \times \phi_k$ are self adjoint we also have

$$\|f \times \phi_k\|_{2n/(n+1)} \leqslant C\|f\|_{2n/(n-1)}.$$

Using this one immediately gets

$$\|f \times \mu_r\|_{2n/(n+1)} \leqslant C \sum_{k=0}^{\infty} \frac{k!\,(n-1)!}{(k+n-1)!} |\phi_k(r)| \, \|f \times \phi_k\|_{2n/(n+1)}$$

$$\leqslant C\left(\sum_{k=0}^{\infty} (2k+n)^{-(n/2)+(1/4)}\right)\|f\|_{2n/(n-1)}.$$

As $n \geqslant 3$ the series converges and this proves Proposition 5.2. So it remains to prove Proposition 5.3.

We have proved this proposition in [18]. We will briefly indicate the proof for the sake of completeness. The definition of the Laguerre polynomials L_k^α can be extended even for complex values of α, $\operatorname{Re}\alpha > -1/2$. We then consider the functions

$$\psi_k^\alpha(z) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} L_k^\alpha\left(\frac{1}{2}|z|^2\right) e^{-1/4|z|^2}$$

and define a family of operators $G_k^\alpha f = f \times \psi_k^{n\alpha}$. One verifies that this is an admissible analytic family of operators. By Stein's interpolation theorem [10] the estimate (5.11) will follow from the two estimates

$$\|G_k^{i\tau} f\|_\infty \leqslant C(1+|\tau|)^{1/2}\|f\|_1.$$
$$\|G_k^{1+i\tau} f\|_2 \leqslant C(1+|\tau|)^n k^{-n}\|f\|_2.$$

These estimates can be proved using certain bounds for the Laguerre function ψ_k^α. We refer to [18] for details.

We will now complete the proof of Theorem 4 by proving the assertion (iii) namely,

$$\|R_r f\|_q \leqslant C\|f\|_p \quad \text{for} \quad 1 \leqslant p \leqslant \frac{2n}{n+1} \quad \text{where} \quad q = \frac{n-1}{n+1}p'.$$

When $p = 2n/(n+1)$, $q = p$ and we already have the inequality $\|R_r f\|_{2n/(n+1)} \leqslant C\|f\|_{2n/(n+1)}$. Interpolating with the estimate $\|R_r f\|_\infty \leqslant C\|f\|_1$ we complete the proof.

We conclude the paper with the following remarks. The estimate $\|R_r f\|_q \leqslant C\|f\|_p$ was established in the interval $1 \leqslant p \leqslant 2n/(n+1)$. By increasing q we can extend the interval of validity. For example, by interpolating with the estimate $\|R_r f\|_{4n/(2n+1)} \leqslant C\|f\|_{4n/(2n+1)}$ we can prove

$$\|R_r f\|_q \leqslant C\|f\|_p, \qquad q = \frac{2n-1}{2n+1}p'$$

in the range $1 \leqslant p \leqslant \frac{4n}{2n+1}\cdot$ Similarly by decreasing the interval $1 \leqslant p \leqslant \frac{2n}{n+1}$ we can obtain estimates valid with $q = \gamma p'$ where $\gamma < \frac{n-1}{n+1}\cdot$ Another remark we would like to make is regarding the assumption $n \geqslant 3$. It would be interesting to see if the Theorem 4 remains true for $n = 1$ and $n = 2$ also.

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Fonctions a support compact dans les analyses multi-résolutions

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Résumé

Nous nous intéressons aux fonctions à support compact dans une analyse multi-résolution, en particulier à celles de support minimum. Nous montrons que ces classes de fonctions sont stables par dérivation et primitivation et indiquons une méthode pour de nombreux calculs numériques.

Abstract

The main topic of this paper is the study of compactly supported functions in a multi-resolution analysis and especially of the minimally supported ones. We will show that this class of functions is stable under differentiation and integration and how to compute basic quantities with them.

1. Propriétés de base des analyses multi-résolutions

Nous rappelons dans cette section les propriétés des analyses multi-résolutions dont nous aurons besoin par la suite. La notion d'analyse multi-résolution a été introduite en 1986 par S. Mallat [10] et la plupart des propriétés que nous utiliserons sont démontrées dans le livre d'Y. Meyer [11] et la thèse d'A. Cohen [4].

Une *analyse multi-résolution* est une suite de sous-espaces fermés $(V_j)_{j\in\mathbb{Z}}$ de $L^2(\mathbb{R})$ qui vérifie les propriétés suivantes:

(1.1) $V_j \subset V_{j+1}$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ et $\bigcup_{j\in\mathbb{Z}} V_j$ est dense dans $L^2(\mathbb{R})$,

(1.2) $f(x) \in V_j$ si et seulement si $f(2x) \in V_{j+1}$,

(1.3) V_0 a une base de Riesz de la forme $g(x-k)$, $k \in \mathbb{Z}$, avec g à valeurs réelles. Le plus souvent on impose à g d'être à décroissance rapide à l'infini,

(1.4) Pour tout $k \in \mathbb{N}$, $x^k g \in L^2$.

À quelles conditions une fonction g vérifiant (1.4) engendre-t-elle une analyse multi-résolution? La réponse est fournie par le théorème suivant, essentiellement dû à A. Cohen [4].

Proposition 1. *Soit $g \in L^2(\mathbb{R})$ à valeurs réelles. Alors les propriétés suivantes sont équivalentes*:

(i) *g vérifie* (1.3) *et* (1.4) *pour une analyse multi-résolution* $(V_j)_{j\in\mathbb{Z}}$;

(ii) *la transformée de Fourier de g*

$$\hat{g}(\xi) = \int g(x)\, e^{-ix\xi}\, dx,$$

s'écrit

$$\hat{g}(\xi) = \hat{g}(0) \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

où $\hat{g}(0) \neq 0$, m_0 est C^∞, 2π-périodique et vérifie

(2.1) $m_0(0) = 1$,

(2.2) $\sup_{n\in\mathbb{N}} \left\| \prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right) \right\|_\infty < +\infty$,

(2.3) *il existe un compact K réunion finie d'intervalles disjoints tel que*

(j) $\sum_{k\in\mathbb{Z}} \chi_K(\xi + 2k\pi) = 1$ *p.p.*

(jj) *pour tout $\xi \in K$ et $j \in \mathbb{N}^*$,* $m_0\left(\frac{\xi}{2^j}\right) \neq 0$.

DÉMONSTRATION. La démonstration repose sur le lemme suivant:

Lemme 1. *Si α et $\beta \in L^2(\mathbb{R})$ vérifient* (1.4) *alors le série*

$$\sum_{k \in \mathbb{Z}} \hat{\alpha}(\xi + 2k\pi) \times \bar{\hat{\beta}}(\xi + 2k\pi)$$

converge en tout point vers une fonction C^∞.

Le lemme est immédiat, puisque d'après la formule sommatoire de Poisson on a

$$\sum_{k \in \mathbb{Z}} \alpha(x-k)\, e^{i(k-x)\xi} = \sum_{k \in \mathbb{Z}} \hat{\alpha}(\xi + 2k\pi)\, e^{2ik\pi x}$$

et donc

$$\sum_{k \in \mathbb{Z}} \hat{\alpha}(\xi + 2k\pi)\bar{\hat{\beta}}(\xi + 2k\pi) = \int_0^1 \left\{ \sum_{k \in \mathbb{Z}} \alpha(x-k)\, e^{ik\xi} \right\} \left\{ \sum_{k \in \mathbb{Z}} \bar{\beta}(x-k)\, e^{-ik\xi} \right\} dx.$$

(1.4) implique la convergence pour tout N de

$$\int_0^1 \left\{ \sum_{k \in \mathbb{Z}} |k|^N |\alpha(x-k)| \right\}^2 dx,$$

d'où la convergence et la régularité de la série étudiée.

Si g vérifie (1.3) et (1.4), alors l'hypothèse que les $g(x-k)$ forment une base de Riesz implique l'existence d'une constante $A \geqslant 1$ telle que

$$\text{pour tout} \quad \xi \in \mathbb{R} \qquad \frac{1}{A} \leqslant \sum |\hat{g}(\xi + 2k\pi)|^2 \leqslant A.$$

De plus, puisque $g\left(\dfrac{x}{2}\right) \in V_{-1} \subset V_0$, on a $g\left(\dfrac{x}{2}\right) = \sum_{k \in \mathbb{Z}} a_k g(x-k)$ avec $(a_k) \in l^2(\mathbb{Z})$. D'où $\hat{g}(2\xi) = m_0(\xi)\hat{g}(\xi)$ avec

$$m_0(\xi) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k\, e^{-ik\xi}$$

2π périodique et localement de carré intégrable. En particulier, on a

$$m_0(\xi) \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} \hat{g}(2\xi + 4k\pi)\bar{\hat{g}}(\xi + 2k\pi)$$

et donc, puisque $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule pas, m_0 est C^∞. De plus

$$\sum_{k \in \mathbb{Z}} |\hat{g}(4k\pi)|^2 = |m_0(0)|^2 \sum_{k \in \mathbb{Z}} |\hat{g}(2k\pi)|^2$$

et donc $|m_0(0)| \leqslant 1$; d'où

$$|\hat{g}(\xi)|^2 = \prod_{j=1}^{N} \left| m_0\left(\frac{\xi}{2^j}\right)\right|^2 \left|\hat{g}\left(\frac{\xi}{2^N}\right)\right|^2 = |\hat{g}(0)|^2 \prod_{j=1}^{\infty} \left| m_0\left(\frac{\xi}{2^j}\right)\right|^2,$$

et donc nécessairement $\hat{g}(0) \neq 0$ et $m_0(0) = 1$.

Comme $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule jamais, il existe autour de tout point ξ de $[-\pi, \pi]$ un intervalle ouvert I_ξ et un nombre entier $k(\xi)$ tels que $\hat{g}$ ne s'annule pas sur $I_\xi + 2k(\xi)\pi$. Par compacité de $[-\pi, \pi]$, on peut extraire de ces I_ξ un recouvrement fini de $[-\pi, \pi]$. La construction de K est alors immédiate. De plus, sur K, $\inf_{\xi\in K} |\hat{g}(\xi)|$ est positif, d'où

$$\prod_{j=0}^{N-1} m_0\left(\frac{\xi}{2^j}\right) = \frac{\hat{g}(2^N\xi)}{\hat{g}(\xi)}$$

est borné indépendamment de N sur K d'où sur $\mathbb{R}$ tout entier.

Réciproquement supposons que $g \in L^2$ et

$$\hat{g}(\xi) = \hat{g}(0) \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

où m_0 vérifie (2.1) à (2.3). Alors $\hat{g}$ est C^∞ et toutes ses dérivées appartiennent à L^2: on a en effet

$$\hat{g}^{(N+1)}(\xi) = \sum_{\alpha=1}^{N+1} \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} m_0\left(\frac{\xi}{2^k}\right) \frac{1}{2^{j\alpha}} m_0^{(\alpha)}\left(\frac{\xi}{2^j}\right) C_{N+1}^{\alpha} \frac{1}{2^{j(N+1-\alpha)}} \hat{g}^{(N+1-\alpha)}\left(\frac{\xi}{2^j}\right)$$

d'où

$$\|\hat{g}^{(N+1)}\|_2 \leqslant C_N \left(\sum_{j=1}^{\infty} \frac{1}{2^{j(N+1/2)}}\right)\left(\sum_{\alpha=1}^{N+1} \|\hat{g}^{(N+1-\alpha)}\|_2\right).$$

On en conclut que g vérifie (1.4); en particulier $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ est borné. De plus, $\sum |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule pas, car sur $K |\hat{g}(\xi)|$ est minoré par un nombre positif. Les $g(x - k)$ sont alors une base de Riesz d'un sous-espace fermé V_0 de $L^2(\mathbb{R})$. On définit V_j par

$$f(x) \in V_j \quad \text{si et seulement si} \quad f\left(\frac{x}{2^j}\right) \in V_0.$$

Puisque $\hat{g}(2\xi) = m_0(\xi)\hat{g}(\xi)$, on a facilement que $V_j \subset V_{j+1}$. De plus la série $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ converge uniformément sur tout compact (série de fonctions positives continues et de somme continue) et donc il existe $A > 0$ tel que

$$\text{pour tout} \quad f \in V_0, \qquad \int_{|\xi|>A} |\hat{f}(\xi)|^2 \, d\xi < \frac{1}{2} \|f\|_2^2.$$

Si $f \in \bigcap_{j\in\mathbb{Z}} V_j$ alors

$$\text{pour tour} \quad j \in \mathbb{Z}, \qquad \int_{|\xi|>A2^j} |\hat{f}(\xi)|^2\, d\xi < \frac{1}{2}\, \|f\|_2^2$$

d'où $f = 0$ (en faisant tendre j vers $-\infty$). Par ailleurs si $\hat{f} \in L^2$ est à support compact, alors

$$\hat{f} = \lim_{j\to+\infty} \Big(\sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2^j 2k\pi)\Big)\hat{g}\Big(\frac{\xi}{2^j}\Big)\hat{g}(0)^{-1}$$

dans L^2 et donc $\bigcup_j V_j$ est dense dans L^2: en effet $\hat{f}(\xi)\hat{g}(\xi/2^j)\hat{g}(0)^{-1}$ tend vers $\hat{f}(\xi)$ et, pour j assez grand,

$$\Big\|\Big(\sum_{k\neq 0} \hat{f}(\xi + 2^j 2k\pi)\Big)\hat{g}\Big(\frac{\xi}{2^j}\Big)\Big\|_2^2 = \int |\hat{f}(\xi)|^2 \sum_{k\neq 0}\Big|\hat{g}\Big(\frac{\xi}{2^j} + 2k\pi\Big)\Big|^2 d\xi,$$

ce qui tend vers

$$\Big(\int |\hat{f}(\xi)|^2\, d\xi\Big)\Big(\sum_{k\neq 0} |\hat{g}(2k\pi)|^2\Big)\cdot$$

Or on a nécessairement pour $k \neq 0$

$$\hat{g}(2k\pi) = \hat{g}(2^N 2k\pi) = \lim_{N\to+\infty} \hat{g}(2^N 2k\pi) = 0.$$

La proposition est donc démontrée.

Corollaire 1. *Si g vérifie* (1.3) *et* (1.4) *pour une analyse multi-résolution, alors*

(3.1) $\hat{g}(0) \neq 0$ *et* $\hat{g}(2k\pi) = 0$ *pour* $k \in \mathbb{Z}^*$,

(3.2) $m_0(0) = 1$ *et* $m_0(\pi) = 0$,

(3.3) $\sum g(x-k) = \hat{g}(0)$.

Il reste à verifier que $m_0(\pi) = 0$. Or, d'après (3.1), on a

$$0 = \sum_{k\in\mathbb{Z}} |\hat{g}(2\pi + 4k\pi)|^2 = |m_0(\pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{g}(\pi + 2k\pi)|^2;$$

comme $\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule jamais, $m_0(\pi) = 0$. Quant à (3.3) cela vient de la formule sommatoire de Poisson

$$\sum_{k\in\mathbb{Z}} g(x-k) = \sum_{k\in\mathbb{Z}} \hat{g}(2k\pi)\, e^{2ik\pi x}.$$

Remarques.

(i) Il est clair que dans (2.3) on peut supposer que $0 \in \mathrm{Int}\, K$, puisque $m_0(0) = 1$ et que donc m_0 ne s'annule pas sur un voisinage de 0. De plus, si m_0 vérifie (2.1) à (2.3) avec 0 intérieur à K, le produit infini

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

est dans L^2 si et seulement si on a

$$\sup_N \left\| \prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right) \chi_K\left(\frac{\xi}{2^N}\right) \right\|_2 < \infty;$$

cela est immédiat puisque

$$\prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right) \chi_K\left(\frac{\xi}{2^N}\right) = \chi_K\left(\frac{\xi}{2^N}\right) \frac{\hat{g}(\xi)}{\hat{g}(\xi/2^N)}.$$

De plus on a alors

$$\prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right) \chi_K\left(\frac{\xi}{2^N}\right) \to \hat{g} \quad \text{dans } L^2.$$

(ii) Si m_0 vérifie (2.1), (2.3) et

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1,$$

alors si

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right),$$

g est automatiquement dans L^2 et les $g(x - k)$ sont orthonormées. (Il suffit de vérifier que les $\theta_N(x - k)$ sont orthonormées, où

$$\hat{\theta}_N(\xi) = \prod_{j=1}^{N} m_0\left(\frac{\xi}{2^j}\right) \chi_K\left(\frac{\xi}{2^N}\right).\Big)$$

2. Produits infinis de polynômes trigonométriques

Si m_0 est un polynôme trigonométrique tel que

$$m_0(0) = 1 \quad \text{et si} \quad \hat{g}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right),$$

alors g est une distribution à support compact. Si de plus m_0 vérifie (2.3), et $g \in L^2$, alors (2.2) est immédiat, puisque $\hat{g}$ est bornée (g est intégrable) et que $|\hat{g}|$ est minorée sur K.

Dans le cas d'un polynôme trigonométrique, la condition (2.3) s'exprime plus algébriquement de la manière suivante:

Proposition 2. *Si m_0 est un polynôme trigonométrique tel que $m_0(0) = 1$ alors la condition* (2.3) *est équivalente à*:

(i) *pour tout* $\xi \in \mathbb{R}$, $\quad |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 > 0$,

(ii) *il n'existe pas de* $\xi_0 \in]0, 2\pi[$ *tel que*:

$$\text{pour tout} \quad N \in \mathbb{N}, \qquad m_0(2^N \xi_0 + \pi) = 0.$$

DÉMONSTRATION. Le sens direct est immédiat. (2.3) signifie que si on note

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

alors pour $\xi \in \mathbb{R}$, il existe $k \in \mathbb{Z}$, $\hat{g}(\xi + 2k\pi) \neq 0$. Si on avait

$$m_0(\xi_1) = m_0(\xi_1 + \pi) = 0$$

pour un $\xi_1 \in \mathbb{R}$ alors on aurait $\hat{g}(2\xi_1 + 2k\pi) = 0$ quel que soit k. De même supposons que (ii) ne soit pas vérifié; comme m_0 n'a qu'un nombre fini de zéros modulo 2π, on doit avoir pour deux entiers N et M: $2^N\xi_0 - 2^M\xi_0 \in 2\pi\mathbb{Z}$; on ne peut avoir $2^M\xi_0 \in 2\pi\mathbb{Z}$ (sinon pour un $M' \geqslant 0$ on aurait $2^{M'}\xi_0 \in \pi + 2\pi\mathbb{Z}$ et $m_0(2^{M'}\xi_0 + \pi) = 1$; quitte à changer ξ_0 en $2^M\xi_0 \bmod (2\pi)$, on peut supposer $M = 0$. On va montrer que $\hat{g}(\xi_0 + 2k\pi) = 0$ pour tout k. Il revient au même de considérer $\hat{g}(2^N\xi_0 + 2k\pi)$. Or

$$\hat{g}(2^N\xi_0 + 2k\pi) = \hat{g}\left(\xi_0 + \frac{2k\pi}{2^N}\right) \prod_{j=1}^{N} m_0\left(\frac{2^N\xi_0 + 2k\pi}{2^j}\right);$$

si k n'est pas divisible par 2^N on obtient 0; si $k = 2^N k'$, alors on a $\xi_0 = 2^N\xi_0 - 2k_0\pi$ où $1 \leqslant k_0 \leqslant 2^N - 2$, et on est ramené à étudier $\hat{g}(2^N\xi_0 + 2(k' - k_0)\pi)$; si $k' = 0$, $k' - k_0$ n'est pas divisible par 2^N et $\hat{g}(2^N\xi_0 + 2(k' - k_0)\pi) = 0$; si $k' \neq 0$, alors $|k' - k_0| \leqslant |k'| + 2^N - 2 < 2^N|k'| = |k|$ et ce cas ne peut donc indéfiniment se reproduire. (2.3) entraîne donc bien (i) et (ii).

Réciproquement supposons que (2.3) ne soit pas vérifié et que (i) soit vrai; on va montrer que (ii) est faux. On peut supposer que

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

est dans L^2, quitte à remplacer m_0 par $\left(\frac{1+e^{i\xi}}{2}\right)^M m_0$ avec M suffisamment grand $\Big($en effet $\prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$ est borné sur $[-\pi, \pi]$ par continuité sur un compact et on en déduit facilement que $|\hat{g}(\xi)| \leqslant C(1+|\xi|)^\alpha$ avec

$$\alpha = \frac{\log \|m_0\|_\infty}{\log 2};$$

il suffit de prendre $M > \alpha + 1$ car on a

$$\prod_{j=1}^{\infty}\left(\frac{1+e^{i\xi/2^j}}{2}\right)^M m_0\left(\frac{\xi}{2^j}\right) = \left(\frac{e^{i\xi}-1}{i\xi}\right)^M \hat{g}(\xi)\Big):$$

cette substitution n'affecte ni (i) ni (ii). Si (2.3) n'est pas vérifié, alors

$$Q(\xi) = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$$

a au moins un zéro ξ_1; de plus

$$Q(\xi_1) = Q\left(\frac{\xi_1}{2}\right)m_0\left(\frac{\xi_1}{2}\right) + Q\left(\frac{\xi_1}{2}+\pi\right)m_0\left(\frac{\xi_1}{2}+\pi\right);$$

quitte à changer ξ_1 en $\xi_1 + 2\pi$ on peut supposer $m_0\left(\frac{\xi_1}{2}\right) \neq 0$ et donc $Q\left(\frac{\xi_1}{2}\right) = 0$. On trouve de même $Q\left(\frac{\xi_1}{4}\right) = 0$ quitte à changer ξ_1 en $\xi_1 + 4\pi$, et ainsi de suite. Or g est à support compact et donc

$$Q(\xi) = \sum_{k\in\mathbb{Z}} \left(\int g(x)\bar{g}(x-k)\,dx\right) e^{-ik\xi}$$

est un polynôme trigonométrique et donc n'a qu'un nombre fini de racines modulo 2π. De sorte qu'on a

$$Q(\xi_1) = Q\left(\frac{\xi_1}{2}\right) = \cdots = Q\left(\frac{\xi_1}{2^N}\right) = 0 \quad \text{et} \quad \frac{\xi_1}{2^N} - \xi_1 \in 2\pi\mathbb{Z}.$$

Par ailleurs $Q\left(\frac{\xi_1}{2}+\pi\right), \ldots, Q\left(\frac{\xi_1}{2^N}+\pi\right)$ sont tous non nuls; en effet supposons que

$$Q\left(\frac{\xi_1}{2^j}+\pi\right) = 0$$

et posons

$$Z = e^{-i(\xi_1/2^j + \pi)};$$

le même raisonnement que ci-dessus nous fournit $Z^{2^M} = Z$ pour un $M \geqslant 1$ tandis que $(-Z)^{2^N} = -Z$ d'où $Z^{2^{M+N}} = Z^{2^N} = -Z$ et $Z^{2^{M+N}} = (-Z)^{2^M} = Z$, ce qui est absurde. On peut donc écrire $Q(\xi) = |A(\xi)|^2 R(\xi)$ avec

$$A(\xi) = \prod_{j=1}^{N} (e^{-i\xi} - e^{-i\xi_1/2^j})$$

(où N est choisi comme le plus petit entier $\geqslant 1$ tel que $\xi_1/2^N - \xi_1 \in 2\pi\mathbb{Z}$) et $R(\xi)$ est un polynôme trigonométrique tel que $R(\xi_1/2^j + \pi) \neq 0$ pour $1 \leqslant j \leqslant N$. On pose alors

$$\hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{A(\xi)}$$

et nous allons voir que $\gamma \in L^2$ et est à support compact. Notons $P(\xi)$ le polynôme trigonométrique

$$P(\xi) = \sum_{k \in \mathbb{Z}} \frac{1}{2} \int \gamma\left(\frac{x}{2}\right) \bar{\gamma}(x-k)\, dx\, e^{-ik\xi} = \sum_{k \in \mathbb{Z}} \hat{\gamma}(2\xi + 4k\pi)\bar{\hat{\gamma}}(\xi + 2k\pi).$$

On a alors

$$\begin{aligned} A(2\xi)P(\xi) &= \sum_{k \in \mathbb{Z}} \hat{g}(2\xi + 4k\pi)\bar{\hat{\gamma}}(\xi + 2k\pi) \\ &= m_0(\xi)A(\xi)R(\xi). \end{aligned}$$

Or $A(2\xi) = A(\xi)A(\xi + \pi)$ et les zéros de $A(\xi + \pi)$ ne sont pas des zéros de $R(\xi)$. D'où

$$m_0\left(\frac{\xi_1}{2} + \pi\right) = \cdots = m_0\left(\frac{\xi_1}{2^N} + \pi\right) = 0,$$

et on obtient

$$\text{pour tout} \quad k \in \mathbb{N}, \qquad m_0(2^k \xi_1 + \pi) = 0,$$

tandis que $\xi_1 \notin 2\pi\mathbb{Z}$ puisque $Q(0) \geqslant |\hat{g}(0)|^2 = 1$. La Proposition 2 est donc démontrée, pourvu que γ soit bien dans L^2 et à support compact. C'est l'objet du lemme suivant.

Lemme 2. *Si $g \in L^2$ est à support compact et si*

$$\sum_{k \in \mathbb{Z}} |\hat{g}(\xi_0 + 2k\pi)|^2 = 0$$

alors γ définie par

$$\hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{e^{-i\xi} - e^{-i\xi_0}}$$

est dans L^2 et à support compact.

Le lemme est classique. D'abord $\hat{\gamma} \in L^2$ puisque, en notant à nouveau

$$Q(\xi) = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2,$$

on a

$$\sum_{k \in \mathbb{Z}} |\hat{\gamma}(\xi + 2k\pi)|^2 = \frac{Q(\xi)}{|e^{-i\xi} - e^{-i\xi_0}|^2};$$

comme Q est $\geqslant 0$, ξ_0 est zéro au moins double et $Q(\xi)/|e^{-i\xi} - e^{-i\xi_0}|^2$ est un polynôme trigonométrique.

Ensuite la formule sommatoire de Poisson nous donne:

$$\sum_{k \in \mathbb{Z}} g(x-k)\, e^{i(k-x)\xi_0} = \sum_{k \in \mathbb{Z}} \hat{g}(\xi_0 + 2k\pi)\, e^{2ik\pi x} = 0,$$

d'où la fonction

$$\alpha(x) = \sum_{k=-\infty}^{-1} g(x-k)\, e^{i(k-1)\xi_0} = -\sum_{k=0}^{\infty} g(x-k)\, e^{i(k-1)\xi_0}$$

est à support compact et dans L^2, or $\alpha(x) = g(x+1) + \alpha(x+1)\, e^{-i\xi_0}$ d'où $\hat{\alpha}(\xi) = \hat{g}(\xi)\, e^{i\xi} + \hat{\alpha}(\xi)\, e^{i\xi - i\xi_0}$ et donc $\alpha = \gamma$. Le lemme est donc démontré.

Corollaire 2. *Si m_0 est un polynôme trigonométrique à coefficients réels tel que*

$$m_0(0) = 1 \quad \textit{et} \quad \hat{g}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

soit dans L^2, alors il existe une et une seule analyse multi-résolution (V_j) de $L^2(\mathbb{R})$ telle que $g \in V_0$.

En effet, supposons que $m_0(\xi)$ et $m_0(\xi + \pi)$ ait une racine commune ξ_0; alors $-\xi_0$ est racine puisque m_0 est à coefficients réeles et on a $m_0(\xi) = (\cos 2\xi - \cos 2\xi_0) M(\xi)$ avec M polynôme trigonométrique. De plus, si

$$Q(\xi) = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2,$$

alors il est clair que $Q(2\xi_0) = Q(-2\xi_0) = 0$ et donc, d'après le lemme précédent, si

$$\hat{\gamma}(\xi) = \hat{g}(\xi) \frac{1 - \cos 2\xi_0}{\cos \xi - \cos 2\xi_0}$$

alors γ est dans L^2 et à support compact. De plus

$$\begin{aligned} \hat{\gamma}(2\xi) &= \frac{1}{\cos 2\xi - \cos 2\xi_0} (\cos \xi - \cos 2\xi_0) m_0(\xi) \hat{\gamma}(\xi) \\ &= M(\xi)(\cos \xi - \cos 2\xi_0)\hat{\gamma}(\xi); \end{aligned}$$

d'où

$$\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} m_1\left(\frac{\xi}{2^j}\right) \quad \text{avec} \quad m_1(\xi) = M(\xi)(\cos \xi - \cos 2\xi_0).$$

Si on appelle degré d'un polynôme trigonométrique $\sum_{k \neq N_1}^{N_2} a_k e^{-ik\xi}$ avec $a_{N_1} \neq 0$, $a_{N_2} \neq 0$ le nombre $N_2 - N_1$, alors il est clair que $\deg m_1 = \deg m_0 - 2$. Au bout d'un nombre fini d'opérations, on peut supposer que m_0 et $m_0(\xi + \pi)$ n'ont pas de racines communes.

Si maintenent (ii) n'est pas vérifié, on a vu que $m_0(\xi)$ admettait un facteur

$$\prod_{j=0}^{N-1} (e^{-i\xi} + e^{-i\xi_0 2^j})$$

avec $2^N\xi_0 - \xi_0 \in 2\pi\mathbb{Z}$ et $\xi_0 \notin 2\pi\mathbb{Z}$. Il admet également le facteur

$$\prod_{j=0}^{N-1} (e^{-i\xi} + e^{i\xi_0 2^j})$$

qui est soit confondu soit premier avec le premier facteur. De plus, on a vu que si $Q(\xi) = \sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ alors $Q(\xi_0) = 0$ et il en va de même pour $2^j\xi_0$, $1 \leqslant j \leqslant N - 1$. On pose alors

$$\hat{\gamma}(\xi) = \frac{A(0)}{A(\xi)} \hat{g}(\xi)$$

où

$$A(\xi) = \prod_{j=0}^{N-1} (e^{-i\xi} - e^{-i\xi_0 2^j})$$

où

$$\prod_{j=0}^{N-1} (e^{-i\xi} - e^{-i\xi_0 2^j}) \prod_{j=0}^{N-1} (e^{-i\xi} - e^{i\xi_0 2^j})$$

suivant que les deux facteurs sont égaux ou distincts; d'après le lemme, $\hat{\gamma}$ est dans L^2 et à support compact; de plus on a

$$\hat{\gamma}(2\xi) = \frac{A(\xi)m_0(\xi)}{A(2\xi)}\hat{\gamma}(\xi) = \frac{m_0(\xi)}{A(\xi+\pi)}\hat{\gamma}(\xi)$$

où m_1 est le polynôme trigonométrique à coefficients réels $m_0(\xi)/A(\xi + \pi)$. A nouveau $\deg m_1 < \deg m_0$.

Au bout d'un nombre fini d'opérations, on obtient alors que

$$\hat{g}(\xi) = P(\xi)\prod_{j=1}^{\infty} M\left(\frac{\xi}{2^j}\right)$$

où P et M sont deux polynômes trigonométriques à coefficients réels,

$$\prod_{j=1}^{\infty} M\left(\frac{\xi}{2^j}\right) \in L^2,$$

$M(0) = 1$ et M vérifie les points (i) et (ii) de la Proposition 2. La fonction g est donc dans l'espace V_0 de l'analyse multi-résolution engendrée par la fonction γ définie par

$$\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} M\left(\frac{\xi}{2^j}\right).$$

L'existence de l'analyse multi-résolution a été démontrée. Pour l'unicité, il suffit de remarquer que si $g \in V_0$ pour une analyse multi-résolution alors la fonction γ obtenue à la fin est encore dans V_0 et que les $\gamma(x - k)$ forment alors une base de Riesz de V_0.

Lemme 3. *Si $g \in V_0$ (où V_0 correspond à une analyse multi-résolution) est à support compact et si*

$$\sum_{k\in\mathbb{Z}} |\hat{g}(\xi_0 + 2k\pi)|^2 = 0$$

alors la fonction γ définie par

$$\hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{e^{-i\xi} - e^{i\xi_0}}$$

est encore dans V_0.

En effet, on a vu que

$$\gamma = \sum_{k=-\infty}^{-1} g(x-k)\, e^{i(k-1)\xi_0}$$

et il est clair que si $\alpha \in L^2$ est à support compact

$$\langle \gamma | \alpha \rangle = \sum_{k=-\infty}^{-1} e^{i(k-1)\xi_0} \langle g(x-k) | \alpha \rangle .$$

De plus

$$\left\| \sum_{k=-N}^{-1} g(x-k)\, e^{i(k-1)\xi_0} \right\|_2 = \frac{1}{\sqrt{2\pi}} \left\| \frac{\hat{g}(\xi)}{1-e^{-i(\xi_0-\xi)}} \left(1-e^{-iN(\xi_0-\xi)}\right) \right\|_2 \leqslant 2 \|\gamma\|_2 .$$

On en conclut que la série converge vers γ faiblement dans L^2 et donc que γ appartient à l'adhérence faible de V_0 et donc à V_0.

Lemme 4. *Si $g \in V_0$ est à support compact et si $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule pas alors les $g(x-k)$ forment une base de Riesz de V_0.*

Si les $h(x-k)$ forment une base de Riesz de V_0, alors $\hat{g}(\xi) = M(\xi)\hat{h}(\xi)$ avec $M \in L^2_{\text{loc}}$ 2π-périodique; d'où

$$|M(\xi)|^2 \sum_{k \in \mathbb{Z}} |\hat{h}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$$

et donc M et $1/M$ sont essentiellement bornées. Le lemme est alors immédiat.

3. Fonctions de V_0 à support compact

Théorème 1. *Soit $(V_j)_{j \in \mathbb{Z}}$ une analyse multi-résolution de $L^2(\mathbb{R})$. Si V_0 contient des fonctions à support compact non nulles, alors il existe une fonction $\gamma \in V_0$ à support compact et à valeurs réelles telle que*

(4.1) *Les fonctions $\gamma(x-k)$, $k \in \mathbb{Z}$, forment une base de Riesz de V_0,*
(4.2) *Toute fonction de V_0 à support compact s'écrit comme combinaison linéaire finie des $\gamma(x-k)$.*

DÉMONSTRATION. Si h est dans L^2 et à support compact, on sait que

$$\sum_{k \in \mathbb{Z}} |\hat{h}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} \int h(x)\bar{h}(x-k)\, dx\, e^{-ik\xi}$$

est un polynôme trigonométrique. Nous l'appellerons le *polynôme d'autocorrélation* de h et le noterons P_h. Si h est à valeurs réelles, P_h est un polynôme en $\cos \xi$ à coefficients réels. On choisit $\gamma \in V_0$ à valeurs réelles et à support compact de sorte que P_γ soit de degré minimal.

Alors $P_\gamma(\xi)$ ne s'annule pas. Si on avait $P_\gamma(\xi_0) = 0$, on poserait

$$\hat{g}(\xi) = \frac{\hat{\gamma}(\xi)}{e^{-i\xi} - e^{-i\xi_0}}$$

si $\xi_0 \in \pi\mathbb{Z}$ et

$$\hat{g}(\xi) = \frac{\hat{\gamma}(\xi)}{\cos\xi - \cos\xi_0}$$

sinon. D'après les Lemmes 2 et 3, g serait dans V_0, à valeurs réelles et de support compact et $\deg P_g$ serait strictement inférieur à P_γ. On en conclut que P_γ ne s'annule pas et donc que les $\gamma(x-k)$ forment une base de Riesz de V_0 (d'après le Lemme 4).

Considérons maintenant $h \in V_0$ à support compact. On peut supposer h à valeurs réelles, quitte à raisonner sur $\operatorname{Re} h$ et $\operatorname{Im} h$. On sait que h s'exprime en fonction des $\gamma(x-h)$ par: $\hat{h}(\xi) = U(\xi)\hat{\gamma}(\xi)$ avec U 2π-périodique et dans L^2_{loc}. En fait $U(\xi)$ est une fraction rationnelle en $e^{-i\xi}$ puisque

$$\sum \hat{h}(\xi + 2k\pi)\hat{\gamma}(\xi + 2k\pi) = U(\xi)P_\gamma(\xi),$$

d'où

$$U(\xi) = \frac{C(\xi)}{P_\gamma(\xi)}$$

où C est un polynôme trigonométrique à coefficients réels. De plus on a:

$$\sum |\hat{h}(\xi + 2k\pi)|^2 = |C(\xi)|^2 \frac{1}{P_\gamma(\xi)}$$

d'où

$$|C(\xi)|^2 = P_\gamma(\xi) \sum |\hat{h}(\xi + 2k\pi)|^2.$$

Le théorème de Riesz nous permet de trouver deux polynômes $A, B \in \mathbb{R}[X]$ avec $A(0) \neq 0$, $B(0) \neq 0$, $C(\xi) = e^{-ip\xi}\overline{A(e^{-i\xi})}B(e^{-i\xi})$ pour un $p \in \mathbb{Z}$ et $P_\gamma(\xi) = |A(e^{-i\xi})|^2$. On a alors

$$\hat{h} = e^{-ip\xi}\frac{B(e^{-i\xi})}{A(e^{-i\xi})}\hat{\gamma}.$$

On divise B en $B = AQ + R$. Alors si $\hat{k} = \hat{h} - e^{-ip\xi}Q(e^{-i\xi})\hat{\gamma}$, $k \in V_0$ est à support compact et à valeurs réelles; de plus on a:

$$\begin{aligned} P_k(\xi) &= P_k(\xi) + |Q(e^{-i\xi})|^2 P_\gamma(\xi) - 2\operatorname{Re} \sum_{l\in\mathbb{Z}} \hat{h}(\xi + 2l\pi)\bar{\hat{\gamma}}(\xi + 2l\pi)e^{ip\xi}\overline{Q(e^{-i\xi})} \\ &= |B(e^{-i\xi})|^2 + |Q(e^{-i\xi})A(e^{-i\xi})|^2 - 2\operatorname{Re} B(e^{-i\xi})\overline{A(e^{-i\xi})}\,\overline{Q(e^{-i\xi})} \\ &= |R(e^{-i\xi})|^2. \end{aligned}$$

Comme P_γ est de degré minimal, nécessairement $k = 0$ et donc h s'écrit comme une combinaison linéaire finie des $\gamma(x - k)$.

Corollaire 3.

(i) *On a* $\hat{\gamma}(2\xi) = m_0(\xi)\hat{\gamma}(\xi)$ *pour* m_0 *un polynôme trigonométrique à coefficients réels tel que* $m_0(0) = 1$.
(ii) γ *est de support minimal.*
(iii) *Si* $h \in V_0$ *est de support compact, les bornes inférieure et supérieure de son support sont entières.*
(iv) *Si on impose que* $\hat{\gamma}(0) = 1$ *et que la borne inférieure du support de* γ *soit* 0, *alors* γ *est unique.*

DÉMONSTRATION. Comme $\gamma(x/2) \in V_{-1} \subset V_0$ et que $\gamma(x/2)$ est à support compact, $\gamma(x/2)$ s'exprime comme une combinaison linéaire finie des $\gamma(x - k)$. Le point (i) est donc démontré. Si on a

$$\gamma\left(\frac{x}{2}\right) = \sum_{k=N_1}^{N_2} b_k \gamma(x - k)$$

alors nécessairement la borne inférieure du support de γ est N_1 et la borne supérieure est N_2. De même si

$$h(x) = \sum_{k=M_1}^{M_2} h_k \gamma(x - k)$$

alors la borne inférieure de son support est $M_1 + N_1$ et la borne supérieure $M_2 + N_2$; en particulier la longueur de l'enveloppe convexe de ce support est $N_2 - N_1 + M_2 - M_1$ et donc supérieure à celle du support de γ (avec égalité pour les seules fonctions multiples d'un $\gamma(x - k)$). Les points (ii), (iii), (iv) sont alors démontrés.

Corollaire 4. *Il existe une fonction* $\theta \in W_0$ (*où* W_0 *est le complémentaire orthogonal de* V_0 *dans* V_1) *à support compact et à valeurs réelles telle que*

(5.1) *Les fonctions* $\theta(x - k)$, $k \in \mathbb{Z}$ *forment une base de Riesz de* W_0;
(5.2) *Toute fonction de* W_0 *à support compact s'écrit comme une combinaison linéaire finie des* $\theta(x - k)$.
θ est alors de support minimum dans W_0.

DÉMONSTRATION. Si $h \in W_0$ est à support compact, alors h peut s'écrire comme $\hat{h}(\xi) = e^{-ip\xi}R(e^{-i\xi/2})\hat{\gamma}(\xi/2)$ pour un polynôme $R(z)$ et un entier $p \in \mathbb{Z}$, puisque $W_0 \subset V_1$. Notons Q le polynôme de degré minimum tel que $Q(e^{-i\xi/2})\hat{\gamma}(\xi/2)$ soit la transformée de Fourier d'un élément (normal) de W_0. On peut supposer Q à coefficients réels (car si $\hat{w} = Q(e^{-i\xi/2})\hat{\gamma}(\xi/2)$ alors Re w et Im w sont dans w_0). De même, on supposera h à valeurs réelles (et donc $R \in \mathbb{R}[X]$).

On remarque d'abord que $Q(z)$ et $Q(-z)$ sont premiers entre eux, car si $Q(z) \wedge Q(-z) = A(z^2)$ alors $[Q(e^{-i\xi/2})/A(e^{-i\xi})][\hat{\gamma}(\xi/2)]$ est encore la transformée de Fourier d'un élément de W_0, ce qui contredit la minimalité de Q. On a alors, en posant $\hat{\theta} = Q(e^{-i\xi/2})\hat{\gamma}(\xi/2)$.

$$\sum_{k\in\mathbb{Z}} |\hat{\theta}(\xi + 2k\pi)|^2 = |Q(e^{-i\xi/2})|P_\gamma(\xi/2) + |Q(-e^{-i\xi/2})|P_\gamma\left(\frac{\xi}{2} + \pi\right) > 0$$

ce qui entraîne que les $\theta(x - k)$ forment une base de Riesz de W_0 (car on sait que W_0 a une base de Riesz de la forme $w(x - k)$, $k \in \mathbb{Z}$).

Maintenant si $\hat{h} = e^{-ip\xi}R(e^{-i\xi/2})\hat{\gamma}(\xi/2)$ est la transformée de Fourier de $h \in V_1$, alors $h \in W_0$ si et seulement si on a

$$R(e^{-i\xi})\overline{m_0(\xi)}P_\gamma(\xi) + R(e^{-i(\xi+\pi)})\overline{m_0(\xi + \pi)}P_\gamma(\xi + \pi) = 0.$$

d'où si $\overline{m_0(\xi)}P_\gamma(\xi) = e^{-2iq\xi}A(e^{-i\xi})$, $R(z)A(z) + R(-z)A(-z) = 0$. En particulier $\deg R + \deg A$ est impair. On en conclut que $\deg R$ et $\deg Q$ ont la même parité. De proche en proche, on obtient que $R(z) = B(z^2)Q(z) + C(z)$ avec $\deg C < \deg Q$, d'où $R(z) = B(z^2)Q(z)$. On a alors $\hat{h} = e^{-ip\xi}B(e^{-i\xi})\hat{\theta}(\xi)$.

Remarque. La fonction θ décrite par le Corollaire 4 a été d'abord étudiée par C. K. Chui et J. Z. Wang [3]. Cependant les propriétés décrites dans ce corollaire sont nouvelles, ces deux auteurs n'ayant pas démontré (4.2) et donc pas (5.2) ni la minimalité du support de θ. Par contre, ils donnent le calcul de Q: si $A(z) \wedge A(-z) = B(z^2)$ et si $A(z) = B(z^2)C(z)$ (où $A(e^{-i\xi}) = e^{2iq\xi}\overline{m_0(\xi)}P_\gamma(\xi)$) alors $Q(z) = zC(-z)$.

4. La fonction de V_0 de support minimal

Théorème 2. *Soit $G \in V_0$ non nulle, à support compact et à valeurs réelles. Alors les propriétés suivantes sont équivalentes*:

(6.1) *G est de support minimal.*

(6.2) *Le polynôme d'auto-corrélation de G est de degré minimum.*

(6.3) *Tout élément de V_0 à support compact s'écrit comme une combinaison linéaire finie des $G(x - k)$.*

(6.4) *Il existe $H \in L^2$ de support compact telle que $\langle H \mid G(x - k)\rangle = \delta_k$.*

(6.5) *$\hat{G}$ se décompose en*

$$\hat{G}(\xi) = \hat{G}(0) \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right)$$

où m_0 *est un polynôme trigonométrique* $m_0(\xi) = e^{-ip\xi}P(e^{-i\xi})$ *avec* $p \in \mathbb{Z}$ et $P \in \mathbb{R}(X)$ *où P vérifie*:

(i) $P(1) = 1$.
(ii) $P(z)$ et $P(-z)$ *sont premiers entre eux.*
(iii) $P(z)$ *n'a pas de facteurs de la forme*

$$\prod_{k=1}^{N-1} (z + z_0^{2^k}) \quad avec \quad z_0^{2^N} = z_0 \quad et \quad z_0 \neq 1.$$

(6.6) *Les restrictions à* $[0, 1]$ *des fonctions*

$$G(x + k), \qquad \inf \operatorname{Supp} G \leqslant k < \sup \operatorname{Supp} G,$$

sont linéairement indépendantes.

DÉMONSTRATION. Le Théorème 1 donne l'équivalence de (6.1), (6.2) et (6.3).

(6.1) ⇒ (6.5). On sait d'après le Corollaire 3 que $\hat{G}(2\xi) = m_0(\xi)\hat{G}(\xi)$ avec m_0 un polynôme trigonométrique. De plus $\hat{G}(0) \neq 0$ d'aprés (3.1) et donc $m_0(0) = 1$. On peut donc écrire $m_0(\xi) = e^{-ip\xi}P(e^{-i\xi})$ avec $P(1) = 1$ et $P(0) \neq 0$. D'après la Proposition 2, $P(z)$ et $P(-z)$ n'ont pas de racines communes sur le cercle unité et $P(z)$ n'a pas de facteurs

$$\prod_{k=1}^{N-1} (z + z_0^{2^k}) \quad \text{avec} \quad z_0 \neq 1, \qquad z_0^{2^N} = z_0.$$

Il ne reste donc à prouver que $P(z) \wedge P(-z) = 1$. Si $P(z) \wedge P(-z) = R(z^2)$, on pose

$$\hat{h}(\xi) = \frac{\hat{G}(\xi)}{R(e^{-i\xi})},$$

alors

$$\hat{h}(2\xi) = e^{-ip\xi} \frac{P(e^{-i\xi})}{R(e^{-2i\xi})} R(e^{-i\xi})\hat{h}(\xi)$$

et on a $\sup \operatorname{Supp} h - \inf \operatorname{Supp} h = \deg P - \deg R \leqslant \sup \operatorname{Supp} G - \inf \operatorname{Supp} G$ avec égalité si et seulement si $P(z) \wedge P(-z) = 1$. Puisque G est de support minimal, on a bien $P(z) \wedge P(-z) = 1$.

(6.5) ⇒ (6.1). Quitte à translater G et γ (où γ est décrite par le Théorème 1), on peut supponer $\inf \operatorname{Supp} G = \inf \operatorname{Supp} \gamma = 0$. On a alors $\hat{G}(2\xi) = P(e^{-i\xi})\hat{G}(\xi)$, $\hat{\gamma}(2\xi) = P_0(e^{-i\xi})\hat{\gamma}(\xi)$ et enfin, par (4.2), $\hat{G}(\xi) = Q(e^{-i\xi})\hat{\gamma}(\xi)$. Remarquons que $Q(0) \neq 0$ (puisque $\inf \operatorname{Supp} G = \inf \operatorname{Supp} \gamma$) et que Q n'a pas de racines sur le cercle unité (puisque $P_G(\xi) = |Q(e^{-i\xi})|^2 P_\gamma(\xi)$ et que P_G ne s'annule pas, d'après la Proposition 2). De plus, on a: $P(z)Q(z) = Q(z^2)P_0(z)$ et donc $\deg P = \deg P_0 + \deg Q$; comme $P(z) \wedge P(-z) = 1$, il est nécessaire que P_0 divise P, d'où il existe A tel que: $Q(z^2) = Q(z)A(z)$. Si z est une racine de Q, il en va de même pour z^2, et donc $z^4, z^8, \ldots$ Comme ni 0 ni les complexes de module 1 ne sont racines de Q et que Q doit avoir un nombre fini de zéros, alors on trouve que Q est une constante et donc $G = C^{te}\gamma$.

(6.1) $\Rightarrow$ (6.6). Il suffit de reproduire la démonstration du théorème d'Yves Meyer [12] sur la restriction à [0, 1] des bases d'I. Daubechies. On va donc montrer que, pour $j \geqslant 0$, la dimension de l'espace des restrictions à [0, 1] des fonctions de V_j est exactement $2^j + \sup \operatorname{Supp} G - \inf \operatorname{Supp} G - 1$.

Quitte à translater G on peut supposer $\inf \operatorname{Supp} G = 0$; on pose $\sup \operatorname{Supp} G = N$. Commençons par remarquer que si

$$\sum_{k=-M}^{N-1} \alpha_k G(x+k)$$

est nulle sur $[0, M+1]$ et si $M + 1 \geqslant 2\,(\sup \operatorname{Supp} \theta - \inf \operatorname{Supp} \theta)$ (où θ est la fonction décrite par le Corollaire 4) alors les α_k sont nuls; en effet

$$f(x) = \sum \alpha_k G(x+k)$$

est portée par $[-N+1, 0] \cap [M+1, M+N]$; dire que $f \in V_0$ revient à dire que f est orthogonale à toutes les fonctions $\theta(2^j x - k)$, $j \geqslant 0$, $k \in \mathbb{Z}$ (car $V_0^\perp = \bigoplus_{j \geq 0} W_j$), mais cela est alors vrai de $f|_{(-N+1,0)}$ et de $f|_{(M+1,M+N)}$ puisque $\theta(2^j x - k)$ est portée par un intervalle de longueur $2^{-j}(\sup \operatorname{Supp} \theta - \inf \operatorname{Supp} \theta)$. Puisque G est de support minimal dans V_0, on obtient $f = 0$. Par dilatation, cela donne que la dimension des restrictions à [0, 1] des fonctions de V_j est exactement $2^j + N - 1$ pour j assez grand ($2^j \geqslant 2(\sup \operatorname{Supp} \theta - \inf \operatorname{Supp} \theta)$). Pour passer de j à $j - 1$, on suit la démonstration de Y. Meyer, basée sur le seule propriété que $P(z) \wedge P(-z) = 1$.

(6.6) $\Rightarrow$ (6.4). On peut supposer $\inf \operatorname{Supp} G = 0$. Comme $G|_{[0,1]}$ est indépendante des $G(x+k)|_{[0,1]}$ $(k \neq 0)$, il existe $H \in L^2([0,1])$ avec $\langle H \mid G \rangle = 1$ et $\langle H \mid G(x+k) \rangle = 0$ pour $k \neq 0$.

(6.4) $\Rightarrow$ (6.1). Comme H et G sont à support compact, on en déduit immédiatement que les $G(x-k)$ sont une base de Riesz de V_0 et que tout élément de V_0 se représente comme

$$h = \sum_{k \in \mathbb{Z}} \langle h \mid H(x-k) \rangle G(x-k).$$

(6.3) (et donc (6.1)) est alors immédiat.

5. Calculs fondamentaux dans une analyse multi-résolution

Nous considérons une analyse multi-résolution $(V_j)_{j\in\mathbb{Z}}$ où V_0 admet une base de Riesz $\gamma(x-k)$ avec γ à valeurs réelles et de support compact. La fonction γ est choisie de manière à ce que son support soit minimal, que $\inf \operatorname{Supp} \gamma = 0$ et que $\hat{\gamma}(0) = 1$. On appellera γ la *base normalisée* de V_0.

On a

$$\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} P_0(e^{-i\xi/2^j})$$

avec $P_0 \in \mathbb{R}[X]$, $P_0(1) = 1$ et $P_0(0) \neq 0$. Si $\hat{\gamma}(\xi) \in L^1$, alors

$$\prod_{j=1}^{N} P_0(e^{-i\xi/2^j})\chi_K\left(\frac{\xi}{2^N}\right)$$

tend vers $\hat{\gamma}$ en norme L^1 (où K est décrit dans la Proposition 1). On a alors le procédé d'approximation suivant [5]:

Proposition 3. *On pose*

$$h_0(x) = \chi_{[-1/2,1/2]}(x) \quad et \quad h_{k+1}(x) = \sum_{l=0}^{N} \alpha_l h_k(2x-l)$$

(où

$$P_0(z) = \frac{1}{2}\sum_{l=0}^{N} \alpha_l z^l\bigg)\cdot$$

Si $\hat{\gamma} \in L^1$, alors $h_k \to \gamma$ en norme L^∞, pour $k \to +\infty$.

En effet h_k est constante par morceaux sur des intervalles de longueur $1/2^k$. De plus sur les points de $1/2^k\mathbb{Z}$, h_k coincide avec θ_k où

$$\hat{\theta}_k = \prod_{j=1}^{k} P_0(e^{-i\xi/2^j})\chi_K\left(\frac{\xi}{2^k}\right)$$

(par récurrence sur k). Enfin γ est uniformément continue puisque continue et à support compact. Comme $\|\theta_k - \gamma_k\|_\infty \to 0$, on a bien $\|h_k - \gamma_k\|_\infty \to 0$.

Néanmoins la convergence n'est pas rapide. Prenons l'exemple d'un spline cubique:

$$\hat{\gamma}(\xi) = \left(\frac{1-e^{-i\xi}}{-i\xi}\right)^4.$$

On pose

$$\hat{G}(\xi) = \frac{\hat{\gamma}(\xi)}{P_\gamma(\xi)}$$

de sorte que $\langle G(x) \mid \gamma(x-k)\rangle = \delta_k$. Alors il est facile de voir que

$$h_k\left(\frac{p}{2^k}\right) = \left\langle \gamma\left(\frac{x}{2^k}\right) \middle| \, G(x-p) \right\rangle.$$

En particulier, on a

$$h_k(p) = \left\langle \gamma\left(\frac{x}{2^k} + p\right) \middle| \, G(x) \right\rangle = \gamma(p) + \frac{1}{2^k}\gamma'(p) \int xG(x)\,dx + O\left(\frac{1}{4^k}\right).$$

Or

$$\int xG(x)\,dx = 2 \quad \text{et} \quad \gamma'(1) = \frac{1}{2},$$

d'où

$$|h_k(1) - \gamma(1)| \sim \frac{1}{2^k}.$$

L'erreur n'est divisée que par 2 à chaque itération. Si l'on veut calculer γ aux points entiers, on utilisera plutôt la remarque suivante.

Proposition 4. *On note M la transformation linéaire définie sur $\mathbb{R}^{N-1}$ par*

$$M\begin{bmatrix} x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} \quad \textit{où} \quad y_j = \sum_{l=0}^{N} \alpha_l x_{2j-l}$$

(en prolongeant x_j par 0 *en dehors de* $\{1, \ldots, N-1\}$*) où*

$$P_0(z) = \frac{1}{2}\sum_{l=0}^{N} \alpha_l z^l.$$

Si $\hat{\gamma}(\xi) \in L^1$, alors Ker $(M - \mathrm{Id})$ *est de dimension* 1 *et*

$$\begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(N-1) \end{bmatrix}$$

est déterminé uniquement par

$$M\begin{bmatrix}\gamma(1)\\ \vdots\\ \gamma(N-1)\end{bmatrix} = \begin{bmatrix}\gamma(1)\\ \vdots\\ \gamma(N-1)\end{bmatrix}$$

et

$$\sum_{j=1}^{N-1} \gamma(j) = 1.$$

DÉMONSTRATION. Soit

$$\begin{bmatrix}x_1\\ \vdots\\ x_{N-1}\end{bmatrix}$$

une solution de $MX = X$. On pose

$$h_0(x) = \sum_{j=1}^{N-1} x_j \theta_0(x-j)$$

où $\hat{\theta}_0(\xi) = \chi_K(\xi)$ et

$$h_{k+1}(x) = \sum_{l=0}^{N} \alpha_l h_k(2x-l).$$

Alors

$$\widehat{h_{k+1}}(\xi) = P_0(e^{-i\xi/2})\widehat{h_k}\left(\frac{\xi}{2}\right)$$

et donc, en posant

$$Q(\xi) = \sum_{j=1}^{N-1} x_j e^{-ij\xi},$$

on a

$$\widehat{h_k}(\xi) = \prod_{j=1}^{k} P_0(e^{-i\xi/2^j})\chi_K\left(\frac{\xi}{2^k}\right)Q\left(\frac{\xi}{2^k}\right),$$

ce qui tend en norme L^1 vers $\hat{\gamma}(\xi)Q(0)$. En particulier $h_k(p)$ tend vers $Q(0)\gamma(p)$; or $h_k(p) = x_p$ (par récurrence sur k) et donc

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \left(\sum_{j=1}^{N-1} x_j\right) \begin{bmatrix} \gamma(1) \\ \vdots \\ \gamma(N-1) \end{bmatrix}.$$

Proposition 6. *Les intégrales*

$$\int x^k \gamma(x)\, dx, \qquad k \in \mathbb{N},$$

se calculent récursivement par les formules:

- $\int x^k \gamma(x)\, dx = i^k \hat{\gamma}^{(k)}(0)$.
- $\hat{\gamma}(0) = 1$.
- $(2^{k+1} - 1)\hat{\gamma}^{(k+1)}(0) = \sum_{\alpha=0}^{k} C_k^\alpha \hat{\gamma}^{(\alpha)}(0) m_0^{(k+1-\alpha)}(0)$.

Il s'agit juste d'appliquer la formule de Leibnitz à $\hat{\gamma}(2\xi) = m_0(\xi)\hat{\gamma}(\xi)$ avec $m_0(\xi) = P_0(e^{-i\xi})$.

Proposition 7.

(i) *Si $\gamma \in H^1$ (c'est-à-dire si $\gamma' \in L^2$) alors il existe une analyse multi-résolution V_j' de $L^2(\mathbb{R})$ de base normalisée M_1 telle que*

$$\gamma'(x) = M_1(x) - M_1(x-1).$$

(ii) *Il existe de même une analyse multi-résolution V_j'' de $L^2(\mathbb{R})$ de base normalisée M_2 telle que $\gamma(x) - \gamma(x-1) = M_2'(x)$.*

La Proposition 7 permet donc de dériver les fonctions de V_0 et d'intégrer les fonctions de V_0 d'intégrales nulles. Le fait que la dérivation dans V_0 revient à appliquer un opérateur de différence finie dans un autre espace V_0' m'a été signalé par G. Malgouyres [9].

DÉMONSTRATION.

(i) Comme γ' est à support compact et que

$$\sum_{k \in \mathbb{Z}} \gamma'(x-k) = 0$$

$\left(\text{puisque } \sum_{k \in \mathbb{Z}} \gamma(x-k) = \hat{\gamma}(0)\right)$, on a

$$\gamma'(x) = M_1(x) - M_1(x-1) \quad \text{où} \quad M_1(x) = \sum_{k=-\infty}^{-1} \gamma'(x-k)$$

est dans L^2 et à support compact. De plus, on a

$$\widehat{M_1}(\xi) = \frac{i\xi}{1 - e^{-i\xi}}\hat{\gamma}(\xi) \quad \text{d'où} \quad \widehat{M_1}(2\xi) = \frac{2}{1 + e^{-i\xi}} P_0(e^{-i\xi})\widehat{M_1}(\xi).$$

Comme

$$P_0(-1) = 0, \qquad P_1(z) = \frac{2}{1+z} P_0(z)$$

est un polynôme avec $P_1(1) = 1$ d'où

$$\widehat{M_1}(\xi) = \prod_{j=1}^{\infty} P_1(e^{i\xi/2^j}).$$

Il est clair que P_1 vérifie les conditions de la Proposition 2 et donc que les $M_1(x-k)$, $k \in \mathbb{Z}$, forment une base de Riesz d'un espace V_0' pour une analyse multi-résolution $(V_j')_{j\in\mathbb{Z}}$. De plus, il est clair également que P_1 vérifie (6.5) et donc M_1 est la base normalisée de V_0'.

(ii) Il suffit de poser

$$\widehat{M_2}(\xi) = \frac{1 - e^{-i\xi}}{i\xi}\hat{\gamma}(\xi);$$

alors il est clair que $M_2 \in L^2$ et que

$$M_2'(x) = \gamma(x) - \gamma(x-1)$$

et donc M_2 est à support compact. De plus on a

$$\widehat{M_2}(2\xi) = \frac{1 + e^{-i\xi}}{2} P_0(e^{-i\xi})\widehat{M_2}(\xi) \quad \text{d'où} \quad \widehat{M_2}(\xi) = \prod_{j=1}^{\infty} P_2(e^{-i\xi/2^j})$$

où

$$P_2(z) = \frac{1+z}{2} P_0(z),$$

et la conclusion (voir à nouveau la Proposition 2 et (6.5)) est immédiate.

Corollaire 5. *Si $\gamma \in H^k$ (c'est-à-dire si pour $j \in \{0, \dots, k\}$, $\gamma^{(j)} \in L^2$) alors $\hat{\gamma}^{(j)}(2l\pi) = 0$ pour $l \neq 0$ et $0 \leqslant j \leqslant k$. En particulier, tout polynôme de degré $\leqslant k$ se représente comme*

$$Q(x) = \sum_{l\in\mathbb{Z}} R(l)\gamma(x-l)$$

où R est également un polynôme de degré $\leqslant k$.

En effet, si $\gamma \in H^k$ alors $\gamma'(x) = M_1(x) - M_1(x-1)$ où $M_1 \in H^{k-1}$. De plus si

$$\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} P_0(e^{-i\xi/2^j}) \quad \text{alors} \quad \widehat{M_1}(\xi) = \prod_{j=1}^{\infty} P_1(e^{-i\xi/2^j})$$

où

$$P_1(z) = \frac{2}{1+z} P_0(z).$$

D'où si $\gamma \in H^k$ alors $(2/(1+z))^k P_0(z)$ est un polynôme qui vérifie de plus que sa valeur en -1 est nulle (d'après (3.2)). On a donc

$$P_0(z) = \left(\frac{1+z}{2}\right)^{k+1} Q_0(z);$$

comme

$$\hat{\gamma}(2k\pi + \xi) = P_0(e^{-i\xi/2 - il\pi})\hat{\gamma}\left(l\pi + \frac{\xi}{2}\right),$$

on montre que $\hat{\gamma}^{(j)}(2l\pi) = 0$ par récurrence sur M tel que $l/2^M \in 2\mathbb{Z} + 1$.

De plus, si on note M_j la base normalisée de l'analyse multi-résolution où se trouve $\gamma^{(j)}$, alors on a:

$$\begin{aligned}\left(\sum a_l \gamma(x-l)\right)' &= \sum a_l (M_1(x-l) - M_1(x-1-l)) \\ &= \sum (a_l - a_{l-1}) M_1(x-l)\end{aligned}$$

d'où

$$\left(\sum a_l \gamma(x-l)\right)^{(k)} = \sum \Delta^k a_l M_k(x-l) \quad \text{où} \quad \Delta(a_l) = (a_l - a_{l-1}).$$

Si $a_l = R(P)$ avec $\deg R \leqslant k$ alors $\Delta^k a_l$ est constante et

$$\left(\sum a_l \gamma(x-l)\right)^{(k)} = c^{te}\widehat{M_k}(0).$$

On obtient que $R \to \sum R(l)\gamma(x-l)$ envoie $\mathbb{C}_k[X]$ dans $\mathbb{C}_k[X]$; de plus cette transformation est injective donc surjective.

Proposition 8. *Si V_j et V'_j sont deux analyses multi-résolutions de bases normalisées α et β, alors $\alpha * (\beta(-x))$ est dans un espace V''_0 d'une analyse multi-résolution, et on peut donc calculer $\int \alpha(x)\beta(x-k)\,dx$ à l'aide de la Proposition* 4.

En effet, si

$$\gamma(x) = \int \alpha(y)\beta(y-x)\,dy,$$

alors on a

$$\hat{\gamma}(\xi) = \hat{\alpha}(\xi)\bar{\hat{\beta}}(\xi) \quad \text{d'où} \quad \hat{\gamma}(2\xi) = P_0(e^{-i\xi})\overline{P_1(e^{-i\xi})}\hat{\gamma}(\xi)$$

(où $\hat{\alpha}(2\xi) = P_0(e^{-i\xi})\hat{\alpha}(\xi)$ et $\hat{\beta}(2\xi) = P_1(e^{-i\xi})\hat{\beta}(\xi)$). Le Corollaire 2 permet alors de conclure. (Il se peut que les $\gamma(x-k)$ ne forment toutefois pas une base de Riesz de V_0''.)

En combinant les Propositions (7) et (8), on voit qu'on peut calculer pour un opérateur différentiel $\sum a_\alpha \left(\frac{d}{dx}\right)^\alpha$ à coefficients constant les quantités $\left\langle \sum a_\alpha \left(\frac{d}{dx}\right)^\alpha \psi_{j,k}^{(\epsilon)} \,\middle|\, \psi_{j,k}^{(\eta)} \right\rangle$ où $\epsilon, \eta \in \{0, 1\}$, $\psi^{(0)} = \varphi$ et $\psi^{(1)} = \psi$, φ et ψ étant les père et mère des ondelettes orthonormées d'I. Daubechies. Or ces calculs interviennent dans l'analyse de l'opérateur $\sum a_\alpha \left(\frac{d}{dx}\right)^\alpha$ par l'algorithme de Beylkin-Coifman-Rokhlin [2]. Ce genre de calculs est actuellement développé par M. Lahzami [7] pour la programmation numérique.

6. Exemples

(i) *Les fonctions splines.* L'analyse multi-résolution des splines de degré k est engendrée par l'espace V_0 des splines de degré k à noeuds dans $\mathbb{Z}$ et de carré intégrable (c'est-à-dire que $f \in V_0$ si et seulement si $f \in L^2 \cap C^{k-1}$ et pour tout $l \in \mathbb{Z}$, $f|_{]l,l+1[}$ est polynomiale de degré $\leqslant k$). La base normalisée de V_0 est la fonction $\gamma = \chi_{[0,1]} * \cdots * \chi_{[0,1]} = \chi_{[0,1]}^{(*)k+1}$.

L'echelle des fonctions splines est évidemment stable par dérivation et primitivation et la Proposition 7 ne nous apprend rien.

Remarquons que la Proposition 7(ii) est un cas particulier de la Proposition 8, puis-qu'en fait $M_2 = \chi_{[0,1]} * \gamma$.

L'existence d'ondelettes splines à support compact (*cf.* Corollaire 2) a été signalée par Lemarié en 1987 [8] et systématiquement étudiée dans la thèse de P. Auscher [1].

(ii) *Les bases d'I. Daubechies.* I. Daubechies a construit des fonctions ${}_N\varphi$ telles que: $\text{Supp}_N\varphi \subset [0, 2N-1]$, les ${}_N\varphi(x-k)$ sont orthonormées quand k décrit $\mathbb{Z}$ et ${}_N\varphi$ est de classe $C^{\lambda N}$ pour un $\lambda > 0$ [5]. Il est clair que ${}_N\varphi$ est la base normalisée de l'espace V_0 associé (puisque (6.3) est immédiat). Par contre l'échelle de ces fonctions ${}_N\varphi$ n'est stable ni par primitivation ni par dérivation et les calculs comme ceux développés à la suite de la Proposition 8 font intervenir d'autres analyses multi-résolution.

(iii) *Les analyses multi-résolution non orthogonales de J. C. Fauveau.* Pour construire des analyses multi-échelle à analyse et synthèse rapides et à filtre à phase linéaire, J. C. Fauveau a été amené à introduire la notion d'analyse multi-résolution non orthogonale [6]. Sa construction repose sur deux fonctions γ_1 et γ_2 à support compact telles que:

$$\hat{\gamma}_1(\xi) = \prod_{j=1}^{\infty} P_1(e^{-i\xi/2^j}), \qquad \hat{\gamma}_2(\xi) = \prod_{j=1}^{\infty} P_2(e^{-i\xi/2^j}),$$

$$P_1(z)P_2(z) + P_1(-z)P_2(-z) = 1$$

et les $\gamma_i(x-k)$ sont la base de Riesz d'un espace $V_{0,i}$ lié à une analyse multi-résolution. Il est alors clair que γ_i est la base normalisée de $V_{0,i}$ (d'après (6.5)).

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Multiplicative structure of de Branges's spaces

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1. Introduction

L. de Branges has originated a viewpoint one of whose repercussions has been the detailed analysis of certain Hilbert spaces of holomorphic functions contained within the Hardy space H^2 of the unit disk. The initial study of the spaces was made by de Branges and J. Rovnyak [4] about 25 years ago. Although neglected for a while, the spaces are now attracting considerable attention because of their beautiful internal structure and their relevance to function theory [21]. Our aim in this paper is to investigate their multipliers.

The starting point is a nonconstant function b in $B(H^\infty)$, the unit ball in the space H^∞ of all bounded holomorphic functions in the open unit disk, D, of the complex plane. The de Branges space $\mathsf{H}(b)$ consists by definition of the range of the operator $(1 - T_b T_{\bar{b}})^{1/2}$ (where, for ϕ in L^∞ of the unit circle, T_ϕ denotes the Toeplitz operator on H^2 with symbol ϕ). The space $\mathsf{H}(b)$ is given the Hilbert space structure that makes the operator $(1 - T_b T_{\bar{b}})^{1/2}$ a coisometry of H^2 onto $\mathsf{H}(b)$. By a multiplier of $\mathsf{H}(b)$ we mean a holomorphic function m in D such that mh is in $\mathsf{H}(b)$ whenever h is. Since the evaluation functionals on $\mathsf{H}(b)$ at the points of D are bounded, one sees from the closed graph theorem that the multiplication operator on $\mathsf{H}(b)$ induced by such an m is bounded, from which it follows that m must be in H^∞ [23].

There are two extreme cases. If $\|b\|_\infty < 1$, then $\mathsf{H}(b)$ is just a renormed version of H^2 and every function in H^∞ is a multiplier of it. At the other extreme, if b is an inner function, then $\mathsf{H}(b)$ is an ordinary subspace of H^2, namely, the orthogonal complement of the Beurling invariant subspace bH^2.

It is thus the typical invariant subspace of S^*, the adjoint of the unilateral shift operator, S, on H^2 ($(Sf)(z) = zf(z)$). In this case, $H(b)$ has no nonconstant multipliers. (Proof: If b is an inner function and m is a multiplier of $H(b)$ then, because S^*b is in $H(b)$ [21], we have, for all f in H^2, the equality $0 = \langle mS^*b, bf\rangle$. One easily sees that the right side equals $\langle S^*m, (1 - \overline{b(0)}b)f\rangle$. Setting $f = S^*m/(1 - \overline{b(0)}b)$, we find that $S^*m = 0$).

The spaces $H(b)$ break naturally into two classes according to whether b is or is not an extreme point of $B(H^\infty)$, or, what is equivalent, according to whether the function $1 - |b|^2$ is not or is log-integrable on ∂D [14]. A few results in the latter case can be found in [18]. It is shown there, for example, that if b is not an extreme point of $B(H^\infty)$ then any function holomorphic in a neighborhood of $\bar{D}$ is a multiplier of $H(b)$, and those b for which every function in H^∞ is a multiplier of $H(b)$ are characterized. Further progress has recently been made by B. M. Davis and J. E. McCarthy [1] who, among other things, have characterized the functions that are multipliers of every space $H(b)$ with b nonextreme. For the case where b is an extreme point, on the other hand, next to nothing has been known up to now beyond the negative result for inner functions cited above. In particular, it has been an open question whether there is any extreme point b such that $H(b)$ has nonconstant multipliers.

In this paper we shall concentrate mainly on the case where b is an extreme point but not an inner function. The main thrust of our results is that $H(b)$ has an abundance of multipliers in that case.

A space closely related to $H(b)$, called $H(\bar{b})$, arises naturally in the search for multipliers. By definition, $H(\bar{b})$ is the range of the operator $(1 - T_{\bar{b}}T_b)^{1/2}$, with the Hilbert space structure that makes this operator a coisometry of H^2 onto $H(\bar{b})$. The space $H(\bar{b})$ is trivial if b is an inner function, but otherwise it is infinite dimensional. It turns out that every multiplier of $H(b)$ differs by at most a constant from a function in $H(\bar{b})$. The culmination of our efforts will be a proof, for the case where b is an extreme point of $B(H^\infty)$, that the multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.

In Section 2 the place of the spaces $H(b)$ and $H(\bar{b})$ in the general scheme of de Branges is described. A lemma concerning that scheme is established and used to obtain information about $H(b)$ and $H(\bar{b})$. (Some of the results here can be found in the literature, but the present proofs seem particularly apt.).

Section 3 explains the relation between $H(b)$ and $H(\bar{b})$ and certain spaces of Cauchy integrals. The multipliers of $H(b)$ coincide with the multipliers of its related space of Cauchy integrals. Cauchy integrals in the unit disk have been studied extensively beginning with V. P. Havin [12], but from a viewpoint rather different from ours. In Section 4 we show how our methods provide a simple proof of a theorem of S. A. Vinogradov.

The remainder of our paper addresses mainly the case where b is an extreme point of $B(H^\infty)$. Section 5 contains two negative results for that case that say, very roughly speaking, that nonconstant multipliers cannot behave too nicely. Section 6 pertains to decompositions of the space $H(b)$ and Section 7 to the case where b is invertible. In the latter case b is shown to be a multiplier of $H(b)$, and the converse is shown to hold when b is an extreme point. In Section 8 the multiplication operator on $H(b)$ induced by a multiplier is discussed, for the extreme point case.

In Section 9, again for the extreme point case, we introduce two conjugations, one on $H(b)$ and another on the one-dimension extension of $H(\bar{b})$ by the constant functions. It is shown that to each multiplier m of $H(b)$ there corresponds a conjugate multiplier, m_*. The multipliers of $H(b)$ thus form a $*$-algebra, although not a C^*-algebra. A certain algebra of Cauchy integrals is introduced which contains all the multipliers of $H(b)$.

Section 10 contains two needed lemmas on Cauchy integrals. They are used in Section 11 to obtain more information on the space $H(\bar{b})$ and in Section 12 to establish a criterion that, among other things, enables us to construct a set of multipliers of $H(b)$ that is dense in $H(\bar{b})$ (again, for the extreme point case).

If u is an inner function, then, as is explained in Section 6, every multiplier of $H(ub)$ is a multiplier of $H(b)$. In Section 13, in the extreme point case, a criterion is obtained for a multiplier of $H(b)$ to be a multiplier of $H(ub)$. The functions that are multipliers of $H(ub)$ for every inner function u are characterized. A sufficient condition on u is found for $H(b)$ and $H(ub)$ to have the same multipliers.

In Section 14 we give a complete description of the multipliers of $H(b)$ for a certain class of extreme points b. This result is related to a well-known theorem of H. Helson and G. Szegö. We also give an example to show that, even when b is an extreme point, an inner function u can exist such that not every multiplier of $H(b)$ is one of $H(ub)$. (Such an example with b not an extreme point can be extracted from [18].)

The concluding Section 15 contains a short list of open questions.

Besides the notations already introduced, the following additional ones are needed.

L^2 denotes the L^2 space of normalized Lebesgue measure on $\partial\mathrm{D}$ and P_+ denotes the orthogonal projection in L^2 with range H^2. The norm and inner product in L^2 are denoted by $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle$.

The norm and inner product in $H(b)$ are denoted by $\|\cdot\|_b$ and $\langle\cdot,\cdot\rangle_b$, and those in $H(\bar{b})$ by $\|\cdot\|_{\bar{b}}$ and $\langle\cdot,\cdot\rangle_{\bar{b}}$.

The kernel function in H^2 for the point w of D is denoted by

$$k_w \; k_w(z) = (1 - \bar{w}z)^{-1}.$$

The kernel functions in $H(b)$ and $H(\bar{b})$ for w are denoted by k_w^b, as a simple argument shows. For k_w^b one has the expression $k_w^b = (1 - \overline{b(w)}b)k_w$ [4], [19].

The term «operator» will always mean «bounded operator».

The following two simple properties of $H(b)$, the first of which was mentioned earlier, can be found in [19].

1. S^*b belongs to $H(b)$.
2. b belongs to $H(b)$ if and only if b is not an extreme point of $B(H^\infty)$.

Alternative proofs of some of the results below have recently been found by A. V. Lipin (private communication).

2. Relations between $H(b)$ and $H(\bar{b})$

It is helpful to fit the spaces $H(b)$ and $H(\bar{b})$ into the general scheme promulgated by de Branges (for example, in [2]). If H and H_1 are Hilbert spaces and A is an operator in $L(H_1, H)$, then de Branges's space $M(A)$ consists of the range of A, with the Hilbert space structure that makes A into a coisometry of H_1 onto $M(A)$. Thus, for example, if y is in H_1 and is orthogonal to the kernel of A, then $\|Ay\|_{M(A)} = \|y\|_{H_1}$. If $\|A\| \leq 1$, then the space $M((1 - AA^*)^{1/2})$ is called by de Branges the complementary space of $M(A)$ and denoted by $H(A)$. Our spaces $H(b)$ and $H(\bar{b})$, therefore, coincide with $H(T_b)$ and $H(T_{\bar{b}})$, respectively. We shall denote $M(T_b)$ by $M(b)$.

A factorization criterion of R. G. Douglas [5] is often useful in establishing containment relations between de Branges's spaces, and in showing a given operator maps one of these spaces into another one.

Douglas's criterion. *Let H, H_1, and H_2 be Hilbert spaces and A and B operators in $L(H_1, H)$ and $L(H_2, H)$, respectively. Then the operator inequality $BB^* \leq AA^*$ is necessary and sufficient for the existence of a factorization $B = AR$ with R in $L(H_2, H_1)$ and $\|R\| \leq 1$.*

This tells us, for example, that the two spaces $M(A)$ and $M(B)$ coincide as Hilbert spaces if and only if $AA^* = BB^*$. In virtue of the operator inequality $1 - T_{\bar{b}}T_b \leq 1 - T_bT_{\bar{b}}$, it tells us also that $H(\bar{b})$ is contained in $H(b)$, with the inclusion map a contraction.

If A is a contraction in $L(H_1, H)$, then $M(A)$ is an ordinary subspace of H if and only if A is a partial isometry, in which case $H(A)$ is the ordinary orthogonal complement of $M(A)$. In the contrary case the intersection $M(A) \cap H(A)$, which we call an overlapping space, is nontrivial (de Branges and Rovnyak [3] use the term «overlapping space» in a slightly different way). A simple lemma establishes the relation between $H(A)$, $H(A^*)$, and their overlapping spaces.

Lemma 2.1. *Let H and H_1 be Hilbert spaces and A a contraction in $L(H_1, H)$. The vector x in H belongs to $H(A)$ if and only if A^*x belongs to $H(A^*)$, in which case.*

$$\|x\|_{H(A)}^2 = \|x\|_H^2 + \|A^*x\|_{H(A^*)}^2.$$

The overlapping space $M(A^*) \cap H(A^*)$ coincides with $A^*H(A)$.

The inclusion $A^*H(A) \subset H(A^*)$ follows from the operation identity

$$A^* (1 - AA^*)^{1/2} = (1 - A^*A)^{1/2} A^*,$$

which goes back at least to a paper of P. R. Halmos [10]. Suppose x is a vector in H such that A^*x is in $H(A^*)$, say $A^*x = (1 - A^*A)^{1/2}y$ with y in H_1 and orthogonal to the kernel of $(1 - A^*A)^{1/2}$. Then $AA^*x = (1 - AA^*)^{1/2} Ay$ (by the same identity used above), from which one concludes that

$$x = (1 - AA^*)^{1/2}[(1 - AA^*)^{1/2}x + Ay],$$

showing that x is in $H(A)$. As Ay is easily seen to be orthogonal to the kernel of $(1 - AA^*)^{1/2}$, we have

$$\|x\|_{H(A)} = \|(1 - AA^*)^{1/2}x + Ay\|_H.$$

The square of the right side equals

$$\begin{aligned} &\langle (1 - AA^*)\, x,x\rangle_H + \|Ay\|_H^2 + 2Re\langle (1 - A^*A)^{1/2}\, A^*\, x,\, y\rangle_{H_1} \\ &= \|x\|_H^2 - \|A^*x\|_{H_1}^2 + \|Ay\|_H^2 + 2\langle (1 - A^*A)y,y\rangle_{H_1} \\ &= \|x\|_H^2 - \|A^*x\|_{H_1}^2 + \|y\|_{H_1}^2 + \|(1 - A^*A)^{1/2}y\|_{H_1}^2 \\ &= \|x\|_H^2 + \|y\|_{H_1}^2 \\ &= \|x\|_H^2 + \|A^*x\|_{H(A^*)}^2, \end{aligned}$$

which gives the desired expression for $\|x\|_{H(A)}$. This completes the proof of the lemma.

For the situation of interest in this paper, we obtain the following immediate consequences of Lemma 2.1.

Lemma 2.2. *The H^2 function h belongs to $H(b)$ if and only if $T_{\bar{b}}h$ is in $H(\bar{b})$. If h is in $H(b)$, then*

$$\|h\|_b^2 = \|h\|_2^2 + \|T_{\bar{b}}h\|_{\bar{b}}^2.$$

Lemma 2.3. *The overlapping space $M(b) \cap H(b)$ equals $T_b H(\bar{b})$. The operator T_b acts as a contraction from $H(\bar{b})$ to $H(b)$.*

Two corollaries of the last lemma are worth recording.

Corollary 2.4. *Every multiplier of $H(b)$ is a multiplier of $H(\bar{b})$.*

Corollary 2.5. *The overlapping space $M(b) \cap H(b)$ is dense in $H(b)$ if and only if b is an outer function.*

Corollary 2.4 is immediate. To establish Corollary 2.5 it suffices to note that $T_b H(\bar{b})$ is the range of the operator $(1 - T_b T_{\bar{b}})^{1/2} T_b$, which is dense in $H(b)$ if and only if the range of T_b is dense in H^2, in other words, if and only if b is an outer function.

Lemma 2.6. *If ϕ is a function in H^∞ then the spaces $H(b)$ and $H(\bar{b})$ are invariant under the Toeplitz operator $T_{\bar{\phi}}$, whose norm as an operator in each of them does not exceed $\|\phi\|_\infty$.*

For the proof, we can assume with no loss of generality that $\|\phi\|_\infty = 1$. To settle the case of $H(\bar{b})$ it will be enough, by Douglas's criterion, to verify the operator inequality

$$T_{\bar{\phi}}(1 - T_{\bar{b}} T_b) T_\phi \le 1 - T_{\bar{b}} T_b.$$

One easily sees that the difference between the right and left sides equals

$$1 - T_{|b|^2} - T_{|\phi|^2} + T_{|\phi b|^2},$$

which is the Toeplitz operator with symbol $(1 - |\phi|^2)(1 - |b|^2)$, hence positive semidefinite, as desired. The case of $H(b)$ follows immediately from the case of $H(\bar{b})$ in conjunction with Lemma 2.2.

3. Cauchy integrals

For μ a finite complex Borel measure on ∂D, we let $K\mu$ denote its Cauchy integral, that is, the holomorphic function in $C \backslash \partial D$ defined by

$$(K\mu)(z) = \int_{\partial D} \frac{1}{1 - e^{-i\theta} z} \, d\mu(e^{i\theta}).$$

If μ is absolutely continuous and σ is its Radon-Nikodym derivative with respect to normalized Lebesgue measure, we write $K\sigma$ in place of $K\mu$. (What we are calling Cauchy integrals are often referred to as integrals of Cauchy-Stieltjes type.)

If the measure μ is positive, we define the transformation K_μ on $L^2(\mu)$ by $K_\mu q = K(q\mu)$. The function $K_\mu q$ vanishes identically in D if and only if q is orthogonal to $H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$. We denote by

$K^2(\mu)$ the space of all functions $K_\mu q$ with q in $L^2(\mu)$ and give it the Hilbert space structure that makes K_μ an isometry of $H^2(\mu)$ onto it. As before, if μ is absolutely continuous with Radon-Nikodym derivative σ, we write K_σ and $K^2(\sigma)$ in place of K_μ and $K^2(\mu)$.

We let μ_b denote the measure on ∂D whose Poisson integral is the real part of the function $(1 + b)/(1 - b)$. For q in $L^2(\mu_b)$, we define the function $V_b q$ in D by

$$(V_b q)(z) = (1 - b(z))(K_{\mu_b} q)(z).$$

A proof of the following representation for $H(b)$ can be found in [20].

Lemma 3.1. *The transformation V_b is an isometry of $H^2(\mu_b)$ onto $H(b)$. It maps the function k_w ($|w| < 1$) to the function $(1 - \overline{b(w)})^{-1} k_w^b$.*

Thus, the problem of finding the multipliers of $H(b)$ is the same as the problem of finding the multipliers of $K^2(\mu_b)|D$. We note for future reference that the equality $H^2(\mu_b) = L^2(\mu_b)$ holds if and only if the Radon-Nikodym derivative with respect to Lebesgue measure of the absolutely continuous component of μ_b fails to be log-integrable [14, p. 50]. That Radon-Nikodym derivative equals $(1 - |b|^2)/|1 - b|^2$ and so is not log-integrable if and only if b is an extreme point of $B(H^\infty)$.

The operator on $H^2(\mu_b)$ of multiplication by $e^{i\theta}$ will be denoted by Z_b.

Lemma 3.2. *The transformation $V_b|H^2(\mu_b)$ intertwines the operators $Z_b^*[1 - (1 - b(0))(1 \otimes 1)]$ and S^*.*

In the proof, we shall denote the inner product in $L^2(\mu_b)$ by $\langle\cdot,\cdot\rangle_{\mu_b}$. Let q be any function in $L^2(\mu_b)$, and let $g = K_{\mu_b} q$, so that $V_b q = (1 - b)g$. Since $(K_{\mu_b} Z_b^* q)(z) = \langle Z_b^* q, k_z\rangle_{\mu_b} = \langle q, Z_b k_z\rangle_{\mu_b}$, we have, for $z \neq 0$,

$$\begin{aligned}(K_{\mu_b} Z_b^* q)(z) &= \int_{\partial D} \frac{e^{-i\theta} q(e^{i\theta})}{1 - ze^{-i\theta}}\, d\mu_b(e^{i\theta}) \\ &= \frac{1}{z}\int_{\partial D}\left(\frac{1}{1 - ze^{-i\theta}} - 1\right) q(e^{i\theta})\, d\mu_b(e^{i\theta}) \\ &= \frac{g(z) - g(0)}{z}.\end{aligned}$$

Therefore,

$$(V_b Z_b^* q)(z) = (1 - b(z))\,\frac{g(z) - g(0)}{z}$$

$$= \frac{(1 - b(z))g(z) - (1 - b(0))g(0)}{z} + g(0)\,\frac{b(z) - b(0)}{z}$$

$$= \frac{(V_b q)(z) - (V_b q)(0)}{Z} + \frac{(V_b q)(0)}{1 - b(0)}\,(S^*b)(z)$$

$$= (S^* V_b q)(z) + \langle V_b q,\ V_b 1\rangle_{\mu_b}(S^*b)(z).$$

In the last line we have used the equality $V_b 1 = (1 - \overline{b(0)})^{-1}k_0^b$ from Lemma 3.1. The same equality shows that, when q is the constant function 1, the function g equals $(1 - \overline{b(0)}b)/(1 - \overline{b(0)})(1 - b)$. Inserting these expressions into the equality above (the one that gives $K_{\mu_b} Z_b^* q$ in terms of g) one obtains, after a few lines of calculation, the formula

$$(K_{\mu_b} Z_b^* 1)(z) = \frac{1}{(1 - b(0))(1 - b(z))}\left(\frac{b(z) - b(0)}{z}\right),$$

implying that

$$V_b Z_b^* 1 = (1 - b(0))^{-1} S^* b.$$

The expression for $V_b Z_b^* q$ can thus be rewritten as

$$V_b Z_b^* q = S^* V_b q + (1 - b(0))\langle V_b q,\ V_b 1\rangle_b\ V_b Z_b^* 1,$$

or as

$$S^* V_b q = V_b Z_b^* [q - (1 - b(0))\langle q,\ 1\rangle_b 1],$$

which is the desired conclusion.

For $H(\bar{b})$, the situation is simpler than for $H(b)$. We let ϱ denote the function $1 - |b|^2$ on $\partial\mathrm{D}$.

Lemma 3.3. *The transformation* K_ϱ is an isometry of $\mathrm{H}^2(\varrho)$ onto $H(\bar{b})$. *It* maps k_w *to* $k_w^{\bar{b}}$. *Hence* $H(\bar{b}) = K^2(\varrho)|\mathrm{D}$.

We denote the inner product in $L^2(\varrho)$ by $\langle\cdot,\cdot\rangle_\varrho$. For any points z and w of D we have

$$\begin{aligned}\langle k_w, k_z\rangle_\varrho &= \langle(1 - |b|^2)k_w,\ k_z\rangle\\ &= \langle(1 - T_{\bar{b}}T_b)k_w,\ k_z\rangle\\ &= \langle k_w^{\bar{b}},\ k_z\rangle\\ &= k_w^{\bar{b}}(z)\\ &= \langle k_w^{\bar{b}},\ k_z^{\bar{b}}\rangle_b.\end{aligned}$$

But $(K_\varrho k_w)(z) = \langle k_w, k_z\rangle_\varrho$, so it follows that $K_\varrho k_w = k_w^{\bar{b}}$ and that $\langle K_\varrho k_w, K_\varrho k_z\rangle_{\bar{b}} = \langle k_w, k_z\rangle_\varrho$. Thus K_ϱ maps the linear manifold in $L^2(\varrho)$ spanned by the functions k_w isometrically onto a dense linear manifold in $H(b)$ (the one spanned by the functions $k_w^{\bar{b}}$). One can now complete the proof by a standard limit argument.

The operator on $H^2(\varrho)$ of multiplication by $e^{i\theta}$ will be denoted by Z_ϱ.

Lemma 3.4. *The transformation $K_\varrho|H^2(\varrho)$ intertwines the operators Z_ϱ^* and S^*.*

This is a standard property of Cauchy integrals. It is established, except for a difference in notation, as the first step in the proof of Lemma 3.2.

Corollary 3.5. *If ϕ is a function in H^∞, then the transformation $K_\varrho|H^2(\varrho)$ intertwines the operators $\phi(Z_\varrho)^*$ and $T_{\bar{\phi}}$.*

In fact, the case where ϕ is a polynomial follows immediately from Lemma 3.4. To handle the general case one takes a sequence of polynomials that is uniformly bounded on ∂D and converges almost everywhere on ∂D to ϕ. The obvious limit argument yields the conclusion.

Our first theorem implies, in virtue of Corollary 2.4, that any multiplier of $H(b)$ differs by a constant from a function in $H(\bar{b})$. (If b is not an extreme point of $B(H^\infty)$ then $H(\bar{b})$ contains the constants, so one gets the stronger conclusion that the multipliers of $H(b)$ lie in $H(\bar{b})$. The theorem itself is trivial in that case.)

Theorem 3.6. *If b is not an inner function, then every multiplier of $H(\bar{b})$ differs by a constant from a function in $H(\bar{b})$.*

As noted above, the theorem is trivial if b is not an extreme point of $B(H^\infty)$, so we assume it is an extreme point. Let m be a multiplier of $H(\bar{b})$ and let h be any function in $H(\bar{b})$ such that $h(0) \neq 0$. By Lemma 2.6, the functions mS^*h and $S^*(mh)$ belong to $H(\bar{b})$. Since $S^*(mh) = mS^*h + h(0)S^*m$, it follows that S^*m is in $H(\bar{b})$. Because b is an extreme point, the function ϱ is not log-integrable, which implies that $H^2(\varrho) = L^2(\varrho)$ [14, p. 50]. Therefore, by Lemma 3.4, the operator $S^*|H(\bar{b})$ is unitary, so in particular $S^*H(\bar{b}) = H(\bar{b})$. The function in $H(\bar{b})$ sent to S^*m by S^* differs from m by a constant.

4. Vinogradov's theorem

If μ is a finite complex Borel measure on ∂D then its Cauchy integral, $K\mu$, as a function in D, belongs to H^p for $0 < p < 1$ and so has an inner-outer factorization [7, p. 39]. The theorem of Vinogradov [24] states that if the inner function u divides the inner factor of $K\mu$, then the quotient $K\mu/u$ is a Cauchy integral; in fact, it is the Cauchy integral of a measure whose norm does not exceed that of μ. A simple and natural proof of this can be based on Lemmas 2.6 and 3.1.

For simplicity we assume $\|\mu\| = 1$, and we choose b so that $|\mu| = \mu_b$. By Lemma 3.1 the function $(1 - b)K\mu$ is in $H(b)$ and has norm at most 1. Thus, by Lemma 2.6, if u is an inner function, then

$$T_{\bar{u}}[(1 - b)K\mu] = (1 - b)K(q\mu_b),$$

where q is in $H^2(\mu_b)$ and has norm at most 1. But if u divides the inner factor of $K\mu$ then it divides the inner factor of $(1 - b)K\mu$, so that

$$T_u[(1 - b)K\mu] = (1 - b)K\mu/u.$$

In that case $K\mu/u = K(q\mu_b)$, which proves Vinogradov's theorem since the measure $q\mu_b$ has norm at most 1.

5. Nonmultipliers

Our concern from now on will be with the case where b is an extreme point of $B(H^\infty)$. In this section we obtain two negative results about multipliers.

It was mentioned in Section 1 that, if b is not an extreme point of $B(H^\infty)$, then every function holomorphic in a neighborhood of $\bar{\mathrm{D}}$ is a multiplier of $H(b)$. If b is an extreme point, exactly the opposite is true: no nonconstant multiplier can be continued analytically across all of $\partial\mathrm{D}$. This is a consequence of the next theorem together with Theorem 3.6 and Corollary 2.4.

Theorem 5.1. *If b is an extreme point of $B(H^\infty)$, then no nonzero function in $H(\bar{b})$ can be continued analytically across all of $\partial\mathrm{D}$.*

In fact, suppose the function h in $H(\bar{b})$ can be continued analytically across all of $\partial\mathrm{D}$. By Lemma 3.3 we can write $h = K(q\varrho)$ with q in $L^2(\varrho)$. The function $q\varrho$ is in L^2, being the product of the L^2 function $q\varrho^{1/2}$ and the bounded function $\varrho^{1/2}$. This enables us to write $h = P_+(q\varrho)$. (Recall that P_+ is the orthogonal projection in L^2 with range H^2.) Because b is an extreme point of $B(H^\infty)$, the function ϱ is not log-integrable, and therefore neither is $q\varrho$, because

$$\log|q\varrho| \leq \log^+|q\varrho^{1/2}| + \frac{1}{2}\log\varrho.$$

But the forward Fourier coefficients of $q\varrho$ coincide with the Taylor coefficients of h, which tend to zero exponentially since h is holomorphic across $\partial\mathrm{D}$. It is known [16, p. 12] that a function on $\partial\mathrm{D}$ whose forward Fourier coefficients tend to 0 exponentially is log-integrable unless it vanishes identically. Hence $q_\varrho = 0$, which means $h = 0$, as desired.

Corollary 5.2. *If b is an extreme point of $B(H^\infty)$ and an outer function, then no nonzero function in $H(b)$ can be continued analytically across all of* $\partial\mathrm{D}$.

In fact, suppose the function h in $H(b)$ can be continued analytically across all of $\partial\mathrm{D}$. Its Taylor coefficients then tend to 0 exponentially, and a simple estimate shows that then the forward Fourier coefficients of $\bar{b}h$ exhibit the same behavior. Hence $T_{\bar{b}}h$ can be continued analytically across all of $\partial\bar{\mathrm{D}}$. By Lemma 2.2, $T_{\bar{b}}h$ is in $H(\bar{b})$, so it is 0 by Theorem 5.1. Since b is outer it follows that $h = 0$, as desired.

The noncyclic vectors of the backward shift operator, S^*, have been characterized by R. G. Douglas, H. S. Shapiro, and A. L. Shields [6] as the functions in H^2 that possess pseudocontinuations to the complement of $\bar{\mathrm{D}}$. Our next theorem implies that, if b is an extreme point of $B(H^\infty)$ then, just as is the case with an ordinary continuation, the possession of a pseudocontinuation disqualifies a nonconstant function from being a multiplier of $H(b)$. (Davis and McCarthy [1] have obtained this independently.) In particular, if b is an extreme point, then no nonconstant inner function is a multiplier of $H(b)$, a result from [15].

Theorem 5.3. *If b is an extreme point of $B(H^\infty)$, then the nonzero functions in $H(\bar{b})$ are cyclic vectors of S^*.*

This theorem is nearly disjoint from Theorem 5.1: the only functions in H^2 that possess both ordinary continuations across $\partial\mathrm{D}$ and pseudocontinuations to the complement of $\bar{\mathrm{D}}$ are the rational functions [6].

To prove the theorem, let h be a nonzero function in $H(\bar{b})$. As in the proof of Theorem 5.1, we have $h = P_+(q\varrho)$ where q is a function in $L^2(\varrho)$. Also as in the proof of Theorem 5.1, the function $q\varrho$ is not log-integrable.

Let M be the invariant subspace of S^* generated by h and let $N = M + (H^2)^\perp$. Then N is an invariant subspace of the adjoint of the bilateral shift operator on L^2. By the known structure of these subspaces [14, p. 111], either $N = \chi_E L^2$ with E a measurable subset of $\partial\mathrm{D}$ or $N = v\overline{H^2}$ with v a unimodular function in L^∞. The latter possibility is precluded because N contains the function $q\varrho$, which fails to be log-integrable (and is not the zero function). Thus N is of the form $\chi_E L^2$, and since it contains the function h, which is nonzero almost everywhere, it must actually be all of L^2. That means $M = H^2$, so h is a cyclic vector of S^*, as desired.

Corollary 5.4. *If b is an extreme point of $B(H^\infty)$ and an outer function, then the nonzero functions in $H(b)$ are cyclic vectors of S^*.*

In fact, suppose h is a nonzero function in $H(b)$. Then $T_{\bar{b}}h$ is in $H(\bar{b})$ by Lemma 2.2 and is nonzero because b is outer. By Theorem 5.3, then, $T_{\bar{b}}h$ is a cyclic vector of S^*. But $T_{\bar{b}}h$ lies in the S^*-invariant subspace generated by h, so h also is a cyclic vector of S^*.

6. Decompositions of $H(b)$

Let u_0 be the inner part and b_0 the outer part of the function b. Then, as de Branges and Rovnyak [4, p. 32] first pointed out, the space $H(b)$ is the orthogonal direct sum of the two subspaces $H(u_0)$ and $u_0H(b_0)$. Moreover, the inclusion map of $H(u_0)$ into $H(b)$ is an isometry, and T_{u_0} acts as an isometry of $H(b_0)$ into $H(b)$. To verify these statements, it suffices to rewrite the equality

$$1 - T_bT_{\bar{b}} = 1 - T_{u_0}T_{\bar{u}_0} + T_{u_0}(1 - T_{b_0}T_{\bar{b}_0})T_{\bar{u}_0}$$

as $1 - T_bT_{\bar{b}} = AA^*$, where $A = (A_1\ A_2)$, an operator from $H^2 \oplus H^2$ to H^2, with

$$A_1 = (1 - T_{u_0}T_{\bar{u}_0})^{1/2} \qquad \text{and} \qquad A_2 = T_{u_0}(1 - T_{b_0}T_{\bar{b}_0})^{1/2}.$$

The equality tells us that $H(b) = M(A_1) + M(A_2)$, and this is an orthogonal direct sum, with the inclusion map of each summand into $H(b)$ isometric, because $\ker A = \ker A_1 \oplus \{0\}$. Since $M(A_1) = H(u_0)$ and $M(A_2) = u_0H(b_0)$, the decomposition of $H(b)$ follows. One immediate consequence of the decomposition is that every multiplier of $H(b)$ is a multiplier of $H(b_0)$. More generally, the same reasoning shows that if u is any inner function, then $H(ub)$ is the orthogonal direct sum of $H(u)$ and $uH(b)$. Thus, every multiplier of $H(ub)$ is a multiplier of $H(b)$.

When b is an extreme point of $B(H^\infty)$, there is a companion orthogonal decomposition of $H(b)$.

Theorem 6.1. *Let b be an extreme point of $B(H^\infty)$. Then $H(b)$ is the orthogonal direct sum of $H(b_0)$ and $b_0H(u_0)$. The inclusion map of $H(b_0)$ into $H(b)$ is an isometry, and the operator T_{b_0} acts as an isometry from $H(u_0)$ into $H(b)$.*

The situation when b is not an extreme point is completely different. In that case, $H(b_0)$ is dense in $H(b)$ [18, p. 87].

Theorem 6.1 can be established by a slight modification of the argument in the discussion preceding it. We suppose that b is not inner, since otherwise the theorem reduces to a triviality. This time we use the factorization $1 - T_b T_{\bar{b}} = AA^*$, where $A = (A_1\ A_2)$, but with

$$A_1 = (1 - T_{b_0} T_{\bar{b}_0})^{1/2} \quad \text{and} \quad A_2 = T_{b_0}(1 - T_{u_0} T_{\bar{u}_0})^{1/2}.$$

We assert that $\ker A = \{0\} \oplus \ker A_2$. Clearly, once this has been verified, the previous reasoning applies. To establish the assertion, let $f_1 \oplus f_2$ belong to $\ker A$, and write $g = (1 - T_{b_0} T_{\bar{b}_0})^{1/2} f_1$ and $h = (1 - T_{u_0} T_{\bar{u}_0}) f_2$. Then g is in $H(b_0)$, while h is in $H(u_0)$, and $g = -b_0 h$, implying by Lemma 2.3 that h is in $H(\bar{b}_0)$ (which is the same as $H(\bar{b})$). Since b is an extreme point, Theorem 5.3 implies that $h = 0$, and hence also that $g = 0$. It follows that f_2 is in $\ker A_2$ and $f_1 = 0$, the latter because $\ker (1 - T_{b_0} T_{\bar{b}_0})$ is trivial, b_0 being a nonconstant outer function. This concludes the proof of the theorem.

The next theorem clarifies the relation of $H(b)$ and $H(\bar{b})$ in the extreme point case.

Theorem 6.2. *Let b be an extreme point of $B(H^\infty)$. Then the othogonal complement of $H(\bar{b})$ in $H(b)$ is $b_0 H(u_0)$. The closure of $H(\bar{b})$ in $H(b)$ is $H(b_0)$.*

Again, the situation is completely different when b is not an extreme point. In that case $H(\bar{b})$ is always dense in $H(b)$ [18, p. 87].

The second assertion in Theorem 6.2 follows immediately from the first assertion together with Theorem 6.1. It will thus be enough to establish the first assertion. Some new notations are needed.

As in Section 3, we let ϱ denote the function $1 - |b|^2$ on ∂D and $\langle \cdot, \cdot \rangle_\varrho$ the inner product in $L^2(\varrho)$. Let J_ϱ denote the natural injection of H^2 into $L^2(\varrho)$. One easily verifies that K_ϱ is the adjoint of J_ϱ and that $K_\varrho J_\varrho = 1 - T_{\bar{b}} T_b$. If h is a function in $H(b)$ then $T_b h$ belongs to $H(\bar{b})$, by Lemma 2.2 and so is the image under K_ϱ of a function in $L^2(\varrho)$, by Lemma 3.3. The latter function is unique (also by Lemma 3.3, since $H^2(\varrho) = L^2(\varrho)$); we denote it by $W_\varrho h$.

That $b_0 H(u_0)$ is contained in the orthogonal complement of $H(\bar{b})$ in $H(b)$ is an immediate consequence of Theorem 6.1. To establish the opposite containment, let h be any function in $H(b)$ that is orthogonal to $H(\bar{b})$. Let g be any function in $H(\bar{b})$ and, using Lemma 3.3, write $g = K_\varrho q$ with q in $L^2(\varrho)$. Corollary 3.5 tells us that $T_{\bar{b}} g = K_\varrho(\bar{b} q)$, showing that $W_\varrho g = \bar{b} q$. Thus, by Lemma 2.2,

$$\begin{aligned} 0 &= \langle h, g \rangle_b \\ &= \langle h, g \rangle + \langle T_{\bar{b}} h, T_{\bar{b}} g \rangle_{\bar{b}} \end{aligned}$$

$$= \langle h, g\rangle + \langle W_\varrho h, W_\varrho g\rangle_\varrho$$
$$= \langle h, K_\varrho q\rangle + \langle W_\varrho h, \bar{b}q\rangle_\varrho$$
$$= \langle J_\varrho h + bW_\varrho h, q\rangle_\varrho .$$

This equality holds for all q in $L^2(\varrho)$, so $J_\varrho h + bW_\varrho h$ is the zero function in $L^2(\varrho)$, in other words, $W_\varrho h = -h/b$ on the set where $1 - |b|^2$ is nonzero. Multiplying the last equality by $1 - |b|^2$, we conclude that

$$(1 - |b|^2)W_\varrho h = -\frac{h}{b} + \bar{b}h \, ;$$

in particular, the function h/b belongs to L^2. Projecting both sides of the preceding equality onto H^2, we obtain

$$K_\varrho W_\varrho h = P_+(-h/b) + T_{\bar{b}}h.$$

But $T_{\bar{b}}h = K_\varrho W_\varrho h$ by the definition of W_ϱ, so the function h/b is orthogonal to H^2. However, the function h/b_0 is in H^2 since it is in L^2 and b_0 is an outer function. Since $h/b = \bar{u}_0 h/b_0$, we conclude that h/b_0 is in $H(u_0)$, which means that h is in $b_0H(u_0)$, as desired.

7. Consequences of invertibility

Theorem 7.1. *If b is an extreme point of $B(H^\infty)$, then the following conditions are equivalent*

(i) *b is invertible in H^∞,*

(ii) *$H(b) = H(\bar{b})$,*

(iii) *b is a multiplier of $H(b)$,*

(iv) *$S^*|H(b)$ is similar to a unitary operator.*

An analogous result for the case where b is not an extreme point of $B(H^\infty)$ can be found in [18]. In condition (ii), by the equality $H(b) = H(\bar{b})$ we mean to say that the two spaces are equal as vector spaces but not that their Hilbert space structures coincide. If they are equal as vector spaces then their norms are equivalent, by the closed graph theorem.

The equivalence of conditions (ii) and (iii) in the theorem is an immediate consequence of the equality $M(b) \cap H(b) = bH(\bar{b})$ from Lemma 2.3. To see that (i) implies (ii), assume b is invertible and write $H(b) = T_{\bar{b}^{-1}}T_{\bar{b}}H(b)$. By Lemma 2.2, $T_{\bar{b}}H(b) \subset H(\bar{b})$, and, by Lemma 2.6, $T_{\bar{b}^{-1}}H(\bar{b}) \subset H(\bar{b})$, so it follows that $H(b) = H(\bar{b})$, as desired. This much does not involve the hypothesis that b is an extreme point.

We complete the proof by showing that (ii) implies (iv) and (iv) implies (i). Actually, the first of these implications follows immediately from Lemma 3.4, which says that the operator $S^*|H(\bar{b})$ is unitarily equivalent to the operator Z_ϱ^*; the last operator is unitary when b is an extreme point (since then $H^2(\varrho) = L^2(\varrho)$). It only remains to prove that (iv) implies (i).

Assume that b is not invertible in H^∞. We shall show that then $S^*|H(b)$ is not similar to a unitary operator. The noninvertibility of b implies the noninvertibility of $T_{\bar{b}}$. Hence, given $\epsilon > 0$, there is an f in H^2 with $\|f\|_2 = 1$ and $\|T_{\bar{b}}f\|_2 < \epsilon$. Let $h = (1 - T_bT_{\bar{b}})^{1/2}f$. Then h is in $H(b)$ with $\|h\|_b \leq 1$, and

$$\begin{aligned} \|h\|_b^2 \geq \|h\|_2^2 &= \langle(1 - T_bT_b)f,f\rangle \\ &= \|f\|_2^2 - \|T_bf\|_2^2 \\ &\geq 1 - \epsilon^2. \end{aligned}$$

One consequence of the assumption that b is an extreme point is (in the terminology of de Branges and Rovnyak) the identity for difference quotients:

$$\|S^*g\|_b^2 = \|g\|_b^2 - |g(0)|^2 \qquad (g \in H(b))$$

[19, p. 162]. From this it follows that

$$\lim_{n\to\infty} \|S^{*n}h\|_b^2 = \|h\|_b^2 - \|h\|_2^2 \leq 1 - (1 - \epsilon^2) = \epsilon^2.$$

As ϵ is arbitrary, the desired conclusion, that $S^*|H(b)$ is not similar to a unitary operator, follows, and the proof of Theorem 7.1 is complete.

Later, in Section 11, we shall see that the condition that b be a multiplier of $H(\bar{b})$ is equivalent to the conditions of Theorem 7.1.

Corollary 7.2. *If b is an extreme point of $B(H^\infty)$ and is invertible in H^∞ then b^{-1} is a multiplier of $H(b)$.*

In fact, if b is invertible, then Lemma 2.3 and Theorem 7.1 combine to give $bH(\bar{b}) = H(b)$. (The same result holds, and the same reasoning applies, when b is not an extreme point. The corollary uses only the implication (i) implies (ii) from Theorem 7.1, which, as noted in the proof, holds for nonextreme points as well.)

Corollary 7.3. *If b is not an inner function then $H(b)$ has nonconstant multipliers.*

We need only to treat the case where b is an extreme point. Under the assumption that b is not an inner function, there is a factorization $b = b_1b_2$, where b_1 and b_2 are in $B(H^\infty)$ and b_1 is nonconstant and invertible in H^∞. (For example, one can take b_1 to be the outer function whose modulus on $\partial\mathrm{D}$ is the maximum of $|b|$ and $1/2$). Using the reasoning at the beginning of Section 6 we obtain the decompositions

$$H(b) = H(b_1) + b_1H(b_2) = H(b_2) + b_2H(b_1).$$

Thus $b_1H(b_2) \subset H(b)$ and $b_2H(b_1) \subset H(b)$, and

$$b_1H(b) = b_1H(b_2) + b_2b_1H(b_1).$$

By Theorem 7.1 $b_1H(b_1) \subset H(b_1)$, and hence b_1 is a multiplier of $H(b)$.

As in the last section, we let u_0 denote the inner part and b_0 the outer part of b.

Theorem 7.4. *If b is an extreme point of $B(H^\infty)$ and b_0 is invertible in H^∞, then b_0 and $1/b_0$ are multipliers of $H(b)$ and one has the decompositions*

$$H(b) = H(u_0) + H(b_0) = b_0H(u_0) + u_0H(b_0).$$

The proof depends on the decompositions

$$H(b) = H(u_0) + u_0H(b_0) = H(b_0) + b_0H(u_0)$$

from Section 6. If b_0 is invertible then, as seen above, we have $b_0H(b_0) = H(b_0)$, so that

$$\begin{aligned} b_0H(b) &= b_0H(u_0) + u_0b_0H(b_0) \\ &= b_0H(u_0) + u_0H(b_0) \\ &\subset H(b), \end{aligned}$$

and

$$\begin{aligned} b_0^{-1}H(b) &= b_0^{-1}H(b_0) + H(u_0) \\ &= H(b_0) + H(u_0) \\ &\subset H(b), \end{aligned}$$

Thus b_0 and $1/b_0$ are multipliers of $H(b)$, so the preceding inclusions must actually be equalities, and the desired decompositions of $H(b)$ follow.

8. Multiplication operators

For m a multiplier of $H(b)$, we let M_m denote the corresponding multiplication operator on $H(b)$. For w in D, the kernel function k_w^b is an eigenvector of M_m^*

with eigenvalue $\overline{m(w)}$ (since it is orthogonal to the range of $M_m - m(w)$). Conversely, if M is an operator on $H(b)$ such that each kernel function k_w^b is an eigenvector of M_m^*, then M is a multiplication operator. This well-known property of reproducing kernel Hilbert spaces can be found in [23].

It will be convenient to denote the operator $S^*|H(b)$ by X; it is a contraction by Lemma 2.6. The adjoint X^* is given by

$$X^*h = Sh - \langle h, S^*b \rangle_b b$$

[4], [19]. If b is an extreme point of $B(H^\infty)$, then b is not in $H(b)$, and one can draw the following conclusion.

Lemma 8.1. *If b is an extreme point of $B(H^\infty)$ and h is in $H(b)$, then Sh is in $H(b)$ if and only if $\langle h, S^*b \rangle_b = 0$.*

When b is an extreme point and m is a multiplier of $H(b)$, the operator M_m^* has unexpected eigenvectors.

Theorem 8.2. *Let b be an extreme point of $B(H^\infty)$ and m a multiplier of $H(b)$. Then S^*b is an eigenvector of M_m^*. If $\bar{\alpha}$ is the corresponding eigenvalue, then $(m - \alpha)b$ belongs to $H(b)$ and $m - \alpha$ belongs to $H(\bar{b})$, and the commutation relation*

$$M_m^*X - XM_m^* = S^*b \otimes (m - \alpha)b$$

holds.

In fact, Lemma 8.1 implies that the orthogonal complement of S^*b in $H(b)$ is in variant under M_m, so that S^*b is an eigenvector of M_m^*. To obtain the commutation relation, consider a point w in D and the corresponding kernel function k_w^b. From the expresion $k_w^b = (1 - \overline{b(w)}b)k_w$ one easily obtains the equality $Xk_w^b = wk_w^b - \overline{b(w)}S^*b$. Thus

$$\begin{aligned}(M_m^*X - XM_m^*)k_w^b &= \bar{w}\,\overline{m(w)}k_w^b - \bar{\alpha}\overline{b(w)}S^*b - \overline{m(w)}(wk_w^b - \overline{b(w)}S^*b) \\ &= (\overline{m(w)} - \alpha)\overline{b(w)}S^*b.\end{aligned}$$

As the functions k_w^b span $H(b)$ it follows that $M_m^*X - XM_m^*$ is an operator of rank 1 with range spanned by S^*b:

$$M_m^*X - XM_m^* = S^*b \otimes \phi,$$

where ϕ is some function in $H(b)$. But by the preceding equality,

$$\overline{\phi(w)} = \langle k_w^b, \phi \rangle_b = (\overline{m(w)} - \alpha)\overline{b(w)},$$

in other words, $\phi = (m - \alpha)b$. In particular, $(m - \alpha)b$ is in $H(b)$. It now follows from Lemma 2.3 than $m - \alpha$ is in $H(\bar{b})$, and the proof of the theorem is complete.

For w in D we define the operator Q_w on H^2 by

$$(Q_w f)(z) = \frac{f(z) - f(w)}{z - w}.$$

A simple argument produces the alternative expression $Q_\omega = (1 - wS^*)^{-1}S^*$. In particular, $Q_w b = (1 - wX)^{-1}S^*b$, showing that $Q_w b$ is in $H(b)$.

Corollary 8.3. *If b is an extreme point of $B(H^\infty)$ and m is a multiplier of H(b), then each function $Q_w b$ is an eigenvector of M_m^*.*

The case $w = 0$ is given by Theorem 8.2 so, for the proof, assume $w \neq 0$. The communication relation gives

$$(1 - wX)M_m^* - M_m^*(1 - wX) = wS^*b \otimes (m - \alpha)b.$$

Applying both sides to $Q_w b$, we obtain

$$(1 - wX)M_m^* Q_w b - \bar{\alpha}S^*b = \omega\langle Q_w b, (m - \alpha)b\rangle_b S^*b.$$

It follows that

$$M_m^* Q_w b = [\bar{\alpha} + w\langle Q_w b, (m - \alpha)b\rangle_b] Q_w b,$$

the desired conclusion.

The properties of M_m given by Theorem 8.2 characterize multiplication operators in the extreme point case.

Theorem 8.4. *If b is an extreme point of $B(H^\infty)$ and if M is an operator on $H(b)$ such that $M^*S^*b = \bar{\alpha}S^*b$ and*

$$M^*X - XM^* = S^*b \oplus \phi,$$

then $M = M_m$ for a multiplier m of $H(b)$.

To prove this it will suffice to show that the hypotheses imply k_w^b is an eigenvector of M^* whenever $b(w) \neq 0$. Assuming the last condition and applying the commutation relation to k_w^b, we obtain

$$\begin{aligned}\overline{\phi(w)}S^*b &= M^*(\bar{w}k_w^b - \overline{b(w)}S^*b) - XM^*k_w^b \\ &= (\bar{w} - X)M^*k_w^b - \bar{\alpha}\overline{b(w)}S^*b,\end{aligned}$$

so that

$$(\bar{w} - X)M^*k_w^b = (\overline{\phi(w)} + \bar{\alpha}\overline{b(w)})S^*b.$$

But also $(\bar{w} - X)k_w^b = \overline{b(w)}S^*b$, and the operator $\bar{w} - X$ is injective because of the assumption that $b(w) \neq 0$ (which implies that k_w is not in $H(b)$ [19]). We can conclude that

$$M^*k_w^b = \frac{\overline{\phi(w)} + \bar{\alpha}\overline{b(w)}}{\overline{b(w)}}\, k_w^b,$$

and the proof is complete.

9. Conjugations

We assume throughout this section that b is an extreme point of $B(H^\infty)$. As we mentioned earlier, in Section 3, one consequence of this assumption is the equality $H^2(\mu_b) = L^2(\mu_b)$, which enables us to define a conjugation on $H(b)$ by transferring via the map V_b a conjugation on $L^2(\mu_b)$. This conjugation and another on a space related to $H(\bar{b})$ that we shall introduce a little later are intimately connected with the structure of the multipliers of $H(b)$.

The conjugation on $H(b)$ that turns out to be useful is the one that corresponds to the conjugation $q \to e^{-i\theta}\bar{q}$ on $L^2(\mu_b)$. We denote it by C:

$$Ch = V_b(Z_b^*\overline{V_b^{-1}h}) \qquad (h \in H(b)).$$

That C is a conjugation (an anti-unitary involution) is obvious.

Lemma 9.1. *For w in* D, $Ck_w^b = Q_w b$.

This is a straightforward calculation. By the way μ_b is defined, the function $(1 + b)/(1 - b)$ differs by an imaginary constant from the Herglotz integral of μ_b. Using the equality $k_w^b = (1 - \overline{b(w)})V_b k_w$ from Lemma 3.1, we obtain

$$\begin{aligned}
(Ck_w^b)(z) &= (1 - b(w))(1 - b(z)) \int_{\partial D} \frac{e^{-i\theta}}{(1 - e^{-i\theta}w)(1 - e^{-i\theta}z)}\, d\mu_b(e^{i\theta}) \\
&= \frac{(1 - b(w))(1 - b(z))}{z - w} \int_{\partial D} \left[\frac{1}{1 - e^{-i\theta}z} - \frac{1}{1 - e^{-i\theta}w}\right] d\mu_b(e^{i\theta}) \\
&= \frac{(1 - b(w))(1 - b(z))}{2(z - w)} \int_{\partial D} \left[\frac{e^{i\theta} + z}{e^{i\theta} - z} - \frac{e^{i\theta} + w}{e^{i\theta} - w}\right] d\mu_b(e^{i\theta}) \\
&= \frac{(1 - b(w))(1 - b(z))}{2(z - w)} \left[\frac{1 + b(z)}{1 - b(z)} - \frac{1 + b(w)}{1 - b(w)}\right]
\end{aligned}$$

$$= \frac{b(z) - b(w)}{z - w},$$

as desired.

The conjugation C intertwines the operator X ($= S^*|H(b)$) and its adjoint.

Lemma 9.2. *$CXC = X^*$*

It will suffice to show that $CXk_w^b = X^*Ck_w^b$ for all w. We transform both sides with the aid of Lemma 9.1. First,

$$\begin{aligned} CXk_w^b &= C(wk_w^b - \overline{b(w)}S^*b) \\ &= wQ_wb - b(w)k_0^b. \end{aligned}$$

Next, by the formula for X^* mentioned in Section 8,

$$\begin{aligned} X^*Ck_w^b &= X^*Q_wb \\ &= SQ_wb - \langle Q_wb, S^*b\rangle_b b \\ &= SQ_wb - \langle k_0^b, k_w^b\rangle_b b \\ &= SQ_wb - (1 - \overline{b(0)}b(w))b. \end{aligned}$$

Now

$$\begin{aligned} SQ_w &= (S - w)S^*(1 - wS^*)^{-1} + wQ_w \\ &= -(1 - SS^*)(1 - wS^*)^{-1} + 1 + wQ_w. \end{aligned}$$

The operator $(1 - SS^*)(1 - wS^*)^{-1}$ is easily seen to equal $1 \otimes k_w$; in fact, its adjoint applied to the H^2 function f gives

$$(1 - \bar{w}S)^{-1}(1 - SS^*)f = f(0)(1 - \bar{w}S)^{-1}1 = f(0)k_w.$$

Thus $SQ_wb = -b(w) + b + wQ_wb$. Inserting this into the expression above for $X^*Ck_w^b$, we get

$$\begin{aligned} X^*Ck_w^b &= -b(w) + b + wQ_wb - (1 - \overline{b(0)}b(w))b \\ &= wQ_wb - b(w)(1 - \overline{bh(0)}b), \end{aligned}$$

as desired.

We now introduce our second conjugation. It will act on the space $K_+^2(\varrho)$, by which we mean the space of functions that are sums of functions in $K^2(\varrho)$ and constant functions. The functions in $K^2(\varrho)$, and here those in $K_+^2(\varrho)$, are defined in the complement of $\partial\mathrm{D}$, and we define them at ∞ in the obvious way (namely, those in $K^2(\varrho)$ are assigned the value 0 at infinity). For f in $K^2_+(\varrho)$ we define its conjugate, f_*, by

$$f_*(z) = \overline{f(1/\bar{z})}.$$

Straightforward calculations show that, if $f = K_p q + c$ with q in $L^2(\varrho)$ and c a constant, then $f_* = -SK_\varrho(Z_\varrho^*\bar{q}) + \bar{c}$, and also $f_* = -K_\varrho\bar{q} + (K_\varrho\bar{q})(0) + \bar{c}$. The latter expression shows that f_* is in $K^2_+(\varrho)$.

We let $K^\infty(\varrho)$ denote the space of bounded functions in $K^2_+(\varrho)$. (Here, by bounded we mean bounded in the entire complement of ∂D, not merely in D.) It is obvious that the conjugation on $K^2_+(\varrho)$ maps $K^\infty(\varrho)$ into itself.

The next lemma gives a relation between our two conjugations.

Lemma 9.3. *If f is in $H(\bar{b})$, then $C[(1 - b)f] = (b - 1)S^*f_*$.*

To prove this, let q be the function in $L^2(\varrho)$ such that $f = K_\varrho q$. Because $(1 - |b|^2)/|1 - b|^2$ is the Radon-Nikodym derivative with respect to normalized Lebesgue measure of the absolutely continuous component of μ_b, we can also write $f = K_{\mu_b}(|1 - b|^2 q)$, provided we regard $|1 - b|^2 q$ as vanishing on the singular component of μ_b, if there is one. Thus

$$(1 - b)f = V_b(|1 - b|^2 q),$$

and we obtain

$$\begin{aligned} C[(1 - b)f] &= V_b Z_b^*(|1 - b|^2\bar{q}) \\ &= (1 - b)K_{\mu_b} Z_b^*(|1 - b|^2\bar{q}) \\ &= (1 - b)K_\varrho(Z_\varrho^*\bar{q}). \end{aligned}$$

As mentioned above, $f_* = -SK_\varrho(Z_\varrho^*\bar{q})$, so that $K_\varrho(Z_\varrho^*\bar{q}) = -S^*f_*$, and the desired equality follows.

We are now able to determine the effect of conjugation on multiplication operators.

Theorem 9.4. *If m is a multiplier of $H(b)$, then CM_mC is a multiplication operator, namely, it equals M_{m_*}.*

Corollary 9.5. *The multipliers of $H(b)$ are in $K^\infty(\varrho)$.*

The corollary follows immediately from the theorem. To prove the theorem we note first that, because C is a conjugation, $(CM_mC)^* = CM_m^*C$. This in conjunction with Lemma 9.1 and Corollary 8.3 implies that if m is a multiplier of $H(b)$ then each of the functions $\dot{k}_w^b$ is an eigenvector of $(CM_mC)^*$ and hence that CM_mC is a multiplication operator. It remains to determine the corresponding multiplier. Let it be denoted by m'.

From the proof of Corollary 8.3 we have

$$M_m^*Q_wb = [\bar{\alpha} + w\langle Q_wb, (m - \alpha)b\rangle_b]Q_wb,$$

where a is the eigenvalue of S^*b as an eigenvector of M_m^*. Consequently

$$CM_m^*Ck_w^b = [\alpha + \bar{w}\langle (m - \alpha)b, Q_wb\rangle_b]k_w^b,$$

from which we conclude that

$$\begin{aligned} m^1(z) &= \bar{\alpha} + z\langle Q_zb, (m - \alpha)b\rangle_b \\ &= \bar{\alpha} + z\langle C[(m - \alpha)b], k_z^b\rangle_b \\ &= \bar{\alpha} + zC[(m - \alpha)b](z). \end{aligned}$$

Hence $m^1(0) = \bar{\alpha}$, and

$$S^*m^1 = C[(m - \alpha)b].$$

By Lemma 9.3,

$$C[(m - \alpha)(b - 1)] = (1 - b)S^*m_*.$$

(The lemma applies because $m - \alpha$ belongs to $H(\bar{b})$, by Theorem 8.2) In view of the last two equalities, we seek an expression for $C(m - \alpha)$ in terms of m^1.

Let $\bar{\beta}$ be the eigenvalue of S^*b as an eigenvector of $M_{m^1}^*$. Because m and m^1 play symmetric roles, we have $\beta = \overline{m(0)}$ and $C[(m^1 - \beta)b] = S^*m$. We can rewrite the last equality as

$$X(m - \alpha) = C[(m^1 - \beta)b].$$

Since $CX = X^*C$, it follows that

$$X^*C(m - \alpha) = (m^1 - \beta)b.$$

Using the formula for X^* mentioned in Section 8, we can rewrite the left side here as

$$\begin{aligned} SC(m - \alpha) - \langle C(m - \alpha), S^*b\rangle_b b &= SC(m - \alpha) - \langle k_0^b, m - \alpha\rangle_b b \\ &= SC(m - \alpha) - (\overline{m(0)} - \bar{\alpha})b \\ &= SC(m - \alpha) - (\bar{\alpha} - \beta)b. \end{aligned}$$

Applying S^*, we find that

$$C(m - \alpha) + (\bar{\alpha} - \beta)S^*b = S^*[(m^1 - \beta)b]$$

$$= bS^*m^1 + (m^1(0) - \beta)S^*b$$
$$= bS^*m^1 + (\bar{\alpha} - \beta)S^*b.$$

Hence $C(m - \alpha) = bS^*m^1$.

Combining the last equality with the previously obtained expressions for $C[(m - \alpha)b]$ and $C[(m - \alpha)(b - 1)]$, we find that $S^*m^1 = S^*m_*$, so m^1 and m_* differ by at most a constant. But from the way m_* is defined one easily sees that $m_* - \beta$ $(= m_* - \overline{m(0)})$ belongs to $H(\bar{b})$. Since $m^1 - \beta$ also belongs to $H(\bar{b})$, but the constant functions do not, we must have $m^1 = m_*$, and the proof of the theorem is complete.

10. Lemmas on Cauchy integrals

We need two simple facts about Cauchy integrals. For μ a finite complex Borel measure on $\partial \mathrm{D}$, we let $P_*\mu$ denote the Poisson integral of μ and $Q_*\mu$ the conjugate Poisson integral of μ.

Lemma 10.1. *If μ is a finite complex Borel measure on $\partial \mathrm{D}$ and $f = K\mu$, then, in* D,

$$f(z) - f(1/\bar{z}) = (P * \mu)(z)$$

$$f(z) = \frac{1}{2}[(P * \mu)(z) + i(Q * \mu)(z) + (P * \mu)(0)]$$

Lemma 10.2. *If μ is a finite complex Borel measure on $\partial \mathrm{D}$, and if f and g are holomorphic functions in* D *such that $f - \bar{g} = P * \mu$, then, in* D,

$$f(z) = (K\mu)(z) + \overline{g(0)}$$
$$\overline{g(z)} = (K\mu)(1/\bar{z}) + \overline{g(0)}.$$

Lemma 10.1 is a straightforward consequence of the relation between the Cauchy kernel and the Poisson and conjugate Poisson kernels, and Lemma 10.2 follows easily from Lemma 10.1.

To illustrate the use of these lemmas we show here that, when b is an extreme point of $B(H^\infty)$, the space $K^\infty(\varrho)$ is closed under multiplication. Let f and g be functions in $K^\infty(\varrho)$. The function $fg - \bar{f}_*\bar{g}_*$ is then bounded and harmonic in D, so it is the Poisson integral of its boundary function. (The function is defined in $\bar{\mathrm{C}}\backslash\partial \mathrm{D}$, but by its boundary function we mean the interior boundary function, that is, the boundary function from D.) By Lemma 10.2, to prove fg is in $K^\infty(\varrho)$ it will suffice to prove that the interior boundary function of $fg - \bar{f}_* \bar{g}_*$ has the form $q\varrho$ with q in $L^2(\varrho)$. Let q_1 and q_2 be the functions in $L^2(\varrho)$ such that $f = f(\infty) + K_\varrho(q_1)$ and $g = g(\infty) + K_\varrho(q_2)$. Then, by Lemma 10.1, the interior boundary function of $f - \bar{f}_*$ is $q_1\varrho$ and the interior boundary function of $g - \bar{g}_*$ is $q_2\,\varrho$. Writing

$$fg - \bar{f}_*\bar{g}_* = (f - \bar{f}_*)g + \bar{f}_*(g - \bar{g}_*),$$

we see that the interior boundary function of $fg - \bar{f}_*\bar{g}_*$ is $(q_1 g + q_2\bar{f}_*)\varrho$. (In the last expression, of course, g and $\bar{f}_*$ denote interior boundary functions.) Since g and $\bar{f}_*$ are bounded, the function $q_1 g + q_2\bar{f}_*$ is in $L^2(\varrho)$, the desired conclusion.

Thus, $K^\infty(\varrho)$ is an algebra, and by reasoning like that above one easily sees that the spectrum of a function f in this algebra equals the closure of $f(\mathsf{C}\backslash\partial\mathsf{D})$. In fact, that the spectrum of f contains the closure of $f(\mathsf{C}\backslash\partial\mathsf{D})$ is obvious, so one only needs to show that f is invertible in $K^\infty(\varrho)$ if it is bounded away from 0 in $\mathsf{C}\backslash\partial\mathsf{D}$. If the latter happens, and if q_1 is the function in $L^2(\varrho)$ such that $f = f(\infty) + K_\varrho q_1$, then by Lemma 10.1 the interior boundary function of $f^{-1} - \bar{f}_*^{-1}$ is $-q_1\,\varrho/f\bar{f}_*$, which is of the form $q\varrho$ with q in $L^2(\varrho)$. Lemma 10.2 thus guarantees that f^{-1} is in $K^\infty(\varrho)$.

11. More on $H(\bar{b})$

We return in this section to the assumption that b is an extreme point of $B(H^\infty)$. The functions in $H(\bar{b})$ are restrictions to D of functions in $K^2(\varrho)$, so they have natural extensions to the exterior of ∂D. The next lemma states the process of extension preserves multiplication, to the extent that it can. (This fails when b is not an extreme point, except in the trivial case where ϱ is constant.) For f in $H(\bar{b})$, we let f_* denote the restriction to D of the conjugate of the extension of f. (It differs by a constant from a function in $H(\bar{b})$.)

Lemma 11.1. *If the function f, g, and fg belong to $H(\bar{b})$, then $(fg)_* = f_* g_*$.*

For the case where f and g are in $K^\infty(\varrho)$ this is established at the end of the preceding section. The argument for the general case is similar but slightly more elaborate.

Let q_1, q_2, and q be the functions in $L^2(\varrho)$ such that $f = K_\varrho q_1$, $g = K_\varrho q_2$, and $fg = K_\varrho q$. By Lemma 10.1, the boundary functions of $f - \bar{f}_*$ and $\bar{g} - \bar{g}_*$ are $q_1\,\varrho$ and $g_2\,\varrho$ respectively. The function $fg - \bar{f}_*\bar{g}_*$ is the sum of an H^2 function and the conjugate of an H^1 function, so it is the Poisson integral of its boundary function. Writing

$$fg - \bar{f}_*\bar{g}_* = (f - \bar{f}_*)g + \bar{f}_*(g - \bar{g}_*),$$

we see that its boundary function is $(gq_1 + \bar{f}_* q_2)\varrho$. (In the usual way, we are identifying functions in D with their boundary functions.) Therefore, by Lemma 10.2,

$$fg = K[(gq_1 + \bar{f}_* q_2)\varrho].$$

Hence

$$K[(gq_1 + \bar{f}_* q_2 - q)\varrho] = 0.$$

As the functions $q_1\varrho^{1/2}$, $q_2\varrho^{1/2}$, and $q\varrho^{1/2}$ are in L^2, the function $(gq_1 + \bar{f}_* q_2 - q)\varrho^{1/2}$ is in L^1, which implies that the function $(gq_1 + \bar{f}_* q_2 - q)\varrho$ fails to be log-integrable (the reasoning can be found in Section 5). Since the Cauchy integral of the latter function vanishes so do its forward Fourier coefficients, and hence it is the zero function. Thus $fg - \bar{f}_*\bar{g}_*$ is actually the Poisson integral of $q\varrho$, and we can conclude by Lemma 10.2 that $(fg)_* = f_*g_*$, as desired.

Lemma 11.1 enables us to obtain the analogue of Theorem 9.4 and its corollary for multipliers of $H(\bar{b})$.

Theorem 11.2. *If m is a multiplier of $H(\bar{b})$ then m is in $K^\infty(\varrho)$ and m_* is a multiplier of $H(\bar{b})$.*

To prove this we use the conjugation on $H(\bar{b})$ that corresponds, under the transformation K_ϱ, to the conjugation $q \to -Z_\varrho^*\bar{q}$ on $L^2(\varrho)$. We shall not introduce a special notation for it because we shall not have occasion to use it again. A straightforward calculation shows that it is given by $f \to S^*f_*$. The important property for us is that the preceding map sends $H(\bar{b})$ onto itself, which also follows from the unitarity of $S^*|H(\bar{b})$ (used before in the proof of Theorem 3.6).

Let m be a multiplier of $H(\bar{b})$, and let f be any function in $H(\bar{b})$. By Theorem 3.6, m is in $K^2_+(\varrho)|D$, so Lemma 11.1 can be applied to give $(mf)_* = m_*f_*$. Also $f_*(0) = 0$, so $S^*(mf)_* = m_*S^*f_*$. In view of the remark at the end of the last paragraph we can conclude that m_* is a multiplier of $H(\bar{b})$. In particular, m_* is bounded in D, and thus m is in $K^\infty(\varrho)$.

The next result enables us to supplement Theorem 7.1.

Theorem 11.3. *The function b belongs to $K^2_+(\varrho)|D$ if and only if $1/b$ is in H^2. In that case $b_* = 1/b$.*

For the proof, suppose first that $1/b$ is in H^2. Then $1/b$ is also in $L^2(\varrho)$, and we have

$$K_\varrho(1/\bar{b}) = \mathrm{P}_+\left(\frac{1-|b|^2}{\bar{b}}\right) = 1/\overline{b(0)} - b,$$

showing that $1/\overline{b(0)} - b$ is in $K^2(\varrho)|\mathrm{D}$ and hence that b is in $K^2_+(p)|\mathrm{D}$. Moreover, because of the way the transformation K_ϱ interacts with the conjugation on $K^2_+(\varrho)$ (as was pointed out in Section 9), we have

$$\begin{aligned}(1/\overline{b(0)} - b)_* &= -SK_\varrho(Z^*_\varrho(1/b))\\ &= -SS^*K_\varrho(1/b)\\ &= -SS^*P_+\left(\frac{1-|b|^2}{\bar{b}}\right)\\ &= -SS^*\left(\frac{1}{b} - \overline{b(0)}\right)\\ &= \frac{1}{b(0)} - \frac{1}{b},\end{aligned}$$

which gives $b_* = 1/b$.

Suppose, conversely, that b is in $K^2_+(\varrho)|\mathrm{D}$, in other words, that $b - c$ is in $H(\bar{b})$, where c is a constant. Then $c \neq 0$, since b is not in $H(b)$. Also, because $1 - \overline{b(0)}b$ $(= k^b_0)$ is in $H(b)$, we must have $b(0) \neq 0$ and $c = 1/\overline{b(0)}$. Let q be the function in $L^2(\varrho)$ that maps to $b - 1/\overline{b(0)}$ under K_ϱ. Then

$$\begin{aligned}K_\varrho(\bar{b}q) &= \mathrm{T}_{\bar{b}}\mathrm{K}_\varrho q = T_{\bar{b}}(b - 1/\overline{b(0)})\\ &= T_{\bar{b}}b - 1 = -P_+(1-|b|^2)\\ &= K_\varrho(-1).\end{aligned}$$

Since K_ϱ has a trivial kernel, it follows that $q = -1/\bar{b}$ (modulo the measure $\varrho d\theta$). Therefore $(1-|b|^2)/|b|^2$ is in L^1, implying that $1/b$ is in L^2. In addition,

$$1/\overline{b(0)} - b = K_\varrho(1/\bar{b}) = P_+\left(\frac{1-|b|^2}{\bar{b}}\right) = P_+(1/\bar{b}) - b,$$

so $P_+(1/\bar{b}) = 1/\overline{b(0)}$. Therefore $1/\bar{b}$ is in $\overline{H^2}$, in other words, $1/b$ is in H^2, and the proof is complete.

Corollary 11.4. *If b is a multiplier of $H(\bar{b})$ then b is invertible in H^∞.*

The corollary is an immediate consequence of Theorems 11.2. and 11.3.

12. Construction of multipliers

We retain the assumption that b is an extreme point of $B(H^\infty)$. Our next main result, Theorem 12.2, is a criterion for a function in $K^\infty(\varrho)$ to be a multiplier of $H(b)$. The criterion enables us to show that $H(b)$ has an abundance of multipliers; in particular, the multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.

Lemma 12.1. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho(q)$. Let g be a function in H^2 such that $g(0) = 0$. Then $T_{\bar{g}}m = K(\bar{g}q\varrho)|\mathrm{D}$. The function $T_{\bar{g}}m$ is in $H(b)$ if and only if gq is in $L^2(\varrho)$.*

In fact, by Lemma 10.1, the interior boundary function of $m - \overline{m}_*$ is $q\varrho$, so

$$T_{\bar{g}}m = P_+(\bar{g}\overline{m}_*) + P_+(\bar{g}q\varrho).$$

The first term on the right is 0 because $g(0) = 0$, and the second term is $K(\bar{g}q\varrho)|\mathrm{D}$. This proves the first assertion in the lemma. It is obvious that $T_{\bar{g}}m$ is in $H(\bar{b})$ if $g\varrho$ is in $L^2(\varrho)$, which is one direction in the second assertion. For the other direction, suppose $T_{\bar{g}}m$ is in $H(\bar{b})$, say $T_{\bar{g}}m = K_\varrho(q_1)|\mathrm{D}$ with q_1 in $L^2(\varrho)$. Then $K((\bar{g}q - q_1)\varrho)|\mathrm{D} = 0$. But $(\bar{g}q - q_1)\varrho$ is not log-integrable since it is the product of the L^1 function $gq\varrho^{1/2} - q_1\varrho^{1/2}$ and the function $\varrho^{1/2}$, which is not log-integrable. It follows that $q_1 = \bar{g}q$, and the proof is complete.

Theorem 12.2. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$.*

(i) *The function m is a multiplier of $H(\bar{b})$ if and only if fq is in $L^2(\varrho)$ for every f in $H(\bar{b})$.*

(ii) *The function m is a multiplier of $H(b)$ if and only if hq is in $L^2(\varrho)$ for every h in $H(b)$.*

To prove (i), let f be any function in $H(\bar{b})$, and let q_1 be the function in $L^2(\varrho)$ such that $f = K_\varrho q_1|\mathrm{D}$. By Lemma 11.1, if mf is in $H(\bar{b})$ then $(mf)_* = m_*f_*$. This in conjunction with Lemmas 10.1 and 10.2 implies that mf is in $H(\bar{b})$ if and only if the boundary function of $mf - \overline{m}_*\bar{f}_*$ has the form $q_2\varrho$ with q_2 in $L^2(\varrho)$. On $\partial\mathrm{D}$ we have

$$\begin{aligned} mf - \overline{m}_*\bar{f}_* &= (m - \overline{m}_*)f + \overline{m}_*(f - \bar{f}_*) \\ &= fq\varrho + \overline{m}_*q_1\varrho. \end{aligned}$$

Since m_* is bounded the function $\overline{m}_* q_1$ is in $L^2(\varrho)$. Hence $mf - \overline{m}_*\bar{f}_*$ has the required form if and only if fq is in $L^2(\varrho)$, which proves (i).

Because of (i), in proving (ii) we can assume, without loss of generality, that m is a multiplier of $H(\bar{b})$. Let h be any function in $H(b)$. By Lemma 2.2, mh is in $H(b)$ if and only if $T_{\bar{b}}(mh)$ is in $H(\bar{b})$. We have

$$\begin{aligned} T_{\bar{b}}(mh) &= mT_{\bar{b}}h + P_+((\bar{b}h - P_+\bar{b}h)m) \\ &= mT_{\bar{b}}h + T_{\bar{g}}m, \end{aligned}$$

where $g = \overline{(1 - P_+)(\bar{b}h)}$. The first term on the right is in $H(\bar{b})$ since we have assumed that m is a multiplier of $H(\bar{b})$. Hence mh is in $H(b)$ if and only if the second term on the right, $T_{\bar{g}}m$, is in $H(\bar{b})$. By Lemma 12.1, that happens if and only if gq is in $L^2(\varrho)$. The function $qP_+(\bar{b}h)$ $(= qT_{\bar{b}}h)$ is in $L^2(\varrho)$ by (i) (since $T_{\bar{b}}h$ is in $H(\bar{b})$ and m is a multiplier of $H(\bar{b})$). Hence mh is in $H(b)$ if and only if bhq is in $L^2(\varrho)$, in other words, if and only if $|b|^2 |h|^2|q|^2\varrho$ is in L^1. But $(1 - |b|^2)|h|^2|q|^2\varrho$ $(= |h|^2|q|^2\varrho^2)$ is in L^1 since $q\varrho$ $(= m - \overline{m}_*)$ is bounded. Hence mh is in $H(b)$ if and only if $|h|^2|q|^2\varrho$ is in L^1, in other words, if and only if hq is in $L^2(\varrho)$. This proves (ii).

Corollary 12.3. *If m is a multiplier of $H(b)$ then the spectrum of M_m is the closure of $m(\mathbb{C}\backslash\partial\mathbb{D})$.*

As shown in Section 10, the closure of $m(\mathbb{C}\backslash\partial\mathbb{D})$ equals the spectrum of m in the algebra $K^\infty(\varrho)$, and this set is obviously contained in the spectrum of M_m. To establish the opposite containment it will suffice to show that the invertibility of m in $K^\infty(\varrho)$ implies that $1/m$ is a multiplier of $H(b)$. Assume m is invertible in $K^\infty(\varrho)$, and let q and q_1 be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$ and $1/m = 1/m(\infty) + K_\varrho q_1$. Since

$$\frac{1}{m} - \frac{1}{\overline{m}_*} = \frac{\overline{m}_* - m}{m\overline{m}_*}$$

we conclude by Lemmas 10.1 and 11.1 that $q_1 = -q/\overline{m\overline{m}_*}$. If h is in $H(b)$ then Theorem 12.2(ii) tells us that hq is in $L^2(\varrho)$, and therefore so is hq_1, since $1/m$ and $1/m_*$ are bounded. Theorem 12.2.(ii) now implies that $1/m$ is a multiplier of $H(b)$, as desired.

Corollary 12.4. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$. If $q\varrho^{1/2}$ is bounded, then m is a multiplier of $H(b)$.*

This corollary is an immediate consequence of Theorem 12.2.

Corollary 12.5. *If m is an invertible function in H^∞ such that $(1 - |m|^2)^2/\varrho$ is bounded on $\partial\mathbb{D}$, then m is a multiplier of $H(b)$, and $m_* = 1/m$.*

We remark that if b is not an inner function, in other words, if ϱ does not vanish identically, then nonconstant functions satisfying the hypotheses of Corollary 12.5 can be constructed by standard means. (One example is the outer function with modulus $\max\{|b|, 1/2\}$ on ∂D.)

To establish Corollary 12.5 it suffices to note that the bounded harmonic function $m - 1/\overline{m}$ is the Poisson integral of its boundary function, which equals $(|m|^2 - 1)/\overline{m}$. From Lemma 10.2. it follows that m is in $K^\infty(\varrho)$ with $m - 1/\overline{m(0)} = K_\varrho((|m|^2 - 1)/\varrho\overline{m})$ in D, and $m_* = 1/m$. That m is a multiplier of $H(b)$ is now immediate from Corollary 12.4.

Corollary 12.6. *If m is a function in H^∞ such that $|Re\ m|/\varrho^{1/2}$ is bounded on ∂D, then m is a multiplier of $H(b)$ and $m_* = -m$.*

It is not completely obvious that there are nonzero functions satisfying the hypotheses of the corollary in all cases where b is not an inner function. That there are will be pointed out below in connection with the proof of Corollary 12.8.

To establish Corollary 12.6, it suffices to use Lemma 10.2 in the same way as in the preceding proof and earlier ones to obtain

$$m + \overline{m(0)} = K_\varrho\left(\frac{2Re\ m}{\varrho}\right)$$

in D and $m_* = -m$. Corollary 12.4 now applies to show that m is a multiplier of $H(b)$.

Corollary 12.7. *If the outer factor, b_0, of b is invertible in H^∞, then all of the functions $k_w^{b_0}$ and $Q_w b_0$ are multipliers of $H(b)$, and $(Q_w b_0)_* = Sk_w^{b_0}/b_0$ and $(k_w^{b_0})_* = -b_0(w)SQ_w(1/b_0)$.*

To simplify the notation slightly in the proof, we shall assume that b itself is invertible. This is not a genuine loss of generality, because the criterion in Corollary 12.4, upon which the proof of Corollary 12.7 is based, is insensitive to the inner factor of b.

Assuming then that b is invertible, we note that

$$\begin{aligned}
K_\varrho(Z_\varrho^*\bar{k}_w/\bar{b}) &= S^*K_\varrho(\bar{k}_w/\bar{b}) \\
&= S^*P_+\left(\frac{(1-|b|^2)\bar{k}_w}{b}\right) \\
&= -S^*P_+(b\bar{k}_w) \\
&= -S^*(1-wS^*)^{-1}b \\
&= -Q_w b.
\end{aligned}$$

Therefore, by the relation between the transformation K_ϱ and the conjugation on $K_+^2(\varrho)$ (noted in Section 9),

$$\begin{aligned}(Q_w b)_* &= SK_\varrho(k_w/b)\\ &= SP_+(k_w/b - \bar{b}k_w)\\ &= S(k_w/b - T_{\bar{b}}k_w)\\ &= S(1 - \overline{b(0)}b)k_w/b\\ &= Sk_w^b/b.\end{aligned}$$

Thus $Q_w b$ is in $K^\infty(\varrho)$, and Corollary 12.4 implies that it is a multiplier of $H(b)$. We see also that Sk_w^b/b is a multiplier of $H(b)$. Since the space of multipliers is invariant under S^* (by Lemma 2.6), and since b is a multiplier of $H(b)$ (Theorem 7.1), it follows that k_w^b is a multiplier of $H(b)$. To determine $(k_w^b)_*$ one verifies that $k_w^b = \overline{b(w)}K_\varrho(k_w/b)$, which gives the formula

$$(k_w^b)_* = -b(w)SS^*K_\varrho(\bar{k}_w/b).$$

The right side is easily reduced to the desired expression. The details are similar to those above and we omit them.

Corollary 12.8. *The multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.*

To prove this, let q be a real function in $L^\infty(\varrho)$ and let $f = K_\varrho q$. As such functions f clearly span $H(\bar{b})$, it will suffice to show that f can be approximated in the norm of $H(\bar{b})$ by multipliers of $H(b)$. From Lemma 10.1 one sees that the real part of f is bounded in modulus by $\|q\varrho\|_\infty$ in $\mathbf{C}\backslash\partial\mathbf{D}$. For ϵ a positive number smaller than $1/\|q\varrho\|_\infty$, the functions $f/(1 + \epsilon f)$ and $f_*/(1 + \epsilon f_*)$ are then in H^∞, and we have

$$\frac{f}{1 + \epsilon f} - \frac{\bar{f}_*}{1 + \epsilon \bar{f}_*} = \frac{f - \bar{f}_*}{(1 + \epsilon f)(1 + \epsilon \bar{f}_*)}.$$

The interior boundary function of $f - \bar{f}_*$ is by Lemma 10.1 equal to $q\varrho$, so the interior boundary function of the preceding function is $q_\epsilon\varrho$, where $q_\epsilon = q/(1 + \epsilon f)(1 + \epsilon f_*)$. By Lemma 10.2 we conclude that the function $m_\epsilon = f/(1 + \epsilon f)$ equals $K_\varrho(q_\epsilon)$ in $\mathbf{D}$ and that $(m_\epsilon)_* = f_*/(1 + \epsilon f_*)$. Thus m_ϵ is in $K^2(\varrho)$ and in $K^\infty(\varrho)$. It now follows immediately from Corollary 12.4 that m_ϵ is a multiplier of $H(b)$. Finally, since $q_\epsilon \to q$ in $L^2(\varrho)$ as $\epsilon \to 0$, we have $\|f - m_\epsilon\|_{\bar{b}} \to 0$ as $\epsilon \to 0$, completing the proof.

A comment on the preceding proof: Suppose for simplicity that $\|q\varrho\|_\infty = 1$, and let $g = f - f(0)/2$ (which makes $g_* = -g$). The functions

$$m_\epsilon^1 = \frac{1}{2}\left(\frac{g}{1 + \epsilon g} + \frac{g}{1 - \epsilon g}\right)$$

are then in H^∞ for $0 < \epsilon < 1$, and a simple estimate shows that $|Re m_\epsilon^1| \leq |q|\varrho/(1 - \epsilon)^2$ on ∂D. The functions m_ϵ^1 thus satisfy the hypothesis of Corollary 12.6, and the functions $m_\epsilon^1 - m_\epsilon^1(\infty)$ could have been used in the proof of Corollary 12.8 in place of the functions m_ϵ. One can also deduce Corollary 12.8 by combining Corollary 12.6 with the following nice lemma of A. M. Gleason and H. Whitney [9, Lemma 3.1] (slightly rephrased): If k is a nonnegative function in L^∞, then there is a sequence in H^∞ whose real parts lie between 0 and k on ∂D and converge almost everywhere to k.

13. The effect of the inner factor

We continue to asume that b is an extreme point of $B(H^\infty)$. As we observed earlier, the multiplication criterion in Corollary 12.4 is insensitive to the inner factor of b: if a function m passes that test then it is a multiplier not only of $H(b)$ but of $H(ub)$ for every inner function u. We shall show that the condition of Corollary 12.4 characterizes multipliers of the preceding kind.

Lemma 13.1. *Let m be a multiplier of $H(b)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$. Let u be an inner function. Then m is a multiplier of $H(ub)$ if and only if gq is in $L^2(\varrho)$ for every g in $H(u)$.*

This lemma follows immediately from Theorem 12.2 and the decomposition $H(ub) = H(u) + uH(b)$ (explained in Section 6).

Corollary 13.2. *If u is a finite Blaschke product, then every multiplier of $H(b)$ is a multiplier of $H(ub)$.*

Indeed, if u is a finite Blaschke product, then the functions in $H(u)$ are bounded (in fact, they are rational functions), so the condition in Lemma 13.1 is satisfied.

Theorem 13.3. *Let m be a function in $K^\infty(\varrho)$ and let q be the function in $L^2(\varrho)$ such that $m = m(\infty) + K_\varrho q$. Then m is a multiplier of $H(ub)$ for every inner function u if and only if $q\varrho^{1/2}$ is bounded.*

The «if» part is Corollary 12.4. The «only if» part is an immediate consequence of the preceding lemma and the following one.

Lemma 13.4. *Let σ be a nonnegative essentially unbounded measurable func-*

tion on ∂D. *Then there is a function* g *in* H^2 *that is noncyclic for* S^* *such that* $g\sigma$ *is not in* L^2.

The function g that we shall produce lies in $H(u)$ for an interpolating Blaschke product u. We shall let $|E|$ stand for the unnormalized Lebesgue measure of the measurable subset E of ∂D.

Since σ is unbounded there is a sequence $\{t_n\}_1^\infty$ of positive numbers such that $t_{n+1} > 2t_n$ for all n and such that each set $E_n = \{t_n \le \sigma \le 2t_n\}$ has positive measure. For each n let λ_n be a point of density of E_n. The points λ_n are distinct so, passing to a subsequence, we can assume they converge to a point distinct from all of them. That being the case, we can find disjoint arcs $I_1, I_2, \ldots$ such that I_n has center λ_n for each n. Shrinking these arcs successively, if need be, we can assume $|I_{n+1}| < |I_n|/2$ and $|I_n \cap E_n| > |I_n|/2$ for each n.

Let w_n be the point in D such that $w_n/|w_n| = \lambda_n$ and $1 - |w_n| = |I_n|/2$, and let $g_n = (1 - |w_n|^2)^{1/2} k_{w_n}$, the normalized kernel function for the point w_n. Since $1 - |w_{n+1}| < (1 - |w_n|)/2$, the sequence $\{w_n\}_1^\infty$ is an interpolating sequence. Therefore, by a theorem of H. S. Shapiro and A. L. Shields [22], the functions g_n form a Riesz basis for their span in H^2, that span being $H(u)$, where u is the Blaschke product with zero sequence $\{w_n\}_1^\infty$.

We need to estimate the size of g_n on I_n. For that, fix an n and let $r = |w_n|$. We have

$$g_n(\lambda_n e^{i\theta}) = \frac{(1 - r^2)^{1/2}}{1 - re^{i\theta}}$$

$$= \frac{(1 - r^2)^{1/2}(1 - r\cos\theta + ir\sin\theta)}{(1 - r)^2 + 4r\sin^2(\theta/2)}.$$

Thus $Re g_n > 0$ on ∂D, and for $\lambda_n e^{i\theta}$ in I_n, that is, for $|\theta| < 1 - r$,

$$Re g_n(\lambda_n e^{i\theta}) \ge \frac{(1 - r^2)^{1/2}(1 - r)}{(1 - r)^2 + 4r\sin^2((1 - r)/2)}$$

$$\ge \frac{(1 + r)^{1/2}(1 - r)^{3/2}}{2(1 - r)^2}$$

$$\ge \frac{1}{2|I_n|^{1/2}}.$$

Since $t_n \to \infty$ we can find a sequence $\{c_n\}_1^\infty$ of positive numbers such that $\Sigma c_n^2 < \infty$ but $\Sigma c_n^2 t_n^2 = 2$. Let $g = \Sigma c_n g_n$. By the theorem of Shapiro and Shields mentioned above, g is in H^2 and is not a cyclic vector of S^*. On I_n we have

$$|g| \geq Re g \geq Re g_n \geq \frac{c_n}{2|I_n|^{1/2}}.$$

Hence

$$\begin{aligned}\int_{\partial D} |g\sigma|^2 \, d\theta &\geq \sum \int_{I_n \subset E_n} |g\sigma|^2 \, d\theta \\ &\geq \sum \frac{c_n^2}{4|I_n|} t_n^2 |I_n \cap E_n| \\ &\geq \frac{1}{8} \sum c_n^2 t_n^2 = \infty,\end{aligned}$$

which proves the lemma.

Up to now we have not given an example of a multiplier that fails to satisfy the criterion in Corollary 12.4. That will come in the next section. In the other direction, one sees from Theorem 13.3 that if ϱ is bounded away from 0 on the set where it is nonzero, then $H(b)$ and $H(ub)$ have the same multipliers for all inner functions u.

The next result, which identifies a class of inner functions u for which $H(ub)$ and $H(b)$ have the same multipliers, does not require the assumption that b is an extreme point.

Theorem 13.5. *If u is an inner function such that dist$(b, uH^\infty) < 1$, then every multiplier of $H(b)$ is a multiplier of $H(ub)$.*

We first show that the distance inequality is equivalent to the equality $H(u) = (1 - T_u T_{\bar{u}})H(b)$, or, what amounts to the same thing, to the inclusion $H(u) \subset (1 - T_u T_{\bar{u}})H(b)$. By the criterion of Douglas we used earlier (in Section 2), the inclusion is equivalent to the operator inequality

$$1 - T_u T_{\bar{u}} \leq c\,(1 - T_u T_{\bar{u}})(1 - T_b T_{\bar{b}})(1 - T_u T_{\bar{u}})$$

for some $c \geq 1$. The operator inequality means that

$$\|h\|_2^2 \leq c\,(\|h\|_2^2 - \|T_{\bar{b}} h\|_2^2)$$

for all h in $H(u)$, in other words, that

$$\|T_{\bar{b}} h\|_2^2 \leq \frac{c-1}{c} \|h\|_2^2$$

for all h in $H(u)$, in other words, that $\|T_{\bar{b}}|H(u)\| < 1$. Since it is known [17] that $\|T_{\bar{b}}|H(u)\| = \text{dist}(b, uH^\infty)$, the equivalence is established.

Thus, assuming u satisfies the condition in the theorem, we have $H(u) = (1 - T_u T_{\bar{u}})H(b)$. Suppose m is a multiplier of $H(b)$. Then, because $H(ub) = H(u) + uH(b)$, to show m is a multiplier of $H(ub)$ we need only show $mH(u) \subset H(ub)$. Let g be any function in $H(u)$. Then, because $H(u) = (1 - T_u T_{\bar{u}}) H(b)$, there is a function h in $H(b)$ whose projection onto $H(u)$ is g. The difference $h - g$ is then in $H(ub)$ and in uH^2, so it is in $uH(b)$. Hence $m(h - g)$ is in $uH(b)$ and thus in $H(ub)$. Since also mh is obviously in $H(ub)$, it follows that mg is in $H(ub)$, and the theorem is established.

14. Helson-Szegö weights

For certain extreme points b of $B(H^\infty)$, those for which the conjugation operator behaves in a decent manner relative to μ_b, we are able to describe the multipliers of $H(b)$ completely. By a Helson-Szegö weight we shall mean a nonnegative function σ on ∂D that has the form $\sigma = \exp(\phi + \tilde{\psi})$, where ϕ and ψ are real functions in L^∞ with $\|\psi\|_\infty < \pi/2$, and $\tilde{\psi}$ denotes the conjugate function of ψ. The following properties hold.

1. If σ is a Helson-Szegö weight then so is $1/\sigma$.
2. A Helson-Szegö weight is in $L^{1+\epsilon}$ for sufficiently small positive numbers ϵ.
3. If σ is a Helson-Szegö weight then the conjugation operator is bounded on $L^2(\sigma)$. This property characterizes Helson-Szegö weights.

Property 1 is trivial and property 2 is a well-known result of V. I. Smirnov [7, p. 34]. Property 3 is the basic theorem of H. Helson and G. Szegö [13]. A thorough discussion of these and related matters can be found in the book [8].

If μ_b is absolutely continuous and its Radon-Nikodym derivative is a Helson-Szegö weight then, as Davis and McCarthy show [1] (on the basis of the Helson-Szegö theorem), every function in H^∞ is a multiplier of $H(b)$. (They prove the converse also.) Such a b of course is not an extreme point of $B(H^\infty)$. The next theorem says that an analogous result holds for extreme points whose corresponding measures are made in a simple way from Helson-Szegö weights.

Theorem 14.1. *If μ_b is absolutely continuous with Radon-Nikodym derivative $\chi_E \sigma$, where σ is a Helson-Szegö weight and E is a subset of ∂D of positive measure whose complement has positive measure, then the following spaces coincide:*

1. The space of multipliers of $H(b)$, *2.* $K^{\infty}(\varrho)$,

3. $K^{\infty}(\chi_E)$.

A lemma is needed.

Lemma 14.2. *Let σ be a Helson-Szegö weight and let q be a function in $L^2(\sigma)$. Then, in* D, *the Cauchy integral $K_\sigma q$ belongs to H^1, and its interior boundary function has the form $q_1\sigma$ with q_1 in $L^2(\sigma)$.*

To see that $K_\sigma q$ is in H^1, choose a positive number ϵ such that σ is in $L^{1+\epsilon}$. Then $q\sigma$ is the product of the L^2 function $q\sigma^{1/2}$ and the $L^{2+2\epsilon}$ function $\sigma^{1/2}$, so it is in $L^{(2+2\epsilon)/(2+\epsilon)}$, by Hölder's inequality. By M. Riesz's theorem, the conjugate function of $q\sigma$ lies in the same space. Hence (by Lemma 10.1), the Cauchy integral $K_\sigma q$ is in $H^{(2+2\epsilon)/(2+\epsilon)}$ and a fortiori in H^1.

To see that the interior boundary function of $K_\sigma q$ has the required form we note that, because $q\sigma$ is in $L^2(1/\sigma)$, the Helson-Szegö theorem implies that the interior boundary function of $K_\sigma q$, its Cauchy integral, is in $L^2(1/\sigma)$. Thus, if q_2 is that boundary function, then the function $q_1 = q_2/\sigma$ is in $L^2(\sigma)$, which is the desired conclusion.

As for the theorem, we already know that every multiplier of $H(b)$ is in $K^{\infty}(\varrho)$, and one easily sees that $K^{\infty}(\varrho)$ is contained in $K^{\infty}(\chi_E)$. It only remains to show that every function in $K^{\infty}(\chi_E)$ is a multiplier of $H(b)$, or, equivalently, of $K^2(\chi_E\sigma)$.

The argument is similar to several we have already given. Let f be a function in $K^2(\chi_E\sigma)$, say $f = K(q\chi_E\sigma)$, where q is in $L^2(\chi_E\sigma)$. Let m be a function in $K^{\infty}(\chi_E)$. By Lemma 14.2 the functions f and f_*, in D, belong to H^1. Hence mf and m_*f_* are in H^1, implying that the harmonic function $mf - \overline{m}_*\overline{f}_*$ is the Poisson integral of its boundary function. By Lemma 10.2, to prove mf is in $K^2(\chi_E\sigma)$ it will suffice to prove that the boundary function of $mf - \overline{m}_*\overline{f}_*$ is of the form $q_1\chi_E\sigma$ with q_1 in $L^2(\chi_E\sigma)$. For this we write, as usual,

$$mf - \overline{m}_*\overline{f}_* = (m - \overline{m}_*)f + \overline{m}_*(f - \overline{f}_*).$$

In the first summand on the right, the boundary function of the first factor, $m - \overline{m}_*$, is bounded and vanishes off E (Lemma 10.1), while the boundary function of the second factor is in $L^2(\sigma)$, by Lemma 14.2. The boundary function of the first summand is thus of the required form. In the second summand, the boundary function of the first factor, $\overline{m}_*$, is bounded, and the boundary function of the second factor, $f - \overline{f}_*$, is $q\chi_E\sigma$. The boundary function of the second summand thus also has the required form, and the proof is complete.

We are now able to given an example of an extreme point b, a multiplier m of $H(b)$, and an inner function u, such that m is not a multiplier of $H(ub)$. Fix δ in (0,1), and let the function σ on ∂D be defined by $\sigma(e^{i\theta}) = |\theta|^{\delta}$, $(-\pi \leq \theta \leq \pi)$. This is a Helson-Szegö weight by a result of G. H. Hardy and J. E. Littlewood [11]. One can prove that nowadays by verifying that σ satisfies B. Muckenhoupt's condition (A_2), which characterizes Helson-Szegö weights. (Details are in [8].) Let E be the right half of ∂D, and let b be the function such that $(1 + b)/(1 - b)$ is the Herglotz integral of $\chi_E\sigma$. Theorem 14.1 applies, telling us that $K^{\infty}(\chi_E)$ is the space of multipliers of $H(b)$.

Let q_0 be a C^1 function on D that vanishes off E and is nonzero at the point 1. Since q_0 is of class C^1 its conjugate function is continuous, and this implies by Lemma 10.1. that the Cauchy integral $m = Kq_0$ is bounded in C$\backslash\partial$D and hence belongs to $K^{\infty}(\chi_E)$. Thus m is a multiplier of $H(b)$.

We also have $m = K_{\varrho}q$ where $q = q_0/\varrho$. The function $q\varrho^{1/2}$ $(= q_0\,\varrho^{-1/2})$ is unbounded because $q_0\sigma^{-1/2}$ is and $\varrho = |1 - b|^2\chi_E\sigma \leq 4\chi_E\,\sigma$. Hence Theorem 13.3 guarantees the existence of an inner function u such that m is not a multiplier of $H(ub)$. The proof of Lemma 13.4 provides an explicit example of such a u, a certain interpolating Blaschke product. By using estimates similar to those in the proof of Lemma 13.4 it is not hard to show that the Blaschke product with zero sequence $\{1 - 2^{-n}\}_1^{\infty}$ also has the required property.

15. Questions

Many questions puzzle us.

1. If b is an extreme point of $B(H^{\infty})$, must every function in $K^{\infty}(\varrho)$ be a multiplier of $H(\bar{b})$? An answer most likely will involve subtleties of the conjugation operator (although we may be overlooking something simple).
2. If b is an outer function, must $H(b)$ and $H(\bar{b})$ have the same multipliers? Results in [18] show that the answer can be negative when b is not an extreme point. What about the extreme point case?
3. To understand better the multipliers of $H(b)$, one needs examples, in addition to those given by Theorem 14.1, where they can be described completely. As a very special query: Suppose in the example in Section 14 one lets $\delta = 1$, thus passing beyond the realm of Helson-Szegö weights. What are the multipliers of $H(b)$ for the corresponding b?
4. Davis and McCarthy [1] prove that if μ is a finite positive Borel measure on ∂D and μ_a is its absolutely continuous component, then every multiplier of $K^2(\mu)$ is a multiplier of $K^2(\mu_a)$. In case the singular component

of μ is a finite sum of point masses and the Radon-Nikodym derivative of μ_a is log-integrable, they are able to specify precisely which multipliers of $K^2(\mu_a)$ are also multipliers of $K^2(\mu)$. Can one describe in more general cases, or perhaps even in general, how the singular component of μ influences the space of multipliers of $K^2(\mu)$? Progress on this will undoubtedly lead to a better understanding of the structure of the corresponding space $H(b)$.

5. In case b is an extreme point, the algebra $K^\infty(\varrho)$ appears to be an interesting object of study. It becomes a Banach algebra when equipped with the norm $\|f\| = \|f\|_\infty + \|q\|_{L^2(\varrho)}$, where q is the function in $L^2(\varrho)$ such that $f = f(\infty) + K_\varrho q$, and $\|f\|_\infty$ stands for the supremum of $|f|$ over $\mathrm{C}\backslash\partial\mathrm{D}$. As shown in Section 10, the spectrum of a function f in $K^\infty(\varrho)$ is the closure of $f(\mathrm{C}\backslash\partial\mathrm{D})$. What can one say about the maximal ideal space of $K^\infty(\varrho)$? Is $\mathrm{C}\backslash\partial\mathrm{D}$ dense in it?

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Covering Lemmas and BMO Estimates for Eigenfunctions on Riemannian Surfaces

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Abstract

The principal aim of this note is to prove a covering Lemma in $\mathbf{R}^2$. As an application of this covering lemma, we can prove the BMO estimates for eigenfunctions on two-dimensional Riemannian manifolds (M^2, g). We will get the upper bound estimate for $\|\log |u|\|_{BMO}$, where u is the solution to $\Delta u + \lambda u = 0$, for $\lambda > 1$ and Δ is the Laplacian on (M^2, g). A covering lemma in homogeneous spaces is also obtained in this note.

1. Introduction

Let M be a smooth, compact and connected Riemannian manifold without boundary. Let Δ denote the Laplacian on M. Assume that u is the solution to $\Delta u + \lambda u = 0$, $\lambda > 1$, *i.e.*, u is an eigenfunction with eigenvalue λ.

Many authors have studied the estimates of the BMO norm and the Nodal sets of eigenfunctions, see [DF1], [DF2], [C], [B], and [CM]. In [C], Cheng proved that u vanishes at most to order $c\lambda$ in the two-dimensional case. In [B], Brüning showed the lower bound for the volume of the nodal set for C^∞ metrics on Riemannian surfaces. Donnelly and Fefferman, see [DF1] and [DF2], obtained the growth property, estimates of the BMO norm and bounds for the volume of the Nodal set of eigenfunctions for all $n \geq 2$. Recently, Chanillo and Muckenhoupt, see [CM], improved the results of [DF2] for $n \geq 3$.

The purpose of this note is to get a better BMO estimate of eigenfunctions in the two-dimensional case. The main result is the following

Theorem 1. ***(BMO estimate for*** log $|u|$ *) For u, λ as above, and for* $\epsilon > 0$

$$\|\log |u|\|_{BMO} \leq c\lambda^{15/8+\epsilon}$$

where $c = c(\alpha, M)$ *is independent of* λ.

The proof of Theorem 1 is based on the following covering lemma which is of independent interest.

Lemma 1. ***(Covering Lemma)*** *Let* $\delta > 0$ *be small enough, then given any finite collection of balls* $\{B_\alpha\}_{\alpha \in I}$, *one can select a subcollection* $B_1, \ldots, B_N$ *such that*

(i) $$\bigcup_\alpha B_\alpha \subset \sum_{i=1}^{N} (1 + \delta) B_i$$

and

(ii) $$\sum_{i=1}^{N} \chi_{B_i}(x) \leq c\delta^{-7/4} \log \frac{1}{\delta}$$

where c is an constant independent of δ *and the given collection of balls.*

The motivation of this note is from [CM]. In [CM], a covering lemma plays an important role. Lemma 1 is an improvement of the covering lemma for the two-dimensional case in [CM] so that we can get a better estimate of the BMO norm for eigenfunctions by using Lemma 1 and adapting the proof given in [CM] for $n \geq 3$. We would like to point out that it is quite possible to prove a covering lemma in case $n \geq 3$ which is better than that in [CM] by modifying the proof of Lemma 1.

This note is organized as follows: Section 2 explains why we should use a new selection of balls in order to get a better covering lemma than that in[CM]; Section 3 is devoted to the proof of a covering lemma in homogeneous spaces which is of independent interest; Section 4 and 5 deal with the proof of Lemma 1; Section 6 is devoted to the proof of Theorem 1.

One world about notations: Throughout this note, C and c will always denote generic positive constants independent of the given balls $\{B_\alpha\}_{\alpha \in I}$ and $\delta > 0$; $\varrho(B)$ will denote the radius of the ball B; $B(x, r)$ will denote the ball centered at x and of radius r.

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2. A Covering Lemma due to Chanillo-Muckenhoupt

In this section, we first recall a covering lemma in [CM].

Lemma 2.1 *(Chanillo-Muckenhoupt) Fix* $0 < \delta < 1/2$. *Given any finite collection of balls* $\{B_\alpha\}_{\alpha \in I}$ *in* $\mathbf{R}^n$, *one can select a subcollection* $B_1, \ldots, B_N$ *such that*

(*i*)
$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (1 + \delta) B_i$$

(*ii*)
$$\sum_{i=1}^{N} \chi_{B_i}(x) \leq 4^n \delta^{-n},$$

for all $x \in \mathbf{R}^n$.

In the proof of Lemma 2.1, one first selects a ball B_1 with the largest radius in $\{B_\alpha\}_{\alpha \in I}$. Having selected $B_1, \ldots, B_{k-1}$, one selects B_k so that

$$B_k \not\subset \bigcup_{i=1}^{k-1} (1 + \delta) B_i \tag{2.2}$$

and B_k has the largest possible radius in the collection $\{B_\alpha\}_{\alpha \in I} \setminus \{B_i\}_{i=1}^{k-1}$. Here we want to point out that (ii) of Lemma 2.1 is the best possible result which can be obtained by the above selection of balls. We show this by giving the following:

EXAMPLE: Consider a family of unit balls centered inside a cube

$$Q = \{(x, y): 0 \leq x \leq 1/2,\ 0 \leq y \leq 1/2\} \subset \mathbf{R}^2$$

and the centers of these balls have coordinates

$$\{(2k\delta, 4l\delta)\},\ 0 \leq k \leq [1/4\delta^{-1}],\ 0 \leq l \leq [1/8\delta^{-1}],$$

where $[\ \cdot\ ]$ denotes the largest integer part. We also denote the unit ball with center $(2k\delta, 4l\delta)$ by $B_{k,l}$. We now select the balls in the following order:

$B_{0,0}, B_{1,0}, \ldots, B_{[1/4\delta^{-1}],0}, B_{0,1}, B_{1,1}, \ldots, B_{[1/4\delta^{-1}],1}, \ldots, B_{0,[1/8\delta^{-1}]}, B_{1,[1/8\delta^{-1}]}, \ldots,$
$B_{[1/4\delta^{-1}],[1/8\delta^{-1}]},$

Then an easy calculation shows (2.2) holds. This selection exactly follows the method used in the proof of Lemma 2.1. But

$$\sum_{i=1}^{N} \chi_{B_i}(x, y) \approx \delta^{-2}, \quad \text{for} \quad 0 \leq x \leq 1/2, \ 0 \leq y \leq 1/2.$$

Therefore the above example tells us that we need to use a new selection of balls in order to get a better result than that in Lemma 2.1.

3. A Covering Lemma in Homogeneous Spaces

In this section, we are going to prove a covering lemma in homogeneous metric spaces with a doubling Borel measure. This section is independent of the others. We will apply the technique of partitioning the radii of balls to reduce the proof of the main lemma to a certain basic case. The proof as given below is an adaptation to homogeneous spaces of the proof on $\mathbf{R}^n$ due to S. Chanillo.

We say a pair (X, ϱ) is a homogeneous metric space in the sense of Coifman and Weiss, if the following hold:

(*i*) $\varrho : X \times X \to \mathbf{R}^+$ satisfies the following conditions:

$$\varrho(x, y) = 0 \quad \text{if and only if} \quad x = y$$

$$\varrho(x, y) = \varrho(y, x)$$

$$\varrho(x, y) \leq K[\varrho(x, z) + \varrho(z, y)]$$

where K is a constant independent of x, y, z and

(*ii*) there is a Borel measure μ such that

(3.1) $$0 < \mu(B(x, r)) \leq A\mu(B(x, r/2)) < +\infty,$$

where A is a constant independent of the ball $B(x, r)$ centered at x and with radius r.

An easy consequence of (3.1) is

(3.2) $$\mu(B(x, r)) \leq A^{\log_2(r/r')+1}\mu(B(x, r'))$$

for any $x \in X$ and $0 < r' < r$.

According to [CoW], both Vitali type and Whitney type covering lemmas are true. It is well-known that Besicovitch covering lemma may not be true in homogeneous spaces as pointed out in [SW]. Sometimes a Vitali covering lemma is not good enough for applications as in [CM], but the following covering lemma could be a replacement of both Vitali and Besicovitch covering lemmas.

Lemma 3.3. *Let $\delta > 0$ be small enough and $\{B_\alpha\}_{\alpha \in I}$ be a finite collection of balls in X. If there exists a doubling Borel measure μ on X satisfying* (3.1), *then one can select a subcollection $B_1, \ldots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (K + \delta) B_i \tag{3.4}$$

$$\sum_{i=1}^{N} \chi_{B_i}(x) \le C\delta^{-d} \log \frac{1}{\delta} \tag{3.5}$$

where C depends only on K and A, and $d = \log_2 A$.

Proof. Select a ball B with the largest radius in $\{B_\alpha\}$. Having selected balls $B_1, \ldots, B_{k-1}$, select B_k such that

$$B_k \not\subset \bigcup_{i=1}^{k-1} (K + \delta) B_i \tag{3.6}$$

and B_k has the largest radius out of the collection

$$\{B_\alpha\}_{\alpha \in I} \setminus \{B_i\}_{i=1}^{k-1}$$

The subcollection $B_1, \ldots, B_M$ chosen by the above selection obviously satisfies

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{M} (K + \delta) B_i$$

Now we prove (3.5) in the lemma. We first fix any point $x_0 \in X$. With no loss of generality, we may assume $x_0 \in \bigcap_{i=1}^{l} B_i$. We also assume $B_i = B_i(z_i, r_i)$. By the selection of $\{B_\alpha\}_{i=1}^{M}$, we know $r_l \le r_{l-1} \le \ldots \le r_1$. We note that there exists $\sigma = \sigma(x_0)$ such that

$$(3.7) \qquad \sum_{i=1}^{l} \chi_{B_i}(x_0) = \sum_{k=1}^{\sigma} \sum_{2^k r_l \le r_i < 2^{k+1} r_l} \chi_{B_i}(x_0)$$

We now have the following claims.
Claim (1): $\delta 2^\sigma \le 2K^2$
If not, then $\delta r_1 \ge \delta 2^\sigma r_l > 2K^2 r_l$.
We note that for $y \in B_l = B(z_l, r_l)$,

$$\begin{aligned} \varrho(y, z_1) &\le K[\varrho(y, x_0) + \varrho(x_0, z_1)] \\ &\le K\{K[\varrho(y, z_l) + \varrho(z_l, x_0)] + \varrho(x_0, z_1)\} \\ &\le K[2Kr_l + r_1] = 2K^2 r_l + Kr_1 \\ &< \delta r_1 + Kr_1 \\ &= (K + \delta) r_1 \end{aligned}$$

Thus $B_l \subset (K + \delta) B_1$ which is a contradiction to (3.6). Thus claim (1) holds, *i.e.,* $\sigma \le C \log \dfrac{1}{\delta}$, where C only depends on K.

Claim (2): For the subcollection $\{B_{i_j}\}_{j=1}^{N_k}$ of $\{B_i\}_{i=1}^{l}$ with $2^k r_l \le \varrho(B_{i_j}) < 2^{k+1} r_l$, we have

$$\varrho(z_{i_j}, z_{i_h}) > \frac{2^k r_l \delta}{K} \quad \text{for} \quad j \ne h,\ 1 \le j,\ h \le N_k$$

For simplicity, we drop the subscripts and denote B_{i_j} and B_{i_h} by B_i and B_h respectively. We also assume $j > h$. If the claim were not true, we would have for $y \in B_j = B(z_j, r_j)$

$$\begin{aligned} \varrho(y, z_h) &\le K[\varrho(y, z_j) + \varrho(z_j, z_h)] \\ &\le K\left[r_j + \frac{2^k \delta}{K}\right] \\ &\le Kr_j + 2^k r_l \delta \le Kr_j + \delta r_j \\ &= (K + \delta) r_j \\ &\le (K + \delta) r_h. \end{aligned}$$

This implies $(K + \delta) B_h \supset B_j$ which is again a contradiction to (3.6).

Claim (3): The balls $\left\{B\left(z_j, \dfrac{2^k r_l \delta}{2K^2}\right)\right\}_{j=1}^{N_k}$ are mutually disjoint.

We set $R_1 = 2^k r_l \delta / 2K^2$. If the claim were not true, there would exist $y \in B(z_j, R_1) \bigcap (z_h, R_1)$ for some $j \neq h$. Thus

$$\varrho(z_j, z_h)] \leq K[\varrho(z_j, y) + \varrho(y, z_h) \leq \left[\frac{2^k r_l \delta}{2K^2} + \frac{2^k r_l \delta}{2K^2}\right] = \frac{2^k r_l \delta}{K},$$

which is a contradiction to the claim (2).

Claim (4): The balls $\{B_j\}_{j=1}^{N_k}$ are all contained in $B(x_0, R_2)$, where $R_2 = K 2^{k+2} r_l$.

For $y \in B(z_j, r_j)$, we have

$$\begin{aligned} \varrho(y, x_0) &\leq K[\varrho(y, z_j) + \varrho(z_j, x_0)] \\ &\leq 2Kr_j \leq 2K \cdot 2^{k+1} \mathrm{r}_l \\ &= K 2^{k+2} r_l. \end{aligned}$$

Thus the claim holds.

Claim (5): $B(x_0, R_2) \subset B(z_j, R_3)$ for each $1 \leq j \leq N_k$, where $R_3 = K(2K + 1) 2^{k+1} r_l$.

In fact, for $y \in B(x_0, R_2)$, we have

$$\begin{aligned} \varrho(y, z_j) &\leq K[\varrho(y, x_0) + \varrho(x_0, z_j)] \\ &\leq K[R_2 + r_j] \leq K[2^{k+2} r_l k + 2^{k+1} r_l] \\ &= K(2K + 1) 2^{k+1} r_l = R_3 . \end{aligned}$$

This proves claim (5).

Now by claims (3) and (4), we have

$$\sum_{j=1}^{N_k} \mu(B(z_j, R_1)) \leq \mu(B(x_0, R_2)), \tag{3.8}$$

and by claim (5) and (3.2), we have

$$\mu(B(x_0, R_2)) \leq \mu(B(z_j, R_3)) \leq A^{\log_2(R_3/R_1)+1} \mu(B(z_j, R_1)). \tag{3.9}$$

Therefore, from (3.8) and (3.9), it is easy to see

$$N_k \leq A^{\log_2(R_3/R_2)+1} = A^{\log_2[4K^3(2K+1)/\delta]}.$$

By using $A = 2^d$ and an easy calculation, we get

$$N_k \leq [8K^3(2K + 1)]^d \delta^{-d}$$

Now,

$$\sum_{i=1}^{l} \chi_{B_i}(x_0) = \sum_{k=0}^{\sigma} \sum_{2^k r_l \leq r_j \leq 2^{k+1} r_l} \chi_{B_j}(x_0)$$

$$\leq \sum_{k=0}^{\sigma} N_k \leq \sum_{k=1}^{\sigma} [8K^3(2K + 1)]^d \delta^{-d}$$

$$\leq C \left(\log \frac{1}{\delta}\right) \delta^{-d}$$

where C only depends on K and A. This shows (3.5) and thus completes the proof of lemma (3.3).

4. A Basic Covering Lemma

The purpose of this section is to prove a covering lemma for balls which centered in any given cube in R^2 with sidelenght $\sqrt{\delta}$. Moreover, these balls have radii whose values are close to one another.

Lemma 4.1. *Let $\delta > 0$ be given small enough. Given any cube Q in $\mathbf{R}^2$ with sidelength $\sqrt{\delta}$ and given any finite collection of balls $\{B_\alpha\}_{\alpha \in I}$ with $r \leq \varrho(B_\alpha) \leq \leq r + \delta$, for some $1 \leq r \leq 2$ and centered in this cube Q, one can select a subcollection of balls $B_1, \ldots, B_N$ such that*

(i) $$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + c\delta) B_i,$$

(ii) $$N \leq c\delta^{-1/4},$$

where c is an absolute constant independent of δ and the given balls.

In order to prove Lemma 4.1, we need the following propositions.

Proposition 4.1. *The sum of all exterior angles of any convex polygon is 2π.*

This a well-known formula in plane geometry. One may also deduce this fact from the Gauss-Bonnet formula in differential geometry.

Proposition 4.2. *The perimeter of any convex plane polygon contained inside a cube Q with sidelength $\sqrt{\delta}$ is less than $2\pi\sqrt{2\delta}$.*

The proof of the above Proposition 4.2 uses the Cauchy-Crofton formula in $\mathbf{R}^2$. It states that for a given regular plance curve C with length l, the measure of the set of straight lines (counted with multiplicities) which meet C is equal to $2l$. A proof of this assertion may be found on page 41 in [Do]. A higher dimentional version of Cauchy-Crofton formula is proved in [F].

PROOF OF PROPOSITION 4.2. Let ∂P be the boundary of the convex polygon P inside Q, and let S be the set of straight lines which meet P. Then if we denote by ϱ the distance from the origin to the lines and by θ the angle between the positive x-axis and the line, and assume without loss of generality,

$$Q = \{(x, y): 0 \leq x \leq \sqrt{\delta},\ 0 \leq y \leq \sqrt{\delta}\},$$

we have

$$2 \text{ length } (\partial P) = \int_S d\pi d\theta \leq \int_0^{2\pi}\int_0^{\sqrt{2\delta}} 2d\varrho d\theta = 4\pi\sqrt{2\delta}.$$

Thus, length $(\partial P) \leq 2\pi\sqrt{2\delta}$.

Remark: After this note was prepared, Prof. B. Muckenhoupt pointed out that the proof of Proposition 4.2 can be simplified by projecting ∂P to ∂Q.

Proposition 4.3. *Given any $\delta > 0$ small enough and oriented rectangle R in $\mathbf{R}^2$ with sidelength $\sqrt{\delta}$ and δ. Let $\{B_\alpha\}_{\alpha\in I}$ be a finite collection of balls centered in R and with radii $r \leq \varrho(B_\alpha) \leq r + \delta$ for some $1 \leq r \leq 2$. Then we only need to select at most two balls B_1 and B_2 such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{2} (1 + c\delta) B_i$$

PROOF. If there are no more than two balls in the given collection, then there is nothing to prove. If there are more than two centers in R, then we select two balls B_1 and B_2 with centers $O_1 = (x_{o_1}, y_{o_1})$ and $O_2 = (x_{o_2}, y_{o_2})$ respectively such that one of the centers is on the extreme left, the other is on the extreme right (as shown in Figure 1). We claim

$$\bigcup_\alpha B_\alpha \leq \bigcup_{i=1}^{2} (1 + c\delta) B_i$$

We will assume with no loss of generality that both B_1 and B_2 have the smallest possible radius in our collection, that is r. We may also assume with no loss of generality that $y_{o_1} = y_{o_2}$, then the ball B_1' with radius r and center (x_{o_1}, y_{o_2}) is contained in $(1 + 2\delta)B_1$ and thus if we prove our claim with B_1 replaced by B_1', our proposition will be proved.

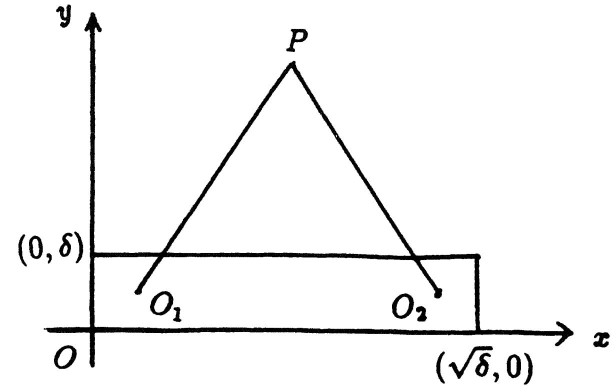

Figure 1

Let P denote the intersection point of $(1 + c\delta)B_1$ and $(1 + c\delta)B_2$. We now show that dist $(P, \partial R) > r + 2\delta$. This will prove our claim. Using the fact that $y_{o_1} = y_{o_2}$, we see that

$$\begin{aligned}\operatorname{dist}(P, \partial R) &\geq \operatorname{dist}(P, \overline{OO_2}) - \delta \\ &\geq [(1 + c\delta)^2 r^2 - (\sqrt{\delta}/2)^2]^{1/2} - \delta.\end{aligned}$$

By choosing $c \geq 4$, we get

$$[(1 + c\delta)^2 r^2 - \delta/4]^{1/2} - \delta > r + 2\delta$$

since $r \geq 1$. This proves our claim and the proposition.

We are also going to need the following:

Proposition 4.4. *Let $O_1O_2O_3$ be a triangle with sidelength less than $c\sqrt{\delta}$. Suppose $\{B_i\}_{i=1}^3$ are three balls centered at $\{O_i\}_{i=1}^3$ and with $r \leq \varrho(B_i) \leq r + \delta$ for some $1 \leq r \leq 2$. Then any ball B with $r \leq \varrho(B) \leq r + \delta$ and centered at some O inside the triangle $O_1O_2O_3$ can be covered by $\bigcup_{i=1}^3 (1 + c\delta)B_i$ for some absolute constant c.*

PROOF.

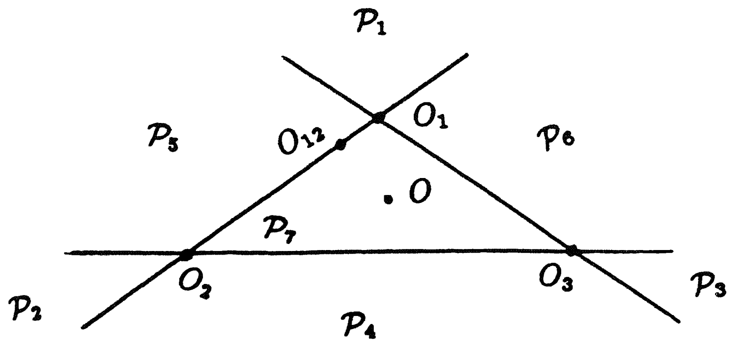

Figure 2

As shown in Fig. 2, we extend segments O_1O_2, O_2O_3, O_3O_1 to lines. Then these three lines subdivide $\mathbf{R}^2$ into seven pieces $\{P_i\}_{i=1}^7$. Obviously, $B \cap P_1$, $B \cap P_2$ and $B \cap P_3$ can be covered by $(1 + c\delta)B_1$, $(1 + c\delta)B_2$ and $(1 + c\delta)B_3$ respectively. Now let O_{12} denote the point O_1O_2 nearest to O and let B' be the ball centered at O_{12} of radius $r + \delta$. Then $B' \cap P_5 \supset B \cap P_5$ by the above choice of O_{12}. But by Proposition 4.3, $(1 + c\delta)B_1 \cup (1 + c\delta)B_2 \supset B'$. Thus

$$B \cap P_5 \subset \bigcup_{i=1}^{3} (1 + c\delta)B_i.$$

A similar argument shows that

$$B \cap P_j \subset \bigcap_{i=1}^{3} (1 + c\delta)B_i \quad \text{for} \quad j = 4, 6.$$

This completes the proof of the proposition.

Proposition 4.5. *Let $\{B_\alpha\}_{\alpha\in I}$ be a finite collection of balls with $r \leq \varrho(B_\alpha) \leq r + \delta$ for some $1 \leq r \leq 2$. If there exists a subcollection $\{B_i\}_{i=1}^m$ of $\{B_\alpha\}$ and another further subcollection $\{B_{i_k}\}_{k=1}^n$ of $\{B_i\}$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{m} (1 + c\delta)B_i$$

$$B_i \subset \bigcup_{k=1}^{n} (1 + c\delta) B_{i_k}$$

for each i. Then

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{k=1}^{n} (1 + c'\delta) B_{i_k}$$

for some $c' > c$ independent of δ and the given balls.

PROOF. It suffices to prove

$$(1 + c\delta) B_i \subset \bigcup_{k=1}^{n} (1 + c'\delta) B_{i_k}$$

for each $1 \leq i \leq m$. Now fix i, let $B_i' = (1 + c\delta) B_i$, O_i be the center of B_i, ∂B_i be the boundary of B_i. Then it is enough to show

$$B_i' \setminus B_i \subset \bigcup_{i=1}^{n} (1 + c'\delta) B_{i_k}.$$

Let $P \subset \partial B_i$, we denote by Q the intersection point between B_i' and the half line starting with the point O_i and passing through P. Then the length of the segment PQ is

$$(1 + c\delta)\, \varrho(B_i) - \varrho(B_i) = c\delta \varrho(B_i) \leq 3c\delta.$$

Assume $P \in (1 + c\delta) B_{i_k}$. In fact, $\operatorname{dist}(P, O_{i_k}) \leq (1 + c\delta)\, \varrho(B_{i_k})$, and then for any $z \in PQ$, we have

$$\begin{aligned} \operatorname{dist}(z, O_{i_k}) &\leq \operatorname{dist}(z, P) + \operatorname{dist}(P, O_{i_k}) \\ &\leq 3c\delta + (1 + c\delta)\, \varrho(B_{i_k}) \\ &\leq (1 + c'\delta)\, \varrho(B_{i_k}). \end{aligned}$$

The last inequality follows from $\varrho(B_{i_k}) \geq 1$. Thus the claim follows. We move P along ∂B_i and note that the union of all such segments PQ cover $B_i' \setminus B_i$, this shows that

$$B_i' \setminus B_i \subset \bigcup_{k=1}^{n} (1 + c'\delta) B_{i_k}.$$

For the remainder of this section, all balls that we will consider have radii ϱ such that $r_0 \leq \varrho \leq r_0 + \delta$ for a fixed r_0, $1 \leq r_0 \leq 2$.

We now begin the proof of Lemma 4.1.

Let S be the collection of centers of the balls $\{B_\alpha\}_{\alpha \in I}$. Let V be the convex hull of S. Then the boundary of V must be a convex plane polygon and the vertices of V consist of centers, say, $O_1, O_2, \ldots, O_M$, enumerated in a clockwise order.

Let us consider any three vertices of V, say, O_i, O_j, O_k, $(1 \leq i < j < k \leq M)$. We introduce the following:

Definition. *If $O_iO_k \leq \delta^{3/4}$, we say the triple $O_iO_jO_k$ is of type* I. *Otherwise, the triple $O_iO_jO_k$ is of type* II. *We further split the type II triples into two cases: If $O_jA_{ijk} \geq \delta$, we say the triple $O_iO_jO_k$ is of type* II_1, *otherwise, of type* II_2 *(see Figure 3 below).*

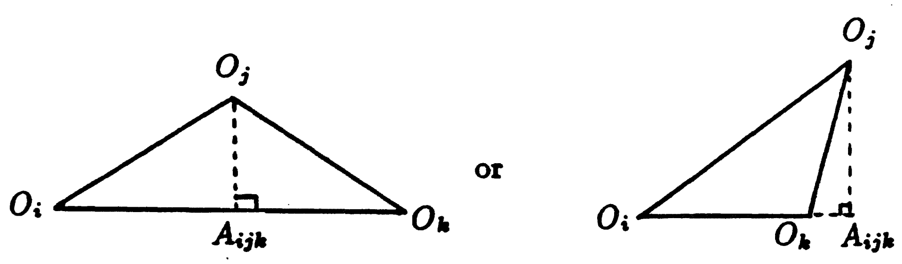

Figure 3

Now we select the vertices $\{O_{i_k}\}$. The balls $\{B_{i_k}\}$ with centers at $\{O_{i_k}\}$ will be the subcollection that shall be used to prove our lemma.

We first make the following observation:

Proposition 4.6. *If for some $k \geq 3$, every triple $O_1O_iO_k$ is of type II_2 for $2 \leq i \leq k - 1$, then all balls centered inside the polygon with vertices $\{O_i\}_{i=1}^k$ can be covered by $(1 + c\delta)B_{i_1} \cup (1 + c\delta)B_{i_2}$ for some $1 \leq i_1 \leq i_2 \leq k$.*

PROOF. The polygon W whose vertices are $\{O_i\}_{i=1}^k$ is contained inside a rectangle R with sidelengths $\sqrt{2\delta}$ and δ and O_1O_k is a part of one side with length $\sqrt{2\delta}$ of the rectangle R. Then by applying Proposition 4.3, we see that $\bigcup_{i=1}^k B_i \subset \bigcup_{m=1}^2 (1 + c\delta)B_{i_m}$. Now any ball B centered in W has its center in a triangle formed by some three vertices of W. By Proposition 4.4, $B \subset \bigcup_{i=1}^k (1 + c\delta)B_i$, thus by Proposition 4.6, we have $B \subset \bigcup_{m=1}^2 (1 + c'\delta)B_{i_m}$.

We now give the selection procedure. We start with the triple $O_1O_2O_3$.

Case 1: If the triple $O_1O_2O_3$ is of type I or II_1, then we select vertices O_1, O_2 and O_3 and pass to the next triple, $O_4O_5O_6$.

Case 2: If the triple $O_1O_2O_3$ is of type II_2, then either

(*i*) Every triple $O_1O_iO_M$ is of type II_2 for $2 \leq i \leq M - 1$.

Then by Proposition 4.6, We can select B_{i_1} and B_{i_2} as our subcollection and the proof of Lemma 4.1 is complete in this case. Or

(*ii*) There exists some k, $3 \leq k \leq M - 1$ such that every triple $O_1O_iO_k$ is of type II_2 for $2 \leq i \leq k - 1$, but there is some triple $O_1O_{i_0}O_{k+1}$ $(2 \leq i_0 \leq k)$ which is not of type II_2, *i.e.,* of type I or II_1.

Then by Proposition 4.6 again, there are two vertices O_{i_1} and O_{i_2} for $1 \leq i_1, i_2 \leq k$ such that all balls centered inside the polygon with vertices $\{O_i\}_{i=1}^k$ can be covered by $(1 + c\delta)B_{i_1} \cup (1 + c\delta)B_{i_2}$. Thus, we select O_1, O_{i_1}, O_{i_2}, O_k and O_{k+1} in this case (We note that some overlap may occur among the above five vertices since O_{i_1}, O_{i_2} may be O_1, O_k). We then pass to the next triple $O_{k+1}O_{k+2}O_{k+3}$. We continue this selection as before with O_{k+1} playing the role of O_1. Because there are only finite vertices, this process stops when O_1 appears again in a new triple. We thus arrive a subcollection of vertices $O_{i_1}, O_{i_2}, \ldots, O_{i_L}$ and a new polygon Z whose vertices are formed by O_{i_1}, $O_{i_2}, \ldots, O_{i_L}$.

We now claim that

(i) Z is a convex polygon.

(ii) Any ball B_α in the original collection $\{B_\alpha\}_{\alpha \in \mathrm{I}}$ is contained in $\bigcup_{k=1}^L (1 + c\delta)B_{i_k}$.

By noticing that Z is the intersection of the convex polygon V and the half spaces formed by the lines $O_{i_k}O_{i_{k+1}}$, $1 \leq k \leq L - 1$, and the intersection of convex regions is convex, this show that Z is convex.

To prove (ii), we note that each center O_α of the balls B_α is in V since V is the convex hull of S. Furthermore, recalling that $\{O_i\}$ denotes the vertices of V, we see that the center O_α must be in one of the triangles $O_1O_mO_{m+1}$ $(2 \leq m \leq M - 1)$. Thus by Propositions 4.4, B_α is contained in $\bigcup_{i=1}^M (1 + c\delta)B_i$. But as observed above by the selection procedure, $B_i \subset \bigcup_{k=1}^L B_{i_k}$. Thus $B_\alpha \subset \bigcup_{k=1}^L (1 + c\delta)B_{i_k}$, for all $\alpha \in I$, by Proposition 4.5.

In order to complete the proof of Lemma 4.1, we need the following proposition.

Proposition 4.7. *Suppose* $\{(O_{i_m}, O_{i_m+1}, O_{i_m+2})\}_{m=1}^N$ *be the family of type* I *triples selected from* $O_1, O_2, \ldots, O_M$ *above, then* $N \leq c\delta^{-1/4}$.

PROOF. Since $\overline{O_{i_m}O_{i_m+2}} \geq \delta^{3/4}$ by definition of type I triple. It follows by considering the perimeter of the triangle $O_{i_m}O_{i_m+1}O_{i_m+2}$, that

$$\overline{O_{i_m}O_{i_m+1}} + \overline{O_{i_m+1}O_{i_m+2}} \geq \overline{O_{i_m}O_{i_m+2}} \geq \delta^{3/4}.$$

Summing over m, we get

$$N\delta^{3/4} \leq \sum_{m=1}^{N} (\overline{O_{i_m}O_{i_m+1}} + \overline{O_{i_m}O_{i_m+2}})$$

$$\leq \text{perimeter of the polygon } Z.$$

Since Z is convex and contained in the cube Q, the perimeter of Z is no more than $c\sqrt{\delta}$ by Proposition 4.2. Thus $N\delta^{3/4} \leq c\sqrt{\delta}$ and consequently $N \leq c\delta^{-1/4}$.

Proposition 4.8. *Suppose* $\{(O_{i_m}, O_{i_m+1}, O_{i_m+2})\}_{m=1}^{N}$ *be the family of type* II_1 *triples selected from* $O_1, O_2, \ldots, O_M$ *above, then* $N \leq c\delta^{-1/4}$.

PROFF. Given a type II_1 triple $(O_{i_m}, O_{i_m+1}, O_{i_m+2})$, one has $\overline{O_{i_m}O_{i_m+2}} \leq \delta^{3/4}$ and $\overline{O_{i_m+1}A_m} \geq \delta$ by the definition of type II_1 triple (see Figure 4).

With no loss of generality, we consider those triples such that $\theta_m \leq \pi/4$ since the number of m such that $\theta_m > \pi/4$ is no more than $\dfrac{2\pi}{\pi/4} = 8$ by Proposition 4.1. We will show that the exterior angle θ_m is always bounded below by $\delta^{1/4}/2$ for those $\theta_m \leq \pi/4$.

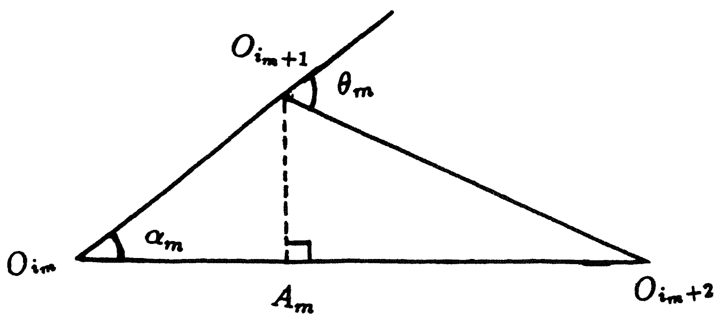

Figure 4

In fact, we note that for $\theta_m \leq \pi/4$, we have $\tan\theta_m \leq 2\theta_m$, thus for $\theta_m \leq \pi/4$, we get

$$\begin{aligned}\theta_m \geq \alpha_m \geq \frac{1}{2}\tan\alpha_m &= \frac{1}{2}\frac{\overline{O_{i_m+1}A_m}}{\overline{O_{i_m}A_m}}\\ &\geq 1/2\,\frac{\overline{O_{i_m+1}A_m}}{\overline{O_{i_m}O_{i_m+2}}} \geq 1/2\,\frac{\delta}{\delta^{3/4}}\\ &= \frac{1}{2}\,\delta^{1/4}.\end{aligned}$$

Applying Proposition 4.1 again, $\sum_{m=1}^{N}\theta_m \leq 2\pi$, where the sum is being taken over the exterior angles which arise in type II_1 triples. But $\theta_m \geq 1/2\delta^{1/4}$, thus $N \leq c\delta^{-1/4}$.

Q.E.D.

Finally, we note that the selected vertices $O_{i_1}, \ldots, O_{i_L}$ are from type I, type II_1 or type II_2 triples. In the convex polygon Z, we consider all maximal chains, where a maximal chain is a union of successive sides of the polygon Z which come from type I or type II_1 triples. Then by the selection procedure, between any two maximal chains, there are no more than five vertices which are probably from type II_2 (see case 2 (ii) at the beginning of the selection procedure in this section). Thus the number of these type II_2 vertices are less than $c\delta^{-1/4}$ also. Therefore, the number of all vertices $O_{i_1}, \ldots, O_{i_L}$ is less than $c\delta^{-1/4}$, and this shows (ii) of Lemma 4.1.

We end this section with the following example which shows (ii) in Lemma 4.1 is the best possible result in each cube with sidelength $\sqrt{\delta}$.

EXAMPLE. Assume that the centers of the unit balls $\{B_\alpha\}$ are on the circle centered at the origin O and of radius $\sqrt{\delta}$. Furthermore, let the arclength between any two centers be $c\delta^{3/4}$. We claim that we exactly need $N = c\delta^{-1/4}$ unit balls $B_1, \ldots, B_N$ such that

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N}(1+\delta)B_i.$$

We consider two balls B_1 and B_2 centered at O_1 and O_2 respectively (as shown on Figure 5).

Let $OA \perp O_1O_2$, $OF \perp AO_2$, $AE = h$, $AO_2 = AO_1 = l$, and let $\angle AOF = \theta$, thus $\angle EO_2A = \theta$. Moreover,

$$\sin\theta = \sin(\angle EO_2A) = \frac{\overline{AE}}{\overline{AO_2}} = h/l,$$

also

$$\sin\theta = \sin(\angle AOF) = \frac{\overline{AF}}{\overline{AO}} = \frac{l/2}{\sqrt{\delta}}.$$

Thus $\dfrac{l/2}{\sqrt{\delta}} = h/l$, *i.e.*,

$$l^2 = 2\sqrt{\delta}h.$$

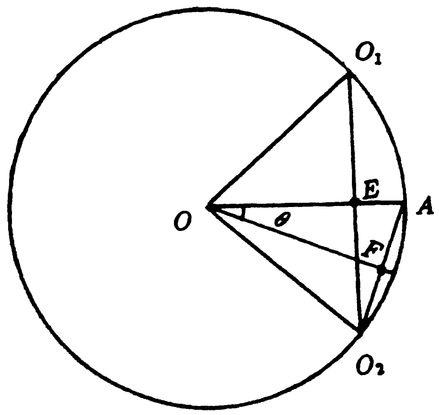

Figure 5

But by Proposition 4.3, $(1+\delta)B_1 \cup (1+\delta)B_2$ can cover all unit balls $\{B_\alpha\}$ centered between O_1 and O_2 if and only if $h \le c\delta$, *i.e.*, $l \le c\delta^{3/4}$. We also note that the arclength of $O_1O_2 \le c\delta^{3/4}$ if and only if $l \le c\delta^{3/4}$. Hence, we exactly need $N = c\sqrt{\delta}/\delta^{3/4} = c\delta^{-1/4}$ balls.

5. Proof of the Main Covering Lemma

We now prove the main covering lemma (Lemma 1) stated in the introduction. We will need the following lemmas and propositions.

Lemma 5.1. *Let δ be small enough. Given any cube Q in $\mathbf{R}^2$ with sidelength $2^k\sqrt{\delta}$ and let $\{B_\alpha\}_{\alpha\in I}$ be any finite collection of balls with $r \le \varrho(B_\alpha) \le r + 2^k\delta$ for some $2^k \le r \le 2^{k+1}$ and centered in Q, where k is an integer. Then one can select a subcollection of balls $B_1, \ldots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (1 + c\delta)B_i,$$

$$N \le c\delta^{-1/4}.$$

The proof of Lemma 5.1 is straightforward if we use Lemma 4.1 and the scaling property.

Lemma 5.2. *Let δ be small enough. Given any cube Q in $\mathbf{R}^2$ with sidelength $2^k\sqrt{\delta}$ and let $\{B_\alpha\}_{\alpha\in I}$ be any finite collection of balls with $2^k \le \varrho(B_\alpha) \le 2^{k+1}$. Then one can select a subcollection $B_1, \ldots, B_N$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (1 + c\delta)B_i,$$

$$N \le c\delta^{-3/4}.$$

PROOF. With no loss of generaliy, we may assume the largest ball B in our collection is of radius 2^{k+1}. Any other ball B_α with $\varrho(B_\alpha) \le 2^{k+1} - 2^k\sqrt{2\delta}$ is contained in B. Thus the balls B_α with $\varrho(B_\alpha) \le 2^{k+1} - 2^k\sqrt{2\delta}$ may be ignored. Now we partition the radii into the following intervals:

$$2^{k+1} - 2^k\sqrt{2\delta} + 2^k\delta l \le \varrho(B_\alpha) \le 2^{k+1} - 2^k\sqrt{2\delta} + 2^k\delta(l + 1)$$

for $0 \le l \le \sigma$.

Thus $\sigma \approx \sqrt{2/\delta}$. Applying Lemma 5.1 to these balls $\{B_\alpha^l\}$ whose radii are in the interval corresponding to l, $0 \le l \le \sigma$, we can select $\{B_i^l\}_{i=1}^{N_l}$ such that

$$\bigcup_\alpha B_\alpha^l \subset \bigcup_{i=1}^{N_l} (1 + c\delta)B_i^l,$$

$$N_l \le c\delta^{-1/4}.$$

Thus

$$\bigcup_{\alpha} B_{\alpha} \subset \bigcup_{\alpha,l} B_{\alpha}^{l} \bigcup B \subset \bigcup_{i,l} (1 + c\delta) B_{i}^{l} \bigcup B$$

and the number of balls $\{B_i^l\}_{i,l}$ is less than

$$\sum_{l=1}^{\sigma} N_l \leq \sigma \cdot c\delta^{-1/4} \leq c\delta^{-1/2}\delta^{-1/4} = c\delta^{-3/4}.$$

Therefore the subcollection $\{B_i^l\}$ will be the one in our lemma.

Now we can show the following:

Lemma 5.3. *Let δ be small enough and $\{B_\alpha\}_{\alpha\in I}$ be a finite collection of balls with $2^k \leq \varrho(B_\alpha) \leq 2^{k+1}$, where k are integers. Then one can select balls $B_1, \ldots, B_N$ such that*

$$\bigcup_{\alpha} B_{\alpha} \subset \bigcup_{i=1}^{N} (1 + c\delta) B_i$$

and

$$\sum_{i=1}^{N} \chi_{B_i}(x) \leq c\delta^{-7/4}$$

for all $x \in \mathbf{R}^2$.

PROOF. We subdivide $\mathbf{R}^2$ into a dyadic grid of cubes $\{Q_j\}_{j=1}^{\infty}$ whose side-lengths are all $2^k\sqrt{\delta}$. Let $\{B_\alpha^j\}$ be the subcollection of $\{B_\alpha\}$ with centers inside Q_j. For each $\{B_\alpha^j\}$ and Q_j, we apply Lemma 5.2 to select $\{B^j_i\}_{i=1}^{N_j}$ such that

$$\bigcup_{\alpha} B^{j}_{\alpha} \subset \bigcup_{i=1}^{N_j} (1 + c\delta) B_i^j,$$

$$N_j \leq c\delta^{-3/4}.$$

where c is independent of δ, k, j.

Then

$$\bigcup_{\alpha} B_{\alpha} \subset \bigcup_{\alpha,j} B_{\alpha}^{j} \subset \bigcup_{j} \bigcup_{i=1}^{N_j} (1 + c\delta) B_i^j = \bigcup_{i,j} (1 + c\delta) B_i^j,$$

which shows (5.4). Now let $x_0 \in \mathbf{R}^2$ be a fixed point and let $B(x_0, 2^{k+1})$ denote the ball centered at x_0 and of radius 2^{k+1}. Since $\varrho(B_i^j) \leq 2^{k+1}$, $\chi_{B_i^j}(x_0)$ vanishes for those balls $\{B_i^j\}$ centered outside $B(x_0, 2^{k+1})$. Now the cardinality of $\{j\}$ such that $Q_j \bigcap B(x_0, 2^{k+1}) \neq \emptyset$ is no more than $c\delta^{-1}$. Thus

$$\sum_{i,j} \chi_{B_i^j}(x_0) = \sum_j \sum_{i=1}^{N_j} \chi_{B_i^j}(x_0)$$

$$\leq \sum_j c\delta^{-3/4} \leq c\delta^{-1}\delta^{-3/4} = c\delta^{-7/4},$$

this shows (5.5). Since x_0 is arbitrary, the proof is complete.

We now prove the main covering lemma.

We first select a ball B_1 with the largest radius. Having selected $B_1, \ldots, B_{k-1}$ select B_k such that

$$B_k \not\subset \bigcup_{i=1}^{k-1} (1 + \delta)B_i \tag{5.6}$$

and B_k has the largest possible radius out of the balls in the collection $\{B_\alpha\}_{\alpha \in I} \setminus \{B_i\}_{i=1}^{k-1}$. Thus clearly, we have

$$\bigcup_\alpha B_\alpha \subset \bigcup_{i=1}^{N} (1 + \delta)B_i.$$

Now let $B_i' = (1 + \delta)B_i$ and $\{B_{i,k}'\}$ be the subcollection of $\{B_i'\}_{i=1}^N$ such that $2^k \leq \varrho(B_{i,k}') \leq 2^{k+1}$. Then by using Lemma 5.3, we can select a subcollection $\{B_{i_j,k}'\}$ of $\{B_{i,k}'\}$ such that

$$\bigcup_i B_{i,k}' \subset \bigcup_{j=1}^{N_k} (1 + c\delta)B_{i_j,k}',$$

$$\sum_{j=1}^{N_k} \chi_{B_{i_j,k}'}(x) \leq c\delta^{-7/4},$$

for all $x \in \mathbf{R}^2$, where c is independent of δ, k and $k = 0, \pm 1 \pm 2, \ldots$ Thus, we have

$$\begin{aligned}
\bigcup_{\alpha} B_\alpha &\subset \bigcup_{i=1}^{N} (1+\delta)B_i \\
&= \bigcup_{i=1}^{N} B_i' = \bigcup_k \bigcup_i B_{i,k}' \\
&\subset \bigcup_k \bigcup_{j=1}^{N_k} (1+c\delta)B_{i_j,k}' \\
&= \bigcup_{k,j} (1+c\delta)(1+\delta)B_{i_j,k} \\
&\subset \bigcup_{j,k} (1+2c\delta)B_{i_j,k}
\end{aligned}$$

for $c > 2$ and δ small enough. Thus, the subcollection $\{B_{i_j,k}\}$ satisfies (i) of Lemma 1.

Fix $x_0 \in \mathbf{R}^2$. There exists $k_0 = k(x_0)$ and $\sigma_0 = \sigma(x_0)$ such that

$$\begin{aligned}
\sum_{j,k} \chi_{B_{i_j,k}}(x_0) &= \sum_{k=k_0}^{k_0+\sigma_0} \sum_{j=1}^{N_k} \chi_{B_{i_j,k}}(x_0) \\
&\le \sum_{k=k_0}^{k_0+\sigma_0} \left(\sum_{j=1}^{N_k} \chi_{B_{i_j,k}'}(x_0) \right) \\
&\le \sum_{k=k_0}^{k_0+\sigma_0} c\delta^{-7/4} = c\sigma_0\delta^{-7/4}.
\end{aligned}$$

We claim $2^{\sigma_0} \le 8\delta^{-1}$. For otherwise there would exist a ball $B_{i_j,k_0+\sigma_0}$ for some j with $x_0 \in B_{i_j,k_0+\sigma_0}$ such that

$$\varrho(B_{i_j,k_0+\sigma_0}) \ge 1/2\varrho(B_{i_j,k+\sigma_0}') \ge 1/2 \cdot 2^{k_0+\sigma_0} \ge 2\delta^{-1}2^{k_0+1}.$$

The first inequality above is due to the fact that

$$\varrho(B_{i_j,k_0+\sigma_0}') = (1+\delta)\,\varrho(B_{i_j,k_0+\sigma_0}) \le 2\varrho(B_{i_j,k_0+\sigma_0}) \quad \text{for} \quad \delta < 1.$$

Let $x_0 \in B_{i_h,k_0}$ for some h be the ball with

$$\varrho(B_{i_h,k_0}) \le \varrho(B_{i_h,k_0}') \le 2^{k_0+1}.$$

Then $(1+\delta)B_{i_j,k_0+\sigma_0} \supset B_{i_h,k_0}$ which is a contradiction to (5.6). Thus $2^{\sigma_0} \le 8\delta^{-1}$, and the claim follows. That is, $\sigma_0 \le c \log \delta^{-1}$. Therefore,

$$\sum_{j,k} \chi_{B_{i_j,k}}(x_0) \le c\sigma_0\delta^{-7/4} \le c\log(\delta^{-1})\delta^{-7/4}.$$

Since x_0 is arbitrary, we are done.

6. Proof of Theorem 1

This section is devoted to the proof of the main theorem in this note (Theorem 1). We begin with recalling the following theorem in [DF2].

Theorem 6.1 *(Donnelly-Fefferman). Let M, u, λ be as in the introduction. Let $B(x, \delta)$ denote the ball centered at x of radius δ. Then*

$$\int_{B(x,\delta(1+\lambda^{-1/2}))} |u|^2 \le c \int_{B(x,\delta)} |u|^2 \tag{6.2}$$

$$\left[\int_{B(x,\delta)} |\nabla u|^2\right]^{1/2} \le c\,\frac{\sqrt{\lambda}}{\delta}\left[\int_{B(x,\delta)} |u|^2\right]^{1/2} \tag{6.3}$$

Now we start the proof of Theorem 1 by showing the following lemmas.

Lemma 6.4. *Let u, λ as above, $1 \le q < \infty$, then u satisfies the Reverse-Hölder inequality*

$$\left[\frac{1}{|B|}\int_B |u|^q\right]^{1/q} \le c\sqrt{\lambda}\left[\frac{1}{|B|}\int_B |u|^2\right]^{1/2} \tag{6.5}$$

where c depends on q.

PROOF. By the Poincaré inequality, for any ball B, we have

$$\left[\frac{1}{|B|}\int_B |u-u_B|^q\right]^{1/q} \le c|B|^{1/2}\left[\frac{1}{|B|}\int_B |\nabla u|^p\right]^{1/p}, \tag{6.6}$$

where $u_B = \frac{1}{|B|}\int_B u$ and $1 < p < 2$, $1/q = 1/p - 1/2$, and $c = c(p, q)$. Applying Hölder inequality and (6.3) to the right side of (6.6), we obtain

$$\left[\frac{1}{|B|}\int_B |u-u_B|^q\right]^{1/q} \le c\sqrt{\lambda}\left[\frac{1}{|B|}\int_B |u|^2\right]^{1/2}.$$

By Minkowski's inequality, Lemma 6.4 follows for $2 < q < \infty$, for the case $1 < q \le 2$, we can apply Hölder inequality again.

Our theorem will follow from the following

Lemma 6.7. *Suppose* $w > 0$, $q > 2$, $\epsilon > 0$ *and* $1 + \epsilon \geq q'$ *where* $1/q' = 1 - 1/q$, *also assume that*

$$\int_{B(x,\delta(1+\lambda^{-1/2}))} w \leq c_0 \int_{B(x,\delta)} w$$

and

$$\left(\frac{1}{|B|}\int_B w^q\right)^{1/q} \leq c_1\lambda \frac{1}{|B|}\int_B w, \tag{6.8}$$

then

$$\|\log w\|_{BMO} \leq c\lambda^{15/8+\epsilon}$$

where $c = c(c_0, c_1)$.

Theorem 1 will follow if we choose $w = |u|^2$.

In order to prove Lemma 6.7, we need the following

Lemma 6.9. *Let* w, q, $0 < \epsilon < 1$ *satisfy the hypothesis of Lemma* 6.7, *let* B *be a fixed ball,* $E \subset B$, *then there exist* c_2, c_3 *such that if*

$$|E| \geq (1 - c_2\lambda^{-15/8-\epsilon}(\log\lambda)^{-1})^k \int_B w$$

then

$$\int_E w \geq (c_3\lambda^{-7/8}(\log\lambda)^{-1})^k \int_E w$$

where $c_2 = c(c_1)$, $c_3 = c(c_0)$.

PROOF. We proceed as the proof given in [CM]. The method is to use induction on k. We first verify the lemma for $k = 1$. To do so, we claim that if $\epsilon > 0$, $|E| \geq (1 - \bar{c}\lambda^{-(1+\epsilon)})|B|$ for some appropriate $\bar{c} = \bar{c}(c_1)$, then

$$\int_E w \geq 1/2 \int_B w.$$

To show this, we first note that $|B \setminus E| \leq \bar{c}\lambda^{-(1+\epsilon)}|B|$. If we choose $q > 2$ such that $\dfrac{1+\epsilon}{q'} \geq 1$, thus by (6.8),

$$\int_{B\setminus E} w \leq \left(\int_B w^q\right)^{1/q} |B\setminus E|^{1/q'}$$

$$\leq c_1\bar{c}^{1/q'}\lambda^{-\frac{1+\epsilon}{q'}+1}\int_B w$$

$$\leq c_1\bar{c}^{1/q'}\int_B w$$

If we choose $\bar{c}$ such that $c_1\bar{c}^{1/q'} < 1/2$, then $\int_{B\setminus E} w \leq 1/2 \int_B w$, this implies $\int_E w > \frac{1}{2}\int_B w$. Here we want to point out the choice of $\bar{c}$ is dependent on ϵ since $c_1 = c_1(q)$ and q is dependent on ϵ.

Thus if $c_2 \leq \bar{c}$, and $|E| \geq (1 - c_2\lambda^{-15/8-\epsilon}(\log\lambda)^{-1})|B|$, then

$$\int_E w \geq 1/2\int_B w \geq c_3(\lambda^{-7/8}(\log\lambda)^{-1})\int_B w,$$

and we are done for the case $k = 1$. Now we assume the statement is true for $k - 1$. Obviously we can assume $|E| \leq (1 - \bar{c}\lambda^{-(1+\epsilon)})|B|$, otherwise, there is nothing to prove. Thus for each density point x of E, we can select a ball $B_x \subset B$ such that $x \in B_x$, and

$$\frac{|B_x \cap E|}{|B_x|} = 1 - \bar{c}\lambda^{-(1+\epsilon)}.$$

Applying Lemma 1 to the balls B_x with the choice $\delta = \lambda^{-1/2}$, and with no loss of generality, assume $\{B_x\}$ are finite, and define

$$E_1 = \left[\bigcup_{i=1}^{N}(1 + \lambda^{-1/2})B_i\right] \cap B.$$

Then $E_1 \subset B$, and as the proof given in [CM], we can show

$$|E| \leq (1 - c_2\lambda^{-15/8-\epsilon}(\log\lambda)^{-1})|E_1|$$

$$\int_E w \geq c_3\lambda^{-7/8}(\log\lambda)^{-1}\int_{E_1} w$$

for some $c_2 = c(c_1)$, $c_3 = c(c_0)$.

Now we prove Lemma 6.7. The proof will be almost the same as that given in [CM]. In order to get the precise estimate, we like to show the details. It will be enough to assume $\frac{1}{|B|}\int_B w = 1$. It is also sufficient to show

$$|\{x \in B : w^{-1}(x) > t\}| \le \frac{|B|}{t^{c\lambda^{-15/8-\epsilon}(\log \lambda)^{-2}}}.$$

It is equivalent to show

$$|\{x \in B : w(x) < t\}| \le t^{c\lambda^{-15/8-\epsilon}(\log \lambda)^{-2}} |B|.$$

Let us denote by $E = \{x \in B : w(x) < t\}$. Select k_0 such that

$$|E| \approx [1 - c_2\lambda^{-15/8-\epsilon}(\log \lambda)^{-1}]^{k_0} |B|.$$

Thus

$$k_0 \approx c(\lambda^{15/8+\epsilon} \log \lambda) \log\left(\frac{|B|}{|E|}\right)$$

Then by Lemma 6.9, and the normalization $\frac{1}{|B|}\int_B w = 1$, we have

$$\begin{aligned} |B| = \int_B w &\le (c_3^{-1}\lambda^{7/8} \log \lambda)^{k_0} \int_E w \\ &\le (c_3^{-1}\lambda^{7/8} \log \lambda)^{k_0} t |E|. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|B|}{|E|} &\le t\, e^{k_0 \log(c_3^{-1}\lambda^{7/8} \log \lambda)} \\ &\le t \left(\frac{|B|}{|E|}\right)^{(c\lambda^{15/8+\epsilon} \log \lambda) \log(c_3^{-1}\lambda^{7/8} \log \lambda)} \\ &\le t \left(\frac{|B|}{|E|}\right)^{c'\lambda^{15/8+\epsilon} (\log \lambda)^2} \end{aligned}$$

Hence

$$|E| \leq t^{c'\lambda^{-15/8-\epsilon}(\log \lambda)^{-2}} |B| ,$$

where c' is dependent on the constant c in (6.4). Since ϵ is arbitrary, we can even have

$$|E| \leq t^{c''\lambda^{-15/8-\epsilon}} |B| .$$

Remark. Since for any $0 < \epsilon < 1$, we can choose q so that $1 + \epsilon \geq q'$, and note that for such q, Lemma 6.7 holds. Unfortunately, the constant c on the right side of (6.5) depends on q (and then depends on ϵ), and is unbounded when $q \to \infty$. Thus we have proved the Theorem 1 for any $\epsilon > 0$. However we can not obtain the theorem 1 for replacing $\lambda^{15/8+\epsilon}$ by $\lambda^{15/8}$ since when $\epsilon \to 0$, c_ϵ is not bounded.

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Added in Proof: After this paper had been accepted for publication, the author has also shown a higher dimensional covering lemma in $\mathbf{R}^n$ ($n \geq 3$) by which one can refine the BMO norm estimate of [CM] by reducing the power of λ to $n - 1/8$ [L].

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Mean value and Harnack inequalities for a certain class of degenerate parabolic equations

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Introduction

In this paper we study the behavior of solutions of degenerate parabolic equations of the form

$$(1.1) \qquad v(x)u_t(x, t) = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x, t) D_{x_j} u(x, t)),$$

where the coefficients are measurable functions, and the coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

$$(1.2) \qquad w_1(x) \sum_{j=1}^{n} \lambda_j^2(x) \xi_j^2 \le \sum_{i,j=1}^{n} a_{ij}(x, t) \xi_i \xi_j \le w_2(x) \sum_{j=1}^{n} \lambda_j^2(x) \xi_j^2$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$ and $(x, t) \in \Omega \times (a, b)$, Ω a bounded open set in $\mathbf{R}^n$.

We are going to assume some conditions on the weights (non-negative functions that are locally integrable) v, w_1, w_2 and on the functions λ_j, $j = 1, \dots, n$, in order to be able to derive mean value and Harnack inequalities for solutions of (1.1). The assumptions on λ_j, which we list below, are the ones stated in [FL2].

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(1.3) $\lambda_1 \equiv 1$, $\lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1})$, $j = 2, \ldots, n$, $x \in \mathbf{R}^n$.

(1.4) Let $\Pi = \{x \in \mathbf{R}^n \colon \Pi x_k = 0\}$. Then $\lambda_j \in C(\mathbf{R}^n) \cap C^1(\mathbf{R}^n \setminus \Pi)$ and $0 < \lambda_j(x) \le \Lambda$, $x \in \mathbf{R}^n \setminus \Pi$, $j = 1, \ldots, n$.

(1.5) $\lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, -x_i, \ldots, x_{j-1})$, for $j = 2, \ldots, n$ and $i = 1, \ldots, j-1$.

(1.6) There is a family of $n(n-1)/2$ non-negative numbers $\varrho_{j,i}$ such that $0 \le x_i(D_{x_i}\lambda_j)(x) \le \varrho_{j,i}\lambda_j(x)$, for $2 \le j \le n$, $1 \le i \le j-1$ and all $x \in R^n \setminus \Pi$.

Denote $\Gamma = \Omega \times (a, b)$ and define $H = H(\Gamma)$ to be the closure of $\mathrm{Lip}(\Gamma)$ under the norm

$$\|u\|^2 = \iint_\Gamma u^2(x,t)(v(x) + w_2(x))\,dxdt + \iint_\Gamma |\nabla_\lambda u(x,t)|^2 w_2(x)\,dxdt + \iint_\Gamma u_t^2(x,t)v(x)\,dxdt, \tag{1.7}$$

where $\nabla_\lambda u = (\lambda_1 D_{x_1}u, \ldots, \lambda_n D_{x_n}u)$. Thus, $H(\Gamma)$ is the collection of all $(n+2)$-triples (u, β, B) such that there exists $u_k \in \mathrm{Lip}(\Gamma)$ with $u_k \to u$, $\nabla_\lambda u_k \to \beta$, $(u_k)_t \to B$, the convergence being in the appropriate L^2 space. We need to verify that β is uniquely determined and for this it is enough to show that for every $F \in C_0^\infty(\Gamma)$,

$$\int_\Gamma u\nabla_\lambda F = -\int_\Gamma \beta F.$$

In order to prove this, note that since $u \in H$, there exists $\{u_k\} \subset \mathrm{Lip}(\Gamma)$ such that $u_k \to u$ in H. Then, by (1.3),

$$\int_\Gamma u_k\lambda_i \frac{\partial F}{\partial x_i} = -\int_\Gamma \frac{\partial}{\partial x_i}(u_k\lambda_i)F = -\int_\Gamma \lambda_i \frac{\partial u_k}{\partial x_i} F.$$

Therefore,

$$\int_\Gamma u_k \nabla_\lambda F = -\int_\Gamma (\nabla_\lambda u_k)F.$$

By Schwarz's inequality and assuming that $w_2^{-1} \in L^1_{\mathrm{loc}}$,

$$\left| \int_\Gamma u_k \nabla_\lambda F - \int_\Gamma u \nabla_\lambda F \right| \le \int_\Gamma | u_k - u | w_2^{1/2} | \nabla_\lambda F | w_2^{-1/2}$$
$$\le \| u_k - u \|_{L^2_{w_2}} \left(\int_\Gamma | \nabla_\lambda F |^2 w_2^{-1} \right)^{1/2}$$
$$\le c \| u_k - u \|_{L^2_{w_2}}.$$

Hence,

$$\int_\Gamma u_k \nabla_\lambda F \to \int_\Gamma u \nabla_\lambda F$$

and similarly we can show

$$\int_\Gamma (\nabla_\lambda u_k) F \to \int_\Omega \beta F.$$

In the same way we prove B is uniquely determined, if $\nu^{-1} \in L^1_{\mathrm{loc}}$. We also define $H_0(\Gamma)$ to be the closure of $\mathrm{Lip}_0(\Gamma)$, Lipschitz functions with compact support in Γ, under the norm defined in (1.7). It is easy to see that the bilinear form b on $\mathrm{Lip}(\Gamma) \cap H(\Gamma)$ defined by

$$b(u, \phi) = \int\int_\Gamma \{u_t \phi \nu + \langle A \nabla u, \nabla \phi \rangle\} dxdt$$

can be continued to all of $H(\Gamma)$ (here and in the rest of the paper the vector ∇u is understood to be the vector $\left(\frac{1}{\lambda_1} \beta_1, \ldots, \frac{1}{\lambda_n} \beta_n \right)$ where $\nabla_\lambda u = (\beta_1, \ldots, \beta_n)$). We say $u \in H(\Gamma)$ is a solution of (1.1) if $b(u, \phi) = 0$ for any $\phi \in H_0$; $u \in H(\Gamma)$ is a subsolution if $b(u, \phi) \le 0$ for any $\phi \in H_0(\Gamma)$, ϕ positive in the H-sense, *i.e.*, ϕ can be approximated in $H(\Gamma)$ by positive functions with compact support in Γ; $u \in H(\Gamma)$ is a supersolution if $b(u, \phi) \le 0$ for any $\phi \in H_0$, ϕ positive in the H-sense.

We also define $\tilde{H} = \tilde{H}(\Omega)$ to be the closure of $\mathrm{Lip}(\Omega)$ under the norm

$$|||u|||^2 = \int_\Gamma u^2(x)(\nu(x) + w_2(x))dx + \int_\Gamma | \nabla_\lambda u(x) |^2 w_2(x) dx,$$

and $\tilde{H}_0(\Omega)$ to be the closure of $\mathrm{Lip}_0(\Omega)$ under the norm defined above.

Next we will define a natural distance (associated with the functions λ_j, $j = 1, \ldots, n$) and state some of its properties. This metric was first introduced by [FL1].

A vector $\nu \in \mathbf{R}^n$ is called a λ-subunit vector at a point x if $\langle \nu, \xi \rangle^2 \le \Sigma \lambda_j^2(x) \xi_j^2$, for every $\xi \in \mathbf{R}^n$. An absolutely continuous curve $\gamma: [0, T] \to \mathbf{R}^n$ is called a λ-subunit curve if $\dot{\gamma}(t)$ is a λ-subunit vector at $\gamma(t)$ for a.e. $t \in [0, T]$.

For any $x, y \in \mathbf{R}^n$ we define $d: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ by

$$d(x, y) = \inf\{T \in \mathbf{R}^+ : \text{there exists a } \lambda\text{-subunit curve } \gamma: [0, T] \to \mathbf{R}^n \text{ with } \gamma(0) = x, \gamma(T) = y\}.$$

One can check that this is a well-defined metric. There is a quasi-metric δ (a function $\delta: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ is called a quasi-metric if there exists $M \geq 1$ such that $\delta(x, y) \leq M\{\delta(x, z) + \delta(z, y)\}$ for all $x, y, z \in \mathbf{R}^n$) equivalent to d, and sometimes easier to work with than d (see [FL2]). If $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ put $H_0(x, t) = x$ and $H_{k+1}(x, t) = H_k(x, t) + t\lambda_{k+1}(H_k(x, t))e_{k+1}$ for $k = 0, \ldots, n - 1$, where $\{e_k\}$ is the standard basis in $\mathbf{R}^n$. Define $\varphi_j(x^*, .) = (F_j(x^*, .))^{-1}$, the inverse function of $F_j(x^*, .)$, where $F_j(x, s) = s\lambda_j(H_{j-1}(x, s))$, for $j = 1, \ldots, n$ and $x^* = (|x_1|, \ldots, |x_n|)$.

We define δ: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^+$ as

$$\delta(x, y) = \operatorname{Max}_{j=1,\ldots,n} \varphi_j(x^*, |x_j - y_j|).$$

Note that

(1.8) $\delta(x, y) < s$ is equivalent to $|x_j - y_j| < F_j(x^*, s)$, $1 \leq j \leq n$.

In (1.9), (1.10), (1.11) below we state some basic facts concerning δ and d (see also [FL2].

(1.9) There exists $a \geq 1$ such that for any $x, y \in \mathbf{R}^n$,

$$a^{-1} \leq \frac{d(x, y)}{\delta(x, y)} \leq a.$$

Consequently, δ is a quasi-metric with $\delta(x, y) \leq a^2[\delta(x, y) + \delta(z, y)]$ and $\delta(x, y) \leq a^2\delta(y, x)$.

(1.10) For any $x \in \mathbf{R}^n$, $s > 0$ and $\theta \in]0, 1[$

$$\theta^{G_j} \leq \frac{F_j(x^*, \theta s)}{F_j(x^*, s)} \leq \theta$$

where $G_1 = 1$ and $G_j = 1 + \Sigma_{l=1}^{j-1} G_l \varrho_{j,l}$, for $j = 2, \ldots, n$.

(1.11) We denote $S(x, r) = \{y \in \mathbf{R}^n : d(x, y) < r\}$ and $Q(x, r) = \{y \in \mathbf{R}^n : \delta(x, y) < r\}$ and we will call $S(x, r)$ a d-ball and $Q(x, r)$ a δ-ball. Note that there is a constant $A > 1$ such that $|S(x, 2r)| \leq A|S(x, r)|$ and $|Q(x, 2r)| \leq A|Q(x, r)|$, where $|\cdot|$ denotes Lebesgue measure. Also, by (1.8), $|Q(x, r)| = \Pi_{j=1}^n F_j(x^*, r)$. If $Q = Q(x, r)$, we write $r = r(Q)$.

In general we say that a non-negative and locally integrable function $w(x)$ is a doubling weight ($w \in D$) if there exists a constant $A > 1$ such that $w(2Q) \leq Aw(Q)$ for any δ-ball Q, where $2Q = Q(x, 2r)$, if $Q = Q(x, r)$ and

$$w(Q) = \int_Q w(x)\,dx.$$

(1.12) If $w \in D$ then there exists $\alpha > 0$ such that, for all $r > 0$, $\theta \in]0, 1]$, and $x \in R^n$, $w(Q(x, \theta r)) \geq \theta^\alpha w(Q(x, r))$.

Given $1 < p < \infty$, we say $w \in A_p$ if there is a constant $c > 0$ such that for all δ-balls Q in $\mathbf{R}^n$.

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/p-1}\,dx\right)^{p-1} \leq c. \tag{1.13}$$

Note that if we have the A_p condition with respect to δ, we have the same condition holding for the metric d, *i.e.* (1.13) holds with Q replaced by S (using doubling and the equivalence between d and δ). If v is a weight, $w \in A_p(v)$ means an analogous inequality holds with dx and $|Q|$ replaced by $v(x)dx$ and $v(Q)$, respectively. We use the notation $A_\infty(v) = \cup_{p>1} A_p(v)$. The theory of weights in homogeneous spaces was studied by A. P. Calderón in [C] and frequently we refer to this paper.

If $x, y \in \mathbf{R}^n$, we shall denote by $H(t, x, y) = (H_1(t, x, y), \ldots, H_n(t, x, y))$ the solution at time t of the Cauchy problem $\dot{H}_j(., x, y) = y_j\lambda_j(H(., x, y))$, $H_j(0, x, y) = x_j$, $j = 1, \ldots, n$.

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $0 < \epsilon_j < \alpha_j$, $j = 1, \ldots, n$, we denote $\Delta_\epsilon^\alpha = \{y \in \mathbf{R}^n: \epsilon_j \leq y_j \leq \alpha_j, j = 1, \ldots, n\}$. If $\sigma \in \{-1, 1\}^n$, we put $T_\sigma y = (\sigma_1 y_1, \ldots, \sigma_n y_n)$, $Q^\sigma(x, r) = \{y \in Q(x, r): \sigma_j(y_j - x_j) \geq 0, j = 1, \ldots, n\}$ and $\Delta_\epsilon^\alpha(\sigma) = T_\sigma(\Delta_\epsilon^\alpha)$.

Now we can state two results proved in [FS].

Let $\gamma \in]0, 1[$ and $\sigma \in \{-1, 1\}^n$ be fixed. Then there exists $\epsilon, \alpha \in \mathbf{R}^n$ as above such that, for all $r > 0$ and $x \in \mathbf{R}^n$,

$$|H(r, x, \Delta_\epsilon^\alpha(\sigma)) \cap Q^\sigma(x, r)| \geq (1 - \gamma)\,|Q^\sigma(x, r)|, \tag{1.14}$$

where $H(r, x, \Delta_\epsilon^\alpha(\sigma)) = \{H(x, r, y): y \in \Delta_\epsilon^\alpha(\sigma)\}$.

Also, there are positive constants c_1, c_2 depending only on ϵ, α and $\varrho_{j,i}$ such that

$$c_1 \,|\, S(x, r) \,|\, \leq \prod \int_0^r \lambda_j(H(t, x, y))dt \leq c_2 \,|\, S(x, r) \,| \tag{1.15}$$

for each $x \in \mathbf{R}^n$, $r > 0$ and $y \in \Delta_\epsilon^\alpha(\sigma)$.

If $q \geq 2$, we say that Sobolev inequality holds for w_1, [illegible] for any $u \in \tilde{H}_0(Q)$, Q a δ-ball in $\mathbf{R}^n$,

$$\left(\frac{1}{w_2(Q)} \int_Q |\, u \,|^q w_2 dx\right)^{1/q} \leq cr(Q) \left(\frac{1}{w_1(Q)} \int_Q |\, \nabla_\lambda u \,|^2 w_1 dx\right)^{1/2}. \tag{1.16}$$

Given $q \geq 2$, we say the Poincaré inequality holds for w_1, w_2 and μ if there are constant $c > 0$ and $a > 0$ (see (1.9)) such that for any δ ball Q and every $u \in \tilde{H}(a^2 Q)$ we have

$$\left(\frac{1}{w_2(Q)} \int_Q |\, u - av_{\mu,Q} u \,|^q w_2 dx\right)^{1/q} \leq \tag{1.17}$$

$$\leq cr(Q) \left(\frac{1}{w_1(Q)} \int_{a^2 Q} |\, \nabla_\lambda u \,|^2 w_1 dx\right)^{1/2},$$

where $av_{\mu,Q} u = \dfrac{1}{\mu(Q)} \displaystyle\int_Q u d\mu$ and $a^2 Q = Q(x, a^2 r)$ if $Q = Q(x, r)$.

The reason that we have $a^2 Q$ on the right side of (1.17) is that we do not have a Kohn type argument (see also [J]) for the quasi-metric δ. In the d-metric, we can state (1.17) with equal balls on both sides. For the metric δ, however, we have convenient cut-off functions (see [FL1]) that are important in order to get Caccioppoli estimates for solutions of (1.1) (see C.1), (C.2) and (C.3)). This explains the reason that we work with δ instead of d.

We can now state our main results.

Theorem A (Harnack's inequality).

Suppose that:

(i) v, w_1, $w_2 \in A_2$,

(ii) *the Poincaré inequality holds for* w_1, w_2 *and* w_1, v *with* $\mu = 1$ *and some* $q > 2$,

(iii) $w_2 v^{-1} \in A_\infty(v)$.

If u is a non-negative solution of (1.1) *in the cylinder* $R = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta)$, *then*

$$\text{ess sup}_{R^-} u \leq c_1 \exp\{c_2[\alpha^{-2}\beta\Lambda(Q(x_0, \alpha)) + \alpha^2\beta^{-1}(\lambda(Q(x_0, \alpha)))^{-1}]\}\text{ess inf}_{R^+} u,$$

where $R^- = Q(x_0, \alpha/2) \times (t_0 - 3\beta/4, t_0 - \beta/4)$, $R^+ = Q(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta)$, $\Lambda(Q) = w_2(Q)/v(Q)$, $\lambda(Q) = w_1(Q)/v(Q)$, *for a δ-ball Q. Here the constants c_1, c_2 depend only on the constants which arise in* (i), (ii), (iii).

We write

$$\int\!\!\!\!-\!\!\int_R f(x, t)m(x, t)dxdt = \int\!\!\int_R f(x, t)m(x, t)dxdt \Big/ \int\!\!\int_R m(x, t)dxdt.$$

Theorem B *(Mean value inequality). Assume that hypotheses* (i), (ii), (iii) *of Theorem A hold. Let* $0 < p < \infty$, $\alpha, \beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$ *and let* $Q = Q(x_0, \alpha)$, $Q' = Q(x_0, \alpha')$ *and* $R = Q \times (t_0 - \beta, t_0 + \beta)$, $R'_+ = Q' \times (t_0 - \beta', t_0 + \beta)$. *If u is a solution of* (1.1) *in R, then u is bounded in R'_+ and*

$$\text{ess sup}_{R'_+} |u|^p \leq D(\alpha^2\beta^{-1}\lambda(Q)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q) + 1)^{h/(h-1)} \int\!\!\!\!-\!\!\int_R |u|^p(\alpha^{-2}\beta w_2 + v)dxdt,$$

where $D \leq C^{1/(h-1)}$ *if* $p \geq 2$, *and* $D \leq c^{\log(3/p)}C^c$ *if* $0 < p < 2$, *and* $C = c\dfrac{\alpha^{2+b}\beta}{(\alpha - \alpha')^{2+b}(\beta - \beta')}$. *Here* $h > 1$, $c > 0$ *and* $b > 0$ *are constants which are independent of* u, p, α, α', β, β'.

The organization of the paper is as follows. In Section 2 we prove the following Sobolev interpolation inequality:

Theorem D: *Let w_1, w_2 be doubling weights, $v \in A_2$ and suppose* (1.17) *holds with w_1, w_2, $\mu = 1$ and some $q > 2$. If $w_2v^{-1} \in A_\infty(v)$ then there exists $h > 1$ and constants $c > 0$, $b > 0$ such that for every ϵ satisfying* $0 < \epsilon \leq 1$,

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h}w_2dx \leq c\epsilon^{-b}\left(\frac{1}{v(Q)}\int_{(1+\epsilon)Q} u^2vdx\right)^{h-1} \times\left(\frac{r(Q)^2}{w_1(Q)}\int_{(1+\epsilon)Q} |\nabla_\lambda u|^2w_1dx + \frac{1}{v(Q)}\int_{(1+\epsilon)Q} u^2vdx\right)$$

for all $u \in \tilde{H}((1 + \epsilon)Q)$.

In section 3 we prove Theorem B. First we show, for $p \geq 2$, the following mean value inequality for subsolutions of (1.1):

(*) $\text{ess sup}_{R'_+} u_+^p \leq$

$$(p^2 C)^{\frac{h}{h-1}} (1 + \alpha^2 \beta^{-1} \lambda(Q)^{-1})^{1/(h-1)} (1 + \alpha^{-2} \beta \Lambda(Q))^{h/(h-1)} \int\int_R u_+^p (\alpha^{-2} \beta w_2 + v) dx dt,$$

where C is as in Theorem B and $u_+ = \max\{u, 0\}$. This inequality is less precise than the one we will eventually obtain because of the presence of the factor p^2 on the right. In order to prove the above inequality we apply Theorem D to the function $H_M(u(., r))$ where

$$H_M(s) = \begin{cases} s^{p/2} \text{ if } s \in [0, M] \\ M^{p/2} + \frac{p}{2} M^{(p-2)/2}(s - M) \text{ if } s \geq M \\ 0 \text{ if } s < 0, \end{cases}$$

and therefore $H_M(u(., \tau))$ is an element of $\tilde{H}(Q(x_0, \alpha)$ for a.e. $\tau \in (t_0 - \beta', t_0 + \beta)$. The first idea would be to apply Theorem D to the function $u_+^{p/2}(., \tau)$ but at this point we do not know if $u_+^{p/2}(., \tau)$ belongs to $\tilde{H}(Q(x_0, \alpha)$. Hence we have to work with $H_M(u)$, and in order to proceed with the proof of (*) we show the following Caccioppolli inequality for $H_M(u)$.

(C.1) Let $2 \leq p < \infty$ and u be a subsolution of (1.1) in R. Let $w_2 \in A_2$ and $\alpha, \alpha', \beta, \beta'$ satisfy $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$. Then

$$\text{ess sup}_{\tau \in (t_0 - \beta', t_0 + \beta)} \int_Q H_M(u(x, r))^2 v(x) dx$$

$$+ \int\int_{R'_+} |\nabla_\lambda (H_M(u))|^2 w_1(x) dx dt$$

$$\leq c \int\int_R u^2 H'_M(u)^2 \left(\frac{w_2}{(\alpha - \alpha')^2} + \frac{v}{\beta - \beta'} \right) dx dt$$

with c independent of all parameters.

The next step is to apply (*) for $p = 2$ to deduce that u_+ is locally bounded. This fact allow us to apply Theorem D to the function $u_+^{p/2}(., \tau)$ for a.e. $\tau \in (t_0 - \beta', t_0 + \beta)$. The Caccioppoli inequality we can deduce from (C.1) for the function $u_+^{p/2}$ is not precise enough since it will have a factor p^2 in the right hand side (note that $uH'_M(u) \leq pu_+^{p/2}/2$) and this is the term we want to eliminate from (*). But with a different test function from the one used in the proof of (C.1), namely, $\phi(x, t) = \eta^2 g(u)\chi(t, \tau_1, \tau_2)$ where

$$g(s) = \begin{cases} s^{p-1} & \text{if} \quad s \in [0, M], \\ M^{p-2}s & \text{if} \quad s \geq M, \\ 0 & \text{if} \quad s < 0, \end{cases}$$

and η is a convenient C^∞ function with compact support, we can deduce the following Caccioppoli inequality for subsolutions of (1.1):

(C.2) Let $2 \leq p < \infty$ and u be a subsolution of (1.1) in R. Let $w_2 \in A_2$ and $\alpha, \alpha', \beta, \beta'$ satisfy $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$. Then

$$\text{ess sup}_{\tau \in (t_0 - \beta', t_0 + \beta)} \int_Q u_+(x, \tau)^p v(x)dx + \int\int_{R'_+} |\nabla_\lambda u_+^{p/2}|^2 w_1(x)dxdt$$

$$\leq c \int\int_R u_+^p \left(\frac{w_2}{(\alpha - \alpha')^2} + \frac{v}{\beta - \beta'} \right) dxdt,$$

with c independent of all parameters.

Now following the steps of the proof of (*) using (C.2) instead of (C.1) we can prove that for $p \geq 2$

(**) $\text{ess sup}_{R'_+} u_+^p \leq$

$$C^{\frac{h}{h-1}}(\alpha^2\beta^{-1}\lambda(Q)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(Q) + 1)^{h/(h-1)} \int\int_R u_+^p(\alpha^{-2}\beta w_2 + v)dxdt,$$

and Theorem B will follow from (**) and an iteration argument like the one given in Lemma 3.4 of [GW2]. Finally we conclude Section 3 by making some comments about the proof of mean value inequalities for u^p, when $p < 0$, where u is a positive solution of (1.1). These inequalities will be necessary in the proof of Theorem A and in order to show them we need the following generalization of (C.2):

(C.3) Let $-\infty < p < +\infty, p \neq 0, 1$, u satisfy $0 < m < u(x, t) < M < \infty$ in R, $w_2 \in A_2$. Then if $p > 1$ and u is a subsolution in R, or if $p < 0$ and u is a supersolution in R,

$$\text{ess sup}_{\tau \in (t_0-\beta', t_0+\beta)} \int_{Q'} u(x, \tau)^p v(x) dx + \frac{p-1}{p} \int\int_{R'_+} |\nabla_\lambda u^{p/2}|^2 w_1(x) dx dt$$

$$\leq c \int\int_R u^p \left(\frac{p}{p-1} \cdot \frac{w_2(x)}{(\alpha - \alpha')^2} + \frac{v(x)}{\beta - \beta'} \right) dx dt.$$

Moreover, if $0 < p < 1$ and u is a supersolution in R, then

$$\text{ess sup}_{\tau \in (t_0-\beta, t_0+\beta')} \int_{Q'} u(x, \tau)^p v(x) dx + \left| \frac{p-1}{p} \right| \int\int_{R'_-} |\nabla_\lambda u^{p/2}|^2 w_1 dx dt$$

$$\leq c \int\int_R u^p \left(\left| \frac{p}{p-1} \right| \frac{w_2}{(\alpha - \alpha')^2} + \frac{v}{\beta - \beta'} \right) dx dt.$$

In this paper we do not present the proofs of (C.2) and (C.3) since their proofs are similar to the ones given in Section 2 of [GW2].

In Section 4, we prove

Theorem E: *Let v and w_1 be weights such that there exists $s > 1$ with*

$$\left(\frac{r(I)}{r(B)} \right)^2 \left(\frac{1}{|I|} \int_I \left(\frac{v}{v(B)} \right)^s dx \right)^{1/s} \left(\frac{1}{|I|} \int_I \left(\frac{w_1}{w_1(B)} \right)^{-s} dx \right)^{1/s} \leq c \tag{1.18}$$

for all δ-balls I, B with $I \subset 2a^2 B$ (a as in (1.9)), where c is a constant independent of the balls. Let $Q = Q(\xi, r)$ and φ be a C^1 function such that $\varphi \equiv 1$ in $Q(\xi, kr)$, $1/2 \leq k < 1$, $0 \leq \varphi \leq 1$, supp $\varphi \subset Q$ and

$$\varphi(x)\varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y))$$

for all x, y, t, t_0 with $0 \leq t \leq t_0$. Then, if $u \in \mathrm{Lip}(Q)$,

$$\int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx \leq c \frac{v(Q)}{w_1(Q)} r(Q)^2 \int_Q |\nabla_\lambda u(x)|^2 \varphi(x) w_1(x) dx,$$

where $A_Q = \dfrac{1}{\varphi(Q)} \displaystyle\int_Q u(x)\varphi(x) dx.$

Finally, in Section 5, we prove Theorem A. This theorem follows as an application of Bombieri's lemma ([GW2]). In order to verify the hypotheses of Bombieri's lemma we need Theorem B and Theorem F, which we state next. We write

$$(v \otimes 1)(A) = \iint_A v(x)dxdt,$$

where $v = v(x)$, $x \in \mathbf{R}^n$, and $A \subset \mathbf{R}^{n+1} = \{(x, t): x \in \mathbf{R}^n, t \in R\}$.

Theorem F: *Suppose v is a doubling weight, $w_2 \in A_2$, (1.18) holds and $w_2 v^{-1} \in A_\infty(v)$. Let Q_R be a δ-ball of radius R, $t_0 \in (a, b)$ and $\tilde{w}_2 = w_2/w_2(Q_R)$ and $\tilde{v} = v/v(Q_R)$. If u is a solution of (1.1) in $Q_{3R/2} \times (a, b)$ which is bounded below by a positive constant, then there are constants c_1, M_2, $\varkappa$ and V such that if for $s > 0$ we define*

$$E^+ = \{(x, t) \in Q_R \times (t_0, b): \log u < -s - M_2(b - t_0) - V\}$$
$$E^- = \{(x, t) \in Q_R \times (a, t_0): \log u > s - M_2(a - t_0) - V\},$$

then

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E^+) \le c_1 \left(\frac{1}{s} \frac{v(Q_R)}{w_1(Q_R)} \frac{R^2}{b - t_0}\right)^{\varkappa} (b - t_0)$$

and

$$((\tilde{v} + \tilde{w}_2) \otimes 1)(E^-) \le c_1 \left(\frac{1}{s} \frac{V(Q_R)}{w_1(Q_R)} \frac{R^2}{t_0 - a}\right)^{\varkappa} (t_0 - a).$$

Here c_1 and $\varkappa$ depend only on the constants in the conditions on v and w_2, $M_2 \approx \dfrac{w_2(Q_R)}{R^2 v(Q_R)}$, and V is a constant which depends on u.

In order to prove this theorem, if we follow the steps of Lemma 4.9 of [GW2], we just have to verify that a certain test function (see [FL1]) satisfies the conditions of Theorem E. This will be done in Lemma 5.4.

2. Interpolation Inequality

In this section we prove Theorem D. We start with

Theorem 2.1. *Let w_1, w_2, and μ be doubling weights and suppose* (1.17) *holds for w_1, w_2 with any μ, and for some $q > 2$. If $Q = Q(\xi, r)$ and $w_2 v^{-1} \in A_\infty(v)$ then there exist $h > 1$ and a constant $c > 0$, independent of Q and u, such that*

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx$$

$$\leq c\left(\frac{1}{v(Q)}\int_Q u^2 v dx\right)^{h-1}\left(\frac{r^2}{w_1(Q)}\int_{Q(\xi,a^2 r)} |\nabla_\lambda u|^2 w_1 dx + (av_{\mu,Q}|u|)^2\right)$$

for all $u \in \tilde{H}(a^2Q)$, *and a as in* (1.9). *Also if* (1.17) *is replaced by* (1.16), *then*

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx \leq c\left(\frac{1}{v(Q)}\int_Q u^2 v dx\right)^{h-1}\left(\frac{r^2}{w_1(Q)}\int_Q |\nabla_\lambda u|^2 w_1 dx\right)$$

for all $u \in \tilde{H}_0(Q)$.

Proof: The proof follows as in [GW1], Theorem 3; the only differences are that we obtain $Q(\xi, a^2 r)$ in the second integral on the right when we apply Poincaré's inequality and in the end we use the results of Calderón for weights in homogeneous spaces (see [C]).

Corollary 2.2. *Let* w_1, w_2 *be doubling weights and suppose* (1.17) *holds with* w_1, w_2, $\mu = 1$ *and some* $q > 2$. *If* $w_2 v^{-1} \in A_\infty(v)$, *then there exists* $h > 1$ *and a constant* $c > 0$ *such that*

$$\frac{1}{w_2(Q)}\int_Q |u|^{2h} w_2 dx$$

$$\leq c\left(\frac{1}{v(Q)}\int_Q u^2 v dx\right)^{h-1}\left(\frac{r^2}{w_1(Q)}\int_{a^2Q} |\nabla_\lambda u|^2 w_1 dx + \frac{1}{v(Q)}\int_Q u^2 v dx\right)$$

for all $u \in \tilde{H}(a^2Q)$, $Q = Q(\xi, r)$.

Proof: The conclusion of Theorem 2.1 holds for $\mu = 1$. But, by Schwarz's inequality,

$$\begin{aligned} av_Q|u| &= \frac{1}{|Q|}\int_Q |u|\,dx \\ &= \frac{1}{|Q|}\int_Q u v^{1/2} v^{-1/2} dx \leq \frac{1}{|Q|}\left(\int_Q u^2 v dx\right)^{1/2}\left(\int_Q \frac{1}{v} dx\right)^{1/2} \\ &\leq \left(\frac{1}{v(Q)}\int_Q u^2 v dx\right)^{1/2}, \end{aligned}$$

where in the last inequality we used the fact that $\nu \in A_2$.

In the next section we prove mean value inequalities. In order to be able to iterate a certain inequality as was done in [GW2] we need a refinement of the above corollary. This refinement is Theorem D and to prove it we need the following lemmas.

Lemma 2.3. *Given $Q = Q(\xi, s)$ and $0 < r < s$, there exists $x_1, \dots, x_{m(r,s)}$ in Q, and $k \geq 1$ independent of ξ, r, s, such that*

(i) $$Q(x_j, r/k) \bigcap Q(x_h, r/k) = \emptyset, \; h \neq j$$

(ii) $$Q(\xi, s) \subset \cup_{j=1}^{m(r,s)} Q(x_j, r).$$

Moreover, $m(r, s) \leq c \left(\dfrac{s}{r}\right)^{\nu'}$ *for some constant ν' depending only on the dimension.*

PROOF: If we apply Theorem 1.2, page 69, of [CoW] to the open covering of Q given by $(S(x, r/4a)_{x \in Q}$, there exist $x_1, \dots, x_{m(r,s)}$ in Q such that: $S(x_h, r/4a) \bigcap S(x_j, r/4a) = \emptyset$ if $j \neq h$ and $Q(\xi, s) \subset \cup_{j=1}^{m(r,r)} S(x_j, r/a)$. By (1.9), $S(x_j, r/4a) \supset Q(x_j, r/4a^2)$ and $S(x_j, r/a) \subset Q(x_j, r)$. Therefore, if we choose $k = 4a^2$, (i) and (ii) follow. It remains to find an upper bound for $m(r, s)$. First, we note that $Q(x_j, r/k) \subset Q(\xi, a^2(k + 1)s/k)$. But

$$\frac{r}{k} = \frac{2a^4(k + 1)s}{k} \; \frac{r}{2a^4(k + 1)s},$$

and so by (1.10), there exists $\nu' > 0$, such that

$$\left| Q\left(x_j, \frac{r}{k}\right) \right| \geq \left(\frac{r}{2a^4(k + 1)s}\right)^{\nu'} \left| Q\left(x_j, \frac{2a^4(k + 1)s}{k}\right) \right|,$$

and since the $Q(x_j, r/k)$ are disjoint,

$$\left| Q\left(\xi, \frac{a^2(k+1)s}{k}\right) \right| \geq \sum_j \left| Q\left(x_j, \frac{r}{k}\right) \right|$$
$$\geq c\left(\frac{r}{s}\right)^\nu \sum_j \left| Q\left(x_j, \frac{2a^4(k+1)s}{k}\right) \right|.$$

But,

$$Q\left(x_j, \frac{2a^4(k+1)s}{k}\right) \supset Q\left(\xi, \frac{a^2(k+1)s}{k}\right)$$

and so

$$\left| Q\left(\xi, \frac{a^2(k+1)s}{k}\right) \right| \geq c\left(\frac{r}{s}\right)^\nu m(r, s) \left| Q\left(\xi, \frac{a^2(k+1)s}{k}\right) \right|.$$

Therefore, $m(r, s) \leq c\,(s/r)^\nu$.

Lemma 2.4. *If $\delta(y, z) < s$ then $F_j(z^*, s) \leq (2a^2)^{G_j} F_j(y^*, s)$, G_j as in* (1.10).

Proof: Since $Q(z, s) \subset Q(y, 2a^2 s)$, $F_j(z^*, s) \leq F_j(y^*, 2a^2 s)$. By (1.10), it follows that

$$F_j(z^*, s) \leq F_j(y^*, 2a^2 s) \leq (2a^2)^{G_j} F_j(y^*, s).$$

Lemma 2.5. *If $0 < \epsilon < 1$ and $\eta \in Q = Q(\xi, s)$, then $Q(\eta, \epsilon s/(2a^2)^\zeta) \subset Q(\xi, (1+\epsilon)s)$, where $\zeta = \max_{j=1,\dots,n} G_j$.*

Proof: If $y \in Q(\eta, \epsilon s/(2a^2)^\zeta)$ then by (1.8), $|y_j - \eta_j| \leq F_j(\eta^*, \epsilon s/(2a^2)^\zeta)$ and by (1.10) and Lemma 2.4

$$F_j\left(\eta^*, \frac{\epsilon s}{(2a^2)^\zeta}\right) \leq \frac{\epsilon}{(2a^2)^\zeta} F_j(\eta^*, s) \leq \epsilon F_j(\xi^*, s).$$

Therefore

$$\begin{aligned} |y_j - \xi_j| &\leq |y_j - \eta_j| + |\eta_j - \xi_j| \leq \epsilon F_j(\xi^*, s) + F_j(\xi^*, s) \\ &= (1+\epsilon) F_j(\xi^*, s) \\ &\leq F_j(\xi^*, (1+\epsilon)s), \end{aligned}$$

where in the last inequality we used (1.10).

Proof of Theorem D.

Let $Q = Q(\xi, s)$. By Lemma 2.5, $\delta(Q, \partial(1+\epsilon)Q) \geq \epsilon s/(2a^2)^\zeta$. Apply Lemma 2.3 to $r = \dfrac{\epsilon s}{(2a^2)^\zeta a^2}$ to find $x_1, \dots, x_{m(r,s)} \in Q$ such that: $Q(x_j, r/k) \bigcap Q(x_h, r/k) = \varnothing$ if $j \neq h$, $Q(\xi, s) \subset \bigcup_{j=1}^{m(r,s)} Q(x_j, r)$ and $m(r, s) \leq c(s/r)^\nu$.

Note that, by (2.5), $Q(x_j, a^2 r) = Q\left(x_j, \frac{\epsilon s}{(2a^2)^\zeta}\right) \subset Q(\xi, (1+\epsilon)s) = (1+\epsilon)Q$.

Then using Corollary 2.2, doubling for w_2, doubling for v and w_1 and the fact that $Q(x_j, 2a^2 s) \supset Q(\xi, s)$ and $Q(\xi, 2a^2 s) \supset Q(x_j, s)$,

$$\begin{aligned}
\int_Q |u|^{2h} w_2\,dx &\le \sum_{j=1}^{m(r,s)} \int_{Q(x_j,r)} |u|^{2h} w_2\,dx \\
&\le c \sum_{j=1}^{m(r,s)} w_2(Q(x_j, r)) \left(\frac{1}{v(Q(x_j, r)} \int_{Q(x_j,r)} u^2 v\,dx\right)^{h-1} \\
&\quad \cdot \left\{\frac{r^2}{w_1(Q(x_j, r))} \int_{Q(x_j,a^2 r)} |\nabla_\lambda u|^2 w_1\,dx + \frac{1}{v(Q(x_j, r))} \int_{Q(x_j,r)} u^2 v\,dx\right\} \\
&\le c \left(\frac{s}{r}\right)^{\nu'} w_2(Q(\xi, s)) \left[\left(\frac{r}{2a^2 s}\right)^{-\alpha} \frac{1}{v(Q(\xi, s))} \int_{(1+\epsilon)Q} u^2 v\,dx\right]^{h-1} \\
&\quad \cdot \left\{\frac{s^2}{w_1(Q(\xi, s))} \left(\frac{r}{2a^2 s}\right)^{-\alpha} \int_{(1+\epsilon)Q} |\nabla_\lambda u|^2 w_1\,dx \right. \\
&\quad \left. + \left(\frac{r}{2a^2 s}\right)^{-\alpha} \frac{1}{v(Q(\xi, s))} \int_{(1+\epsilon)Q} u^2 v\,dx\right\}.
\end{aligned}$$

The theorem follows if we choose $b = \nu + 2\alpha$, since $s/r = c\epsilon^{-1}$.

3. Mean value inequalities.

In this section we prove Theorem B and some other mean value inequalities. Since the proofs are similar to the ones given by [GW2], we just point out the differences. Basically, we have to be a little more careful in the iteration argument since there is a factor ϵ in Theorem D.

We asume throughout this section that:

(3.1)
- (a) $w_1, w_2, v \in A_2$
- (b) Poincaré's inequality, (1.17), holds for both of the pairs w_1, w_2 and w_1, v with some $q > 2$ and $\mu = 1$
- (c) $w_2 v^{-1} \in A_\infty(v)$.

Denote $R_{r,s} = Q(x_0, r) \times (t_0 - s, t_0 + s)$ and let $R = R_{r,s}$, $R' = R_{\varrho,\sigma}$ with $r/2 < \varrho < r$ and $s/2 < \sigma < s$ and define

$$C = c\,\frac{r^{2+b}s}{(r-\varrho)^{2+b}(s-\sigma)} \tag{3.2}$$

where b is given by Theorem D and c is a constant that may vary, but which only depends on the weights and on h, where $h > 1$ is the index for which Theorem D holds for both w_2 and v on the left hand side.

We also write $\lambda(Q) = w_1(Q)/v(Q)$ and $\Lambda(Q) = w_2(Q)/v(Q)$. We start this section with the proof of (C.1). This estimate will be important in deducing a mean value inequality for subsolutions of (1.1).

PROOF OF (C.1): If $u \in H$ define

$$\varphi(x, t) = \eta^2(x, t)\left[\int_0^{u(x,t)} H_M'(s)^2 ds + u(x, t)H_M'(u(x, t))^2\right]\chi(t, \tau_1, \tau_2),$$

where $\eta \in C_0^\infty(R)$ will be specified later, $t_0 - s < \tau_1 < \tau_2 < t_0 + s$ and $\chi(t, \tau_1, \tau_2)$ denotes the characteristic function of (τ_1, τ_2). The fact that the function φ is in H_0 follows as a consequence of the following result: if f is a piecewise smooth function on the real line with $f' \in L^\infty(-\infty, \infty)$ and if $u \in H$, then $f \circ u \in H$. Here we use the convention that $f'(u) = 0$ if $u \in L$ where L denotes the set of corner points of f (the proof follows the steps of Theorem 7.8 of [GT] and it also shows that $\nabla_\lambda(f \circ u) = f'(u)\nabla_\lambda u$ and $(f(u))_t = f'(u)u_t$). The proof of the above fact also verifies that in our case $\varphi \geq 0$ in the H_0-sense since $H_M(s) = 0$ for $s < 0$.

Since u is a subsolution, we have

$$\int\int_R (\langle A\nabla u, \nabla\varphi\rangle + u_t\varphi v)\,dxdt \leq 0. \tag{3.3}$$

Note that by another limiting argument

$$u_t\left[\eta^2\int_0^u H_M'(s)^2 ds\right] = \left[u\eta^2\int_0^u H_M'(s)^2 ds\right]_t - u(\eta^2)_t\int_0^u H_M'(s)^2 ds - \eta^2 H_M'(u)^2 u_t u,$$

and then by definition of φ, for $\tau_1 < t < \tau_2$,

$$u_t\varphi = \left[u\eta^2\int_0^u H_M'(s)^2 ds\right]_t - (\eta^2)_t u\int_0^u H_M'(s)^2 ds$$

and

$$\nabla \varphi = 2\eta \nabla \eta \left[\int_0^u H_M'(s)^2 ds + uH_M'(u)^2\right] + \eta^2[H_M'(u)^2 \nabla u + f_M'(u) \nabla u],$$

where $f_M(s) = sH_M'(s)^2$ (note that $\nabla (f_M(u)) = f_M'(u) \nabla u$, since f_M is piecewise smooth with $f_M' \in L^\infty$). If we substitute the two last equations in (3.3) we get, with $Q = Q(x_0, r)$,

$$\int_Q \int_{\tau_1}^{\tau_2} \left[u\eta^2 \int_0^u H_M'(s)^2 ds\right]_t vdxdt + \int_Q \int_{\tau_1}^{\tau_2} \eta^2 H_M'(u)^2 \langle A \nabla u, \nabla u\rangle dxdt$$

$$\leq \int_Q \int_{\tau_1}^{\tau_2} \left[(\eta^2)_t u \int_0^u H_M'(s)^2 ds\right] vdxdt$$

$$-2 \int_Q \int_{\tau_1}^{\tau_2} \eta \langle A \nabla u, \nabla \eta\rangle \left[\int_0^u H_M'(s)^2 ds + uH_M'(u)^2\right] dxdt$$

$$- \int_Q \int_{\tau_1}^{\tau_2} \eta^2 \langle A \nabla u, \nabla u\rangle f_M'(u) dxdt.$$

We can drop the last term on the right since the integrand is non-negative. The second term on the right is majorized in absolute value by

$$4 \int_Q \int_{\tau_1}^{\tau_2} |\langle A \nabla u, \nabla \eta\rangle| \, \eta H_M'(u)^2 u dxdt$$

$$= 4 \int_Q \int_{\tau_1}^{\tau_2} |\langle A H_M'(u)\eta \nabla u, uH_M'(u) \nabla \eta\rangle| \, dxdt$$

$$\leq 2\epsilon \int_Q \int_{\tau_1}^{\tau_2} \langle A \nabla (H_M(u)), \nabla (H_M(u))\rangle, \eta^2 dxdt$$

$$+ \frac{2}{\epsilon} \int_Q \int_{\tau_1}^{\tau_2} \langle A \nabla \eta, \nabla \eta\rangle u^2 H_M'(u)^2 dxdt$$

where we used the fact that $|\langle Ax, y\rangle| \leq \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2} \leq \frac{\epsilon}{2} \langle Ax, x\rangle + \frac{1}{2\epsilon} \langle Ay, y\rangle$. If we pick $\epsilon = \frac{1}{4}$ we get

$$(3.4)\quad \int_Q \int_{\tau_1}^{\tau_2} \left[u\eta^2 \int_0^u H_M'(s)^2 ds \right]_t v\,dxdt$$

$$+ \frac{1}{2} \int_Q \int_{\tau_1}^{\tau_2} \eta^2 \langle A \nabla (H_M(u)),\ \nabla (H_M(u)) \rangle dxdt$$

$$\leq 8 \int_Q \int_{\tau_1}^{\tau_2} \langle A \nabla \eta,\ \nabla \eta \rangle u^2 H_M'(u)^2 dxdt + \int_Q \int_{\tau_1}^{\tau_2} \left[(\eta^2)_t u \int_0^u H_M'(s)^2 ds \right] v\,dxdt.$$

Choosen η to be zero in a neighborhood of $\{\partial Q \times (t_0 - s, t_0 + s)\} \cup \{Q \times (t = t_0 - s)\}$, $\eta \equiv 1$ in R'_+, $0 \leq \eta \leq 1$, $|\nabla_\lambda \eta| \leq c/(r - \varrho)$, $|\eta_t| \leq c/(s - \sigma)$ (see page 537 of [FL1]). If we pick τ_1 so close to $t_0 - s$ that $\eta(x, \tau_1) = 0$ for all $x \in Q$, drop the second term on the left of (3.4) (which is non-negative) and use Lemma 5 of [AS] it follows that

$$(3.5)\quad \text{ess sup}_{\tau_2 \in (t_0 - \sigma, t_0 + s)} \int_{Q'} u(x, \tau_2) \int_0^{u(x,\tau_2)} H_M'(s)^2 ds\, v\,dx$$

$$\leq c \int \int_R u^2 H_M'(u)^2 \left[\frac{w_2}{(r - \varrho)^2} + \frac{v}{s - \sigma} \right] dxdt.$$

If we fix $\tau_2 \in (t_0 - \sigma, t_0 + s)$ and τ_1 as before and if we drop the first term on the left of (3.4) (which we can see is non-negative after performing the integration) we obtain

$$(3.6)\quad \int_Q \int_{\tau_1}^{\tau_2} \eta^2 \langle A \nabla (H_M(u)),\ \nabla (H_M(u)) \rangle dxdt$$

$$\leq c \int \int_R u^2 H_M'(u)^2 \left[\frac{w_2}{(r - \varrho)^2} + \frac{v}{s - \sigma} \right] dxdt.$$

Letting $\tau_2 \to t_0 + s$ and using (1.2) we get

$$(3.7)\quad \int \int_{R'_+} |\nabla_\lambda (H_M(u))|^2 w_1 dxdt \leq c \int \int_R u^2 H_M'(u)^2 \left[\frac{w_2}{(r - \varrho)^2} + \frac{v}{s - \sigma} \right] dxdt.$$

Finally note that

$$H_M(u)^2 = \int_0^u (H_M(s)^2)'\,ds = \int_0^u 2H_M(s)H_M'(s)\,ds$$

$$\leq 2\int_0^u sH_M'(s)^2\,ds \leq 2u\int_0^u H_M'(s)^2\,ds,$$

since $H_M(s) \leq sH_M'(s)$. Combining this with (3.5) and (3.7), (C.1) follows with α, β, α', β' taken there to be r, s, ϱ, σ.

Lemma 3.8. *Let $p \geq 2$, R, R' be as defined above and assume* (3.1) *holds. If u is a subsolution of* (1.1) *in R, then u_+ is bounded in $R'_+ = Q(x_0, \varrho) \times (t_0 - \sigma, t_0 + s)$ and*

$$\operatorname{ess\,sup}_{R'_+} u_+^p$$

$$\leq (p^2C)^{\frac{h}{h-1}}\left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q)}\right)^{\frac{1}{h-1}}\left(1 + \frac{s}{r^2}\Lambda(Q)\right)^{\frac{h}{h-1}}\int\int_R u_+^p\left(\frac{s}{r^2}w_2 + v\right)dxdt,$$

with C as in (3.2).

PROOF: $H_M(u)$ is a function in H since $u \in H$ and H_M is a C^1 function with bounded derivative. Then by Fubini's theorem we have that $H_M(u(.,\tau)) \in \tilde{H}$ for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$. If we apply Theorem D to the function $F(x) = H_M(u(x,\tau))$, $Q = Q_\varrho$ and $\epsilon > 0$ such that $(1+\epsilon)\varrho < r$ and combine this with (C.1) we obtain

$$\frac{1}{w_2(Q_\varrho)}\int_{Q_\varrho} H_M(u(x,\tau)^{2h}w_2(x)dx$$

$$\leq c\epsilon^{-b}\left\{\frac{1}{v(Q_\varrho)}\int\int_R u^2H_M'(u)^2\left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right)dxdt\right\}^{h-1}$$

$$\cdot\left\{\frac{\varrho^2}{w_1(Q_\varrho)}\int_{Q_{(1+\epsilon)\varrho}} |\nabla_\lambda(H_M(u(x,\tau)))|^2 w_1(x)dx\right.$$

$$\left. + \frac{1}{v(Q_\varrho)}\int\int_R u^2H_M'(u)^2\left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right)dxdt\right\}$$

for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$.

Integrate with respect to τ over $(t_0 - \sigma,\ t_0 + s)$ and apply (C.1) to get

$$\frac{1}{w_2(Q_\varrho)} \iint_{R'_+} H_M(u(x,\ t))^{2h} w_2(x)dxdt$$

$$\le c\frac{\epsilon^{-b}}{v(Q_\varrho)^{h-1}}\left(\frac{\varrho^2}{w_1(Q_\varrho)} + \frac{s+\sigma}{v(Q_\varrho)}\right)\left(\iint_R u^2 H_M'(u)^2 \frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right) dxdt\Bigg)^h$$

Since $(r/2) < \varrho < r$ and $(s/2) < \sigma < s$, by the doubling property of the weights and the definitions of λ and Λ, it follows that

$$\frac{1}{w_2(Q_r)} \iint_{R'_+} H_M(u(x,\ t))^{2h} w_2(x)dxdt$$

$$\le c\frac{\epsilon^{-b}}{v(Q_r)^h}\left(\frac{r^2}{\lambda(Q_r)} + s\right)\left(\iint_R u^2 H_M'(u)^2 \left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right) dxdt\right)^h.$$

A similar inequality holds with w_2 replaced by v on the left, and if we add the two inequalities, we obtain

$$\iint_{R'_+} H_M(u)^{2h}\left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right)dxdt \tag{3.9}$$

$$\le c\frac{\epsilon^{-b}}{v(Q_r)^h}\left(\frac{r^2}{\lambda(Q_r)} + s\right)\left(\iint_R u^2 H_M'(u)^2 \left(\frac{w_2}{(r-(1+\epsilon)\varrho)^2} + \frac{v}{s-\sigma}\right) dxdt\right)^h$$

for any ϵ such that $(1 + \epsilon)\varrho < r$.

Now note that

$$\frac{w_2}{(r - (1 + \epsilon)\varrho)^2} + \frac{v}{s-\sigma} \le \frac{r^2}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left\{\frac{s}{r^2} w_2 + v\right\},$$

$$\iint_{R'_+}\left\{\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right\} dxdt \approx s,$$

$$\iint_R \left\{\frac{s}{r^2} w_2 + v\right\} dxdt \approx s\left\{\frac{s}{r^2} w_2(Q_r) + v(Q_r)\right\} \approx sv(Q_r)\left\{\frac{s}{r^2}\Lambda(Q_r) + 1\right\},$$

$$\frac{sr^{-2}w_2(x) + v(x)}{sr^{-2}w_2(Q_r) + v(Q_r)} \le \frac{w_2(x)}{w_2(Q_r)} + \frac{v(x)}{v(Q_r)}.$$

Thus, by raising both sides of (3.9) to the power $1/h$, normalizing and using the fact that $\epsilon^{-b/h} \leq \epsilon^{-b}$, we obtain

$$(3.10)\quad \left(\int\int_{R'_+} H_M(h)^{2h} \left(\frac{s}{r^2} w_2 + v\right) dxdt\right)^{1/h}$$

$$\leq c\epsilon^{-b} \frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right)\left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot \int\int_R u^2 H'_M(u)^2 \left(\frac{s}{r^2} w_2 + v\right) dxdt$$

for any ϵ such that $(1+\epsilon)\varrho < r$. Since $u_+^{p/2}\chi_{\{0<u<M\}} \leq H_M(u)$ and $uH'_M(u) \leq pu_+^{p/2}/2$, if we let $M \to \infty$ it follows by Fatou's lemma that

$$(3.11)\quad \left(\int\int_{R'_+} u_+^{ph} \left(\frac{s}{r^2} w_2 + v\right) dxdt\right)^{1/h}$$

$$\leq cp^2\epsilon^{-b} \frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right)\left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt.$$

Now, we have to iterate (3.11). Fix r, s, ϱ, σ with $r/2 < \varrho < r$ and $s/2 < \sigma < s$. For $k = 1, 2, \ldots$ define sequences $\{s_k\}_{k\in\mathbf{N}}$ and $\{r_k\}_{k\in\mathbf{N}}$ and $\{\epsilon_k\}_{k\in\mathbf{N}}$ by $s_1 = s$, $s_k - s_{k+1} = \dfrac{s-\sigma}{2^k}$ for $k \geq 1$, $r_1 = r$, $r_k - r_{k+1} = (r-\varrho)/2^k$ for $k \geq 1$, and $\epsilon_k = \dfrac{r-\varrho}{2^k r_k} = \dfrac{r_k - r_{k+1}}{r_k}$ for $k \geq 1$. Also, define $R_k = Q_k \times (t_0 - s_k, t_0 + s)$ for $k \geq 1$, where $Q_k = Q(x, r_k)$. Note that $R_1 = R$ and $\bigcap_{k=1}^\infty R_k = R'_+$. Since

$$\frac{1}{2} sr^{-2} \leq s_k r_k^{-2} \leq 4sr^{-2},$$

if we apply (3.11) with p replaced by ph^{k-1}, $p \geq 2$, and $r = r_k$, $\varrho = r_{k+1}$ and $\epsilon = \epsilon_{k+1}$ (note that $(1+\epsilon_{k+1})r_{k+1} < r_k$), we obtain

$$\left(\iint_{R_{k+1}} u_+^{ph^k}\left(\frac{s}{r^2}w_2+v\right)dxdt\right)^{1/h^k}$$

$$\leq \left\{c(ph^{k-1})^2\epsilon_{k+1}^{-b}\frac{r_k^2 s_k}{(r_k-(1+\epsilon_{k+1})r_{k+1})^2(s_k-s_{k+1})}\left(1+\frac{s}{r^2}\Lambda(Q_r)\right)\right.$$

$$\left.\cdot\left(1+\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}\right\}^{1/(h^{k-1})}\cdot\left\{\iint_{R_k} u_+^{ph^{k-1}}\left(\frac{s}{r^2}w_2+v\right)dxdt\right\}^{1/(h^{k-1})}.$$

But note that

$$\epsilon_{k+1}^{-b}\frac{r_k^2 s_k}{[r_k-(1+\epsilon_{k+1})r_{k+1}]^2(s_k-s_{k+1})}$$

$$=2^{(k+1)b}\frac{r_{k+1}^b}{(r-\varrho)^b}\frac{r_k^2 s_k}{\left(\frac{r-\varrho}{2^k}-\frac{r-\varrho}{2^{k+1}}\right)^2\left(\frac{s-\sigma}{2^k}\right)}$$

$$\leq c2^{(3+b)k}\frac{r^{2+b}s}{(r-\varrho)^{2+b}(s-\sigma)}$$

$$\leq C2^{(3+b)k},$$

where C is given by (3.2). Thus,

$$\left(\iint_{R_{k+1}} u_+^{ph^k}\left(\frac{s}{r^2}w_2+v\right)dxdt\right)^{1/h^k} \tag{3.12}$$

$$\leq\left\{C(ph^{k-1})^2 2^{(3+b)k}\left(1+\frac{s}{r^2}\Lambda(Q_r)\right)\left(1+\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}\right\}^{1/h^{k-1}}$$

$$\cdot\left\{\iint_{R_k} u_+^{ph^{k-1}}\left(\frac{s}{r^2}w_2+v\right)dxdt\right\}^{1/h^{k-1}}.$$

If we iterate (3.12), we obtain

ess $\sup_{R'_+} u_+^p$

$$\leq \prod_{k=1}^{\infty} \left\{ C(ph^{k-1})^2 2^{(3+b)k} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right) \cdot \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h} \right\}^{1/h^{k-1}} \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt.$$

Since $\sum_{k=1}^{\infty} \frac{1}{h^{k-1}} = \frac{h}{h-1}$ and $\sum_{k=1}^{\infty} \frac{k}{h^{k-1}} = \left(\frac{h}{h-1}\right)^2$, it follows that

ess $\sup_{R'_+} u_+^p$

$$\leq (p^2 C)^{\frac{h}{h-1}} \left(1 + \frac{s}{r^2}\Lambda(Q_r)\right)^{\frac{h}{h-1}} \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{\frac{1}{h-1}} \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt,$$

and this proves the lemma. Note that if we apply the above result for $p = 2$, it follows that u_+ is bounded on R'_+.

PROOF OF THEOREM B: By Lemma 3.8 we know that u_+ is bounded in $Q_{(1+\epsilon)\varrho} \times (t_0 - \sigma, t_0 + s)$ for all ϵ such that $(1 + \epsilon)\varrho < r$. If we define $F(x) = u_+^{p/2}(x, \tau)$ then $F \in \tilde{H}(Q_{(1+\epsilon)\varrho})$ for a.e. $\tau \in (t_0 - \sigma, t_0 + s)$ and if we follow the proof of Lemma 3.8 using (C.2) instead of (C.1), we get (see the comments in the introduction)

ess $\sup_{R'_+} u_+^p$

$$\leq C^{\frac{h}{h-1}} \left(1 + \frac{r^2}{s}\frac{1}{\lambda(Q)}\right)^{\frac{1}{h-1}} \left(1 + \frac{s}{r^2}\Lambda(Q)\right)^{\frac{h}{h-1}} \int\int_R u_+^p \left(\frac{s}{r^2} w_2 + v\right) dxdt$$

for $p \geq 2$. For $0 < p < 2$, define I_p and I_∞ as in Lemma 3.4 of [GW2]. The only difference in our case is that

$$I_\infty(\alpha', \beta')^2 \leq c \left[\frac{1}{(\alpha - \alpha')^{2+b}(\beta - \beta')}\right]^{\frac{h}{h-1}} I_2(\alpha, \beta)^2$$

if $1/2 < \alpha' < \alpha < 1$ and $1/2 < \beta' < \beta < 1$. Thus, arguing as in Lemma 3.4 of [GW2] we prove that if u is a solution of (1.1) and $p > 0$ then

(3.13) $\operatorname{ess\,sup}_{R'_+} u_+^p \leq$

$$D\left(1+\frac{r^2}{s}\frac{1}{\lambda(Q)}\right)^{\frac{1}{h-1}}\left(1+\frac{s}{r^2}\Lambda(Q)\right)^{\frac{h}{h-1}}\int\int_R u_+^p\left(\frac{s}{r^2}w_2+v\right)dxdt,$$

where D is as in Theorem B.

If we apply (3.13) to both u and $-u$, we obtain Theorem B of the introduction, with α, β, α', β' taken there to be r, s, ϱ, σ.

In order to prove Harnack's inequality we need a mean value inequality for u^p when $-\infty < p < \infty$ and u is a non-negative solution.

We begin by noting that if we use (C.3) instead of (C.1) we can prove the following analogue of (3.11):

Lemma 3.14. *Suppose* (3.1) *holds,* $0 < m < u(x,t) \leq M < \infty$ *in* $R = R_{r,s}$, $r/2 < \varrho < r$, $s/2 < \sigma < s$ *and* $\epsilon > 0$, $(1+\epsilon)\varrho < r$. *Then, if* $p > 1$ *and* u *is a subsolution in* R, *or if* $p < 0$ *and* u *is a supersolution in* R,

$$\left(\int\int_{R'_+} u^{ph}\left(\frac{w_2}{w_2(Q_r)}+\frac{v}{v(Q_r)}\right)dxdt\right)^{1/h}$$

$$\leq c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left(1+\frac{p}{p-1}\frac{s}{r^2}\Lambda(Q_r)\right)\left(1+\frac{p}{p-1}\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot\int\int_R u^p\left(\frac{p}{p-1}\frac{s}{r^2}w_2+v\right)dxdt.$$

Moreover, if $0 < p < 1$ *and* u *is a supersolution in* R, *then*

$$\left(\int\int_{R'_-} u^{ph}\left(\frac{w_1}{w_2(Q_r)}+\frac{v}{v(Q_r)}\right)dxdt\right)^{1/h}$$

$$\leq c\epsilon^{-b}\frac{r^2 s}{(r-(1+\epsilon)\varrho)^2(s-\sigma)}\left(1+\frac{p}{|p-1|}\frac{s}{r^2}\Lambda(Q_r)\right)\left(1+\frac{p}{|p-1|}\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{1/h}$$

$$\cdot\int\int_R u^p\left(\frac{p}{|p-1|}\frac{s}{r^2}w_2+v\right)dxdt.$$

Both inequalities are still true if we replace the integral averages on the right by the larger integral average

$$\iint_R u^p \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right) dxdt.$$

Theorem 3.15. *Assume* (3.1) *holds,* $r, s > 0$, $r/2 < \varrho < r$, $s/2 < \sigma < s$. *If u is a non negative solution of* (1.1) *in R, then for* $p > 0$

$$\operatorname{ess\,sup}_{R'} u^p$$

$$\leq C^c \left(1 + p\frac{s}{r^2}\Lambda(Q_r)\right)^{\frac{h}{h-1}} \left(1 + p\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{\frac{1}{h-1}} \iint_R u_+^p \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right) dxdt,$$

and for $p < 0$

$$\operatorname{ess\,sup}_{R'_+} u^p \leq C^{\frac{h}{h-1}} \left(1 + |p|\frac{s}{r^2}\Lambda(Q_r)\right)^{\frac{h}{h-1}}$$

$$\cdot \left(1 + |p|\frac{r^2}{s}\frac{1}{\lambda(Q_r)}\right)^{\frac{h}{h-1}} \iint_R u^p \left(\frac{w_2}{w_2(Q_r)} + \frac{v}{v(Q_r)}\right) dxdt,$$

where C is given by (3.2).

PROOF: In Lemma 3.17 of [GW2] we replace (3.20) by the result given here in Lemma 3.14 and then argue as in Lemma 3.17 of [GW2].

4. Proof of Theorem E

We start with the following lemma.

Lemma 4.1. *Suppose* $Q = Q(\xi, r)$ *and* φ *is a* C^1 *function such that* $\varphi \equiv 1$ *in* $kQ = Q(\xi, kr)$, $0 < k < 1$, $0 \leq \varphi \leq 1$, $\operatorname{supp} \varphi \subset Q$ *and*

$$\varphi(x)\varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y)) \tag{4.2}$$

for all x, y, t, t_0 *with* $0 \leq t \leq t_0$. *If u is a Lipschitz function,*

$E = \{x \in Q(\xi, kr): u(x) = 0\}$ and $|E| \geq \beta |Q|$ for some $0 < \beta < 1$, then if $x \in Q$,

$$|u(x)| \sqrt{\varphi(x)} \leq c \int_Q |\nabla_\lambda u(z)| \sqrt{\varphi(z)} \frac{\delta(x, z)}{|Q(x, \delta(x, z))|} dz, \tag{4.3}$$

where c is independent of Q, u, x.

PROOF: (The general outline of this proof follows the steps of the proof of Lemma 4.3 in [FS].) If $x \in Q = Q(\xi, r)$ then $Q(\xi, r) \subset Q(x, 2a^2 r)$ and $Q(x, r) \subset Q(\xi, 2a^2 r)$. Therefore, by doubling, $|Q(x, r)| \simeq |Q|$. Now, we note that there exists $\sigma \in \{-1, 1\}^n$ such that $|E \cap Q^\sigma(x, 2a^2 r)| \geq c\beta |Q^\sigma(x, 2a^2 r)|$. In fact, $E = \cup_\sigma (Q^\sigma(x, 2a^2 r) \cap E)$ and so there exists σ such that

$$|Q^\sigma(x, 2a^2 r) \cap E| \geq \beta 2^{-n} |Q| \geq c\beta |Q^\sigma(x, 2a^2 r)|. \tag{4.4}$$

We also claim that there exist $\alpha, \epsilon \in \mathbf{R}^n$, independent of x and r, $0 < \epsilon_j < \alpha_j$, $j = 1, \ldots, n$, such that

$$|E \cap Q^\sigma(x, 2a^2 r) \cap H(2a^2 r, x, \Delta_\epsilon^\alpha(\sigma))| \geq \frac{c\beta}{2} |Q^\sigma(x, 2a^2 r)|. \tag{4.5}$$

To prove this fact, apply (1.14) to $\gamma = \frac{c\beta}{2}$ and find $\alpha, \epsilon \in \mathbf{R}^n$, $0 < \epsilon_j < \alpha_j$, $j = 1, \ldots, n$, such that

$$|H(2a^2 r, x, \Delta_\epsilon^\alpha(\sigma)) \cap Q^\sigma(x, 2a^2 r)| \geq \left(1 - \frac{c\beta}{2}\right) |Q^\sigma(x, 2a^2 r)|.$$

Then,

$$\begin{aligned}
&|Q^\sigma(x, 2a^2 r)| \geq |(Q^\sigma(x, 2a^2 r) \cap E) \cup (Q^\sigma(x, 2a^2 r) \cap H(\ldots))| = \\
&|Q^\sigma(x, 2a^2 r) \cap E| + |Q^\sigma(x, 2a^2 r) \cap H(\ldots)| - |E \cap Q^\sigma(x, 2a^2 r) \cap H(\ldots)| \\
&\geq |Q^\sigma(x, 2a^2 r)| \left(c\beta + 1 - \frac{C\beta}{2}\right) - |E \cap Q^\sigma(x, 2a^2 r) \cap H(\ldots)|
\end{aligned}$$

and therefore the claim follows.

We can assume $x \notin E$ and define $\Sigma = \{y \in \Delta_\epsilon^\alpha(\sigma): H(2a^2 r, x, y) \in E\}$. Let K be a smooth function supported in $\Delta_{\epsilon/2}^{2\alpha}(\sigma)$, $0 \leq K \leq 1$, $K \equiv 1$ on $\Delta^\alpha{}_\epsilon(\sigma)$. Suppose $u \in \mathrm{Lip}(Q)$. If $y \in \Sigma$ then

$$|u(x)| \sqrt{\varphi(x)} = |u(x) - u(H(2a^2 r, x, y))| K(y) \sqrt{\varphi(x)},$$

and if we integrate on Σ, we obtain

$$|u(x)|\sqrt{\varphi(x)}\,|\Sigma| = \int_{\Sigma} |u(x) - u(H(2a^2r, x, y))|\, K(y)\sqrt{\varphi(x)}dy.$$

Now we note that $\varphi(H(2a^2r, x, y)) = 1$ if $y \in \Sigma$ and using (4.2) we get $\varphi(x) \leq \varphi(H(t, x, y))$ for any $0 \leq t \leq 2a^2r$. Therefore,

$$|u(x)|\sqrt{\varphi(x)}\,|\Sigma| \leq \int_{\text{supp}K} \left|\int_0^{2a^2r} \frac{d}{dt}(u(H(t, x, y)))dt\right| \sqrt{\varphi(H(t, x, y))}dy$$

$$\leq \int_{\text{supp}K} \left|\int_0^{2a^2r} \langle \nabla u(H(t, x, y)), \dot{H}(t, x, y)\rangle dt\right| \sqrt{\varphi(H(t, x, y))}dy$$

$$\leq \int_0^{2a^2r} \int_{\text{supp}K} |\nabla_\lambda u(H(t, x, y))|\; |y|\; \sqrt{\varphi(H(t, x, y))}dydt.$$

If we make change of variables $z = H(t, x, y)$ in $\Delta^{2\alpha}_{\epsilon/2}(\sigma)$, then

$$\left|\det \frac{\partial z}{\partial y}(t, x, y)\right| = \Pi_{j=1}^n \int_0^t \lambda_j(H(s, x, y))ds.$$

For $y \in \Delta^{2\alpha}_{\epsilon/2}(\sigma)$, the last product is equivalent to $|Q(x, t)|$ by (1.15). Hence

$$(4.6)\quad |u(x)|\sqrt{\varphi(x)} \leq \frac{c}{|\Sigma|}\int_0^{2a^2r} \frac{1}{|Q(x, t)|}\int_{H(t,x,\Delta^{2\alpha}_{\epsilon/2}(\sigma))} |\nabla_\lambda u(z)|\sqrt{\varphi(z)}dzdt.$$

Note that there exists $c > 0$ such that $H(t, x, \Delta^{2\alpha}_{\epsilon/2}(\sigma)) \subset Q(x, ct)$. In fact, if we define $\gamma(s) = H(s/|y|, x, y)$ then

$$\langle \dot{\gamma}(s), \xi\rangle^2 = \left\{\sum_{j=1}^n \lambda_j\left(H\left(\frac{s}{|y|}, x, y\right)\right) y_j\xi_j\right\}^2 \frac{1}{|y|^2}$$

$$\leq \sum_{j=1}^n \lambda_j^2\left(H\left(\frac{s}{|y|}, x, y\right)\right)\xi_j^2$$

$$= \sum_{j=1}^n \lambda_j(\gamma(s))\,\xi_j^2$$

for every $\xi \in R^n$. So, γ is a λ-subunit curve starting from x and attaining $H(t, x, y)$ at the time $s = t|y|$. Therefore by (1.9),

$$\delta(x, H(t, x, y)) \leq ad(x, H(t, x, y)) \leq at|y| \leq ct$$

where $c = 2\alpha a$

Thus, from (4.6), we obtain

$$| u(x) | \sqrt{\varphi(x)} \leq \frac{c}{|\Sigma|} \int_0^{2a^2r} \frac{1}{|Q(x,t)|} \int_{Q(x,ct)} | \nabla_\lambda u(z) | \sqrt{\varphi(z)}\, dzdt$$

and, interchanging the order of integration and using the fact that supp $\varphi \subset Q$ (the argument we are going to present next is due to Chanillo, Sawyer and Wheeden), we get

$$(4.7) \quad | u(x) | \sqrt{\varphi(x)} \leq \frac{c}{|\Sigma|} \int_Q | \nabla_\lambda u(z) | \sqrt{\varphi(z)} \left(\int_{c\delta(x,z)}^{\infty} \frac{dt}{|Q(x,t)|} \right) dz.$$

We claim that $\int_{ch}^{\infty} \frac{dt}{|Q(x,t)|} \leq c \frac{ch}{|Q(x,h)|}$. To prove this we note that, by (1.8),

$$\frac{|Q(x,t)|}{t} = \Pi_{j=2}^n F_j(x^*, t),$$

and consequently by (1.10), there exists $\epsilon > 0$ such that if $t > \tau$ then

$$\frac{|Q(x,t)|}{t} \geq c \left(\frac{t}{\tau}\right)^\epsilon \frac{|Q(x,\tau)|}{\tau}.$$

Hence,

$$\int_{ch}^{\infty} \frac{dt}{|Q(x,t)|} = \int_{ch}^{\infty} \frac{t}{|Q(x,t)|} \frac{dt}{t} \leq \int_{ch}^{\infty} \frac{h}{|Q(x,h)|} \left(\frac{h}{t}\right)^\epsilon \frac{dt}{t} = c \frac{h}{|Q(x,h)|}.$$

Finally, we note that $|\Sigma| \geq c > 0$, with c independent of x, since, by the change of variables $z = H(2a^2r, x, y)$,

$$\begin{aligned} |\Sigma| &= \int_\Sigma dy \simeq \int_{H(2a^2r,x,\Sigma)} \frac{1}{|Q(x, 2a^2r)|} dz \\ &= \frac{|H(2a^2r, x, \Sigma)|}{|Q(x, 2a^2r)|} = \frac{|E \cap H(2a^2r, x, \Delta_\epsilon^\alpha(\sigma))|}{|Q(x, 2a^2r)|} \\ &\geq c\beta \frac{|Q^\sigma(x, 2a^2r)|}{Q(x, 2a^2r)|} \geq c > 0. \end{aligned}$$

The lemma follows by combining the last two last estimates with (4.7).

PROOF OF THEOREM E.

Define $Tf(x) = \int_{\mathbf{R}^n} f(y)K(x, y)dy$, where $K(x, y) = \dfrac{\delta(x, y)}{|Q(x, \delta(x, y))|}$.

Fix S a d-ball. In order to show that for a pair of weights $\tilde{v}$, $\tilde{w}$ we have $\|Tf\|_{L^2(S,\tilde{v})} \leq \|f\|_{L^2(S,\tilde{w})}$ (where $\|f\|_{L^2(S,\tilde{v})} = \left(\int_S f^2\tilde{v}\right)^{1/2}$) for all $f \geq 0$, supp $f \subset S$, according to [SW], we need to verify that the following conditions hold:

(a) there exists $s > 1$ such that

$$\varphi(I)\,|I| \left(\frac{1}{|I|}\int_I \tilde{v}^s dx\right)^{\frac{1}{2s}} \left(\frac{1}{|I|}\int_I \tilde{w}^{-s} dx\right)^{\frac{1}{2s}} \leq c$$

for all d-balls $I \subset 2S$, where $\varphi(I)$ is defined to be

$$\varphi(I) = \sup\left\{K(x, y) : x, y \in I,\ d(x, y) \geq \frac{1}{2}r(I)\right\};$$

(b) there is $\epsilon > 0$ such that

$$\frac{|I|}{|I|} \leq c_\epsilon \frac{\varphi(I)}{\varphi(I')}\left(\frac{r(I')}{r(I)}\right)^\epsilon$$

for all pairs of d-balls $I' \subset I$.

Note that it is convenient to work with d since the results of [SW] hold for pseudo-metrics (a pseudo-metric d is a quasi-metric satisfying $d(x, y) = d(y, x)$ for all $x, y \in \mathbf{R}^n$).

Define $\tilde{v} = \dfrac{v}{v(S)}$ and $\tilde{w} = \dfrac{w_1}{w_1(S)} r(S)^2$. Note that if $x, y \in I$ and $d(x, y) \geq \dfrac{1}{2}r(I)$, then by (1.9)

$$K(x, y) = \frac{\delta(x, y)}{|Q(x, \delta(x, y))|} \leq \frac{2ar(I)}{\left|Q\left(x, \frac{1}{2a}r(I)\right)\right|} \leq c\frac{r(I)}{|Q(x, r(I))|},$$

and since $x \in I$, $|Q(x, r(I))| \simeq |I|$. Therefore,

$$\varphi(I) \leq c\frac{r(I)}{|I|}.$$

So, the expression in (a) is bounded by

$$c\frac{r(I)}{|I|}|I|\left(\frac{1}{|I|}\int_I\left(\frac{v}{v(S)}\right)^s dx\right)^{\frac{1}{2s}}\left(\frac{1}{|I|}\int_I\left(\frac{w_1}{w_1(S)}r(S)^2\right)^{-s}dx\right)^{\frac{1}{2s}}$$

$$\leq c\frac{r(I)}{r(S)}\left(\frac{1}{|I|}\int_I\left(\frac{v}{v(S)}\right)^s dx\right)^{\frac{1}{2s}}\left(\frac{1}{|I|}\int_I\left(\frac{w_1}{w_1(S)}\right)^{-s}dx\right)^{\frac{1}{2s}},$$

which is equivalent to the expression in condition (1.18) (if we use doubling and (1.9)). This proves (a).

To show (b) we note that if $x, y \in I$ and $d(x, y) \geq \frac{1}{2}r(I)$ then

$$K(x, y) \geq \frac{(2a)^{-1}r(I)}{|Q(x, 2ar(I))|} \geq c\frac{r(I)}{|I|}.$$

Thus $\varphi(I) \simeq \frac{r(I)}{|I|}$. Then, if $I' \subset I$, $\frac{\varphi(I)}{\varphi(I')} \simeq \frac{r(I)}{r(I')}\frac{|I'|}{|I|}$ and we obtain (b) with $\epsilon = 1$.

By doubling and (1.9), it follows that

$$\|Tf\|_{L^2(Q,\tilde{v})} \leq c\|f\|_{L^2(Q,\tilde{w})}$$

for all $f \geq 0$, $\operatorname{supp} f \subset Q$, where $\tilde{v} = \frac{v}{v(Q)}$ and $\tilde{w} = \frac{w_1}{w_1(Q)}r(Q)^2$.

Suppose u is a Lipschitz function in Q and $|E| = |\{x \in Q(\xi, kr): u(x) = 0\}| \geq \beta|Q|$, $1/2 < k < 1$. If we combine Lemma 4.1 and the fact that $\|Tf\|_{L^2(Q,\tilde{v})} \leq c\|f\|_{L^2(Q,\tilde{w})}$ we obtain

$$(4.8)\qquad \left(\frac{1}{v(Q)}\int_Q |u(x)|^2\varphi(x)v(x)dx\right)^{1/2} \leq cr(Q)\left(\frac{1}{w_1(Q)}\int_Q |\nabla_\lambda u(z)|^2\varphi(z)w_1(z)dz\right)^{1/2}$$

Given Q and a general Lipschitz function u, there is a number $\mu = \mu(u, Q)$, the media of u in Q, such that if $Q^+ = \{x \in Q: u(x) \geq \mu\}$ and $Q^- = \{x \in Q: u(x) \leq \mu\}$ then $|Q^+| \geq |Q|/2$ and $|Q^-| \geq |Q|/2$. Hence, $u_1 = \max\{u - \mu(u, kQ), 0\}$ and $u_2 = \max\{\mu(u, kQ) - u, 0\}$ satisfy the hypohesis of Lemma (4.1) for some β depending on k and so if we apply (4.8) to u_1 and u_2 and add both inequalities, we get

$$\int_Q |u(x) - \mu|^2 \varphi(x)v(x)dx \leq cr(Q)^2 \frac{v(Q)}{w_1(Q)} \int_Q |\nabla_\lambda u(z)^2 \varphi(z) w_1(z)dz. \tag{4.9}$$

Finally, it is easy to see that in (4.9) μ can be replaced by the average A_Q of u defined in Theorem E. In fact,

$$\begin{aligned} \int_Q |u(x) - A_Q|^2 \varphi(x)v(x)dx &\leq 2\int_Q |u(x) - \mu|^2 \varphi(x)v(x)dx \\ &+ 2\int_Q |\mu - A_Q|^2 \varphi(x)v(x)dx, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \int_Q |\mu - A_Q|^2 \varphi(x)v(x)dx &= (\varphi v)(Q)\,|\mu - A_Q|^2 \\ &= (\varphi v)(Q) \left| \mu - \frac{1}{\varphi(Q)} \int_Q u(x)\varphi(x)dx \right|^2 \\ &\leq (\varphi v)(Q) \left(\frac{1}{\varphi(Q)} \int_Q |u(x) - \mu|\, \varphi(x)\,dx \right)^2 \\ &\leq \frac{(\varphi v)(Q)}{(\varphi(Q))^2} \int_Q |u(x) - \mu|^2 \varphi^2(x)v(x)dx \int_Q \frac{1}{v(x)}\,dx, \end{aligned}$$

where in the last inequality we used Schwarz's inequality. Since $v \in A_2$ and $0 \leq \varphi \leq 1$, it follows from (4.9) and (4.10) that

$$\begin{aligned} &\int_Q |u(x) - A_Q|^2 \varphi(x)v(x)dx \\ &\qquad \leq cr(Q)^2 \left[1 + \left(\frac{|Q|}{\varphi(Q)} \right)^2 \right] \frac{v(Q)}{w_1(Q)} \int_Q |\nabla_\lambda u(z)|^2 \varphi(z) w_1(z)dz. \end{aligned}$$

This finishes the proof of Theorem E if we note that $\varphi(Q) \simeq |Q|$ since $1/2 \leq k \leq 1$.

The next corollary is also helpful.

Corollary 4.11. *Theorem E is also true with* $A_Q = \frac{1}{(\varphi\nu)(Q)} \int_Q u\varphi\nu dx.$

Just note that

$$\int_Q |\mu - A_Q|^2 \varphi\nu dx = (\varphi\nu)(Q) |\mu - A_Q|^2$$

$$\leq (\varphi\nu)(Q) \left(\frac{1}{(\varphi\nu)(Q)} \int_Q |\mu - u| \varphi\nu dx \right)^2$$

$$\leq \int_Q |\mu - u|^2 \varphi\nu dx,$$

where the last inequality follows by Schwarz's inequality.

5. Harnack's inequality

The proof of Theorem A follows as an application of Bombieri's lemma which we state next. For its proof see Section 5 of [GW2].

Lemma 5.1. *Let $R(\varrho)$ be a one parameter family of rectangles in $\mathbf{R}^{n+1}$, $R(\sigma) \subset R(\varrho)$, $1/2 \leq \sigma \leq \varrho \leq 1$ and let ν be a doubling measure in $\mathbf{R}^{n+1}$. Let A, μ, M, m, θ and $\varkappa$ be positive constants such that $M \geq 1/\mu$ and suppose that f is a positive measurable function defined in a neighborhood of $R(1)$ satisfying*

$$\text{ess sup}_{R(\sigma)} f^p \leq \frac{A}{(\varrho - \sigma)^m} \iint_{R(\varrho)} f^p \nu(x) dxdt \tag{5.2}$$

for all σ, ϱ, p, $1/2 \leq \theta \leq \sigma < \varrho < 1$, $0 < p < M$ and

$$\nu(\{(x, t) \in R(1) : \log f > s\}) \leq \left(\frac{\mu}{s}\right)^\varkappa \nu(R(1)) \tag{5.3}$$

for all $s > 0$. *Then there is a constant* $\gamma = \gamma(A, m, \varkappa) > 0$ *such that*

$$\log(\operatorname{ess\,sup}_{R(\theta)} u) \leq \frac{\gamma}{(1-\theta)^{2m}} \mu.$$

Hence, in order to prove Theorem A, we need a mean value inequality (that we proved in Section 3) and a logarithm estimate which is given by Theorem F (some steps of its proof we will present in this section). The next lemma shows that the test function described on page 537 of [FL1] satisfies the conditions of Theorem E. Then, as we said before, the proof of Theorem F follows as Lemma 4.9 of [GW2].

Lemma 5.4. *Given* $Q = Q(\xi, r)$ *and* $0 < k < 1$, *there exists* $\varphi \in C^1$ *such that* $\varphi \equiv 1$ *in* kQ, $0 \leq \varphi \leq 1$, $\operatorname{supp}\varphi \subset Q$, $|\nabla_\lambda \varphi| \leq \dfrac{c}{r(1-k)}$ *and* $\varphi(x) \cdot \varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y))$ *for all* x, y, t, t_0 *with* $0 \leq t \leq t_0$.

PROOF: Consider the function φ given by [FL1], page 537:

$$\varphi(x) = \prod_{j=1}^{n} \psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right),$$

where $\psi \in C^\infty(\mathbf{R})$, $0 \leq \psi \leq 1$, $\psi(t) = \psi(-t)$, $\psi \equiv 1$ on $[-k, k]$, $\psi = 0$ outside $]-1, 1[$, $|\psi'(t)| \leq 2(1-k)^{-1}$, for all $t \in \mathbf{R}$. Here, we show that φ satisfies the last condition since all the others are proved in [FL1], page 537.

Fix t, $0 < t < t_0$, x and y. Define $z = H(t, x, y)$. Then,

$$z_j = x_j + y_j \int_0^t \lambda_j(H(s, x, y))ds.$$

Suppose $z_j - \xi_j \geq 0$. If $y_j \geq 0$ then

$$|z_j - \xi_j| \leq x_j - \xi_j + y_j \int_0^{t_0} \lambda_j(H(s, x, y))ds = H_j(t_0, x, y) - \xi_j.$$

On the other hand, if $y_j < 0$,

$$|z_j - \xi_j| \leq |x_j - \xi_j|.$$

Thus, if $z_j - \xi_j \geq 0$ then $|z_j - \xi_j| \leq |H_j(t_0, x, y) - \xi_j|$ or $|z_j - \xi_j| \leq |x_j - \xi_j|$. The same holds if $z_j - \xi_j < 0$. Since $\psi(t)$ can be chosen to be non-increasing for positive t, then $\varphi(z) \geq a_1 \ldots a_n$, where

$$a_j = \psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right)$$

or

$$a_j = \psi\left(\frac{|H_j(t_o, x, y) - \xi_j|}{F_j(\xi^*, r)}\right).$$

Since $0 \le \psi \le 1$,

$$a_j \ge \psi\left(\frac{|H_j(t_0, x, y) - \xi_j|}{F_j(\xi^*, r)}\right)\psi\left(\frac{|x_j - \xi_j|}{F_j(\xi^*, r)}\right)$$

for $1 \le j \le n$. Therefore,

$$\varphi(z) \ge \varphi(x)\varphi(H(t_0, x, y)).$$

The next three lemmas are needed in order to show that the hypothesis in Theorem A imply those in Theorems D and E.

Lemma 5.5. *Assume that Poincaré's inequality holds for* w_1, w_2 *with* $q = 2$ *and* $\mu = 1$. *Then*

$$\left(\frac{r(I)}{r(B)}\right)^2 \frac{w_2(I)}{w_2(B)} \le c\,\frac{w_1(I)}{w_1(B)}$$

for any pair of δ*-balls* I, B, *with* $I \subset 2B$.

PROOF: Suppose $I = Q(u_0, r(I))$ and $B = Q(x, r(B))$ and define

$$F(u) = \sum_{j=1}^{n} \frac{|u_j - (u_0)_j|}{F_j(u_0^*, r(I))}\, r(I)\varphi(u)$$

where φ is the function described in lemma (5.4) associated with I (as opposed to B) and $k = 1/2$. If $u \in I$, by (1.8)

$$\left|\frac{\partial F}{\partial u_k}(u)\right| \le \frac{r(I)}{F_k(u_0^*, r(I))} + \frac{\partial \varphi}{\partial u_k}(u)nr(I),$$

for $k \in \{1, \dots n\}$, and using the fact that $\lambda_k(u) = \lambda_k(u^*) \le \lambda_k(H(u^*, r(I)))$ if $u \in I$ we get

$$\left|\lambda_k(u)\frac{\partial F}{\partial u_k}(u)\right| \le \frac{F_k(u^*, r(I))}{F_k(u_0^*, r(I))} + nr(I)\lambda_k(u)\frac{\partial \varphi}{\partial u_k}(u)$$

and by Lemma 2.4 and the fact that $|\nabla_\lambda \varphi| \leq c/r(I)$ we have $|\nabla_\lambda F(u)| \leq c\chi_I$.

We have Poincaré's inequality for F, *i.e.*,

$$\left(\frac{1}{w_2(B)}\int_{n4^{\eta+1}B} |F(u) - av_{n4^{\eta+1}B}F|^2 w_2(u)du\right)^{1/2} \leq cr(B)\left(\frac{1}{w_1(B)}\int_{na^2 4^{\eta+1}B} |\nabla_\lambda F(u)|^2 w_1(u)du\right)^{1/2}, \tag{5.6}$$

where $\eta = \max_{j=1,\ldots,n}\{G_j\}$. The right side of (5.6) is bounded by $cr(B)\left(\frac{w_1(I)}{w_1(B)}\right)^{1/2}$ by doubling and the fact that $|\nabla_\lambda F| \leq c\chi_I$. Now, if $u \notin \frac{1}{4}I$ there exists $k \in \{1, \ldots, n\}$ such that

$$|u_k - (u_0)_k| \geq F_k\left(u_0^*, \frac{1}{4}r(I)\right)$$

and then if $u \in \frac{1}{2}I \setminus \frac{1}{4}I$ (note that $\varphi(u) = 1$)

$$F(u) \geq \frac{F_k\left(u_0^*, \frac{1}{4}r(I)\right)}{F_k(u_0^*, r(I))} r(I) \geq \left(\frac{1}{4}\right)^{G_k} r(I) \geq \frac{1}{4^\eta} r(I). \tag{5.7}$$

Also, if $u \in I$, $F(u) \leq nr(I)$ and therefore

$$av_{n4^{\eta+1}B}F \leq \frac{|I|}{|n4^{\eta+1}B|} nr(I).$$

But, by (1.10), $F_j(x_B^*, n4^{\eta+1}r(B)) \geq 2n4^\eta F_j(x_B^*, 2r(B))$, and by (1.11),

$$|n4^{\eta+1}B| \geq (2n4^\eta)^n |2B| \geq 2n4^\eta |2B|.$$

Hence, since $I \subset 2B$, $av_{n4^{\eta+1}B}F \leq r(I)/2 \cdot 4^\eta$ and then if $u \in \frac{1}{2}I \setminus \frac{1}{4}I$ (using also 5.7)),

$$|F(u) - av_{n4^{\eta+1}B}F| \geq cr(I).$$

Therefore, the left hand side of (5.6) is larger than a constant times

$$\left[\frac{(r(I))^2}{w_2(B)}\, w_2\left(\frac{1}{2} I \setminus \frac{1}{4} I\right)\right]^{1/2} \geq cr(I)\left(\frac{w_2(I)}{w_2(B)}\right)^{1/2},$$

where in the last inequality we used the fact that $w_2\left(\frac{1}{2} I \setminus \frac{1}{4} I\right) \simeq w_2(I)$,

which is shown in the next lemma.

Lemma 5.8. *If w is a doubling weight then $W(Q(u, 2s) \setminus Q(u, s))$ is equivalent to $w(Q(u, s))$.*

PROOF: Choose $\eta \in Q(u, 2s)$ such that $\delta(u, \eta) = \frac{3s}{2}$. By Lemma 2.5,

$$Q\left(\eta, \frac{3\epsilon s}{2(2a^2)^\zeta}\right) \subset Q\left(u, (1 + \epsilon)\frac{3s}{2}\right)$$

for any $0 < \epsilon < 1$.

Choose j such that $\delta(u, \eta) = \varphi_j(u^*, |\eta_j - u_j|)$. Then, if $y \in Q\left(\eta, \frac{3\epsilon s}{2(2a^2)^\zeta}\right)$,

$$\begin{aligned} F_j\left(u^*, \frac{3s}{2}\right) &= |\eta_j - u_j| \leq |\eta_j - y_j| + |y_j - u_j| \\ &\leq F_j\left(\eta^*, \frac{3\epsilon s}{2(2a^2)^\zeta}\right) + |y_j - u_j|, \end{aligned}$$

By (1.10) and Lemma 2.4,

$$F_j\left(u^*, \frac{3s}{2}\right) \leq \epsilon F_j\left(u^*, \frac{3s}{2}\right) + |y_j - u_j|.$$

Thus,

$$|y_j - u_j| \geq (1 - \epsilon) F_j\left(u^*, \frac{3s}{2}\right) \geq F_j\left(u^*, (1 - \epsilon)\frac{3s}{2}\right).$$

If we choose $\epsilon = 1/3$ we have proved that

$$Q\left(\eta, \frac{s}{2(2a^2)^\zeta}\right) \subset Q(u, 2s) \setminus Q(u, s).$$

The lemma follows by doubling.

Lemma 5.9. *If $w_1 \in A_2$, $v \in A_\infty$ and Poincaré's inequality holds for w_1, v with $q = 2$ and $\mu = 1$, then condition* (1.21) *holds.*

PROOF: If $v \in A_\infty$ there exists $s > 1$ such that

$$\left(\frac{1}{|I|}\int_I \left(\frac{v}{v(B)}\right)^s dx\right)^{1/s} \le \frac{1}{|I|}\frac{v(I)}{v(B)}.$$

So, since Poincaré's inequality holds for w_1, v with $q = 2$, by Lemma 5.5

$$\left(\frac{r(I)}{r(B)}\right)^2 \left(\frac{1}{|I|}\int_I \left(\frac{v}{v(B)}\right)^s dx\right)^{1/s} \le c\,\frac{1}{|I|}\,\frac{w_1(I)}{w_1(B)},$$

and the above condition is equivalent to condition (1.18) since $w_1 \in A_2$.

Now we are ready to prove Theorem A.

PROOF OF THEOREM A

Let u be a non-negative solution of (1.1) in the cylinder $R_{\alpha,\beta} = R_{\alpha,\beta}(x_0, t_0) = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta)$. If we define $T(x, t) = (x, \beta t + t_0)$ and $\bar{u}(x, t) = u(T(x, t))$ then u is a solution in $R_{\alpha,1}(x_0, 0)$ of the equation

$$v(x)\bar{u}_t = \operatorname{div}(\bar{A}(x, t)\,\nabla \bar{u}),$$

where the coefficients matrix $\bar{A} = (\bar{a}_{ij})$ are defined by $\bar{a}_{ij}(x, t) = \beta a_{ij}(x, \beta t + t_0)$ and satisfies the degeneracy condition

$$\bar{w}_1(x)\sum_{j=1}^{n} \lambda_j^2(x)\xi_j^2 \le \sum_{j=1}^{n} \bar{a}_{ij}(x, t)\,\xi_i\xi_j \le \bar{w}_2(x)\sum_{j=1}^{n} \lambda_j^2(x)\xi_j^2,$$

if we put $\bar{w}_i = \beta w_i$, for $i = 1, 2$.

Suppose $|p| < [\alpha^{-2}\bar{\Lambda}(Q(x_0, \alpha)) + \alpha^2/\bar{\lambda}(Q(x_0, \alpha))]^{-1}$, where $\bar{\Lambda}(Q) = \bar{w}_2(Q)/v(Q)$, $\bar{\lambda}(Q) = \bar{w}_1(Q)/v(Q)$. Write

$$R^-(\varrho) = Q\left(x_0, \frac{(\varrho + 1)\alpha}{3}\right) \times \left(-\frac{1}{2} - \frac{\varrho}{2}, -\frac{1}{2} + \frac{\varrho}{2}\right)$$

$$R^+(\varrho) = Q\left(x_0, \frac{(\varrho + 1)\alpha}{3}\right) \times \left(\frac{1}{2} - \frac{\varrho}{2}, 1\right)$$

If we take $1/2 < \varrho < r \leq 1$ then the mean value inequalities in Theorem 3.15 applied to u give

$$\text{(5.10)}\quad \operatorname{ess\,sup}_{R^-(\varrho)} \bar{u}^p \leq c \frac{1}{(r-\varrho)^m} \iint_{R^-(r)} \bar{u}^p \left(\frac{\overline{w}_2}{\overline{w}_2(Q_\alpha)} + \frac{v}{v(Q_\alpha)} \right) dxdt,$$

for some $m > 0$, if $p > 0$, where $Q_\alpha = Q(x_0, \alpha)$, and

$$\text{(5.11)}\quad \operatorname{ess\,sup}_{R^+(\varrho)} \bar{u}^p \leq c \frac{1}{(r-\varrho)^m} \iint_{R^+(r)} \bar{u}^p \left(\frac{\overline{w}_2}{\overline{w}_2(Q_\alpha)} + \frac{v}{v(Q_\alpha)} \right) dxdt,$$

if $p < 0$. Moreover, by Theorem B, $\bar{u}$ is locally bounded and by adding $\epsilon > 0$, we may assume by letting $\epsilon \to 0$ at the end of the proof that $\bar{u}$ is bounded below in $R_{\alpha,1}(x_0, 0)$ by a positive constant.

Now, by Theorem F, we have

$$\text{(5.12)}\quad \left[\left(\frac{v}{v(Q_\alpha)} + \frac{\overline{w}}{\overline{w}_2(Q_\alpha)} \right) \otimes 1 \right] (E^+)$$
$$\leq \left\{ \frac{1}{s} \frac{v(Q_\alpha)}{\overline{w}_1(Q_\alpha)} \alpha^2 \right\}^\varkappa$$
$$\leq c \left\{ \frac{1}{s} \left[\alpha^{-2} \overline{\Lambda}(Q_\alpha) + \alpha^2 \frac{1}{\overline{\lambda}(Q_\alpha)} \right] \right\}^\varkappa,$$

and the same inequality holds for E^-, where E^+, E^- are defined in Theorem F with $u = \bar{u}$, $R = 2/3\alpha$, $a = -1$, $b = 1$, $t_0 = 0$, $M_2 \simeq \overline{\Lambda}(Q_\alpha)/\alpha^2$.

By (5.10) and (5.12), we can apply Bombieri's lemma to the family of rectangles $R^-(\varrho)$ with $\mu = \alpha^{-2}\overline{\Lambda}(Q_\alpha(x_0)) + \alpha^2/\overline{\lambda}(Q_\alpha(x_0))$, $M = 1/\mu$ and $f = e^{-M_2 + V(0)}\bar{u}$, obtaining

$$\operatorname{ess\,sup}_{R^-(1/2)} f \leq C \exp\{ c[\alpha^{-2}\overline{\Lambda}(Q_\alpha) + \alpha^2/\overline{\lambda}(Q_\alpha)]\},$$

and this implies that

$$\text{(5.13)}\quad \operatorname{ess\,sup}_{R^-(1/2)} \bar{u} \leq C \exp\{c[\alpha^{-2}\overline{\Lambda}(Q(x_0,\alpha)) + \alpha^2/\overline{\lambda}(Q(x_0,\alpha))] - V(0)\}.$$

Also, by (5.11) and (5.12), we can apply Bombieri's lemma to the family of rectangles $R^+(\varrho)$, $f = e^{-M_2 - V(0)}\bar{u}^{-1}$, with μ, M, M_2 and $V(0)$ as before, and we obtain

$$\text{ess sup}_{R^+(1/2)} f \leq C \exp\{c[\alpha^{-2}\overline{\Lambda}(Q_\alpha) + \alpha^2/\overline{\lambda}(Q_\alpha)]\},$$

which implies that

$$e^{-V(0)} \leq C\, e^{c[\alpha^{-2}\overline{\Lambda}(Q(x_0,\alpha)) + \alpha^2/\overline{\lambda}(Q(x_0,\alpha))]} \text{ess inf}_{R^+(1/2)} \overline{u}. \tag{5.14}$$

Combining (5.13) and (5.14) it follows that

$$\text{ess sup}_{R^-(1/2)} \overline{u} \leq c_1 e^{c[\alpha^{-2}\overline{\Lambda}(Q(x_0,\alpha)) + \alpha^2/\overline{\lambda}(Q(x_0,\alpha))]} \text{ess inf}_{R^+(1/2)} \overline{u}.$$

Since, $T(R^-(1/2)) = R^-$, $T(R^+(1/2)) = R^+$ and $\alpha^{-2}\overline{\Lambda}(Q_\alpha) + \alpha^2/\overline{\lambda}(Q_\alpha) = \alpha^{-2}\beta\Lambda(Q_\alpha) + \alpha^2\beta^{-1}/\lambda(Q_\alpha)$, Theorem A follows.

REMARK: Using the equivalence between d and δ we can prove the following analogues of Theorem A and B for the metric d.

Theorem A′: *Assume* (i), (ii), (iii) *of Theorem* A. *If u is a non-negative solution of* (1.1) *in the cylinder* $R = S(x_0, \alpha a^2) \times (t_0 - \beta, t_0 + \beta)$, *then*

$$\text{ess sup}_{R^-} u \leq c_1 \exp\{c_2[\alpha^{-2}\beta \Lambda (S(x_0, \alpha)) + \alpha^2\beta^{-1}\lambda(S(x_0, \alpha))^{-1}]\} \text{ ess inf}_{R^+} u$$

where $R^- = S(x_0, \alpha/2) \times (t_0 - 3\beta/4, t_0 - \beta/4)$, $R^+ = S(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta)$, $\Lambda(S) = w_2(S)/v(S)$ *and* $\lambda(S) = w_1(S)/v(S)$ *for a d-ball S. Here the constants* c_1, c_2 *depend only on the constants which arise in* (i), (ii), (iii).

Theorem B′: *Assume hypothesis* (i), (ii), (iii) *of Theorem* A *hold. Let* $0 < p < \infty$, $\alpha, \beta > 0$, $\alpha/2 < \alpha' < \alpha$, $\beta/2 < \beta' < \beta$ *and let* $S(x_0, \alpha) = S$, $S(x_0, \alpha') = S'$ *and* $R(\alpha, \beta) = S \times (t_0 - \beta, t_0 + \beta)$, $R'_+(\alpha, \beta) = S' \times (t_0 - \beta', t_0 + \beta)$. *If u is a solution of* (1.1) *in* $R(a^2\alpha, \beta)$, *then u is bounded in* $R'_+(\alpha, \beta)$ *and*

$$\text{ess sup}_{R'_+(\alpha,\beta)} |u|^p \leq$$

$$D(\alpha^2\beta^{-1}\lambda(S)^{-1} + 1)^{1/(h-1)}(\alpha^{-2}\beta\Lambda(S) + 1)^{h/(h-1)} \int\int_{R(a^2\alpha,\beta)} |u|^p(\alpha^{-2}\beta w_2 + v)\,dx\,dt$$

where D is as in Theorem B, and $C = c\dfrac{\alpha^{2+b}\beta}{(\alpha - \alpha')^{2+b}(\beta - \beta')}$. *Here* $h > 1$, *constants which are independent of* u, p, α, α', β, β'.

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Analytic Dependence of Orthogonal Polynomials

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Introduction

One way we can define and construct orthogonal polynomials is by applying the Gram-Schmidt orthogonalization process to the sequence of powers $1, x, x^2, \ldots$ More precisely, we embed these power-functions in $L^2(d\mu)$, where $d\mu$ is a probability measure living on some space (usually the closed interval $[-1, 1]$, the real line **R**, or the unit circle **T**), and then we orthogonalize with respect to the scalar product induced by the measure $d\mu$. In the classical theory, $d\mu$ is usually absolutely continuous with respect to the Lebesgue Measure, and given by some very explicit weight function, but recently, in [CM], R. R. Coifman and M. Murray have proposed a different analytic approach to the study of general orthogonal polynomials, based on a «perturbation theory» of the orthogonalization process.

The approach of R. R. Coifman and M. Murray allows one to study orthogonal polynomials within the framework of non-linear Fourier Analysis. As in similar studies of other non-linear problems (dependence of the Riemann mapping on the domain, shape of water waves, Navier-Stokes equation) the main result consists in showing that some relevant operator quantities are analytically dependent on a B.M.O. parameter. In this kind of analysis, the theory of weighted norm inequalities, A_p weights, and the relationships between good weights and BMO, play a central role (an excellent source of reference for this is [GCRF]).

This particular problem belongs also to the context of Toda flows (see [DLT]) where a certain non-linear matrix differential equation admits closed form solutions. These solutions are a «flow» of infinite tridiagonal matrices that can be interpreted as the three-term recursion relation attached to the orthogonal polynomials.

In [CM] the program is carried out in detail starting from two situations: polynomials on the circle **T** orthogonal with respect to Lebesgue measure, and polynomials on $[-1, 1]$ orthogonal with respect to a Jacobi weight. In this work we carry out the same program starting from a set of polynomials on **T** first introduced by Szegö, in fact used by him as a tool for the proof of his equiconvergence theorems.

We then show some new results. Using the closed subspace VMO, we give a better characterization of the BMO region where the analyticity result holds. This can be seen as the definition of a very general set of polynomials that can be «analytically perturbed». The relationship with the theory of Toda flows is exploited to derive new identities that can be used interchangeably with the Kerzman-Stein formula, shedding a new light on the connections among the different operators involved. The flow itself is expressed in terms of conjugation by a self-adjoint projection Q whose dependence from the parameter gives rise to interesting formulae. In fact, starting from Chebychev polynomials, the first Gateaux differential of Q can be expressed via Hilbert transforms and a bilinear singular integral operator first considered by Calderón (see Section 7). The L^2-boundedness of this operator is still an open problem.

We do not deal with numerical applications, but let us note that this theory can provide some useful computer tools. The Kerzman-Stein formula can be used to efficiently compute orthogonal polynomials relative to standard and non-standard weights. The analyticity results suggest the possibility of building fast algorithms (complexity $n \log n$) to convert an orthogonal expansion relative to a fixed set of polynomials into the expansion relative to another set of polynomials, provided they are within the «analyticity range» (see Theorem 1 in Section 6, and se also [AR] for an example of numerical conversion from Chebychev polynomials expansions into Legendre polynomials expansions).

2. Orthogonal Polynomials Dependent on a BMO Parameter

Consider a «perturbed» space $L^2(u^2 d\mu)$ where $u(x) = e^{b(x)}$ is a suitable function. Notice that if $b(x)$ is close to zero (in some Banach Space norm which will be chosen later), then the new measure

$$u^2(x)\,d\mu(x) = e^{2b(x)}\,d\mu(x) \tag{1}$$

is «close» to the original one $d\mu$.

Now, given $f \in L^2(u^2 d\mu)$, we can develop it into a series of orthogonal polynomials (the polynomials which are orthogonal with respect to $u^2 d\mu$) and consider the partial sums $S_n f$ of this series. Notice that, for a fixed measure $u^2 d\mu$ living on a fixed space, the orthogonal polynomials are uniquely determined (up to a normalization), and so is S_n. Notice also that the partial sums operator S_n can be thought of as an operator-valued function of b. In fact, we want to study the mapping

$$b \to S_n(b). \tag{2}$$

It is hard to study (2) directly because $S_n(b)$, in spite of being a well-defined projection from $L^2(u^2 d\mu)$ into polynomials of degree equal or less than n, acts on different L^2 spaces as b varies. On the other hand, we can look at

$$b \to V_n(b) \tag{3}$$

where

$$V_n(b) = uS_n(b)u^{-1} = e^b S_n(b) e^{-b}. \tag{4}$$

With a little abuse of notation, in the above definition we use a function to denote the operator of multiplication by that function. It is easy to check that $V_n(b)$ always acts on $L^2(d\mu)$, that the $L^2(d\mu)$-boundedness of $V_n(b)$ is equivalent to the $L^2(u^2 d\mu)$-boundedness of $S_n(b)$, and that the two operator norms are equal.

The mapping (3) can now be seen in the context of calculus on Banach Spaces, see for example [B], where notions like continuity, differentiability, and analyticity and well defined.

We claim that the «natural» Banach Space for b is BMO (Bounded Mean Oscillation) and that in «many» situations, depending on the choice of the measure space we start with, the dependence (3) is in fact analytic. We will make this claim precise later, and we will prove it.

Let us notice at this point that a further simplification in the study of this problem is brought about by a remarkable formula due to Kerzman and Stein [KS]. The formula is

$$V = P(I + (P - P^*))^{-1} \tag{5}$$

where V is a self-adjoint projection sending a Hilbert Space H into a proper closed subspace K and P is an oblique projection (non self-adjoint) also sending H into K.

The first step in the proof of (5) is to show that the operator $I + (P - P^*)$

is invertible, but in fact, since $(P - P^*)^* = P^* - P = -(P - P^*)$, we see that $P - P^*$ is skew adjoint, which means that its spectrum is purely imaginary. In particular $-1 \notin \text{spec}(P - P^*)$, which shows the invertibility of $I + (P - P^*)$. We can now rewrite (5) as

$$V(I + (P - P^*)) = P$$

and this is the same as

$$P = V + VP - VP^* = V + P - VP^*,$$

which simplifies into

$$VP^* = V.$$

Finally, we prove this last operator equality using an elementary lemma from Functional Analysis (see for example [R] p. 296) and the following chain of scalar-product equalities, which hold for any element $h \in H$:

$$(VP^*h, h) = (P^*h, Vh) = (h, PVh) = (h, Vh) = (Vh, h).$$

Having proven (5), we now apply it to our situation by choosing

$$V = V_n(b) = e^b S_n(b) e^{-b}$$

and

$$P = P_n(b) = e^b S_n(0) e^{-b}. \tag{6}$$

Notice that (with the same abuse of notation) we have used e^b to indicate the operator of multiplication by e^b. Both $V_n(b)$ and $P_n(b)$ are projection operators from $L^2(d\mu)$ into the same closed subspace; $V_n(b)$ is self-adjoint while $P_n(b)$ is an oblique projection that satisfies

$$P_n(b)^* = P_n(-b). \tag{7}$$

The Kerzman-Stein formula (5) tells us that the analytic properties of $P_n(b)$ as an operator-valued function of b are inherited by $V_n(b)$. This is a great simplification because it is easier to deal explicitly with the integral kernel of $S_n(0)$ than that of $S_n(b)$. So, once we have proven the uniform analycity of $P_n(b)$, we have proven it also for $V_n(b)$. To be self-contained, let us state here the following

Definition 1 *of Uniform Analyticity for a sequence* $\{P_n(b)\}$ *of operator-valued functions of* $b \in B$ *(some Banach Space).*

The sequence $\{P_n(b)\}$ is uniformly (real) analytic in a neighborhood of $0 \in B$ if and only if there exists $\delta > 0$ such that for every $f \in L^2(d\mu)$ and all b with $\|b\|_B < \delta$ we have

$$(8) \qquad P_n(b)f = \sum_{k=0}^{\infty} \Lambda_{nk}\,(b, \ldots, b, f) \qquad \textit{for every } n$$

where Λ_{nk} is a bounded $(k+1)$-multilinear operator sending $B^k \times L^2(d\mu)$ into $L^2(d\mu)$ and satisfying the estimate

$$(9) \qquad \|\Lambda_{nk}(b, \ldots, b, f)\|_{L^2(d\mu)} \leq c^{k+1}\|b\|_B^k\|f\|_{L^2(d\mu)}.$$

The constant c in (9) does not depend on n. Notice that $\Lambda_{nk}(b, \ldots, b, \bullet)$ is the k^{th} Gateaux differential of $P_n(b)$ at 0 in the direction of b.

A similar definition can be given, when b belongs to a complex Banach Space, for the uniform holomorphy of $\{P_n(b)\}$ in a neighborhood of 0. Actually, as in the case of one complex variable, the existence of one derivative in an open domain (*e.g.*, a neighborhood of 0) implies holomorphy in the same domain (see [B]), so we have

Definition 2 *$\{P_n(b)\}$ is uniformly holomorphic in a neighborhood U of $0 \in \mathbf{B}$ (complex Banach Space) if for every n, and all $b \in U$ we have*

$$(10) \qquad \|P_n(b)\|_{L^2 \to L^2} \leq c \quad \textit{and} \quad P_n(b) \textit{ is Gateaux-differentiable.}$$

We also have a notion of the «biggest space» in which the variable b can live and maintain the dependence $b \to P_n(b)$ holomorphic. It is given by

Definition 3. *B is the space of uniform holomorphy at 0 for $\{P_n(b)\}$ if $\{P_n(b)\}$ is uniformly holomorphic in a neighborhood of 0 and*

$$(11) \qquad \sup \|\Lambda_{n,1}(b, \bullet)\|_{L^2 \to L^2} < \infty \textit{ if and only if } \; b \in B$$

We will see later that, for the particular sequence of projection in our problem, proving conditions like (10) or (11) amounts to the proof of some suitable weighted norm inequalities.

3. Connections with Infinite-Dimensional Toda Flows

The set-up we have outlined in the previous paragraph is intimately connected with the theory of Toda flows. This theory has been studied independently, at first without any reference to orthogonal polynomials (see for example [DLT]).

Let us consider again the «perturbed» space $L^2(e^{2tb}d\mu)$ where we have added the real parameter $t \in (-\epsilon, \epsilon)$ in the exponential and where we consider the function $b(x)$ fixed.

If we apply the Gram-Schmidt orthogonalization process to the sequence $1, x, x^2, \ldots$ embedded in this L^2 space, we get a sequence $\{p_{j,t}(x)\}$ of orthogonal polynomials which depends on the parameter t. These polynomials satisfy a three-term recurrence

$$(1) \qquad xp_{j,t}(x) = A_j(t)p_{j+1,t}(x) + B_j(t)p_{j,t}(x) + A_{j-1}(t)p_{j-1,j}(x)$$

with $j = 0, 1, \ldots$ and $p_{-1,t}(x) = 0$. Since, for each t, they are a complete orthonormal system in $L^2(e^{2tb}d\mu)$, the map $f(x) \to xf(x)$ induces a bounded linear map $T(t)$ on l^2_+ that can be represented with an infinite tri-diagonal matrix

$$(2) \qquad T(t) = \begin{bmatrix} B_0(t) & A_0(t) & 0 & 0 & \ldots \\ A_0(t) & B_1(t) & A_1(t) & 0 & \ldots \\ 0 & A_1(t) & B_2(t) & A_2(t) & \ldots \\ 0 & 0 & A_2(t) & B_3(t) & A_3(t)\ldots \\ \vdots & \vdots & \vdots & & \end{bmatrix}.$$

If our measure $d\mu$ (and our polynomials) live on the closed interval $[-1, 1]$, we have the following explicit formulae for the recursion coefficients (see, *e.g.*, Szegö [Sz])

$$(3) \qquad A_j(t) = \int_{-1}^{1} xp_{j,t}(x)p_{j+1,j}(x)e^{2tb(x)}d\mu(x).$$

$$(3') \qquad B_j(t) = \int_{-1}^{1} x p_{j,t}^2(x)e^{2tb(x)}d\mu(x).$$

We *claim* that, in general, the operator $T(t)$ is a solution of the (infinite dimensional, non-linear) *Toda equation,* which can be written as

$$(4) \qquad \dot{T} = [T, \tilde{b}(T)]$$

where the square brackets stand for the commutator of two operators and $\tilde{b}(T)$ is a sort of Hilbert Transform obtained in the following way: first we use operator calculus to define the function b evaluated at T, getting some symmetric infinite matrix

$$(5)\qquad b(T) = \begin{bmatrix} \ddots & & b_+ \\ & d & \\ b_- & & \ddots \end{bmatrix}$$

which we have separated in (5) into lower triangular block (b_-), diagonal (d), and upper triangular block (b_+), then we define

$$(6)\qquad \tilde{b}(T) = \begin{bmatrix} \ddots & & -b_-^T \\ & 0 & \\ b_- & & \ddots \end{bmatrix}$$

i.e., we put zeros on the diagonal and replace the upper triangular block (b_+) with the transpose of the lower triangular block (b_-) multiplied by -1.

PROOF of the *claim*

First we want to show that the solution of (4) is expressed by

$$(7)\qquad T(t) = Q^T(t)\,T(0)\,Q(t)$$

where the infinite orthogonal matrix $Q(t)$ comes from

$$(8)\qquad e^{tb(T(0))} = Q(t)R(t)$$

The right hand side of (8) is the QR decomposition of the l.h.s.; this decomposition is uniquely determined by asking that Q be orthogonal and R upper triangular (see [DLT]). After showing that (7) is indeed a solution of (9), we will show that the $T(t)$ so obtained coincides with the $T(t)$ in (2).

In the computations that follow, we will assume implicitly that matrices denoted by capital letters are functions of t, we will denote differentiation with respect to t with a dot, and use the notation $T_0 = T(0)$. By differentiating both members of (7) we get

$$(9)\qquad \begin{aligned} \dot{T} &= \dot{Q}^T T_0 Q + Q^T T_0 \dot{Q} \\ &= \dot{Q}^T Q T Q^T Q + Q^T Q T Q^T \dot{Q} \\ &= \dot{Q}^T Q T + T Q^T \dot{Q}. \end{aligned}$$

The second equality is obtained by plugging in the expression for $T(0)$ that comes from (7), the third equality holds just because $Q^T Q = I$.

We want to check that

$$\dot{T} = [T, \tilde{b}(T)] = T\tilde{b}(T) - \tilde{b}(T)T$$

but, looking at the last term of (9), this amounts to showing that

$$\tilde{b}(T) = Q^T \dot{Q} \tag{10}$$

Notice that, by the definition of $\tilde{b}(T)$, both terms of this equality must be skew-adjoint, therefore if the equality holds we must have

$$(Q^T \dot{Q})^T = \dot{Q}^T Q = -Q^T \dot{Q}.$$

and this is true, as we can see differentiating the identity $Q^T Q = I$.

To prove (10) we differentiate (8) and use the fact that the exponential of an operator commutes with that operator. We get

$$b(T_0)QR = \dot{Q}R + Q\dot{R} \tag{11}$$

and multiplying both terms by Q^T on the left and by R^{-1} on the right we get

$$Q^T b(T_0) Q = Q^T \dot{Q} + \dot{R}R^{-1}$$

which, after noticing that $Q^T b(T_0) Q = b(Q^T T_0 Q) = b(T)$, can be rewritten as

$$b(T) = Q^T \dot{Q} + \dot{R}R^{-1}. \tag{12}$$

Let us now go back to the block notation for $b(T)$ that we used in (5), and show that, in that notation, our last equality (12) becomes

$$\begin{pmatrix} \diagdown & & b_+ \\ & d & \\ b_- & & \diagdown \end{pmatrix} = \begin{pmatrix} \diagdown & & -b_-^T \\ & 0 & \\ b_- & & \diagdown \end{pmatrix} + \begin{pmatrix} \diagdown & & b_+ + b_-^T \\ & d & \\ 0 & & \diagdown \end{pmatrix}. \tag{13}$$

In fact, we know that the first matrix on the right hand side of (12) is skew-adjoint, and we know that the second one is upper triangular (because differentiating, inverting, or multiplying triangular matrices we still get triangular matrices). This knowledge induces a chain of deductions that allows us to «fill in» with the proper blocks the two matrices on the right hand side of (13). The diagonal of the first matrix on the right hand side of (13) must be 0 and this forces the diagonal of the second matrix to be d. Since this second matrix is upper triangular, we know that the lower triangular block is 0 and this forces the lower triangular block of the first matrix to be b_-. By skew-adjointness, the upper triangular block of the first matrix is $-b_-^T$ and this forces the same block in the other matrix to be $b_+ + b_-^T$, completing the picture.

The equality (12) written in the block notation (13) shows in particular that

$$Q^T\dot{Q} = \begin{bmatrix} \diagdown & & -b_+^T \\ & 0 & \\ b_- & & \diagdown \end{bmatrix} = \tilde{b}(T)$$

but this is the identity (10) we wanted to prove, so (7) is indeed a solution of (4).

Now we want to check that the solution $T(t)$ of the Toda equation defined by (7) is the same $T(t)$ defined in (2) as the representation in the orthogonal polynomial basis $\{P_{j,t}(x)\}$ of the operator $M_x\colon f(x) \to xf(x)$. To do that, notice that we can write

$$e^{bt}S_n(bt)e^{-bt} = e^{bt}R^{-1}RS_n(bt)R^{-1}Re^{bt} = QS_n(0)Q^{-1}. \tag{14}$$

Here, as in the first paragraph, $S_n(bt)$ is the partial sum projection operator for the orthogonal polynomials $P_{j,t}(x) \in L^2(e^{2bt}d\mu)$, while R^{-1} is the upper triangular infinite matrix representing the coefficients of the «perturbed» basis of $\{P_{j,t}(x)\}$ in terms of the «unperturbed» one $\{P_{j,0}(x)\}$, and finally $Q(t)$ is an orthogonal transformation defined by

$$Q(t) = e^{bt}R^{-1}(t). \tag{15}$$

Let us point out that the action of R is given by

$$\sum_{j=0}^{\infty} c_j p_{j,0}(x) \overset{R^{-1}}{\longrightarrow} \sum_{j=0}^{\infty} c_j p_{j,t}(x) \overset{R}{\to} \sum_{j=0}^{\infty} c_j p_{j,0}(x). \tag{16}$$

while the action of Q is given by

$$\sum_{j=0}^{\infty} c_j p_{j,0}(x) \overset{Q}{\to} \sum_{j=0}^{\infty} c_j p_{j,t}(x)e^{b(x)t} \overset{Q^{-1}}{\to} \sum_{j=0}^{\infty} c_j p_{j,0}(x). \tag{17}$$

These last two diagrams show that $S_n(0) = RS_n(bt)R^{-1}$, an identity that we have implicitly used in (14). Also they show the orthogonality of Q.

We observe now that the matrix of $T(0)$, defined as in (2), represents the multiplication operator M_x in the orthonormal basis $\{P_{j,0}\}$. We also have

$$b(T(0)) = M_{b(x)} \tag{18}$$

and

$$e^{tb(T(0))} = e^{tM_b} = M_{e^{tb}} = e^{tb} \tag{19}$$

In the last step, we have simply used again the shorthand of indicating a multiplication operator directly by the multiplicator function.

We observe that (15), because of (19), give us the QR decomposition of $e^{tb(T(0))}$, and therefore the Q appearing in (15) is the same one that appears in the solution of the Toda equation (4). Because of the action of Q, we see that the $T(t)$ in (2) is the same as in (7).

4. The Role of Weighted Norm Inequalities

According to Definition 2 in Section 2, the sequence of operators $\{P_n(b)\}$ is uniformly holomorphic in a neighborhood U of $0 \in B$ if $P_n(b)$ is uniformly bounded in operator norm and Gateaux-differentiable for $b \in U$.

It turns out that, for the particular sequence of projections $P_n(b) = e^b S_n(0) e^{-b}$ which we are considering, the uniform boundedness in U implies also Gateaux-differentiability. The idea that leads to this simplification is due to Coifman, Rochberg, and Weiss [CR], and goes as follows:

We can write

$$\{P_n(b)f\}(x) = \int e^{b(x)-b(y)} D_n(x, y) f(y) d\mu(y) \tag{1}$$

where D_n is the Dirichlet Kernel for the partial sums operator $S_n(0)$ relative to the orthogonal polynomials on $L^2(d\mu)$. Let us fix b and multiply it by a complex parameter z. We then have

$$\left\{\frac{d}{dz} P_n(zb)f\right\}(x) = \int (b(x) - b(y)) e^{z(b(x)-b(y))} D_n(x, y)\, f(y) d\mu(y). \tag{2}$$

Notice then that

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \exp\{e^{i\theta}(b(x) - b(y))\} e^{-i\theta} d\theta \\ &= \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_0^{2\pi} \frac{((b(x) - b(y))e^{i\theta})^k}{k!} e^{-i\theta} d\theta \\ &= b(x) - b(y) \end{aligned} \tag{3}$$

Therefore we can write

$$(4)\quad \left\{\frac{d}{dz}P_n(zb)f\right\} = \frac{1}{2\pi}\int\int_0^{2\pi} \exp\{(z+e^{i\theta})\,(b(x)-(b(y))\}\,D_n(x,y)f(y)\,d\mu(y)e^{-i\theta}\,d\theta$$

which, using the notation $P_{n\theta} = P_n((z + e^{i\theta})b)$ can be written as

$$(5)\qquad \left\{\frac{d}{dz}P_n(zb)f\right\}(x) = \frac{1}{2\pi}\int_0^{2\pi}\{P_{n\theta}f\}(x)e^{-i\theta}\,d\theta.$$

We are assuming by hypothesis

$$(6)\qquad \|b\| < \delta \text{ implies } \|P_n(b)\|_{L^2\to L^2} \leq c$$

Therefore, choosing $|z| < 1$ and $\|b\| < \delta/2$, we have

$$(7)\qquad \left\|\frac{d}{dz}P_n(zb)\right\|_{L^2\to L^2} \leq \frac{1}{2\pi}\int_0^{2\pi}\|P_{n\theta}\|_{L^2\to L^2}\,d\theta \leq c.$$

So the uniform boundedness expressed by (6) implies locally also uniform analicity.

Notice that (6) can be seen as a weighted norm inequality for $S_n(0)$. In fact, the assumption that for $\|b\| < \delta$ one has

$$\|P_n(b)f\|_{L^2(d\mu)} \leq c\|f\|_{L^2(d\mu)} \quad \text{for all} \quad f \in L^2(d\mu).$$

just by setting $f = e^b g$, is seen to be equivalent to

$$(8)\qquad \|S_n(0)g\|_{L^2(e^{2b}d\mu)} \leq c\|g\|_{L^2(e^{2b}d\mu)} \quad \text{for all} \quad g \in L^2(e^{2b}d\mu).$$

We will see that in many cases the partial sums operator looks essentially like a Hilbert Transform, while e^{2b} is a good A_p weight when b has a small BMO norm. This allows us to prove weighted norm inequalities like (8) using the results of Hunt, Muckenhaupt, and Wheeden (see [HMW] and also [CF]).

In order to identify BMO also as the space of uniform analyticity at 0, we need to prove, according to Definition 3 in §2, that

$$(9)\qquad b \in \text{BMO if and only if } \sup_n \|\Lambda_{n1}(b,\bullet)\|_{L^2\to L^2} < \infty.$$

Let us compute the Gateaux differential that appears in (9). We have

$$\frac{d}{dt} P_n(tb)\Big|_{t=0} = \frac{d}{dt}(e^{tb} S_n(0) e^{-tb})\Big|_{t=0}$$
$$= be^{tb} S_n(0) e^{-tb} + e^{tb} S_n(0)\,(-be^{tb})\Big|_{t=0}$$
$$= [b, S_n(0)]$$

So condition (9) can be rewritten asking that there exist $c_1, c_2 > 0$ such that

$$c_1 \|b\|_* \le \sup_n \|[b, S_n(0)]\|_{L^2 \to L^2} \le c_2 \|b\|_* \tag{10}$$

where we indicate with a * subscript the BMO norm.

Actually, the inequality on the right is a consequence of uniform holomorphy (which implies the boundedness of all derivatives), so we only need to prove the one on the left.

5. Perturbation Theory of a Family of Orthogonal Polynomials Introduced by Szegö

In Szegö's classic book [Sz] on orthogonal polynomials, the starting point for the proof of his equiconvergence results is the study of a particular family of orthogonal polynomials in $L^2(\mathbf{T}, d\mu)$. The measure $d\mu$ on the unit circle is given by

$$d\mu(\theta) = \frac{1}{g(\theta)} \frac{d\theta}{2\pi} \tag{1}$$

with g trigonometric polynomial of degree m and $g(\theta) > 0$ for all $\theta \in [0, 2\pi]$. It turns out that for any fixed g, even though it is not easy to find a closed formula for the first m orthogonal polynomials, there is in fact a simple expression for them when the degree is greater than or equal to m. This restriction does not affect Szegö's proofs, and will not affect ours either.

We have the following:

Proposition 1 The sequence $\{\phi_j(z)\}$ of complex polynomials (normalized with a strictly positive coefficient for z^j) satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_j(z)\overline{\phi_k(z)} \frac{1}{g(\theta)}\, d\theta = \delta_{jk} \tag{2}$$

($z = e^{i\theta}$ in the integral) is given, when $j = m, m+1, \ldots$, by

$$\phi_j(z) = z^{j-m}h^*(z) \tag{3}$$

where $h(z)$ is a complex polynomial of degree m such that

$$g(\theta) = |h(z)|^2 \qquad \text{when } z = e^{i\theta} \tag{4}$$

and h^* is the *reciprocal* polynomial of h, *i.e.*,

$$h^*(z) = z^m \bar{h}\left(\frac{1}{z}\right). \tag{5}$$

PROOF. The existence of representation (4) for the positive polynomial g is a special case of a more general one, from H^p theory, that holds in fact for all f such that $\log f \in L^1(\mathbf{T})$. Notice that we can choose $h(0) > 0$ and $h(z) \neq 0$ for $|z| < 1$.

We can rewrite (3) as

$$\phi_j(z) = z^j \bar{h}(z^{-1}). \tag{6}$$

Here, as before in (5), we use the convention that if $h(z)$ is a polynomial in z, $\bar{h}$ is the polynomial obtained from h by conjugating the coefficients (not z).

Plugging (6) into (2) we can easily verify the case $j = k$ just remembering that on the unit circle we have $\bar{z} = z^{-1}$.

To prove the case $j \neq k$ we actually show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_j(z)\overline{\varrho(z)} \frac{1}{g(\theta)} \, d\theta = 0 \qquad \text{when } z = e^{i\theta}. \tag{7}$$

for any polynomial ϱ of degree $j - 1$. In fact we get

$$\frac{1}{2\pi i} \int_\Gamma z^j \bar{h}(z^{-1}) \frac{\bar{\varrho}(z^{-1})}{h(z)\bar{h}(z^{-1})} \frac{dz}{z} = \frac{1}{2\pi i} \int_\Gamma \frac{z^{j-1}\bar{\varrho}(z^{-1})}{h(z)} \, dz$$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{\varrho^*(z)}{h(z)} \, dz = 0.$$

The last equality holds because we are integrating along the unit circle Γ a meromorphic function with no poles inside.

Given a closed expression of the polynomials, one can give an expression of the Dirichlet kernel of the partial sums operator (we refer to [Sz], p. 292, for the algebraic details). In our case we have

$$(8)\qquad D_n(\theta, \varphi) = \frac{h(e^{i\theta})\overline{h(e^{i\psi})} - e^{i(n+1)(\theta-\psi)}\overline{h(e^{i\theta})}h(e^{i\psi})}{1 - e^{i(\theta-\psi)}}.$$

This expression is valid whenever $n \geq m$, and shows explicitly the dependence of the kernel on the weight. Following our general plan we look at $P_n(b)$ which is now given by

$$(9)\ \{P_n(b)f\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{b(\theta)}D_n(\theta, \psi)e^{-b(\psi)}\frac{f(\psi)}{g(\psi)}\,d\psi$$

$$= \frac{1}{2\pi}\int_{-\pi 1}^{\pi} e^{b(\theta)-b(\psi)}\frac{f(\psi)}{1-e^{i(\theta-\psi)}}\left\{\frac{h(e^{i\theta})}{h(e^{i\psi})} - e^{i(n+1)(\theta-\psi)}\frac{\overline{h(e^{i\theta})}}{\overline{h(e^{i\psi})}}\right\}d\psi$$

$$= \{(P_n'(b) - P_n''(b))f\}(\theta)$$

where

$$(10')\qquad \{(P_n'(b)f\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{b(\theta)}h(e^{i\theta})\frac{f(\psi)}{1-e^{i(\theta-\psi)}}e^{-b(\psi)}h(e^{i\psi})^{-1}\,d\psi$$

and

$$(10'')\ \{(P_n''(b)f\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n+1)\theta+b(\theta)}\overline{h(e^{i\theta})}\frac{f(\psi)}{1-e^{i(\theta-\psi)}}e^{-i(n+1)\psi-b(\psi)}\overline{h(e^{i\psi})}^{-1}d\psi.$$

The operators in (10′) and (10″) consist of a singular integral (essentially a Hilbert transform) conjugated by a multiplication operator on the left and the inverse of the same multiplication on the right. Let us look at a third operator

$$(10''')\quad \{(\mathrm{P}_n'''(b)f\}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{Re}\,b(\theta)}g(\theta)^{1/2}\frac{f(\psi)}{1-e^{i(\theta-\psi)}}e^{-\mathrm{Re}\,b(\psi)}g(\psi)^{-1/2}\,d\psi$$

which, by the result in [HMW], is bounded for $e^{\mathrm{Re}\,b(\theta)}g(\theta)^{1/2} \in A_2$. This last condition is satisfied when the real part of $b(\theta)$ has BMO-norm smaller than some suitable δ. (See [GCRF], chapter 4, and notice that $g(\theta)$ is a polynomial without zeroes for every θ).

The same condition on $b(\theta)$ implies uniform boundedness also for (10′) and (10″), just noticing that they are both obtained from (10‴) multiplying on the left by a complex function of modulus 1 and on the right by the inverse of that same function. By what we have seen in the previous paragraph, for this particular kind of operators uniform boundedness implies uniform holomorphy, so the family of projections $\{P_n(b)\}$ is uniformly holomorphic for b in a neighborhood of $0 \in$ BMO.

By Kerzman-Stein formula (2.5), or also using the Toda Flow identities of Section 3, the same uniform result holds for the self adjoint projections $V_n(b)$.

The next step of the plan consists in showing that BMO is in fact the space of uniform holomorphy of the $\{P_n(b)\}$, and, as we have seen in general, that amounts to proving the inequality

$$(11) \qquad c_1 \| b \|_{\#} \leq \sup_n \| [b, S_n(0)] \|_{L^2 \to L^2}.$$

To do that we define

$$(12) \qquad f_\theta(\psi) = \chi_I(\psi)\,(1 - e^{i(\theta-\psi)})\,h(e^{i\psi})$$

where I is a closed subinterval of $[0, 2\pi]$.

We have

$$(13)\ \{[b, S_n(0)]f_\theta\}(\theta) = \frac{1}{2\pi}\int_I (b(\theta) - b(\psi))\left\{h(e^{i\theta}) - e^{i(n+1)(\theta-\psi)}\frac{h(e^{i\psi})\overline{h(e^{i\theta})}}{\overline{h(e^{i\psi})}}\right\} d\psi.$$

By Riemann-Lebesgue lemma we have

$$(14)\ \lim_{n\to\infty}\{[b, S_n(0)]f_\theta\}(\theta) = \frac{1}{2\pi}\int_I (b(\theta) - b(\psi))h(e^{i\theta})\,d\psi = \frac{|I|}{2\pi}h(e^{i\theta})(b(\theta) - m_I(b))$$

where $m_I(b)$ is the average of the function b over I. By Fatou's lemma we have

$$(15) \quad |I| \int_I |b(\theta) - m_I(b)|\,d\theta \leq c \limsup_{n\to\infty} \int_I |\{[b, S_n(0)]f_\theta\}(\theta)| \frac{d\theta}{|h(e^{i\theta})|}.$$

Therefore (11) will hold if we can bound the right hand side of this by

$$c\,|\,I\,|^2 \sup_n \|[b, S_n(0)]\|$$

where c does not depend on n and I.

Let α be the midpoint of I; a straightforward computation allows us to rewrite (12) in the form

$$(16)\quad f_\theta(\psi) = e^{i(\theta-\alpha)/2} \sin\frac{\theta-\alpha}{2} f_1(\psi) + e^{i(\theta-\alpha)/2} \cos\frac{\theta-\alpha}{2} f_2(\psi)$$

where f_1 and f_2 do not depend on θ, and are given by

$$(16')\qquad f_1(\psi) = -2i\chi_I(\psi) h(e^{i\psi}) e^{i(\alpha-\psi)/2} \cos\frac{\alpha-\psi}{2}.$$

$$(16')\qquad f_2(\psi) = -2i\chi_I(\psi) h(e^{i\psi}) e^{i(\alpha-\psi)/2} \sin\frac{\alpha-\psi}{2}.$$

Using (16) and Schwarz's inequality, we get

$$(17)\quad \int_I |\{[b, S_n(0)]f_\theta\}(\theta)| \frac{d\theta}{|h(e^{i\theta})|}$$
$$\le \left(\int_I \left|\sin\frac{\theta-\alpha}{2}\right|^2 d\theta\right)^{1/2} \left(\int_I |\{[b, S_n(0)]f_1\}(\theta)|^2 \frac{d\theta}{|h(e^{i\theta})|^2}\right)^{1/2}$$
$$+ \left(\int_I \left|\cos\frac{\theta-\alpha}{2}\right|^2 d\theta\right)^{1/2} \left(\int_I |\{[b, S_n(0)]f_2\}(\theta)|^2 \frac{d\theta}{|h(e^{i\theta})|^2}\right)^{1/2}.$$

This last inequality holds for every n, so we can take the upper limit of the left hand side. Also, we notice that the right factors in the two terms on the right hand side are, for $k = 1, 2$, exactly the $L^2(d\mu)$ norms of $[b, S_n(0)]f_k$, since $d\mu(\theta) = |h(e^{i\theta})|^{-2}d\theta$.

We have

$$(18)\quad \limsup_{n\to\infty} \int |\{[b, S_n(0)]f_\theta\}(\theta)| \frac{d\theta}{|h(e^{i\theta})|} \le c_1 |I|^{3/2} A \|f_1\|_{L^2(d\mu)}$$
$$+ c_2 |I|^{1/2} A \|f_2\|_{L^2(d\mu)},$$

where c_1, c_2 de not depend on n or I and

(19) $$A = \sup \| [b, S_n(0)] \|_{L^2(d\mu) \to L^2(d\mu)} .$$

To complete our proof we only need to check that

(20) $$\|f_1\|_{L^2(d\mu)} \leq c_3 \, | I |^{1/2} \quad \text{and} \quad \|f_2\|_{L^2(d\mu)} \leq c_4 \, | I |^{3/2}$$

and this is easily seen to be true by our choice of f_1 and f_2.

6. Generalizations and the Role of VMO

In the previous paragraph we have proven that the family of orthogonal projections

$$V_n(b) = e^b S_n(b) e^{-b}: L^2(d\mu) \to L^2(d\mu)$$

is uniformly holomorphic (Definition 2.2) for b belonging to a neighborhood of 0 in BMO. Also we have shown that BMO is the space of uniform holomorphy at 0 (Definition 2.3). Notice that our $d\mu$ is a measure on $\mathbf{T}$ expressed via a *non-vanishing* weight of the form (5.1).

In fact, as long as we satisfy the proper estimates, we could have started from a measure $d\mu$, given by a weight with zeroes. The intuitive idea is that a BMO function can contain unbounded logarithmic spikes (and the exponential of a negative spike can be 0); here we want to make this idea more precise, getting an analyticity result that holds for a «starting set» of orthogonal polynomials more general than those of the form (5.2).

We need first to introduce an important closed Banach subspace of BMO It is called VMO (Vanishing Mean Oscillation) and is defined by

(1) $$\mathrm{VMO} = \{f \in \mathrm{BMO}: \lim_{a \to 0} M_a(f) = 0\}$$

where

(2) $$M_a(f) = \sup_{|I| \leq a} \frac{1}{|I|} \int_I | f(x) - f_I | \, dx.$$

As usual I denotes an interval and f_I the average of f on I. Notice that, with this notation, the usual BMO norm is given by

(3) $$\|f\|_* = \lim_{a \to \infty} M_a(f)$$

This space has been studied by Sarason, and in [Sa] one can find the statement and proof of some basic properties; we are going to use, in particular, the fact that VMO is the closure, in BMO-norm, of the subspace of continuous functions.

Lemma 1 *Given a real-valued function* $\gamma \in$ V.M.O. $(\mathbf{T})$ *one can find a sequence of strictly positive trigonometric polynomials* $g_j(\theta)$ *such that*

$$\lim_{j\to\infty} \left\| \gamma(\theta) - \log\left(\frac{1}{g_j(\theta)}\right) \right\|_* = 0$$

PROOF. Continous functions are BMO-dense in VMO and since $\|f\|_* \leq 2\|f\|_\infty$ we only need to show that for any real-valued $\beta(\theta) \in C(\mathbf{T})$ one can find a sequence of strictly positive trigonometric polynomials $g_j(\theta)$ such that

$$\lim_{j\to\infty} \left\| \beta(\theta) - \log\left(\frac{1}{g_j(\theta)}\right) \right\|_\infty = 0 \tag{4}$$

we have

$$\left| \beta(\theta) - \log\left(\frac{1}{g_j(\theta)}\right) \right| = \left|\log(g_j(\theta)e^{\beta(\theta)})\right| = \left|\log(1 + e^{\beta(\theta)}(g_j(\theta) - e^{-\beta(\theta)}))\right|. \tag{5}$$

And the compactness of $\mathbf{T}$ implies the existence of two positive constants c_1, c_2 such that

$$c_1 \leq e^{\beta(\theta)} \leq c_2 . \tag{6}$$

But this implies that the quantity in (5) can be made uniformly small just by uniform approximation of the positive continuous function $e^{-\beta(\theta)}$ with the positive polynomials $g_j(\theta)$.

Lemma 2 *The radius of analycity* δ *for the mapping relative to the orthogonal polynomials* (5.2) *does not depend on the particular* $g(\theta)$ *contained in the weight.*

PROOF. Going back to the previous paragraph, we notice that we have analyticity for those functions $b \in$ BMO wich satisfy

$$e^{\operatorname{Re} b(\theta)} \sqrt{g(\theta)} \in A_2 . \tag{7}$$

It is a property of A_2 weights to be invariant under multiplication by functions bounded away from 0 and infinity (see [GCRF], chapter 4). On the other hand the first factor in (7) is an A_2 weight in general only if $\|\operatorname{Re} b\|_* < \delta$.

Remark. If one makes additional assumptions on the nature of b, one can find functions with large BMO norm whose exponential is still in A_2. This surprising fact is well illustrated by a theorem of Garnett and Jones which says the following

$$A(b) = \sup\{\lambda > 0 \colon e^{\lambda b} \in A_2\} \approx \Big(\operatorname*{dist}_{\mathrm{BMO}} \{b, L^\infty\}\Big)^{-1}, \tag{8}$$

where the symbol « ≈ » means «same order of magnitude» and where

$$\operatorname*{dist}_{\mathrm{BMO}} \{b, L^\infty\} = \inf\{\|b - f\|_* \colon f \in L^\infty\}. \tag{9}$$

In particular, the exponential of any VMO function (regardless of the BMO norm) is an A_2 weight.

We can now state our general results as follows

Theorem 1. *The uniform analicity of the mapping $b \to V_n(b)$ holds for all orthogonal polynomials on $L^2(d\mu)$ where $d\mu$ is a measure on* $\mathbf{T}$ *of the form*

$$d\mu(\theta) = e^{2(\gamma(\theta) + \beta(\theta))} \frac{d\theta}{2\pi} \tag{10}$$

with $\gamma(\theta)$ any real-valued VMO *function and $\beta(\theta)$ a complex-valued* BMO *function such that*

$$\|\operatorname{Re} \beta\|_* < \delta \tag{11}$$

where δ is the radius of analyticity for the Szegö polynomials (5.2).

The new radius of analyticity, if we start from the polynomials relative to the measure (10), *is given by*

$$\delta' = \delta - \|\operatorname{Re} \beta\|_* \tag{12}$$

PROOF. Let us fix some BMO function β satisfying (11). We know that starting from a measure of the form (5.1) we can expand $V_n(b)f$ into a series of multilinear operators (2.8) satisfying an estimate (2.9). Let us define a new «starting space» for our perturbation

$$L^2(d\mu) = L^2\left(e^{2\beta(\theta)} \frac{1}{g(\theta)} \frac{d\theta}{2\pi}\right). \tag{13}$$

Then let us denote, according to the notations of Section 2, by S_n and V_n the projections relative to the L^2 space in (13), while we denote the same projections relative to the $\beta = 0$ situation by S_n^0 and V_n^0. We have

$$\begin{aligned} S_n(0) &= S_n^0(\beta) \\ S_n(b) &= S_n^0(b + \beta) \\ V_n(b) &= e^b S_n^0(b + \beta)e^{-b} \end{aligned} \tag{14}$$

where the operators in the «new» space are expressed in terms of those in the «old» space. Using these identities we can write

$$\| V_n(b)\|_{L^2(d\mu)\to L^2(d\mu)} = \|e^{b+\beta}S_n^0(b + \beta)e^{-(b+\beta)}\|_{L^2(g)\to L^2(g)} \tag{15}$$

where we denote by $L^2(g)$ the space relative to the measure $\dfrac{1}{g(\theta)} \dfrac{d\theta}{2\pi}$.

By the results of Section 5, we have uniform holomorphy for $V_n(b)$ if

$$\|\mathrm{Re}(b + \beta)\|_* < \delta \tag{16}$$

and this holds for any b whose real part has BMO norm less than the δ' in (12).

Finally, lemma 1 and lemma 2 imply that we can substitute $1/g(\theta)$ in the right hand side of (13) with $e^{2\gamma(\theta)}$ where $\gamma \in$ VMO. We remark that weights of the form (10) can have any number of zeros (only with restrictions on the rate of decay of the weight around each zero).

7. The ~ Operation on Infinite Matrices Expressed Via the Hilbert Transform and a Remarkable Bilinear Operator

Starting from our knowledge of the operator-valued function

$$b \to V_n(b)$$

and given the identities (3.14-3.15), *i.e.*,

$$V_n(b) = e^b S_n(b) e^{-b} = Q(b) S_n(0) Q(b)^{-1}$$

where

$$Q(b) = e^b R^{-1}(b)$$

is an orthogonal transformation depending on $b \in$ BMO, it is a natural problem to study the mapping

$$b \to Q(b). \tag{1}$$

The first step of this study is to compute and understand the Gateaux differential of (1) at 0. We claim that

$$\frac{d}{dt} Q(bt) \,|_{t=0} = \tilde{M}_b \tag{2}$$

where M_b is the operator of multiplication by b and $\tilde{M}_b$ is defined as in (3.5-3.6).

In fact, by the Kerzman-Stein formula (2.5) we have

$$V_n(b) = P_n(b)\,(I + P_n(b) - P_n(-b))^{-1}$$

therefore

$$\begin{aligned} \frac{d}{dt} V_n(bt) \,|_{t=0} &= \frac{d}{dt} P_n(tb) \,|_{t=0} - P_n(0) \frac{d}{dt} \{P_n(tb) - P_n(-tb)\} \,|_{t=0} \\ &= [b, S_n(0)] - S_n(0) \{[b, S_n(0)] + [b, S_n(0)]\} \\ &= \{I - 2S_n(0)\}\, [b, S_n(0)] \\ &= [\tilde{M}_b, S_n(0)]\,. \end{aligned} \tag{3}$$

The last equality can be checked by writing the operators as infinite matrices, remembering that $S_n(0)$ is the identity on the first $(n + 1) \times (n + 1)$ entries while

$$b = M_b = (m_{nk}) \quad \text{implies} \quad \tilde{M}_b = (\operatorname{sgn}(n - k) m_{nk}) \tag{4}$$

Because of identity (3.14), we have

$$[\dot{Q}(0), S_n(0)] = [\tilde{M}_b, S_n(0)] \tag{5}$$

and since this commutator equality holds for $n = 0, 1, 2, \ldots$, and $\dot{Q}^* = -\dot{Q}$, we can conclude that

$$\dot{Q}(0) = \tilde{M}_b.$$

While studying the explicit form that the operator $\tilde{M}_b$ takes for some explicit sets of orthogonal polynomials, we discovered a remarkable formula. It is an operator identity expressing the operation $\sim$ on infinite matrices via Hilbert Transforms on the circle, and via the bilinear operator

$$B(b, f)(x) = \frac{1}{2\pi} \int b(x - \theta) f(x - 2\theta) \cot \frac{\theta}{2} \, d\theta . \tag{6}$$

Notice that it is still an open problem to establish whether operators like (6), or their analogues on the real line, are L^2-bounded for $b \in L^\infty$ (or $b \in \mathrm{BMO}$).

The context in which the formula arises is cosine polynomials (Chebychev polynomials after a change of variable).

Consider the L^2 space of functions of the form

$$f(x) = \sum_{j=0}^{\infty} f_j \cos jx \qquad (\mathrm{b}_j \in \mathbf{C}) \tag{7}$$

and the multiplication operator

$$\{M_b f\}(x) = b(x) f(x) \tag{8}$$

where

$$b(x) = \sum_{j=0}^{\infty} b_j \cos jx \qquad (b_j \in \mathbf{C}). \tag{9}$$

The operator M_b can be represented in the basis $\{\cos jx\}$ by an infinite matrix

$$M_b = (m_{nk}) \qquad n, k = 0, 1, 2, \ldots \tag{10}$$

and we can define

$$\tilde{M}_b = (-i \operatorname{sgn}(n - k) m_{nk}) \tag{11}$$

which is the usual definition, apart from the constant factor $-i$.

Introducing a new variable θ, we can write an integral expression for $\tilde{M}_b$, in fact

$$\tilde{M}_b = \frac{1}{2\pi} \cdot \int_1^{2\pi} U_\theta M_b U_{-\theta} \cot \frac{\theta}{2} \, d\theta \tag{12}$$

where

$$U_\theta f = \sum_{j=0}^{\infty} e^{-ij\theta} f_j \cos jx, \tag{13}$$

$$U_{-\theta} f = \sum_{j=0}^{\infty} e^{ij\theta} f_j \cos jx.$$

Now, we have

$$\begin{aligned} U_{-\theta} f &= \sum_{j=0}^{\infty} f_j (\cos j\theta \cos jx + i \sin j\theta \cos jx) \\ &= \sum_{j=0}^{\infty} f_j \left(\frac{\cos j(x+\theta) + \cos j(x-\theta)}{2} + \frac{\sin j(x+\theta) - \sin j(x-\theta)}{2} \right) \\ &= \frac{1}{2} \{ T_{-\theta} f + T_\theta f + i(T_{-\theta} Hf - T_\theta Hf) \} \end{aligned} \tag{14}$$

where

$$\{T_\theta f\}(x) = f(x - \theta) \tag{15}$$

and Hf is the Hilbert Transform of f (on the circle).

A similar computation shows that

$$U_\theta f = \frac{1}{2} \{ T_{-\theta} f + T_\theta f - i(T_{-\theta} Hf - T_\theta Hf) \}. \tag{16}$$

Using (14) and (16), let us compute the operator $U_\theta M_b U_{-\theta}$. In what follows, in order to make the notation less cumbersome, we will simply write M for M_b. We have

$$\begin{aligned}(17)\ 4U_\theta MU_{-\theta} &= \\ &= U_\theta\{MT_{-\theta} + MT_\theta + i(MT_{-\theta}H - MT_\theta H)\} \\ &= T_{-\theta}MT_{-\theta} + T_{-\theta}MT_\theta + iT_{-\theta}MT_{-\theta}H - iT_{-\theta}MT_\theta H \\ &\quad + T_\theta MT_{-\theta} + T_\theta MT_\theta + iT_{-\theta}MT_{-\theta}H - iT_\theta MT_\theta H \\ &\quad - iT_{-\theta}HMT_{-\theta} - iT_{-\theta}HMT_\theta + T_{-\theta}HMT_{-\theta}H - T_{-\theta}HMT_\theta H \\ &\quad + iT_\theta HMT_{-\theta} + iT_\theta HMT_\theta - T_\theta HMT_{-\theta}H + T_\theta HMT_\theta H.\end{aligned}$$

If we plug these sixteen terms back into (12), we notice that the eight terms giving the real part cancel after integration against $\cot\frac{\theta}{2}d\theta$. In fact, we have

$$\begin{aligned}(18)\quad \{(T_{-\theta}MT_{-\theta} + T_{-\theta}MT_\theta + T_\theta MT_{-\theta} + T_\theta MT_\theta)f\}(x) &= \\ &= b(x+\theta)f(x+2\theta) + b(x+\theta)f(x) \\ &\quad + b(x-\theta)f(x) + b(x-\theta)\,(x-2\theta),\end{aligned}$$

and these four terms define an even function of θ, which in the integral is multiplied by the odd function $\cot\theta/2$.

Similarly, the other four real terms applied to f give (using the notation $Hf = \tilde{f}$)

$$(19)\quad H(b(x+\theta)\tilde{f}(x+2\theta) - b(x+\theta)\tilde{f}(x) - b(x-\theta)\tilde{f}(x) + b(x-\theta)\tilde{f}(x-2\theta))$$

and these cancel, too, if we assume that the integration against $\cot\frac{\theta}{2}d\theta$ can be done before the Hilbert Transform H in the x variable.

Let us now look at the other eight terms in (17); they are

$$\begin{aligned}(20)\quad i(T_{-\theta}MT_{-\theta}H - T_{-\theta}MT_\theta H + T_\theta MT_{-\theta}H - T_\theta MT_\theta H \\ - T_{-\theta}HMT_{-\theta} - T_{-\theta}HMT_\theta + T_\theta HMT_{-\theta} + T_\theta HMT_\theta)\end{aligned}$$

and applying them to $f(x)$, we get

$$\begin{aligned}(21)\ &4\mathrm{Im}\{(U_\theta MU_{-\theta})f\}(x) \\ &= b(x+\theta)\tilde{f}(x+2\theta) - b(x+\theta)\tilde{f}(x) + b(x-\theta)\tilde{f}(x) - b(x-\theta)\tilde{f}(x-2\theta) \\ &\quad + H\{-b(x+\theta)f(x+2\theta) - b(x+\theta)f(x) \\ &\quad + b(x-\theta)f(x) + b(x-\theta)f(x-2\theta)\}.\end{aligned}$$

Plugging this expression into (12), and making the change of variable $\theta \to -\theta$ in the four integrals that contain $x+\theta$ or $x+2\theta$, we get

$$(22)\quad \tilde{M}f = \frac{i}{4}\left\{2\tilde{f}(x)\int b(x-\theta)\,dm(\theta)\right.$$
$$-2\int b(x-\theta)\tilde{f}(x-2\theta)\,dm(\theta)$$
$$+\left(2f(x)\int b(x-\theta)\,dm(\theta)\right)^{\sim}$$
$$\left.+\left(2\int b(x-\theta)f(x-2\theta)\,dm(\theta)\right)^{\sim}\right\}$$

where we have used the notation

$$(23)\qquad dm(\theta) = \cot\frac{\theta}{2}\,\frac{d\theta}{2\pi}$$

and $dm(\theta)$ lives on $[0, 2\pi]$.

The final formula is

$$(24)\quad \{\tilde{M}f\}(x) = \frac{i}{2}\left\{\tilde{f}(x)\tilde{b}(x) - \frac{1}{2\pi}\int_0^{2\pi} b(x-\theta)\tilde{f}(x-2\theta)\cot\frac{\theta}{2}\,d\theta\right.$$
$$+\,(f\tilde{b})^{\sim}(x)$$
$$\left.+\frac{1}{2\pi}\left(\int_0^{2\pi} b(x-\theta)f(x-2\theta)\cot\frac{\theta}{2}\,d\theta\right)^{\sim}\right\}.$$

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Harnack Inequality and Green Function for a Certain Class of Degenerate Elliptic Differential Operators

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Introduction

The main purpose of this work is to obtain a Harnack inequality and estimates for the Green function for the general class of degenerate elliptic operators described below. Let

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju), \tag{0.1}$$

where $A = [a_{ij}]$ is symmetric, measurable and satisfies the following ellipticity condition

$$v(x)\sum_{i=1}^{n}\lambda_i^2(x)\xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq w(x)\sum_{i=1}^{n}\lambda_i^2(x)\xi_i^2, \tag{0.2}$$

for every $\xi \in \mathbf{R}^n$ and almost every x in an open bounded set Ω of $\mathbf{R}^n$. The functions λ_i are defined on $\mathbf{R}^n$ and satisfy

(0.3) $\quad \lambda_1 \equiv 1 \quad \lambda_j(x) = \lambda_j(x_1, \dots, x_{j-1}) \in C(\mathbf{R}^n) \bigcap C^1(\mathbf{R}^n - \Pi)$ where

$$\Pi = \left\{x \in \mathbf{R}^n : \prod_{i=1}^{n} x_i = 0\right\} \quad \text{for} \quad j = 2, \dots, n$$

(0.4) $\lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, -x_i, \ldots, x_{j-1}),\ i = 1, \ldots, j-1$

(0.5) $0 < \lambda_j(x) \leq \Lambda$ for every $x \in \mathbf{R}^n - \Pi$, $j = 1, \ldots, n$. Moreover, there exist non-negative numbers b_{ji} such that

$$0 \leq x_i(D_i\lambda_j)\,(x) \leq b_{ji}\lambda_j(x)$$

for $i = 1, \ldots, j-1$, $j = 2, \ldots, n$ and for every $x \in \mathbf{R}^n - \Pi$.

A vector $(\lambda_1, \ldots, \lambda_n)$ satisfying these properties generates a distance d and a quasi-distance δ on $\mathbf{R}^n$ in such a way that $(\mathbf{R}^n, d)$ and $(\mathbf{R}^n, \delta)$ become spaces of homogeneous type with the Lebesgue measure (see [CG], [CW] and [C]) and, moreover, there exists a constant $a > 1$ such that $a^{-1}\delta < d < a\delta$ (see [FL1]). The conditions on the pair of weights (v, w) can now be stated in terms of this geometry. Given $\alpha \in (0, 1]$ and $\sigma > 1$, we introduce the class $S_{\sigma,\alpha}$ as the class of pairs (v, w) such that satisfy

(0.6) $0 < v(Q),\ w(Q) < \infty$ for every δ-ball $Q \subset \Omega$, where

$$w(Q) = \int_Q w, \quad v(Q) = \int_Q v,$$

(0.7) there exists $C > 0$ such that

$$\left(\frac{w(Q_0 \cap Q)}{w(Q_0)}\right)^{1/2\sigma} (v^{-1}(Q_0 \cap Q)\, v(Q_0))^{1/2} \leq C\,|Q_0|^{1-\alpha}\,|Q|^{\alpha}$$

for every Q_0 and Q δ-balls in Ω such that radious $(Q) \leq 8\,a^2$ radious (Q_0).

Examples of operators satisfying the preceeding conditions are the following

(0.8) $Lu = -\operatorname{div}(d(0, x)^{\beta} D_1 u,\ d(0, x)^{-\beta}\,|x_1|^{\gamma} D_2 u)$

for $x = (x_1, x_2) \in \mathbf{R}^2$, $\gamma > 0$ and $\beta > 0$. Since our results will apply when $\alpha \in (1 - (\sum_j G_j)^{-1}, 1)$, ($G_1 = 1$ and $G_j = 1 + \sum_{i=1}^{j-1} b_{ji} G_i$, $j = 2, \ldots, n$), we get Harnack's inequality and estimates for Green's function for the operator L in (0.8) when

$$1 - \frac{1}{2+\gamma} < \frac{1}{4}\,\frac{(4+\beta)\,(2-\beta)}{2+\beta}.$$

We point out that our results contain as special cases those in Moser ([M]), Fabes, Kenig and Serapioni ([FKS]), Fabes, Jerison and Kenig ([FJK]), Chanillo and Wheeden ([ChW2]), Franchi and Lanconelli ([FL2]) and ([FL3]) and Franchi and Serapioni ([FS]).

In Section 1, we present a brief survey of results of the particular geometry introduced by Franchi and Lanconelli. Section 2 is devoted to the construction of a family of δ-balls which resembles the dyadic cubes. In Section 3, we prove Sobolev and Poincaré inequalities. Section 4 contains an analysis of the relations among our conditions on (v, w) and those in the work of Chanillo and Wheeden. Finally, Section 5 contains the statements of the results about Harnack's inequality and estimates of Green's function.

1.

In this section we give the definitions of the natural distance d and the quasi-distance δ and state its basic properties.

Let us start introducing the notions of λ-subunit vector and λ-subunit curve: a vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{R}^n$ is a λ-subunit vector at a point x if

$$\left(\sum_{j=1}^{n} \gamma_j \xi_j\right)^2 \leq \sum_{j=1}^{n} \lambda_j^2(x)\xi_j^2, \quad \text{for every} \quad \xi \in \mathbf{R}^n;$$

we say that $\gamma: [0, T] \to \mathbf{R}^n$ is a λ-subunit curve if it is an absolutely continuous curve and $\dot{\gamma}(t)$ is a λ-subunit vector at $\gamma(t)$ for a.e. $t \in [0, T]$.

Definition 1.1. *For any $x, y \in \mathbf{R}^n$ we define d: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_0^+$ as $d(x, y) = \inf\{T \in \mathbf{R}^n$: there exists a λ-subunit curve $\gamma: [0, T] \to \mathbf{R}^n$, $\gamma(0) = x$, $\gamma(T) = y\}$.*

Remark 1.2 ([FL1], [FL3]): d is a well defined distance. In fact our hypotheses on $\lambda = (\lambda_1, \ldots, \lambda_n)$ guarantee the existence of a λ-subunit curve joining x and y, for any pair of points x and y.

Four our purposes it is useful to introduce a quasi-distance δ, more explicitely defined and sometimes easier than d to work with.

If $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$ put $H_0(x, t) = x$ and $H_{k+1}(x, t) = H_k(x, t) + t\lambda_{k+1}(H_k(x, t))e_{k+1}$ for $k = 0, \ldots, n-1$. Here $\{e_k\}_{k=1}^n$ is the usual canonical basis in $\mathbf{R}^n$. It is clear that the function $s \to F_j(x, s) = s\lambda_j(H_{j-1}(x, s))$, is strictly increasing on $(0, \infty)$ for any $x = (x_1, \ldots, x_n)$ such that $x_k \geq 0$, $k = 1, \ldots, j-1$, and for $j = 1, \ldots, n$. Hence it is possible to define the inverse function of $F_j(x, \cdot)$, that is $\phi_j(x, \cdot) = (F_j(x, \cdot))^{-1}$ for $j = 1, \ldots, n$.

Definition 1.3. *For any $x, y \in \mathbf{R}^n$ we define δ: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_0^+$ as*

$$\delta(x, y) = \max_{j=1,\ldots,n} \phi_j(x^*, |x_j - y_j|)$$

where $x^* = (|x_1|, \ldots, |x_n|)$.

The following two Lemmas contain the basic properties of the functions F_j, ϕ_j, d and δ.

Lemma 1.4. *Put* $G_1 = 1$ *and* $G_j = 1 + \sum_{i=1}^{j-1} G_i b_{ji}$ *for* $j = 2, \ldots, n$. *Then*

(1.5) *for every* $x \in \mathbf{R}^n$, $s > 0$, $\theta \in (0, 1)$ *we have*

$$\theta^{G_j} \leq \frac{F_j(x^*, \theta s)}{F_j(x^*, s)} \leq \theta,$$

$$\theta \leq \frac{\phi_j(x^*, \theta s)}{\phi_j(x^*, s)} \leq \theta^{1/G_j};$$

(1.6) *if* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *verifies* $|y_i| + F_i(y^*, \theta s) \leq |x_i| + KF_i(x^*, s)$, $i = 1, \ldots, j$, *for some* $s > 0$, $\theta \in (0, 1]$ *and* $K \geq 1$, *we have*

$$\frac{F_{j+1}(j^*, \theta s)}{F_{j+1}(x^*, s)} \leq \theta K^{G_{j+1}-1}.$$

PROOF: For (1.5) see Proposition 4.3 of [FL2]. Let us prove (1.6) from (0.5) we get that λ_{j+1} is increasing in each variable on $\{x \in \mathbf{R}^n : x_k \geq 0 \ k = 1, \ldots, j\}$, then

$$\begin{aligned} F_{j+1}(y^*, \theta s) &= \theta s \lambda_{j+1}(|y_1| + F_1(y^*, \theta s), \ldots, |y_j| + F_j(y^*, \theta s)) \\ &\leq \theta s \lambda_{j+1}(|x_1| + F_1(x^*, Ks), \ldots, |x_j| + F_j(x^*, Ks)) \\ &\leq \theta K^{G_{j+1}-1} F_{j+1}(x^*, s), \end{aligned}$$

the last inequality follows from (1.5).

In the sequel we shall use the following notation for d-balls, δ-balls and their dilations

$$\begin{aligned} S(x, r) &= \{y \in \mathbf{R}^n : d(x, y) < r\}, \\ Q(x, r) &= \{y \in \mathbf{R}^n : \delta(x, y) < r\} \\ \alpha S(x, r) &= S(x, \alpha r), \alpha Q(x, r) = Q(x, \alpha r), \alpha > 0. \end{aligned}$$

Lemma 1.7. *There exist constants a, b, $A \in (1, \infty)$, depending only on n and the constants in* (0.5), *such that*

(1.8) $$\frac{1}{a} \le \frac{d(x, y)}{\delta(x, y)} \le a, \text{ for all } x, y;$$

(1.9) $$\frac{1}{b} \, |x - y| \le d(x, y) \le b \, |x - y|^{\eta}$$

if $|x - y| \le 1$, *where* $\eta = \min_j \{1/G_j\}$;

(1.10) $|2S| \le A\,|S|$, $|2Q| \le A\,|Q|$ *for any d-ball S and any δ-ball Q.*

PROOF: For (1.8) see Theorems 2.6 and 2.7 in [FL1]. (1.9) and (1.10) follow immediately from the above Lemma and (1.8).

2.

Here we shall construct families of δ-balls that resembles the family of dyadic cubes. Let τ be the set of all n-tuples $s = l_1 \dots l_n$ with $l_i = -1, 0, 1$; $i = 1, \dots, n$. For $k \in \mathbf{Z}$; $l_i = -1, 0, 1$; $j_i \in \mathbf{Z}$ and $i = 1, \dots, n$, define

$$T^k_{l_1} \colon \mathbf{R}^n \to \mathbf{R}^n;\ T^k_{l_1}(x) = x + l_1 2^k e_1,$$
$$T^k_{l_1 \dots l_{i-1} l_i} \colon \mathbf{R}^n \to \mathbf{R}^n;\ T^k_{l_1 \dots l_{i-1} l_i}(x) = T^k_{l_1 \dots l_{i-1}}(x) + l_i F_i(T^k_{l_1 \dots l_{i-1}}(x)^*, 2k) e_i,$$

and

$$x_{j_1} = (2j_1 - 1) 2^k e_1,$$
$$x_{j_1 \dots j_i} = x_{j_1} + \dots + x_{j_1 \dots j_{i-1}} + (2j_i - 1) F_i(x^*_{j_1 \dots j_{i-1}}, 2^k) e_i.$$

For $k \in \mathbf{Z}$ and $s = l_1 \dots l_n \in \tau$ given, the family of δ-balls

$$D^{k,s} = \{Q(T^k_{l_1 \dots l_n}(x_{j_1 \dots j_n}), 2^k) : j_k \in \mathbf{Z};\quad i = 1, \dots, n\}$$

is an a.e. covering of $\mathbf{R}^n$.

The following Lemma states the main property of these families.

Lemma 2.2. *For* $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \mathbf{R}^n$ *and* $r > 0$, *there exists* $s \in \tau$ *and* $Q_0 \in D^{k,s}$ *with* $2^{k-1} < 2r \le 2^k$ *such that*

(2.3) $Q(\tilde{x}, r) \subset Q_0$,

(2.4) $|Q_0| \leq C|Q(\tilde{x}, r)|$, *where C is independent of* $\tilde{x}$, r, k *and* s.

PROOF. It is obvious that there exists a center c_1 of a δ-ball belonging to D^{k,s_1}, with $s_1 = 1\ldots 1$, and a l_1 in $\{-1, 0, 1\}$ such that

$$x_1(T^k_{l_1 1\ldots 1}(c_1)) \in [\tilde{x}_1 - F_1(\tilde{x}^*, r), \tilde{x}_1 + F_1(\tilde{x}^*, r)]$$
$$\subseteq [x_1(T^k_{l_1}(c_1)) - F_1(T^k_{l_1}(c_1)^*, 2^k), x_1(T^k_{l_1}(c_1)) + F_1(T^k_{l_1}(c_1)^*, 2^k)],$$

where $x_1(T^k_{l_1}(c_1))$ denote the first component of $T^k_{l_1}(c_1)$. Now, let us suppose that we have determined $l_2, \ldots, l_m$ and c_i, with $Q(c_i, 2^k) \in D^{k,s_i}$, $s_i = l_1\ldots l_{i-1}1\ldots 1$, $i = 1, \ldots, m$ in such a way that

(2.5) $x_j(c_i) = x_j(\mathrm{T}^k_{1\ldots 1 l_j}(c_j))$, $j = 1, \ldots, i-1$,

$$x_i(T^k_{l_1\ldots l_i}(c_i)) \in [\tilde{x}_i - F_i(\tilde{x}^*, r), \tilde{x}_i + F_i(\tilde{x}, r)]$$
$$\subset [x_i(T^k_{l_1\ldots l_i}(c_i)) - F_i(T^k_{1\ldots 1 l_i}(c_i)^*, 2^k), x_i(T^k_{l_1\ldots l_i}(c_i)) + F_i(T^k_{1\ldots 1 l_i}(c_i)^*, 2^k)].$$

Then

$$|\tilde{x}_i| + F_i(\tilde{x}^*, r) \leq |x_i(T^k_{1\ldots 1 l_i}(c_i))| + F_i(T^k_{1\ldots 1 l_i}(c_i)^*, 2^k),$$

for $i = 1, \ldots, m$. Now, by using Lemma 1.4, we get

$$F_{m+1}(\tilde{x}^*, r) \leq 2F_{m+1}(T^k_{1\ldots 1 l_m}(c_m)^*, 2^k).$$

From this inequality it follows inmediately that there exists a center c_{m+1} of a δ-ball in $D^{k,s_{m+1}}$, where $s_{m+1} = l_1\ldots l_m 1\ldots 1$, and a value of l_{m+1} such that (2.5) holds with $i = m+1$. The inductive process continues to obtain (2.5) for $i = n$. Then, taking $c = T^k_{1\ldots 1 l_n}(c_n)$, $Q_0 = Q(c, 2^k)$ and $s = l_1\ldots l_n$, we get (2.3). Finally (2.4) follows from the choice of k and the doubling property (1.10).

3.

The main results of this section are the following.

Theorem 3.1. *Let* $\beta > 0$, *fixed, let* $Q = Q(\tilde{x}, r)$ *be a* δ*-ball such that* $Q \subset \Omega$ *and let* $(v, w) \in S_{\sigma,\alpha}$ *for a given* $\alpha \in \left(1 - \left(\sum_j G_j\right)^{-1}, 1\right]$. *Then, for each* $u \in C^1(\overline{Q})$ *such that verifies*

(3.2) $\quad |\{x \in Q : u(x) = 0\}| \geq \beta |Q|,$

we get

$$\left(\frac{1}{w(Q)} \int_Q |u|^{2\sigma} w dx\right)^{1/2\sigma} \leq Cr \left(\frac{1}{v(Q)} \int_Q |\nabla_\lambda u|^2 v dx\right)^{1/2}, \tag{3.3}$$

where C depends only on n and the constants in (0.5) and (0.7), and $\nabla_\lambda u = (\lambda_1 D_1 u, \ldots, \lambda_n D_n u)$.

Theorem 3.4 (Sobolev inequality). *Let Q and (v, w) be as in the above theorem. Then, for any $u \in C_0^1\left(\frac{1}{2} Q\right)$ we get*

$$\left(\frac{1}{w(Q)} \int_{1/2Q} |u|^{2\sigma} w dx\right)^{1/2\sigma} \leq Cr \left(\frac{1}{v(Q)} \int_{1/2Q} |\nabla_\lambda u| \, v dx\right)^{1/2}, \tag{3.5}$$

where C depends only on n and the constants in (0.5) *and* (0.7).

Theorem 3.6 (Poincaré inequality): *Let Q and (v, w) be as in Theorem* 3.1. *Then, for any $u \in C^1(\overline{Q})$ we get*

$$\left(\frac{1}{w(Q)} \int_Q |u - u_Q|^{2\sigma} w dx\right)^{1/2\sigma} \leq Cr \left(\frac{1}{v(Q)} \int_Q |\nabla_\lambda u|^2 v dx\right)^{1/2}, \tag{3.7}$$

where $u_Q = \int_Q u \, w dx$ and C depends only on n and the constants in (0.5) *and* (0.7).

The proof of Theorem 3.1 is based on an estimate of u in terms of certain fractional integral operators applied to $\nabla_\lambda u$ and on a norm inequality with two weights for these operators.

Let us start with the definition of these operators.

Definition 3.8. *Let $k \in \mathbf{Z}$, $s \in \tau$ and $\mu \in (0, 1]$. We define*

$$(P_\mu^{k,s} f)(x) = \begin{cases} \dfrac{1}{|Q|^\mu} \displaystyle\int_Q |f(y)| \, dy, & \text{if } x \in Q \in D^{k,s} \\ 0 & \text{if } x \in \mathbf{R}^n - \displaystyle\bigcup_{Q \in D^{k,s}} Q \end{cases}$$

for $f \in L^1_{loc}(\mathbf{R}^n)$.

Now, with these operators, we can prove

Lemma 3.9. *Let* $\beta > 0$, *fixed, and let* $Q = Q(\tilde{x}, r)$ *be a* δ*-ball. If* $u \in C^1(\overline{Q})$ *verifies*

$$|\{x \in Q : u(x) = 0\}| \geq \beta |Q|$$

then, for each $\mu \in \left(1 - \left(\sum_j G_j\right)^{-1}, 1\right]$ *there exist a sequence* $\{a_i\}_{i=1}^{\infty} \subset \mathbf{R}^+$ *depending only on* μ *with* $\sum a_i < \infty$, *a sequence of integers* $\{k_i\}$ *depending only on* r *and a constant* C *such that*

$$|u(x)| \leq Cr|Q|^{\mu-1} \sum_{i=1}^{\infty} a_i \sum_{s \in \tau} (\mathrm{P}_{\mu}^{k_i,s}(\chi_Q |\nabla_\lambda u|))(x)$$

for all $x \in Q$.

PROOF. By using similar techniques than those in Lemma 4.3 in [FS] and a dyadic partition, we get

$$(3.10) \quad |u(x)| \leq C_0 \int_0^{2a^2r} \frac{1}{|S(x,t)|} \int_{S(x,C_0t)} |\nabla_\lambda u(y)| \chi_Q(y) dy dt$$

$$\leq C_0 r \sum_{i=1}^{\infty} \left(\int_{\frac{a^2r}{2^{i-1}}}^{\frac{a^2r}{2^{i-2}}} \frac{dt}{|S(x,tr)|^{1-\mu}} \right)$$

$$\frac{1}{\left|S\left(x, \frac{a^2r}{2^{i-1}}\right)\right|^{\mu}} \int_{S(x,\, C_0a^2r/2^{i-2})} |\nabla_\lambda u(y)| \chi_Q(y) dy,$$

for all $x \in Q$. From Lemmas 1.7 and 2.2 follows that for each i there exists $k_i \in \mathbf{Z}$, $s_i \in \tau$ and $Q_i \in D^{k_i,s_i}$ such that

$$2^{k_i-1} < \frac{C_0 a^3 r}{2^{i-3}} \leq 2^{k_i},$$

$$S\left(x, \frac{C_0 a^2 r}{2^{i-1}}\right) \subset Q_i,$$

$$|Q_i| \leq C \left| S\left(x, \frac{C_0 a^2 r}{2^{i-1}}\right) \right|,$$

On the other hand, by (1.5) and (1.8), we get $|S(x, tr)| \geq Ct^{\Sigma G_j} |S(x, r)|$ for $t \in (0, 1]$. Then, taking $i_0 \in \mathbf{N}$ such that $a^2/2^{i_0-1} < 1 \leq a^2/2^{i_0-2}$, we obtain from (3.10)

$$|u(x)| \leq Cr \sum_{i=1}^{i_0} \left(\int_{\frac{a^2}{2^{i-1}}}^{\frac{a^2}{2^{i-2}}} \frac{dt}{|S(x, tr)|^{1-\mu}} \right) \frac{1}{|Q_i|^{\mu}} \int_{Q_i} |\nabla_\lambda u(y)| \chi_Q(y) dy$$

$$+ \frac{1}{|S(x, r)|^{1-\mu}} \sum_{i=i_0+1}^{\infty} \left(\int_{\frac{a}{2^{i-1}}}^{\frac{a^2}{2^{i-1}}} t^{-(\sum_j G_j)(1-\mu)} dt \right) \frac{1}{|Q_i|^{\mu}} \int_{Q_i} |\nabla_\lambda u(y)| \chi_Q(y) dy$$

$$\leq Cr |S(x, r)|^{\mu-1} \sum_{i=1}^{\infty} a_i \sum_{s \in \tau} (P_\mu^{k_i, s}(|\nabla_\lambda u| \nu_Q))(x)$$

where $a_i = 1$ for $i = 1, \ldots, i_0$ and $a_i = 2^{(i-1)[(\mu-1)(\sum_j G_j)+1]}$. Thus, since $|Q| \simeq |S(x, r)|$ we get the thesis.

Remark 3.11. From the proof is easy to see that $2^{k_i} \geq 8a^2r$ then $Q \subset S(\tilde{x}, ar) \subset S(x, 2ar) \subset S(x, C_0 ar/2^{i-2})$ and thus

$$P_\mu^{k_i, s_i}(|\nabla_\lambda u| \chi_Q)(x) = \frac{1}{|Q_i|^{\mu}} \int_Q |\nabla_\lambda u| \, dy.$$

In the following Lemma we prove a two weight norm inequality for the operators $P_\mu^{x,s}$. The proof is based on techniques of E. T. Sawyer (see [S]).

Lemma 3.12. *Suppose $1 < p \leq q < \infty$. Let $E \subset \mathbf{R}^n$ be a bounded open set and let (v, w) be a pair of non negative integrable functions defined on E. Then*

$$\text{(3.13)} \qquad \left(\int_E |P_\mu^{k,s} f|^q w dx \right)^{1/q} \leq C_0 \left(\int_E |f|^p v dx \right)^{1/p},$$

for all $f \in L^p(E, vdx)$ with supp $f \subset E$, *if and only if*

$$(3.14)\quad w(E \cap Q)^{1/q}\left(\frac{v^{1/p-1}(E \cap Q)}{|Q|^{\mu}}\right)^{1-1/p} \le C_0$$

for all $Q \in D^{k,s}$. *The constant* C_0 *in* (3.13) *and* (3.14) *is the same.*

PROOF. For sake of simplicity P_μ and D denote $P_\mu^{k,s}$ and $D^{k,s}$ respectively. Now assume that (3.13) holds. Let us first show that $v^{-1/(p-1)}\chi_{E \cap Q} \in L^p(E, vdx)$ for $Q \in D$. Suppose this is not the case, then since

$$\int_{E \cap Q} v^{-1/(p-1)}dx = \int_{E \cap Q} v^{-p/(p-1)}vdx = \infty,$$

we can find a g in $L^p(E, vdx)$ such that

$$\int_{E \cap Q} gv^{-1}vdx = \infty,$$

which is a contradiction with (3.13) taking $f = g$ on E for every $Q \in D$. We get (3.14) by taking $f = v^{-1/(p-1)}\chi_{Q_0 \cap Q}$ in (3.13). Conversely assume that (3.14) holds. Then

$$\int_E |P_\mu f|\, wdx = \sum_{Q \in D} \int_{E \cap Q}\left(\frac{1}{|Q|^{\mu}}\int_Q |f|\, dy\right)^q wdx$$

$$\le \sum_{Q \in D} w(E \cap Q)\left(\frac{v^{-1/(p-1)}(E \cap Q)}{|f|^{\mu}}\right)^{q(p-1)/p}\left(\int_Q |f|^p vdx\right)^{q/p}.$$

Finally, from (3.14) and the fact that,

$$\sum_{Q \in D}\left(\int_Q |f|^p vdx\right)^{q/p} \le \left(\sum_{Q \in D}\int_Q |f|^p vdx\right)^{q/p}$$

for $q \ge p$, we obtain (3.13).

PROOF OF THEOREM (3.1). From Lemma (3.9) we get

$$\left(\frac{1}{w(Q)} \int_Q |u|^{2\sigma} w dx\right)^{1/2\sigma}$$

$$\leq \frac{Cr}{|Q|^{1-\mu}} \sum_{i=1}^{\infty} a_i \sum_{s \in \tau} \left(\frac{1}{w(Q)} \int_Q |P_\mu^{k_i,s}(|\nabla_\lambda u| \chi_Q)|^{2\sigma} w dx\right)^{1/2\sigma}.$$

Now, the inequality (3.3) follows by applying (0.7), (0.8) and Lemma 3.12 for the k_i such that $2^{k_i} \leq 8a^2 r$ and the Remark 3.11, the Schwartz inequality and (0.8) for the remainders.

PROOF OF THEOREM 3.4. We need only to apply the above theorem in the ball Q, keeping in mind that, by the doubling property (1.10), it follow inmediately that $|Q - (1/2)Q| \simeq |Q|$.

PROOF OF THEOREM 3.6. With Q and u given it is always possible to find a number $b = b(Q, u)$ sich that $Q^+ = \{x \in Q : u(x) \geq b\}$ and $Q^- = \{x \in Q : u(x) \leq b\}$ verifies

$$(3.15) \quad |Q^+| \geq \frac{1}{2}|Q| \quad \text{and} \quad |Q^-| \geq \frac{1}{2}|Q|.$$

Assume this fact, then both functions $(u - b)^+$ and $(u - b)^-$ satisfy the hipotheses of Theorem (3.1) with $\beta = 1/2$. By that Theorem we get

$$\frac{1}{w(Q)} \int_{Q^\pm} |u - b|^{2\sigma} w dx \leq (Cr)^{2\sigma} \left(\frac{1}{w(Q)} \int_{Q^\pm} |\nabla_\lambda u|^2 v dx\right)^\sigma,$$

adding these two inequalities we have

$$\frac{1}{w(Q)} \int_Q |u - b|^{2\sigma} w dx \leq 2(Cr)^{2\sigma} \left(\frac{1}{v(Q)} \int_Q |\nabla_\lambda u|^2 v dx\right)^\sigma.$$

Then, since

$$\left(\int_Q |u-u_Q|^{2\sigma}wdx\right)^{1/2\sigma} \le \left(\int_Q |u-b|^{2\sigma}wdx\right)^{1/2\sigma}$$

$$+\left(\frac{1}{w(Q)}\int_Q |u-b|\,wdx\right) w(Q)^{1/2\sigma}$$

$$\le 2\left(\int_Q |u-b|^{2\sigma}wdx\right)^{1/2\sigma},$$

we obtain the thesis. Let us prove (3.15). Observe that the two functions $\phi(t) = |\{x \in Q: u(x) \le t\}|$ and $\psi(t) = |\{x \in Q: u(x) \ge t\}|$ are respectively increasing, right-continuous and decreasing, left-continuous. Define $b = \inf\{t: \phi(t) \ge 1/2\,|Q|\}$ then by the right-continuity $\phi(b) \ge 1/2\,|Q|$. Suppose now, by contradiction, that $\psi(b) < 1/2\,|Q|$. Then by the left-continuity there is $t < b$ such that $\psi(t) < |Q|/2$, so that $\phi(t) > |Q|/2$ and this contradicts the definition of b. Finally (3.15) holds.

4.

In [FS], B. Franchi and R. Serapioni prove inequalities of type (3.3), (3.5) and (3.7) for the case $v = Cw$. The assumption on the weight is that $w \in A_2$ respect to the d-balls, *i.e.:* $w(S)w^{-1}(S) \simeq |S|^2$ for all d-ball S. Inequalities of the same type for the euclidean case, *i.e.:* $\lambda_i = 1$ for all i, have been proved by S. Chanillo and R. Wheeden in [ChW1]. The hypotheses on the pair of weights in that work are the euclidean case of

(4.1) $w \in D_\infty$ respect to δ- balls, *i.e.:* $w(Q) \simeq w(2Q)$ for every δ-ball Q

(4.2) $v \in A_2$ respect to δ-balls, *i.e.:* $v(Q)v^{-1}(Q) \simeq |Q|^2$ for every δ-ball Q

(4.3) there exists $\sigma > 1$ and $C > 0$ such that

$$\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{1/2\sigma} \le C\left(\frac{v(\theta Q)}{v(Q)}\right)^{1/2}, \text{ for every } \delta\text{-ball } Q, \text{ and } \theta \in (0,1],$$

We say that (v, w) belongs to C_σ if (v, w) satisfy the conditions (4.1), (4.2) and (4.3). The main purpose of this section is to find relations among the conditions of type A_2 and C_σ and the condition $S_{\sigma,\alpha}$. We begin with the following result.

Lemma 4.4. *Let* $(v, w) \in S_{\sigma,\alpha}$ *for some* $\alpha \in [1 - (\Sigma_j G_j)^{-1}, 1]$ *and* $\Omega = \mathrm{R}^n$. *The* (v, w) *verifies* (4.2) *and* (4.3).

PROOF. By taking $Q_0 = Q$ in (0.7), we get $v(Q_0)v^{-1}(Q_0) \leq C|Q_0|^2$ and thus v satisfy (4.2). Now assume that $\alpha = 1 - (\Sigma_j G_j)^{-1}$ then, from (0.7), we get

$$\left(\frac{w(\theta Q)}{w(Q)}\right)^{1/2\sigma} (v^{-1}(\theta Q)v(Q))^{1/2} \leq C|Q|^{(\Sigma_j G_j)^{-1}} |\theta Q|^{1-(\Sigma_j G_j)^{-1}}$$

for any δ-ball Q and any $\theta \in (0, 1]$. From this and Hölder inequality follows (4.3), in fact

$$\left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{1/2\sigma} \leq C \frac{|\theta Q|}{(v^{-1}(\theta Q)v(Q))^{1/2}}$$

$$\leq C\left(\frac{v(\theta Q)}{v(Q)}\right)^{1/2}.$$

Now, to complete the proof, it is sufficient to prove that if $\alpha_1, \alpha_2 \in (0, 1]$ and $\alpha_1 > \alpha_2$ then $S_{\sigma,\alpha_1} \subset S_{\sigma,\alpha_2}$. Note that only is necessary to prove that (0.7) with $\alpha = \alpha_2$ holds. This is trivial if $w(Q_0 \cap Q)$ or $v^{-1}(Q_0 \cap Q)$ is zero. Assume that both $w(Q_0 \cap Q)$ are positive. The inequality in (0.7) with $\alpha = \alpha_1$ is equivalent to

$$\left(\frac{|Q_0|}{|Q|}\right)^{\alpha_1} \leq C \frac{|Q_0|}{(v^{-1}(Q_0 \cap Q)v(Q_0))^{1/2}} \left(\frac{w(Q_0)}{w(Q_0 \cap Q)}\right)^{1/2\sigma}. \tag{4.5}$$

On the other hand, since $Q_0 \cap Q \neq \emptyset$ and radious $(Q) \leq 8a^2$ radious (Q_0), the doubling property (1.10) allow us to write

$$\left(\frac{|Q_0|}{|Q|}\right)^{\alpha_1} \leq C\left(\frac{|Q_0|}{|Q|}\right)^{\alpha_1}$$

with C independent of Q_0 and Q. From this and (4.5) follows that $(v, w) \in S_{\sigma,\alpha_2}$.

We shall next show that condition C_σ implies a condition $S_{\sigma,\alpha}$. In the proof of this fact we shall use the following result.

Lemma 4.6. *Let* (v, w) *be a pair of non negative weights satisfying* (4.1) *and* (4.3). *Then there exist* $\eta \in (0, 1)$ *and* $\sigma' \in (1, \sigma)$ *such that the inequality* (4.3) *holds with* $(\Sigma_j G_j)/\eta$ instead of $\Sigma\, G_j$ and σ' instead of σ.

PROOF. Since $w \in D_\infty$ we get

$$w(Q) = w\left(\frac{1}{2}\, Q\right) + w\left(Q - \frac{1}{2}\, Q\right)$$

$$\geq (1 + C)\, w\left(\frac{1}{2}\, Q\right)$$

for every δ-ball Q. By iteration we have a $\beta \geq 1$ such that

$$\text{(4.7)} \qquad \frac{w(\theta Q)}{w(Q)} \geq C\theta^\beta \quad \text{for every} \quad \theta \in (0, 1], \quad \text{and every} \quad \delta\text{-ball } Q.$$

On the other hand, from (1.5) it follows that

$$|\theta Q| \geq \theta^{\Sigma_j G_j} |Q| \quad \text{for every} \quad \theta \in (0, 1], \quad \text{and every} \quad \delta\text{-ball } Q.$$

Then, this inequality (4.7) and (4.4) allow us to obtain the inequality

$$\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}(1-\epsilon\beta)} \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}+\epsilon} \leq C\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}(1-\epsilon\beta)} \theta^{\epsilon\beta}\left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}}$$

$$\leq C\left(\frac{|\theta Q|}{|Q|}\right)^{(\sum_j G_j)^{-1}} \left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}}$$

$$\leq C\left(\frac{v(\theta Q)}{v(Q)}\right)^{1/2},$$

for all $\theta \in (0, 1]$, $\epsilon > 0$, and all δ-ball Q. Finally, by taking ϵ in $(0, \min\{1/\beta, (1-1/\sigma)/2\sigma\})$, we get the thesis with $\eta = 1 - \epsilon\beta$ and $\sigma' = \dfrac{\sigma}{1 + 2\epsilon\sigma}$.

Lemma 4.8. *Let $(v, w) \in C_\sigma$. Then $(v, w) \in S_{\sigma',\alpha}$ for some $\alpha \in ((\Sigma_j G_j)^{-1}, 1]$ and some $\sigma' \in (1, \sigma)$.*

PROOF. We only need to prove (0.7). Let $Q_0 = Q(x_1, r_1)$ and $Q = Q(x_2, r_2)$ be two δ-balls such that $r_2 \leq 8a^2 r_1$. If $Q_0 \cap Q = \emptyset$ there is nothing to prove. Assume $Q_0 \cap Q \neq \emptyset$, then there exists $C_1 = C_1(a) > 1$ and $C_2 = C_2(a)$ such that $Q_0 \subset \tilde{Q} = (C_1 r_1/r_2)\,Q \subset C_2 Q_0$. Now, let σ' and η be as in the above Lemma and $\theta = r_2/C_1 r_1$. Then, from (4.1), (4.2), (1.10) and Lemma 4.6 it follows that

$$\left(\frac{w(Q_0 \cap Q)}{w(Q_0)}\right)^{1/2\sigma'} (v^{-1}(Q_0 \cap Q)v(Q_0))^{1/2}$$

$$\leq C\left(\frac{w(\theta\tilde{Q})}{w(\tilde{Q})}\right)^{1/2\sigma'} (v^{-1}(\theta\tilde{Q})v(\tilde{Q}))^{1/2}$$

$$\leq C\left(\frac{|\tilde{Q}|}{|\theta\tilde{Q}|}\right)^{(\sum_j G_j)^{-1}\eta} (v^{-1}(\theta\tilde{Q})v(\theta\tilde{Q}))^{1/2}$$

$$\leq C|\tilde{Q}|^{(\sum_j G_j)^{-1}\eta}\,|\theta\tilde{Q}|^{1-(\sum_j G_j)^{-1}\eta}$$

$$\leq C|Q_0|^{(\sum_j G_j)^{-1}\eta}\,|Q|^{1-(\sum_j G_j)^{-1}\eta}.$$

Thus, $(v, w) \in S_{\sigma',\alpha}$ with $\alpha = 1 - (\Sigma_j G_j)^{-1}\eta$.

Now, by using the above result, we get

Lemma 4.9. *Let $w \in A_2$ with respect to d or δ-balls. Then $(w, w) \in S_{\sigma,\alpha}$ for some $\sigma > 1$ and some $\alpha \in (1 - (\Sigma_j G_j)^{-1}, 1]$.*

PROOF. From the previous Lemma, we only need to prove that $(w, w) \in C_\sigma$ for some $\sigma > 1$. We know that

$$\frac{w(\theta Q)}{w(Q)} \geq C\frac{w(\theta Q)w^{-1}(Q)}{|Q|^2} \geq C\frac{w(\theta Q)w^{-1}(\theta Q)}{|Q|^2} \geq C\left(\frac{|\theta Q|}{|Q|}\right)^2$$

for all $\theta \in (0, 1]$ and all δ-ball Q. Then, by taking $\sigma > 1$ such that $1/\sigma > 1 - (\Sigma_j G_j)^{-1}$, we get

$$\left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}}\left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}} = \left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}}\left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2\sigma}+\frac{1}{2}-\frac{1}{2}}$$

$$\leq C\left(\frac{w(\theta Q)}{W(Q)}\right)^{\frac{1}{2}}\left(\frac{|\theta Q|}{|Q|}\right)^{(\Sigma_j G_j)^{-1}+\frac{1}{\sigma}-1}$$

$$\leq C\left(\frac{w(\theta Q)}{w(Q)}\right)^{\frac{1}{2}}.$$

Thus $(w, w) \in C_\sigma$.

Let us describe some examples of pairs (v, w) that satisfy the hypotheses (0.6) and (0.7) for any $\lambda_1, \ldots, \lambda_n$ in the conditions (0.3) to (0.5).

EXAMPLE 4.10. In [FS], Franchi and Serapioni prove that $d(0, x)^\beta$ for $\beta \in (-n, n)$ is a weight in A_2 with respect to d-balls. In particular they prove the following inequalities

$$(4.11)\qquad \int_{S(y,r)} d(0, x)^\beta dx \leq [d(0, y) + r\beta|\beta|]^\beta |S(y, r)|$$

if $d(0, y) \geq 2r$,

$$\int_{S(y,r)} d(0, x)^\beta dx \leq Cr^\beta |S(y, r)|,$$

if $d(0, y) \leq 2r$.

These facts allows us to prove that there exists values of β in $(0, n)$ such that $v(x) = d(0, x)^{\beta}$ and $w(x) = v(x)^{-1}$ belong to a class $S_{\sigma,\alpha}$ for some $\sigma > 1$ and some α in $(1 - (\Sigma_j G_j)^{-1}, 1]$. Since both v and w belong to A_2 we only need to show that there exists σ and α such that (0.7) holds with S instead of Q_0 and θS instead of Q for any θ in (0.1], where S is any d-ball. Let us now prove this fact. First, note that, by the A_2 condition, we get

$$\left(\frac{w(\theta S)}{w(S)}\right)^{\frac{1}{2\sigma}} (v^{-1}(\theta S)v(S))^{1/2} \leq C \frac{(w(\theta S)w^{-1}(S))^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/2}} \tag{4.12}$$

for all $\sigma > 1$. Let $S = S(y, r)$. If $d(0, y) \geq 2r$, from (4.11) we get

$$\frac{(w(\theta S)w^{-1}(S))^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/\sigma}} \leq 3^{\beta} |S|^{1-\frac{1}{2}\left(1+\frac{1}{\sigma}\right)} |\theta S|^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}.$$

On the other hand, if $d(0, y) \leq 2\theta r$, from the same inequalities and Lemmas 1.4 and 1.7 it follows that

$$\frac{(w(\theta S)w^{-1}(S))^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/\sigma}} \leq C \frac{(\theta^{-\beta}|S|\,|\theta S|)^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)}}{|S|^{1/\sigma}}$$

$$\leq C|S|^{1-\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)} |\theta S|^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)}.$$

The case $2\theta r < d(0, y) < 2r$ follows in a similar way, so we get that (v, w) belongs $S_{\sigma,\alpha}$ for

$$\alpha = \frac{1}{2}\left(1 + \frac{1}{\sigma}\right)\left(1 - \frac{\beta}{n}\right).$$

Then, by taking $\sigma = 1 + \beta/n$, we can choose $\beta_0 \in (0, n)$ such that $\alpha \in (1 - (\Sigma_j G_j)^{-1}, 1]$ for all $\beta \in (0, \beta_0]$.

EXAMPLE 4.13. Let $w(x) = d(0, x)^{-\beta}$ for $\beta \in (0, n)$. From (4.11) and Lemmas 1.4 and 1.7 we get

$$\frac{w(\theta S)}{w(S)} \leq C\left(\frac{|\theta S|}{|S|}\right)^{1-\frac{\beta}{n}} \text{ for every } d\text{-ball } S \text{ and all } \theta \in (0, 1]. \tag{4.14}$$

The for $\nu \equiv 1$ we get that for any $\sigma > 1$ and $\epsilon \in (0, 1)$, it holds

$$\left(\frac{w(\theta S)}{w(S)}\right)^{\frac{1}{2\sigma}} \leq C\left(\frac{|\theta S|}{|S|}\right)^{\left(1-\frac{\beta}{n}\right)\frac{1}{2\sigma}}\left(\frac{|\theta S|}{\nu^{-1}(\theta S)}\right)^{\frac{1}{2}}$$

$$\leq C\,\frac{|S|^{1-\left(\frac{\epsilon}{2}+\left(1-\frac{\beta}{n}\right)\frac{1}{2\sigma}\right)}\,|\theta S|^{\frac{\epsilon}{2}+\left(1-\frac{\beta}{n}\right)\frac{1}{2\sigma}}}{(\nu^{-1}(\theta S)\nu(S))^{1/2}}.$$

Thus, $(\nu, w) \in S_{\sigma,\alpha}$ for $\alpha = \epsilon/2 + (1 - \beta/n)/2\sigma$. By taking $\epsilon = 1 - \mathrm{b}/n$ and $\sigma = 1 + \beta/n$, we can choose $\beta_0 \in (0, n)$ such that $\alpha \in (1 - \Sigma_j G_j)^{-1}, 1]$ for every $\beta \in (0, \beta_0]$.

EXAMPLE 4.15. Let $\nu(x) = d(0, x)^\beta$, $\beta \in (0, n)$, and $w(x) \equiv 1$. Then, from (4.14), we get

$$\left(\frac{w(\theta S)}{w(S)}\right)^{\frac{1}{2\sigma}}(\nu^{-1}(\theta S)\nu(S))^{1/2} \leq C|S|^{1-\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)}|\theta S|^{\frac{1}{2}\left(1+\frac{1}{\sigma}\right)\left(1-\frac{\beta}{n}\right)}$$

for all $\sigma > 1$. Thus, by reasoning as in the preceding examples, we get that there exists $\beta_0 \in (0, n)$ such that $(\nu, w) \in S_{\sigma,\alpha}$, for some $\sigma > 1$ and some $\alpha \in (1 - (\Sigma\, G_j)^{-1}, 1]$, for all $\beta \in (0, \beta_0]$.

5.

Let S be a d-ball such that $2a^2S \subset \Omega$. For ϕ and ψ in Lip $(\bar{S})$ we define

(5.1) $$a_0(\phi, \psi) = \int_S \langle A\nabla\phi, \nabla\psi\rangle$$

(5.2) $$a(\phi, \psi) = a_0(\phi, \psi) + \int_S \phi\psi w.$$

It is easy to prove that (5.1) defines a scalar product in $\mathrm{Lip}_0(S)$ and that (5.2) defines a scalar product in $\mathrm{Lip}(\bar{S})$.

Definition 5.3. *We denote with* $H_0(S)$ *and* $H(S)$ *to the completion of* $\mathrm{Lip}_0(S)$ *and* $\mathrm{Lip}(\bar{S})$ *respect to the norms* $\|\cdot\|_0 = a_0(\cdot\,,\ \cdot)^{1/2}$ *and* $\|\cdot\| = a(\cdot\,,\ \cdot)^{1/2}$, *respectively.*

Remark 5.4. From Sobolev inequality (Theorem 3.4) we get $H_0(S) \subset H(S)$.

Remark 5.5. It is possible to associate a function in $L^2(S, wdx)$ to each element in $H(S)$ and define its derivative as functions in $L^2(S, vdx)$.

Definition 5.6. *Let* f *be such that* $f/w \in L^{2\sigma/(2\sigma-1)}(S, wdx)$ *and let* $\psi \in H(S)$. *We say that* $u \in H(S)$ *is a solution of*

$$\begin{aligned} Lu &= f \quad in \quad S \\ u &= \psi \quad in \quad \partial S \end{aligned}$$

if

$$a_0(u, \phi) = \int_S u\phi \quad for\ all \quad \phi \in H_0(S),$$

and $u - \psi \in H_0(S)$.

Definition 5.7. *Let* $F = (f_1, \ldots, f_n)$ *be such that* $|F|/v \in L^2(S, vdx)$ *and let* $\psi \in H(S)$. *We say that* $u \in H(S)$ *is solution of*

$$\begin{aligned} Lu &= -\mathrm{div}_\lambda F \quad in \quad S \\ u &= \psi \qquad\qquad in \quad \partial S \end{aligned}$$

if

$$a_0(u, \phi) = \int_S \langle F, \nabla_\lambda \phi\rangle \quad for\ all \quad \phi \in H_0(S),$$

and $u - \psi \in H_0(S)$.

Remark 5.8. We can prove, by the representation theorem for continuous linear functional on Hilbert spaces, the existence and uniqueness of solutions for the above Dirichlet problems.

Remark 5.9. The above definitions and remarks hold if we change d-balls by δ-balls.

By using the results of Sections 3 and 4, and the technique in [ChW2], we get

Theorem 5.10 (Harnack inequality). *Let $Q_0 = Q(\tilde{x}, 4R)$ be a δ-ball in Ω. If $u \in H(Q_0)$ is a non-negative solution of $Lu = 0$ and $Q = (1/4)Q_0$ then*

$$\sup_Q u \leq \exp\left\{C\left(\frac{w(2Q)}{w((1/2)Q)}\right)^\gamma \left(\frac{w(2Q)}{v(2Q)}\right)^{1/2}\right\} \inf_Q u,$$

where $\gamma = (3\sigma^2 - 2\sigma + 1)/(\sigma - 1)$ and C depends only on the constants in (0.5) *and* (0.7).

Let S be a d-ball such that $2aS \subset \Omega$. For $y \in S$ and $\varrho > 0$ fixed such that $Q_\varrho = Q(y, \varrho) \subset S$, we define the mapping

$$\psi \to \frac{1}{w(Q_\varrho)} \int_{Q_\varrho} \psi w, \quad \psi \in H_0(S).$$

From Sobolev inequality (3.4) follows that the above mapping is a continuous linear functional on $H_0(S)$. Then, there is a unique $G_y^\varrho \in H_0(S)$ such that

$$a_0(G_y^p, \psi) = \frac{1}{w(Q_\varrho)} \int_{Q_\varrho} \psi w, \quad \text{for all} \quad \psi \in H_0(S).$$

In the next, $G_y^\varrho = G^\varrho(\cdot, y)$ will be called the «ϱ-approximate Green function for S with pole y». For the sake of simplicity we often will use the notation G^ϱ.

Lemma 5.11. *G^ϱ is non negative on S.*

PROOF. Follows the line of the euclidean case. (See Section 3 of [ChW3].)

Lemma 5.12. *There exists a constant C such that*

$$w(\{G_y^\varrho > t\}) \leq C\left(\frac{R^2}{v(2Q)}\right)^\sigma \frac{w(2Q)}{t^\sigma} \quad \textit{for all } y \in S,\ \varrho > 0 \textit{ with } Q(y, \varrho) \subset S$$

where R is the radious of S and Q is the δ-ball with the same centre that S and radious aR.

PROOF. The technique is the same that Chanillo and Wheeden have used for the euclidean case (see Section 3 of [ChW3]) but with Sobolev inequality for our particular geometry (Theorem 3.4).

Then, with the above result we have

Lemma 5.13. *For each $p \in (0, \sigma)$ there exists C such that*

$$\sup_{r/2<d(y,x)<r} G_y^\varrho \le C\left(\frac{w(S(y,4r/3))}{v(S(y,r)}\right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{r^2}{v(S(y,r))}\left(\frac{w(S(y,8a^2r/3))}{\inf\limits_{r/2<d(y,z)<r} w(S(z,r/(4a^2))}\right)^{1/p}$$

for all $\varrho \in (0, r/4a)$ and for all y and r such that $S(y, 3a^4 r) \subset \Omega$.

PROOF. Let $x \in \{x \in S : r/2 < d(y, x) < 3r/4\}$ and $\varrho \in (0, r/4a)$, then $S(x, r/4) \subset S(y, r) \setminus S(y, a\varrho)$. Note that G^ϱ satisfies $Lu = 0$, then for each $p \in (0, \sigma)$, we have

$$\text{(5.14)}\quad \sup_{Q(x,r/8a)} G^\varrho \le C\left(\frac{w(Q(x,r/4a))}{v(Q(x,r/4a))}\right)^{\frac{\sigma^2}{p(\sigma-1)}}\left(\frac{1}{w(Q(x,r/4a))}\int_{Q(x,r/4a)}(G^\varrho)^p w\right)^{1/p},$$

where C depends only on the constants of (0.5) and (0.7) (see Lemmas 3.1 and 3.11 of [ChW2]). On the other hand, from Lemma 6.2 follows

$$\int_{Q(x,r/4a)} (G^\varrho)^p w \le Cw(S(y,2a^2r))\left(\frac{r^2}{v(S(y,2r))}\right)^p$$

for each $p \in (0, \sigma)$. From this and (6.4) we have

$$\sup_{Q(x,r/8a)} G^\varrho \le C\left(\frac{w(Q(x,r/4a))}{v(Q(x,r/4a))}\right)^{\frac{\sigma^2}{p(\sigma-1)}}\left(\frac{w(s(y,28^2r))}{w(Q(x,r/4a))}\right)^{1/p}\frac{r^2}{v(S(y,2r))}$$

for all p, σ and x. Then, for each $p \in (0, \sigma)$ the inequality

$$(5.15) \qquad \sup_{r/2 < d(y,x) < 3r/4} G^{\varrho} \leq C \left(\frac{w(S(y, r))}{v(S(y, r))} \right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{r^2}{v(S(y, r))} \frac{w(S(y, 2a^2 r))}{\inf_{r/2 < d(y,z) < 3r/4} w(S(x, r/4a^2))}$$

holds for all $\varrho \in (0, r/4a)$, and all y and r such that $S(y, 3a^4 r) \subset \Omega$.

The above inequality allows us to obtain a similar one but on $S \setminus (1/2)\, S$. Indeed, if G_0^{ϱ} denotes the ϱ-approximate Green function for $S_0 = S(y, 4r/3)$ with pole y then, by the weak maximum principle (the proof is similar that the Lemma 2.6 of [ChW3]), se have $G^{\varrho} \leq G_0^{\varrho}$ on S, and from this and (5.15)

$$\sup_{2r/3 < d(y,x) < r} G^{\varrho} \leq C \left(\frac{w(S(y, 4/3r))}{v(S(y, 4r/3))} \right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{r^2}{v(S(y, 4r/3))} \left(\frac{w(S(y, 8a^2 r/3))}{\inf_{2r/a < d(y,z) < r} w(S(2, r/3a^2))} \right)^{1/p}$$

Then, the thesis follows from (5.15) and the last inequality.

Lemma 5.16. *Let $S(x_0, R)$ be a d-ball such that $S(x_0, 13a^4 R) \subset \Omega$. Then, for each $p \in (0, \sigma)$, there exists a constant C, independent of x_0 and R such that*

$$(5.17) \qquad \sup_{r/2 < d(y,x) < r} G_y^{\varrho}(x) \leq C \int_r^R \frac{t^2}{v(S(y, t))} (F_1(y, t)^{\frac{\sigma^2}{\sigma-1}} F_2(y, t))^{1/p} \frac{dt}{t},$$

for all $y \in S(x_0, R/2)$, $r \in (0, R/2)$ and $\varrho \in (0, r/4a)$, where

$$F_1(y, t) = \frac{w(S(y, 12a^2 t))}{v(S(y, t))}$$

and

$$F_2(y, t) = \frac{w(S(y, 12a^2 t))}{\inf_{d(y,z) < 12a^2 t} w\left(S\left(z, \frac{t}{128a^5}\right)\right)}.$$

PROOF. First, let us consider the case $y = x_0$. For $s > 0$. Let us denote with S_s to $S(y, s)$ and with G_s^ϱ to the approximate Green function for S_s with pole y. Now, for $(t_1, t_2) \in \mathbf{R}^+ \times \mathbf{R}^+$, we define

$$g(t_1, t_2) = \left(\frac{w(S_{a^2 t_1})}{v(S_{t_2})}\right)^{\frac{\sigma^2}{p(\sigma-1)}} \frac{t_1^2}{v(S_{t_2})} \left(\frac{w(S_{a^2 t_1})}{\inf\limits_{d(y,z)<t_1} w(S(z, t_2))}\right)^{1/p}$$

Note that this function depend on p, is increasing on t_1 and decreasing on t_2. Let $r < R/2$ and $m \in \mathbf{N}$ such that $(3/2)^{m-1} r \leq R < (3/2)^m r$. Then on $S_r - S_{r/2}$, holds

$$(5.18) \qquad G_R \leq G^\varrho_{(3/2)^m r} = G_r^\varrho + \sum_{j=1}^{m} \left(G^\varrho_{(3/2)^j r} - G^\varrho_{(3/2)^{j-1} r}\right).$$

From the above lemma follows

$$(5.19) \quad \sup_{S_r - S_{r/2}} G_r^\varrho \leq Cg\left(\frac{8a^2 r}{3}, \frac{r}{4a^2}\right), \quad \text{for all} \quad \varrho \in \left(0, \frac{r}{4a}\right),$$

for each $p \in (0, \sigma)$. By the other hand, by using a similar argument to the Lemma 2.7 of [ChW3] and the above lemma, we get

$$\sup_{S_s} \left(G^\varrho_{(3/2)s} - G_s^\varrho\right) \leq Cg\left(4a^2 s, \frac{3s}{8a^2}\right), \quad \text{for all} \quad \varrho \in \left(0, \frac{s}{4a}\right),$$

for each $p \in (0, \sigma)$. Then, from this inequality, (5.19) and (5.18) follows

$$\begin{aligned}
(5.20) \quad \sup_{S_r - S_{r/2}} G_R^\varrho &\leq \sum_{j=1}^{m-1} g\left(6a^2 \left(\frac{3}{2}\right)^{j-1} r, \left(\frac{3}{2}\right)^{j-1} \frac{r}{4a^2}\right) \\
&\leq C \sum_{j=1}^{m-1} \int_{(3/2)^{j-1} r}^{(3/2)^j r} g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t} \\
&\leq C \int_r^R g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}, \quad \text{for all} \quad \varrho \in \left(0, \frac{r}{4a}\right)
\end{aligned}$$

for each $p \in (0, \sigma)$. Finally, since any $y \in S(x_0, R/2)$ verifies $S(x_0, R) \subset S(y, 2R)$, we get $G^\varrho \leq G^\varrho_{2R}$ on $S(x_0, R)$ and then, from (5.20), follows the thesis

$$\sup_{r/2<d(y,x)<r} G^\varrho \leq C \int_r^{2R} g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}$$

$$= C \int_r^{R} g\left(12a^2 t, \frac{t}{3a^2}\right) \frac{dt}{t} + \int_r^{2r} g\left(6a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}$$

$$\leq C \int_r^{R} g\left(12a^2 t, \frac{t}{6a^2}\right) \frac{dt}{t}, \quad \text{for all } \varrho \in \left(0, \frac{r}{4a}\right).$$

Corollary 5.21. *With the same hypothesis as in Lemma* 5.16 *for a.e. $y \in S(x_0, R/2)$ there is a constant $C = C(y, x_0, R, w, v) > 0$ such that*

$$(5.22) \qquad \sup_{r/2<d(y,x)<r} G^\varrho_y(x) \leq C \min\left\{\int_r^R \frac{t^{\,2}}{|S(y,t)|}\frac{dt}{t}, \int_r^R \frac{1}{|S(y,t)|}\frac{dt}{t}\right\},$$

for all $r \in (0, R/2)$ and all $\varrho \in (0, r/4a)$.

PROOF. We set $C_1(y) = \sup_{t\leq R} \dfrac{w(S(y, 12a^2 t))}{v(S(y,t))}$, $C_2(y) = \sup_{t\leq R} \dfrac{S(y,t)}{v(S(y,t))}$ and $C_3(y) = \sup_{t\leq R} \dfrac{S(y,t)}{w(S(y,t))}$. Now from (5.17) follows

$$(5.23) \qquad \sup_{r/2<d(y,x)<r} G^\varrho_y(x) \leq C C_1(y)^{\left(\frac{\sigma^2}{\sigma-1}+1\right)\frac{1}{p}} C_2(y) \int_r^R \frac{t^2}{|S(y,t)|}\frac{dt}{t},$$

for each $y \in S(x_0, R/2)$ and for all $r \in (0, R/2)$ and $\varrho \in (0, r/4a)$. On the other hand, from (1.5), (1.8) and (1.10), we get

$$\left(\frac{t}{a^2 R}\right)^{\sum_j G_j} \leq \frac{|S(y,t)|}{|S(y,R)|} \qquad 0 < t \leq R,$$

then from Lemma 4.4, it follows that

$$\frac{t^2}{v(S(y,\,t))} \le CR^2 \left(\frac{w(S(y,\,aR))}{w(S(y,\,t/a))}\right)^{1/\sigma} \frac{1}{v(S(y,\,R))}, \quad 0 < r \le R.$$

This inequality and (5.17) allows us to obtain

$$\sup_{r/2<d(y,x)<r} G_y^\varrho(x) \le CC_1(y)^{\left(\frac{\sigma^2}{\sigma-1}+1\right)\frac{1}{p}} C_3(y)^{1/\sigma} R^2 \left(\frac{w(S(y,\,aR))}{v(S(y,\,R))}\right)^{1/\sigma},$$
$$\int_r^R \frac{1}{|(S(y,\,t)|^{1/\sigma}} \frac{dt}{t}.$$

Now, from (5.23) and the above inequality, by taking infinum on p, (5.22) follows.

The next Lemma gives us an estimate of $\nabla_\lambda G^\varrho$ in terms of G^ϱ.

Lemma 5.24. *Let $S = S(x_0,\,R)$ be a d-ball such that $2S \subset \Omega$. Then there exists a constant C such that*

$$\int_{S\setminus Q(y,r)} \langle \Delta\nabla G_y^\varrho,\ \nabla G_y^\varrho\rangle \le \frac{C}{r^2}\int_{Q(y,r)\setminus Q(y,r/2)} (G_y^\varrho)^2 w,$$

for all $y \in (1/2)\,S$, $r \in (0,\,R/2a)$ and $\varrho \in (0,\,r/2)$.

PROOF. The proof is very similar to that of Lemma 4.2 of [ChW3] by keeping in mind that exists $\eta \in C^\infty(\mathbf{R}^n)$ such that $\eta \equiv 0$ on $Q(y,\,r/2)$, $\eta \equiv 1$ outside $Q(y,\,r)$ and $|\nabla_\lambda \eta| \le C/r$ (see [FL2], p. 537).

Now, the above lemma and Corollary 5.21 allows us to obtain the following result.

Lemma 5.25. *Let $S = S(x_0,\,R)$ be a d-ball such that $13a^4 S \subset \Omega$. If $n > 2$, then, for a.e. $y \in (1/2)\,S$ there exists a constant $C = C(y,\,x_0,\,R,\,w,\,v) > 0$ such that*

$$\int_{S\setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v \le C \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t},$$

for all $r \in (0, R/2a)$ *and for* $\varrho \in (0, R/2a)$.

PROOF. Let $\varrho \in \left(0, \frac{r}{8a^2}\right)$, then

$$(5.26)\quad \int_{S\setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v \le \frac{C}{r^2} w(Q(y,r)) \left(\sup_{r/2a<d(y,x)<ar} G_y^\varrho(x) \right)^2$$

$$\le \frac{C}{r^2} w(Q(y,r)) \left(\sum_{j=1}^{j\le \log_2 a^2+1} \sup_{\frac{ar}{2^{j+1}}<d(y,x)<\frac{ar}{2^j}} G_y^\varrho(x) \right)^2$$

$$\le \frac{Cw(Q(y,r))}{r^2} \left(\min \int_{r/2a}^{R} \frac{t^2}{|S(y,t)|} \frac{dt}{t}, \int_{r/2a}^{R} \frac{1}{|S(y,r)|^{1/\sigma}} \frac{dt}{t} \right)^2$$

$$\le \frac{C}{r^2} w(Q(y,r)) \left(\int_{r/2a}^{R} \frac{t^2}{|S(y,t)|} \frac{dt}{t} \right) \left(\int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t} \right).$$

On the other hand, from Lemmas 1.4 and 1.7 it follows that

$$\int_{r/2a}^{R} \frac{t^2}{|S(y,t)|}\frac{dt}{t} \le \sum_{j=1}^{\infty} \int_{\frac{2^j r}{2a}}^{\frac{2^{j+1} r}{2a}} \frac{t^2}{|S(y,t)|}\frac{dt}{t}$$

$$\le \sum_{j=0}^{\infty} \left(\frac{2^j r}{2a}\right) \frac{1}{\left|S\left(y, \frac{2^j r}{a}\right)\right|}$$

$$\le \frac{C\, r^2}{|S(y,r)|} \sum_{j=0}^{\infty} 2^{j(2-n)}$$

$$= \frac{C\, r^2}{|S(y,r)|}\,.$$

Then, from this and (5.26), we get

$$\int_{S \setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v \le C \frac{w(Q(y,r))}{|Q(y,r)|} \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t}\,. \tag{5.27}$$

Now, if $\varrho \in [r/8a^2, R/2a)$, by applying Sobolev inequality in $Q = Q(x_0, aR)$ we have

$$a_0(G_j^\varrho, G_j^\varrho) = \frac{1}{w(Q(y,\varrho)} \int_{Q(y,\varrho)} G_y^\varrho w$$

$$\le CR \left(\frac{w(2Q)}{w(Q(y,\varrho))}\right)^{1/2\sigma} \left(\frac{1}{w(2Q)} \int_S |\nabla_\lambda G_y^\varrho|^2 v\right)^{1/2}$$

$$\le \frac{1}{w(Q(y,\varrho))}\, a_0(G_y, G_y)^{1/2}.$$

Then

$$\int_S |\nabla_\lambda G_y^\varrho|^2 v \le a_0(G_y^\varrho, G_y^\varrho) \le \frac{C}{w(Q(y,\varrho))^{1/\sigma}}$$

$$\le \left(\frac{|S(y,r)|}{w(Q(y,r/8a^2))}\right)^{1/\sigma} \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t}.$$

Finally, from (5.27) and the above inequality, we get our thesis.

With Lemma 5.25 we can prove the following integrability property of $|\nabla_\lambda G^\varrho|$.

Lemma 5.28. *Let $S = S(x_0, R)$ be a d-ball such that $13a^4 S \subset \Omega \subset \mathbf{R}^n$ with $n > 2$. Then for each $q \in (0, 2\sigma/(\sigma+1))$ and a.e. $y \in (1/2)S$ there exists $C = C(q, y, x_0, R, w, v) > 0$ such that*

$$\int_S |\nabla_\lambda G_y^\varrho|^q \le C \quad \textit{for all} \quad \varrho \in \left(0, \frac{R}{2a}\right). \tag{5.29}$$

PROOF. Let $y \in \dfrac{1}{2} S$ and $\varrho \in \left(0, \dfrac{R}{2a}\right)$. From Lemma 5.25 we get

$$v(\{|\nabla_\lambda G_y^\varrho| > s\}) \le \frac{1}{s^2} \int_{S \setminus Q(y,r)} |\nabla_\lambda G_y^\varrho|^2 v + v(Q(y,r))$$

$$\le \frac{C}{s^2} \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t} + |Q(y,r)| \sup_{t \le R} \frac{v(Q(y,t))}{|Q(y,t)|}$$

$$\le C\left(\frac{1}{s^2} \int_{r/2a}^{R} \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t} + |Q(y,r)|\right)$$

for a.e. $y \in (1/2)\,S$ and for all $r \in (0, R/2a)$ and $s > 0$, where $C = C(y, S, w, v)$. Then, by using the properties of d and δ, it follows that

$$\nu(\{\,|\nabla_\lambda G_y^\varrho| > s\}) \le C\left(|Q(y,r)| + \frac{1}{s^2}\sum_{j=0}^{\infty}\int_{\frac{2^j r}{2a}}^{\frac{2^{j+1} r}{2a}} \frac{1}{|S(y,t)|^{1/\sigma}}\frac{dt}{t}\right)$$

$$\le C\left(|Q(y,r)| + \frac{1}{s^2\,|S(y,r)|^{1/\sigma}}\sum_{j=0}^{\infty} 2^{-nj/\sigma}\right)$$

$$\le C\left(|Q(y,r)| + \frac{1}{s^2\,|Q(y,r)|^{1/\sigma}}\right).$$

Now, by choosing r such that $|Q(y,r)| = s^{-\frac{2\sigma}{\sigma+1}}$, holds

$$\nu(\{\,|\nabla_\lambda G_y^\varrho| > s\}) \le C\, s^{-\frac{2\sigma}{\sigma+1}}$$

for all $s > |Q(y, R/2a)|^{-(\sigma+1)/2\sigma}$ for all $\varrho \in (0, R/2a)$ and for a.e. $y \in (1/2)\,S$. From this inequality (5.29) follows inmediately.

Let S be a d-ball such that $2aS \subset \Omega \subset \mathbf{R}^n$ with $n \ge 3$ and $(p, q) \in [1, \sigma) \times [1, 2\sigma/(\sigma+1))$. Let $X_{p,q}$ be the completion of $\mathrm{Lip}_0(S)$ with the norm

$$\|\phi\|_{p,q} = \left(\int_S |\phi|^p w\right)^{1/p} + \left(\int_S |\nabla_\lambda \phi|^q v\,dx\right)^{1/q}.$$

Now the above results about G_y^ϱ and $\nabla_\lambda G_y^\varrho$, allow to prove existence of the Green functions $G_y(\,\cdot\,) = G(\,\cdot\,, y)$ for S with pole y.

Theorem 5.30. *Let $S = S(x_0, R)$ be a d-ball such that $13a^4 S \subset \Omega$. For a.e. $y \in (1/2)\,S$ exists a sequence $\{\varrho_k\} \subset (0, R/2a)$ and a function G_y such that*

(5.31) $\varrho_k \downarrow 0$ *and* $\{G_y^{\varrho_k}\}$ *converges weakly to* G_y *on* $X_{p,q}$

for all $(p, q) \in [1, \sigma) \times \left[1, \frac{2\sigma}{\sigma + 1}\right)$;

(5.32) $\|G_y\|_{p,q} \leq C$

for every $(p, q) \in [1, \sigma) \times \left[1, \frac{2\sigma}{\sigma + 1}\right)$, *where C is independent of* y;

(5.33) *if* (v, w) *satisfies* $\left(\frac{w}{v}\right)^{\frac{q_0}{q_0-1}} v \in L^1(S, dx)$ *for some* $q_0 \in \left(1, \frac{2\sigma}{\sigma + 1}\right)$

then

$$\int_S \langle A \nabla G_y, \nabla \phi \rangle = \phi(y)$$

for all $\phi \in \mathrm{Lip}_0(S)$, *and a.e.* $y \in \frac{1}{2} S$;

(5.34) *if u is solution of* (5.6) *with* $\psi = 0$, *then*

$$u(y) = \int_S G_y(x) f(x) dx \quad a.e. \quad y \in \frac{1}{2} s;$$

(5.35) *if u is solution of* (5.7) *with* $\psi = 0$, *then*

$$u(y) = \int_S \langle \nabla_\lambda G_y(x), F(x) > dx \quad a.e. \quad y \in \frac{1}{2} S.$$

PROOF. Follows from the above results about G_y^ϱ and $\nabla_\lambda G_y^\varrho$ with the arguments in sections 5 and 6 of [ChW3].

Now, we prove another estimate for G_y^ϱ. Then we shall to use this and the above to prove some estimates of size for the Green function.

Lemma 5.37. *Let* $S(x_0, R)$ *be a d-ball such that* $13a^4 S \subset \Omega$. *Then exists a constant C depending only on the constants of* (0.5) *and* (0.7) *such that*

$$(5.37) \qquad \inf_{r/2 < d(y,x) < r} G_y^\varrho(x) \geq C \int_r^R \frac{t^2}{w(S(y, 2at))} e^{-CF_2(y,t)^\gamma F_1(y,t)^{1/2}} \frac{dt}{t}$$

for all $y \in S(x_0, R/2)$, $r \in (0, R/8a^2)$ *and* $\varrho \in 0, r/4a^2)$, *where* $F_1(y, t)$ *and* $F_2(y, t)$ *are the functions of* (5.17) *and* $\gamma = \dfrac{3\sigma^2 - 2\sigma + 1}{\sigma - 1}$.

PROOF. First, let us consider the case $y = x_0$. Let $r \in (0, R/4a^2)$. For $s > 0$, let us denote with G_y^ϱ the approximate Green function for $S(y, s)$ with pole y. From Lemma 5.24, we get

$$(5.38)\quad \int_{S(y,2ar)\setminus Q(y,r)} \langle A \nabla G_{2ar}^\varrho, \nabla G_{2ar}^\varrho \rangle \leq \frac{C}{r^2}\, w(Q(y, r)) \left(\sup_{S(y,ar)-S(y,r/2a)} G_{2ar}^\varrho \right)^2$$

On the other hand, since $LG_{2ar}^\varrho = 0$ on $S(y, 2ar) \setminus S(y, r/4a)$, the Harnack inequality (Theorem 5.10) allow us to obtain

$$\sup_{Q(x,r/8a^2)} G_{2ar}^\varrho \leq \exp\left\{ C \left(\frac{w(S(y,2ar))}{\inf_{d(y,z)<ar} w(S(z, r/16a^3))} \right)^\gamma \left(\frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\} \inf_{Q(x,r/8a^2)} G_{2ar}^\varrho$$

for all $\varrho \in (0, r/4a^2)$ and all $x \in S(y, ar) \setminus S(y, r/2a)$. From this inequality and (5.38) follows that

$$(5.39)\quad \left(\inf_{S(y,ar)\setminus S(y,r/2a)} G_{2ar}^\varrho \right)^2$$

$$\leq \frac{Cr^2}{w(Q(y, r))} \exp\left\{ -C \left(\frac{w(S(y, 2ar))}{\inf_{d(y,z)<ar} w(S(z, r/16a^3))} \right)^\gamma \left(\frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\}$$

$$\int_{S(y,2ar)\setminus Q(y,r)} \langle A \nabla G_{2ar}^\varrho, \nabla G_{2ar}^\varrho \rangle .$$

Now, by taking $\eta \in C_0^\infty(S(y, 2ar))$ such that $\eta \equiv 1$ on $Q(y, r)$ and $|\nabla_\lambda \eta| \leq \dfrac{C}{r}$ and applying the ellipticity condition, we get

$$1 = a_0(G^{\varrho}_{2ar}, \eta) \le \frac{C}{r} w(y, 2ar))^{1/2} \left(\int_{S(y,2ar) \setminus Q(y,ar)} \langle A \nabla G^{\varrho}_{2ar}, \nabla G^{\varrho}_{2ar} \rangle \right)^{1/2}.$$

Then, from (5.39), holds

$$(5.40) \quad \inf_{S(y,ar) \setminus S(y,r/2a)} G^{\varrho}_{2ar} \ge \frac{Cr^2}{w(S(y,2ar))} \exp\left\{ -C\left(\frac{w(S(y, 2ar))}{\inf_{d(y,z)<ar} w(S(z, r/16a^3))} \right)^{\gamma} \left(\frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\}$$

for all $\varrho \in (0, r/4a^2)$. Now, by applying the weak maximum principle, it follows that

$$(5.41) \quad G^{\varrho}_{2ar} - G^{\varrho}_{r} \ge \frac{Cr^2}{w(S(y, 2ar))} \exp\left\{ -C\left(\frac{w(S(y, 2ar))}{\inf_{d(y,z)<ar} w(S(z, r/16a^3))} \right)^{\gamma} \left(\frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\}$$

a.e. in $S(y, r)$ and for all $\varrho \in (0, r/4a^2)$. On the other hand, if $m \in \mathbf{N}$ is such that $(2a)^m r \le R < (2a)^{m+1} r$, we get

$$(5.42) \quad G^{\varrho}_{R} \ge G^{\varrho}_{(2a)^m r} = G^{\varrho}_{2ar} + \sum_{j=1}^{m-1} (G^{\varrho}_{(2a)^{j+1} r} - G_{(2a)^j r}), \text{ a.e. in } S(y, 2ar).$$

Then, with $S_t = S(y, t)$ and

$$g(t_1, t_2) = \frac{t_1^2}{w(S_{t_2})} \exp\left\{ -C\left(\frac{w(S_{t_2})}{\inf_{d(y,z)<ar} w(S(z, t_1/16a^3))} \right)^{\gamma} \left(\frac{w(S_{t_2}))}{v(S_{t_1}))} \right)^{1/2} \right\}, \; t_1, t_2 \in \mathbf{R}^+.$$

from (5.40) and (5.41) the inequality

$$(5.43)\quad G_R^\varrho \geq C \sum_{j=0}^{m-1} g((2a))^j r, (2a)^{j+1} r)$$

$$\geq C \int_{2ar}^{R} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t}$$

$$= C\left[\int_{4a^2r}^{R} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t} + \int_{2ar}^{4a^2r} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t}\right]$$

$$\geq C\left[\int_{2ar}^{R} g\left(\frac{t}{4a^2}, t\right) \frac{dt}{t} + \frac{1}{2}\int_{r}^{2ar} g\left(\frac{t}{2a}, 2at\right) \frac{dt}{t}\right]$$

$$\geq C \int_{r}^{R} g\left(\frac{t}{4a^2}, 2at\right) \frac{dt}{t},$$

holds for a.e. in $S(y, ar) \setminus S(y, r/2a)$ and for all $\varrho \in (0, r/4a^2)$. Finally, the result for $y \neq x_0$ follows easily from the fact that $G_y^\varrho \geq G_{R/2}^\varrho$ a.e. in $S(y, R/2)$ and (5.43).

In the following we prove a result of functional analysis which shall be used to prove the size estimates.

Lemma 5.44. *Let $E \subset \mathbf{R}^n$ be a measurable set, $h \in L_{\mathrm{loc}}(E, dx)$ be positive a.e. in E and $p \in (1, \infty)$. If $\{f_k\} \subset L^p(E, dx)$ converges weakly to a function f and satisfies* $\sup_k f_k \leq C_0$ *a.e. in a bounded set $F \subset E$, for some constant C_0, then $f \leq C_0$ a.e. in F.*

PROOF. From the hypothesis we get

$$\int_E (C_0 - f_k)\, gh\, dx \geq 0,$$

for every k *and for all* $g \in L^p(E, hdx)$ such that supp $g \subset F$ and $g \geq 0$ a.e. in F. Then, by letting $k \to \infty$, the inequality

$$\int_E (C_0 - f)\, ghdx \geq 0,$$

holds. Now, by taking $g = \chi_{F \cap \{f > C_0\}}$, we get the thesis.

Finally, we are in position to prove

Theorem 5.45. *Let* $S = S(x_0, R)$ *be a d-ball such that* $13a^4 S \subset \Omega$. *Then for a.e.* $y \in (1/2)\, S$. G_y *is non-negative a.e. in* S *and satisfies*

$$\text{(5.46)} \qquad \sup_{r/2 < d(y,x) < r} G_y(x) \leq C \int_r^R \frac{t^2}{v(S(y, t))} ((F_1(y, t))^{\gamma_1} F_2(y, t))^{1/p} \frac{dt}{t}$$

for each $p \in (0, \sigma)$

$$\text{(5.47)} \qquad \inf_{r/2 < d(y,x) < r} G_y(x) \geq C \int_r^R \frac{t^2}{w(S(y, 2at))} e^{-CF_1(y,t)^{1/2} F_2(y,t)^{\gamma_2}} \frac{dt}{t}$$

for all $r \in (0, R/8a^2)$, where $\gamma_1 = \dfrac{\sigma^2}{\sigma - 1}$, $\gamma_2 = \dfrac{3\sigma^2 - 2\sigma + 1}{\sigma - 1}$ and F_1 and F_2 are define as in (5.17).

PROOF. It follow from Lemmas 5.11 and 5.36 and Theorem 5.30 by using the above lemma.

Finally, we shall prove a result about the integrability without weights of the Green function.

Theorem 5.48. *Let* $S = S(x_0, R)$ *be a d-ball such that* $17a^4 S \subset \mathbf{R}^n$ *with* $n \geq 3$. *Then* $G_y \in L^p(S, dx)$ *for a.e.* $y \in (1/2)\, S$ *and for all*

$$p \in \left(1, \max\left\{\frac{\sum G_j}{\sum G_j - 2}, 1 + \frac{n}{\sum G_j}(\sigma - 1)\right\}\right).$$

PROOF. From (5.46) by applying a similar argument to that Corollary 5.21 follows

$$\sup_{r/2<d(y,x)<r} G_y(x) \le C \min\left\{\int_r^R \frac{t^2}{|S(y,t)|}\frac{dt}{t}, \int_r^R \frac{1}{|S(y,t)|^{1/\sigma}}\frac{dt}{t}\right\} \tag{5.49}$$

for a.e. $y \in (1/2)S$ and all $r \in (0, R/8a^2)$ where $C = C(y, x_0, R, w, v)$. By denoting with G_{16a^2R} the Green function for $S(y, 16a^2R)$ with pole y, the weak maximum principle, (5.49) and the properties of d allow us to get

$$\begin{aligned}
\int_{S(x_0,R)} (G_y)^p\,dx &\le \int_{S(x_0,R)} (G_{16a^2R})^p\,dx \\
&\le \sum_{i=0}^{\infty} \int_{S(y,2R/2^i)-S(y,2R/2^{i+1})} (G_{16a^2R})^p\,dx \\
&\le C \sum_{i=0}^{\infty} \left(\int_{\frac{R}{2^{i-1}}}^{16a^2R} \frac{t^2}{|S(y,t)|}\frac{dt}{t}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \int_{\frac{R}{2^{i-1}}2^j}^{\frac{R}{2^{i-1}}2^{j+1}} \frac{t^2}{|S(y,t)|}\frac{dt}{t}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{(R\,2^{1-i}2^{j+1})^2}{\left|S\left(y, \frac{R}{2^{i-1}}2^j\right)\right|}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \left(\frac{(R\,2^{1-i})^2}{\left|S\left(y, \frac{R}{2^{i-1}}\right)\right|} \sum_{j=0}^{\infty} 2^{j(2-n)}\right)^p \left|S\left(y, \frac{R}{2^{i-1}}\right)\right| \\
&\le C \sum_{i=0}^{\infty} \frac{(R\,2^{i-1})^{2p}}{\left|S\left(y, \frac{R}{2^{i-1}}\right)\right|^{p-1}}
\end{aligned} \tag{5.50}$$

$$\leq C\left(\frac{(2R)^p}{|S(y,2R)|^{p-1}}+\frac{R^{2p}}{|S(y,R)|^{p-1}}\sum_{j=1}^{\infty}2^{i((\Sigma_jG_j-2)p-\Sigma_jG_j)}\right).$$

Note that the right hand side is finite if and only if $p < (\Sigma_j G_j)/(\Sigma_j G_j - 2)$. On the other hand, by applying (5.46) again and a similar argument, we get

$$\int_{S(x_0,R)}(G_j)^p dx \leq \frac{C}{|S(y,R)|^{p/\sigma-1}}\left(1+\sum_{i=1}^{\infty}2^{i1/\sigma((p-1)\Sigma_jG_j+n(1-\sigma))}\right),$$

where the right member is finite iff $p < 1 + \dfrac{n}{\Sigma_j G_j}(\sigma - 1)$. Finally, from this and (5.50), we obtain the thesis.

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