Pointwise and Spectral Control of Plate Vibrations

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Résumé

On considère le problème du contrôle ponctuel (c'est-à-dire au moyen d'une masse de Dirac située en un point fixé) des vibrations d'une plaque $\Omega$. Sous des conditions aux limites générales, incluant les plaques posées ou encastrées, mais excluant (et pour cause) le cas où existent des vibrations propres multiples, nous montrons la contrôlabilité des combinaisons linéaires finies des fonctions propres en tout point de $\Omega$ qui n'est zéro d'aucune fonction propre et en tout temps strictement supérieur à la moitié de la surface de la plaque. Ce résultat est optimal car aucune combinaison linéaire finie non nulle de fonctions propres n'est ponctuellement contrôlable en un temps strictement inférieur à la moitié de la surface de la plaque. Sous la même condition sur le temps, mais pour un domaine $\Omega$ quelconque de $\mathbb{R}^2$, on résout le problème du contrôle spectral interne, c'est-à-dire que pour tout disque ouvert $\omega \subset \Omega$, une combinaison linéaire finie quelconque des fonctions propres peut être ramenée à l'équilibre au moyen d'un contrôle $h \in \mathcal{D}((0, T) \times \Omega)$ tel que $\text{supp}(h) \subset (0, T) \times \omega$.

Abstract

We consider the problem of controlling pointwise (by means of a time dependent Dirac measure supported by a given point) the motion of a vibrating plate $\Omega$. Under general boundary conditions, including the special cases of simply
supported or clamped plates, but of course excluding the cases where some multiple eigenvalues exist for the biharmonic operator, we show the controllability of finite linear combinations of the eigenfunctions at any point of \( \Omega \) where no eigenfunction vanishes at any time greater than half of the plate's area. This result is optimal since no finite linear combination of the eigenfunctions other than 0 is pointwise controllable at a time smaller than half of the plate's area. Under the same condition on the time, but for an arbitrary domain \( \Omega \) in \( \mathbb{R}^2 \), we solve the problem of internal spectral control, which means that for any open disk \( \omega \subset \Omega \), any finite linear combination of the eigenfunctions can be set to equilibrium by means of a control function \( h \in \mathcal{D}((0, T) \times \omega) \) supported in \( (0, T) \times \omega \).

1. Introduction and Functional Setting

In order to make the theory more transparent, we shall consider the general case of a second order conservative evolution equation and apply only at the end our abstract results to the specific case of a 2-dimensional vibrating plate. Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) (or a compact \( N \)-dimensional manifold without boundary) and \( A \) a positive self-adjoint operator in \( H = L^2(\Omega) \). We assume that \( A \) satisfies the following properties

(1.1) \( A \) is coercive on \( H \).
(1.2) \( D(A^{1/2}) \subset C(\overline{\Omega}) \) with continuous imbedding.

Given \( T > 0 \), \( \xi \in \Omega \) and \( [y^0, y^1] \in D(A^{1/2}) \times L^2(\Omega) \), we are interested in the existence of a control function \( h \in L^2([0, T]) \) such that \( \operatorname{supp}(h) \subset [0, T] \) and for which the unique generalized solution \( y \) of

\[
\begin{aligned}
  y'' + Ay &= h(t)\delta(x - \xi) \quad \text{in} \quad [0, T], \\
  y(0, x) &= y^0(x), \\
  y'(0, x) &= y^1(x) \quad \text{in} \quad \Omega,
\end{aligned}
\]

(1.3)

satisfies \( y(T, x) = y'(T, x) = 0 \) in \( \Omega \). If such a control \( h \) exists, we shall say that the state \( [y^0, y^1] \) is «pointwise exactly \( L^2 \)-controllable in \( \xi \) at time \( T \)».

The possibility of solving this «pointwise exact controllability problem» is related to the amount of information revealed by the restriction to \( [0, T] \) of \( t \mapsto \phi(t, \xi) \) where \( \phi \) is an arbitrary solution of the homogeneous equation

\[
\phi'' + A\phi = 0 \quad \text{in} \quad [0, T], \quad \phi \in C(0, T; V) \cap C^1(0, T; H)
\]

(1.4)

with \( V = D(A^{1/2}), H = L^2(\Omega) \). (Note that as a consequence of (1.2) we have \( \phi(t, \xi) \in C([0, T]) \) for any such solution \( \phi \). In fact if any \( [y^0, y^1] \) from a dense subset of \( V \times H \) is exactly \( L^2 \)-controllable in \( \xi \) at time \( T \), then any solution
\[ \phi \] of the homogeneous equation (1.4) such that \( \phi(t, \xi) \) vanishes identically on \( [0, T[ \) is the trivial solution \( \phi = 0 \). Conversely, if any solution \( \phi \) of the homogeneous equation (1.4) such that \( \phi(t, \xi) \) vanishes identically on \( [0, T[ \) is the trivial solution \( \phi = 0 \), then for each \( (\phi^0, \phi^1) \in V \times H \), we consider the (clearly well-defined) norm

\[ p(\phi^0, \phi^1) = \left( \int_0^T \phi^2(t, \xi) \, dt \right)^{1/2}, \]

where \( \phi \) is the solution of equation (1.4) with initial data \( (\phi^0, \phi^1) \). The following result then follows from the general HUM method of J. L. Lions ([17, 18, 19]).

**Proposition 1.1.** A given state \( [y^0, y^1] \in V \times H \) is exactly \( L^2 \)-controllable at \( \xi \) in time \( T \) if and only if there exists a constant \( C \geq 0 \) such that for every \( (\phi^0, \phi^1) \in V \times H \),

\[ \left| \int_0^T (\phi^0 y^1 - \phi^1 y^0) \, dx \right| \leq Cp(\phi^0, \phi^1). \]

As was clearly established in [6], the set of pointwise exactly \( L^2 \)-controllable states (always a dense subset of \( V \times H \) when \( p \) is a norm) is usually complicated and more precisely depends on the observation point \( \xi \) in a very complicated and unstable way, even in the simplest case of the standard vibrating string with fixed end! The only reasonable thing to be expected in general is that (1.6) might hold true when both \( y^0 \) and \( y^1 \) are finite linear combinations of the eigenfunctions of \( A \), assuming that no eigenfunction vanishes at \( \xi \). This implies in particular that all eigenvalues of \( A \) are simple, a condition that we shall assume in most of this text (Sections 2, 3 and 5). The controllability of all states for which \( y^0 \) and \( y^1 \) are finite linear combinations of the eigenfunctions of \( A \) is what we shall call «pointwise spectral controllability». Taking account of the form of the general solutions to the homogeneous equation, it is natural to apply the methods of harmonic analysis to solve this problem. Indeed, any solution of (1.4) can be written as a series

\[ \phi(t, x) = \sum \left\{ \phi_n(x) \cos \sqrt{\lambda_n} t + \psi_n(x) \sin \sqrt{\lambda_n} t \right\} \]

where the functions \( \phi_n, \psi_n \) are eigenfunctions of \( A \) associated to the eigenvalues \( \lambda_n \), or in complex form

\[ \phi(t, x) = \sum \varphi_n(x)e^{i\mu_n t} \]

where the \( \mu_n \) stand for the (positive or negative) square roots of the eigenvalues \( \lambda_n \). Thus for fixed \( x \), it is a linear combination of some complex exponentials, the properties of which will be the key point of this work. Our main result
will, maybe surprisingly, turn out to be a consequence of one among the deepest classical results on harmonic analysis from the «sixties», namely the Beurling Malliavin criterion for computing the completeness radius of a family of complex exponentials. The application of this powerful machinery to our problem is the object of Sections 2 and 3 of this paper. The case of a vibrating plate with constant Lamé coefficients is a special case of our abstract result obtained for $N = 2$ and $\Delta = \Delta^2$ with relevant boundary conditions. In Section 4, we shall combine the result of Section 2 with, essentially, a biorthogonality technique in the spirit of [2, 16, 22] to solve the easier problem of «internal spectral controllability» under slightly relaxed conditions on the domain. However we feel that much more should be done in this last direction, as already strongly suggested by the special cases considered in [6, 11, 15]. Finally in Section 5, we consider some additional examples and we discuss the relationship between spectral controllability and some uniqueness questions.

2. Some Properties of the Completeness Radius of a Family of Complex Exponentials

The main tool from harmonic analysis that we shall use in this paper is the notion of completeness radius and its characterization by some estimates.

2.1. Definition and some properties of the Completeness Radius

Definition 2.1.1. Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of distinct real numbers. Consider all the functions of the form

$$f(t) = \sum_{k \in J} f_k e^{i\lambda_k t},$$

(2.1)

$J$ being any finite subset of $\mathbb{Z}$. The completeness radius of $\Lambda$ is defined as $R(\Lambda) = \sup \{A > 0: \text{the functions } f \text{ of the form (2.1) are dense in } C([-A, A])\}$.

In particular, if the functions $f$ of the form (2.1) are dense in $C([-A, A])$ for all $A > 0$, we set $R(\Lambda) = \infty$. On the other hand, if the density fails for all $A > 0$, we set $R(\Lambda) = 0$.

Remark 2.1.2. A classical result from the theory of nonharmonic Fourier series (cf. e.g. [24, Theorem 8 p. 129]) asserts that either the functions of the form (2.1) are dense in $C([a, b])$, or no complex exponential of the form $e^{i\nu t}$ with $\nu$ different from all $\lambda_n$ can be obtained as a limit of functions of the form (2.1) in $C([a, b])$. This interesting alternative is the main idea for the proof of Proposition 2.2.1 below.
Remark 2.1.3. For all $p \in [1, +\infty)$ we also have

$$R(\Lambda) = \sup \{ A > 0: \text{the functions } f \text{ of the form (2.1) are dense in } L^p([-A, A]) \},$$

with the same conventions in the limiting cases $R(\Lambda) = 0, \infty$ (see [24]). In the sequel we shall be especially concerned by the case $p = 2.$

2.2. A «Density-Controllability» Alternative

The main result of this section is the following

Proposition 2.2.1. Let $\Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Then we have the following properties

1) For each $T > 2R(\Lambda)$ and for each $n \in \mathbb{Z}$ there exists a constant $C_n$ such that

$$|f_n| \leq C_n \left( \int_0^T |f(t)|^2 \, dt \right)^{1/2},$$

for each function $f$ of the form (2.1) with $n \in J.$

2) On the other hand for each $T < 2R(\Lambda)$ and for each finite sequence $\{ \alpha_n \}_{n \in F}$ of complex numbers having a non zero term, there exists no constant $C > 0$ such that

$$\left| \sum_{n \in F} \alpha_n f_n \right| \leq C \left( \int_0^T |f(t)|^2 \, dt \right)^{1/2}$$

for each $f$ of the form (2.1) with $F \subset J.$

As a first step we will establish the following lemma.

Lemma 2.2.2. Let $I = (0, T)$ and let $\Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that the set of functions of the form (2.1) is not dense in $L^2(I).$ Then, for each $n \in \mathbb{Z}$ there exist a constant $C_n$ such that (2.2) holds for each function $f$ of the form (2.1) with $n \in J.$

Proof. If (2.2) is not satisfied for some $n,$ we can find a sequence of functions $\{ f^p \}$ of the form (2.1) such that $|f_n^p| = 1$ and $\int_I |f^p(x)|^2 \, dx \to 0$ as $p \to +\infty.$ It follows that the constant 1 is the limit in $L^2(I)$ of some functions $g$ of the form

$$g(t) = \sum_{k \in J} g_k e^{ik \lambda t},$$
where $J$ is a finite subset of $\mathbb{Z} - \{n\}$, and: $\mu_k = \lambda_k - \lambda_n$. By repeated integration in $t$, we deduce that all polynomials of $t$ with complex coefficients are also limits in $L^2(I)$ of some functions $g$ of the form (2.3). Indeed, let $p \in \mathbb{N}$, $\epsilon > 0$ and $g$ of the form (2.3) be such that

$$\left| \int_0^T - \sum_{k \in J} g_k e^{i\mu_k t} \right|_2 \leqslant \epsilon$$

where $\| \cdot \|_2$ stands for the norm in $L^2(I)$. By integrating in $t$, we deduce easily the estimate

$$\left| t^{p+1} - (p+1) \sum_{k \in J} g_k \frac{e^{i\mu_k t}}{\mu_k} + (p+1) \sum_{k \in J} \frac{g_k}{\mu_k} \right|_\infty \leqslant (p+1)\epsilon T^{1/2}$$

where $\| \cdot \|_\infty$ stands for the norm in $L^\infty(I)$. Hence, in particular

$$\left| t^{p+1} - (p+1) \sum_{k \in J} g_k \frac{e^{i\mu_k t}}{\mu_k} + (p+1) \sum_{k \in J} \frac{g_k}{\mu_k} \right|_2 \leqslant (p+1)\epsilon T.$$

Then by approximating the constant $(p+1) \sum_{k \in J} (g_k/\mu_k)$ in $L^2(I)$ by functions of the form (2.3), we find a sequence of coefficients $\{g_k'\}_{k \in J'}$ for which

$$\left| t^{p+1} - \sum_{k \in J'} g_k' e^{i\mu_k t} \right|_2 \leqslant 2(p+1)\epsilon T.$$

This proves the claim by induction on $p$ since it has been proved already for $p = 0$. Finally by the Stone-Weierstrass density theorem, the functions $g$ of the form (2.3) are dense in $L^2(I)$, and the same property follows at once for functions $f$ of the form (2.1).

**Proof of Proposition 2.2.1.** It follows clearly from the definition of $R(\Lambda)$ that for each $T > 2R(\Lambda)$ the functions $f$ of the form (2.1) are not dense in $L^2(0, T)$, and therefore assertion 1) is an immediate consequence of Lemma 2.2.2. On the other hand for each finite set $F \subset \mathbb{Z}$, if we denote by $\Lambda^F$ the set $\{\lambda_n\}_{n \in \mathbb{Z} - F}$, then classically $R(\Lambda^F) = R(\Lambda)$. As a consequence for each $T < 2R(\Lambda)$, the set of functions $f$ of the form (2.1) with $J \cap F = \emptyset$ is dense in $L^2(0, T)$, and therefore for each non trivial sequence $\{\alpha_n\}_{n \in F}$ of complex numbers, the function

$$a(t) = \sum_{k \in F} \alpha_k e^{i\lambda_k t}$$

can be approached in $L^2(0, T)$ by functions $f$ of the form (2.1) with $J \cap F = \emptyset$. By taking the difference we find a sequence of functions of the form (2.1) tending to 0 in $L^2(0, T)$ and for which the left-hand side in (2.2)' is constant and positive. This clearly establishes assertion 2).
2.3. Computation of a Beurling-Malliavin Density

The main result of this section is the following

**Theorem 2.3.1.** Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d^+, d^- \geq 0$ and $0 \leq \alpha < 1$

\[(2.4) \quad \# \{\lambda \in \Lambda : 0 \leq \lambda \leq t\} = d^+ t + O(t^\alpha)\]

and

\[\# \{\lambda \in \Lambda : -t \leq \lambda \leq 0\} = d^- t + O(t^\alpha).\]

Then we have

\[(2.5) \quad R(\Lambda) = \pi d, \quad d = \max\{d^+, d^-\}.\]

Theorem 2.3.1. will be a consequence of the famous Beurling-Malliavin Theorem. In the important special case where $d^+ = d^-$, it will be sufficient to verify the following lemma.

**Lemma 2.3.2.** Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be a sequence of real numbers. Assume that we have for some $d \geq 0$ and $0 \leq \alpha < 1$

\[(2.6) \quad \# \{\lambda \in \Lambda : 0 \leq \lambda \leq t\} = dt + O(t^\alpha) \quad \text{and} \quad \# \{\lambda \in \Lambda : -t \leq \lambda \leq 0\} = dt + O(t^\alpha).\]

Let us represent the generic compact interval of $\mathbb{R}$ by $\omega = [\omega_1, \omega_2]$ and define for each $\varepsilon > 0$ the set

\[(2.7) \quad \Omega_\varepsilon = \{\omega : \|\omega\|^{-1} \# (\Lambda \cap \omega) - d \geq \varepsilon\}.\]

Then if we represent each interval $\omega$ by a point in the upper half-plane through the formulas

\[(2.8) \quad T(\omega) = (x, y) \quad \text{with} \quad x = (\omega_1 + \omega_2)/2 \quad \text{and} \quad y = |\omega| = \omega_2 - \omega_1,

we have

\[(2.9) \quad \forall \varepsilon > 0, \quad \int_{T(\Omega_\varepsilon)} \frac{dx \, dy}{1 + x^2 + y^2} < \infty,\]

for every $\varepsilon > 0$.

**Proof.** As a consequence of hypothesis (2.6) we have immediately

\[\# (\Lambda \cap \omega) - d|\omega| = O\{||\omega_1|^\alpha + |\omega_2|^\alpha\}.\]
therefore we only need to check that for each \( K \geq 0 \), the set

\[ A(K) = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq K(|x + y|^\alpha + |x - y|^\alpha)\} \]

satisfies

\[ \iint_{A(K)} \frac{dx \, dy}{1 + x^2 + y^2} < \infty. \]

But obviously

\[ A(K) \subset B = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq M(1 + x^2)^{\alpha/2}\} \]

for some constant \( M \) related to \( K \). Finally we have

\[ \int_{A(K)} \frac{dx \, dy}{1 + x^2 + y^2} \leq \int_{-\infty}^{+\infty} dx \int_{0}^{M(1 + x^2)^{\alpha/2}} dy \frac{dy}{1 + x^2 + y^2} \]

\[ \leq M \int_{-\infty}^{+\infty} (1 + x^2)^{\alpha/2 - 1} \, dx < \infty. \]

The result follows immediately.

In order to complete the proof of Theorem 2.3.1, it will be useful to recall the main concepts required to formulate the general Beurling-Malliavin Theorem.

**Definition 2.3.3.** A sequence of distinct real numbers \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) is said to be regular with Beurling-Malliavin density equal to \( d \geq 0 \) if for each \( \varepsilon > 0 \), the set \( \Omega_{\varepsilon} \), given by (2.7) satisfies (2.9) with \( T \) given by (2.8).

We now recall the main result of [1].

**Theorem 2.3.4.** (Beurling-Malliavin.) Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) be a sequence of distinct real numbers. Then

(a) If \( \Lambda \) is regular with Beurling-Malliavin density equal to \( d \geq 0 \), we have

\[ R(\Lambda) = \pi d. \]

(b) If \( \Lambda \) is not regular, then

\[ R(\Lambda) = \pi d, \]

where \( d \) is the infimum of all Beurling-Malliavin densities of regular sequences of distinct real numbers containing \( \Lambda \).
Proof of Theorem 2.3.1. (a) If \( d^+ = d^- \), the result of Lemma 2.3.2. precisely means that \( \Lambda \) is regular with Beurling-Malliavin density equal to \( d \), and (a) from the statement of Theorem 2.3.4. gives exactly (2.5).

(b) Otherwise, one easily finds that any regular sequence of distinct real numbers containing \( \Lambda \) has a density at least equal to \( d \). On the other hand by «completing» \( \Lambda \) it is rather straightforward to build a sequence of distinct real numbers containing \( \Lambda \) and satisfying (2.6). As a consequence of Lemma 2.3.2., such a sequence must be regular with Beurling-Malliavin density equal to \( d \). Then (2.5) follows at once from (b) in the statement of Theorem 2.3.4.

By combining the results of Proposition 2.2.1 and Theorem 2.3.1, we obtain

Corollary 2.3.5. Let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) be a sequence of real numbers. Assume that we have for some \( d^+ \geq 0, d^- \geq 0 \) and \( 0 \leq \alpha < 1 \)

\[
\begin{align*}
\# \{ \lambda \in \Lambda : 0 \leq \lambda \leq t \} &= d^+ t + O(t^\alpha) \quad \text{and} \\
\# \{ \lambda \in \Lambda : -t \leq \lambda \leq 0 \} &= d^- t + O(t^\alpha).
\end{align*}
\]

Then letting \( d = \max \{ d^+, d^- \} \), we have the following properties

1. For each \( T > 2\pi d \) and for each \( n \in \mathbb{Z} \) there exists a constant \( C_n \) such that

\[
|f_n| \leq C_n \left( \int_0^T |f(t)|^2 \, dt \right)^{1/2},
\]

for each function \( f \) of the form (2.1) with \( n \in J \).

2. On the other hand for each \( T < 2\pi d \) and for each finite sequence \( \{ \alpha_n \}_{n \in F} \) of complex numbers having a non zero term, there exists no constant \( C > 0 \) such that

\[
\left| \sum_{n \in F} \alpha_n f_n \right| \leq C \left( \int_0^T |f(t)|^2 \, dt \right)^{1/2}
\]

for each \( f \) of the form (2.1) with \( F \subset J \).

The special case where \( d^+ = d^- \) is especially important for the sequel (Sections 3 and 4) and therefore we state it separately for the reader’s convenience

Corollary 2.3.6. Let \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) be a sequence of real numbers. Assume that we have for some \( d \geq 0 \) and \( 0 \leq \alpha < 1 \)

\[
\begin{align*}
\# \{ \lambda \in \Lambda : 0 \leq \lambda \leq t \} &= dt + O(t^\alpha) \quad \text{and} \\
\# \{ \lambda \in \Lambda : -t \leq \lambda \leq 0 \} &= dt + O(t^\alpha).
\end{align*}
\]

Then we have the following properties
(1) For each \( T > 2 \pi d \) and for each \( n \in \mathbb{Z} \) there exists a constant \( C_n \) such that

\[
|f_n| \leq C_n \left( \int_0^T |f(t)|^2 \, dt \right)^{1/2},
\]

for each function \( f \) of the form (2.1) with \( n \in J \).

(2) On the other hand for each \( T < 2 \pi d \) and for each finite sequence \( \{\alpha_n\}_{n \in F} \) of complex numbers having a non zero term, there exists no constant \( C > 0 \) such that

\[
\left| \sum_{n \in F} \alpha_n f_n \right| \leq C \left( \int_0^T |f(t)|^2 \, dt \right)^{1/2}
\]

for each \( f \) of the form (2.1) with \( F \subset J \).

3. Application to Spectral Pointwise Control of some Plate Models

3.1. An Abstract Controllability Result

The main result of the section is the following.

**Theorem 3.1.1.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) (or a compact \( N \)-dimensional manifold without boundary) and \( A \) a positive self-adjoint operator in \( H = L^2(\Omega) \) satisfying conditions (1.1) and (1.2) with \( A^{-1} \) compact. We denote by \( \Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty} \) the increasing sequence of eigenvalues of \( A^{1/2} \). We assume that all the eigenvalues \( \lambda_j \) are simple and that we have for some \( d \geq 0 \) and \( 0 \leq \alpha < 1 \)

\[
(3.1) \quad |\{\lambda \in \Lambda^+ : \lambda \leq t\}| = dt + O(t^\alpha).
\]

Let \( \xi \in \Omega \) be any point at which no eigenfunction of \( A \) vanishes, and let us denote by \( D \) the vector space of all (finite) linear combinations of the eigenfunctions of \( A \). Then

1. or every \( T > 2 \pi d \), and \( (y^0, y^1) \in D \times D \), there exists \( h = h(t) \in L^2(0, T) \) with \( \text{supp}(h) \subset [0, T] \) and such that the unique solution \( y \) of (1.3) satisfies \( y(T, x) = y^1(T, x) = 0 \) in \( \Omega \).

2. This result is optimal: as soon as \( T < 2 \pi d \), there is no \( (y^0, y^1) \in D \times D \) except the trivial state \( (0, 0) \) for which such a control \( h \) exists.

**Proof.** This is a straightforward consequence of Proposition 1.1 and Corollary 2.3.6 applied with \( \Lambda = \Lambda^+ \cup (-\Lambda^+) \). Indeed any solution of (1.4) with
initial data in $D \times D$ has the form

$$u(t, x) = \sum_{j \in J} \{u_j \cos \lambda_j t + v_j \sin \lambda_j t \} \varphi_j(x),$$

where the functions $\varphi_j$ denote an orthonormal sequence of eigenfunctions of $A$ and the coefficients $u_j$ and $v_j$ and given by the formulas

$$u_j = \int_{\Omega} u(0, x) \varphi_j(x) \, dx, \quad v_j = \frac{1}{\lambda_j} \int_{\Omega} u'(0, x) \varphi_j(x) \, dx.$$

It is then clear that a direct application of Corollary 2.3.6 to the function $f(t) = u(t, \xi)$ with $u$ as above provides exactly the result by taking into account Proposition 1.1.

### 3.2. Application to Simply Supported Plates

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. We denote by $\Lambda^+ = \{\lambda_j\}_{1 \leq j \leq +\infty}$ the increasing sequence of eigenvalues of $(-\Delta)$ in $H^1_0(\Omega)$; it is known that under very general assumptions on $\Omega$, for instance if $\partial\Omega$ is smooth, the counting function $n(t) = \# \{\lambda \in \Lambda^+ : \lambda \leq t\}$ where each $\lambda \in \Lambda^+$ is repeated according to its multiplicity satisfies the so called Weyl formula:

$$n(t) = dt + O(t^{1/2}) \quad \text{with} \quad d = (1/4\pi) \text{vol} (\Omega). \quad (3.2)$$

As a special consequence of Theorem 3.1.1 we find

**Theorem 3.2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ satisfying the «Weyl formula», assume that all eigenvalues of $(-\Delta)$ in $H^1_0(\Omega)$ are simple and let us denote by $D$ the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta)$ in $H^1_0(\Omega)$. Let finally $\xi = (\xi_1, \xi_2) \in \Omega$ be any point at which no eigenfunction of $(-\Delta)$ in $H^1_0(\Omega)$ vanishes. Then

For every $T > (1/2) \text{vol} (\Omega)$, and $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with supp $(h) \subset [0, T]$ such that the unique solution $\Psi$ of

$$\begin{cases}
\Psi_{tt} + \Delta^2 \Psi = h(t) \delta_\xi(x, y) & \text{in } [0, T] \times \Omega, \\
\Psi = \Delta \Psi = 0 & \text{on } [0, T] \times \partial \Omega, \\
\Psi(0; x, y) = \Psi^0(x, y) & \text{in } \Omega, \\
\Psi_t(0; x, y) = \Psi^1(x) & \text{in } \Omega,
\end{cases}$$

satisfies $\Psi(T, \cdot) = \Psi_T(T, \cdot) = 0$.

This result is optimal: more precisely if $T < (1/2) \text{vol} (\Omega)$, no non-zero finite linear combination of the eigenfunctions of $(-\Delta)$ in $H^1_0(\Omega)$ is pointwise $L^2$-controllable.
When $\Omega$ is a rectangle of the form $(0, \pi) \times (0, L)$, with $(L/\pi)^2 \not\in \mathbb{Q}$, let $D$ denote the vector space of finite linear combinations of the basic eigenfunctions $\sin mx \sin (n\pi y/L)$, $m \in \mathbb{N}$, $n \in \mathbb{N}$. We have the following result:

**Proposition 3.2.2.** Let $\xi = (\xi_1, \xi_2) \in \Omega$ be fixed with $\xi_1/\pi \notin \mathbb{Q}$, $\xi_2/L \notin \mathbb{Q}$. For each $T > (1/2)\pi L$ and each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\text{supp}(h) \subset [0, T]$ and such that the unique solution $\Psi$ of

$$
\begin{align*}
\Psi_{tt} + \Delta^2 \Psi &= h(t) \delta_\xi(x, y) & \text{in} & \quad [0, T] \times \Omega, \\
\Psi &= \Delta \Psi = 0 & \text{on} & \quad [0, T] \times \partial \Omega, \\
\Psi(0; x, y) &= \Psi^0(x, y) & \text{in} & \quad \Omega, \\
\Psi_t(0; x, y) &= \Psi^1(x) & \text{in} & \quad \Omega,
\end{align*}
$$

satisfies $\Psi(T, \cdot) = \Psi, (T, \cdot) = 0$.

3.3. Application to the Case of Clamped Plates and Other Boundary Conditions

Let $\Omega$ be a rectangle or a bounded domain in $\mathbb{R}^2$ with a smooth boundary. We denote by $\Lambda^+ = \{\lambda_j\}_{j \geq 1}$ the increasing sequence of the square roots of the eigenvalues of $\Delta^2$ with relevant homogeneous boundary conditions: under very general assumptions on these boundary conditions, the counting function

$$n^*(t) = \# \{\lambda \in \Lambda^+ : \lambda \leq t\}
$$

where each $\lambda \in \Lambda^+$ is repeated according to its multiplicity still satisfies the Weyl formula (3.1) with the same value of $d$. As a consequence of Theorem 3.1.1 we find for instance

**Theorem 3.3.1.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ for which all the eigenvalues of $\Delta^2$ in $H^2_0(\Omega)$ are simple, and let us denote by $D$ the vector space of all (finite) linear combinations of the eigenfunctions of $\Delta^2$ in $H^2_0(\Omega)$. Let finally $\xi = (\xi_1, \xi_2) \in \Omega$ be any point at which no eigenfunction of $\Delta^2$ in $H^2_0(\Omega)$ vanishes. Then

For every $T > (1/2) \text{vol}(\Omega)$, and $(\Psi^0, \Psi^1) \in D \times D$, there exists $h = h(t) \in L^2(0, T)$ with $\text{supp}(h) \subset [0, T]$ such that the unique solution $\Psi$ of

$$
\begin{align*}
\Psi_{tt} + \Delta^2 \Psi &= h(t) \delta_\xi(x, y) & \text{in} & \quad [0, T] \times \Omega, \\
\Psi &= |\nabla \Psi| = 0 & \text{on} & \quad [0, T] \times \partial \Omega, \\
\Psi(0; x, y) &= \Psi^0(x, y) & \text{in} & \quad \Omega, \\
\Psi_t(0; x, y) &= \Psi^1(x) & \text{in} & \quad \Omega,
\end{align*}
$$

satisfies $\Psi(T, \cdot) = \Psi, (T, \cdot) = 0$. 

This result is optimal: more precisely if \( T < (1/2) \text{vol} (\Omega) \), no non zero finite linear combination of the eigenfunctions of \( \Delta^2 \) in \( H^2_0(\Omega) \) is pointwise \( L^2 \)-controllable.

**Proof.** Let \( \Omega \) be a bounded smooth domain or a rectangle in \( \mathbb{R}^2 \) for which all the eigenvalues of \( \Delta^2 \) in \( H^2_0(\Omega) \) are simple. Then Ivrii [10] asserts that under general positivity conditions, the fact that \( \Delta^2 \) with the given boundary conditions is elliptic in the sense of Shapiro-Lopatinskii implies that the counting function \( n'(t) \) satisfies (3.1). It is rather easy to check (cf. e.g. Wloka [23]) that the operator \( \Delta^2 \) in \( H^2_0(\Omega) \) satisfies the Shapiro-Lopatinskii condition, therefore Theorem 3.1.1. is applicable. (For a related weaker property cf. also Pleel [21].)

**Remark 3.3.2.** Of course the difficulty in general will be to determine the «strategic points» \( \xi = (\xi_1, \xi_2) \) at which no eigenfunction of \( \Delta^2 \) in \( H^2_0(\Omega) \) vanishes. Even when \( \Omega \) is a rectangle of the form \((0, \pi) \times (0, L)\), the eigenfunctions of \( \Delta^2 \) in \( H^2_0(\Omega) \) become more complicated than in the case of simply supported plates, and it is probably not so easy to find the strategic points. We know, however, that in the absence of multiple eigenvalues, almost every point is strategic.

**Remark 3.3.3.** In Section 5, the case of variable Lamé coefficients will be treated.

### 4. Some Applications to Spectral Internal Control

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) (or a compact \( N \)-dimensional manifold without boundary) and \( A \) a positive self-adjoint operator with compact resolvent in \( H = L^2(\Omega) \). Let \( \{\lambda_j\}_{j \in \mathbb{N}} \) be the increasing (without taking care of multiplicity) sequence of eigenvalues of \( A \) and for each \( j \), let

\[
F_j = \{ u \in L^2(\Omega); Au = \lambda_j u \}.
\]

Then we have the following

**Theorem 4.1.** Assume that \( A \) has the following properties

1. For every \( j \), the conditions \( u \in F_j \) and \( u = 0 \) on some non-empty open set imply \( u = 0 \).
2. There is a finite \( T_0 > 0 \) for which the functions

\[
\sum_{j \in J} \left[ u_j e^{i\sqrt{\lambda_j} t} + v_j e^{-i\sqrt{\lambda_j} t} \right], \quad J \text{ finite subset of } \mathbb{N} - \{0\}
\]

where the \( u_j \) and \( v_j \) are complex coefficients that are not dense in \( L^2(0, T_0; \mathbb{C}) \).
Then for each $T > T_0$, there are functions $\{f_j\}_{1 \leq j \leq +\infty}$ of class $C^\infty$ with compact support in $(0, T)$ such that we have the following properties

(a) For every $\Psi \in L^2(\Omega)$, the unique solution $u$ of

$$u'' + Au = f_j(t)\Psi(x) \text{ in } \mathbb{R} \times \Omega, \quad u(0, x) = u'(0, x) = 0 \text{ in } \Omega$$

fulfills $u(T, \cdot) \in F_j$ and $u'(T, \cdot) = 0$.

(b) For every non-empty open subset of $\Omega$, and every $\varphi \in F_j$, there exists $\Psi \in D(\Omega)$ with compact support in $\omega$ such that the unique solution $u$ of (3.1) satisfies $u(T) = \varphi$ and $u'(T) = 0$. Moreover the solution $v$ of

$$v'' + Av = f_j(t)\Psi(x) \text{ in } \mathbb{R} \times \Omega, \quad u(0, x) = u'(0, x) = 0 \text{ in } \Omega$$

fulfills $v(T) = 0$ and $v'(T) = -\lambda_j \varphi$.

PROOF. Let $\mu_j = \lambda_j^{1/2}$ for all $j$. We shall prove the result in five steps.

(1) First of all it follows from (2) that for each $j$ fixed, the function $\sin(\mu_j t)$ is not a limit in $L^2(0, T_0; \mathbb{C})$ of finite linear combinations of the functions $\exp(\pm i\mu_k t)$ for $k \neq j$. Indeed in such a case, the function $\exp(i\mu_j t)$ would be a limit in $L^2(0, T_0; \mathbb{C})$ of finite linear combinations of the functions $\exp(\pm i\mu_k t)$ for $k \neq j$ and of the function $\exp(-i\mu_j t)$, which by an argument similar to the proof of Lemma 2.2.2 would contradict property (2).

(2) In particular, there exists $h_j \in L^2(0, T_0; \mathbb{C})$ for which

$$\int_0^T h_j(t)e^{\pm i\mu_k t} dt = 0 \quad \text{for } k \neq j; \quad \int_0^T h_j(t)\sin(\mu_j t) dt \neq 0.$$

Replacing $h_j$ by either its real or its imaginary part, we may assume $h_j \in L^2(0, T_0; \mathbb{R})$.

(3) Let now $T > T_0$, $0 < \eta \leq (T - T_0)/2$ and

$$h_j(t, \eta) = 0 \quad \text{on } (-\infty, \eta),$$
$$h_j(t, \eta) = h_j(t - \eta) \quad \text{on } (\eta, T_0 + \eta),$$
$$h_j(t, \eta) = 0 \quad \text{on } (T_0 + \eta, +\infty).$$

For every $k \neq j$, we clearly have

$$\int_0^T h_j(t, \eta)e^{\pm i\mu_k t} dt = 0.$$
On the other hand, if $\eta$ is small enough we have
\[
\int_0^T h_j(t, \eta) \sin (\mu_j t) \, dt \neq 0.
\]
For such a fixed $\eta$, let $h_j(t, \eta) = h_j(t)$. Define also $\rho_\epsilon \in D(0, \epsilon)$ with $\rho_\epsilon \geq 0$ and $\int \rho_\epsilon(x) \, dx = 1$, and let us introduce $h_j \ast \rho_\epsilon = h_{j, \epsilon}$. Then for each $\epsilon \in (0, \eta)$ we have $h_{j, \epsilon} \in D(0, T)$. In addition,
\[
\int_0^T h_{j, \epsilon}(t) e^{\pm i \omega_k t} \, dt = 0 \quad \text{for} \quad k \neq j,
\]
\[
\lim_{\epsilon \to 0} \int_0^T h_{j, \epsilon}(t) \sin (\lambda_j t) \, dt = \int_0^T h_j(t) \sin (\lambda_j t) \, dt \neq 0.
\]
By selecting $\epsilon > 0$ small enough and replacing $h_{j, \epsilon}$ by some proportional function, we obtain $g_j \in D(0, T)$ such that
\[
\int_0^T g_j(t) e^{\pm i \omega_k t} \, dt = 0 \quad \text{for} \quad k \neq j \quad \text{and} \quad \int_0^T g_j(t) \sin (\mu_j t) \, dt = 1.
\]
We can, in fact, also assume
\[
\int_0^T g_j(t) \cos \mu_j t \, dt = 0.
\]
As a matter of fact, if
\[
\int_0^T g_j(t) \cos \mu_j t \, dt = I \neq 0,
\]
leth $g_j(t) = g_j(t + \alpha) + c g_j(t)$, $\alpha \neq 0$ being taken small enough. Then
\[
\int_0^T g_j(t) \cos \mu_j t \, dt = c I + \int_0^T g_j(t) \cos (\mu_j(t - \alpha)) \, dt
\]
\[
= (c + \cos (\alpha \mu_j) I + \sin (\alpha \mu_j),
\]
vanishes for
\[
c = -\cos (\alpha \mu_j) - \sin (\alpha \mu_j) / I.
\]
Taking $c$ as above we have
\[
\int_0^T g_j(t) \sin \mu_j t \, dt = c + \int_0^T g_j(t) \sin \mu_j(t - \alpha) \, dt
\]
\[
= c + \cos (\alpha \mu_j) - I \sin (\alpha \mu_j)
\]
\[
= -(I + 1/I) \sin (\alpha \mu_j) \neq 0
\]
for $\alpha$ small.

We can then replace $g_j$ by $\lambda g_j^\ast$ with $\lambda \neq 0$ properly chosen.
(4) Let \( f_j(t) = g_j(T - t) \) for \( t \in [0, T] \). The solution of (4.1) is given by

\[
  u(t, x) = \int_0^t f_j(t - s) \left\{ \sum_m \sin (\mu_m s) \Psi_m(x) \right\} ds
\]

with

\[
  \Psi_m = \text{Proj}_{P_m} (\Psi) = P_m \Psi, \quad \text{for} \quad m \in \{1, 2, \ldots\}.
\]

In particular we have

\[
  u(T, x) = \int_0^T g_j(s) \left\{ \sum_m \sin (\mu_m s) \Psi_m(x) \right\} ds
  = \sum_m \left\{ \int_0^T g_j(s) \sin (\mu_m s) \right\} \Psi_m(x)
  = \Psi_j(x).
\]

On the other hand for \( t \) close to \( T \) we have

\[
  u'(t, x) = \int_0^t f_j(t - s) \left\{ \sum_m \sin (\mu_m s) \Psi_m(x) \right\} ds.
\]

Therefore

\[
  u'(T, x) = \int_0^T f_j(T - s) \left\{ \sum_m \sin (\mu_m s) \Psi_m(x) \right\} ds
  = \sum_m \left\{ \int_0^T g'_j(s) \sin (\mu_m s) ds \right\} \Psi_m(x) = 0,
\]

since integration by parts gives

\[
  \int_0^T g'_j(s) \sin (\mu_m s) ds = -\lambda_m \int_0^T g_j(s) \cos (\mu_m s) ds = 0.
\]

for every \( m \in \mathbb{N} - \{0\} \). This establishes (a) with \( u(T, \cdot) = \Psi_j \). Moreover we notice that \( u' = v \) is the solution of (4.2) with initial data \((0, 0)\) and satisfies \( v(T, \cdot) = 0; \quad v'(T, \cdot) = u''(T, \cdot) = -\lambda_j \Psi_j \).

(5) To establish (b), we now use hypothesis (1). Indeed, for a fixed integer \( j \), we consider an orthonormal basis \( \{ \varphi_1, \ldots, \varphi_j \} \) of \( F_j \). To finish the proof we just need to show that we can find \( \Psi \in \mathcal{D}(\Omega) \) with support in \( \omega \) such that \( P_j \Psi = \varphi_1 \) (say). In the opposite case, the linear form defined by

\[
  \Psi \in \mathcal{D}(\omega) \mapsto \int_\omega \Psi(x) \varphi_1(x) dx
\]
would vanish on the intersection of the kernels of the linear forms defined by
\[
\Psi \in \mathcal{D}(\omega) \mapsto \int_\omega \Psi(x)\varphi_k(x) \, dx, \quad k \geq 2.
\]

By a standard result of linear algebra we would deduce the existence of real coefficients \(\{\alpha_k\}_{k \geq 2}\) for which
\[
\int_\omega \Psi(x)\varphi_1(x) \, dx = \sum_{k \geq 2} \alpha_k \int_\omega \Psi(x)\varphi_k(x) \, dx,
\]
for all \(\Psi \in \mathcal{D}(\omega)\). This immediately implies that for every \(x \in \omega\),
\[
\varphi_1(x) = \sum_{k \geq 2} \alpha_k \varphi_k(x).
\]
This is in contradiction with hypothesis (1) and the linear independence of \(\{\varphi_k\}_{k \geq 1}\).

Let us now denote by \(D\) the vector space of all (finite) linear combinations of the eigenfunctions of \(A\) and assume that all hypotheses of Theorem 4.1 are satisfied. Then by an immediate calculation we obtain the following result.

**Corollary 4.2.** For each \((y^0, y^1) \in D \times D\), there exists \(h \in \mathcal{D}((0, T) \times \Omega)\) with \(\text{supp} \, (h) \subset (0, T) \times \omega\) and such that the unique solution \(y\) of
\[
\begin{align*}
y'' + Ay &= h(t, x) \quad \text{in} \quad (0, T) \times \Omega, \\
y(0, x) &= y^0(x) \quad \text{in} \quad \Omega, \\
y'(0, x) &= y^1(x) \quad \text{in} \quad \Omega
\end{align*}
\]  
(4.5)

satisfies \(y(T, \cdot) = y'(T, \cdot) = 0\).

In particular, in the case of vibrating plates we obtain

**Corollary 4.3.** Let \(\Omega\) be a bounded smooth domain or a rectangle in \(\mathbb{R}^2\) and let us denote by \(D\) the vector space of all (finite) linear combinations of the eigenfunctions of \((-\Delta)\) in \(H^1_0(\Omega)\). Then for any \(T > (1/2)\text{vol}(\Omega)\) and each \((\Psi^0, \Psi^1) \in D \times D\), there exists \(h \in \mathcal{D}((0, T) \times \Omega)\) with \(\text{supp} \, (h) \subset (0, T) \times \omega\) and such that the unique solution \(\Psi\) of
\[
\begin{align*}
\Psi_{tt} + \Delta^2 \Psi &= h(t, x, y) \quad \text{in} \quad (0, T) \times \Omega, \\
\Psi &= \Delta \Psi = 0 \quad \text{on} \quad [0, T] \times \partial \Omega, \\
\Psi(0; x, y) &= \Psi^0(x, y) \quad \text{in} \quad \Omega, \\
\Psi_t(0; x, y) &= \Psi^1(x) \quad \text{in} \quad \Omega,
\end{align*}
\]  
(4.6)
satisfies $\Psi(T, \cdot) = \Psi_0(T, \cdot) = 0$. In addition the same result is valid for the equation

\[
(4.6)' \quad \begin{cases} 
\Psi_{tt} + \Delta^2 \Psi = h(t, x, y) & \text{in} \quad (0, T) \times \Omega, \\
\Psi = |\nabla \Psi| = 0 & \text{on} \quad [0, T] \times \partial \Omega, \\
\Psi(0; x, y) = \Psi_0(x, y) & \text{in} \quad \Omega, \\
\Psi_t(0; x, y) = \Psi_1(x) & \text{in} \quad \Omega.
\end{cases}
\]

Proof. Property (1) is clearly satisfied in both cases. It is therefore sufficient to check (2) for all $T_0 > (1/2) \text{vol} \Omega$. In the case of (4.6), when all the eigenvalues of $(-\Delta)$ in $H^1_0(\Omega)$ are simple, this follows at once from the Weyl formula and Corollary 2.3.6. In fact, by considering for instance some artificial additional frequencies or by using a generalization of the Beurling-Malliavin theory for exponential-polynomial series, it is possible to extend Corollary 2.3.6 in the more general situation of a counting function allowing arbitrary finite repetitions of the frequencies. Then the Weyl formula implies (2) without requiring the eigenvalues of $(-\Delta)$ in $H^1_0(\Omega)$ to be simple. The rest is clear. The same proof works for (4.6)'.

Remark 4.4. In the case of (4.6) in a rectangle, by using some results of J. P. Kahane [13], S. Jaffard [11] established the (in a sense stronger) result of exact internal controllability of any state with a finite energy, for any $T > 0$. However this result does not seem to imply immediately the existence of a $C^\infty$ control for states $(\Psi^0, \Psi^1) \in D \times D$. The result of [11] has been recently generalized in arbitrarily many dimensions (for a product of intervals) by V. Komornik [15]. On the other hand, no such internal controllability result seems to be known for the clamped plate equation.

This theory is also applicable to cases where the open set $\Omega$ is replaced by a compact manifold without boundary. We obtain for instance the following result, valid in any dimension $N \geq 1$.

**Corollary 4.5.** Let $\Sigma$ be the unit sphere of $\mathbb{R}^N$, and let us denote by $(-\Delta_\Sigma)$ the Laplace-Beltrami operator on $\Sigma$ and by $D$ the vector space of all (finite) linear combinations of the eigenfunctions of $(-\Delta_\Sigma)$. For all $T > 0$, and all $\omega$ non-empty open subset of $\Sigma$, and for each $(\Psi^0, \Psi^1) \in D \times D$, there exists $h \in \mathcal{D}'((0, T) \times \Sigma)$ with $\text{supp} \, h \subset (0, T) \times \omega$ and such that the unique solution $\Psi$ of

\[
(4.7) \quad \begin{cases} 
\Psi_{tt} + \Delta_\Sigma \Psi = h(t, \sigma) & \text{in} \quad (0, T) \times \Sigma, \\
\Psi(0; \sigma) = \Psi_0(\sigma) & \text{on} \quad \Sigma, \\
\Psi_t(0; \sigma) = \Psi_1(\sigma) & \text{on} \quad \Sigma
\end{cases}
\]

satisfies $\Psi(T, \cdot) = \Psi_0(T, \cdot) = 0$. 

Proof. Property (1) is clearly satisfied. Property (2) is a rather immediate consequence of the fact that the inverses of the positive eigenvalues of \((-\Delta_c)\) are summable. (Cf. e.g. [24], Theorem 15 p. 139.)

Remark 4.6. The control functions constructed here are of a special type. Their construction ultimately relies on the existence of a sequence of functions «bior-thogonal» to some complex exponentials, a technique already widely used (cf. e.g., H. O. Fattorini [2], J. Lagnese [16], D. L. Russel [22]) in control theory.

Remark 4.7. It seems rather reasonable to conjecture that the internal spectral controllability for the above plate equation is valid for general 2-dimensional domains and for every \(T > 0\). The study of this conjecture and some related problems will be the object of further research.

5. Possible Extensions and Additional Remarks

5.1. One-Dimensional Vibrating Systems

We consider first the equation of vibrating strings

\[
\begin{align*}
\int u_t - (a(x)u_x)_x &= 0 & \text{on } \mathbb{R} \times (0, L) \\
u(t, 0) = u(t, L) &= 0 & \text{on } \mathbb{R},
\end{align*}
\]

where \(a\) is smooth and bounded from below by a positive constant. Let \(A\) be the (strongly elliptic) unbounded operator on \(L^2(0, L)\) defined by

\[
D(A) = H^2 \cap H^1_0(0, L); \quad Av = -(a(x)v'_x)_x \quad \text{for } v \in D(A).
\]

The solutions are of the form

\[
u(t, x) = \sum_{n \in \mathbb{Z} - \{0\}} u_n e^{i\omega_n t} w_n(x)
\]

where \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) is given by \(\Lambda^+ \cup (-\Lambda^+)\) and \(\Lambda^+ = \{\lambda_j\}_{j \neq 0} \cup \omega\) is the increasing sequence of eigenvalues of \(A^{1/2}\) in \(H^1_0(0, L)\). Here Weyl's formula implies (cf. e.g. Hörmander [8], p. 273)

\[
n(t) = ct + O(t^{1/2})
\]

for some \(c > 0\). Hence, by Theorem 3.1.1 we obtain that, apart from the nodal points (zeroes of the eigenfunctions of \(A\)) pointwise spectral controllability holds true for all times \(T > 2\pi c\). This result extends to pointwise control some previous result of J. Lagnese [16] concerning internal exact controllability of strings.
Similarly we can consider vibrating beams given by

\[ u_{tt} + (a(x)u_{xx})_{xx} = 0 \quad \text{on} \quad \mathbb{R} \times (0, L) \]

with either of the following boundary conditions

\[ u(t, 0) = u(t, L) = u_t(t, 0) = u_x(t, L) = 0 \quad \text{on} \quad \mathbb{R}, \]

or

\[ u(t, 0) = u(t, L) = u_{xx}(t, 0) = u_{xx}(t, L) = 0 \quad \text{on} \quad \mathbb{R}. \]

Here of course \( a(x) \) is assumed smooth and bounded from below by some positive constant. Here Theorem 3.1.1 provides pointwise spectral controllability for all times \( T' > 0 \) since the completeness radius of the corresponding complex exponentials is obviously 0. Actually in such cases the result can also be deduced by means of a variant of Ingham’s Lemma (cf. [9.5]).

### 5.2. The Case of Plates with Nonconstant Lamé Coefficients

We deal with a similar case as in Section 3.3 except that the bilaplacian is now replaced by

\[ A = \sum_{i,j=1,2} \partial_{i,j} m_{i,j} \quad \text{with} \quad m_{i,j} = \frac{4\mu}{3} \left( \partial_{i,j} + \frac{\lambda}{\lambda + 2\mu} \delta_{i,j} \Delta \right). \]

The functions \( \lambda \) and \( \mu \) are the Lamé coefficients which we suppose to be nonconstant in the plate, but \( C^\infty \). In order to obtain a «Weyl's formula» for this operator, we have to check that the assumptions given in [10] are fulfilled. The operator is symmetric since

\[ \langle Au, v \rangle = \sum_{i,j} \int \frac{4\mu}{3} \partial_{i,j} u \partial_{i,j} v + \frac{4\lambda\mu}{3(\lambda + 2\mu)} \int \Delta u \Delta v. \]

The principal symbol of \( A \) is \((4\mu/3)(1 + \lambda/(\lambda + 2\mu))\) which is positive definite. Thus we only have to check the Shapiro-Lopatinskii condition on the boundary. It is a condition on the principal part of the operator which must hold at each point \( x_0 \) of the boundary. Here, the principal part of \( A \) is a bilaplacian multiplied by the smooth function \( a = (4\mu/3)(1 + \lambda/(\lambda + 2\mu)) \). Thus, up to the multiplicative factor \( a(x_0) \), the condition to check is exactly the same as if we had \( A = \Delta^2 \) with the corresponding boundary conditions, and the conclusion will be the same as for the bilaplacian; namely, for simply supported or clamped plates, formula (3.1) will hold with \( d = (1/4\pi) \text{vol}(\Omega) \), and thus, also the conclusion of the analog of Theorem 3.2.1.
5.3 Uniqueness and the Schrödinger Equation

In Section 3, we have given precisely the minimal time for pointwise spectral controllability. It is clear that a time $T$ of pointwise spectral controllability is also a uniqueness time in the sense that the trace of a solution at the observation point on $(0, T)$ determines the solution. In the case of second order problems (1.3)-(1.4), it is conjectured (cf. Kahane [14]) that the minimal uniqueness time is equal to the minimal time for pointwise spectral controllability.

Now let $\Omega$ be a bounded domain of $\mathbb{R}^2$ and $A$ a positive self-adjoint operator in $H = L^2(\Omega)$. We assume that $A$ satisfies the properties (1.1) and (1.2). Given $T > 0$, $\xi \in \Omega$ and $y^0$ a finite linear combination of the eigenfunctions of $A$, we are interested in the existence of a control function $h \in L^2([0, T])$ such that $\text{supp}(h) \subset [0, T]$ and for which the unique generalized solution $y$ of the Schrödinger type equation

$$y - iAy = h(t)\delta(x - \xi) \quad \text{in} \quad J0, T[, \quad y(0, x) = y^0(x) \quad \text{in} \quad \Omega,$$

satisfies

$$y(T, x) = y'(T, x) = 0 \quad \text{in} \quad \Omega.$$

The solutions of the homogeneous equation $\phi' - iA\phi = 0$ are here given by

$$\phi(t, x) = \sum_{n \geq 1} \phi_n e^{i\lambda_n t} w_n(x),$$

where the numbers $\lambda_n$ are the eigenvalues of $A$ and the functions $w_n$ are the associated eigenfunctions. For a given $\xi = (\xi_1, \xi_2) \in \Omega$, let

$$f(t) = \phi(t, \xi) = \sum_{n \geq 1} \alpha_n e^{i\lambda_n t},$$

Assume that the Weyl formula (3.1) holds for $A$ with $d > 0$: then by Corollary 2.3.5 the minimal time for spectral controllability is easily seen to be also positive (more precisely equal to $2\pi d$).

On the other hand, let us show that any positive time $T$ is in fact a «uniqueness time». From the Weyl formula (3.1), the properties of the initial data and the standard estimates on $\|w_n\|_\infty$ we deduce that the coefficients $\alpha_n$ have at most polynomial growth. Suppose that $f$ vanishes identically on $[0, T]$ and let $\varphi$ be a $C^\infty$ nonnegative function supported inside $[-T/2, 0]$ with integral 1. Then the convolution product $\varphi * f$ vanishes identically on $[0, T/2]$ and we have

$$(\varphi * f)(t) = \sum_{n \geq 1} \alpha_n \varphi(\lambda_n) e^{i\lambda_n t} = \sum_{n \geq 1} c_n e^{i\lambda_n t},$$
where the sequence \((c_n)\) is quickly decreasing, hence \(\varphi \ast f\) is \(C^\infty\). Let
\[
\psi(t) = (\varphi \ast f)(t)\omega(t)
\]
where \(\omega(t)\) is in the Schwartz class with a compactly supported Fourier transform. Then \(\psi\) is in the Schwartz class with
\[
\hat{\psi}(\xi) = [(\varphi \ast f) \ast \hat{\omega}](\xi) = \left[ \sum_{n \geq 1} c_n \delta_{\lambda_n} \right] \ast \hat{\omega}(\xi).
\]
Since the numbers \(\lambda_n\) are all positive, and \(\omega\) has a compactly supported Fourier transform, the Fourier transform of \(\psi\) vanishes on \((\pm \infty, \beta]\) for some \(\beta\). Hence \(g(t) = e^{\beta t} \psi(t)\) is a \(C^\infty\) function which vanishes identically on \([0, T/2]\) and whose Fourier transform is supported by \([0, +\infty)\). A well known theorem of Helson and Szegö (cf. [7]) asserts that if \(g \in L^2(\mathbb{R})\) has its Fourier transform supported in \([0, +\infty)\), then either \(g \equiv 0\) or \(\log |g(t)|/(1 + t^2) \in L^1(\mathbb{R})\): in particular if \(g\) vanishes on an interval we must conclude that \(g \equiv 0\). In our case we conclude that \(\psi \equiv 0\) for any choice of functions \(\varphi, \omega\) as above. It then follows immediately that \(f \equiv 0\), hence uniqueness is established for any \(T > 0\).

As a conclusion, in the case of the above Schrödinger equation, for all times between 0 and the minimal spectral controllability time \(T_\eta = 2\pi d\), there exists a dense family of pointwise controllable states, but none of them is a finite linear combination of the eigenfunctions of \(A\). It would be, of course, of interest to decide what happens for our plate models for small positive times, and in particular to settle Kahane's conjecture.

5.4. The Plate Equation in Higher Dimensions

In dimensions higher than or equal to 3, the calculations are very similar to those in dimension 2 and we shall not repeat them. The results, on the other hand, are quite different: for instance, for a 3-dimensional «plate» the Weyl formula now gives
\[
N(\lambda) = c\lambda^{3/2} + O(\lambda),
\]
hence the Beurling-Malliavin density of the \(\lambda_n\) is infinity and for no finite \(T > 0\) we have pointwise spectral controllability. The uniqueness problem is also open in this case. However, if we consider the associated Schrödinger equation, then the eigenvalues are positive and the proof of Section 5.2 is still applicable. Thus here, any positive time is a uniqueness time, while there is no finite time for pointwise spectral controllability!

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On Pseudospheres

John L. Lewis and Andrew Vogel
Dedicated to the memory of Allen Shields

1. Introduction

Denote points in Euclidean space, $\mathbb{R}^n$, by $x = (x_1, \ldots, x_n)$ and let $\overline{E}$, $\partial E$, denote the closure and boundary of $E \subseteq \mathbb{R}^n$, respectively. Put $B(x, r) = \{y : |y - x| < r\}$ when $r > 0$. Define $k$ dimensional Hausdorff measure, $1 \leq k \leq n$, in $\mathbb{R}^n$ as follows: for fixed $\delta > 0$ and $E \subseteq \mathbb{R}^n$, let $L(\delta) = \{B(x_i, r_i)\}$ be such that $E \subseteq \bigcup B(x_i, r_i)$ and $0 < r_i < \delta$, $i = 1, 2, \ldots$ Set

$$\phi^k_\delta(E) = \inf_{L(\delta)} \sum \alpha(k) r_i^k,$$

where $\alpha(k)$ denotes the volume of the unit ball in $\mathbb{R}^k$. Then

$$H^k(E) = \lim_{\delta \to 0} \phi^k_\delta(E), \quad 1 \leq k \leq n.$$

Let $D$ be a bounded domain in $\mathbb{R}^n$ with $0 \in D$ and $H^{n-1}(\partial D) < +\infty$. We shall say $D$ is a pseudo sphere if

(a) $\partial D$ is homeomorphic to the unit sphere, $S$, in $\mathbb{R}^n$
(b) $g(0) = a^2 \int_D g \, dH^{n-1}$, whenever $g$ is harmonic in $D$ and continuous on $\overline{D}$.

In (b), $a$ denotes a constant. The construction of pseudo spheres in $\mathbb{R}^2$, which are not circles, was first done by Keldysh and Lavrentiev to show the existence of domains not of Smirnov type (see [11, Ch. 3]). Also a completely different proof of existence has been given by Duren, Shapiro, and Shields in [3] (see also [2, Ch. 10]). Both proofs are heavily reliant on conformal mapping and $\mathbb{R}^2$ facts, such as: the logarithm of the gradient of a harmonic function is subharmonic.
In [12, p. 347], Shapiro asked whether there exists a pseudo sphere in $\mathbb{R}^n$ which is not a sphere. In this paper we answer Shapiro's question in the affirmative and even prove a little more:

**Theorem 1.** There exists a pseudo sphere $D$ in $\mathbb{R}^n$, $n \geq 3$, which is not a sphere. In fact $D$ can be chosen so that there is a homeomorphism $f$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f(S) = \partial D$ and

$$c(\beta)^{-1}|x - y|^{1/\beta} \leq |f(x) - f(y)| \leq c(\beta)|x - y|^{\beta},$$

whenever $\beta \in (0, 1)$ and $|x - y| \leq 1/2$.

In Theorem 1, as in the sequel, $c(\beta)$ denotes a positive constant depending only on $\beta$ and $n$. Also, $c$ will denote a positive constant depending only on $n$, not necessarily the same at each occurrence. Our method of proof is inspired by the proof of Keldysh and Lavrentiev in [9]. Here though conformal mapping techniques are not available. We outline our proof with $a = 1$ in (b). Let $\Omega$ be a bounded domain with $0 \in \Omega$ and let $G$ be Green’s function for $\Omega$ with pole at 0. That is,

$$G(x) = \frac{1}{n(n - 2)c(n)}|x|^{2 - n}, \quad x \in \mathbb{R}^n,$$

is harmonic in $\Omega$ and $G$ has boundary value 0 in the sense of Perron-Wiener-Brelot. It is known that if $\partial \Omega$ is sufficiently smooth, then

$$\nabla G(x) = \left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}\right)$$

extends continuously to $\tilde{\Omega} = \{0\}$. Under this assumption suppose that $|\nabla G| \geq 1$ on $\partial \Omega$. In Section 2, given $\epsilon$, $0 < \epsilon \leq \epsilon_0$, we add smooth bumps to $\partial \Omega$ by «pushing out» $\partial \Omega$ along certain small surface elements in $\{x \in \partial \Omega: |\nabla G(x)| > 1 + \epsilon\}$ of approximate side length $r$, $0 < r \leq \epsilon_0$. Let $\Omega'$, $G'$ be the smooth domain, and Green’s function with pole at 0, obtained from this process. Then $\Omega \subset \Omega'$ and we shall choose the bumps so that for $\epsilon \leq t \leq 1$,

$$H^{n - 1}(\partial \Omega') \geq H^{n - 1}(\partial \Omega) + \eta(t)H^{n - 1}\{x: |\nabla G(x)| > 1 + t\},$$

where $\eta$ is a positive function on $(0, \infty)$. It turns out that $\eta$ can be chosen independent of $\Omega, \Omega'$. We note from the Hopf boundary maximum principle (see [6, Lemma 3.4]) and $|\nabla G| \geq 1$ on $\partial \Omega$, that $|\nabla G'| \geq 1$ on $\partial \Omega \cap \partial \Omega'$. Also from Schauder type estimates, it will follow that $|\nabla G'| \geq 1$ on the bumps. Hence,

$$|\nabla G'(x)| \geq 1, \quad x \in \partial \Omega'.$$
Next we modify the identity mapping slightly in a neighborhood of each bump, to get $h$, a homeomorphism from $\mathbb{R}^n$ into $\mathbb{R}^n$, with $h(\partial D) = \partial \Omega'$. In Section 3 using a lemma of Wolff ([14, Lemma 2.7]) we will show the bumps can be chosen so that

$$
\int_{\partial \Omega'} |\nabla G'| \log |\nabla G'| \, dH^{n-1} \lesssim \int_{\partial \Omega} |\nabla G| \log |\nabla G| \, dH^{n-1}.
$$

The proof of (1.3) is somewhat involved, but luckily much of the hardwork has been done for us by Wolff.

In Section 4 we use (1.1)-(1.3) and induction to construct $D$. More specifically put $D_0 = B(0, \rho)$ and let

$$
G_0(x) = \frac{1}{n(n-2)\alpha(n)} (|x|^2 - \rho^2)^{-n}, \quad x \in B(0, \rho),
$$

be Green’s function for $B(0, \rho)$, where $\rho$ is chosen so that if $x \in \partial B(0, \rho)$, then

$$
|\nabla G(x)| = \frac{1}{\alpha(n)} \rho^{1+n} = 2.
$$

We put $\Omega = D_0$ and modify $\Omega$ as above to obtain $\Omega' = D_1$, $G' = G_1$, with $\epsilon$ replaced by $\epsilon_1$ and $h$ by $h_1$. Suppose $D_k$ has been constructed for $0 \leq k \leq m$. Again we put $\Omega = D_m$ and modify $\Omega$ as above to obtain $\Omega' = D_{m+1}$, $G' = G_{m+1}$, with $\epsilon$ replaced by $\epsilon_{m+1} = 2^{-m} \epsilon_m$, and $h$ by $h_{m+1}$. By induction we get $(D_k)^{n_0}, (h_k)^{n_0}, (G_k)^{n_0}$, satisfying (1.1), (1.2), with $\Omega'$, $\Omega$, replaced by $D_{k+1}$, $D_k$, respectively. Let $h_0(x) = \rho x$, and let $f_k = h_k \circ h_{k-1} \circ \cdots \circ h_0$, where $\circ$ denotes composition. Then it will follow from our construction for $k = 1, 2, \ldots$, that

$$
c(\beta)^{-1} |x - y|^{1/\beta} \leq |f_k(x) - f_k(y)| \leq c(\beta) |x - y|^{1/\beta},
$$

when $x, y \in \mathbb{R}^n$ and $|x - y| \leq 1/4$. Moreover, each $f_k$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f_k(S) = \partial D_k$. Set $D = \bigcup_k D_k$, and note from (1.5) that there exists a subsequence $(f_n)$ of $(f_k)$ which converges to a homeomorphism $f$ of $\mathbb{R}^n$, satisfying the conclusions of Theorem 1. Thus (a) in the definition of a pseudo sphere is valid. To prove (b) we first note from Green’s Theorem and (1.2) that

$$
1 = \int_{\partial D_k} |\nabla G_k| \, dH^{n-1} \geq H^{n-1}(\partial D_k),
$$

for $k = 0, 1, \ldots$ Second, observe for each $\delta > 0$ that

$$
\lim_{k \to \infty} H^{n-1}\{x \in \partial D_k : |\nabla G_k(x)| > 1 + \delta\} = 0,
$$
since otherwise we could use (1.1) and iteration to get a contradiction to (1.6) for large \( k \). Next from (1.2), (1.3), and iteration we deduce that for \( \alpha > 1 \), \( k = 0, 1, \ldots \).

\[ (1.8) \log \alpha \sum_{|\nabla G_k| > \alpha} |\nabla G_k| dH^{n-1} \leq \int_{\partial D_k} |\nabla G_k| \log |\nabla G_k| dH^{n-1} \leq c < +\infty. \]

Also in Section 4 we show that as \( k \to \infty \),

\[ (1.9) H^{n-1}|_{\partial D_n} \to H^{n-1}|_{\partial D}, \]

weakly as measures on \( \mathbb{R}^n \). Let \( g \geq 0 \) be a harmonic function in \( D \) which is continuous on \( \bar{D} \). Then from (1.2), (1.9), and Green’s Theorem we get

\[ (1.10) g(0) = \int_{\partial D_n} g|\nabla G_{n_k}| dH^{n-1} \geq \int_{\partial D_n} g dH^{n-1} - \int_{\partial D} g dH^{n-1}, \]

as \( k \to \infty \). To obtain the reverse inequality for fixed \( \delta < 10^{-3} \) and \( \alpha > 10^3 \), put

\[ E_k = \{ x \in \partial D_{n_k} : 1 \leq |\nabla G_{n_k}(x)| \leq 1 + \delta \}, \]

\[ F_k = \{ x \in \partial D_{n_k} : 1 + \delta < |\nabla G_{n_k}(x)| \leq \alpha \}, \]

\[ L_k = \{ x \in \partial D_{n_k} : |\nabla G_{n_k}(x)| > \alpha \}, \]

for \( k = 0, 1, 2, \ldots \). Then

\[ g(0) = \int_{\partial D_{n_k}} g|\nabla G_{n_k}| dH^{n-1} = \int_{E_k} \cdots + \int_{F_k} \cdots + \int_{L_k} \cdots = I_1 + I_2 + I_3. \]

Clearly,

\[ |I_1| \leq (1 + \delta) \int_{\partial D_{n_k}} g dH^{n-1}. \]

Also from (1.7) we find that

\[ |I_2| \leq \alpha \| g \|_\infty H^{n-1} \{ x \in \partial D_{n_k} : 1 + \delta < |\nabla G_{n_k}| \} \to 0, \]

as \( k \to \infty \). Here, \( \| g \|_\infty \) denotes the maximum of \( g \) in \( \bar{D} \). Using (1.8) we get

\[ |I_1| \leq \| g \|_\infty \int_{|\nabla G_{n_k}| > \alpha} |\nabla G_{n_k}| dH^{n-1} \leq \frac{c}{\log \alpha} \| g \|_\infty. \]

Letting \( k \to \infty \) we obtain from the above estimates and (1.9) that

\[ g(0) \leq (1 + \delta) \int_{\partial D} g dH^{n-1} + \frac{c}{\log \alpha} \| g \|_\infty. \]

Finally letting \( \delta \to 0, \alpha \to \infty \), we have

\[ g(0) \leq \int_{\partial D} g dH^{n-1}. \]
In view of (1.10) we conclude that

\begin{equation}
(1.11) \quad g(0) = \int_{\partial D} g \, dH^{n-1}
\end{equation}

when \( g \geq 0 \) is continuous on \( \bar{D} \) and harmonic in \( D \). From (1.11) with \( g = 1 \) we note that, \( H^{n-1}(\partial D) = 1 \). If \( g_1 \) is continuous on \( \bar{D} \), harmonic in \( D \), and \( g_1 - m \geq 0 \) in \( \bar{D} \), then from (1.11) and the above note we deduce

\[ g_1(0) = (g_1 - m)(0) + m = \int_{\partial D} (g_1 - m) \, dH^{n-1} + m = \int_{\partial D} g_1 \, dH^{n-1}. \]

Thus, \( D \) is a pseudo sphere. The initial bumps on \( D \) will be chosen to have low peaks relative to those added to form \( D_k, k \geq 2 \), in order to guarantee that \( D \) is not a ball.

We remark that \( D \) will be regular for the Dirichlet problem, so each continuous function on \( \partial D \) will have a harmonic extension to \( D \) which is continuous on \( D \). From (1.11) it follows that harmonic measure and \( H^{n-1} \) measure on \( \partial D \) are equal (see [7, Ch. 8] for the Dirichlet problem). Moreover, since \( H^{n-1}(\partial D) = 1 \), it follows (see [4, Section 5.8]) that \( D \) is of finite perimeter. Thus several other measures are equal to \( H^{n-1} \) measure on \( \partial D \) (see [5, Thm. 4.5.19, (16)] and [5, Thm. 3.2.26]). Also \( D \) will be a nontangentially accessible (NTA) domain in the sense of Kenig and Jerison [8]. Using the corkscrew condition for NTA domains (i) in Section 3) it is easily deduced that every point in \( \partial D \) lies in the measure theoretic boundary of \( D \) (see [4, Section 5.8]). Hence \( D \) satisfies the hypotheses of Theorem 1 in [10], from which we conclude

\[ \sup \{ |\nabla G^*(x)| : x \in D - B(0, \rho/2) \} = +\infty, \]

where \( G^* \) is Green's function for \( D \) with pole at 0. Next we remark that this paper leaves open the very interesting question as to whether \( f \) in Theorem 1 can also be chosen for some \( K > 1 \) to be a \( K \) quasiconformal mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n, n \geq 3 \). In \( \mathbb{R}^2 \) it follows from a criteria of Ahlfors (see [1, Ch. 4]) and the Keldysh-Lavrentiev construction that the answer to the above question is yes, and in fact \( K \) can be chosen arbitrary near 1.

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2. Preliminary reductions

If \( x \in \mathbb{R}^n \), we let \( x' = (x_1, \ldots, x_{n-1}) \) and shall write, \( x = (x', x_n) \). We assume throughout this section that \( \Omega \) is a bounded domain of class \( C^4 \) with \( \theta \in \Omega \).
More specifically, for each \( y \in \partial \Omega \) there exists \( s > 0 \) such that \( B(y, s) \cap \partial \Omega \) is a part of the graph of a four-times continuously differentiable function, defined on a hyperplane in \( \mathbb{R}^n \), and \( B(y, s) \cap \Omega \) lies above the graph. From compactness and a standard covering argument it follows for each \( r > 0 \) that there exists, \( y^1, y^2, \ldots, y^N \in \partial \Omega \), such that

\[
\partial \Omega \subset \bigcup_{i=1}^N B(y^i, 100r) \quad \text{and} \quad B(y^i, 10r) \cap B(y^j, 10r) = \emptyset, \quad i \neq j.
\]

Moreover, if \( 0 < r < r_0, \ r_0 \) sufficiently small, and \( y = (y', y_n) \in \{ y^j \}_{j=1}^N \), then from the implicit function theorem we see there exists \( \theta = \theta(\cdot, y) \), four-times continuously differentiable on \( \mathbb{R}^{n-1}(\theta \in C^4(\mathbb{R}^{n-1})) \), with \( \theta(0) = 0 \), \( \nabla \theta(0) = 0 \), such that after a possible rotation of axes:

\[
\partial \Omega \cap B(y, 1000r^{1/2}) \subset \{ (x' + y', \theta(x') + y_n): x' \in \mathbb{R}^{n-1} \},
\]

\[
\Omega \cap B(y, 1000r^{1/2}) \subset \{ (x' + y', x_n): x_n - y_n > \theta(x'), x' \in \mathbb{R}^{n-1} \}
\]

Here \( \nabla \) denotes the \( \mathbb{R}^{n-1} \) gradient. Put

\[
M_1 = \max_{y \in \{ y^j \}_{j=1}^N} \left\{ \max_{x \in \partial \Omega \cap B(y, 1000r^{1/2})} \sum |\partial_\alpha \theta(x', y)| \right\}
\]

where the sum is taken over all multi-indexes \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \) with \( |\alpha| = \sum_{j=1}^{n-1} \alpha_j \), and \( 0 \leq |\alpha| \leq 4 \). Also, \( \partial_\alpha \) denotes the corresponding partial derivative with respect to \( (x')^\alpha, x' \in \mathbb{R}^{n-1} \). Given \( \epsilon, 0 < \epsilon < \epsilon_0 \leq 10^{-3} \), choose \( r_0 > 0 \) so small that for \( 0 < r \leq r_0 \)

(2.1) \[
M_1 r^{1/2} \leq 10^{-3} \epsilon^{1/4} \leq 10^{-9} \epsilon^4.
\]

Again this choice is possible by compactness of \( \partial \Omega \). In this section and the next section we allow \( r_0 \) at a number, satisfying several conditions, which depends only on \( n \). \( r_0 \) will depend on \( \epsilon, M_1, n, \) and \( M_2 \), defined below.

As in Section 1 let \( G \) be Green's function for \( \Omega \) with pole at 0 and assume \( |\nabla G| \geq 1 \) on \( \partial \Omega \). Let

\[
M_2 = \max_{y \in \{ y^j \}_{j=1}^N} \left\{ \max_{x \in \partial \Omega \cap B(y, 1000r^{1/2})} \sum |\partial_\alpha G(x)| \right\},
\]

where now \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( 0 \leq |\alpha| \leq 4 \), and \( \partial_\alpha \) denotes the corresponding partial derivative with respect to \( x^\alpha, x \in \Omega \). From Schauder's Theorem (see [6, Ch. 6]), it is clear that \( M_2 < +\infty \). We choose \( r_0 \) still smaller, if necessary, so that in addition to the above conditions, we have

(2.2) \[
M_2 r^{1/2} \leq 10^{-3} \epsilon^{1/4} \leq 10^{-9} \epsilon^4.
\]
Let \( l \) be the largest nonnegative integer such that \( 2^{-l} \alpha_0 > \varepsilon \) and put \( \sigma_k = 2^{-k} \alpha_0 \), for \( k = 0, 1, \ldots \). Set

\[
E_k = \{ x \in \partial \Omega : 1 + \sigma_k < |\nabla G(x)| \leq 1 + \sigma_{k-1} \}, \quad 1 \leq k \leq l + 1, \\
E_0 = \{ x \in \partial \Omega : |\nabla G(x)| > 1 + \alpha_0 \}.
\]

Let \( \psi, \ 0 \leq \psi \leq 1 \), be a fixed \( C^\infty \) function on \( \mathbb{R}^{n-1} \) with \( \max_{\mathbb{R}^{n-1}} \psi = 1 \) and support in the unit ball of \( \mathbb{R}^{n-1} \), to be specified in Section 3. We form a domain \( \Omega' \) of class \( C^4 \) by adding smooth bumps to \( \partial \Omega \). More specifically, let \( L \) be the set of all \( y \in \{ y^i \}_{i=1}^N \) for which

\[
B(y, 100r) \cap \bigcup_{k=0}^{l+1} E_k \neq \emptyset.
\]

For fixed \( y = (y', y_n) \in L \), let \( j \) be the smallest nonnegative integer with

\[
(2.3) \quad B(y, 100r) \cap E_j \neq \emptyset.
\]

Put

\[
\xi(x') = \theta(x') - a_j^2 r \lambda_j^{-1} \psi(\lambda_j r x'/r) + y_n, \quad x' \in \mathbb{R}^{n-1},
\]

where \( (\lambda_j)_0^\infty \) is an increasing sequence of positive numbers with \( \lambda_j \geq 1/\sigma_j \), \( j = 0, 1, \ldots \), which will be defined explicitly in Section 3. Also \( (\lambda_j)_0^\infty \) will depend only on \( \alpha_0 \). Define \( \Omega' \) by

(i) \( \Omega' = \bigcup_{z \in L} B(z, 10r) \),

(ii) \( \partial \Omega' \cap B(y, 10r) = \{ (x' + y', \xi(x')) : x' \in \mathbb{R}^{n-1} \} \cap B(y, 10r) \),

(iii) \( \Omega' \cap B(y, 10r) = \{ (x' + y', \xi(x')) : x > \xi(x') \} \cap B(y, 10r) \).

Thus for each \( y \in L \) and smallest \( j, \ 0 \leq j \leq l + 1 \), satisfying (2.3), we add a bump to \( \Omega \) under \( y \), as defined above, to get \( \Omega' \). Clearly \( \Omega' \) is of class \( C^4 \). Moreover, if \( r_0 \) is small enough, we claim as in (1.2) that

\[
(2.4) \quad |\nabla G'(x)| \geq 1, \quad x \in \partial \Omega'.
\]

Indeed, if \( x \in \partial \Omega' \cap \partial \Omega \), then it follows from the Hopf boundary maximum principle that (2.4) is true. To prove (2.4) for \( x \in \partial \Omega' - \partial \Omega \), we let \( \tilde{B}(t) = \{ x' \in \mathbb{R}^{n-1} : |x'| < t \} \). We shall need the following lemma of Schauder type.

In Lemma 1, \( \phi, \psi, \gamma \), are \( C^5 \) functions on \( \tilde{B}(2), \ k \geq 3 \). Moreover, \( \phi < 1/4, \) and \( |\cdot|_k \) denotes the \( C^k \) norm on \( \tilde{B}(2) \). Also, \( c^* = c^*(\cdot, k) \), is an increasing function on \( (0, \infty) \) which depends only on \( k \).
Lemma 1. Let
\[ H = \{(x', x_n) : |x'| < 1 \text{ and } |\phi(x')| < x_n < 1\}. \]

Let \( u \) be harmonic in \( H \), with \( |u| \leq M_3 < +\infty \), and suppose that \( u = \gamma \) continuously on \( \{(x', \phi(x'))\} \cap \partial H \). Then for \( k \geq 3 \)
\[ \sum_{0 \leq |\alpha| \leq k} |\partial^\alpha u(x)| = c'\|\phi\|_k(\|\gamma\|_k + M_3), \quad x \in B(0, 1/2) \cap \tilde{H}. \]

Lemma 1 is given in [6, Corollary 6.7] for \( C^{2,\alpha} \) domains with a constant depending on \( H \). However, the proof is essentially unchanged if \( C^{2,\alpha} \) is replaced by \( C^k \), and \( c'(\cdot) \) can be used for the resulting constant (see the remark following Lemma 6.5 in [6]). To prove (2.4) on a bump, we first let
\[ Z(y, t) = \{(x', x_n) : x_n - y_n < t, |x' - y'| < t\} \]
and note that since \( \psi \) has support in \( \tilde{B}(1), \)
\[ \Omega^* \cap B(y, 10r) \subseteq Z(y, r\lambda_j^{-1}), \]
whenever \( y \in L \) and \( j \) is the smallest integer satisfying (2.3). Second, observe from the Hopf boundary maximum principle and (2.5) that to prove (2.4) on a bump it suffices to show
\[ |\nabla G^*(x)| \geq 1, \quad x \in \tilde{Z}(y, r\lambda_j^{-1}) \cap \tilde{\Omega}^*, \]
where \( \Omega^* \) is obtained from \( \Omega \) by adding just one bump at \( y \) as above, and \( G^* \) is the Green's function for \( \Omega^* \) with pole at 0. To prove (2.6) let
\[ F = \tilde{Z}(y, r\lambda_j^{-1}) \cap \tilde{\Omega}^* \]
and
\[ M_d = \max_{x \in F} |\nabla G^*(x)|. \]

Then from the mean value theorem of calculus and the fact that \( G = 0 \) on \( \partial \Omega \), we deduce
\[ 0 \leq G^* - G \leq cM_d \sigma_j^3 \lambda_j^{-1} r \]
on \( \partial \Omega \). Since \( G^* - G \) is harmonic in \( \Omega \), we see from the maximum principle for harmonic functions that (2.7) also holds in \( \Omega \). From (2.1), (2.2), (2.7), and the fact that
\[ \nabla G(y) = \begin{pmatrix} 0, \ldots, \frac{\partial G(y)}{\partial y_n} \end{pmatrix} \]
we get for \( x \) in \( \tilde{\Omega} \cap \tilde{B}(y, 20r\lambda_j^{-1}) \),
(2.8) $|G^*(x) - |\nabla G(y)||x_n - y_n||$
\[
\leq cM_4 \sigma_j^2 \lambda_j^{-1} r + |G(x) - |\nabla G(y)||x_n - y_n||
\]
\[
\leq cM_4 \sigma_j^2 \lambda_j^{-1} r + \int_{\gamma_{x_n + \theta(x')} - y} \left| \frac{\partial G}{\partial t_n}(x', t_n) - \frac{\partial G}{\partial t_n}(y', y_n) \right| dt_n
\]
\[
+ |\nabla G(y)||\theta(x' - y')|
\]
\[
\leq cM_4 \sigma_j^2 \lambda_j^{-1} r + cM_2 (\lambda_j^{-1} r)^2 + cM_2 M_1 (\lambda_j^{-1} r)^2
\]
\[
\leq c(M_4 \sigma_j^2 + \epsilon^2) \lambda_j^{-1} r.
\]

Put, $\beta = 10r/\lambda_j$,
\[
\phi(x') = \beta^{-1}(\beta x' - y_n), \quad x' \in \tilde{H}(2),
\]
\[
u(x) = \beta^{-1} G^*(\beta x + y) - |\nabla G(y)||x_n, \quad x \in H,
\]
where $H$ is defined relative to $\phi$ as in Lemma 1. Using (2.1) it is easily checked that $||\phi||_4 \leq c \sigma_j^2 ||\psi||_4 + c \epsilon^2$. Since $u = -|\nabla G(y)||\phi$ on $\partial H \cap B(1)$, we find from this inequality, (2.8), and Lemma 1 with $k = 4$ that
\[
|\nabla u(x)| \leq c'(||\phi||_4)(M_4 \sigma_j^2 + c \sigma_j^2 |\nabla G(y)| + c \epsilon^2)
\]
\[
x \in B(0, 1/2) \cap H,
\]
where
\[
c'(||\phi||_4) \leq c'(||\psi||_4 + 1) = c_0.
\]
From this inequality and the fact that $\epsilon \leq 2\sigma_j$, $|\nabla G(y)| \geq 1$, we deduce
(2.9) $||\nabla G^*(x) - |\nabla G(y)|| \leq c_0 M_4 \sigma_j^2 + c_1 \sigma_j^2 |\nabla G(y)|$

for $x \in \tilde{Z}(y, \lambda_j^{-1}) \cap \tilde{H}_*$. Let $\sigma_0$, $0 < \sigma_0 \leq 10^{-3}$, be so small that
(2.10) $c_0 + c_1 < 10^{-3} \sigma_0^{-1}$.

Choosing $x$ so that
\[
|\nabla G^*(x)| = M_4,
\]
we conclude from the triangle inequality and (2.9) that
\[
M_4(1 - c_0 \sigma_j^2) \leq (1 + c_1 \sigma_j^2)|\nabla G(y)|.
\]
Hence,
(2.11) $M_4 \leq (1 + 2c_0 \sigma_j^2)(1 + c_1 \sigma_j^2)|\nabla G(y)|$. 
Now from (2.2), (2.3), we see that $|\nabla G(y)| \geq 1 + \sigma_j/2$. Using this fact, (2.10), and (2.11), in (2.9), we deduce

$$|\nabla G(x)| \geq (1 - 2(c_0 + c_1\sigma_j^2)|\nabla G(y)| \geq 1 + \frac{1}{4} \sigma_j.$$  

Hence (2.6) is valid. From our earlier remarks it now follows that (2.4) is valid.

If

$$c_2 = \int_{R_n} |\nabla^r \psi(x')|^2 \, dx',$$

and

$$\sigma_0 \leq c_2 \leq \alpha(n - 1)\left(\max_{R_n} |\nabla^r \psi|\right)^2 \leq \sigma_0^{-1}10^{-6},$$

then from (2.1), it follows that

$$H^{n-1}(Z(y, r\lambda_j^{-1}) \cap \partial \Omega) = \int_{B(\lambda_j^{-1})} \sqrt{1 + |\nabla^r \xi|^2} \, dx'$$

$$\geq \int_{B(\lambda_j^{-1})} \sqrt{1 + \sigma_j^4(|\nabla^r \psi(y, r^{-1}x')|^2 \, dx' - \epsilon^8 \alpha(n - 1)(r/\lambda_j)^{(n-1)}}$$

$$= \left(\int_{B(1)} \sqrt{1 + \sigma_j^4(|\nabla^r \psi(x')|^2 \, dx'}\right)(r/\lambda_j)^{(n-1)} - \epsilon^8 \alpha(n - 1)(r/\lambda_j)^{(n-1)}$$

$$\geq \left(1 + \frac{1}{4} \sigma_j^4 c_2 - \epsilon^8 \right) \alpha(n - 1)(r/\lambda_j)^{(n-1)}$$

$$\geq \frac{1}{\lambda_j} \sigma_j^4 c_2 \alpha(n - 1)(r\lambda_j^{-1})^{(n-1)} + H^{n-1}(Z(y, r\lambda_j^{-1}) \cap \partial \Omega).$$

Given $t \geq \epsilon$, let $k$ be the least nonnegative integer such that $t \geq \sigma_k$, $0 \leq k \leq l + 1$. Let $J = J(k)$, be the set of all $i$ such that (2.3) holds with $y = y^i$ and $j \leq k$. From (2.1) it is clear that

$$H^{n-1}\{x \in \partial \Omega: |\nabla G(x)| \geq 1 + t\} \leq H^{n-1}\left(\bigcup_{i \in J} B(y^i, 100r) \cap \partial \Omega\right)$$

$$\leq 2 \sum_{i \in J} \alpha(n - 1)(100r)^{n-1}.$$

Using (2.13), (2.14), and (2.5) we deduce

$$H^{n-1}(\partial \Omega) \geq H^{n-1}(\partial \Omega) + \frac{c_3 \sigma_k^4}{\lambda_k^{n-1}} H^{n-1}\{x \in \partial \Omega: |\nabla G(x)| > 1 + t\}.$$
where $c_3 > 0$ depends only on $n$. Let

$$
\eta(t) = \begin{cases} 
\frac{c_3 \sigma_0^4}{\lambda_0} - 1, & \sigma_0 \leq t \\
\frac{c_3 \sigma_k^4}{\lambda_k} - 1, & \sigma_k \leq t < \sigma_{k-1}, \quad k = 1, 2, \ldots
\end{cases}
$$

Clearly $\eta$ does not depend on $\Omega$ or $\Omega'$. Rewriting (2.15) in terms of $\eta$ we obtain (1.1).

Next we define the homeomorphism $h$ mentioned in Section 1. If $y \in L$ and $j$ is the smallest positive integer for which (2.3) holds, define $h$ on $Z(y, r)$ by $h(x) = (x', h^*(x))$, where

$$
h^*(x', x_n) = \begin{cases} 
(x_n - \xi(x' - y'))(x_n - y_n) + r + y_n, & x \in Z(y, r) \cap \Omega \\
\frac{(x_n - \xi(x' - y'))(x_n + y_n) - r + y_n}{r + \theta(x' - y')}, & x \in Z(y, r) \cap (\mathbb{R}^n - \Omega)
\end{cases}
$$

Define $h(x) = x$ in the complement of the union of all $Z(y, r)$ for which (2.3) holds. We note that $h$ restricted to $Z(y, r) = Z$ is simply a projection by lines parallel to the $x_n$ axis of $Z \cap (\mathbb{R}^n - \Omega)$, $Z \cap \Omega$, respectively onto $Z \cap (\mathbb{R}^n - \Omega')$, $Z \cap \Omega'$, which keeps $\partial Z(y, r)$ fixed. Thus, $h$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $h(\Omega) = \Omega'$. Moreover, using (2.1) it is easily checked that

$$(1 - c_4 \sigma_0^2)|x - z| \leq |h(x) - h(z)| \leq (1 + c_4 \sigma_0^2)|x - z|,$$

when $x, z \in \mathbb{R}^n$ and

$$(2.17) \quad |x - z| - c_4 \sigma_0^2 r \leq |h(x) - h(z)| \leq |x - z| + c_4 \sigma_0^2 r,$$

when $|x - z| > r$. Also for use in proving (1.9) we shall show for $x, z \in \partial \Omega$, that

$$(2.18) \quad |h(x) - h(z)| \geq (1 - c_4 r^{1/2})|x - z|.$$

Indeed, suppose $x, z \in \partial \Omega$, $5r \leq |x - z| < 100r^{1/2}$, $x \in B(w, 100r)$, and $z \in B(y, 100r)$, where $w, y \in \{ y \}^n_1$. Let $\theta$ be defined relative to $y$ as previously and recall that $B(y, 1000r^{1/2}) \cap \partial \Omega$ can be expressed in terms of $\theta$. Let $\nu(p)$ denote the outer unit normal to $p$ in $\partial \Omega$ and let $\cdot$ denote inner product. Then

$$|\nu(y) \cdot \nu(w)| = (1 + |\nabla \theta(w' - y')|^2)^{-1/2} > 1 - \frac{1}{4} M_1^2 |w' - y'|^2.$$

Thus if $\delta$ denotes the angle between $\nu(y)$ and $\nu(w)$, then

$$\delta < 4 M_1 |w' - y'| < 164 M_1 |x - z|. $$
Next suppose \( h(x) = (v', u_a) \) and \( v' \neq x' \). Then we can draw the right triangle with vertices \( x, h(x) \), and \( P = (v', x_a) \). Let \( l_1, l_2, \) and \( l_3 \) be the sides of this triangle connecting \( x \) to \( h(x) \), \( h(x) \) to \( P \), and \( P \) to \( x \), respectively. Then from the definition of \( h \) we see that \( v(w) \) is parallel to \( l_1 \), and so \( l_1, l_2 \) form an angle \( \delta \) at \( h(x) \). Also, \( |x - h(x)| < r \), so from trigonometry and the above inequality,

\[
|v' - x'| < r \sin \delta < 164M_1 |x - z|.
\]

From this inequality and (2.1) we deduce

\[
|h(z) - h(x)| > |v' - z'|
\geq |x' - z'| - |v' - x'|
> (1 - cM_1^2 r)|x - z|
> (1 - c_2 \epsilon^{1/2})|x - z|.
\]

Hence (2.18) is valid when \( 5r \leq |x - z| < 100r^{1/2} \). If \( |x - z| < 5r \), then (2.18) remains true as follows easily from the fact that the bumps are greater than \( 6r \) apart. If \( 100r^{1/2} \leq |x - z| \) then it follows from (2.17) that (2.18) is true.

Finally in this section we fix \( a_0 \) to be the largest number for which (2.10), (2.12), hold and

\[
(2.19)
\]

\[
c_4 a_0^2 \leq \frac{1}{2}.
\]

Note from (2.12) that \( 0 < a_0 \leq 10^{-3} \).

3. Wolff's lemma

To prove (1.3) in Section 1 we shall need some definitions. Let \( \Omega \), be a bounded domain. If \( \text{diam } \Omega_1 = 1 \), then \( \Omega_1 \) is an NTA domain with constant \( A \) if it has the following properties:

(i) (Corkscrew condition.) For each \( x \in \partial \Omega_1 \), \( 0 < r < A^{-1} \), there are points \( P_r(x) \in \Omega_1, Q_r(x) \in \mathbb{R}^n - \Omega_1 \), with \( |P_r(x) - x| \leq Ar, |Q_r(x) - x| \leq Ar \), and \( \text{dist}(P_r(x), \partial \Omega_1) \geq A^{-1} r \), \( \text{dist}(Q_r(x), \partial \Omega_1) \geq A^{-1} r \),

(ii) (Harnack chain condition.) For each \( x, y \in \Omega_1 \), there is a path \( \gamma \) from \( x \) to \( y \) with length \( (\gamma) \leq A|x - y| \) and \( \text{dist}(\gamma(t), \partial \Omega_1) \geq A^{-1} \min \{|\gamma(t) - x|, |\gamma(t) - y|\} \).

In general \( \Omega_1 \) is an NTA domain with constant \( A \), if a scaling of it with diameter 1 has constant \( A \). \( \Omega_1 \) is said to be Lipschitz on scale \( t \) with constant \( A \), provided for each \( z \in \partial \Omega_1 \), there is a coordinate system such that \( \partial \Omega_1 \cap B(z, t) \) is the
graph of a Lipschitz function defined on $\mathbb{R}^{n-1}$ with Lipschitz norm less than or equal to $A$. Moreover, $\Omega_1 \cap B(x, t)$ lies above the graph of this function.

Now suppose for some $w \in \partial \Omega_1$ and $t > 0$ that after a possible rotation of coordinates,

$$
\partial \Omega_1 \cap B(w, t) = \{ x : x_n = w_n \} \cap B(w, t)
$$
$$
\Omega_1 \cap B(w, t) = \{ x : x_n > w_n \} \cap B(w, t)
$$

Let $p \leq 0$ be a $C^\infty$ function with support in $B(1)$, suppose $\lambda > 2 \max_{\mathbb{R}^{n-1}} |p| + 1$, and define $\Omega_2 \supset \Omega_1$ as follows:

(a) $\Omega_1 - B(w, t) = \Omega_2 - B(w, t)$,
(b) $\partial \Omega_2 \cap B(w, t) = \{ (x' + w', w_n + t\lambda^{-1}p(t^{-1}\lambda x')) : x' \in \mathbb{R}^{n-1} \} \cap B(w, t)$,
(c) $\Omega_2 \cap B(w, t) = \{ (x' + w', x_n) : x_n > w_n + t\lambda^{-1}p(t^{-1}\lambda x') \} \cap B(w, t)$.

Let $\bar{p}$ be the continuous harmonic extension of $p$ to $(\mathbb{R}^n)^+ = \{ (x', x_n) : x_n > 0 \}$ and put

$$
\Lambda(p) = \int_{\mathbb{R}^{n-1}} \left( \left( \frac{\partial \bar{p}}{\partial x_n} \right)^3 - 3|\nabla p|^2 \frac{\partial \bar{p}}{\partial x_n} \right)(x', 0) \, dx'
$$

where $\nabla p$, as in Section 2, is the $\mathbb{R}^{n-1}$ gradient. Next if $d = \text{diam } \Omega_1$, we assume

$$
B(0, d/A) \subseteq \Omega_1 \subseteq B(0, 4d).
$$

Denote Green’s functions for $\Omega_1, \Omega_2$, with pole at 0, by $G_1, G_2$, respectively, and let $\omega_1$ be harmonic measure on $\Omega_1$ with respect to 0. If $\partial \Omega_1$ is sufficiently smooth we observe that

$$
\omega_1(E) = \int_{E \cap \partial \Omega_1} |\nabla G_1| \, dH^{n-1}, \quad E \text{ Borel.}
$$

Then Wolff proved [14, Lemma 2.7].

**Lemma 2.** Let $\Omega_1$ be NTA and Lipschitz on scale $t$ with constant $A$. Suppose $\Omega_1$ satisfies (3.1), (3.2), and $\Omega_2$ is obtained by adding a bump to $\Omega_1$ as in (a)-(c). If $\Lambda(p) < 0$, then there exists $\lambda^* = \lambda^*(A, p)$, $c_6 = c_6(A, p)$, such that for $\lambda \geq \lambda^*$,

$$
\int_{\partial \Omega_2} |\nabla G_2| \log |\nabla G_2| \, dH^{n-1} \leq \int_{\partial \Omega_1} |\nabla G_2| \log |\nabla G_2| \, dH^{n-1} - \frac{c_6}{\lambda^{n-1}} \omega_1(B(w, t)).
$$

Actually Wolff proves this Lemma only in $\mathbb{R}^3$, but the proof for $\mathbb{R}^n, n \geq 3$, is essentially unchanged. To show the existence of $p \leq 0$ for which $\Lambda(p) < 0$, Wolff first shows that $\Lambda(q) < 0$ for $n = 3$ when $q(x') = -|x' + e_3|^{-1}, x' \in \mathbb{R}^3$, where

$$
eq \max_{\mathbb{R}^{n-1}} |p| + 1$$
\( e_3 = (0, 0, 1) \). In view of this function, the natural function to consider for \( n \geq 3 \) is

\[
q(x') = -|x' + e_n|^{2-n}, \quad e_n = (0, \ldots, 0, 1), \quad x' \in \mathbb{R}^{n-1},
\]

for which \( \tilde{q}(x) = -|x + e_n|^{2-n}, x \in (\mathbb{R}^n)^+ \). Then

\[
\Lambda(q) = (n-1)(n-2)^3 \alpha(n-1) \int_0^\infty (r^2 + 1)^{-3n/2}(1 - 3r^2) r^{n-2} dr
\]

\[
= -\frac{(n-1)(n-2)^3 \alpha(n-1) \Gamma(n-1/2) \Gamma(n/2 - 1/2)}{4 \Gamma(3n/2)} < 0,
\]

where \( \Gamma \) denotes the Euler gamma function and the integral was evaluated using the substitution \( r = \tan \theta \), as well as, the beta function. Let \( \Phi, 0 \leq \Phi < 1 \), be a \( C^\infty \) function on \( \mathbb{R}^{n-1} \) with support in \( \bar{B}(2), |\nabla \Phi| \leq 1000 \), and \( \Phi = 1 \) on \( \bar{B}(1) \). Now if

\[
q_m(x') = \Phi(m^{-1} x') q(x'), \quad x' \in \mathbb{R}^{n-1},
\]

then it follows easily from properties of conjugate harmonic functions (see [13, Ch. 6]) that

\[
\Lambda(q_m) \to \Lambda(q) \quad \text{as} \quad m \to \infty.
\]

Taking a suitable dilation of \( q_m \) for large \( m \), we get \( p \leq 0 \) in \( C^\infty(\mathbb{R}^{n-1}) \) with \( \text{supp} \ p \subseteq \bar{B}(1) \), and \( \Lambda(p) < 0 \).

We now define \( \psi \) and \((\lambda_k)_0^\infty \) introduced in Section 2. Let \( \psi, 0 \leq \psi \leq 1 \), be a fixed \( C^\infty(\mathbb{R}^{n-1}) \) function with support in \( \bar{B}(1), \max_{x \in \mathbb{R}^{n-1}} \psi = 1 \), and \( \Lambda(\psi) > 0 \). Recall that \( a_k = 2^{-k} a_0, k = 0, 1, \ldots \), and define \( \lambda_k \) as follows: let \( A = 200 \) in Lemma 2 and \( p = -\sigma_k \psi \). Let \( \lambda_k = \max \{ b_k^{-1}, \lambda_k^*, \lambda_k^* \}, k = 0, 1, \ldots \), where \( b_k = c_k(200, -\sigma_k^2 \psi), \lambda_k^* = \lambda^*(200, -\sigma_k^2 \psi) \). Put \( \lambda_m = \max_{0 \leq k < m} \lambda_k, m = 0, 1, \ldots \) and note that \( \lambda_k \) depends only on \( n \) since \( a_0 \) and \( \psi \) are fixed.

Let \( \Omega, \Omega', \epsilon, r, L, \) and \((E_k)_0^{L+1} \), be as in Section 2 and suppose also that \( \Omega \) is NTA with constant 100. Moreover, we assume \( B(0, \rho) \subseteq \Omega \subseteq B(0, 2) \), where \( \rho \) is as in (1.4). From our choice of \( r \) we see that \( \Omega \) is Lipschitz on scale \( r^{1/2} \) with constant 2. In order to apply Lemma 2, we need to add flat bumps under each \( y \in L \). For fixed \( y \in L \) let \( f \) be the smallest nonnegative integer for which (2.3) holds, i.e.

\[
B(y, 100 r^{1/2}) \cap E_j \neq \emptyset.
\]

Suppose that \( L = \{ z_1, z_2, \ldots, z_m \} \) and put \( L_k = \{ z_1, \ldots, z_k \}, 1 \leq k \leq m \). For fixed \( y \in L \) we assume that \( B(y, 1000 r^{1/2}) \cap \Omega, B(y, 1000 r^{1/2}) \cap \partial \Omega \), can be
expressed as in Section 2 relative to $\Theta$. Let
\[
\hat{\xi}(x') = -100 M_i r^2 \Phi \left( \frac{x'}{r} \right) + \left( 1 - \Phi \left( \frac{x'}{r} \right) \right) \vartheta(x') + y_n', \quad x' \in \mathbb{R}^{n-1},
\]
\[
\hat{\xi}(x') = \hat{\xi}(x') - a^2 r \lambda_j^{-1} \psi(\lambda_j x'/r), \quad x' \in \mathbb{R}^{n-1},
\]
where $\Phi$ was defined earlier in Section 3 and $M_i$ is as in (2.1). Define $\tilde{\Omega}_k$, $1 \leq k \leq m$, as follows:

(1) \hspace{1em} \tilde{\Omega}_k - \bigcup_{z \in L_k} B(z, 10r) = \Omega - \bigcup_{z \in L_k} B(z, 10r), \hspace{1em} \forall y \in L_k.

(\Pi) \hspace{1em} \partial \tilde{\Omega}_k \cap B(y, 10r) = \{(x' + y', \hat{\xi}(x')): x' \in \mathbb{R}^{n-1} \} \cap B(y, 10r),

(\Pi) \hspace{1em} \tilde{\Omega}_k \cap B(y, 10r) = \{(x' + y', x_n): x_n > \hat{\xi}(x') \} \cap B(y, 10r),

for each $y \in L_k$. From (2.1) and the definition of $\Omega'$ we see that $\tilde{\Omega}_m \supseteq \Omega, \tilde{\Omega}_m \supseteq \Omega'$. Using the fact that $\Omega$ is NTA with constant 100 and local smoothness of $\tilde{\Omega}_k$, it is easily checked that $\tilde{\Omega}_k, \tilde{\Omega}_k, 1 \leq k \leq m$, are NTA and Lipschitz on scale $r$ with constant 200. Let $\tilde{\Omega}_0 = \Omega, \tilde{\Omega}_0 = \tilde{\Omega}_m$. We first apply Lemma 2 with $t = r, \Omega_1 = \tilde{\Omega}_0, \Omega_2 = \tilde{\Omega}_1, \ldots$, after a possible rotation. We next apply Lemma 2 with $\Omega_1 = \tilde{\Omega}_1$ and $\Omega_2 = \tilde{\Omega}_2, \ldots$, etc. Let $\hat{G}_k, \hat{\tilde{G}}_k, \hat{\omega}_k, \hat{\omega}_k$, be the Green’s functions and harmonic measures relative to 0 for $\Omega_k, \tilde{\Omega}_k$. Applying the above argument $m$ times we obtain an inequality for $\hat{G}_m = \hat{\tilde{G}}_0$ and $\hat{\tilde{G}}_m$. Using the definition of $(\lambda_k)_m$, we conclude

(3.3) \hspace{1em} \int_{\partial \Omega_m} |\nabla \hat{G}_m| \log |\nabla \hat{G}_m| \, dH^{n-1}

\leq \int_{\partial \Omega_m} |\nabla \hat{G}_m| \log |\nabla \hat{G}_m| \, dH^{n-1} - c(\lambda_{k+1})^{-(n-1)} \sum_{k=0}^{m-1} \hat{\omega}_k(B(z_k+1, 2r)).

Next we define a function $\tau$ on $[0, 1]$ by $\tau(s) = \min \{\lambda_k: s_k \leq s\}, 0 < s \leq 1$. Choosing $r_0$ still smaller, if necessary, we assume, as we may, that for $0 < r \leq r_0$,

(3.4) \hspace{1em} r^{1/16} \leq \tau(e)^{-(n-1)}.

Note that $\tau(e) = \lambda_{k+1}$. 

To prove (1.3) we must show that $\hat{G}_m, \hat{G}_m$, in (3.3) can be replaced by $G, G'$, with an error term at most,

$$c\tau(e)^{-(n-1)} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_k, 2r)).$$

To do so we introduce $\Omega'_k, 0 \leq k \leq m$, defined by $\Omega'_0 = \Omega'$, and for $1 \leq k \leq m$,

(I') $\Omega'_k = \bigcup_{x \in L_k} B(z, 10r) = \Omega' - \bigcup_{x \in L_k} B(z, 10r),$

(II') $\partial \Omega'_k \cap B(y, 10r) = \{(x' + y', \xi(x')) : x' \in \mathbb{R}^{n-1} \} \cap B(y, 10r),$

(III') $\Omega'_k \cap B(y, 10r) = \{(x' + y', x_n) : x_n > \xi(x') \} \cap B(y, 10r),$

for each $y \in L_k$. Denote the corresponding Green’s functions and harmonic measures relative to 0, by $G'_k, \omega'_k$, $0 \leq k \leq m$. We shall also need the following facts about the NTA domain $\Omega_1$ with constant $A$ satisfying (3.2). If $z \in \partial \Omega_1$, then

(3.5) $c(A)^{-1} \omega_1(B(z, t)) \leq t^{n-2} \max_{B(z, t) \cap \Omega_1} G_1$

$$\leq c(A) t^{n-2} G_1(P_t)$$

$$\leq c(A) \omega_1(B(z, t)),$$

for $0 < t < A^{-1}$, where $P_t = P_t(z)$. Moreover,

(3.6) $\omega_1(B(z, 2t)) \leq c(A) \omega_1(B(z, t)).$

(3.6) is called the doubling inequality for harmonic measure. If $z \in \partial \Omega_1$ and $u, v$ are two positive harmonic functions in $\Omega_1$ which vanish continuously on $\partial \Omega_1 - B(z, t)$, and $P_t = P_t(z)$, then for $x \in \Omega_1 - B(z, 2t)$

(3.7) $c(A)^{-1} u(P_t) \leq v(P_t) \leq c(A) u(P_t) \leq c(A) v(x) \leq c(A) u(x),$

Moreover, (3.7) is valid when $u$ and $v$ vanish on $\partial \Omega_1 \cap B(z, 2t)$, and $x \in B(z, t) \cap \Omega_1$. (3.7) is called the rate inequality. Finally there exists $\mu = \mu(A) > 0$ so that for $z$ and $P_t$ as above, and $x \in B(z, t) \cap \Omega_1$,

(3.8) $G_1(x) \leq c(x - z)^{1/t^*} G_1(P_t).$

For the proof of (3.5)-(3.8) see [8, Sections 4 and 5].

From (3.5), (3.6), (3.8) with $t = A^{-1}$, and the fact that $\omega_1(B(z, A^{-1})) \geq c(A)^{-1}$, when $z \in \partial \Omega_1$, we see there exists $v(A)$, $0 < v < 1$, with

(3.9) $c(A)^{-1} t^{1/\nu} \leq \omega_1(B(z, t)) \leq c(A) t^{* - \nu - 2}, \quad 0 < t < A^{-1}.$
We claim that

\[ (3.10) \quad \sum_{k=0}^{m-1} \omega_k(B(z_k+1, 6r)) \leq c \sum_{k=0}^{m-1} \omega_k(B(z_k+1, 6r)), \]

whenever * and + are elements of \{\wedge, -, \vee\}. Indeed from our construction and the maximum principle for harmonic functions we have,

\[ \tilde{\omega}_0(B(z_k+1, 6r) - B(z_k+1, 2r)) \leq \omega_k(B(z_k+1, 6r) - B(z_k+1, 2r)) \leq \tilde{\omega}_m(B(z_k+1, 6r) - B(z_k+1, 2r)), \]

when \(0 \leq j \leq m, 0 \leq k \leq m - 1\), and * \(\in \{\wedge, -, \vee\}\). Summing and using the doubling inequality it follows that

\[ c^{-1} \sum_{k=0}^{m-1} \tilde{\omega}_0(B(z_k+1, 6r)) \leq \sum_{k=0}^{m-1} \omega_k(B(z_k+1, 6r)) \leq c \sum_{k=0}^{m-1} \tilde{\omega}_m(B(z_k+1, 6r)). \]

On the other hand, from the maximum principle we deduce

\[ \sum_{k=0}^{m-1} \tilde{\omega}_m(B(z_k+1, 6r)) \leq \sum_{k=0}^{m-1} \tilde{\omega}_0(B(z_k+1, 6r)). \]

Hence our claim is true. We shall show for \(0 \leq k \leq m - 1\) that

\[ (3.11) \quad \int_{\partial B_{k+1}} |\nabla G_k| \log |\nabla G_k| \, dH^{n-1} \]

\[ \leq \int_{\partial B_{k+1}} |\nabla G_{k+1}| \log |\nabla G_{k+1}| \, dH^{n-1} + cr^{1/2} \tilde{\omega}_k(B(z_k+1, 6r)), \]

\[ (3.12) \quad \int_{\partial B_{k+1}} |\nabla \tilde{G}_{k+1}| \log |\nabla \tilde{G}_{k+1}| \, dH^{n-1} \]

\[ \leq \int_{\partial B_k} |\nabla \tilde{G}_k| \log |\nabla \tilde{G}_k| \, dH^{n-1} + cr^{1/2} \tilde{\omega}_k(B(z_k+1, 6r)). \]

Summing (3.11) and using (3.10), it then follows that

\[ (3.13) \quad \int_{\partial B'} |\nabla G'| \log |\nabla G'| \, dH^{n-1} \]

\[ \leq \int_{\partial B_m} |\nabla \tilde{G}_m| \log |\nabla \tilde{G}_m| \, dH^{n-1} + cr^{1/2} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_k+1, 6r)), \]

where we have used the fact that \(\Omega_0 = \Omega, \Omega'_m = \Omega_m\).
Summing (3.12) and using (3.10), we find

\[
\int_{\partial_0 m} |\nabla \tilde{G}_m| \log |\nabla \tilde{G}_m| \, dH^{n-1} \leq \int_{\partial_0} |\nabla G| \log |\nabla G| \, dH^{n-1} + cr^{1/2} \sum_{k=0}^{m-1} \tilde{\omega}_k(B(z_{k+1}, 6r)),
\]

since $\tilde{\omega}_0 = \Omega$. Putting (3.13), (3.14), into (3.3) and using (3.6) we get (1.3) provided $r_0$ is small enough, thanks to (3.4). Thus (1.3) is true once we prove (3.11)-(3.12).

We prove only (3.11), (3.12), for $k = 0$, since the proof of all the other inequalities is the same. To prove (3.12) for $k = 0$ we first observe from (3.5) that

\[
\max_{B(z_1, 6r) \cap \tilde{\Omega}_1} G_1 \leq cr^{2-n} \tilde{\omega}_1(B(z_1, 6r)).
\]

Using (3.15), (2.1), and applying Lemma 1 with $k = 4$ after scaling $B(x_1, 6r) \cap \tilde{\Omega}_1$, we find for $x, y$ in the closure of $B(z_1, 3r) \cap \tilde{\Omega}_1$,

\[
|\nabla G_1(x) - \nabla G_1(y)| \leq c|x - y| \rho^{-n} \tilde{\omega}_1(B(z_1, 6r)),
\]

while from (3.15), a barrier argument, (3.5)-(3.6) and (ii), we have

\[
c^{-1} r^{1-n} \tilde{\omega}_1(B(z_1, 6r)) \leq |\nabla G_1(x)| \leq cr^{1-n} \tilde{\omega}_1(B(z_1, 6r)).
\]

Clearly (3.17) and (3.9) imply

\[
|\log |\nabla G_1(x)|| \leq -c \log r,
\]

when $x$ is in the closure of $B(z_1, 3r) \cap \tilde{\Omega}_1$. Using (3.16)-(3.18), (3.6), (2.1), and parametrizing $\partial \Omega$ and $\partial \tilde{\Omega}_1$ in terms of $\theta$ and $\tilde{\theta}$, for $y = z_1$, we obtain with $z_1 = (y_1, y_2)$, $\tilde{x} = (x' + y', \tilde{x}(x'))$, $x = (x' + y', \theta(x') + y_2)$,

\[
\int_{\partial \Omega \cap B(x_1, 3r)} |\nabla G_1| \log |\nabla G_1| \, dH^{n-1} \leq \int_{\partial \tilde{\Omega}_1 \cap B(x_1, 3r)} |\nabla G_1| \log |\nabla G_1| \, dH^{n-1}
\]

\[
\leq \int_{B(3\rho)} |\nabla G_1| \log |\nabla G_1| |x| \sqrt{1 + |\nabla \tilde{\theta}(x')|^2} - \sqrt{1 + |\nabla \tilde{x}(x')|^2} \, dx'
\]

\[
+ \int_{B(3\rho)} |\nabla G_1| |x - \nabla G_1(\tilde{x})| \log |\nabla G_1(x)| \sqrt{1 + |\nabla \tilde{x}(x')|^2} \, dx'.
\]
\[
\begin{align*}
&\quad + \int_{\partial(\mathcal{G})} |\nabla \hat{G}_1(\mathcal{G})| \log |\nabla \hat{G}_1(x)| - \log |\nabla \hat{G}_1(\mathcal{G})| \sqrt{1 + |\nabla \hat{\xi}(x')|^2} \, dx' \\
&\quad \leq (-cM_1^2 r^2 \log r - cM_1 r \log r + \log (1 + M_1 r)) \hat{\omega}_1(B(z_1, 6r)) \\
&\quad \leq cr^{1/2} \hat{\omega}_1(B(z_1, 6r)).
\end{align*}
\]

Next from (3.17), (2.1) and the fact that each point of \(B(z_1, 6r) \cap \partial \mathcal{G}_1\) lies within \(200 M_1 r^2\) of a point of \(B(z_1, 6r) \cap \partial \mathcal{G}_1\), we get

(3.20) \((\hat{G}_1 - G)(x) \leq cM_1 r^{1-\frac{n}{2}} \hat{\omega}_1(B(z_1, 6r))\)

for \(x \in \partial \mathcal{G}_1\). From the maximum principle for harmonic functions and the fact that \(\Omega \subseteq \mathcal{G}_1\), we conclude this inequality holds in \(\Omega\). Let \(\phi(x') = \theta(6r x')/6r\), and define \(H\) relative to \(\phi\) as in Lemma 1. Put

\[
\begin{align*}
&\quad u(x) = \frac{1}{6r} (\hat{G}_1(6r x + z_1) - G(6r x + z_1)), \quad x \in \bar{H}, \\
&\quad \phi_1(x') = \frac{1}{6r} (\hat{\xi}(6r x') - y_n), \\
&\quad H_1 = \{x: |x'| < 8, \phi_1(x') < x_n < 2\}, \\
&\quad u_1(x) = \frac{1}{6r} \hat{G}_1(6r x + z_1), \quad x \in \bar{H}_1.
\end{align*}
\]

We note from (2.1) that

(3.21) \(\max \{ \| \phi \|_4, \| \phi_1 \|_4 \} \leq cM_1 r\).

Using (3.20), (3.21), we first apply Lemma 1 with \(u, H\), replaced by \(u_1, H_1\). As in (3.16) we get

(3.22) \(\sum_{0 \leq |\alpha| \leq 4} |\partial_\alpha u_1(x)| \leq cr^{1-\frac{n}{2}} \hat{\omega}_1(B(z_1, 6r)), \quad x \in \bar{H}.
\]

We note that \(u_1 = 0\) on \(\partial H_1 \cap \{(x', \phi_1(x'))\}\) and \(u = u_1 = \gamma\) on \(\partial H \cap \{(x', \phi(x'))\}\). Using these notes and (3.21)-(3.22) we deduce

(3.23) \(\sum_{|\alpha| = 0}^3 |\partial_\alpha \gamma(x', \phi(x'))| = \sum_{|\alpha| = 0}^3 |\partial_\alpha (u_1(x', \phi(x')) - u_1(x', \phi_1(x')))|
\]

\[\leq cM_1 r^{2-\frac{n}{2}} \hat{\omega}_1(B(z_1, 6r)).\]

Applying Lemma 1 to \(u\) and \(H\), with \(k = 3\) we find from (3.20)-(3.23)

\[\sum_{|\alpha| = 0}^3 |\partial_\alpha u(x)| \leq cM_1 r^{2-\frac{n}{2}} \hat{\omega}_1(B(z_1, 6r)).\]
for \( x \in B(0, 1/2) \cap H \). Hence if \( x \in B(z_1, 3r) \cap \Omega \), then

\[
(3.24) \quad |\nabla \hat{G}_1 - \nabla G(x)| \leq cM_1 r^{-n} \hat{\omega}_1(B(z_1, 6r)) \leq c|\nabla \hat{G}_1(x)| M_1 r,
\]

where the last inequality is just (3.17). From (3.24) and (2.1) we obtain

\[
(3.25) \quad \left| \int_{\partial \Omega \cap B(z_1, 3r)} |\nabla G| \log |\nabla G| dH^n - \int_{\partial \Omega \cap B(z_1, 3r)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| dH^n \right|
\]

\[
\leq \left| \int_{\partial \Omega \cap B(z_1, 3r)} \left| \nabla G - |\nabla \hat{G}_1| \right| \log |\nabla G| dH^n \right| + \int_{\partial \Omega \cap B(z_1, 3r)} |\nabla \hat{G}_1| \log \left( \frac{|\nabla G|}{|\nabla \hat{G}_1|} \right) dH^n
\]

\[
\leq -cM_1 r \log r \hat{\omega}_1(B(z_1, 6r)) \hat{\omega}_1(B(z_1, 6r)) \log (1 + cM_1 r)
\]

\[
\leq cr^{1/2} \hat{\omega}_1(B(z_1, 6r)).
\]

Let \( P = P_{3r}(z_1) \) and let \( G(\cdot, Y) \) denote Green’s function with pole at \( Y \in \Omega \). Following Wolff (see [14, (2.7)]) we first note from (3.20) and the rate inequality (3.7) with \( u = \hat{G}_1 - G \), \( v = G(\cdot, P) \), \( t = 2r \), that

\[
G(x, P)^{-1}(\hat{G}_1 - G)(x) \leq cM_1 r \hat{\omega}_1(B(z_1, 6r)), \quad x \in \Omega - B(z_1, 3r).
\]

Second, given \( w \in \partial \Omega - B(z_1, 3r) \), we apply the rate inequality with \( u = G(\cdot, P) \), \( v = G(\cdot, P_t(w)) \), \( t = 2|w - z_1| \) in \( \Omega - B(z_1, t) \), provided \( 0 \in \Omega - B(z_1, 2t) \). We get for \( x = 0 \),

\[
t^n G(P_t(w), P) \leq c G(0, P)/G(0, P_t(w)).
\]

If \( 0 \in B(z_1, 2t) \), then it follows easily from Harnack’s inequality and \( t \geq \rho/2 \) (since \( B(0, \rho) \subseteq \Omega \)) that

\[
G(P_t(w), P) \leq cr^{2 - n} G(0, P).
\]

From the above inequalities, (3.8) and Harnack’s inequality, we find for \( P_t = P_{t}(w) \),

\[
G(P_t, P) \leq cr^{2 - n}(r/t)^{n}.
\]

Third, we use the rate inequality in \( B(w, 10^{-1}t) \cap \Omega \) with \( u = G(\cdot, P) \), \( v = \hat{G}_1(\cdot, 0) \); the above inequalities, (3.5) and (3.6), to obtain

\[
r^{-1}(\hat{\omega}_1(B(z_1, 6r)))^{-1}M_1^{-1}(\hat{G}_1 - G)(x)\hat{G}_1(x, 0)^{-1} \leq c G(x, P)\hat{G}_1(x, 0)^{-1}
\]

\[
\leq c(r/t)^{n}(\hat{\omega}_1(B(z_1, t)))^{-1},
\]
for $x \in B(w, 10^{-3}t) \cap \Omega$. Letting $x \to w$ and using (2.1) we conclude from this inequality that

\begin{equation}
(3.26) \quad \langle |\nabla \hat{G}_1|^{-1} |\nabla \hat{G}_1 - \nabla G| \rangle (w) \leq cr^{3/4 + \kappa} \hat{\omega}_1 (B(z_1, 6r))(\hat{\omega}_1 (B(z_1, t)))^{-1} |z_1 - w|^{-\eta}.
\end{equation}

Now

\begin{equation}
(3.27) \quad \left| \int_{\partial \Omega - B(z_1, 3\rho)} |\nabla G| \log |\nabla G| dH^{n-1} - \int_{\partial \Omega - B(z_1, 3\rho)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| dH^{n-1} \right|
\leq \left| \int_{\partial \Omega - B(z_1, 3\rho)} |\nabla G| - |\nabla \hat{G}_1| \log |\nabla G| dH^{n-1}
+ \int_{\partial \Omega - B(z_1, 3\rho)} |\nabla \hat{G}_1| \log (|\nabla G|/|\nabla \hat{G}_1|) dH^{n-1} \right|
= I_1 + I_2.
\end{equation}

If $F_k = B(z_1, 3^{k+1}r) - B(z_1, 3^k r)$, $k = 1, 2, \ldots$ then from (3.26) we have

\begin{align*}
I_1 & \leq \sum_{k=1}^{\infty} \int_{F_k \cap \partial \Omega} \|\nabla G\| - \|\nabla \hat{G}_1\| \log |\nabla G| dH^{n-1} \\
& \leq -cr^{3/4 + \kappa} \log r \hat{\omega}_1 (B(z_1, 6r)) \left( \sum_{k=1}^{\infty} k3^{-k\eta} \right) r^{-\eta} \\
& \leq cr^{1/2} \hat{\omega}_1 (B(z_1, 6r)).
\end{align*}

A similar estimate holds for $I_2$. Using these estimates in (3.27) we get

\begin{equation}
(3.28) \quad \left| \int_{\partial \Omega - B(z_1, 3\rho)} |\nabla G| \log |\nabla G| dH^{n-1} - \int_{\partial \Omega - B(z_1, 3\rho)} |\nabla \hat{G}_1| \log |\nabla \hat{G}_1| dH^{n-1} \right|
\leq cr^{1/2} \hat{\omega}_1 (B(z_1, 6r)).
\end{equation}

Next, since

\begin{equation}
\hat{\omega}_1 (B(z_1, 6r)) \leq \hat{\omega}_0 (B(z_1, 6r)),
\end{equation}

we can replace $\hat{\omega}_1$ by $\hat{\omega}_0$ in (3.28), (3.25), and (3.19). Doing this and combining (3.28), (3.25), (3.19), we conclude that (3.12) is true for $k = 0$.

To prove (3.11) for $k = 0$, let $j$ be the smallest positive integer such that $E_j \cap B(z_1, 10r) \neq \emptyset$. Put $r' = 10r/\lambda_j$ and let $z \in B(z_1, 6r) \cap \partial \Omega$. Then it is easily checked that (3.16)-(3.18) hold with $\hat{G}_{1j}, \hat{\omega}_1, r, z_1$, replaced by $G'_{1j}, \omega_1, r', z$, respectively, when $x, y \in B(z, 3r')$. Now from (3.4) we have

\begin{equation}
(3.29) \quad \frac{r}{10} \geq \frac{r}{\lambda_j} \geq \frac{r}{\lambda_{j+1}} = \frac{r}{\tau(e)} \geq r^\gamma
\end{equation}
where
\[ \gamma = 1 + \frac{1}{16(n-1)} \leq \frac{33}{32}. \]

Let \( z^* \) be the point in \( \partial \Omega' \) obtained by projecting \( z \) in the rotated \( x_n \) direction onto \( \partial \Omega' \). Then from the new version of (3.16)-(3.18), and the fact that
\[ |z - z^*| < 200M_1r^2 < r', \]
thanks to (2.1), (2.29) we find
\[ |(\nabla G_i) \log |\nabla G'_i|)(z) - (\nabla G'_i)(z^*)| \leq \bigg| |\nabla G'_i(z) - |\nabla G'_i(z^*)| \log r' + |\nabla G'_i(z)| \log |\nabla G'_i(z)|/|\nabla G'_i(z^*)| \bigg| \leq -cM_1r^2 \log r'(|\nabla G'_i(z)|/r'). \]
Using this inequality, (2.29), and parametrizing \( \partial \Omega' \), \( \partial \Omega'_1 \), we get as in (3.19)
\begin{equation}
(3.30)
\left| \int_{\partial \Omega \cap B(z_1, 3r)} |\nabla G'_i| \log |\nabla G'_i| \, dH_{n-1} - \int_{\partial \Omega'_1 \cap B(z_1, 3r)} |\nabla G'_i| \log |\nabla G'_i| \, dH_{n-1} \right| \leq cr^{1/2} \omega'_1(B(z_1, 6r)).
\end{equation}
Next suppose \( z \in \partial \Omega' \) and observe as in (2.30) that
\begin{equation}
(3.31) \quad (G'_i - G)(z) \leq cM_1r^2(r')^{1-n} \omega'_1(B(z_1, 6r')) \leq cM_1r^2(r')^{1-n} \omega'_1(B(z_1, 6r)).
\end{equation}
It follows from the maximum principle for harmonic functions that (3.31) holds in \( \Omega' \). If \( z = (\xi + y', \xi(z)) \in \partial \Omega', \) put
\[ \phi'(x') = \frac{1}{6r'} (\xi(6r'x' + \bar{z}) - \xi(z)), \]
\[ H' = \{ x : |x'| < 1, \phi'(x') < x_n < 1 \}, \]
\[ u'(x) = \frac{1}{6r'} (G'_i(6r'x + z) - G(6r'x + z)), \quad x \in H', \]
\[ \phi'_1(x') = \frac{1}{6r'} (\xi(6r'x' + \bar{z}) - \xi(z)), \]
\[ H'_1 = \{ x : |x'| < 8, \phi'_1(x') < x_n < 2 \}, \]
\[ u'_1 = \frac{1}{6r'} G'_i(6r'x + z), \quad x \in H'_1. \]
We note that
\[
|\phi'|_{4} + |\phi|_{4} \leq c,
\]
\[
|\phi' - \phi|_{4} \leq cM_{1}r.
\]
Using these inequalities in place of (3.21) and Lemma 1 we get
\[
\sum_{0 \leq |\alpha| \leq 4} |\partial_{\alpha}u'(x)| \leq c(r')^{1-n}\omega'(B(z, 6r')) \leq c(r')^{1-n}\omega'(B(z_{1}, 6r))
\]
in \(H'.\) Also, as in (3.23), we see for \(\gamma = \gamma'\) on \(\partial H' \cap \{ (x', \phi'(x')) \}, \) that
\[
\sum_{|\alpha| = 0}^{3} |\partial_{\alpha}^{'\gamma'}(x')| \leq cM_{1}r(r')^{1-n}\omega'(B(z_{1}, 6r)).
\]
From this inequality, (3.31) and Lemma 1 it follows as in (3.24) that
\[
|\nabla G'_{1} - \nabla G'|_{(x)} \leq cM_{1}r^{2}(r')^{1-n}\omega'(B(z_{1}, 6r))
\]
\[
\leq cM_{1}(r^{2}/r')^{\omega'(B(z_{1}, 6r))(\omega'(B(z_{1}, 6r)))^{-1}}|\nabla G'(x)|,
\]
\(x \in B(z, 3r') \cap \hat{U}' .\) We cover \(\partial U' \cap B(z_{1}, 3r)\) by at most \(c(r/r')^{n-1}\) balls, \(B(z, 3r')\), \(z \in \partial U' \cap B(z_{1}, 3r).\) Using (3.32) in each ball and arguing as in (3.25) we have
\[
\left| \int_{\partial U' \cap B(z_{1}, 3r)} |\nabla G'| \log |\nabla G'| dH^{n-1} - \int_{\partial U' \cap B(z_{1}, 3r)} |\nabla G'| \log |\nabla G'| dH^{n-1} \right|
\]
\[
\leq -cM_{1}r(r/r')^{n} \log \omega'(B(z_{1}, 6r))
\]
\[
\leq cr^{1/2}\omega'(B(z_{1}, 6r)),
\]
thanks to (3.29) and (2.1).

At this point we can use (3.31) in place of (3.20) and repeat the argument following (3.25) in the proof of (3.12) (for \(k = 0\), since only NTA estimates were used. From (3.28) with \(G, \hat{G}_{1}, \tilde{\omega}_{1}\), replaced by \(G', \hat{G}'_{1}, \omega'_{0}\) and (3.30), (3.33), with \(\omega'_{1}\) replaced by \(\omega'_{0}\), we conclude that (3.11) holds when \(k = 0.\) From our earlier remarks we now deduce that (1.3) is true.

4. Proof of Theorem 1

Recall that \(\psi, 0 \leq \psi \leq 1,\) is a fixed \(C^{\infty}\) function with support in \(B(1),\)
\(\max_{x \in \mathbb{R}^{n-1}} \psi = 1,\) and \(\Lambda(\psi) > 0.\) Also \(\sigma_{0}, 0 < \sigma_{0} \leq 10^{-3},\) was chosen to be the largest number for which (2.10), (2.12), and (2.19) are true. Finally, given \(\epsilon, 0 < \epsilon \leq \sigma_{0},\) we note that \(r_{0} = r_{0}(\epsilon, M_{1}, M_{2}),\) was chosen so small that the inequalities in Sections 2 and 3 are true for \(0 < r \leq r_{0}.\)
We elaborate on the induction argument for the construction of $D$ which was outlined in Section 1. Let $D_0 = B(0, \rho)$, where $\rho$ satisfies (1.4). Put $\epsilon_0 = \alpha_0$ and $\epsilon_k = 2^{-k} \epsilon_0$, $k = 0, 1, 2, \ldots$. Choose a covering, $L_i = \{ B(\omega_i, t_0) \}$, $1 \leq i \leq k_0$ of $\partial D_0$ such that $t_0 \leq 1/2$, $i = 1, 2, \ldots, k_0$, and

$$\alpha(n - 1) \sum_{i=1}^{k_0} t_0^{n-1} \leq H^{n-1}(\partial D_0) - \frac{1}{2}. $$

By compactness of $D_0$ we may assume $k_0 < \infty$. Let $2r'_1 > 0$ denote the distance from $\partial D_0$ to $\mathbb{R}^n - \cup_i^N B(\omega_i, t_0)$. We set $\Omega = D_0$, $\epsilon = \epsilon_1$, and apply the results in Section 2 with $r = r_1$, where $r_1$ is the smaller of $10^{-9} \rho, r'_1$, and $r_0 = r'_0(\epsilon_1, M_1, M_2)$. Here $M_1, M_2$ are defined relative to $D_0, G_0$. Let $D_1 = \Omega'$ be the domain obtained by adding smooth bumps to $D_0$ and $h_1 = h$ the homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, which satisfies (2.16)-(2.18) with $r = r_1$. Moreover, $h_1(\partial D_0) = \partial D_1$. By induction suppose for some $m \geq 1$ we have a covering of $\partial D_m$ such that $t_mi \leq 2^{-(m+1)}$, $1 \leq i \leq k_m$, and

$$\alpha(n - 1) \sum_{i=1}^{k_m} t_i^{n-1} \leq H^{n-1}(\partial D_m) - 2^{-m+1} $$

Let $2r_{m+1} > 0$ be the distance from $\partial D_m$ to $\mathbb{R}^n - \cup_i^N B(\omega_i, t_{mi})$. Let $\Omega = D_m$, $\epsilon = \epsilon_m$, and $r = r_{m+1}$, where $r_{m+1}$ is the smaller of $10^{-4m} \rho, r'_{m+1}$, and $r_0(\epsilon_{m+1}, M_1, M_2)$. Here $M_1, M_2$ are defined relative to $D_m, G_m$. Adding smooth bumps to $\Omega$ as in Section 2 we obtain $D_{m+1} = \Omega' \supset D_m$ and $h_{m+1} = h$ a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ which satisfies (2.16)-(2.18) with $r = r_{m+1}$. Moreover, $h_{m+1}(\partial D_m) = \partial D_{m+1}$. By induction we get, $(D_k)_{0}^\alpha, (H_k)_{0}^\alpha, (r_k)_{1}^\alpha, (h_k)_{1}^\alpha$. From our work in Section 2 we see that (1.1), (1.2), are true with $\Omega, \Omega', G, G'$, replaced by $D_k, D_{k+1}, G_k, G_{k+1}$, respectively, $k = 0, 1, \ldots$.

We claim that $D_k, k = 1, 2, \ldots$ is NTA with constant 100. Indeed, since $0 \leq \psi \leq 1$ and $r_k \leq 10^{-4k} \rho$, $k = 1, 2, \ldots$, it follows from the definition of $D_k$, by way of the triangle inequality, that

$$B(0, \rho) \subseteq D_k \subseteq B(0, 2\rho), \quad k = 1, 2, \ldots$$

To prove $D_k$ satisfies the corkscrew condition (i) in the definition of an NTA domain, we proceed by induction. If $0 < s < \rho$, and $z \in \partial D_0$, note that $B(z, s) \cap D_0, B(z, s) \cap (\mathbb{R}^n - D_0)$, each contain a ball of radius $s/4$. From this note and the fact that $\partial D_1$ lies within $r_1$ distance of $\partial D_0$, we deduce for $4r_1^{1/2} \leq s < \rho$, and $z \in \partial D_1$ that $B(z, s) \cap D_0, B(z, s) \cap (\mathbb{R}^n - D_0)$, each contain a ball of radius,

$$(1 - r_1) \frac{s}{4} - r_1 \geq \frac{1}{4} s(1 - 2r_1^{1/2}) = s_1.$$
If $0 < s \leq 4r_1^{1/2}$, then from our choice of $r_1 = r$, we have $z \in B(y, 100r_1)$, for some $y \in \{y^j\}_j$. Moreover, $B(y, 1000r_1^{1/2}) \cap D_1$, $B(y, 1000r_1^{1/2}) \cap \partial D_1$, can be expressed as in Section 2 relative to $\xi$. From (2.12) and (2.1) we observe that $|\nabla \xi| \leq 10^{-1}$. Using these facts and a little geometry it is easily seen that the above inequality remains valid when $0 < s \leq 4r_1^{1/2}$. By induction, suppose we have shown for some $m \geq 1$, that if $z \in \partial D_m$ and $0 < s < \rho$, then $B(z, s) \cap D_m$, $B(z, s) \cap (\mathbb{R}^n - D_m)$, each contain a ball of radius

$$
\frac{1}{4} s \left( 1 - 2 \sum_{k=1}^{m} r_k^{1/2} \right) = s_m.
$$

If $4r_1^{1/2} \leq s < \rho$, and $z \in \partial D_{m+1}$, then since $\partial D_{m+1}$ lies within $r_{m+1}$ of $\partial D_m$, we deduce from (4.3) that $B(z, s) \cap D_{m+1}$, $B(z, s) \cap (\mathbb{R}^n - D_{m+1})$, each contain a ball of radius

$$
\frac{1}{4} (s - r_{m+1}) \left( 1 - \sum_{k=1}^{m} r_k^{1/2} \right) - r_{m+1} \geq \frac{1}{4} s \left( 1 - 2 \sum_{k=1}^{m+1} r_k^{1/2} \right) = s_{m+1}.
$$

If $0 < s < 4r_1^{1/2}$, it follows from local smoothness of $D_{m+1}$ that $B(z, s) \cap D_{m+1}$, $B(z, s) \cap (\mathbb{R}^n - D_{m+1})$, each contain a ball of radius $s_{m+1}$. Thus, by induction we have shown for $z \in \partial D_k$, $k = 0, 1, \ldots$, that $B(z, s) \cap D_k$, $B(z, s) \cap (\mathbb{R}^n - D_k)$, both contain a ball of radius

$$
s_k \geq \frac{1}{4} s \left( 1 - 2 \sum_{m=1}^{\infty} r_m^{1/2} \right) \geq \frac{1}{8} s,
$$

when $0 < s < \rho$. Scaling $D_k$ to have diameter 1, we see that (i) in Section 3 holds with $A = 16$.

To prove (ii), we proceed similarly. Suppose by induction, we have shown for some nonnegative integer $m$ that whenever $x, z \in D_m$, we can join $x$ to $z$ by a curve $\gamma$ with parameter interval, $[0, 1]$, in such a way that $\gamma(0) = x$, $\gamma(1) = z$, and

$$
(4.4) \quad (a) \quad \text{dist}(\gamma(t), \partial D_m) \geq \frac{1}{16} \left( 1 - 2 \sum_{k=1}^{m} r_k^{1/2} \right) \min \{|\gamma(t) - x|, |\gamma(t) - z|\},
$$

$$
(4.5) \quad (b) \quad \text{length} \gamma \leq 3 \left( 1 + \sum_{k=1}^{m} r_k^{1/2} \right)|x - z|.
$$

In case $m = 0$, replace the sums in (4.4), (4.5) by 0. From inspection we see that (4.4), (4.5) hold when $m = 0$, since $D_0 = B(0, \rho)$. Next suppose $x, z \in D_{m+1}$ and $4r_1^{1/2} \leq |x - z|$. Since $D_m \subseteq D_{m+1}$, we note that (4.4) and (4.5) hold trivially unless either $x \notin D_m$ or $z \notin D_m$. If $x \notin D_m$, then $x \in B(y, r_{m+1}) \cap D_{m+1}$
for some $y \in \{ y_j \}_{j=1}^N$, $y \in \partial D_m$, and $x = (x', x_n)$ in the corresponding rotated coordinate system. Put $x^* = (x', x_n + r_{m+1})$ and observe that $x^* \in D_m$. If $x \in D_m$, we also let $x^* = x$. Applying the same argument to $z$ we get $x^*, z^* \in D_m$. Let $\gamma^*$ be the curve joining $x^*$ to $z^*$ which satisfies (4.4), (4.5). If $x \neq x^*$, we modify $\gamma^*$ as follows. Let $t_0, 0 < t_0 < 1$, be the largest $t$ with $\gamma^*(t) \in \overline{B}(y, r_{m+1}^{3/4})$. If $\gamma^*(t_0) = w = (w', w_n)$, we join $x$, $w$, to $\overline{x} = (x', y_n + r_{m+1}^{3/4})$, $\overline{w} = (w', y_n + r_{m+1}^{3/4})$, respectively by line segments, $l_1, l_2$. We then join $\overline{x}$ to $\overline{w}$ by a line segment $l_3$. Let $l_1 + l_2 + l_3$ denote the resulting curve from $x$ to $w$ with parameter interval $[0, t_0]$. If $z \notin D_m$, we see there exists $\hat{y} \in \{ y_i \}_{i=1}^N$ and largest $t_1, 0 < t_0 < t_1 < 1$, such that $z \in B(\hat{y}, r_{m+1})$, and

$$\{ \gamma^*(t): 0 \leq t \leq t_1 \} \cap \overline{B}(\hat{y}, r_{m+1}^{3/4}) = \emptyset.$$ 

As above, we get line segments $l_1', l_2', l_3'$, with $\overline{l}_1 + \overline{l}_2 + \overline{l}_3$ joining $\gamma^*(t_1)$ to $z$. Moreover, $\overline{l}_1 + \overline{l}_2 + \overline{l}_3$ has parameter interval $[t_1, 1]$. Let $\hat{\gamma} = \gamma^*$ on $[t_0, t_1]$ and if $x \notin D_m$, then $\hat{\gamma} = l_1 + l_2 + l_3$ on $[0, t_0]$. Otherwise, $\hat{\gamma} = \gamma^*$ on $[0, t_0]$. If $z \notin D_m$, then $\hat{\gamma} = \overline{l}_1 + \overline{l}_2 + \overline{l}_3$ on $[t_1, 1]$, while if $z \in D_m$, then $\hat{\gamma} = \gamma^*$ on $[t_1, 1]$. From (4.5) we deduce

\begin{equation}
\text{length } \hat{\gamma} \leq \text{length } \gamma^* + 10r_{m+1}^{3/4}
\end{equation}

\begin{align*}
\leq 3 \left(1 + \sum_{k=1}^m r_k^{1/4}\right) |x^* - z^*| + 10r_{m+1}^{3/4} \\
\leq 3 \left(1 + \sum_{k=1}^m r_k^{1/4}\right) |x - z| + 12r_{m+1}^{3/4} \\
\leq 3 \left(1 + \sum_{k=1}^{m+1} r_k^{1/4}\right) |x - z|.
\end{align*}

Moreover, from local smoothness of $\partial D_{m+1}$ it is easily checked for $t \in [0, t_0] \cup [t_1, 1]$, that

$$\text{dist } (\hat{\gamma}(t), \partial D_{m+1}) \geq \frac{1}{16} \left(1 - 2 \sum_{k=1}^{m+1} r_k^{1/4}\right) \min \{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\}.$$ 

If $t \in [t_0, t_1]$, then by construction

$$\min \{|\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z|\} \geq r_{m+1}^{3/4} - r_{m+1}$$

$$\geq \frac{1}{2} r_{m+1}^{3/4}.$$ 

Using this inequality, (4.4), and the fact that $\gamma^* = \hat{\gamma}$ on $[t_0, t_1]$ we get for $t \in [t_0, t_1]$,
(4.7) \[ \text{dist} (\hat{\gamma}(t), \partial D_{m+1}) \geq \frac{1}{16} \left( 1 - 2 \sum_{k=1}^{m} r_k^{1/4} \right) \min \{ |\hat{\gamma}(t) - x^*|, |\hat{\gamma}(t) - z^*| \} \]
\[ \geq \frac{1}{16} \left( 1 - 2 \sum_{k=1}^{m} r_k^{1/4} \right) \min \{ |\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z| \} - \frac{r_{m+1}}{16} \]
\[ \geq \frac{1}{16} \left( 1 - 2 \sum_{k=1}^{m+1} r_k^{1/4} \right) \min \{ |\hat{\gamma}(t) - x|, |\hat{\gamma}(t) - z| \}. \]

If \(|x - z| < 4r_{m+1}^{1/2}\), then from local smoothness of \(\partial D_{m+1}\), we see there exists \(\hat{\gamma}\) for which (4.6) and (4.7) hold. Thus by induction, we obtain (4.4), (4.5), for \(m = 0, 1, 2, \ldots\) Since \(\sum_{k} r_k^{1/4} \leq 1/10\), we conclude that \(D_m, m = 0, 1, \ldots\), is NTA with constant 100. From this fact, (4.2), and our work in Section 3 we now find that (1.3) holds with \(\Omega = D_k, \Omega' = D_{k+1}, k = 0, 1, \ldots\).

Next let \(h_0(x) = px\), and \(f_k = h_k \circ h_{k-1} \circ \cdots \circ h_0\). Then \(f_k\) is a homeomorphism from \(\mathbb{R}^n\) to \(\mathbb{R}^n\) with \(f_k(S) = \partial D_k\). From (2.16), (2.19), and iteration, we find

\[ 2^{-k} \rho|x - z| \leq \rho(1 - c_4 \sigma_0^k)|x - z| \]
\[ \leq |f_k(x) - f_k(z)| \]
\[ \leq \rho(1 + c_4 \sigma_0^k)|x - z| \]
\[ \leq \rho 2^k|x - z|, \]

for \(x, z \in \mathbb{R}^n\). If \(r_j < |x - z|\) for some \(j \geq 1\), then from (4.8) and the fact that \(r_{k+1} \leq 10^{-4k}r_k \rho\), we deduce for \(l \geq j\),
\[ r_{l+1} < 2^{-l} \rho|x - z| \leq |f_l(x) - f_l(z)|. \]

From this inequality, (2.17), (2.19) and iteration we find for \(k > j\),
\[ |f_k(x) - f_k(z)| \leq \frac{1}{2} \sum_{m=j+1}^{k} r_m \leq |f_j(x) - f_j(z)| \leq |f_k(x) - f_k(z)| + \frac{1}{2} \sum_{m=j+1}^{k} r_m. \]

Using the above inequality, (4.8) with \(j = k\), and the fact that
\[ \sum_{m=j+1}^{\infty} r_m \leq \rho 10^{-j} r_j \leq \rho 10^{-j}|x - z|, \]
we get

\[ 2^{-(j+1)} \rho|x - z| \leq |f_k(x) - f_k(z)| \leq \rho 2^{j+1}|x - z|. \]

Given \(\beta \in (0, 1)\), we have
\[ 2^{j+1} \leq c(\beta)|x - z|^\beta - 1, \]
when \( r_j \leq |x - z| \leq r_{j-1} \), \( j = 2, 3, \ldots \) for some \( c(\beta) \), independent of \( j \). Here we have used, \( r_m \leq c10^{-m^2}, \) \( m = 1, 2, \ldots \), which follows easily from our choice of \( (r_m)^m \). Using the above inequality in (4.9), we obtain

\[
c(\beta)^{-1} |x - z|^{1/\beta} \leq |f_k(x) - f_k(z)| \leq c(\beta)|x - z|^\delta,
\]

for \( |x - z| \leq 1/4 \). Hence (1.5) is true. As in Section 1 we put \( D = \bigcup_{k=0}^\infty D_k \) and choose a subsequence \((f_{n_k})\) of \((f_k)\) such that \((f_{n_k})\) converges uniformly to \( f \) on compact subsets of \( \mathbb{R}^n \). We claim that \( D \) is not a sphere. Indeed, since \( \max_{\gamma \leq 1} \psi = 1 \) and (2.1), (3.4) hold for \( r_m, \varepsilon_0, D_1 \), we see that if \( \rho_1 = \rho + (2\lambda_0)^{-1} \nu_0^2 r_1 \), then \( D_i \cap (\mathbb{R}^n - B(0, \rho_i)) \neq \emptyset \). Also, by construction, there exists \( x_0 \in \partial D_1 \) with \( |x_0| = \rho \). Using the definition of \( (r_m)^m \) and the triangle inequality we see that \( f(x_0) \in \partial D \) and \( |f(x_0)| < \rho_1 \). Therefore, \( D \) is not a sphere.

It remains only to prove (1.9) in order to obtain Theorem 1 from the remarks in Section 1. To this end let

\[
p_j(x) = f \circ f_{j-1}^{-1}(x) = \lim_{k \to \infty} h_{n_k} \circ \cdots \circ h_{j+1}(x),
\]

when \( x \in \partial D \) and \( j = 1, 2, \ldots \) Iterating (2.18) we deduce that if

\[
e_j = \prod_{m=j+1}^\infty \left( 1 - c_5 r_0^{1/2} \right),
\]

then

\[
e_j |x - y| \leq |p_j(x) - p_j(y)|, \quad x, y \in \partial D_j.
\]

If \( q_j \) denotes the inverse of \( p_j \), it follows that

\[
(4.10) \quad |q_j(x) - q_j(y)| \leq e_j^{-1}|x - y|,
\]

when \( x, y \in \partial D \). Next we use Kirchmann's Theorem (5, 2.10.43)) to extend \( q_j \) to \( \mathbb{R}^n \) (also denoted \( q_j \)) in such a way that (4.10) holds whenever \( x, y \in \mathbb{R}^n \).

From (4.10) it is easily seen by comparing coverings of each set that

\[
(4.11) \quad H^{n-1}(q_j(F)) \leq e_j^{-n}H^{n-1}(F), \quad F \subseteq \mathbb{R}^n.
\]

\( j = 1, 2, \ldots \) Let \( g \geq 0 \) be a continuous function on \( \mathbb{R}^n \), and put \( \nu(E) = H^{n-1}(q_j^{-1}(E) \cap \partial D) \). Then from (4.11) with \( F = q_j^{-1}(E) \cap \partial D \), we have

\[
H^{n-1}(E \cap \partial D) \leq e_j^{-n}\nu(E).
\]

Also from the usual change of variables formula (5, Thm. 2.4.18) and the above inequality we get
(4.12) \[ e_j^{n-1} \int_{\partial D} g \, dH^{n-1} \leq \int_{\mathbb{R}^n} g \, dv = \int_{\partial D} g \circ q_j \, dH^{n-1}. \]

Letting \( j \to \infty, \quad j \in (n_k)^{n^*}, \) we obtain from the definition of \(( r_k)^n_1\) that \( e_j \to 1, \)
while
\[ \int_{\partial D} g \circ q_j \, dH^{n-1} \to \int_{\partial D} g \, dH^{n-1}, \]
since \( q_n (x) \to x, \) uniformly on compact subsets of \( \mathbb{R}^n. \) Hence from (4.12) we have

(4.13) \[ \limsup_{k \to \infty} \int_{\partial D} g \, dH^{n-1} \leq \int_{\partial D} g \, dH^{n-1}. \]

On the other hand from our choice of \(( r_k)^n_1\) we see that \( L_m, \quad m = 1, 2, \ldots, \) is a covering for \( D. \) Thus if \( \phi_k^{n-1} \) is as in Section 1, then
\[ \phi_k^{n-1}(\partial D) \leq H^{n-1}(\partial D_m) - 2^{-m}. \]

Letting \( m \to \infty, \) we find

(4.14) \[ H^{n-1}(\partial D) \leq \liminf_{m \to \infty} H^{n-1}(\partial D_m). \]

From (4.13), (4.14), it follows that if \( 0 \leq g \leq 1 \) on \( \bar{D}, \) then
\[ H^{n-1}(\partial D) \leq \liminf_{k \to \infty} H^{n-1}(\partial D) \]

\[ \leq \liminf_{k \to \infty} \int_{\partial D} g \, dH^{n-1} + \limsup_{k \to \infty} \int_{\partial D} (1 - g) \, dH^{n-1} \]

\[ \leq \limsup_{k \to \infty} \int_{\partial D} g \, dH^{n-1} + \int_{\partial D} (1 - g) \, dH^{n-1} \]

\[ = H^{n-1}(\partial D). \]

Thus equality holds everywhere and so

(4.15) \[ \lim_{k \to \infty} \int_{\partial D} g \, dH^{n-1} = \int_{\partial D} g \, dH^{n-1} \]

when \( 0 \leq g \leq 1. \) In general we can write, \( g = ag_1 + b, \) where \( 0 \leq g_1 \leq 1 \) on \( D, \)
for properly chosen \( a, b \in \mathbb{R}. \) Applying (4.15) to \( g_1, \) 1 we find that (4.15) holds
when \( g \) is continuous on \( \mathbb{R}^n. \) Hence, (1.9) is true.

The proof of Theorem 1 is now complete.
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Hardy Spaces and Oscillatory Singular Integrals

Yibiao Pan

1. Introduction

Consider the oscillatory singular integral operator $T$:

$$(1) \quad Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i(Bx, y)} K(x - y) f(y) \, dy,$$

where $(Bx, y)$ is a real bilinear form, and $K$ is a Calderón-Zygmund kernel, i.e. $K$ is $C^1$ away from the origin, has mean-value zero on each sphere centered at the origin and satisfies

$$|K(x)| \leq C|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \leq C|x|^{-n-1}.$$ 

It is proved by D. H. Phong and E. M. Stein in [PS], that $T$ is a bounded operator on $L^p$ spaces, with bound independent of $B$. They also introduced some variants of the $H^1$ and BMO spaces (denoted by $H^1_B$ and BMO$_B$, to avoid the confusion with the standard $H^1$ and BMO). Analogous to the fact that the classical singular integral operators are bounded from $H^1$ to $L^1$, Phong and Stein showed that $T$ extends as a bounded operator from $H^1_B$ to $L^1$. This fact was then used to prove the $L^p$ boundedness by interpolating between $L^2$ and $L^\infty$, (see [PS]).

The object of our study is a more general class of oscillatory singular integral operators. An operator in this class is obtained when the bilinear form
in (1) is replaced by some real-valued polynomial in $x$ and $y$. These operators have arisen in the study of Hilbert transform along curves, singular integrals supported on lower-dimensional varieties and singular Radon transforms, etc. F. Ricci and E. M. Stein have proved in [RS] that an operator of this kind is bounded on $L^p$ spaces, with bound depending only on the total degree, not on the coefficients of the polynomial. The fact that these operators are of weak-type $(1,1)$ was subsequently proved by S. Chanillo and M. Christ ([CCJ]).

It is our goal in this paper to establish a Hardy space theory for the class of oscillatory singular integral operators with polynomial phase functions. Given such an operator

(2) \[ Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)}K(x-y)f(y)\,dy, \]

where $P(x,y)$ is a real-valued polynomial, we will define the space $H^1_{\text{b}}$ as some variant of the standard $H^1$ space, and this space $H^1_{\text{b}}$ is closely associated with the given polynomial $P(x,y)$. First let us give the definition of the "atoms":

**Definition.** Let $Q$ be a cube with center $x_Q$, an atom is a function $a(x)$ which is supported in $Q$, so that

\[ |a(x)| \leq \frac{1}{|Q|}, \]

and

\[ \int_Q e^{iP(x,y)}a(y)\,dy = 0. \]

The space $H^1_{\text{b}}$ consists of the subspace of $L^1$ of functions $f$ which can be written as $f = \sum \lambda_j a_j$, where $a_j$ are atoms, and $\lambda_j \in \mathbb{C}$, with $\sum |\lambda_j| < \infty$. Consequently, we define BMO$_{\text{b}}$ as the dual space of $H^1_{\text{b}}$. Our main result is

**Theorem 1.** Suppose $H^1_{\text{b}}$, and $T$ are defined as above. Then $T$ is a bounded operator from $H^1_{\text{b}}$ to $L^1$. The bound of this operator can be taken to depend only on the total degree of $P$, (not on the coefficients of $P$).

We notice that in the paper of Phong and Stein, the fact that the phase function is a real bilinear form makes it possible to apply the Plancherel's theorem to the Fourier transform (or partial Fourier transform) associated with $B$. When $(Bx, y)$ is replaced by the polynomial $P(x, y)$, we no longer have this advantage. So we have to take a different approach, using some $L^2$ estimates of certain oscillatory integrals. This will become clear in our proof.

For $p < 1$, the Calderón-Zygmund singular integral operators are still bounded from $H^p$ to $L^p$. However, this is no longer the case for the oscillatory
singular integral operators. At the end of this article, we will present a simple example which shows that this fails even in the bilinear phase function case.

The main result presented in this paper was included in the author’s thesis. The author would like to thank his advisor, Professor E.M. Stein, for his encouragement and many helpful suggestions. The author would like to thank the referee for his comments.

2. Proof of Theorem 1

PROOF. Let us assume that \( a \) is a function supported in the cube \( Q_0 \), which is centered at the origin, and has sidelength 1, and \( a \) satisfies

\[ |a| \leq 1, \quad \int_{Q_0} a(y) \, dy = 0. \]

First we shall prove that if \( P(x, y) \) is a polynomial in \( x, y \), and \( P(0, y) = 0 \), then

\[ \| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x, y)} K(x - y) a(y) \, dy \|_{L^1} \leq C, \]

where \( C \) depends only on the total degree of \( P \), and is otherwise independent of the coefficients of \( P \).

To prove (3), we shall use induction on the degree \( l \) of \( y \) in \( P(x, y) \).

If \( l = 0 \), then \( e^{iP(x, y)} \) is only a function of \( x \), therefore can be taken out of the integral sign, and (3) follows from the classical result of the standard \( H^1 \) theory. (See, for example [CW].)

Next we assume \( l > 0 \), and (3) is true for \( l - 1 \). By the Ricci-Stein theorem on the \( L^p \) boundedness of \( T \), we have

\[ \int_{|x| \leq 2} |T(a)(x)| \, dx \leq C \left( \int_{|x| \leq 2} |T(a)(x)|^2 \, dx \right)^{1/2} \]

\[ \leq C \left( \int_{\mathbb{R}^n} |a|^2 \, dx \right)^{1/2} \leq C. \]

Write

\[ P(x, y) = \sum_{|\alpha| \geq 1, |\beta| = 0} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y), \]

where \( Q(x, y) \) is a polynomial with degree in \( y \) less than or equal to \( l - 1 \), and still satisfies \( Q(0, y) = 0 \). For any \( r > 0 \), we have

\[ \int_{2 < |x| \leq r} |T(a)(x)| \, dx \leq \int_{2 < |x| \leq r} |(e^{iP(x, y)} - e^{iQ(x, y)})K(x - y)a(y) \, dy| \, dx \]

\[ + \int_{2 < |x| \leq r} \left| \int_{\mathbb{R}^n} e^{iQ(x, y)} K(x - y) a(y) \, dy \right| \, dx. \]

(If \( r \leq 2 \), all the above integrals are 0.)
By our inductive hypothesis, the second term is bounded. Also \(|x - y| \geq |x|/2\), if \(|x| > 2\), \(|y| \leq 1\). So we have

\[
\int_{2 < |x| = r} |T(a)(x)| \, dx \leq C + C \int_{|x| = r} \int_{\mathbb{R}^n} \exp \left( \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha \beta} x^\alpha y^\beta \right) - 1 \left| \frac{a(y)}{|x|^n} \right| \, dy
\]

\[
\leq C + C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha \beta}| \int_{|x| = r} |x|^{|\alpha| - n} \, dx
\]

\[
\leq C + C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha \beta}| |r|^{|\alpha|}.
\]

Now, there exists \((\alpha_0, \beta_0)\) such that \(|\alpha_0| \geq 1\), \(|\beta_0| = l\), and

\[
|a_{\alpha_0 \beta_0}|^{1/|\alpha_0|} = \max_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha \beta}|^{1/|\alpha|}.
\]

Put \(r = |a_{\alpha_0 \beta_0}|^{-1/|\alpha_0|}\), we have

\[
\int_{2 < |x| = r} |T(a)(x)| \, dx \leq C,
\]

where \(C\) depends only on the total degree of \(P(x, y)\). Now we turn to the estimate of the remaining part

\[
\int_{|x| > 2, |x| > r} |T(a)(x)| \, dx.
\]

We shall need the following lemmas:

**Lemma 1.** Suppose

\[
\phi(x) = \sum_{|\alpha| \leq k} a_{\alpha} x^\alpha
\]

is a real-valued polynomial in \(\mathbb{R}^n\) of degree \(k\), and \(\psi \in C_0^\infty\). Then for any \(r\), \(|\psi| = k\), \(a_{\alpha} \neq 0\), we have

\[
(4) \quad \left| \int_{\mathbb{R}^n} e^{i\phi(x)} \psi(x) \, dx \right| \leq C |a_{\alpha}|^{-1/k} (\|\psi\|_{L^\infty} + \|\nabla \psi\|_{L^1})
\]

To see this, simply let \(\xi\) be an unit vector, such that

\[
|\langle \xi \cdot \nabla \rangle^k \phi(x) | \geq c|a_{\alpha}|.
\]
This is possible because
\[
\frac{\partial^\nu \phi(x)}{\partial x^\nu} = \nu! a_x.
\]
(See [ST], page 317.) Without loss of generality, we may assume
\[
\xi = (1, 0, \ldots, 0).
\]
Hence
\[
\left| \frac{\partial^k \phi(y)}{\partial y^k_\xi} \right| \geq c |a_x|.
\]
Now apply the one-dimensional Van der Corput's lemma to obtain (4). See also [ST].

**Lemma 2.** Let
\[
P(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha
\]
denote a polynomial in \(\mathbb{R}^n\) of degree \(d\). Suppose \(\epsilon < 1/d\), then
\[
\int_{|x| \leq 1} |P(x)|^{-\epsilon} \, dx \leq A_\epsilon \left( \sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\epsilon}.
\]
The bound \(A_\epsilon\) depends on \(\epsilon\) (and the dimension \(n\)), but not on the coefficients \(|a_\alpha|\).

This is a result of Ricci and Stein. See [RS], page 182.

Now we continue our proof of Theorem 1. Let
\[
R_j = \{ x \in \mathbb{R}^n : 2^j \leq |x| < 2^{j+1} \},
\]
for \(j \geq 0\), and let \(\varphi \in C_0^\infty(\mathbb{R}^n)\) satisfy
\[
\varphi(x) = 1 \quad \text{for} \quad |x| \leq 1, \quad \varphi(x) = 0 \quad \text{for} \quad |x| \geq 2.
\]
Define \(T_j\) by
\[
(T_j f)(x) = \chi_{R_j}(x) \int_{\mathbb{R}^n} e^{iP(x,y)} \varphi(y) f(y) \, dy,
\]
and consider the operator \(T_j T_j^*:\)
\[
T_j T_j^*(f)(x) = \int_{\mathbb{R}^n} L_j(x, z) f(z) \, dz,
\]
where
\[
L_j(x, z) = \chi_{R_j}(x) \chi_{R_j}(z) \int_{\mathbb{R}^n} e^{i(P(x, y) - P(z, y))} |\varphi(y)|^2 \, dy.
\]

Write
\[
P(x, y) - P(z, y) = \sum_{|\alpha| + |\beta| = l} a_{\alpha\beta} y^\alpha (x^\alpha - z^\alpha) + (Q(x, y) - Q(z, y)),
\]
where the degree of \( y \) in \( Q(x, y) - Q(z, y) \) is less than or equal to \( l - 1 \).

Applying Lemma 1, with \( r = \beta_0 \), we obtain
\[
|L_j(x, z)| \leq C \left| \sum_{|\alpha| \geq 1} a_{\alpha\beta_0} (x^\alpha - z^\alpha) \right|^{-1/N} \chi_{R_j}(x) \chi_{R_j}(z).
\]

On the other hand, it is obvious that \( |L_j(x, z)| \leq C \), so let \( N > 0 \) be a large number (to be chosen later), we have
\[
|L_j(x, z)| \leq C \left| \sum_{|\alpha| \geq 1} a_{\alpha\beta_0} (x^\alpha - z^\alpha) \right|^{-1/N} \chi_{R_j}(x) \chi_{R_j}(z).
\]

By rescaling we would obtain the same norm if we were to replace \( L_j(x, z) \) by \( L'_j(x, z) = 2^{|j|} L_j(2^j x, 2^j z) \), so we have
\[
|L'_j(x, z)| \leq C 2^{|j|} \left| \sum_{|\alpha| \geq 1} (a_{\alpha\beta_0} 2^{|j|\alpha}) x^\alpha - \sum_{|\alpha| \geq 1} a_{\alpha\beta_0} 2^{|j|\alpha} z^\alpha \right|^{-1/N} \chi_{R'_j}(x) \chi_{R'_j}(z).
\]

Choosing \( N \) sufficiently large and applying Lemma 2, we get
\[
\sup_{z} \int_{\mathbb{R}^n} |L'_j(x, z)| \, dx \leq C 2^{|j|} \sup_{z} \left( \sum_{|\alpha| \geq 1} \left| a_{\alpha\beta_0} 2^{|j|\alpha} \right| + \left| \sum_{|\alpha| \geq 1} a_{\alpha\beta_0} 2^{|j|\alpha} z^\alpha \right| \right)^{-1/N}
\]
\[
\leq C 2^{|j|} |a_{\alpha\beta_0}|^{-1/N} 2^{-j |\alpha_0| / N}.
\]

Similar estimate hold: \( \sup_{z} \int_{\mathbb{R}^n} |L'(x, z)| \, dx \), therefore we obtain
\[
\|T_j T_j^*\| \leq C 2^{|j|/2} |a_{\alpha\beta_0}|^{-1/2} 2^{-j |\alpha_0| / 2N},
\]
so
\[
\|T_j\|_{L^2 \rightarrow L^2} \leq C 2^{|j|/2} |a_{\alpha\beta_0}|^{-1/2} 2^{-j |\alpha_0| / 2N}.
\]

Now we have
\[
\int_{|x| > r, |x| > r} |T(a)(x)| \, dx \leq \int_{|x| > 2r, |x| > r} dx \int_{\mathbb{R}^n} |K(x - y) - K(x)| \, |a(y)| \, dy
\]
\[
+ \int_{|x| > 2r, |x| > r} |K(x)| \, dx \int_{\mathbb{R}^n} e^{iP(0, y)} a(y) \, dy = I_1 + I_2.
\]
The estimate for $I_1$ is easy

$$I_1 \leq \int_{|x| > 2, |y| > r} dx \int_{|x| > 2} \frac{|y| |a(y)|}{|x|^{n+1}} dy$$

$$\leq C \int_{|x| > 2} \frac{dx}{|x|^{n+1}} < C.$$

As for $I_2$, using our estimate on $T_j$ and assuming $2^{j_0} \leq r < 2^{j_0+1}$, for some $j_0$, we have

$$I_2 \leq C \int_{|x| > 2, |y| > r} \frac{1}{|x|^n} \left| \int_{|x| > 2} e^{ip(x,y)}a(y) dy \right| dx$$

$$\leq C \sum_{j \geq j_0} \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^n} \left| T_j(a)(x) \right| dx$$

$$\leq C \sum_{j \geq j_0} \left( \int_{2^j \leq |x| < 2^{j+1}} \frac{1}{|x|^{2n}} dx \right)^{1/2} \left| T_j(a) \right|_{L^2}$$

$$\leq C \sum_{j \geq j_0} 2^{-n/2} 2^{n/2} |a_{\alpha_0 \beta_0}|^{-1/2} \left| a_{\alpha_0 \beta_0} \right|^{-1/2} \left| a_{\alpha_0 \beta_0} \right|^{-1/2} \leq C,$$

because $2^{j_0} \geq (1/2)|a_{\alpha_0 \beta_0}|^{-1/2}$, and (3) is proved.

To prove the theorem, we only need to prove that $\left| T(a) \right|_{L^1} \leq C$, for all atoms $a$, and $C$ is a constant which depends only on the total degree of $P(x, y)$.

Let $a$ be an atom associated to the cube $Q$, and the center and sidelength of $Q$ are $x_Q$ and $\delta$ respectively. We observe that

$$\mathcal{P}(T(a))(\delta x + x_Q) \equiv \text{p.v.} \int_{|x| > 2} e^{ip(x,y + x_Q)}K(x - y)\delta^n a(dy + x_Q) dy.$$

Write

$$P(\delta x + x_Q, \delta y + x_Q) = R(x, y) + P(x_Q, \delta y + x_Q),$$

where $R(x, y)$ is a polynomial which satisfies $R(0, y) = 0$, and the total degree of $R$ is not greater than that of $P$. Let

$$b(y) = e^{ip(x_Q, \delta y + x_Q)}a(dy + a_Q),$$

by the definition of the atom, we have

$$\text{supp}(b) \subset Q_0 \quad \text{and} \quad |b(y)| \leq 1,$$

also

$$\int_{Q_0} b(y) dy = \int_{Q} e^{ip(x_Q, \delta y)}a(y) dy = 0.$$
Now invoking (3), we have

$$\| T(a) \|_{L^1} = \left\| \text{p.v.} \int_{\mathbb{R}^n} e^{i\xi \cdot (x,y)} K(x-y)b(y) \, dy \right\|_{L^1} \leq C.$$ 

This completes the proof of Theorem 5.

3. An Extension

In [RS], Ricci and Stein pointed out that the $L^p$ boundedness still holds, if the Calderón-Zygmund kernel in the operator is replaced by some more general distribution. For $H^1_E$, the same thing is true, i.e.

**Theorem 2.** If $K(x,y)$ is a distribution and $C^1$ away from the diagonal $\{x = y\}$, and satisfies:

(i) $|K(x,y)| \leq C|x - y|^{-n}$ and $|\nabla K(x,y)| \leq C|x - y|^{-n-1}$.

(ii) The operator

$$f \rightarrow \int K(x,y)f(y) \, dy$$

extends as a bounded operator on $L^2(\mathbb{R}^n)$.

Then the operator

$$(5) \quad Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y)f(y) \, dy$$

is bounded from $H^1_E$ to $L^1$.

The proof of Theorem 2 is essentially the same as Theorem 1.

4. The Dual Space $\text{BMO}_E$

We define the sharp function $f^E_\#$ to be

$$(f^E_\#)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x) - f_Q^E(x)| \, dx,$$

where

$$f_Q^E(x) = e^{-iP(x,y)} \left( \frac{1}{|Q|} \int_Q e^{-iP(x,y)} f(y) \, dy \right)$$
and as the dual space of $H^1_E$, $\text{BMO}_E$ is given by
\[
\text{BMO}_E = \{ f \in L^1_{\text{loc}} : f_E^* \in L^\infty \}
\]
and
\[
\| f \|_{\text{BMO}_E} = \| f_E^* \|_{L^\infty}.
\]

The dual statement of Theorem 2 is

**Theorem 3.** The operator $T^*$ ($T$ given by (5)) extends as a bounded operator from $L^\infty$ to $\text{BMO}_E$.

5. A Counterexample

In this section, we shall give a simple example to show that the $H^1$ theory on the oscillatory singular integral operators cannot be extended to the $H^p$ case, if $p < 1$.

Let $T$ be defined as
\[
(Ta)(x) = \text{p.v.} \int_{\mathbb{R}^1} e^{iyx} \frac{1}{x-y} a(y) \, dy.
\]

Take $\delta > 0$, $\delta$ is very small, and $a$ is a function supported on $I_\delta = [-\delta, \delta]$, given by
\[
a(y) = \begin{cases} \ \ (2\delta)^{-1/p} & \text{if } y \in [\delta/2, \delta], \\ \ -(2\delta)^{-1/p} & \text{if } y \in [-\delta, -\delta/2], \\ \ 0 & \text{otherwise.} \end{cases}
\]

It is easy to see that $a$ satisfies
\[
|a| \leq |I_\delta|^{-1/p}, \quad \int_\delta a(y) \, dy = 0.
\]

Therefore, we have
\[
\text{Im}(Ta)(x) = (2\delta)^{-1/p} \left( \int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x-y} \, dy + \int_{\delta/2}^{\delta} \sin(xy) \frac{1}{x+y} \, dy \right).
\]

Let $x \in (\pi/4\delta, \pi/3\delta)$, then $x-y > 0$, $x+y > 0$ for $y \in [\delta/2, \delta]$. Also $\pi/8 < xy < \pi/3$. 
Hence
\[
\text{Im} \ (Ta)(x) > c_0 (2\delta)^{-1/p} \left( \int_{\delta/2}^{\delta} \frac{1}{x-y} \ dy + \int_{\delta/2}^{b} \frac{1}{x+y} \ dy \right)
\]
\[
= c_0 (2\delta)^{-1/p} \log \left( 1 + \frac{\delta x}{(x^2 - \delta x/2 - \delta^2/2)} \right)
\]
\[
> c'_0 \delta^{1 - 1/p} x^{-1},
\]
for some constant $c'_0 > 0$. Then, we have
\[
\int_{\delta} |Ta(x)|^p \ dx \geq c_0^p \int_{\delta/4}^{\delta/36} (\delta^{1 - 1/p} x^{-1})^p \ dx = c_0^2 (\delta^{2/p} - 1).
\]
This is unbounded as $\delta \to 0$ and $p < 1$.

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Local Properties of Stationary Solutions of some Nonlinear Singular Schrödinger Equations

Bouchaib Guerch and Laurent Veron

Abstract

We study the local behaviour of solutions of the following type of equation

\[ -\Delta u - V(x)u + g(u) = 0 \]

when \( V \) is singular at some points and \( g \) is a non-decreasing function. Emphasis is put on the case when \( V(x) = c|x|^{-2} \) and \( g \) has a power-like growth.

Introduction

In this article we study the local behaviour of a solution \( u \) of the following time-independent, \( N \)-dimensional, nonlinear Schrödinger equation

\[ -\Delta u - V(x)u + g(u) = 0 \quad (0.1) \]

near an isolated singularity of the potential \( V \), \( g \) being some asymptotically nondecreasing real valued function. In many physical examples \( V \) is a Coulombian potential.
\[ V(x) = \sum_{i=1}^{k} z_i |x - a_i|^{-1} \]

in the case of a nucleus in the Thomas-Fermi-Dirac-von Weizsäcker theory [3], [4]. However it is mathematically more exciting when \( V \) can be compared with \( |x - a|^{-2} \) near the isolated singularity \( a \). In that case the interference between the Laplacian, the potential and the nonlinearity is very strong. The model equation is the following

\[ -\Delta u - \frac{c}{|x|^2} u + u|u|^{q-1} = 0 \]

where \( q > 1 \) and \( c \) is some real number. If we look for a specific solution of (0.3) under the form

\[ u_s(r) = \alpha r^\beta, \]

then

\[ \beta = -\frac{2}{q - 1} \]

and

\[ \alpha^{q-1} = c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right). \]

Henceforth the solution \( u_s \) exists if and only if

\[ c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right) > 0. \]

It is worth noticing that if (0.5) does not hold then

\[ c \leq \left( \frac{N - 2}{2} \right)^2, \]

and this condition plays a fundamental role in the description of the fundamental solutions of the equation

\[ \Delta \phi + \frac{c}{|x|^2} \phi = 0. \]

If (0.6) is satisfied let \( \beta \) be \( \sqrt{(N - 2)^2 - 4c} \) and \( \mu \) the two fundamental solutions of (0.7), that is
\[
\mu_1(x) = \begin{cases} 
|x|^{-(N-2+\delta)/2} & \text{if } c < \left(\frac{N-2}{2}\right)^2, \\
|x|^{-(N-2)/2} \ln(1/|x|) & \text{if } c = \left(\frac{N-2}{2}\right)^2, 
\end{cases}
\]

\[
\mu_2(x) = \begin{cases} 
|x|^{-(N-2-\delta)/2} & \text{if } c < \left(\frac{N-2}{2}\right)^2, \\
|x|^{-(N-2)/2} & \text{if } c = \left(\frac{N-2}{2}\right)^2. 
\end{cases}
\]

It is important to notice that \(\mu_2\) is the regular solution of (0.7) in the sense that 
\(c|\cdot|^{-2}\mu_2(\cdot)\) is locally integrable in \(\mathbb{R}^N\) and

\[
\Delta \mu_2 + \frac{c}{|x|^2} \mu_2 = 0
\]

holds in \(D'(\mathbb{R}^N)\) (if \(c \leq 0, u\) is continuous), as the same holds for \(\mu_1\) if and only if \(c > 0\); in any case \(\mu_2 = o(\mu_1)\) near 0. Our first removability result deals with the meaning of the equation in the sense of distributions.

**Theorem 1.1.** Let \(\Omega\) be an open subset of \(\mathbb{R}^N\) containing 0, \(\Omega^* = \Omega \setminus \{0\}\), \(g\) a continuous real valued function satisfying

\[
\begin{cases} 
\liminf_{r \to \infty} \frac{g(r)}{r^q} > 0 \\
\limsup_{r \to -\infty} \frac{g(r)}{(-r)^q} < 0 
\end{cases}
\]

and \(V \in C^0(\Omega^*)\) is such that

\[
-\infty < |x|^2 V(x) \leq c
\]

near 0 for some constants \(q > 1\) and \(c\). If we assume either \(q > N/(N-2)\), or \(1 < q \leq N/(N-2)\) and

\[
c + \frac{2}{q-1} \left(\frac{2q}{q-1} - N\right) \leq 0,
\]

any \(u \in C^1(\Omega^*)\) satisfying

\[
-\Delta u - Vu + g(u) = 0
\]

in \(D'(\Omega^*)\) can be extended as a solution of the same equation in \(D'(\Omega)\).
We must remark that if \((0.5)\) is satisfied with \(q > N/(N - 2)\) there exist singular solutions of the model problem \((0.3)\) with a rather weak singularity; this must be compared with

\[
\Delta u + u^q = 0
\]

for which the same holds when \(q > N/(N - 2)\). Our second removability result is to compare a solution of \((0.14)\) in \(\Omega^*\) with the regular solution of \((0.10)\).

**Theorem 1.2.** Let \(\Omega\) and \(V\) be as in Theorem 1.1 and let \(g\) be a continuous real valued function satisfying \((0.11)\) for some \(q > 1\). Assume also that

\[
0 = g(0) = g^{-1}(0)
\]

and

\[
c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right) \leq 0
\]

hold. Then if \(u\) is any \(C^1(\Omega^*)\) function satisfying \((0.14)\) in \(D'(\Omega^*)\), \(u/\mu_2\) remains locally bounded in \(\Omega\).

It is important to notice that, as \((0.17)\) holds, \((0.6)\) also holds which allows us to have a comparison principle.

Our second section is devoted to the extension of Vázquez-Veron's isotropy theorems [23], [24] to the potential case. Let us introduce some notations: let \(S^{N-1}\) be the unit sphere in \(\mathbb{R}^N\), \((r, \sigma) \in \mathbb{R}_+ \times S^{N-1}\) the spherical coordinates in \(\mathbb{R}^N\setminus \{0\}\) and \(\bar{\rho}(r)\) the spherical average of a function \(\rho(r, \sigma)\), that is

\[
\bar{\rho}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \rho(r, \sigma) \, d\sigma.
\]

**Theorem 2.1.** Assume \(\Omega\) is an open subset of \(\mathbb{R}^N\) containing \(0, \Omega^* = \Omega \setminus \{0\}\), \(g\) is a continuous nondecreasing real valued function and \(u \in C^1(\Omega^*)\) is a solution of \((0.14)\) in \(D'(\Omega^*)\) where \(V \in C^0(\Omega^*)\) is a radial function such that

\[
-\infty < |x|^2 V(x) \leq c \leq \left( \frac{N - 2}{2} \right)^2, \quad \text{for every} \quad x \in \Omega^*.
\]

If \(u\) satisfies

\[
\liminf_{r \to 0} r^{(N - 2 + \sqrt{N^2 - 4\varepsilon})/2} \|u(r, \cdot) - \bar{u}(r)\|_{L^2(S^{N-1})} = 0,
\]

then \(u(x)/\mu_1(x)\) admits a limit in \(\mathbb{R} \cup \{-\infty, \infty\}\) as \(x\) tends to \(0\).
A similar isotropy result holds for a solution \( u \) of (0.14) in an exterior domain of \( \mathbb{R}^N \). An interesting class of solutions of (0.14) in \( \Omega^* \) are those which present a singular linear behaviour near 0. As the equation in the sense of distributions in \( \Omega \) is not very significative except when \( c = 0 \) where the Dirac mass plays a fundamental role [6], [10], the good criterion for the behaviour of linear singularities will be the existence of a finite, not always 0, limit of \( u(x)/\mu_1(x) \) as \( x \) tends to 0, as in [29].

**Theorem 3.1.** Assume \( g \) is a continuous nondecreasing real valued function and

\[
(0.21) \quad c < \left( \frac{N-2}{2} \right)^2.
\]

Then the equation

\[
(0.22) \quad -\Delta u - \frac{c}{|x|^2} u + g(u) = 0
\]

admits solutions \( u \) in

\[\mathcal{B}_1(0) \setminus \{0\} = \{ x \in \mathbb{R}^N : 0 < |x| < 1 \}\]

such that

\[
(0.23) \quad \lim_{x \to 0} \frac{u(x)}{\mu_1(x)} = \gamma,
\]

where \( \gamma \) is any arbitrary real number if and only if

\[
(0.24) \quad \int_{1}^{\infty} \left( g(t) + |g(-t)| \right) t^{2(1 - \alpha_1)/\alpha_1} \, dt < \infty,
\]

where

\[
\alpha_1 = -\left( N - 2 + \sqrt{(N - 2)^2 - 4c} \right)/2.
\]

When \( c = 0 \), condition (0.24) is the one introduced by Brézis and Bénilan [6] for solving equations of type

\[
(0.25) \quad -\Delta u + g(u) = m
\]

where \( m \) is a bounded measure. When

\[
c = \left( \frac{N-2}{2} \right)^2
\]

the situation is quite more complicated (see Vázquez [22] for the case \( N = 2 \)).
We define

\[
\begin{aligned}
&b^+_e = \inf \left\{ b > 0 : \int_0^1 g \left( r^{-2/(N-2)} (N+2) \ln (1/t)/b \right) dt < \infty \right\}, \\
&b^-_e = \inf \left\{ b > 0 : \int_0^1 g \left( r^{-2/(N-2)} (N+2) \ln t/b \right) dt > -\infty \right\},
\end{aligned}
\]

and we prove

**Theorem 3.2.** Assume \( g \) is a continuous nondecreasing real valued function. Then the equation

\[
-\Delta u - \left( \frac{N-2}{2|x|} \right)^2 u + g(u) = 0
\]

admits solutions \( u \) in \( B_1(0) \setminus \{0\} \) such that

\[
\lim_{x \to 0} |x|^{(N-2)/2} u(x)/\ln (1/|x|) = \gamma,
\]

where \( \gamma \) is a real number, if and only if

\[
-(N+2)/(2b^-_e) \leq \gamma \leq (N+2)/(2b^+_e).
\]

The Dirichlet problems corresponding to Theorems 3.1, 3.2 are also solved.

In the last section we study the limit properties of the solutions \( u \) of (0.3) (as \( |x| \) tends to 0 or \( \infty \) as well). If we perform the classical transformation

\[
u(r, \vartheta) = r^{-2/(q-1)} v(t, \vartheta), \quad t = \ln r\]

and denote by \( \Delta_{S^{N-1}} \) the Laplace-Beltrami operator on \( S^{N-1} \), then

\[
u_{tt} + \left( N - 2 + \frac{q+1}{q-1} \right) v_t + \Delta_{S^{N-1}} v + \lambda v - |v|^{q-1} = 0
\]

holds in \( (-\infty, 0) \) or \( (0, \infty) \) with

\[
\lambda = c + \frac{2}{q-1} \left( \frac{2q}{q-1} - N \right),
\]

moreover \( u \) is bounded. When \( q \neq (N+2)/(N-2) \) the study of this equation is an extension of previous results of Veron [26] [27], Chen-Matano-Veron [13] and Bidaud-Veron-Veron [7]. A typical result is the following

**Theorem 4.1.** Assume \( q \in (1, \infty) \setminus \{(N+2)/(N-2)\} \) and \( u \) is a solution of (0.3) in \( B_1(0) \setminus \{0\} \). Then \( r^{2/(q-1)} u(r, \cdot) \) converges in the \( C^1(S^{N-1}) \)-topology
to some compact connected subset $\xi'$ of the set $\xi$ of the $C^1(S^{N-1})$-functions $\omega$ satisfying

\[(0.33)\quad -\Delta_{S^{N-1}} \omega + \omega |\omega|^{q-1} = \lambda \omega\]

on $S^{N-1}$. Moreover there exists precisely one $\omega \in \xi$ such that

\[(0.34)\quad \lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) = \omega(\cdot),\]

at least in the following cases:

(i) $u$ is nonnegative,
(ii) $\lambda \leq N - 1$,
(iii) $q$ is an odd integer,
(iv) $\xi'$ is an hyperbolic limit manifold in the sense of Simon [21],
(v) $N = 2$ and $c \leq 1$.

When $q = (N + 2)/(N - 2)$ the study is more complicated, in particular because of the conformal invariance of (0.3) and the existence of solitary waves satisfying (0.3) (see [7]). Convergence results hold at least for nonnegative solutions [17]. When $\lambda \leq 0$ there always holds

\[(0.35)\quad \lim_{x \to 0} |x|^{2/(q-1)} u(x) = 0\]

for any solution $u$ of (0.3) in $B_1(0) \setminus \{0\}$ and the exact behaviour is given by $\mu_2$ from Theorem 1.2 except in the particular case $\lambda = 0$, $q > (N + 2)/(N - 2)$ and we prove

**Theorem 4.2.** Assume $0 < c \leq ((N - 2)/2)^2$, $\lambda = 0$ and $q > (N + 2)/(N - 2)$. If $u$ is any solution of (0.3) in $B_1(0) \setminus \{0\}$, then the following limit exists

\[(0.36)\quad \lim_{x \to 0} u(x)/(|\mu_2(x) \ln (1/|x|)^{\alpha_2/2}) = l\]

with

\[\alpha_2 = \left(2 - N + \sqrt{(N - 2)^2 - 4c}\right)/2 = -2/(q - 1)\]

and

\[(0.37)\quad l \in \{0, \pm((N(q - 1) - 2(q + 1))/(q - 1)^2)^{1/(q - 1)}\}.

When $\lambda > 0$ it may happen that (0.35) holds. In that case the behaviour of $u$ near 0 is most often described by the solutions of
(0.38) \[-\Delta \xi - \frac{c}{|x|^2} \xi = 0,\]

satisfying (0.35) except when $-2/(q-1)$ is a solution of the algebraic equation

(0.39) \[X^2 + (N - 2)X + c - k(k + N - 2) = 0\]

for some integer $k$. Only when $N = 2$ and $c \leq 0$ this spectral case is understood. In some cases, when the rate of blow-up of $u$ near 0 is of order $|x|^{-(N-2)/2}$, $u$ may behave as a finite superposition of travelling waves near 0 (up to the damping factor $|x|^{(N-2)/2}$).

We also study the asymptotics of the solution of (0.3) in an exterior domain and end this section with some orbit connecting questions where the structure of the set of the stationary solutions of (0.33) plays a fundamental role.

Our paper is organized as follows

(1) Removable singularities.
(2) The isotropy theorems.
(3) Solutions with linear singularities.
(4) The power case.

1. Removable Singularities

In this section we assume that $\Omega \supset \bar{B}_1(0), \Omega^* = \Omega \setminus \{0\}$ and we first prove the following a priori estimate of Osserman's type [19], [10], [31].

Lemma 1.1. Assume $u \in L^\infty_{\text{loc}}(\Omega^*)$ satisfies $\Delta u \in L^\infty_{\text{loc}}(\Omega^*)$ and

(1.1) \[-\Delta u - \frac{c}{|x|^2} u + au^q \leq b\]

a.e. on $\{x \in \Omega: u(x) \geq 0\}$, for some constants $a > 0$, $b$ and $c \geq 0$ and $q > 1$. Then

(1.2) \[u(x) \leq A |x|^{-2/(q - 1)} + B \quad (\text{for all } x \in \bar{B}_{1/2}(0) \setminus \{0\}),\]

where

(1.3) \[A = \sigma(N, q) \left(1 + \frac{c}{a}\right)^{1/(q - 1)}, \quad B = \sigma(N, q) \left(\frac{b}{a}\right)^{1/q},\]

with $\sigma(N, q) > 0$. 

LOCAL PROPERTIES OF STATIONARY SOLUTIONS OF SOME SCHRODINGER EQUATIONS

PROOF. Let \( x_0 \) be such that \( 0 < |x_0| < 1/2 \). Set
\[
G = \left\{ x \in \Omega : |x - x_0| < \frac{1}{2} |x_0| \right\}
\]
and \( G^+ = \{ x \in G : u(x) \geq 0 \} \). The function \( u \) is essentially bounded in \( G \) and
\[
-\Delta u + \frac{a}{2} u^q \leq \beta = \max_{r > 0} \left\{ b + \frac{4c}{|x_0|^2} r - \frac{a}{2} r^q \right\}
\]
a.e. in \( G^+ \). If we compute \( \beta \) we find
\[
\beta = b + \frac{4(q - 1)c}{|x_0|^2} \left( \frac{8c}{aq|x_0|^2} \right)^{1/(q-1)}
\]
As in [10], [31] we consider a function \( v \) under the following form
\[
v(x) = \rho \left( \frac{1}{4} |x_0|^2 - |x - x_0|^2 \right)^{-2/(q - 1)} + \tau.
\]
If
\[
\eta = \max \left\{ \frac{2N}{q - 1}, \frac{4(q + 1)}{(q - 1)^2} \right\}, \quad \rho = \left( \frac{2\eta}{a} \right)^{1/(q - 1)}, \quad \tau = \left( \frac{2\beta}{a} \right)^{1/q},
\]
v satisfies
\[
-\Delta v + \frac{a}{2} v^q \geq \beta
\]
in \( G \). Using Kato’s inequality as in [10], [11], we deduce \( v \geq u \) in \( G \), which implies \( v(x_0) \geq u(x_0) \) and gives (1.3).

Lemma 1.2. Assume \( 1 < q \leq N/(N - 2) \),
\[
c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right) \leq 0
\]
and \( u \in L^\infty_{\text{loc}}(\Omega^*) \) satisfies \( \Delta u \in L^\infty_{\text{loc}}(\Omega^*) \) and (1.1) a.e. on \( \{ x \in \Omega : u(x) \geq 0 \} \) for some constants \( a > 0 \) and \( b > 0 \). Then \( \mu u \in L^q_{\text{loc}}(\Omega) \).

PROOF. From Kato’s inequality we have
\[
-\Delta u^+ - \frac{c}{|x|^2} u^+ + au^+ \leq b
\]
in $D'(\Omega^*)$, and from (1.2) and (1.8) we deduce that $c \leq 0$ and 

$$u^+ \leq K_0 \mu_1$$

near 0, for some $K_0 > 0$. Let $\phi$ be an element of $C^\infty_0(\Omega)$, $\phi \geq 0$, $\eta_n$ be a $C^\infty$-function in $\Omega$ such that $0 \leq \eta_n \leq 1$ and 

$$\eta_n(x) = \begin{cases} 0 & \text{if } 0 < |x| < 1/n \\ 1 & \text{if } |x| \geq 2/n \end{cases}$$

with $|\nabla \eta_n| \leq Kn$, $|\Delta \eta_n| \leq Kn^2$.

Then we claim that 

$$\int (u^+)^q \mu_2 \phi \, dx < \infty.$$ 

As a test function we take $\phi \eta_n \mu_2$ and get 

$$\int u^+ (-\Delta (\phi \eta_n \mu_2) + \frac{c}{|x|^2} (\phi \eta_n \mu_2)) + a \int (u^+)^q \phi \eta_n \mu_2 \leq K(\phi).$$

But 

$$\Delta (\phi \eta_n \mu_2) = \phi \eta_n \Delta \mu_2 + \mu_2 \Delta (\phi \eta_n) + 2 \nabla \eta_n \nabla (\phi \eta_n)$$

and (1.12) becomes 

$$\int u^+ (-\mu_2 \Delta (\phi \eta_n) - 2 \nabla \mu_2 \nabla (\phi \eta_n)) + a \int (u^+)^q \phi \eta_n \mu_2 \leq K(\phi).$$

Let $\Gamma_n = \{ x \in \Omega : 1/n < |x| < 2/n \}$ and let $\chi_{\Gamma_n}$ be the characteristic function of $\Gamma_n$. There exist $K_1, K_2$ such that $K_1 > 0$ and 

$$|\Delta (\phi \eta_n)| \leq K_1 + K_2 n^2 \chi_{\Gamma_n}, \quad |\nabla (\phi \eta_n)| \leq K_1 + K_2 n \chi_{\Gamma_n}.$$ 

Plugging into (1.12) implies 

$$a \int (u^+)^q \phi \eta_n \mu_2 < K(\phi) + K_0 \int \mu_1 \mu_2 (K_1 + K_2 n^2 \chi_{\Gamma_n})$$

$$+ 2K_0 \int \mu_1 |\nabla \mu_2| (K_1 + K_2 n \chi_{\Gamma_n}).$$

As $\mu_1 \mu_2 = |x|^{-N}$ and $\mu_1 |\nabla \mu_2| = K^* |x|^{1-N}$, the right-hand side of (1.15) is bounded independently of $n$. Letting $n$ tend to infinity implies the claim.

**Lemma 1.3.** Under the hypotheses of Theorem 1.1, $g(u)$ and $V u$ are locally integrable in $\Omega$. 

PROOF. We shall treat separately the cases $1 < q \leq N/(N-2)$ and $q > N/(N-2)$ but from (0.11) and Lemma 1.1 in any case $|x|^{2/(q-1)}u(x)$ is locally bounded in $\Omega$.

Case 1. $1 < q \leq N/(N-2)$. From Kato's inequality we have

$$-\Delta u^+ - Vu^+ + \text{sign}^+(u)g(u) \leq 0$$

in $D'(\Omega^*)$. Let $\xi$, be $\mu_2/(\epsilon + \mu_2)$ ($\epsilon > 0$). As a test function we take $\phi \eta_n \xi$, where $\phi$ and $\eta_n$ are as in Lemma 1.2 with $\phi = 1$ in $\bar{B}_{1/2}(0)$. We get

$$\int u^+ ((-\Delta(\phi \eta_n \xi)) - V\phi \eta_n \xi) + \int \text{sign}^+(u)g(u)\phi \eta_n \xi \leq 0.$$  

As

$$\Delta(\phi \eta_n \xi) = \phi \eta_n \Delta \xi + \xi \Delta(\phi \eta_n) + 2 \nabla \xi \nabla (\phi \eta_n)$$

and

$$\nabla \xi = \frac{\epsilon}{(\epsilon + \mu_2)^2} \nabla \mu_2,$$

$$\Delta \xi = \frac{\epsilon}{(\epsilon + \mu_2)^2} \Delta \mu_2 - \frac{2\epsilon}{(\epsilon + \mu_2)^3} |\nabla \mu_2|^2.$$

As

$$\Delta \mu_2 = -\frac{c}{|x|^2} \mu_2,$$

we get

$$-\Delta(\phi \eta_n \xi) - V\phi \eta_n \xi = -\left(\frac{ce \mu_2}{|x|^2(\epsilon + \mu_2)^2} - \frac{2\epsilon}{(\epsilon + \mu_2)^3}|\nabla \mu_2|^2\right)\phi \eta_n$$

$$- V\phi \eta_n \frac{\mu_2}{\epsilon + \mu_2}$$

$$- \xi \Delta(\phi \eta_n)$$

$$- \frac{2\epsilon}{(\epsilon + \mu_2)^3} \nabla \mu_2 \nabla (\phi \eta_n).$$

Henceforth (1.17) implies

$$\int \frac{\mu_2}{\epsilon + \mu_2} \phi \eta_n(-Vu^+ + \text{sign}^+(u)g(u))$$

$$\leq -\int \frac{ce \mu_2 u^+}{|x|^2(\epsilon + \mu_2)^2} \phi \eta_n + \int u^+ \xi \Delta(\phi \eta_n) + 2\epsilon \int \frac{u^+}{(\epsilon + \mu_2)^3} \nabla \mu_2 \nabla (\phi \eta_n).$$
We also have
\begin{align}
(1.19) \quad |\Delta(\phi_{\eta_0})| & \leq K_1 x_{\Gamma_2} + n^2 K_2 x_{\Gamma_n}, \\
(1.20) \quad |\nabla(\phi_{\eta_0})| & \leq K_1 x_{\Gamma_2} + n K_2 x_{\Gamma_n},
\end{align}
and
\begin{equation}
(1.21) \quad \left| \int_{\Gamma_n} u^+ \xi_\varepsilon \Delta(\phi_{\eta_0}) \right| \leq K_1 \int_{\Gamma_2} u^+ + n^2 K_2 \int_{\Gamma_n} u^+ \xi_\varepsilon.
\end{equation}
As
\begin{equation}
\int_{\Gamma_n} u^+ \xi_\varepsilon \leq \left( \int_{\Gamma_n} (u^+)^q \mu_2 \right)^{1/q} \left( \int_{\Gamma_n} \frac{\mu_2}{(\varepsilon + \mu_2)^{(q-1)/q}} \right)^{(q-1)/q},
\end{equation}
and
\begin{equation}
\left( \int_{\Gamma_n} \frac{\mu_2}{(\varepsilon + \mu_2)^{(q-1)/q}} \right)^{(q-1)/q} \leq \frac{c(N)}{\varepsilon} n^{-(\alpha_2 + N)(q-1)/q}
\end{equation}
where
\begin{equation}
\alpha_2 = \frac{2 - N + \sqrt{(N-2)^2 - 4c}}{2}
\end{equation}
is such that \( \mu_2(x) = |x|^{\alpha_2} \); as (1.8) holds \( \alpha_1 \leq -2/(q-1) \) and as \( \alpha_1 + \alpha_2 = 2 - N \), we deduce that
\begin{equation}
-(\alpha_2 + N)(q-1)/q + 2 \leq 0.
\end{equation}
From Lemma 1.2, \( \mu_2(u^+)^{q} \) is locally integrable in \( \Omega \); henceforth
\begin{equation}
\lim_{n \to \infty} n^2 \int_{\Gamma_n} u^+ \xi_\varepsilon = 0.
\end{equation}
In the same way
\begin{equation}
e \int_{\Gamma_2} \frac{u^+}{(\varepsilon + \mu_2)^2} \nabla \mu_2 \nabla(\phi_{\eta_0}) \leq 2eK_1 \int_{\Gamma_2} u^+ \frac{\nabla \mu_2}{\mu_2} + 2Kn \int_{\Gamma_n} \frac{eu^+}{(\varepsilon + \mu_2)^2} |\nabla \mu_2|
\end{equation}
and
\begin{equation}
\int_{\Gamma_n} \frac{eu^+}{(\varepsilon + \mu_2)^2} |\nabla \mu_2| \leq \int_{\Gamma_n} u^+ \xi_\varepsilon \frac{e\alpha_2}{|x|(\varepsilon + \mu_2)} \leq n\alpha_2 \int_{\Gamma_n} u^+ \xi_\varepsilon,
\end{equation}
which yields

\[(1.24) \quad \lim_{n \to \infty} \int_{\Gamma_n} \frac{u^+}{(e + \mu_2)^2} \nabla \mu_2 = 0.\]

For the last right-hand side term of (1.18) we have

\[(1.25) \quad -ce \int \frac{u^+ \mu_2}{|x|^2(e + \mu_2)^2} \Phi \eta_n \leq -cn^2 \int_{\Gamma_n} u^+ \xi.\]

Using (1.23), (1.24) and the facts that \(V\) is negative and \(\text{sign}^+ (u) g(u)\) is bounded below by some constant imply

\[(1.26) \quad \int \Phi (-Vu^+ + \text{sign}^+ (u) g(u)) \leq K\]

for some \(K > 0\). In the same way

\[(1.27) \quad \int \Phi (-Vu^- + \text{sign}^- (u) g(u)) \leq K.\]

Henceforth \(Vu\) and \(g(u)\) are locally integrable in \(\Omega\).

**Case 2.** \(q > N/(N - 2)\). From Lemma 1.1 and (0.12) \(uV\) is locally integrable in \(\Omega\). Taking \(\eta_n \Phi\) as a test function in (1.16) implies

\[(1.28) \quad \int u^+ (-\Delta (\Phi \eta_n) - V\Phi \eta_n) + \int \text{sign}^+ (u) g(u) \Phi \eta_n \leq 0.\]

Using (1.19) yields

\[\left| \int u^+ \Delta (\Phi \eta_n) \right| \leq K_1 \int_{\Gamma_n} u^+ + n^2 K_2 \int_{\Gamma_n} u^+ \leq K_1 \int_{\Gamma_n} u^+ + K_2 n^2 (-N + 2/(q - 1)) \leq K^2.\]

Letting \(n\) tend to infinity implies

\[\int \text{sign}^+ (u) g(u) \Phi < \infty.\]

In the same way

\[\int \text{sign}^- (u) g(u) \Phi < \infty\]

which ends the proof.

**Proof of Theorem 1.1.** **Case 1.** \(1 < q \leq N/(N - 2)\).
As a test function we take $\eta_n \zeta, \phi$ where $\eta_n$ and $\zeta$ are as in Lemma 1.3 and $\phi \in C^0_0(\Omega)$. We have

\begin{equation}
(1.29) \quad \int u(-\Delta(\eta_n \phi \zeta)) - \int V u \eta_n \phi \zeta + \int g(u) \eta_n \phi \zeta = 0
\end{equation}

and

\[ \Delta(\eta_n \phi \zeta) = \eta_n \phi \Delta \zeta + \eta_n \zeta \Delta \phi + \phi \zeta \Delta \eta_n + 2 \eta_n \nabla \phi \nabla \zeta + 2 \phi \nabla \eta_n \nabla \zeta + 2 \zeta \nabla \phi \nabla \eta_n. \]

As in Lemma 1.3, Case 1 it is easy to let $n$ tend to infinity and obtain, for $\epsilon > 0$ fixed,

\begin{equation}
(1.30) \quad \int u(-\phi \Delta \zeta - \zeta \Delta \phi + 2 \nabla \phi \nabla \zeta) + \int (g(u) - Vu) \phi \zeta = 0.
\end{equation}

But

\[ |u \nabla \phi \nabla \zeta| \leq |\nabla \phi| |u| \frac{\epsilon \alpha_2 \beta_2}{|x|(\epsilon + \mu_2)^2} \leq \frac{\alpha_2}{2|x|} |\nabla \phi| |u|, \]

\[ |u \phi \Delta \zeta| \leq |c| |u| |\phi| \frac{\epsilon \mu_2}{|x|^2(\epsilon + \mu_2)^2} + 2 \alpha_2^2 |u| |\phi| \frac{\epsilon \mu_2^2}{|x|^2(\epsilon + \mu_2)^2} \]

\[ \leq \frac{|c|}{2} |\phi| \frac{|u|}{|x|^2} + \frac{8 \alpha_2^2}{27} |\phi| \frac{|u|}{|x|^2}. \]

If $c = 0$ the two terms $|u \nabla \phi \nabla \zeta|$ and $|u \phi \Delta \zeta|$ vanish; if $c < 0$ we know from Lemma 1.3 that $u/|x|^2$ is locally integrable in $\Omega$. Henceforth, from Lebesgue’s theorem, we get

\begin{equation}
(1.31) \quad \lim_{\epsilon \to 0} \int u(-\phi \Delta \zeta - \zeta \Delta \phi + 2 \nabla \phi \nabla \zeta) = \int (-u \Delta \phi)
\end{equation}

and

\begin{equation}
(1.32) \quad \int (-u \Delta \phi) - \int Vu \phi + \int g(u) \phi = 0.
\end{equation}

**Case 2.** $q > N/(N - 2)$.

As a test function we just take $\phi \eta_n$ and we have from Lemma 1.3 and Hölder’s inequality

\begin{equation}
(1.33) \quad \lim_{n \to \infty} \int -u \Delta(\phi \eta_n) - Vu \phi \eta_n + g(u) \phi \eta_n = \int -u \Delta \phi - Vu \phi + g(u) \phi,
\end{equation}

which ends the proof.
Lemma 1.4. Assume $\Omega$ is as above and $V$ is continuous in $\Omega^*$ and satisfies

\begin{equation}
-\infty < |x|^2 V(x) \leq c \leq \left( \frac{N-2}{2} \right)^2
\end{equation}

near 0. If $w \in C^0(\Omega^*)$ is a nonnegative function satisfying

\begin{equation}
\Delta w + Vw \geq 0
\end{equation}

in $D'(\Omega^*)$ and $w = o(\mu_1)$ near 0, then $u/\mu_2$ remains locally bounded in $\Omega$.

**Proof.** Let $M$ be the supremum of $w$ on $\{x: |x| = 1\}$ and, for $\epsilon > 0$, $\Phi_\epsilon = M\mu_2 + \epsilon\mu_1$. We write (1.35) in spherical coordinates and get

\begin{equation}
w_{rr} + \frac{N-1}{r^2} w_r + \frac{c}{r^2} w + \frac{1}{r^2} \Delta_{S^{N-1}} w \geq 0
\end{equation}

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on $S^{N-1}$ (we have used (1.35)). We shall distinguish two cases:

**Case 1.** $c < \left( \frac{N-2}{2} \right)^2$. We write

\begin{equation}
u(s, \alpha) = w(r, \alpha)/\mu_1(r), \quad s = r^\beta, \quad \beta = \sqrt{(N-2)^2 - 4c}
\end{equation}

and get

\begin{equation}s^2 \nu_{ss} + \frac{1}{\beta^2} \Delta_{S^{N-1}} \nu \geq 0.
\end{equation}

We write $\phi_\epsilon(s, \alpha) = \Phi_\epsilon(r, \alpha)/\mu_1(r), \quad t = \ln(1/s)$ and $\psi(t, \alpha) = (v - \phi_\epsilon)(s, \alpha)$. The following relation holds in $D'(\mathbb{R}_+^\times \times S^{N-1})$

\begin{equation}\psi_{tt} + \psi_t + \frac{1}{\beta^2} \Delta_{S^{N-1}} \psi \geq 0.
\end{equation}

By convolution on $t$ we may assume that $\psi \in C^\omega(\mathbb{R}_+^\times \times C^0(S^{N-1}))$ and if we approximate $\psi$ by the solution $\chi_\eta$ ($\eta > 0$) of

\begin{equation}-\eta \Delta_{S^{N-1}} \chi_\eta + \chi_\eta = \psi,
\end{equation}

which converges to $\psi$ in $L^2(S^{N-1})$ as $\eta$ tends to 0, we deduce that

\begin{equation}\frac{d^2}{dt^2} \| \psi^+(t, \cdot) \|_{L^2(S^{N-1})} + \frac{d}{dt} \| \psi^+(t, \cdot) \|_{L^2} \geq 0,
\end{equation}

where $\psi^+ = \max(\psi, 0)$.
which implies that the function $s \mapsto \|v - \phi_s\|^2_{L^2(\mathbb{R}^{N-1})}$ is convex on $(0, 1)$. As it vanishes at 0 and 1, it is always 0. Letting $\epsilon$ tend to 0 implies the claim.

**Case 2.** $c = \left(\frac{N - 2}{2}\right)^2$. We just write

\begin{equation}
    v(t, \sigma) = w(r, \sigma)/r^{-(N - 2)/2} \quad t = \ln(1/r)
\end{equation}

and $v$ satisfies

\begin{equation}
    v_{tt} + \Delta_{S^{N-1}} v \geq 0
\end{equation}

in $D'(\mathbb{R}^+ \times S^{N-1})$. By the same approximation we see that the convexity and the fact that $v = o(t)$ at infinity imply the estimate $w \leq K\mu_2$.

**Proof of Theorem 1.2.** Case 1. We assume that

\begin{equation}
    c + \frac{2}{q - 1} \left(\frac{2q}{q - 1} - N\right) < 0.
\end{equation}

In that case $2/(q - 1) < (N - 2 + \beta)/2$ and $u = o(\mu_1)$ near 0. As we have

\begin{equation}
    \Delta u^+ + \frac{c}{|x|^2} u^+ \geq \text{sign}^+(u) g(u) \geq 0
\end{equation}

in $D'(\Omega^*)$, we deduce $u^+ \leq K\mu_2$. We do the same with $u^-$.

**Case 2.** We assume that $c \leq 0$.

In that case $u^+$ satisfies

\begin{equation}
    \Delta u^+ \geq \text{sign}^+(u) g(u) \geq a(u^+)^q - b.
\end{equation}

From Brézis-Veron's result [11] $u^+$ is locally bounded in $\Omega$; henceforth $u^+ \leq K\mu_2$. The same with $u^-$.

**Case 3.** We assume

\begin{equation}
    0 < c \leq \left(\frac{N - 2}{2}\right)^2,
\end{equation}

\begin{equation}
    c + \frac{2}{q - 1} \left(\frac{2q}{q - 1} - N\right) = 0.
\end{equation}
For $n \in \mathbb{N}^*$ let $\phi_n$ be the solution of

\begin{align*}
\begin{cases}
\Delta \phi_n + \frac{c}{|x|^2} \phi_n - a \phi_n^q = 0 & \text{in } B_1(0) \setminus B_{1/n}(0) \\
\phi_n = \max \{ u^+(x): |x| = 1 \} & \text{on } \partial B_1(0) \\
\phi_n = \max \{ u^+(x): |x| = 1/n \} & \text{on } \partial B_{1/n}(0)
\end{cases}
\end{align*}

(1.49)

($\phi_n$ exists by minimization techniques and it is positive and unique) where $a$ is defined as in (1.46) or Lemma 1.1. Let $\sigma$ be $b/(c + 2N)$. Then $\psi_n = \phi_n + \sigma |x|^2$ satisfies

$$
\Delta \psi_n + \frac{c}{|x|^2} \psi_n - a \psi_n^q \leq b.
$$

(1.50)

We then deduce, as in Lemma 1.4, that $u^+ \leq \psi_n$ in $B_1(0) \setminus B_{1/n}(0)$. As $\phi_n$ remains locally bounded in $B_1(0) \setminus B_{1/n}(0)$, independently of $n$ (Lemma 1.1), we deduce that (up to a subsequence) it converges in the $C^1_{\text{loc}}(\bar{B}_1(0) \setminus \{0\})$-topology to a function $\phi$ which is radial and satisfies

$$
\begin{align*}
\Delta \phi + \frac{c}{|x|^2} \phi - a \phi^q = 0 & \quad \text{in } B_1(0) \setminus \{0\}, \\
\phi = \max \{ u^+(x): |x| = 1 \} & \quad \text{on } \partial B_1(0),
\end{align*}
$$

(1.51)

and

$$
u(x) \leq \phi(x) + \sigma |x|^2
$$

in $\bar{B}_1(0) \setminus \{0\}$. Moreover, in the range (1.48), we have

$$
\phi(x) \leq c \mu_1(x) = c |x|^\alpha_1 = c |x|^{-2/(q-1)}.
$$

(1.53)

If we set $\phi(x) = \tilde{\phi}(r)$ and

$$
\eta(t) = r^{2/(q-1)} \tilde{\phi}(r), \quad t = \text{Ln} \left(1/r\right)
$$

(1.54)

then we get

$$
\eta_{tt} - \left( N - 2 - \frac{q+1}{q-1} \right) \eta_t - a \eta^q = 0
$$

(1.55)

in $(0, +\infty)$.

(i) If $q = (N+2)/(N-2)$ the first order coefficient is 0 and

$$
W(\eta, \eta_t) = \frac{1}{2} \eta_t^2 - \frac{a}{q+1} \eta^{q+1}
$$

(1.56)
is constant. As \( \eta \) is nonnegative and bounded, the only admissible constant is 0 and \( \eta(t) \) tends to 0 as \( t \) tends to infinity.

(ii) If \( q \neq (N+2)/(N-2) \) then

\[
\frac{d}{dt} W(\eta, \eta) = \left( N-2 \frac{q+1}{q-1} \right) \eta_i^2.
\]

From La Salle invariance principle \( \lim_{t \to \infty} \eta(t) = 0 \).

Henceforth

\[
\lim_{x \to 0} u^+(x)/\mu_1(x) = 0.
\]

As the same holds for \( u^- \) we deduce the claim from Lemma 1.4.

Remark 1.1. Using Theorems 2.1 and 3.1 it is possible to extend Theorem 1.2 to the case where \( g \) satisfies

\[
\begin{cases}
\liminf_{r \to \infty} \frac{g(r)}{r} \ln r > 0 \\
\limsup_{r \to -\infty} \frac{g(r)}{|r|^q} \ln r < 0
\end{cases}
\]

and (0.17) (see [24] for the zero potential case and [16]).

2. The isotropy Theorems

In this section \( \Omega \) is an open subset of \( \mathbb{R}^N \) containing \( \bar{B}_1(0) \), \( \Omega^* = \Omega \setminus \{0\} \), \( g \) is a continuous nondecreasing real valued function and \( V \in C(\Omega^*) \) is a radial potential such that

\[
-\infty \leq |x|^2 V(x) \leq c \leq \left( \frac{N-2}{2} \right)^2 , \quad \text{for all} \quad x \in \Omega^*.
\]

We are interested in solutions \( u \in C^1(\Omega^*) \) of

\[
-\Delta u - Vu + g(u) = 0.
\]

Lemma 2.1. Assume \( u \in C^1(\Omega^*) \) satisfies (2.2) in \( D'(\Omega^*) \) and

\[
\lim_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \|u(r, \cdot) - \bar{u}(r)\|_{L^\infty(\mathbb{R}^{N-1})} = 0.
\]
Then there exists a constant $K \geq 0$ such that

$$
\| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} \leq Kn^{2-N+\sqrt{N^2-4c}}/2
$$

for $0 < r \leq 1$.

PROOF. Case 1. $c < \left( \frac{N-2}{2} \right)^2$. In radial coordinates we have

$$
u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_{S^{N-1}} u = g(u).
$$

We write $u$ as in (1.37), that is

$$v(r, \vartheta) = u(r, \vartheta)/\mu_1(r), \quad s = r^\beta, \quad \beta = \sqrt{(N-2)^2 - 4c},$$

and get

$$s^2 v_{ss} + \frac{1}{\beta^2} \Delta_{S^{N-1}} v + \frac{1}{\beta^2} (s^{1-\beta} V(s^{\beta}) - c) v = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} g(s^{\alpha_1/\beta} v)$$

with $\alpha_1 = (2-N-\beta)/2$.

Let $\bar{v}(s)$ be the spherical average of a function $\rho(s, \vartheta)$; then

$$s^2 \bar{v}_{ss} + \frac{1}{\beta^2} (V(s^{1-\beta}) - c) \bar{v} = \frac{1}{\beta^2} s^{(2-\alpha_1)/\beta} \bar{g}(s^{\alpha_1/\beta} \bar{v}).$$

As

$$\int_{S^{N-1}} (-\Delta v - \bar{v}) \, d\sigma \geq (N-1) \int_{S^{N-1}} (v - \bar{v})^2 \, d\sigma,$$

$$\int_{S^{N-1}} (g(s^{\alpha_1/\beta} v) - \bar{g}(s^{\alpha_1/\beta} \bar{v}))(v - \bar{v}) \, d\sigma \geq 0,$$

and (2.1) we get

$$s^2 \int_{S^{N-1}} (v_{ss} - \bar{v}_{ss})(v - \bar{v}) \, d\sigma = \frac{N-1}{\beta^2} \int_{S^{N-1}} (v - \bar{v})^2 \, d\sigma \geq 0.$$

Setting

$$X(s) = \left( \int_{S^{N-1}} (v - \bar{v})^2(s) \, d\sigma \right)^{1/2},$$

we obtain

$$\int_{S^{N-1}} (v - \bar{v})^2 \, d\sigma \geq X(s)^2,$$
we obtain
\[
(2.9) \quad s^2 \mathcal{X}_{ss} - \frac{N - 1}{\beta^2} \mathcal{X} \geq 0
\]
in \(D'(0, 1)\). As \(X(s_n) = o(s_n^{(\beta - \sqrt{N^2 - 4c})/2\beta})\) for some sequence \(\{s_n\}\) converging to 0 we deduce that
\[
(2.10) \quad X(s) \leq Ks^{(\beta + \sqrt{N^2 - 4c})/2\beta},
\]
which is (2.4).

**Case 2.** \(c = \left(\frac{N - 2}{2}\right)^2\). We write \(u\) as in (1.42):
\[
(2.11) \quad w(t, \sigma) = t^{(N - 2)/2}u(r, \sigma), \quad t = \ln(1/r);
\]
then
\[
(2.12) \quad w_{tt} + \Delta_{S^{N-1}} w + (\overline{\varphi}(t) - c)w = e^{(Q - N)\varepsilon/2}g(e^{(N - 2)\mu/2}w)
\]
where
\[
\overline{\varphi}(t) = e^{(Q - N)\varepsilon/2}v(e^{(N - 2)\mu/2}).
\]
If we set
\[
X(t) = \left( \int_{S^{N-1}} (w - \overline{w})^2(t) \, ds \right)^{1/2},
\]
then
\[
(2.13) \quad X_{tt} - (N - 1)X \geq 0
\]
in \(D'(\mathbb{R}^+))\), and \(X(t_n) = o(e^{\varepsilon(N-1)tn})\) for some sequence \(\{t_n\}\) tending to \(\infty\). The maximum principle implies that
\[
(2.14) \quad X(t) \leq Ke^{-\varepsilon(N-1)t}
\]
which is (2.4).

In order to have a \(L^\infty\) estimate we need the following result the proof of which is essentially contained in [24].

**Lemma 2.2.** Assume \(\gamma, a\) and \(b\) are positive numbers such that \(a < b\) and \(\phi, \psi \in L^2(S^{N-1})\). Then there exists a unique function \(\Phi \in C([a, b]; L^2(S^{N-1}))\)
verification of category \( \cap C^\infty((a, b) \times S^{N-1}) \) such that

\[
\begin{aligned}
  s^2 \Phi_{ss} + \frac{1}{\gamma} \Delta_{S^{N-1}} \Phi &= 0 & \text{in} & \quad (a, b) \times S^{N-1}, \\
  \Phi(a, \cdot) &= \phi(\cdot) & & \text{in} \quad S^{N-1}, \\
  \Phi(b, \cdot) &= \psi(\cdot) & & \text{in} \quad S^{N-1}.
\end{aligned}
\tag{2.15}
\]

Moreover there exists a constant \( C_1 > 0 \) such that

\[
\begin{aligned}
  \| \Phi(s, \cdot) \|_{L^\infty(S^{N-1})} &\leq C_1 \left\{ \left( 1 + \frac{1}{\text{Ln}(s/a)} \right)^{(N-1)/2} \| \Phi \|_{L^2(S^{N-1})} \\
  &\quad + \left( 1 + \frac{1}{\text{Ln}(b/s)} \right)^{(N-1)/2} \| \psi \|_{L^2(S^{N-1})} \right\}.
\end{aligned}
\tag{2.16}
\]

This result is, up to change of variable and unknown, essentially an estimate concerning harmonic functions in an annulus.

**Lemma 2.3.** Assume the hypotheses of Lemma 2.1 hold with \( c < (N - 2)^2/4 \). Then the function \( v \) introduced in (2.6) satisfies

\[
\begin{aligned}
  \| v(s, \cdot) - \bar{v}(s) \|_{L^\infty(S^{N-1})} &\leq \bar{c} s^\left( \beta + \sqrt{N^2 - 4c} \right)/2
\end{aligned}
\tag{2.17}
\]

for some \( \bar{c} > 0 \) and any \( s \in (0, 1/2] \).

**Proof.** Let \( y \) be the solution of

\[
\begin{aligned}
  s^2 y_{ss} = &\frac{1}{\beta^2} s^{2-\alpha_1/\beta} g(s^{\alpha_1/\beta} y) & \text{in} & \quad (a, b), \\
  y(a) = &\rho, & y(b) = \tau,
\end{aligned}
\tag{2.18}
\]

with \( 0 < a < b < 1, \rho \) and \( \tau \) real numbers. Let \( w \) be \( v - y \), \( \phi \) be \( (v(a, \cdot) - \rho)^+ \), \( \psi \) be \( (v(b, \cdot) - \tau)^+ \) and \( \Phi \) be the solution of (2.15). Then \( \Phi \geq 0 \). If we define \( h \) as

\[
h = \frac{1}{\beta^2} s^{2-\alpha_1/\beta} (g(s^{\alpha_1/\beta} y) - g(s^{\alpha_1/\beta} \psi))/w,
\]

then \( h \geq 0 \) and

\[
s^2 w_{ss} + \frac{1}{\beta^2} \Delta_{S^{N-1}} w = hw.
\tag{2.19}
\]

Henceforth \( \Phi \) is a super-solution for (2.19) and \( \Phi \geq w \). Using (2.16) with \( \rho = \bar{v}(a), \tau = \bar{v}(b) \) we get
(2.20) \[ v(s, a) - v(s) \leq C_1 \left( 1 + \frac{1}{\ln(s/a)} \right)^{(N-1)/2} \| (v(a, \ast) - \overline{v}(a))^+ \|_{L^2(S^{N-1})} + \left( 1 + \frac{1}{\ln(b/s)} \right)^{(N-1)/2} \| (t(b, \ast) - \overline{v}(b))^+ \|_{L^2(S^{N-1})} \]

for any \( a \in S^{N-1} \) and any \( a < s < b \).

If we take \( \frac{s}{a} = \frac{b}{s} = 2 \) and use estimate (2.4) in the \( s \) variable, we get

(2.21) \[ v(s, a) - v(s) \leq C_2 s^{(\beta + \sqrt{N^2 - 4\epsilon})/2\beta} . \]

In the same way we have

(2.22) \[ v(s, a) - v(s, a) \leq C_2 s^{(\beta + \sqrt{N^2 - 4\epsilon})/2\beta} , \]

which implies the claim.

**Lemma 2.4.** Assume the hypotheses of Lemma 2.1 hold and \( c = (N - 2)^2/4 \).

Then the function \( w \) introduced in (2.12) satisfies

(2.23) \[ \| w(t, \ast) - \overline{w}(t) \|_{L^2(S^{N-1})} \leq \tilde{c} e^{-N^{-1/2} t} \]

for some \( \tilde{c} > 0 \) and any \( t > 0 \).

**Proof.** It is essentially the same as the one of Lemma 2.3 except that Lemma 2.2 is replaced by the following estimate: for \( a > 0 \) and \( \phi \in L^2(S^{N-1}) \) the unique bounded solution \( \Phi \) of

(2.24) \[
\begin{aligned}
\Phi_t + \Delta_S^{N-1} \Phi &= 0 & & \text{in } (a, +\infty) \times S^{N-1} , \\
\Phi(a, \ast) &= \phi(\ast) & & \text{on } S^{N-1} .
\end{aligned}
\]

satisfies

(2.25) \[ \| \Phi(t, \ast) \|_{L^2(S^{N-1})} \leq \tilde{c} \left( 1 + \frac{1}{t - a} \right)^{(N-1)/2} \| \phi \|_{L^2(S^{N-1})} . \]

This is essentially Poisson's formula.

**Proof of Theorem 2.1.** Case 1. \( c < \left( \frac{N - 2}{2} \right)^2 \).

The proof follows the ideas of [24] and we have to distinguish according \( \overline{v} \) is bounded or not near 0.
Local Properties of Stationary Solutions of Some Schrödinger Equations

Case 1.1. \( \bar{v} \) is bounded near 0. There exist a sequence \( \{s_n\} \) tending to 0 and some \( z \) such that \( \bar{v}(s_n) \) converges to \( z \) as \( n \to \infty \). Assuming \( z > 0 \), we write \( \bar{g}(r) = g(r) - g(0) \) and call \( \bar{v} \) the solution of

\[
\begin{align*}
(2.26) \quad & s^2 \bar{v}_{ss} + \frac{1}{\beta^2} (s^{2/\beta} V(s^{1/\beta}) - c) \bar{v} = \frac{1}{\beta^2} s^{(2-\alpha)/\beta} \bar{g}(s^{\alpha/\beta} \bar{v}) & \quad \text{on} \quad (s_n, s_{n_0}), \\
& \bar{v}(s_n) = \bar{v}(s_{n_0}) = z/2,
\end{align*}
\]

where \( n_0 \) is such that \( v(s_n, \sigma) > z/2 \) for all \( n \geq n_0 \), and \( \sigma \in S^{N-1} \). It is clear that \( \bar{v} \geq 0 \). If \( \Lambda \) is the solution of

\[
(2.27) \quad \begin{cases}
s^2 \Lambda_{ss} + \frac{1}{\beta^2} (s^{2/\beta} V(s^{1/\beta}) - c) \Lambda + \frac{1}{\beta^2} s^{(2-\alpha)/\beta} |g(0)| = 0 & \quad \text{on} \quad (s_n, s_{n_0}), \\
\Lambda(s_n) = \Lambda(s_{n_0}) = 0,
\end{cases}
\]

then \( \Lambda \geq 0 \) and \( \Lambda(s) \leq Ks \) for some constant independent of \( n \). If we set \( v^* = \bar{v} - \Lambda \), then \( v^* \) is a sub-solution for (2.7) which implies

\[
(2.28) \quad \forall (s, \sigma) \in [s_n, s_{n_0}] \times S^{N-1},
\]

and \( v_k(s, \sigma) = v(s, \sigma) + Ks \) is nonnegative in \( (0, s_{n_0}] \times S^{N-1} \). As the spherical average \( \bar{v}_K \) of \( v_K \) satisfies

\[
(2.29) \quad s^2 (\bar{v}_K)_{ss} + \frac{1}{\beta^2} (s^{2/\beta} V(s^{1/\beta}) - c) \bar{v}_K \geq \frac{1}{\beta^2} s^{(2-\alpha)/\beta} g(-K) + \frac{K}{\beta^2} s^{(2/\beta) V(s^{1/\beta}) - c)s},
\]

there exist two constants \( M \) and \( N \geq 0 \) such that the function \( E(s) = \bar{v}_K(s) + M s^{(2-\alpha)/\beta} + N(s \ln s - s) \) is convex. As \( E(s_n) \) tends to \( z \) we deduce \( \lim_{s \to 0} E(s) = z \), which yields

\[
(2.30) \quad \lim_{s \to 0} \bar{v}(s) = z = \lim_{s \to 0} v(s, \bullet),
\]

uniformly on \( S^{N-1} \).

If \( z < 0 \) we proceed similarly. If \( z = 0 \), then it is clear by using the technique above that

\[
(2.31) \quad \lim_{s \to 0} \bar{v}(s) = 0 = \lim_{s \to 0} \|v(s, \bullet)\|_{L^\infty(S^{N-1})},
\]

Case 1.2. \( \bar{v} \) is unbounded near 0. Then there exists a sequence \( \{s_n\} \) tending
to 0 such that \( \lim_{t \to 0} \vartheta(s, t) = \infty \) (\(-\infty\) in the same way). We conclude by the same convexity argument as in Case 1.1 that

\[
(2.32) \quad \lim_{s \to 0} \nu(s, \sigma) = \infty
\]

uniformly on \( S^{N-1} \).

**Case 2.** \( c = \left( \frac{N - 2}{2} \right)^2 \). We essentially follow the ideas of Case 1 but use the \( t \) variable \((t > 0)\) and Lemma 2.4. If \( \bar{w} \) is bounded in \( \mathbb{R}^+ \), then

\[
(2.33) \quad \lim_{|x| \to 0} u(x)/\mu_1(x) = 0.
\]

If \( \bar{w} \) is not bounded we deduce from convexity arguments that

\[
(2.34) \quad \text{either } \lim_{t \to -\infty} w(t, \bullet) = +\infty, \quad \text{or } \lim_{t \to +\infty} w(t, \bullet) = -\infty,
\]

uniformly on \( S^{N-1} \). Assuming the first case we also have from convexity the fact that \( \vartheta(t)/t \) admits a limit in \( \mathbb{R}^+ \cup \{ +\infty \} \). This limit is the same as the one of \( u(x)/\mu_1(x) \) as \( x \) tends to zero and this ends the proof.

**Remark 2.1.** It is interesting to notice that (2.3) is automatically satisfied as soon as \( g \) has a fast enough growth, that is

\[
(2.35) \quad \left\{ \begin{array}{l}
\liminf_{r \to \infty} g(r)/r^q = \infty, \\
\limsup_{r \to -\infty} g(r)/(-r)^q = -\infty,
\end{array} \right.
\]

for some \( q > 1 \) such that

\[
(2.36) \quad c + \frac{q + 1}{q - 1} \left( \frac{q + 1}{q - 1} - N \right) = 0.
\]

In that case we have

\[
\frac{2}{q - 1} = \frac{N - 2 + \sqrt{N^2 - 4c}}{2}
\]

and

\[
u(x) = o(|x|^{-2/(q - 1)})
\]

from (2.35) and Lemma 1.1. In the zero potential case the limit exponent \( q \) is \((N + 1)/(N - 1)\).
As we have proved Theorem 2.1 we can prove a similar result for the solution of (2.2) in an exterior domain $G$

**Theorem 2.2.** Assume $G \supset \{ x \in \mathbb{R}^N : |x| \geq 1 \}$ and $u \in C^1(G)$ satisfies (2.2) in $G$ where $V$ is a radial potential defined in $G$ and satisfying

$$-\infty < |x|^2 V(x) \leq c \leq \left( \frac{N-2}{2} \right)^2, \text{ for every } x \in G.$$  

If $g(0) = 0$ and $u$ satisfies

$$\liminf_{r \to \infty} r^{(N-2-\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0,$$

then $u(x)/\mu_2(x)$ admits a limit in $\mathbb{R} \cup \{ \infty, -\infty \}$ as $|x|$ tends to infinity. Moreover, if $\lim_{|x| \to \infty} u(x)/\mu_2(x) = 0$, there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{|x| \to \infty} u(x)/\mu_1(x) = \gamma.$$  

The zero potential case of this result can be found in [31]. We can apply this type of methods to symmetry problems as in [28].

**Corollary 2.1.** Assume $V$ is a radial potential defined in $\mathbb{R}^N \setminus \{ 0 \}$ and satisfying

$$-\infty < |x|^2 V(x) \leq c \leq \frac{N^2}{4}, \text{ for every } x \in \mathbb{R}^N \setminus \{ 0 \},$$

and $g$ is a nondecreasing real valued function. If $u \in C^1(\mathbb{R}^N \setminus \{ 0 \})$ satisfies

$$\liminf_{r \to 0} r^{(N-2+\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0,$$

and

$$\liminf_{r \to \infty} r^{(N-2-\sqrt{N^2-4c})/2} \| u(r, \cdot) - \bar{u}(r) \|_{L^2(S^{N-1})} = 0;$$

then $u$ is a radial function.

It is important to notice that the hypothesis on $V$ is weaker as the proof essentially deals with the study of the following differential inequality

$$X_{rr} + \frac{N-1}{r} X_r + \left( V - \frac{N-1}{r^2} \right) X \geq 0.$$

Other symmetry results for Schrödinger operator with singular radial potentials can be found in [28], [5].
3. Solution with Linear Singularities

We assume that Ω and Ω* are as in Section 1 and g is a continuous nondecreasing real valued function.

Proof of Theorem 3.1. We recall that we assume $c < (N-2)^2/4$.

Step 1. Suppose (0.24) is satisfied, that is

$$\int_{1}^{\infty} (g(t) + |g(-t)|) t^{2(1-\alpha)/\alpha_1} dt < \infty,$$

then we claim that for any $\gamma \in \mathbb{R}$ there exists $u \in C^{1}(\Omega^*)$ satisfying

$$-\Delta u - \frac{c}{|x|^2} u + g(u) = 0$$

in $\Omega^*$ and

$$\lim_{x \to 0} u(x)/\mu_1(x) = \gamma.$$

We take $\gamma \geq 0$ and for $\epsilon > 0$ let $y_\epsilon$ be the solution of

$$\begin{cases}
    s^2(y_\epsilon)_{ss} = \frac{1}{\beta^2} (s + \epsilon) (s - \alpha_1/\beta^2) g((s + \epsilon)^{\alpha_1/\beta} y_\epsilon) & \text{in } (0, 1), \\
y_\epsilon(0) = \gamma, & y_\epsilon(1) = 0.
\end{cases}$$

In order to avoid technical difficulties we suppose $g(0) = 0$. Henceforth $y_\epsilon$ is positive, convex, nondecreasing and

$$y_\epsilon(s) \leq \gamma + s(1 - \gamma), \quad \text{for all } 0 < s < 1.$$

From (3.4) we get

$$\begin{align*}
    (y_\epsilon)_s(s) &= (y_\epsilon)_s(1) - \frac{1}{\beta^2} \int_{s}^{1} (\tau + \epsilon) (\ tau + \epsilon)^{(2-\alpha_1)/\beta} \ g((\ tau + \epsilon)^{\alpha_1/\beta} y_\epsilon) \ d\tau
\end{align*}$$

for $0 < s < 1$, and $(y_\epsilon)_s(1)$ is bounded from (3.4)-(3.5). From (3.6) we deduce

$$\begin{align*}
    |y_\epsilon(s_1) - y_\epsilon(s_2)| \leq a(s_2 - s_1) + \frac{1}{\beta^2} \int_{s_1}^{s_2} \int_{s}^{1} (\tau + \epsilon) (\ tau + \epsilon)^{(2-\alpha_1)/\beta} \ g((\ tau + \epsilon)^{\alpha_1/\beta} y_\epsilon) \ d\tau
\end{align*}$$

for some constant $a > 0$ and $0 < s_1 < s_2 \leq 1$. But
\[(3.8) \quad \int_{s_1}^{s_2} \int_s^1 (\tau + \epsilon)^{q(2-\alpha_1/\beta-2/\gamma)} d\tau \, ds \leq \int_{s_1}^{s_2+\epsilon} \int_s^{s_2} t^{q(2-\alpha_1/\beta-2/\gamma)} \, dt \, ds\]

(we assume \(\epsilon < 1\)). Set
\[
\phi(x) = \int_s^{s_2} \int_s^{s_2+\epsilon} t^{q(2-\alpha_1/\beta-2/\gamma)} \, dt \, ds,
\]
then
\[(3.9) \quad \lim_{x \to 0} \phi(x) = \int_{s_1}^{s_2} \int_s^{s_2+\epsilon} t^{q(2-\alpha_1/\beta-2/\gamma)} \, dt \, ds \]
\[= \int_{\gamma^{2/\alpha_1/\beta}}^{\infty} t^{2(1-\alpha_1/\alpha_1)} g(t) \, dt < \infty\]
from hypothesis \((l = l(\alpha_1, \beta) > 0)\). Henceforth \(\phi\) is extendable to \([0, 2]\) as a uniformly continuous function \(\tilde{\phi}\) and (3.7) reads as
\[(3.10) \quad |\gamma(s_1) - \gamma(s_2)| \leq a|s_2 - s_1| + \frac{1}{\beta^2} (\tilde{\phi}(s_2 + \epsilon) - \tilde{\phi}(s_1 + \epsilon))
\]
which implies the equicontinuity of \(\{\gamma_t\}_{0 < t < 1}\) in \(C([0, 1])\) and the existence of a \(\gamma \in C([0, 1])\) satisfying
\[(3.11) \quad \begin{cases} \frac{s^2}{\gamma_0} s^2 = \frac{1}{\beta^2} s^{q(2-\alpha_1/\beta)} g(s^{\alpha_1/\gamma}) \quad \text{in} \quad (0, 1), \\ \gamma(0) = \gamma, \quad \gamma(1) = 0. \end{cases}
\]
The function \(u_r(x) = |x|^\alpha \gamma(|x|^\beta)\) is a solution of (3.2) satisfying (3.3).

Step 2. We assume that there exists \(\gamma > 0\) such that (3.3) holds for some \(u \in C^1(\Omega^*)\) and that
\[(3.11) \quad \int_1^\infty t^{2(1-\alpha_1/\alpha_1)} g(t) \, dt = \infty.
\]
As \(\lim_{r \to 0} \bar{u}(r)/\mu_1 = \gamma\) and
\[(3.12) \quad \bar{u}_{rr} + \frac{N-1}{r} \bar{u}_r + \frac{c}{r^2} \bar{u} = g(\bar{u}(r)) \quad \text{in} \quad (0, 1)
\]
we deduce, from the monotonicity of $g$, that

\begin{equation}
\bar{u}_{rr} + \frac{N-1}{r}\bar{u}_r + \frac{c}{r^2}\bar{u} \geq g(\gamma r^{\alpha_1}/2).
\end{equation}

Defining $\psi(s)$ by $\bar{u}(r)/r^{\alpha_1}$ with $s = r^\beta$, then

\begin{equation}
s^2 \psi_{ss} \geq \frac{1}{\beta^2} s^{(\alpha_1 - \alpha_1)/\beta} g(\gamma s^{\alpha_1}/\beta)/2).
\end{equation}

Integrating (3.14) twice yields

\begin{equation}
\psi(s) \geq \psi(1) + \psi_1(1) (s - 1) - \frac{1}{\beta^2} \int_s^1 \int_t^1 \sigma^{(\alpha_1 - \alpha_1)/\beta - 2} g(\gamma \sigma^{\alpha_1}/\beta) \, d\sigma \, dt.
\end{equation}

As

\[ \lim_{s \to 0} \int_s^1 \int_t^1 \sigma^{(\alpha_1 - \alpha_1)/\beta - 2} g(\gamma \sigma^{\alpha_1}/\beta) \, d\sigma \, dt = \infty, \]

we derive

\begin{equation}
\lim_{s \to 0} \psi(s) = \lim_{r \to 0} \bar{u}(r)/r^{\alpha_1} = \infty,
\end{equation}

a contradiction.

**Remark 3.1.** With the above techniques it is easy to show that if $\Omega$ is bounded with a regular boundary $\partial \Omega$, for any $\phi \in C(\partial \Omega)$ and any $\gamma \in \mathbb{R}$ there exists a unique $u_\gamma \in C(\overline{\Omega \setminus \{0\}}) \cap C^1(\Omega^*)$ satisfying

\begin{equation}
\begin{cases}
-\Delta u_\gamma - \frac{c}{|x|^2} u_\gamma + g(u_\gamma) = 0 & \text{in } \Omega^*, \\
u_\gamma = \phi & \text{on } \partial \Omega, \\
\lim_{x \to 0} u_\gamma(x)/\mu_1(x) = \gamma.
\end{cases}
\end{equation}

**Proof of Theorem 3.2.** Here we assume that $c = (N - 2)^2/4$. We recall the definition of $b^+_g$, $b^-_g$:

\[ b^+_g = \inf \left\{ b > 0 : \int_0^1 g(t^{-(N-2)/(N+2)}, \ln (1/t)/b) \, dt < \infty \right\}, \]

\[ b^-_g = \inf \left\{ b > 0 : \int_0^1 g(t^{-(N-2)/(N+2)}, \ln t/b) \, dt > -\infty \right\}. \]
Step 1. Existence result. It is clear that if \([- (N + 2)/2 b_\varepsilon^-, (N + 2)/2 b_\varepsilon^+\] = \(\{0\}\) there exists \(u\) satisfying the equation (0.27) with a zero limit in (0.28); so we shall assume \(b_\varepsilon^+ < \infty\), and consider any \(\gamma \in (0, (N + 2)/2 b_\varepsilon^-]\).

Case 1. \(\gamma < (N + 2)/2 b_\varepsilon^+\). For \(\varepsilon > 0\) let \(y_\varepsilon\) be the solution of

\[
(y_\varepsilon)_s = (s + \varepsilon)^{-3} \exp \left( - \frac{N + 2}{2(s + \varepsilon)} \right) \exp \left( \frac{N - 2}{2(s + \varepsilon)} \right) \text{ on } [0, 1],
\]

\[
y_\varepsilon(0) = \gamma, \quad y_\varepsilon(1) = 0.
\]

We assume again that \(g(0) = 0\); \(y_\varepsilon\) is decreasing, positive and convex; therefore

\[
|y_\varepsilon(s_1) - y_\varepsilon(s_2)| \leq a|s_1 - s_2| + \int_{s_1 + \varepsilon}^{s_2 + \varepsilon} \int_s^2 t^{-3} \exp\left(-\frac{N + 2}{2t^2}\right) g(\gamma e^{-\frac{(N - 2)}{2t} t^{-1}}) \, dt \, ds,
\]

for \(0 < s_1 < s_2 < 1\) \((\varepsilon < 1)\). Let \(\phi\) be defined by

\[
\phi(x) = \int_0^x \int_s^2 t^{-3} \exp\left(-\frac{N + 2}{2t^2}\right) g(\gamma e^{-\frac{(N - 2)}{2t} t^{-1}}) \, dt \, ds,
\]

then

\[
\lim_{x \to 0} \phi(x) = \int_0^{-\frac{(N + 2)/4}{(2\gamma/(N + 2)) x^{-\frac{(N - 2)}{2(\gamma - 1)}(N + 2) \ln(1/x))}} dt < \infty.
\]

As in the proof of Theorem 3.1, \(\{y_\varepsilon\}\) is equicontinuous in \([0, 1]\) and there exists \(y \in C([0, 1])\) such that

\[
y_\varepsilon = s^{-3} \exp\left(-\frac{N + 2}{2s}\right) g(\gamma e^{-\frac{(N - 2)}{2s} s^{-1}}) \text{ on } (0, 1),
\]

\[
y(0) = \gamma, \quad y(1) = 0.
\]

If we set

\[
u_\varepsilon(x) = |x|^{-\frac{(N - 2)}{2} \ln(1/|x|)} g(-1/\ln |x|),
\]

then

\[
-\Delta u_\varepsilon - \left(\frac{N - 2}{2|x|}\right)^2 u_\varepsilon + g(u_\varepsilon) = 0 \text{ in } B_{\varepsilon^{-1}}(0) \setminus \{0\},
\]

\[
u_\varepsilon(x) = 0 \text{ on } \partial B_{\varepsilon^{-1}}(0),
\]

\[
\lim_{x \to 0} u_\varepsilon(x)/|x|^\gamma = \gamma.
\]
Case 2. \( \gamma = (N + 2)/2b_s^+ \). Let \( y_n \) be the solution of

\[
\begin{cases}
(y_n)_{ss} = s^{-3} \exp\left(-\frac{(N + 2)}{2s}\right)g(y_ne^{-(N-2)/2s}) & \text{on } (0, 1), \\
y_n(0) = (N + 2)/(2b_s^+) - \frac{1}{n}, & y_n(1) = 0.
\end{cases}
\]

(3.25)

The function \( y_n \) is again decreasing, positive and convex and the sequence \( \{y_n\} \) is increasing and bounded. From Dini’s Theorem it is uniformly convergent on \( (0, 1) \) and its limit \( y \) is continuous on \([0, 1] \) and satisfies (3.23) with \( \gamma = (N + 2)/2b_s^+ \).

Step 2. Assume there exists \( \gamma \) and a solution \( u \) of (0.27) such that

\[
\lim_{x \to 0} \frac{u(x)}{u'(x)} = \gamma
\]

and we assume for example that

\[
\gamma > (N + 2)/2b_s^+.
\]

(3.26)

(we proceed similarly if \( \gamma < -(N + 2)/2b_s^- \)). We define

\[
u(r, \sigma) = \frac{1}{t} \exp\left((N-2)/2t\right)u(t, \sigma), \quad t = -1/\ln r,
\]

(3.27)
and \( v \) satisfies

\[
\begin{cases}
v_{ss} + \Delta_{S^{N-1}}v = s^{-3} \exp\left(-\frac{(N + 2)}{2s}\right)g(v) & \text{in } \Omega^*, \\
\lim_{s \to 0} v(s, \sigma) = \gamma & \text{uniformly on } S^{N-1}.
\end{cases}
\]

(3.28)
We consider \( \epsilon_0 \in (0, \gamma - (N + 2)/2b_s^+ \) and set \( \lambda = \gamma\epsilon_0 > (N + 2)/2b_s^+ \). For \( s \) small enough we have \( v(s, \sigma) \geq \lambda \) (for every \( \sigma \in S^{N-1} \)) and it is the same with the spherical average \( \bar{v}(s) \). Therefore

\[
\bar{v}_{ss} \geq s^{-3} \exp\left(-\frac{(N + 2)}{2s}\right)g(\lambda s^{-1} \exp((N-2)/2s))
\]

(3.29)

Integrating (3.29) twice as in step 1 and using the definition of \( b_s^+ \) implies

\[
\lim_{s \to 0} \bar{v}(s) = +\infty,
\]

a contradiction.

Remark 3.2. The Dirichlet problem is also solvable in the case \( c = (N - 2)^2/4 \).
Remark 3.3. If \( g(r) \) behaves like \( r^q \) \((q > 1)\) at infinity (and \( -|r|^q \) at \(-\infty)\), (3.1) means that

\[
0 < -\alpha_1 < \frac{2}{q - 1} \quad \text{or} \quad 1 < q < \frac{N + 2 + \beta}{N - 2 + \beta}
\]

and

\[
c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right) < 0.
\]

In the critical case \( c = \left( \frac{N - 2}{2} \right)^2 \), the role of the Sobolev exponent \( \frac{N + 2}{N - 2} \) is enlightened:

\[
b_*^+ \quad b_*^- = \begin{cases} 
\infty & \text{if } q \geq (N + 2)/(N - 2), \\
0 & \text{if } 1 < q < (N + 2)/(N - 2).
\end{cases}
\]

Remark 3.4. Let \( u_r \) be the solution on (3.17), for any \( \gamma \) if \( c < (N - 2)^2/4 \) or if \( c = (N - 2)^2/4 \) or if \( c = (N - 2)^2/4 \) and \( b_*^+ = 0 \). Then the mapping \( \gamma \to u_r \) is increasing. If we assume that

\[
\int_A \frac{ds}{\sqrt{sg(s)}} < \infty
\]

for some \( A > 0 \), \( u_r \) is bounded above in \( \mathcal{U} \setminus \{0\} \) by a continuous function in \( \mathcal{U} \setminus \{0\} \). Then \( u_\infty = \lim_{\gamma \to \infty} u_r \) exists. In the case \( g(r) = |r|^{q-1}r \) we shall prove in Section 4 that

\[
\lim_{x \to 0} |x|^{2/(q-1)} u_\infty(x) = \left( c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right) \right)^{1/(q-1)}
\]

Moreover \( u_\infty \) is the unique solution of (3.17) with \( \gamma = \infty \) (see [31] for example).

4. The Power Case

In this Section we study the solutions of (0.3), that is

\[
-\Delta u - \frac{c}{|x|^\gamma} u + |u|^{q-1}u = 0
\]

in \( B_1(0) \setminus \{0\} \) or in \( \bigcup B_t(0) \) or in \( \mathbb{R}^N \setminus \{0\} \). As some of the results are direct extensions of [13] and [7], we shall abbreviate their proof.
Proof of Theorem 4.1. It is clear from the classical energy method that in the case \( q \neq (N + 2)/(N - 2) \), \( r^{2/(q - 1)} u(r, \cdot) \) converges in the \( C^1(S^{N-1}) \)-topology to a compact connected subset of the set \( \xi \) of solutions of the following equation on \( S^{N-1} \):

\[
-\Delta_{S^{N-1}} \omega + |\omega|^{q-1} \omega = \left( c + \frac{2}{q - 1} \left( 2q \left( q - 1 - N \right) \right) \omega, \right.
\]

(4.2)

(see [2], [7], [13], [26], [27]). Set

\[
\lambda = c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right),
\]

then

(i) if \( u \geq 0, \xi \cap C^+ (S^{N-1}) \) is reduced to 0 and \( \lambda^{1/2} (q - 1) \) \( \lambda > 0 \),

(ii) if \( 0 < \lambda \leq N - 1, \xi \) is reduced to 0, \( \lambda^{1/2} (q - 1) \) and \( -\lambda^{1/2} (q - 1) \),

(iii) if \( q \) is an odd integer \( r \rightarrow r^q \) is a real analytic function and we can apply Bidaut-Veron-Veron and Simon’s theorem [7], [8], [21].

(v) if \( \xi' \) is an hyperbolic limit manifold, that is for any \( \omega \in \xi' \) and any \( \psi \in C^2(S^{N-1}) \) satisfying

\[
-\Delta \psi + q|\psi|^{q-1} \psi - \left( c + \frac{2}{q - 1} \left( 2q \left( q - 1 - N \right) \right) \right) \psi = 0
\]

(4.3)

there exists a one-parameter family \( \{ \omega_s \}_{0 \leq s \leq 1} \) of elements of \( \xi' \) such that

\[
\lim_{s \to 0} s^{-1} (\omega_s - \omega) = \psi
\]

(4.4)

in \( C^2(S^{N-1}) \). We can use Simon’s result [21, Theorem 6.6]. Henceforth we are left with (v): \( N = 2, c \leq 1 \).

Lemma 4.1. Assume \( A \) is an open subset strictly included into

\[
\bar{B}_1^+ (0) \setminus \{0\} = \{ (r, \theta) \in \mathbb{R}^2 : 0 < r \leq 1, 0 \leq \theta \leq r \}
\]

and assume \( c \leq 1 \). Then

\[
\lambda_1 (A) = \inf \left\{ \frac{1}{2} \int_A \left( |\nabla \phi|^2 - \frac{c}{|x|^2} f^2 \right) dx : \phi \in W^{1,2}_{0} (A) \right\} > 0.
\]

Proof. Let \( A = \{ (t, \theta) : (e', \theta) \in A \} \) and \( \psi (t, \theta) = \phi (r, \theta) \) with \( r = e' \). Then

\[
\int_A \left( \phi_r^2 + \frac{1}{r^2} \phi_\theta^2 - \frac{c}{r^2} \phi^2 \right) r dr d\theta = \int_A (\psi_r^2 + \psi_\theta^2 - c\psi^2) dt d\theta.
\]
As \( A \subset (\ln a, \ln b) \times (0, \pi) \) for some \( 0 < a < b < 1 \) and as the first eigenvalue of \(-\Delta\) in \( W^{1,2}_0((\ln a, \ln b) \times (0, \pi))\) is \( \pi^2 \left( \frac{1}{\pi^2} + \frac{1}{(\ln b/a)^2} \right) \), we deduce by the monotonicity property that

\[
\lambda_1(A) \geq 1 - c + \frac{\pi^2}{(\ln b/a)^2} > 0,
\]

which is the claim.

We define \( v \) by (0.30); as \( N = 2 \), it satisfies

\[
(4.5) \quad v_{tt} - \frac{4}{q-1} v_t + v_{\theta\theta} + \left( c + \left( \frac{2}{q-1} \right)^2 \right) v - |v|^{q-1} v = 0
\]

in \((-\infty, 0) \times S^1\). As in [13] we are left with the situation where the \( \alpha \)-limit set of the negative trajectory of \( v(t, \cdot) \) defined by

\[
(4.6) \quad \Gamma^- = \bigcap_{t < 0} \bigcup_{\tau \leq t} \overline{v(\tau, \cdot)}^{C^1(S^1)}
\]

is included into one of the non trivial-\( S^1 \)-action connected component of the set of solutions of

\[
(4.7) \quad \omega_{\theta\theta} + \left( c + \left( \frac{2}{q-1} \right)^2 \right) \omega - |\omega|^{q-1} \omega = 0 \quad \text{on} \quad S^1,
\]

that is

\[
(4.8) \quad \Gamma^- \subset \{ \omega(\alpha + \alpha) \colon \alpha \in S^1 \},
\]

where \( \omega \) is a solution of (4.7) with anti-period \( \pi/k \) (\( k \in \mathbb{N} \)). The following result is then an extension of [13 Lemma 1.6].

**Lemma 4.2.** Let \( \omega \) be an element of \( \Gamma^- \). If \( \omega_\theta(\theta_0) > 0 \) (resp. \( < 0 \)) at some \( \theta_0 \in S^1 \), then there exists \( t^* \leq 0 \) such that

\[
(4.9) \quad v_\theta(t, \theta_0) \geq 0 \quad \text{(resp.} \leq 0 \text{)}
\]

for any \( t \leq t^* \).

**Proof.** For proving it we may assume \( \theta_0 = 0 \) and define

\[
(4.10) \quad \bar{u}(r, \theta) = u(r, \theta) - u(r, -\theta);
\]
then \( \tilde{u} \) satisfies

\[
-\Delta \tilde{u} - \frac{c}{|x|^2} \tilde{u} + d(x)\tilde{u} = 0
\]

in \( B_r^+(0) \setminus \{0\} \) where \( d(x) \geq 0 \). Let us set

\[
\theta^+ = \{ x \in \bar{B}_1^+ \setminus \{0\} : \tilde{u}(x) > 0 \}; \quad \theta^- = \{ x \in \bar{B}_1^- \setminus \{0\} : \tilde{u}(x) < 0 \}.
\]

If \( C \) is a connected component of \( \theta^+ \) or \( \theta^- \), we claim that

\[
0 \in \partial C \quad \text{or} \quad C \cap \partial B_1(0) \neq \emptyset.
\]

Assume the contrary; if \( C \) is such a component, there exists \( a, b \) such that

\[
0 < a < b < 1 \quad \text{and}
\]

\[
\bar{C} \subset \{ (r, \theta) : a < r < b, \quad 0 \leq \theta \leq \pi \} = \Gamma_{a, b}^+.
\]

Extending \( \tilde{u} \) by 0 in \( \Gamma_{a, b}^+ \setminus C \) then the new function \( \tilde{u}^a \) belongs to \( \mathcal{W}^{1, 2}_0(\Gamma_{a, b}^+) \) and

\[
\int_{\Gamma_{a, b}^+} \left( |\nabla \tilde{u}^a|^2 - \frac{c}{|x|^2} (\tilde{u}^a)^2 + d(x)(\tilde{u}^a)^2 \right) = 0.
\]

Then \( \bar{u} = 0 \) in \( C \) from Lemma 4.1, contradiction. The remaining of the proof of Lemma 4.2 goes as in [10 Lemma 1.6 (i)].

**Remark 4.1.** Using Lemma 4.2 and comparison principles implies that if \( \omega_\theta(0) > 0 \) for \( \theta \in [\theta_0, \theta_1] \subset S^1 \), then there exists \( t^* \leq 0 \) such that

\[
v_b(t, \theta) \geq 0, \quad \int_{\theta_0}^{\theta_1} v_b(t, \theta) d\theta > 0
\]

for any \( t \leq t^* \), \( \theta \in [\theta_0, \theta_1] \). However it is interesting to notice that the other assertions of [13, Lemma 1.6] do not hold for \( 0 < x \leq 1 \) as they involve Neuman boundary data.

The remaining of the proof of Theorem 4.1 goes exactly as in [13, Theorem 1.1].

**Remark 4.2.** The potential \( c|x|^{-2} \) of Theorem 4.1 can be replaced by a more general potential \( V \) such that \( V \in C^{1, +} (\bar{B}_1(0) \setminus \{0\}) \) and \( r^2 V(r, \cdot) \) converges to \( c \) as \( r \) tends to 0 in the \( C^{1, +}(\mathbb{S}^{N-1}) \)-topology. In the case (iv) we have also to assume: either \( c < 1 \) or \( |x|^2 V(x) \leq 1 \) in some punctured neighborhood of 0.
It is clear that if
\[ \lambda = c + \frac{2}{q - 1} \left( \frac{2q}{q - 1} - N \right) \]
is nonpositive, the set \( \xi \) of the solutions of (4.2) is reduced to \{0\} and from
Theorem 1.2 \( u \) is described by \( \mu_2 \). However, if \( q > (N + 2)/(N - 2) \) and if \( \lambda = 0 \) we have
\[ (4.17) \quad \alpha_1 < \alpha_2 = -2/(q - 1). \]

The superposition of the linear and the nonlinear effect gives rise to the
phenomena described in Theorem 4.2.

**Proof of Theorem 4.2.**  **Step 1.**  *A priori estimate.* We claim that for any \( \epsilon \in (0, 1) \) there exists \( K_{\epsilon} > 0 \) such that
\[ (4.18) \quad |u(x)| \leq L(N, q)(|x|^2 \ln (1/|x|))^{-1/(q - 1)}(1 + K_{\epsilon}(\ln (1/|x|))^\epsilon)^{-1} \]
where
\[ (4.19) \quad L(N, q) = \left( \left( \frac{1}{q - 1} \right) \left( N - 2 \frac{q + 1}{q - 1} \right) \right)^{1/(q - 1)}. \]

We use the function \( \psi(t, \sigma) \) defined in (0.30) and \( \psi \) satisfies
\[ (4.20) \quad \psi_{tt} + \left( N - 2 \frac{q + 1}{q - 1} \right) \psi_t + \Delta_{sN-1} \psi - |\psi|^q - 1 = 0 \]
in \(( -\infty, 0 ) \times S^{N-1}\) and \( \lim_{t \to -\infty} \psi(t, \sigma) = 0 \) uniformly on \( S^{N-1} \).

Let \( \psi(t) = L(N, q)(-t)^{-1/(q - 1)} + M(-t)^{-\rho}, \ M, \rho > 0, \) then
\[ (4.21) \quad \psi_{tt} + \left( N - 2 \frac{q + 1}{q - 1} \right) \psi_t - \psi^q \]

\[ = L(N, q) \frac{q}{(q - 1)^2} (-t)^{-2 - 1/(q - 1)} \]
\[ + \rho(\rho + 1)M(-t)^{-\rho - 2} + L^q(N, q)(-t)^{-q/(q - 1)} \]
\[ + M_\rho \left( N - 2 \frac{q + 1}{q - 1} \right) (-t)^{-\rho - 1} \]
\[ - L^q(N, q)(-t)^{-q/(q - 1)} - qM L^q(N, q)(-t)^{-\rho - 1} \]
\[ + o(t^{-\rho - 1}). \]
If we choose \( \frac{1}{q-1} < \rho < \frac{q}{q-1} \) then \(-2 - \frac{1}{q-1} < -\rho - 1 \) and

\[
(4.22) \quad M_N \left( N - 2 \frac{q+1}{q-1} \right) < qML^{q-1}(N, q).
\]

Henceforth, there exists \( T < 0 \) such that

\[
(4.23) \quad \psi_t + \left( N - 2 \frac{q+1}{q-1} \right) \psi_t - \psi^q \leq 0
\]

in \((-\infty, T) \times S^{N-1}\). Choosing \( M \) large enough we conclude that \( v \leq \psi \). Arguing similarly for the negative part of \( v \) yields (4.18), (4.19).

**Step 2. End of the proof.** Let us define

\[
(4.24) \quad \xi(t, \sigma) = (-t)^{1/(q-1)} u(t, \sigma).
\]

\( \xi \) is bounded in \((-\infty, -1] \times S^{N-1}\) where it satisfies

\[
(4.25) \quad \xi_t + \left( N - 2 \frac{q+1}{q-1} \right) \xi_t + \Delta_{S^{N-1}} \xi - \frac{2}{t(q-1)} \xi_t + \frac{q}{t^2(q-1)} \xi
\]

\[
+ \frac{1}{t} \left( |\xi|^{q-1} \xi - L^{q-1}(N, q) \xi \right) = 0.
\]

From Agmon-Douglis-Niremberg [15] and Schauder theory all the derivatives \((\partial^\alpha / \partial t^\alpha) \nabla_{\sigma} \xi \) up to the order 3 are uniformly bounded in \((-\infty, -1] \times S^{N-1}\). Henceforth the \( \alpha \)-limit set \( \Gamma^- \) of the trajectory of \( \xi(t, \cdot) \), \( t \leq -1 \), is a non-empty compact subset of \( C^3(S^{N-1}) \). Multiplying (4.25) by \( \xi_t \) and integrating over \((-\infty, -1] \times S^{N-1}\) yields

\[
(4.26) \quad \int_{-\infty}^{-1} \int_{S^{N-1}} \xi^2_t \, d\sigma \, dt < \infty,
\]

after some easy integrations by parts. This immediately implies

\[
\int_{-\infty}^{-1} \int_{S^{N-1}} \xi^2_t \, d\sigma \, dt < \infty.
\]

The uniform continuity of \( \xi_t \) and \( \xi_{tt} \) yields

\[
(4.27) \quad \lim_{t \to -\infty} \| \xi_t(t, \cdot) \|_{L^2(S^{N-1})} = \lim_{t \to -\infty} \| \xi_{tt}(t, \cdot) \|_{L^2(S^{N-1})} = 0.
\]
If we multiply (4.25) by \( \phi \in C^\infty(S^{N-1}) \) and take some element \( l \in \Gamma^- \) we get

\[
\int_{S^{N-1}} l \Delta \phi \, d\sigma = 0.
\]

Henceforth \( \Gamma^- \) is reduced to a constant \( l \) which satisfies

\[
l \in [-L(N, q), L(N, q)],
\]

from (4.18). If we integrate (4.25) on \((t, -1) \times S^{N-1}\) we get

\[
\int_{t}^{-1} \int_{S^{N-1}} \frac{1}{T} (L^{q-1}(N, q) - \xi^{q-1}) \xi \, d\sigma \, dt = \Phi(t)
\]

where \( \Phi(t) \) admits a limit as \( t \) tends to \(-\infty\). Henceforth it is the same with the left-hand side of (4.30) and \( l \) must satisfy

\[
(L^{q-1}(N, q) - l^{q-1})l = 0
\]

which ends the proof.

Remark 4.3. A similar argument can be found in [2] for the study of the isolated singularities of the solutions of

\[
-\Delta u = u^{N/(N-2)} \quad (u > 0)
\]

or in [27] for the study of the long range behaviour of the solutions of

\[
-\Delta u + |u|^{2/(N-2)} u = 0
\]

in an exterior domain.

When

\[
\lim_{t \to 0} |x|^{2/(q-1)} u(x) = 0
\]

we are usually in the situation where the behaviour of \( u \) near 0 is essentially of linear type. If we look for solutions of

\[
\Delta \xi + \frac{c}{|x|^2} \xi = 0
\]

in \( B_1(0) \setminus \{0\} \) under the form

\[
\xi(r, \sigma) = y(r) \phi(\sigma),
\]
we find out that \( \phi \) must be an eigenfunction of \(-\Delta_{5N-1}\) with corresponding eigenvalue \( \lambda_k = k(k + N - 2) \) and \( y \) must satisfy

\[
(4.36) \quad r^2 y_{rr} + r(N - 1)y_r + (c - \lambda_k)y = 0
\]

\( k \in \mathbb{N} \); the corresponding characteristic equation is

\[
(4.37) \quad x^2 + (N - 2)x + c - \lambda_k = 0,
\]

with discriminant \( \delta_k = (N - 2 + 2k)^2 - 4c \). If \( \delta_k \geq 0 \) (4.36) admits two fundamental solutions with constant sign

\[
(4.38) \quad \mu_1^k(x) = \begin{cases} |x|^{-\frac{(N - 2 + \beta_k)}{2}} & \text{if } \delta_k > 0, \\ |x|^{-\frac{(N - 2)}{2}} \ln (1/|x|) & \text{if } \delta_k = 0, \end{cases}
\]

\[
(4.39) \quad \mu_2^k(x) = \begin{cases} |x|^{-\frac{(N - 2 - \beta_k)}{2}} & \text{if } \delta_k > 0, \\ |x|^{-\frac{(N - 2)}{2}} & \text{if } \delta_k = 0, \end{cases}
\]

with \( \beta_k = \sqrt{\delta_k} \). If \( \delta_k < 0 \) the space of solutions of (4.36) is generated by

\[
(4.40) \quad \begin{cases} \nu_1^k(x) = |x|^{(2 - N)/2} \cos \left( \sqrt{-\delta_k} \ln \sqrt{r} \right), \\ \nu_2^k(x) = |x|^{(2 - N)/2} \sin \left( \sqrt{-\delta_k} \ln \sqrt{r} \right). \end{cases}
\]

Surprisingly the case \( q > (N + 2)/(N - 2) \) is simpler than the case \( 1 < q < (N + 2)/(N - 2) \).

**Theorem 4.3.** Assume \( q > (N + 2)/(N - 2) \) and that \(-2/(q - 1)\) is not a solution of (4.37) for some \( k \in \mathbb{N} \). If \( u \) is a solution of (4.1) in \( B_1(0) \setminus \{0\} \) such that

\[
(4.41) \quad \lim_{r \to 0} |x|^{2/(q - 1)}u(x) = 0;
\]

then we have the following alternative.

(i) either there exists \( l \in \mathbb{N} \) satisfying \( \delta_l > 0 \) and \( \psi \in \text{Ker} (\Delta_{5N-1} + \lambda_l I) \), \( \psi \neq 0 \), such that

\[
(4.42) \quad \lim_{r \to 0} r^{DN - 2 - \delta_l/2}u(r, \cdot) = \psi(\cdot)
\]

in the \( C^2(5N - 1) \)-topology,

(ii) or \( u \equiv 0 \).

**Proof.** As \(-2/(q - 1)\) is not a root of (4.37) we can apply [13, Lemma 2.1]. Henceforth there exists \( \varepsilon < 0 \) such that
near 0. Let $k_0$ be the smallest integer such that $\delta k_0 > 0$; then, as in [13], we derive the estimate

$$|u(x)| \leq M |x|^{-(N - 2 - \beta_k)/2}$$

near 0 and this estimate yields easily that there exists $\psi_0 \in \text{Ker} (\Delta_{S^{N-1}} + \lambda_{k_0} I)$ such that

$$\lim_{x \to 0} r^{-(N - 2 - \beta_k)/2} u(r, \sigma) = \psi_0.$$ 

If $\psi_0 = 0$, then, as in [13], we obtain

$$|u(x)| \leq M |x|^{-(N - 2 - (\beta_k + 1)/2},$$

etc., and we carry on as above. If we assume that

$$\lim_{x \to 0} |x|^{-(N - 2 - \beta_k)/2} u(x) = 0$$

for any $k \in \mathbb{N}$, we conclude that $u = 0$ from Aronszajn's unique continuation theorem [1].

If $1 < q < (N + 2)/(N - 2)$ we have $(N - 2)/2 < 2/(q - 1)$ and the properties of $u$ will depend on the sign of $(N - 2)^2 - 4c$.

**Theorem 4.4.** Assume $1 < q < (N + 2)/(N - 2)$, that $-2/(q - 1)$ is not a root of (4.37) for some $k \in \mathbb{N}$ and $(N - 2)^2 \geq 4c$. If $u$ is a solution of (4.1) in $B_1(0) \setminus \{0\}$ satisfying (4.41); then let $k_0$ be the largest integer such that

$$\frac{(N - 2 + \beta_k)}{2} < \frac{2}{(q - 1)};$$

(i) either there exist an integer $k \in [0, k_0]$ and a nonzero $\psi \in \text{Ker} (\Delta_{S^{N-1}} + \lambda_k I)$ such that

$$\lim_{r \to 0} u(r, \ast)/\mu_1(r) = \psi(\ast)$$

in the $C^2(S^{N-1})$-topology,

(ii) or there exist an integer $k \geq 0$ and a nonzero $\psi \in \text{Ker} (\Delta_{S^{N-1}} + \lambda_k I)$ such that (4.49) holds with $\mu_2$ instead of $\mu_1$,

(iii) or $u \equiv 0$.

The proof is the same as the one of Theorem 4.3.
Theorem 4.5. Assume $1 < q < (N + 2)/(N - 2)$, that $-2/(q - 1)$ is not a root of (4.37) for some $k \in \mathbb{N}$ and $(N - 2)^2 < 4c$. If $u$ is a solution of (4.1) in $B_1(0) \setminus \{0\}$ satisfying (4.41), let $k_o \geq 1$ be the smallest integer such that $\delta_{k_o} \geq 0$.

Case I. $2/(q - 1) \geq (N - 2 + \beta_{k_o})/2$. Let $k_1 \geq k_o$ be the largest integer such that

$$(N - 2 + \beta_{k_1})/2 < 2/(q - 1).$$

Then

(i) either there exist an integer $k \in [k_o, k_1]$ and a nonzero $\psi \in \text{Ker} (\Delta_{S_{N-1}} + \lambda_k I)$ such that (4.49) holds,

(ii) or there exists $k_0$ couples of functions $(\phi_k, \psi_k)$ both belonging to $\text{Ker} (\Delta_{S_{N-1}} + \lambda_k I)$ for $k \in \mathbb{N} \cap [0, k_0 - 1]$, one of the above functions at least being nonzero, such that

$$(4.51) \lim_{r \to 0} \left[ r^{(N - 2)/2} u(r, \cdot) \right]$$

$$- \sum_{k = 0}^{k_o - 1} \left( \cos \left( \sqrt{-\delta_k} \ln \sqrt{r} \right) \phi_k + \sin \left( \sqrt{-\delta_k} \ln \sqrt{r} \right) \psi_k \right) = 0$$

in the $C^2(S^{N-1})$-topology,

(iii) or there exist an integer $k \geq k_o$ and a nonzero $\psi \in \text{Ker} (\Delta_{S_{N-1}} + \lambda_k I)$ such that (4.49) holds with $\mu^k_2$ instead of $\mu^k_1$,

(iv) or $u = 0$.

Case II. $(N - 2 - \beta_{k_o})/2 > 2/(q - 1)$. Only the parts (ii), (iii) and (iv) of Case I hold.

Proof. As in the proof of Theorem 4.2 the (i) of Case I is clear as $-2/(q - 1)$ is not a root of (4.37). Henceforth we may assume that

$$|u(x)| \leq M|x|^{(2 - N)/2}$$

and define

$$w(t, \sigma) = r^{(N - 2)/2} u(r, \sigma), \quad t = \ln r.$$ 

Therefore $w$ satisfies

$$w_{tt} + \Delta_{S_{N-1}} w + \left( c - \frac{(N - 2)^2}{4} \right) w + e^{(q - 1)(N - 2)/2} |w|^{q - 1} = 0$$

(4.54)
in \((-\infty, 0] \times S^{N-1}\) where it stays bounded. Let \(w^k\) be the projection of \(w\) onto \(\text{Ker}(\Delta_{S^{N-1}} + \lambda_k I)\) for \(0 \leq k \leq k_0 - 1\). Then \(w_k\) satisfies
\[
(4.55) \quad w^k_{tt} + \left(c - \frac{(N - 2 - 2k)^2}{4}\right) w^k + e^{(q - 1)(N - 2)/2} f_k = 0
\]
where \(f_k\) is bounded, and it is easy to check that
\[
(4.56) \quad \lim_{t \to -\infty} \left( w^k(t) - \cos \left(\sqrt{-\delta_k} \frac{t}{2}\right) \phi_k - \sin \left(\sqrt{-\delta_k} \frac{t}{2}\right) \psi_k \right) = 0
\]
for some \(\phi_k, \psi_k\) in \(\text{Ker}(\Delta_{S^{N-1}} + \lambda_k I)\). As the \(\alpha\)-limit set of the trajectory of \((w(t, \cdot), \cdot)_{t \geq 0}\) is included into the direct sum of the \(\text{Ker}(\Delta_{S^{N-1}} + \lambda_k I)\) for \(k = 0, \ldots, k_0 - 1\), we get (ii). The remaining of the proof is as in Theorem 4.2.

**Remark 4.4.** If \(N = 2\) and \(c \leq 0\), Theorem 4.4 holds for any \(q > 1\).

Similar types of results (with some times many cases to examine) hold for the exterior problem. We just give the basic ones corresponding to Theorem 4.1-4.2.

**Theorem 4.6.** Assume \(q \in (1, \infty) \setminus \{(N + 2)/(N - 2)\}\) and \(u\) is a solution of (4.1) in \(G \ni \{x: |x| \geq 1\}\). Then \(r^{2/(q - 1)} u(r, \cdot)\) converges in the \(C^1(S^{N-1})\)-topology to some compact connected subset of the set \(\xi\) of the \(C^1(S^{N-1})\)-functions satisfying (4.2). Moreover there exists precisely one \(\omega \in \xi\) such that
\[
(4.57) \quad \lim_{r \to 0} r^{2/(q - 1)} u'(r, \cdot) = \omega(\cdot),
\]
at least if one of the conditions (i)-(v) of Theorem 4.1 is fulfilled.

**Theorem 4.7.** Assume \(1 < q < (N + 2)/(N - 2)\), \(0 < c \leq (N - 2)/2\) and \(\lambda = 0\). If \(u\) is any solution of (4.1) in \(G \ni \{x: |x| \geq 1\}\), then the following limit exists
\[
(4.58) \quad \lim_{|x| \to \infty} u(x)/(\mu_1(x)(\ln |x|)^{\alpha_1/2}) = \bar{I}
\]
with
\[
\alpha_1 = \frac{2 - N + \sqrt{(N - 2)^2 - 4c}}{2} = -\frac{2}{q - 1}
\]
and
\[
(4.59) \quad \bar{I} \in \{0, \pm((2q + 1) - N(q - 1))/(q - 1)^{1/(q - 1)}\}.
\]
From Theorems 4.1 and 4.6 we know that a global solution of (4.1) in 
\( \mathbb{R}^N \setminus \{0\} \) satisfies

\[
\lim_{r \to 0} r^{2/(q-1)}u(r, \cdot) \in \xi^-, \quad \lim_{r \to \infty} r^{2/(q-1)}u(r, \cdot) \in \xi^+
\]

where \( \xi^- \) and \( \xi^+ \) are two compact connected subsets of \( \xi \). If we define

\[
E(\eta) = \int_{S^{N-1}} \left( \frac{1}{2} |\nabla \eta|^2 + \frac{1}{q+1} |\eta|^{q+1} - \frac{\lambda}{2} \eta^2 \right) d\sigma,
\]

then \( E|_{\xi^-} = E_- \), \( E|_{\xi^+} = E_+ \) and

\[
\left( N - 2 \frac{q+1}{q-1} \right) \int_{-\infty}^{\infty} \int_{S^{N-1}} v^2 d\sigma \delta \tau = E_+ - E_-, \tag{4.62}
\]

where we have used the notations of (0.30). This relation tells us what are the set of elements of \( \xi \) for which a connecting orbit may exist. The way to constructing connecting orbits is to go through a semiflow as in [13] and to constructing such a semiflow we need an existence and uniqueness result for some initial boundary value problem.

**Theorem 4.8.** Assume \( 1 < q \). Then for any \( \phi \in C(\partial B_1(0)) \) there exists a unique \( u \in C(\bar{\mathbb{R}} B_1(0)) \cap C^3(B_1(0)) \) satisfying

\[
-\Delta u - \frac{c}{|x|^q} u + u|u|^{q-1} = 0
\]

in \( \bar{\mathbb{R}} B_1(0) \) and \( u = \phi \) on \( \partial B_1(0) \) if one of the two following conditions is fulfilled

1. \( \lambda > 0 \), \( u \) and \( \phi \) are nonnegative, and either \( 1 < q \leq (N+2)/(N-2) \), or \( q > (N+2)/(N-2) \) and \( c > (N-2)^2/4 \),
2. either \( c \leq 0 \), or \( 0 < c \leq (N-2)^2/4 \) and \( 1 < q \leq (N+2)/(N-2) \).

**Proof.** Case I-Step 1. Uniqueness. If \( \lambda > 0 \) and \( u \geq 0 \), we know, from Theorem 4.1 if \( q \neq (N+2)/(N-2) \) or from [17] when \( q = (N+2)/(N-2) \), that

\[
\lim_{r \to \infty} r^{2/(q-1)}u(r, \cdot) = L \in [0, \lambda^{1/(q-1)}].
\]

If \( \phi \neq 0, u > 0 \) in \( \bar{B}_1(0) \) from the strong maximum principle. If \( L = 0 \) we get

\[
\bar{v}_n + \left( N - 2 \frac{q+1}{q-1} \right) \bar{v} + (\lambda - \epsilon(\ell)) \bar{v} = 0,
\]

where \( \epsilon(\ell) \) is the positive solution of the equation

\[
\lambda^{1/(q-1)} \bar{v} + \left( N - 2 \frac{q+1}{q-1} \right) \bar{v} = 0.
\]
where \( \epsilon(t) \) is a positive function tending to 0 at infinity. Let \( \delta \) be the discriminant of the equation with constant coefficients associated to (4.65)

\[
\delta = \left( N - 2 \frac{q + 1}{q - 1} \right)^2 - 4\lambda = (N - 2)^2 - 4c.
\]

If \( 1 < q < (N + 2)/(N - 2) \), 0 is a source and \( \tilde{v} \) cannot tend to 0 except if it is identically 0; if \( q \geq (N + 2)/(N - 2) \), then \( \delta < 0 \) and any solution of (4.65) tending to 0 at infinity must oscillate around 0, a contradiction. Henceforth \( L = \lambda^{1/(q - 1)} \). If we linearize (0.31) at \( \lambda^{1/(q - 1)} \) we obtain the following equation

\[
\phi_{tt} + \left( N - 2 \frac{q + 1}{q - 1} \right) \phi_t + \Delta_{\mathbb{S}^N - 1} \phi - (q - 1)\lambda \phi = 0.
\]

As this equation satisfies the maximum principle we deduce that for any \( \epsilon > 0 \) there exists \( C\varepsilon > 0 \) such that

\[
\| \phi(t, \cdot) - \lambda^{1/(q - 1)} \|_{L^\infty(\mathbb{S}^N - 1)} \leq C\varepsilon e^{-\tau - \sigma t},
\]

where

\[
\tau = \frac{1}{2} \left\{ \left( N - 2 \frac{q + 1}{q - 1} \right) + \sqrt{\left( N - 2 \frac{q + 1}{q - 1} \right)^2 + 4\lambda(q - 1)} \right\},
\]

which implies

\[
\| u(r, \cdot) - \lambda^{1/(q - 1)} r^{-2/(q - 1)} \|_{L^\infty(\mathbb{S}^N - 1)} \leq C\varepsilon r^{-2/(q - 1) - \tau + \epsilon}.
\]

Assume now \( u \) and \( \tilde{u} \) are two solutions of (4.63) with same initial data \( \phi \geq 0 \), \( u \) and \( \tilde{u} \) \( \geq 0 \). Then for any \( R > 1 \) we have

\[
- \int_{B_R(0)} \left( \frac{\Delta u}{u} - \frac{\Delta \tilde{u}}{\tilde{u}} \right)(u^2 - \tilde{u}^2) + \int_{B_R(0)} (|u|^{q - 1} - |\tilde{u}|^{q - 1})(u^2 - \tilde{u}^2) = 0,
\]

and

\[
\int_{B_R(0)} \left( \frac{\Delta u}{u} - \frac{\Delta \tilde{u}}{\tilde{u}} \right)(u^2 - \tilde{u}^2) = - \int_{\partial B_R(0)} \left( \frac{u}{u} - \frac{\tilde{u}}{\tilde{u}} \right)(u^2 - \tilde{u}^2)
\]

\[
+ \int_{B_R(0)} \left( \nabla u - \frac{u}{\tilde{u}} \nabla \tilde{u} \right)^2 + \left( \nabla \tilde{u} - \frac{\tilde{u}}{u} \nabla u \right)^2.
\]

But from (4.70) we have

\[
\int_{\partial B_R(0)} \left( \frac{u}{u} - \frac{\tilde{u}}{\tilde{u}} \right)(u^2 - \tilde{u}^2) = O(R^{-2 - 4/(q - 1) - \tau + \epsilon})
\]
and
\[ N - 2 - \frac{4}{q - 1} - \tau = \frac{1}{2} \left( N - 2 \frac{q + 1}{q - 1} - \sqrt{\left( N - 2 \frac{q + 1}{q - 1} \right)^2 + 4\lambda(q - 1)} \right) \]
which is negative. For \( \epsilon \) small enough we deduce
\[ (4.73) \int_{B_R(0)} \left( \left\| \nabla u - \frac{u}{\tilde{u}} \nabla \tilde{u} \right\|^2 + \left\| \nabla \tilde{u} - \frac{\tilde{u}}{u} \nabla u \right\|^2 \right) \]
\[ + \int_{\partial B_R(0)} (|u|^{q-1} - |\tilde{u}|^{q-1})(u^2 - v^2) = 0, \]
which implies the uniqueness.

**Step 2. Existence.** We consider the following iterative scheme
\[ \begin{cases} 
-\Delta u_n + u_n^q = \frac{c}{|x|^2} u_{n-1} & \text{in } \bar{B}_1(0) \quad (n \geq 1), \\
u_n = \phi & \text{on } \partial B_1(0) \\
u_0 = 0. 
\end{cases} \]
\[ \{u_n\} \text{ is increasing. For } \Lambda > \lambda^{\frac{1}{2(q-1)}} \text{ the function } \psi_A(r) = \Lambda r^{2/(q-1)} \text{ satisfies} \]
\[ (4.75) \quad -\Delta \psi_A + \psi_A^q \geq \frac{c}{|x|^2} \psi_A. \]
If we choose \( \Lambda \geq \|\phi\|_{L^\infty(\partial B_1(0))} \) we deduce \( \psi_A \geq u_1 \) and finally \( 0 < u_1 < u_2 < \cdots < u_n < \psi_A \). Clearly \( u_n \) converges to a solution \( u \) of (4.63) with initial value \( \phi \).

**Case II.** **Step 1. Uniqueness.** Let \( u \) and \( \tilde{u} \) be two solutions, \( w = u - \tilde{u} \), \( v(s, \cdot) = w(s, \cdot)/\mu_1(r) \), \( s = r^\beta \) (we assume \( c < (N-2)^2/4 \), the case \( c = (N-2)^2/4 \) is treated by the same technique, see Lemma 1.4); then
\[ (4.76) \quad \mathcal{S}(\|w(s, \cdot)\|_{L^\infty(\mathbb{R}^{N-1})}) > 0 \]
in \( D'(0, +\infty) \). As \( \|w(s, \cdot)\|_{L^\infty(\mathbb{R}^{N-1})} = o(s) \) at infinity, \( w \equiv 0 \).

**Step 2. Existence.** We approximate \( u \) by the solution of the following problem in \( B_n(0) \setminus B_1(0) \)
\[ \begin{cases} 
-\Delta u_n - \frac{c}{|x|^2} u_n + |u_n|^{q-1} u_n = 0 & \text{in } B_n(0) \setminus B_1(0), \quad n \geq 2, \\
u_n = \phi & \text{on } \partial B_1(0) \\
u_n = 0 & \text{on } \partial B_n(0). 
\end{cases} \]
$u_\lambda$ is unique (see Step 1), uniformly bounded, therefore it is convergent to the desired $u$.

**Remark 4.5.** Using a phase plane analysis for the radial solutions of (0.31) we can see that Theorem 4.8 is optimal. It is also of some interest to notice that if $\lambda \leq 0$ there exists no positive solution of (4.63) in $\bar{B}_1(0)$: if $u$ were such a solution, then $\lim_{r \to \infty} r^{2/(q-1)}u(r, \cdot) = 0$ from Theorem 4.1 if $q < (N+2)/(N-2)$ and [17] if $q = (N+2)/(N-2)$ and then

$$
(4.78) \quad \left( N - 2 \frac{q+1}{q-1} \right) \int_0^\infty \int_{S^{N-1}} u_0^2 d\sigma dt = -E(\phi) + \frac{1}{2} \int_{S^{N-1}} u_0^2(0, \cdot) d\sigma
$$

where we use the notations of (0.30) and (4.61), and $u(x) = \phi$ on $\partial B_1(0)$. If $q = (N+2)/(N-2)$ we deduce $E(\phi) = 0$ $\Rightarrow$ $\phi = 0$. But we can replace $\phi = u(0, \cdot)$ by $u(T, \cdot)$ for any $T \geq 0$.

**Remark 4.6.** Thanks to Theorem 4.8 we can define a semiflow $\Phi$ on $X = C^+(S^{N-1})$ in Case I or on $C(S^{N-1})$ in Case II by the formula

$$
(4.79) \quad \Phi_t(\phi)(\cdot) = u(t, \cdot) \quad (t \geq 0)
$$

if $u$ satisfies (4.63) in $\bar{B}_1(0)$ and $u = \phi$ on $\partial B_1(0)$. Clearly $\Phi$ satisfies

(i) $\Phi_0 = I$,
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$,
(iii) $(t, \phi) \mapsto \Phi_t(\phi)$ is continuous in $(t, \phi)$.

The proof of those assertions is the same as the one of [13, Proposition 3.2]. Moreover $\Phi$ is strongly order preserving in the sense that being given $\phi_1$ and $\phi_2$ on $X$, $\phi_1 \geq \phi_2$, $\phi_1 \neq \phi_2$, then for any $t > 0$ there exists $\delta > 0$ such that for any $\eta_1, \eta_2 \in X$, satisfying

$$
|\phi_1 - \eta_1|_{C^0(S^{N-1})} \leq \delta,
|\phi_2 - \eta_2|_{C^0(S^{N-1})} \leq \delta
$$

we have

$$
(4.80) \quad \Phi_t(\eta_1) \geq \Phi_t(\eta_2),
\Phi_t(\eta_1) \neq \Phi_t(\eta_2).
$$

Finally, if $B$ is a bounded subset of $X$ and $t > 0$, $\Phi_t(B)$ is relatively compact in $X$. Those results are what we need to apply Matano’s Theorem concerning heteroclinic orbits of $\Phi$ connecting two equilibria $\omega_1$ and $\omega_2$ such that $[\omega_1, \omega_2]$ contains no other equilibria than $\omega_1$ and $\omega_2$ [18].
Remark 4.7. In order to apply Matano’s method we need to know what is the structure of the set $\xi$ of the solutions of

$$-\Delta S^{N-1} \omega + \omega |\omega|^{q-1} = \lambda \omega$$

on $S^{N-1}$. The complete structure is far out of reach, but using the geometric technique we have introduced in [30] one can describe some of the solutions of (4.81) associated to a tessellation of $S^{N-1}$. We first recall that if $G$ is a subgroup of $O(N)$ generated by reflections through hyperplanes containing $0$ and if $G$ is finite, then $G$ contains a finite number of reflections through hyperplanes $(H_k)_{k \in K}$ containing $0$ and those hyperplanes divide $\mathbb{R}^N$ into a finite number of angular polyhedra $(P_i)_{i \in I}$, each of them being limited by at most $N$ faces [12], [9]. Moreover those polyhedra are all equal and $G$ acts transitively on them. The intersections of those angular polyhedra with $S^{N-1}$ are spherical simplices $(S_i)_{i \in I}$ on which $G$ also acts transitively. Henceforth we can consider only model simplex $S$ as a fundamental domain for $G$. The complete description of those finite groups generated by reflections can be found in [9] but on $\mathbb{R}^3$ there exists only five types of subgroups: type I is generated by reflections through two hyperplanes with angle $\pi/n$; type II is generated by the reflections through two hyperplanes with angle $\pi/n$ and a reflection through an hyperplane orthogonal to them; type III, IV and V are associated to Plato’s polyhedra [14] and have respectively 24, 48 and 120 elements. In order to construct a solution of (4.81) with a high degree of complexity we consider a finite subgroup of reflections $G$ with fundamental simplicial domain $S$ on $S^{N-1}$ and we call $\lambda(S)$ the first eigenvalue of $-\Delta S^{N-1}$ in $W^{1,2}(S)$. It is clear that $\lambda(S)$ is an eigenvalue of $\Delta S^{N-1}$ on $S^{N-1}$. If $\lambda > \lambda(S)$ we call $\omega_S$ the unique positive solution of the following equation on $S$

$$-\Delta S^{N-1} \omega_S + \omega_S^q = \lambda \omega_S,$$

$\omega_S$ vanishing on $\partial S$ ($\omega_S$ is a minimizer). We then extend $\omega_S$ by reflection to whole $S^{N-1}$ according the formula

$$\omega_{G|S} = \det (g_i) \omega_S \circ g_i^{-1}$$

if $S_i = g_i(S)$ for some $g_i \in G$. As the vertices have codimension 2 in $S^{N-1}$ and $\omega_G$ is bounded, $\omega_G$ belongs to $\xi$ [20]. For $\lambda > 0$ let $\xi^*$ be the subset of $\xi$ of solutions of (4.81) containing the three constants and the solutions which are of type $\omega_G$ for some finite subgroup of reflections $G$ (as $\omega_G$ is constructed, $\omega_G \circ \tau$, for any $\tau \in O(N)$, is of the same type). If $\omega_G$ and $\omega_G'$ are two non-constant elements of $\xi^*$ associated to $G$ and $G'$ with fundamental simplicial domains $S$ and $S'$ and if $S$ is a disjoint union of a finite number $k$ of $g_j(S')$, $1 \leq j \leq k$, $g_j \in G'$ we shall say that the frequency of $\omega_G'$ is a multiple of the frequency of $\omega_G$. As $\omega_{G|S}$ is energy minimizing we clearly have
\[(4.84) \int_S \left( \frac{1}{2} |\nabla \omega_G|^2 + \frac{1}{q+1} |\omega_G|^{q+1} - \frac{\lambda}{2} \omega_G^2 \right) < \int_S \left( \frac{1}{2} |\nabla \omega_G|^2 + \frac{1}{q-1} |\omega_G|^{q+1} - \frac{\lambda}{2} \omega_G^2 \right) \]

which clearly implies
\[(4.85) \quad E(\omega_G) < E(\omega_{G'}) < 0.\]

Using also the techniques of Theorem 4.8, for example the increasing iterative scheme
\[(4.86) \begin{cases} -\Delta_{S^{N-1}} \omega_n + \omega_n^q = \lambda \omega_{n-1} & \text{in } S \\ \omega_n = 0 & \text{on } \partial S \end{cases} \]

with \(\omega_0 = \omega_{G|S'}^{+}\) \((S' \subset S)\), we deduce
\[(4.87) \quad \omega_G > \omega_{G'}, \quad \text{in } S.\]

The following result the proof of which is an extension of Theorem 4.8 will be useful in the sequel for constructing a semiflow.

**Theorem 4.9.** Assume \(\Omega\) is an open subset of \(S^{N-1}\), \(K_0\) is the piece of cone defined by
\[(4.88) \quad K_0 = \{ \tau \sigma; \tau > 1, \sigma \in \Omega \}, \]
\(\lambda(\Omega)\) is the first eigenvalue of \(-\Delta_{S^{N-1}}\) in \(W_0^{1,2}(\Omega)\) and \(\partial K_0\) is the lateral boundary of \(K_0\); let \(q\) be bigger than 1. Then for any \(\phi \in C_0(\Omega)\) there exists a unique \(u \in C(K_0) \cap C^3(K_0)\) satisfying
\[(4.89) \quad -\Delta u - \frac{c}{|x|^2} u + u|u|^{q-1} = 0 \]

in \(K_0\), \(u = 0\) on \(\partial K_0\), \(u = \phi\) on \(\Omega\) if one of the following two conditions is fulfilled

(I) \(\lambda > \lambda_1(\Omega), u\) and \(\phi\) are nonnegative, and either \(1 < q \leq (N+2)/(N-2)\) or \(q > (N+2)/(N-2)\) and \(c > \lambda_1(\Omega) + (N-2)^2/4,\)

(II) either \(c \leq \lambda_1(\Omega), \) or \(\lambda_1(\Omega) < c \leq \lambda_2(\Omega) + (N-2)^2/4\) and \(1 < q \leq (N+2)/(N-2).\)

With this result we can define a semiflow \(\Phi_t^G\) on \(X = C_0^+(\Omega)\) in Case I or on \(C_0(\Omega)\) in Case II and, if \(\partial \Omega\) is Lipschitz, \(\Phi_t^G\) is a strongly order preserving semiflow on \(X\) mapping bounded subsets of \(X\) into relatively compact subsets.
of \( X \) for \( t > 0 \). We are now able to study the solutions \( u \) of (4.63) in \( \mathbb{R}^N \setminus \{0\} \) such that

\[
\begin{align*}
\lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) &= \omega_1, & \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) &= \omega_2 \quad \text{or} \\
\lim_{r \to 0} r^{2/(q-1)} u(r, \cdot) &= \omega_2, & \lim_{r \to \infty} r^{2/(q-1)} u(r, \cdot) &= \omega_1
\end{align*}
\] (4.90)

and by extension of the notations of Remark 4.7 we shall say that the two constants \( \pm \lambda^{1/(q-1)} \) are of type \( \omega_G \) with \( G = \{ I_d \} \) and \( S = S^{N-1}, \lambda(S) = 0 \).

**Theorem 4.10.** Assume \( q > 1 \), \( \omega_1 \) and \( \omega_2 \in \xi^* \). Then there exists a solution \( u \) of (4.63) in \( \mathbb{R}^N \setminus \{0\} \) satisfying (4.90) if

(A) \( \omega_1 = 0, \omega_2 = \omega_G \) for some \( G \) and one of the following two conditions is fulfilled

(i) \( 1 < q \leq (N + 2)/(N - 2) \), \\
(ii) \( q > (N + 2)/(N - 2) \) and \( c > \lambda(S) + (N - 2)^2/4 \);

(B) \( \omega_1 = \lambda^{1/(q-1)}, \omega_2 = \omega_G \) for some non trivial \( G \) and either \( c \leq 0 \), or \( 0 < c \leq (N - 2)^2/4 \) and \( 1 < q \leq (N + 2)/(N - 2) \);

(C) \( \omega_1 = \omega_G, \omega_2 = \omega_G \), the frequency of \( \omega_G \), is a multiple of the frequency of \( \omega_G \), and either \( c \leq \lambda(S) \) or \( \lambda(S) < c \leq \lambda(S) + (N - 2)^2/4 \) and \( 1 < q \leq (N + 2)/(N - 2) \).

The proof of the Theorem is essentially a consequence of the construction of Remark 4.7 and of Theorem 4.8 for \( B \) and \( C \) and Theorem 4.9 applied in \( K_5 \) for \( A \); in that last case the positive solution constructed in the cone with basis \( 0 \) and vertex \( 0 \) is extended by reflection to be a solution of (4.63) in \( \mathbb{R}^N \setminus \{0\} \). It must also be noticed that the case \( A, B \) and \( C \) imply \( \lambda > \lambda(S) \), \( \lambda > \lambda(S') \) respectively.

**Remark 4.8.** The complete set of the critical values of \( E \) is not known, in particular is it true that all the connected components of \( \xi \) have different energy value (the energy is constant on each connected component from Sard’s theorem)? Such an exclusion principle is valid on \( \xi^* \).

**References**

Local properties of stationary solutions of some Schrödinger equations


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Ondelettes sur l’intervalle

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1. Introduction

En 1873 Dubois-Reymond construisit une fonction continue de la variable réelle \(x\) et 2\(\pi\)-périodique dont la série de Fourier diverge en un point donné. Ce contre-exemple amena A. Haar à se poser, puis à résoudre le problème de l’existence d’une base orthonormée \(h_0(x), h_1(x), \ldots, h_m(x), \ldots\) de \(L^2[0, 1]\) ayant la propriété que, pour toute fonction continue \(f(x)\), la série \(\sum \langle f, h_m \rangle h_m(x)\) converge uniformément vers \(f(x)\). A Haar choisit \(h_0(x) = 1, h_1(x) = h(x)\) où \(h(x) = 1\) sur \([0, 1/2[, h(x) = -1\) sur \([1/2, 1[\) et \(h(x) = 0\) ailleurs. Enfin, pour \(m = 2^j + k, 0 \leq k < 2^j, j \geq 0\), il pose \(h_m(x) = 2^{j/2}h(2^jx - k)\).

Les sommes partielles \(\langle f, h_0 \rangle h_0(x) + \cdots + \langle f, h_m \rangle h_m(x) = S_m(f)(x)\) sont des approximations de \(f(x)\) par des fonctions en escalier. Mais si, pour un certain exposant \(s \in ]0, 1[\), \(f(x)\) vérifiait en outre,

\[
(1.1) \quad f(x + h) - f(x) = o(h^s)
\]

uniformément en \(x \in \mathbb{R}\), les sommes partielles \(S_m(f)\) ne pourraient converger vers \(f(x)\) pour la norme de l’espace de Banach \(C^s\) défini par (1.1).

Peut-on modifier la construction du système de Haar et obtenir une base orthonormée de \(L^2[0, 1]\) convenant à l’analyse et à la synthèse des espaces de Hölder \(C^s\) définis par (1.1)?

Ce problème a été étudié depuis le travail de pionnier de Haar. G. Faber puis J. Schauder ont commencé par remplacer les fonctions \(h_m(x)\) du système de Haar par leurs primitives \(\Delta_m(x)\). En changeant la normalisation, il vient \(\Delta_m(x) = \Delta(2^jx - k)\) où \(m = 2^j + k, 0 \leq k < 2^j\), et où \(\Delta(x) = 2x\) si \(0 \leq x \leq 1/2, 2 - 2x\) si \(1/2 \leq x \leq 1\) et 0 hors de l’intervalle \([0, 1]\).
L’approximation d’une fonction continue sur \([0, 1]\) par les sommes partielles de la série \(a + bx + \sum_1^\infty \alpha_m \Delta_m(x)\) revient à approcher le graphe de \(f(x)\) par des lignes polygonales inscrites et constitue donc une amélioration de l’approximation à l’aide des fonctions en escalier. Si \(f(x)\) appartient à \(C^\infty_0[0, 1]\), on a \(\alpha_m = o(m^{-s})\) et réciproquement, si cette condition est vérifiée, la série \(a + bx + \sum_1^\infty \alpha_m \Delta_m(x)\) converge vers \(f(x)\) en norme \(C^s[0, 1]\). En ce sens, le système de Schauder \(\Delta_m(x), m \geq 1\) (complété par 1 et \(x\)) est une base inconditionnelle de l’espace \(C^s_0[0, 1]\).

En revanche, le système de Schauder ne peut plus servir à l’analyse de l’espace \(L^2[0, 1]\). Une façon de le voir est d’observer que les coefficients \(\alpha_m\) se calculent par

\[
\alpha_m = f\left(\left(\frac{k + 1}{2}\right)2^{-j}\right) - \frac{1}{2} \left[f(k2^{-j}) + f((k + 1)2^{-j})\right]
\]

et que cette formule n’a plus aucun sens si \(f(x)\) appartient à \(L^2[0, 1]\).

Pour corriger ce défaut de la base de Schauder, Ph. Franklin a eu l’idée d’orthonormaliser la suite \(1, x, \Delta_1(x), \ldots, \Delta_m(x), \ldots\) en utilisant le procédé de Gram-Schmidt. On obtient alors une suite \(f_m(x), m \geq -1, o_f(x) = 2\sqrt{3}(x - 1/2), \ldots\) Le système de Franklin est un peu tombé dans l’oubli parce que les fonctions \(f_m(x)\) ne sont pas fournies par un algorithme aussi simple que celui des fonctions \(h_m(x)\) du système de Haar. Cependant en 1963, Ciesielski démontra que tout se passe comme si l’on avait \(f_m(x) = 2^{j/2}\psi(2^jx - k)\) lorsque \(m = 2^j + k\). Il prouve en effet que \(|f_m(x)| \leq C2^{j/2} \exp(-\gamma |2^j - k|)\) pour un certain exposant \(\gamma > 0\). Cette estimation, jointe à

\[
\int_0^1 f_m(x) dx = \int_0^1 x f_m(x) dx = 0,
\]

fournit la caractérisation attendue des espaces \(C^s[0, 1]\) par \(\alpha_m = O(m^{-1/2 - s})\). La différence avec la base de Schauder vient de ce que les normalisations sont différentes.

Nous nous proposons de corriger le défaut du système de Franklin. Plus précisément, pour tout entier \(N \geq 1\), nous allons construire une base orthonormée \(f_m^{(N)}\) de \(L^2[0, 1]\) ayant une structure algorithmique aussi simple que celle du système de Haar. En outre, pour une constante absolue \(\gamma > 0\) qui est estimée dans [1], les espaces \(C^s[0, 1]\) seront caractérisés par \(\alpha_m = O(m^{-1/2 - s})\) lorsque \(0 < s < \gamma(N - 1)\).

Pour l’essentiel notre nouvelle base a exactement la structure du système de Haar puisqu’elle contient toutes les fonctions \(2^{j/2}\psi(2^jx - k), j \geq 0, k \geq 0\), dont le support est inclus dans \([0, 1]\). Ici \(\psi(x)\) désigne «l’ondelette de Daubechies» dont le support est l’intervalle \([0, 2N - 1]\).

Malheureusement les fonctions précédentes engendrent un sous-espace de \(L^2[0, 1]\) de codimension infinie. Ceci est dû aux effets de bord produits par 0 et
1. Pour tenir compte de ces effets de bord, il nous faudra adjoindre aux fonctions précédentes (les \(2^{j/2}\psi(2^j x - k)\)) les fonctions \(2^{j/2}\psi_1^e(2^j x), \ldots, 2^{j/2}\psi_{N-1}^e(2^j x)\), «affectées à 0» et les fonctions \(2^{j/2}\psi_1^b(2^j (1 - x)), \ldots, 2^{j/2}\psi_{N-1}^b(2^j (1 - x))\) «affectées à 1». Nous apprendrons à construire les \(2N - 2\) fonctions \(\psi_i^e\) et \(\psi_i^b\), \(1 \leq i \leq N - 1\). Il manquera alors un ensemble fini explicite pour constituer la base orthonormée cherchée de \(L^2[0, 1]\).

2. Rappels sur les ondelettes d'Ingrid Daubechies et sur les filtres associés

On part d’un entier \(N \geq 1\) et de \(2N\) coefficients \(h_0, h_1, \ldots, h_{2N-1}\) tels que le polynôme trigonométrique

\[
m_0(\xi) = h_0 + h_1 e^{i\xi} + \cdots + h_{2N-1} e^{i(2N - 1)\xi}
\]

vérifie les trois conditions suivantes

\[
(2.1) \quad |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1
\]
\[
(2.2) \quad m_0(0) = 1
\]
\[
(2.3) \quad m_0(\xi) \neq 0 \quad \text{si} \quad -\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}.
\]

Ces trois conditions assurent l’existence d’une suite orthonormée \(\varphi(x - k), k \in \mathbb{Z}\), de fonctions de \(L^2(\mathbb{R})\) telle que l’on ait

\[
(2.4) \quad \frac{1}{2} \varphi(x) = h_0 \varphi(2x) + \cdots + h_k \varphi(2x - k) + \cdots + h_{2N-1} \varphi(2x - (2N - 1))
\]

et

\[
\int_{-\infty}^{\infty} \varphi(x) \, dx = 1.
\]

On construit \(\varphi(x)\) à l’aide de

\[
(2.5) \quad \int_{-\infty}^{\infty} e^{i\xi t} \varphi(x) \, dx = m_0(\xi/2) m_0(\xi/4) \ldots
\]

et le support de \(\varphi\) est inclus dans \([0, 2N - 1]\).

Nous ferons dans tout ce qui suit le choix particulier suivant. On part de la fonction \(c(\sin t)^{2N-1}\) où \(c = c(N) > 0\) est la constante définie par

\[
c \int_0^{\pi} (\sin t)^{2N-1} \, dt = 1.
\]
On désigne par $g(t)$ la primitive de $c(\sin t)^{2N-1}$ qui est nulle en $\pi$. On a alors $g(-t) = g(t), g(t) \geq 0$ pour tout $t$ et $g(t) + g(t + \pi) = 1$. Le lemme de F. Riesz permet alors de choisir

$$m_0(t) = h_0 + h_1 e^{it} + \cdots + h_{2N-1} e^{i(2N-1)t}$$

de sorte que

$$|m_0(t)|^2 = g(t).$$

Notons que

$$|m(t)|^2 = h_0 h_{2N-1} e^{-i(2N-1)t} + \cdots + h_0 h_{2N-1} e^{i(2N-1)t}$$

et que $h_0 h_{2N-1} \neq 0$.

Ces choix de $m_0(\xi)$ conduisent à des ondelettes de régularité $C^{\gamma(N-1)}$ où $\gamma > 0$ est une constante universelle et où l'on suppose $N \geq 2$. Le cas $N = 1$ correspond au système de Haar.

On définit ensuite l'ondelette $\psi(x)$ en introduisant

$$m_1(\xi) = e^{i(2N-1)\xi} \frac{m_0(\xi + \pi)}{m_0(\xi)}$$

$$= g_0 + g_1 e^{i\xi} + \cdots + g_{2N-1} e^{i(2N-1)\xi}$$

et en posant

$$\frac{1}{2} \psi(x) = g_0 \varphi(2x) + g_1 \varphi(2x - 1) + \cdots + g_{2N-1} \varphi(2x - 2N + 1).$$

Les identités (2.4) et (2.6) conduisent à

$$\varphi(2x) = h_0 \varphi(x) + h_3 \varphi(x + 1) + \cdots + h_{2N-2} \varphi(x + N - 1)$$

$$+ \bar{g}_0 \psi(x) + \bar{g}_2 \psi(x + 1) + \cdots + \bar{g}_{2N-2} \psi(x + N - 1)$$

et à

$$\varphi(2x - 1) = \bar{h}_1 \varphi(x) + \bar{h}_3 \varphi(x + 1) + \cdots + \bar{h}_{2N-1} \varphi(x + N - 1)$$

$$+ \bar{g}_1 \psi(x) + \bar{g}_3 \psi(x + 1) + \cdots + \bar{g}_{2N-1} \psi(x + N - 1).$$

Voici l'interprétation géométrique de ces identités. Les fonctions $\varphi(x - k), \ k \in \mathbb{Z},$ sont une base orthonormée d'un espace que l'on note $V_0$. On définit $V_j, \ j \in \mathbb{Z},$ par

$$\forall f \in L^2(\mathbb{R}), \quad f(x) \in V_0 \iff f(2^j x) \in V_j$$

et les fonctions $\psi(x - k), \ k \in \mathbb{Z},$ forment une base orthonormée du complément orthogonal $W_0$ de $V_0$ dans $V_1$. En particulier $\varphi(2x)$ appartient à $V_1$ et
(2.7) fournit sa décomposition \( u + v \) où \( u \in V_0 \) et \( v \in W_0 \). Il en est de même pour \( \varphi(2x - 1) \) qui est décomposé grâce à (2.8). Finalement toutes les fonctions \( \varphi(x - k) \), \( k \in \mathbb{Z} \), se décomposent, si \( k \) est pair, grâce à (2.7) et si \( k \) est impair, grâce à (2.8). Cela signifie que (2.7) et (2.8) sont les formules de changement de base permettant de passer de la base orthonormée \( \sqrt{2} \varphi(2x - k) \), \( k \in \mathbb{Z} \), de \( V_1 \) à la nouvelle base orthonormée que l’on obtient en réunissant les bases \( \varphi(x - k) \), \( k \in \mathbb{Z} \), de \( V_0 \) et \( \psi(x - k) \), \( k \in \mathbb{Z} \), de \( W_0 \).

3. L’analyse multirésolution \( V^{[0,1]}_j \) de \( L^2[0, 1] \)

On part de la suite emboîtée \( V_j \), \( -\infty < j < \infty \), de sous-espaces fermés de \( L^2(\mathbb{R}) \), définis par

\[
\begin{align*}
(3.1) \quad & \varphi(x - k), \ k \in \mathbb{Z}, \text{ est une base orthonormée de } V_0 \\
(3.2) \quad & f(x) \in V_0 \iff f(2^j x) \in V_j, \text{ pour toute } f \in L^2(\mathbb{R}).
\end{align*}
\]

et l’on désigne, pour tout \( j \in \mathbb{N} \), par \( V^{[0,1]}_j \) l’espace des restrictions à \([0, 1]\) des fonctions de \( V_j \). Puisque \( \varphi(x) \) est une fonction à support compact, il est évident que \( V^{[0,1]}_j \) est de dimension finie. En fait, on a un résultat beaucoup plus précis.

Désignons par \( S(j) \) l’intervalle d’entiers \( k \) définis par \(-2^N + 2 \leq k \leq 2^j - 1\) ou, ce qui est équivalent, par la condition que le support de la fonction \( \varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k) \) rencontre l’intervalle \([0, 1]\). Alors on a

**Théorème 1.** Pour tout entier \( j \geq 0 \), les fonctions \( \varphi_{j,k}, \ k \in S(j) \) constituent une base de l’espace \( V^{[0,1]}_j \) des restrictions à \([0, 1]\) des fonctions de \( V_j \).

Cet énoncé peut se formuler de façon équivalente et cette seconde formulation nous sera utile, par la suite.

**Corollaire.** Soit

\[
f(x) = \sum_{-\infty}^{\infty} c_k \varphi(2^j x - k)
\]

une fonction arbitraire de \( V_j \). Supposons \( j \geq 0 \) et \( f(x) = 0 \) si \( 0 \leq x \leq 1 \). Alors \( f(x) \) est formellement nulle sur \([0, 1]\), c’est-à-dire que \( c_k = 0 \) pour tout \( k \in S(j) \).

Nous établirons le théorème par une récurrence descendante sur \( j \). Nous commencerons par traiter le cas facile où \( 2^j \geq 4N - 4 \) en utilisant le lemme suivant.
Lemme 1. Si

\[ f(x) = \sum_{k=-\infty}^{\infty} c_k \varphi(x - k) \]

est nulle sur \([-\infty, 0]\), alors \(f(x)\) est formellement nulle sur \([-\infty, 0]\), c'est-à-dire \(f(x) = c_0 \varphi(x) + c_1 \varphi(x - 1) + \cdots\)

Pour le montrer, on commence par observer que

\[ c_k = \int_{-\infty}^{\infty} f(x) \varphi(x - k) \, dx = 0 \quad \text{si} \quad k \leq -2N + 1. \]

On désigne alors par \(l\) le plus petit des entiers \(k\) tels que \(c_k \neq 0\). Si \(l \geq 0\), il n'y a rien à démontrer et si \(l < 0\), il suffit d'observer que \(f(x)\), restreinte à \([l, l + 1]\) est nulle, par hypothèse, mais est aussi égale à \(c_l \varphi(x - l)\). Puisque le support de \(\varphi\) est exactement \([0, 2N - 1]\), on aboutit à une contradiction.

Pour démontrer le Théorème 1, nous définissons \(j_0\) comme le plus petit entier \(j\) tel que \(2^j \geq 4N - 4\). Nous commençons par établir le Théorème 1 quand \(j \geq j_0\). Dans ce cas l'ensemble \(S(j)\) des entiers \(k\) tels que le support de \(\varphi_{j,k}\) rencontre \([0, 1]\) se divise en trois ensembles disjoints \(S_1(j), S_2(j)\) et \(S_3(j)\) selon que l'intérieur du support de \(\varphi_{j,k}\) contient 0, que le support de \(\varphi_{j,k}\) est inclus dans \([0, 1]\) ou que l'intérieur du support de \(\varphi_{j,k}\) contient 1. Lorsque \(j \geq j_0\) et que \(k \in S_1(j)\), le support de \(\varphi_{j,k}\) est inclus dans \([-\infty, 1/2]\) et lorsque \(j \geq j_0\) et \(k \in S_3(j)\), ce support est inclus dans \([1/2, \infty[. Si \(j \geq j_0\) et si

\[ f(x) = \sum_{k \in S(j)} c_k \varphi_{j,k}(x) \]

est nulle sur \([0, 1]\), alors

\[ c_k = \int_{[0, 1]} f(x) \varphi_{j,k}(x) \, dx = 0 \quad \text{si} \quad k \in S_2(j). \]

Ensuite

\[ f_1(x) = \sum_{k \in S_1(j)} c_k \varphi_{j,k}(x) \]

est nulle sur \([1/2, \infty[,

\[ f_2(x) = \sum_{k \in S_3(j)} c_k \varphi_{j,k}(x) \]

l'est sur \([-\infty, 1/2]\) et \(f_1(x) + f_2(x) + f_3(x) = f(x)\) l'est sur \([0, 1]\). Il en résulte que \(f_1(x)\) est nulle sur \([0, \infty[. Le Lemme 1 s'applique et \(c_k = 0\) si \(k \in S_1(j)\).

On en déduit, de même, que \(c_k = 0\) si \(k \in S_3(j)\).
Nous désignerons par $P_j$ la propriété: les fonctions $\varphi_{j,k}(x)$, $k \in S(j)$, forment une base de $V_j^{[0,1]}$. Nous nous proposons maintenant de démontrer que $P_j$ implique $P_{j-1}$. Pour cela, on utilise le lemme suivant.

**Lemme 2.** Si les $h_0, \ldots, h_{2N-1}$ sont les coefficients utilisés pour construire $m_0(\xi)$, alors toute suite $u_0, \ldots, u_{2N-3}$ vérifiant, pour $0 \leq k \leq N-2$, les relations

$$h_0 u_k + h_2 u_{k+1} + \cdots + h_{2N-2} u_{k+N-1} = 0$$

et

$$h_1 u_k + h_3 u_{k+1} + \cdots + h_{2N-1} u_{k+N-1} = 0$$

est nécessairement la suite nulle.

On pose, en effet,

$$U(z) = h_0 + h_2 z + \cdots + h_{2N-2} z^{N-1}$$

et

$$V(z) = h_1 + h_3 z + \cdots + h_{2N-1} z^{N-1}$$

et l'on a

$$m_0(\xi) = U(e^{2i\xi}) + e^{i\xi} V(e^{2i\xi}).$$

Puisque

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1,$$

il vient

$$|U(e^{i\xi})|^2 + |V(e^{i\xi})|^2 = \frac{1}{2}.$$

Puisque les coefficients $h_k$ sont réels, il en découle que

$$U(z)U(z^{-1}) + V(z)V(z^{-1}) = \frac{1}{2}$$

pour tout $z \neq 0$. Donc $U(z)$ et $V(z)$ n'ont aucun zéro commun $z \neq 0$ et, puisque $h_0 \neq 0$, $U(z)$ et $V(z)$ n'ont aucun zéro commun.

Désignons par $z_1, \ldots, z_{N-1}$ les zéros, supposés simples, de $U(z)$ et par $\tilde{z}_1, \ldots, \tilde{z}_{N-1}$ ceux de $V(z)$, que l'on supposera également simples dans un premier temps. On a alors

$$u_k = c_1 z_1^k + \cdots + c_{N-1} z_{N-1}^k \quad \text{pour} \quad 0 \leq k \leq 2N-3.$$
et, de même,

\[ u_k = \gamma_1 s_1^k + \cdots + \gamma_{N-1} s_{N-1}^k \quad \text{pour} \quad 0 \leq k \leq 2N - 3. \]

Il en découle que

\[ c_1 z_1^k + \cdots + c_{N-1} z_{N-1}^k - \gamma_1 z_1^k - \cdots - \gamma_{N-1} z_{N-1}^k = 0 \]

pour \( 0 \leq k \leq 2N - 3 \). Or ces relations de liaison entraînent la nullité du déterminant de Van der Monde

\[
\begin{vmatrix}
1 & \cdots & 1 \\
z_1 & \cdots & z_{N-1} \\
z_1^{2N-3} & \cdots & z_{N-1}^{2N-3}
\end{vmatrix}
\]

ce qui est absurde, puisque

\[ \{ z_1, \ldots, z_{N-1} \} \cap \{ z_1, \ldots, z_{N-1} \} = \emptyset. \]

Ce raisonnement s'adapte immédiatement au cas de racines multiples de \( U(z) \) ou de \( V(z) \).

Revenons à la preuve du Théorème 1 et montrons que \( P_{j+1} \) implique \( P_j \), pour tout \( j \geq 0 \).

Supposons donc que

\[ \sum_{k \in S(j)} x_k \varphi(2^j x - k) = 0 \]
sur \([0, 1]\). On écrit

\[ \frac{1}{2} \varphi(2^j x - k) = h_0 \varphi(2^{j+1} - 2k) + h_1 \varphi(2^{j+1} x - 2k - 1) + \cdots + h_{2N-1} \varphi(2^{j+1} x - 2k - 2N + 1). \]

On a donc, si \( 0 \leq x \leq 1 \),

\[
0 = \sum_{k \in S(j)} x_k \varphi(2^j x - k) = \sum_{0}^{2N-1} \sum_k x_k h_j \varphi(2^{j+1} x - 2k - l).
\]

La propriété \( P_{j+1} \) nous apprend que toute somme \( \sum_{k \neq 0} y_k \varphi(2^{j+1} x - k) \) qui est identiquement nulle sur \([0, 1]\) est, en fait, formellement nulle sur \([0, 1]\); chaque terme qui la compose est nul sur \([0, 1]\). On a donc les \( 2^{j+1} + 2N - 2 \) relations
\[ y_{-2N+2} = x_{-N+1} h_0 + x_{-N} h_2 + \cdots + x_{-2N+2} h_{2N-2} = 0 \]
\[ y_{-2N+3} = x_{-N+1} h_1 + x_{-N} h_3 + \cdots + x_{-2N+2} h_{2N-1} = 0 \]
\[ y_{-2N+4} = x_{-N+2} h_0 + x_{-N+1} h_2 + \cdots + x_{-2N+3} h_{2N-2} = 0 \]
\[ y_{-2N+5} = x_{-N+2} h_1 + x_{-N+1} h_3 + \cdots + x_{-2N+3} h_{2N-1} = 0 \]
\[ \cdots \cdots \]
\[ y_{2j+1-2} = x_{2j} h_0 + \cdots + x_{2j-N} h_{2N-2} = 0 \]
\[ y_{2j+1-1} = x_{2j} h_1 + \cdots + x_{2j-N} h_{2N-1} = 0. \]

On utilise alors le Lemme 2 et il vient \( x_{-2N+2} = \cdots = x_{2j-1} = 0 \), comme annoncé.

Nous pouvons compléter le Théorème 1 par l’énoncé quantitatif correspondant. A savoir l’existence de deux constantes \( C_2 > C_1 > 0 \), indépendantes de \( j \geq 0 \), telles que l’on ait pour toute suite \( \alpha_{j,k} \), \( k \in S(j) \), de coefficients

\[
C_1 \left( \sum_{k \in S(j)} |\alpha_{j,k}|^2 \right)^{1/2} \leq \left\| \sum_{k \in S(j)} \alpha_{j,k} \varphi_{j,k}(x) \right\|_{L^2(0,1)} \leq C_2 \left( \sum_{k \in S(j)} |\alpha_{j,k}|^2 \right)^{1/2}.
\]

Si \( 0 \leq j < j_0 \), il n’y a rien à démontrer et (3.3) paraphrase le Théorème 1. Si \( j \geq j_0 \), on pose

\[ f(x) = \sum_{k \in S(j)} \alpha_{j,k} \varphi_{j,k}(x) \]

puis

\[ f_1(x) = \sum_{k \in S_1(j)} (\cdots), \quad f_2(x) = \sum_{k \in S_2(j)} (\cdots) \quad \text{et} \quad f_3(x) = \sum_{k \in S_3(j)} (\cdots), \]

L’inégalité de Bessel fournit, tout d’abord,

\[
\sum_{k \in S_2(j)} |\alpha_{j,k}|^2 \leq \int_0^1 |f(x)|^2 \, dx.
\]

Il en découle que

\[
\|f_1 + f_3\|_{L^2(0,1)} \leq 2 \| f \|_{L^2(0,1)}.
\]

Mais les supports de \( f_1 \) et de \( f_3 \) sont disjoints puisque \( j \geq j_0 \). Il en résulte que \( \| f_1 \|_2 + \| f_3 \|_2 \leq 2 \sqrt{2} \| f \|_2 \). Les normes de \( f_1(x) \) et de \( f_3(x) \) s’évaluent enfin par simple changement d’échelle et les estimations ne dépendent pas de \( j \).

L’inégalité de droite de (3.3) est encore plus facile, en utilisant la décomposition \( f = f_1 + f_2 + f_3 \).
Avant de passer à la construction des ondelettes, nous allons déduire du Théorème 1 la construction d’une base orthonormée de $V_{j}^{0,1}$. Nous nous limiterons au cas où $j \geq j_0$. Alors les fonctions $\varphi_{j,k}, k \in S_1(j)$, $k \in S_2(j)$ ou $k \in S_3(j)$ forment une base (non orthogonale) de $V_{j}^{0,1}$. Il importe de remarquer que, pour le produit scalaire de $L^2[0,1]$, les fonctions $\varphi_{j,k}, k \in S_1(j)$ sont orthogonales aux fonctions $\varphi_{j,k}, k \in S_2(j)$. Cette orthogonalité tient à ce que l’on a

$$\int_{-\infty}^{\infty} \varphi(x) \overline{\varphi}(x-k) \, dx = 0 \quad \text{si} \quad k \in \mathbb{Z}, \quad k \neq 0.$$ 

Ensuite les fonctions $\varphi_{j,k}, k \in S_1(j)$ sont orthogonales aux fonctions $\varphi_{j,k}, k \in S_3(j)$ parce que leurs supports sont disjoints.

Pour transformer en une base orthonormée la base des $\varphi_{j,k}, k \in S(j)$, il suffit donc de rendre orthogonales entre elles les fonctions du paquet $\varphi_{j,k}, k \in S_1(j)$ et, de même, de rendre orthogonales entre elles les fonctions du paquet $\varphi_{j,k}, k \in S_2(j)$. Dans chaque cas, il s’agit de $N - 1$ fonctions et les calculs à faire sont invariants par dilatation. On obtient donc, à la place des $N - 1$ fonctions $\varphi_{j,k}(x), k \in S_1(j), N - 1$ nouvelles fonctions

$$2^{j/2} \varphi_1^*(2^j x), \ldots, 2^{j/2} \varphi_{N-1}^*(2^j x)$$

et de même, à la place des $N - 1$ fonctions $\varphi_{j,k}, k \in S_3(j), N - 1$ nouvelles fonctions

$$2^{j/2} \varphi_1^b(2^j(1-x)), \ldots, 2^{j/2} \varphi_{N-1}^b(2^j(1-x))$$

Nous pouvons conclure en énonçant le résultat suivant

**Proposition 1.** *Pour $j \geq j_0$, la collection des fonctions

$$2^{j/2} \varphi_1^*(2^j x), \ldots, 2^{j/2} \varphi_{N-1}^*(2^j x), 2^{j/2} \varphi(2^j x - k), \quad 0 \leq k \leq 2^j - 2N + 1$$

et, finalement

$$2^{j/2} \varphi_1^b(2^j(1-x)), \ldots, 2^{j/2} \varphi_{N-1}^b(2^j(1-x))$$

est une base orthonormée de $V_{j}^{0,1}$.*

Une dernière remarque nous sera utile dans ce qui suit

**Lemme 3.** *L’espace vectoriel $V_{0}^{0,1}$ (de dimension $2N - 1$) contient l’espace vectoriel (de dimension $N$) des restrictions à $[0,1]$ des polynômes de degré $\leq N - 1$.*
Pour le voir, on commence par observer que
\[ \sum_{-\infty}^{\infty} (x - k)^m \varphi(x - k) = c_m \quad \text{pour} \quad 0 \leq m \leq N - 1, \]
on le montre en appliquant la formule sommatoire de Poisson et en observant que
\[ \left( \frac{d}{dx} \right)^m \varphi(x) = 0 \quad \text{si} \quad x = 2k\pi, \quad k \neq 0, \quad \text{et} \quad 0 \leq m \leq N - 1. \]
Il en découle que
\[ \sum_{-\infty}^{\infty} k^m \varphi(x - k) = P_m(x) \]
où \( P_m(x) \) est un polynôme de degré \( m \).
En restreignant cette identité à \([0, 1]\), on obtient le Lemme 3.

La construction des ondelettes sur l'intervalle \([0, 1]\) débute par l'orthonormalisation de la base de \( V_0^{[0, 1]} \) constituée des fonctions \( \varphi(x + 2N - 2), \varphi(x + 2N - 3), \ldots, \varphi(x) \). Compte tenu du Lemme 3, nous commencerons par substituer à cette base celle composée des monômes \( 1, x, \ldots, x^{N-1} \) puis de \( N - 1 \) fonctions \( g_1(x), \ldots, g_{N-1}(x) \) de \( V_0^{[0, 1]} \). L'orthonormalisation de cette nouvelle base fournit donc \( N \) polynômes orthogonaux suivis de \( N - 1 \) fonctions de \( V_0^{[0, 1]} \) dont les moments d'ordre \( \leq N - 1 \) sont tous nuls. Ces fonctions seront donc des ondelettes et il en sera de même de celles que nous allons maintenant construire.

4. La construction des ondelettes sur l'intervalle

Cette construction repose sur l'énoncé suivant qui permet de compléter la base \( \varphi_{j,k}, k \in S(j) \), en une base de \( V_{j+1}^{[0, 1]} \).

**Théorème 2.** Pour tout \( j \geq 0 \), une base de \( V_{j+1}^{[0, 1]} \) est constituée de la réunion de la base \( \varphi_{j,k} \) de \( V_{j+1}^{[0, 1]} \) et des fonctions \( \psi_{j,k} \) telles que \( -N + 1 \leq k \leq 2^j - N \).

Pour établir ce résultat, on commence par démontrer le résultat suivant

**Lemme 4.** Les fonctions \( \psi(2^j x - k), -2N + 2 \leq k \leq -N, \) une fois restreintes à \([0, 1]\), appartiennent à \( V_j^{[0, 1]} \).

Nous poserons \( 2^j x = t \) et définirons \( V_0^{[0, \infty]} \) comme l'espace des restrictions à \([0, \infty[ \) des fonctions de \( V_0 = V_0(\mathbb{R}) \). Le Lemme 4 résultera du résultat plus précis suivant
Lemme 5. \( \text{Les fonctions } \psi(x - k), \ -2N + 2 \leq k \leq -N, \text{ une fois restreintes à } [0, \infty[, \text{ appartiennent à } V^{[0, \infty[}_0. \)

Cela signifie que si plus la moitié du support de \( \psi(x - k) \) tombe à l'extérieur de \([0, \infty[\), la restriction à \([0, \infty[\) de \( \psi(x - k) \) appartient à \( V_0 \), mais, en revanche, que les \( N - 1 \) fonctions \( \psi(x + N - 1), \ldots, \psi(x + 1) \), une fois restreintes à \([0, \infty[\), sont linéairement indépendantes modulo \( V_0 \).

Revenons au Lemme 5. Pour l'établir, on retourne à (2.7) où l'on remplace successivement \( x \) par \( x + 2N - 2 \), puis par \( x + 2N - 3 \), \ldots et, enfin, \( x \) par \( x + N \). Puisque \( \varphi(2x + 2N - 1) = 0 \) si \( x \geq 0 \), il vient successivement, pour \( x \geq 0 \),

\[
\tilde{g}_0 \psi(x + 2N - 2) + \tilde{h}_0 \varphi(x + 2N - 2) = 0
\]

puis

\[
\tilde{g}_0 \psi(x + 2N - 3) + \tilde{g}_2 \psi(x + 2N - 2) + \tilde{h}_0 \psi(x + 2N - 3) + \tilde{h}_2 \varphi(x + 2N - 2) = 0
\]

et enfin

\[
\tilde{g}_0 \psi(x + N) + \cdots + \tilde{g}_{2N-4} \psi(x + 2N - 2) + \tilde{h}_0 \varphi(x + N) + \cdots + \tilde{h}_{2N-4} \varphi(x + 2N - 2) = 0.
\]

A l'aide de ces relations, on démontre successivement que \( \psi(x + 2N - 2) \), restreint à \([0, \infty[\), appartient à \( V^{[0, \infty[}_0 \) puis qu'il en est de même pour \( \psi(x + 2N - 3) \) et, de proche en proche, pour tous les \( \psi(x + k) \), \( 2N - 2 \geq k \geq N \).

Le Lemme 4 résulte du Lemme 5 par simple changement d'échelle.

Pour démontrer le Théorème 2, on observe d'abord que le nombre de fonctions proposées est exactement la dimension \( (2^j + 1 + 2N - 2) \) de \( V_j^{[0, \infty[} \). Pour établir le théorème, il suffit d'établir que ces fonctions constituent un système générateur.

On part donc d'une fonction arbitraire \( f \) de \( V_{j+1} \) et l'on veut montrer que la restriction de \( f \) à \([0, 1]\) s'écrit \( g + h \) où \( g \in V_j^{[0, \infty[} \) et

\[
(4.1) \quad h(x) = \sum_{N+1}^{2^j - N} \alpha(j, k) \psi_{j,k}(x).
\]

En fait, \( f = u + v \) où \( u \in V_j \) et \( v \in W_j \). À ce titre, \( v(x) = \sum_k \beta(j, k) \psi_{j,k}(x). \) Dans cette série, on peut distinguer sept ensembles de valeurs de \( k \). Si \( k \leq -2N + 1 \), la restriction de \( \psi_{j,k} \) à \([0, 1]\) est nulle et on n'a pas à considérer les termes correspondants. Si \( -2N + 2 \leq k \leq -N \), la restriction de \( \psi_{j,k} \) à \([0, 1]\)
appartiennent à $V_j^{[0,1]}$ (Lemme 4) et contribue à la fonction $g(x)$. Les termes tels que $-N + 1 \leq k \leq 2^j - N$ sont ceux qui nous intéressent.

Si $2^j - N + 1 \leq k \leq 2^j - 1$, la restriction de $\psi_{j,k}$ à $[0,1]$ appartient à $V_j^{[0,1]}$. On le démontre par un raisonnement identique à celui qui a conduit au Lemme 4. Enfin si $k \geq 2^j$, la restriction de $\psi_{j,k}$ à $[0,1]$ est nulle.

Le Théorème 2 est donc démontré.

La construction de la base orthonormée d’ondelettes sur l’intervalle $[0,1]$ suit désormais le schéma classique des analyses multirésolutions.

On dispose d’une suite emboîtée de sous-espaces $V_j^{[0,1]}$ de $L^2[0,1]$, $j \geq 0$. La réunion des $V_j^{[0,1]}$ est dense dans $L^2[0,1]$ puisque la réunion des $V_j$ est dense dans $L^2(\mathbb{R})$. On désigne alors par $W_j^{[0,1]}$ le complément orthogonal de $V_j^{[0,1]}$ dans $V_{j+1}^{[0,1]}$. On observera que $W_j^{[0,1]}$ n’est pas l’espace des restrictions à $[0,1]$ des fonctions de $W_j$. En effet le Lemme 4 nous apprend que la restriction à $[0,1]$ de $\psi(2^j x - k)$ apparaît à $V_j^{[0,1]}$ si $-2N + 2 \leq k \leq -N$ ou si $2^j - N + 1 \leq k \leq 2^j - 1$. Lorsque $-N + 1 \leq k \leq -1$ ou $2^j - 2N + 2 \leq k \leq 2^j - N$, les restrictions des fonctions $\psi(2^j x - k)$ n’appartiennent certes pas à $V_j^{[0,1]}$ mais ne sont pas pour autant orthogonales à $V_j^{[0,1]}$.

En tout état de cause, on a

$$L^2[0,1] = V_0^{[0,1]} \oplus W_0^{[0,1]} \oplus W_1^{[0,1]} \oplus \cdots \oplus W_j^{[0,1]} \oplus \cdots$$

Nous disposons déjà d’une base orthonormée de $V_j^{[0,1]}$. Nous nous proposons de construire une base orthonormée de $W_j^{[0,1]}$ pour chaque $j \geq 0$.

À cet effet, on utilise le Théorème 2, en distinguant les cas $0 \leq j < j_0$ et $j \geq j_0$ ($2^{j_0} \geq 4N - 4$).

Dans le premier cas, il suffit de projeter orthogonalement sur $W_j$ les fonctions $\psi_{j,k}$ telles que $-N + 1 \leq k \leq 2^j - N$. Puisque nous disposons déjà d’une base orthonormée de $V_j$, l’opérateur de projection orthogonale sur $V_j$ est explicite. Une fois projetés sur $W_j$, les $\psi_{j,k}$ deviennent des fonctions $\psi_{j,k}$ qu’il convient ensuite d’orthonormaliser entre elles pour $-N + 1 \leq k \leq 2^j - N$.

Lorsque $j \geq j_0$, tout se clarifie. En effet les $\psi_{j,k}$ telles que $0 \leq k \leq 2^j - 2N + 1$ appartiennent de «plein droit» à $W_j^{[0,1]}$. Pour obtenir la base orthonormée de $W_j^{[0,1]}$, il suffit d’adjoindre à ces $2^j - 2N + 2$ fonctions les $2N - 2$ ondelettes manquantes (la dimension de $W_j^{[0,1]}$ est $2^j$). Ces $2N - 2$ ondelettes manquantes se composent de $N - 1$ ondelettes «affectées à 0» et de $N - 1$ ondelettes «affectées à 1».

Pour construire les premières, on part des $N - 1$ fonctions $\psi_{j,k}$ telles que $-N + 1 \leq k \leq -1$. Elles sont orthogonales aux $\varphi_{j,k}$ lorsque $k \geq 0$ tout comme elles le sont aux $\psi_{j,k}$ pour $k \geq 0$. Ce qui manque à ces fonctions $\psi_{j,k}$ est l’orthogonalité aux $N - 1$ fonctions $2^j/2 \varphi^0_{2^j}(2^j x)$, $\ldots$, $2^j/2 \varphi^0_{N-1}(2^j x)$. Ces $N - 1$ fonc-
tions formant une suite orthogonale, les corrections rendant les \( \psi_{j,k} \) \((-N + 1 \leq k \leq -1)\) orthogonales à \( V_j^{[0,1]} \) sont évidentes. On obtient alors \( N - 1 \) fonctions
\( 2^{j/2} h_1(2^j x), \ldots, 2^{j/2} h_{N-1}(2^j x) \) où \( h_1, \ldots, h_{N-1} \) ne dépendent pas de \( j \). Il suffit enfin d’orthonormaliser ces \( N - 1 \) fonctions pour obtenir les \( N - 1 \) ondelettes « affectées à 0 », à savoir \( 2^{j/2} \psi_{1}^a(2^j x), \ldots, 2^{j/2} \psi_{N-1}^a(2^j x) \).

La construction des \( N - 1 \) ondelettes « affectées à 1 », à savoir \( 2^{j/2} \psi_{1}^b(2^j(1 - x)), \ldots, 2^{j/2} \psi_{N-1}^b(2^j(1 - x)) \) est semblable.

Il est plus commode d’indexer les fonctions que nous venons de construire par l’ensemble \( \mathcal{S} \cup E \) où \( \mathcal{S} \) est la collection de tous les intervalles dyadiques inclus dans \([0,1]\) et \( E \) est un ensemble de cardinalité \( 2N - 1 \). On commencera par examiner les intervalles \( I = [k2^{-j}, (k + 1)2^{-j}] \) où \( j \geq j_0, 0 \leq k < 2^j \). On distingue trois cas. Désignons systématiquement par \( (2N - 1)I = \bar{I} \) l’intervalle ayant le même centre que \( I \) et pour longueur \( 2N - 1 \) fois celle (notée \( |I| \)) de \( I \). Le premier cas est celui où \( n_0 \) n’appartient pas à l’intérieur de \( \bar{I} \). Alors \( \bar{I} \) est inclus dans \([0,1]\) et l’on pose
\[
(4.5) \quad \psi_f(x) = 2^{j/2} \psi(2^j x - k + N - 1).
\]

On observera que le support est exactement \( \bar{I} \). Si \( n_0 \) appartient à l’intérieur de \( \bar{I} \), on a \( 0 \leq k \leq N - 2 \) et l’on pose
\[
(4.6) \quad \psi_f(x) = 2^{j/2} \psi^a_{k+1}(2^j x).
\]

Si enfin \( 1 \) appartient à l’intérieur de \( \bar{I} \), on a \( 2^j - (N - 1) \leq k \leq 2^j - 1 \) et l’on pose \( I = 2^j - k \) et
\[
(4.7) \quad \psi_f(x) = 2^{j/2} \psi^b(2^j(1 - x)).
\]

Finalement il reste à indexer les \( 2^{j_0} + 2N - 2 \) fonctions manquantes. Nous disposons pour cela de \( 1 + 2 + \cdots + 2^{j_0 - 1} \) intervalles dyadiques \( I \subset [0,1] \) qui n’ont pas encore servi. Désignons par \( j_0 \) le plus petit entier \( j \) tel que \( 2^j \geq 2N - 1 \); si \( j_0 < j < j_0 - 1 \), on incorpore dans la base orthonormée de \( W_j \) les fonctions \( 2^{j/2} \psi(2^j x - k), 0 \leq k \leq 2^j - 2 \) \( -N + 1 \), dont le support est inclus dans \([0,1]\). On complète ces fonctions (que l’on notera bien évidemment \( \psi_f \) en revenant à (4.5)) en une base orthonormée de \( W_j \) que l’on indexe arbitrairement à l’aide des intervalles \( I \) de longueur \( 2^{-j} \) non encore utilisés.

Nous venons, pour chaque \( j \geq 0 \), de former une base orthonormée de \( W_j \) qui se compose des ondelettes \( \psi_f \), \( I \) intervalle dyadique de longueur \( 2^{-j} \) inclus dans \([0,1]\); l’ensemble de ces intervalles dyadiques sera noté \( E_j \). Enfin \( \psi_f \) est donné par (4.5) chaque fois que \( (2N - 1)I = \bar{I} \) est inclus dans \([0,1]\).

Pour obtenir une base orthonormée de \( L^2[0,1] \), il nous reste à former une base orthonormée \( \varphi_0(x), \ldots, \varphi_{2N-1}(x) \) de \( V_0 \). Remarquons tout d’abord que \( 1, x, \ldots, x^{N-1} \) appartiennent à \( V_0 \) et \( \varphi_0, \ldots, \varphi_{N-1} \) ne sont autres que ces monômes orthogonalisés par le procédé de Gram-Schmidt. On complète ces polynômes orthogonaux dans une base orthonormée de \( V_0 \).
Nous venons d’établir le résultat suivant

**Théorème 3.** L’ensemble des fonctions $\psi_{i}(x)$, $i \in \mathfrak{I}$, et des $2N - 1$ fonctions $\varphi_{0}(x), \ldots, \varphi_{2N-2}(x)$ constitue une base orthonormée de $L^{2}[0, 1]$.

5. **Analyse des espaces de Hölder $C^{s}[0, 1])$, $s > 0$, dans la base précédente**

Rappelons que $C^{s}[0, 1]$, $s > 0$, désigne l’espace des restrictions à l’intervalle $[0, 1]$ des fonctions de $C^{s}(\mathbb{R})$. Nous désignerons par $C^{s}_{0}(\mathbb{R})$ la fermeture, pour la norme de l’espace de Banach $C^{s}$, des fonctions de la classe $\mathcal{D}(\mathbb{R})$ de Schwartz. Si $s = 1$, convenons que $C^{1}$ ne désignera pas l’espace usuel mais la classe de Zygmund $\Lambda_{*}$ définie par la condition

$$
|f(x + h) + f(x - h) - 2f(x)| \leq C|h|.
$$

Si $1 < s \leq 2$, on écrira $s = 1 + r$ et $f \in C^{s}(\mathbb{R})$ signifie que $f^{r}(x)$, la dérivée de $f$, appartient à $C^{r}$ etc...

Alors il vient, pour toute fonction $f(x) \in L^{2}[0, 1]$,

**Proposition 2.** Si $0 < s < N$ et si $f(x)$ appartient à $C^{s}[0, 1]$, alors on a

$$
\int_{0}^{1} f(x)\psi_{i}(x) \, dx = O(|I|^{1/2 + s}).
$$

Réciproquement si $0 < s < \gamma(N - 1)$, cette condition caractérise l’espace $C^{s}[0, 1]$.

Rappelons que $\gamma(N - 1)$ mesure la régularité de l’ondelette $\psi(x)$ et de la fonction $\varphi(x)$ qui lui est associée. La preuve de ce résultat est semblable à celle que le lecteur pourra trouver dans [2].

On a des résultats analogues en ce qui concerne l’espace $H^{s}[0, 1]$, $s \geq 0$, des restrictions à $[0, 1]$ des fonctions $f(x) \in H^{s}(\mathbb{R})$.

Un cas remarquable est celui de l’espace BMO de John et Nirenberg dont nous rappelons la définition.

**Définition 1.** Une fonction $f(x)$, appartenant à $L^{2}[0, 1]$, appartient en outre, à BMO $[0, 1]$ s’il existe une constante $C > 0$ telle que pour tout intervalle $I \subset [0, 1]$, on puisse trouver une constante $\gamma(I)$ de sorte que l’on ait

$$
\left( \frac{1}{|I|} \int_{I} |f(x) - \gamma(I)|^{2} \, dx \right)^{1/2} \leq C.
$$
La borne inférieure de ces constantes $C$ est la norme $\text{BMO}$ de $f$. Mais on peut souhaiter éviter (ce que nous ferons) que les constantes aient pour norme $0$. Alors on ajoute $\left| \int_0^1 f(x) \, dx \right|$ à la norme précédente.

L'espace $\text{BMO} [0, 1]$ se compose des restrictions à $[0, 1]$ des fonctions de $\text{BMO} (\mathbb{R})$. Pour le montrer, il suffit de construire l'opérateur de prolongement d'une fonction $f \in \text{BMO} [0, 1]$ en une fonction $g \in \text{BMO} (\mathbb{R})$. Ce prolongement s'effectue en imposant à $g(x)$ d'être paire et périodique de période $2$.

L'espace $\text{BMO} [0, 1]$ est le dual $E^*$ d'un espace de Banach $E$; $E$ est le sous-espace de l'espace de Hardy $H^1 (\mathbb{R})$ composé de toutes les fonctions de $H^1 (\mathbb{R})$ dont le support est inclus dans $[0, 1]$.

On suppose $N \geq 2$ et l'on a alors (avec les notations du Théorème 3)

**Proposition 3.** Une fonction $f(x)$ de carré sommable sur $[0, 1]$ appartient à $\text{BMO} [0, 1]$ si et seulement si l'on a, pour une certaine constante $C$,

$$\sum_{j \in I} \| (f, \psi_j) \|^2 \leq C |I|$$

où $I$ est un intervalle dyadique arbitraire inclus dans $[0, 1]$ et où la somme porte sur tous les sous-intervalles dyadiques $J \subset I$.

Pour conclure cette section, observons que l'analyse en ondelettes des espaces $C^r [0, 1]$ ou $\text{BMO} [0, 1]$ fournit automatiquement des opérateurs de prolongement à $\mathbb{R}$ tout entier. Explicitons ce point dans le cas de $\text{BMO} [0, 1]$. Les ondelettes permettant de reconstituer $f$ sont soit de la forme $2^{j/2} \psi_1 (2^j x - k)$ et leur support est alors inclus dans $[0, 1]$, soit de la forme $2^{j/2} \psi_1 (2^j x), \ldots, 2^{j/2} \psi_{N-1} (2^j x), \ldots, 2^{j/2} \psi_{N-1} (2^j (1 - x)), \ldots, 2^{j/2} \psi_{N-1} (2^j (1 - x))$, soit enfin l'une des fonctions $\varphi_0, \ldots, \varphi_{2N-2}$. Si bien que $f \in \text{BMO} [0, 1]$ s'écrit canoniquement $f = f_1 + f_2 + f_3 + f_4$ où $f_1$ est une fonction de $\text{BMO} (\mathbb{R})$ dont le support est inclus dans $[0, 1]$,

$$f_2 (x) = \sum_{j \geq J_0} \sum_{1}^{N-1} \alpha (j, m) \psi_{n} (2^{j} x) \quad \text{et} \quad \sup_{j \geq j_0} |\alpha (j, m)| < \infty$$

$$f_3 (x) = \sum_{j \geq J_0} \sum_{1}^{N-1} \beta (j, m) \psi_{n} (2^{j} (1 - x)) \quad \text{et} \quad \sup_{j \geq j_0} |\beta (j, m)| < \infty$$

et finalement

$$f_4 (x) = \gamma_0 \varphi_0 (x) + \cdots + \gamma_{2N-2} \varphi_{2N-2} (x).$$

Le prolongement des fonctions $\psi_{n} (x)$ et $\varphi_{n} (x)$ à $\mathbb{R}$ tout entier est fourni par construction. Naturellement ces fonctions, une fois prolongées, perdent leur caractère oscillant. On a
\[ \int_0^\infty x^k \psi_m(x) \, dx = 0 \]

si \( 0 \leq k \leq N - 1, 1 \leq m \leq N - 1 \) mais l’intégrale correspondante sur \( (-\infty, \infty) \) n’est pas nulle. Néanmoins, les fonctions \( f_j(x) \) et \( f_j^b(x) \) appartiennent à BMO(\( \mathbb{R} \)) car seule la régularité et le support compact des fonctions \( \psi^b_m \) et \( \psi^b_m \) importent.

6. Opérateurs de Calderón-Zygmund sur l’intervalle

Commençons par étendre au carré unité [0, 1] \times [0, 1] = \( Q_0 \) la base orthonormée de \( L^2[0, 1] \) que nous venons de construire. On procède comme dans [2] et l’on observe que la réunion des sous-espaces \( V_j^{[0, 1]} \otimes V_j^{[0, 1]} \) de \( L^2(Q_0) \) est dense dans \( L^2(Q_0) \). On a donc

\[ L^2(Q_0) = (V_0 \otimes V_0) \oplus (W_0 \otimes V_0) \oplus (V_0 \otimes W_0) \oplus (W_0 \otimes W_0) \oplus \cdots \]

Une base orthonormée de \( V_0 \otimes V_0 \) se compose des fonctions \( \varphi_i(x)\varphi_m(y) \), \( 0 \leq i \leq 2N - 2, 0 \leq m \leq 2N - 2 \), déjà utilisées. Nous utiliserons de même une base orthonormée \( \varphi_j \) de \( V_j = V_j^{[0, 1]} \), construite de la façon suivante. La base \( \varphi_j \) est indexée par l’ensemble \( F_j \) des intervalles dyadiques de longueur \( 2^{-j} \) inclus dans \( [-2^{-j} - 2, 2^{-j}] \). Si \( I = (2^{-j} - 1)I \) est inclus dans [0, 1], alors \( \varphi_j(x) = 2^{j/2} \varphi(2^j x - k + N - 1) \) et si 0 ou 1 appartiennent à l’intérieur de \( I \), la construction de \( \varphi_j \) est calquée sur celle que nous avons donnée pour \( \psi_j \).

Une base orthonormée de \( W_j \otimes V_j \) est donc composée des fonctions \( \psi_j(x)\psi_j(y) \) où \( I \in E_j \) (ensemble des intervalles dyadiques \( I \) inclus dans [0, 1]) et de longueur \( 2^{-j} \) et \( J \in E_j \). De même une base orthonormée de \( V_j \otimes W_j \) est composée de fonctions \( \varphi_j(x)\psi_j(y) \), \( I \in F_j \), \( J \in E_j \). Enfin une base orthonormée de \( W_j \otimes W_j \) est composée des fonctions \( \psi_j(x)\psi_j(y) \), \( I \in E_j \), \( J \in E_j \).

Tout ce que nous dit dans le cas de l’intervalle [0, 1] convient au cas de [0, 1] \times [0, 1]. C’est-à-dire que la décomposition dans cette base fournit un prolongement canonique des objets que l’on analyse.

Nous allons vérifier cette assertion en analysant, dans la base que nous venons de construire, les noyaux distributions \( S(x, y) \) des opérateurs de Calderón-Zygmund opérant sur \( L^2[0, 1] \).

Commençons par une forme bilinéaire \( J: C^1[0, 1] \times C^1[0, 1] \to \mathbb{C} \) définie par la distribution correspondante \( S \in \mathcal{D}'(\mathbb{R}^2) \) dont le support est inclus dans [0, 1].

Nous supposerons que la restriction à \( y \neq x \) de \( S(x, y) \) est une fonction \( K(x, y) \) vérifiant les conditions

\[ |K(x, y)| \leq C_0 |x - y|^{-1}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad y \neq x, \]

(6.2)
(6.3) il existe un exposant \( \gamma \in ]0, 1[ \) tel que l'on ait
\[
|K(x', y) - K(x, y)| \leq C_0 |x' - x|^\gamma |x - y|^{-1 - \gamma}
\]
chaque fois que
\[
|x' - x| \leq \frac{1}{2} |x - y|, \quad 0 \leq x \leq 1, \quad 0 \leq x' \leq 1, \quad 0 \leq y \leq 1, \quad y \neq x,
\]
on a de même,
\[
(6.4) \quad |K(x, y') - K(x, y)| \leq C_0 |y' - y|^\gamma |x - y|^{-1 - \gamma}
\]
chaque fois que
\[
|y' - y| \leq \frac{1}{2} |x - y|, \quad 0 \leq x \leq 1, \quad 0 \leq y' \leq 1, \quad 0 \leq y \leq 1, \quad x \neq y.
\]

Définissons maintenant la continuité faible. A cet effet, pour tout intervalle fermé \( I \subset ]0, 1[ \) et toute fonction \( f \in C^1[0, 1] \) à support dans \( I \), nous posons
\[
N_I(f) = |I|^{1/2} \sup_I |f(x)| + |I|^{3/2} \sup_I |f'(x)|;
\]
si \( I = [0, a], a < 1 \), nous ne demandons pas que \( f \) ou \( f' \) s'annulent en 0 et de même si \( I = [b, 1] \).

Nous dirons que \( J \) a la propriété de continuité faible s'il existe une constante \( C \) telle que, pour tout intervalle fermé \( I \subset ]0, 1[ \) et tout couple de fonctions \( f \) et \( g \) de \( C^1[0, 1] \) à supports dans \( I \), on ait
\[
(6.5) \quad |J(f, g)| \leq CN_I(f)N_I(g).
\]

Désignons enfin par \( T \) l'opérateur défini par la forme \( J \) et par
\[
\langle T(f), g \rangle = J(f, g).
\]

Le problème que nous posons est de savoir si un opérateur \( T \) vérifiant (6.2), (6.3), (6.4) et (6.5) se prolonge en un opérateur continu sur \( L^2[0, 1] \). A cet effet, analysons \( S(x, y) \) dans la base d'ondelettes du carré. On obtient, grâce à la continuité faible et aux propriétés (6.2), (6.3) et (6.4)
\[
(6.6) \quad |J(\psi^n, \psi_j)| \leq C_1 (1 + |k - l|)^{-1 - \gamma}
\]
\[
(6.7) \quad |J(\varphi^n, \varphi_j)| \leq C_1 (1 + |k - l|)^{-1 - \gamma}
\]
\[
(6.8) \quad |J(\psi^n, \psi_j)| \leq C_1 (1 + |k - l|)^{-1 - \gamma}
\]
lorsque \( I = [k2^{-i}, (k + 1)2^{-i}] \) et \( J = [l2^{-j}, (l + 1)2^{-j}] \). Inversement ces propriétés caractérisent les distributions \( S \) vérifiant (6.2), (6.3), (6.4) et la propriété de continuité faible.
Désignons par $G_j$ l'ensemble des intervalles dyadiques $I$ de longueur $2^{-j}$ tels que $(2N - 1)I = I \subset [0, 1]$. On part de la décomposition de la distribution $S(x, y)$ dans la base d'ondelettes de $L^2(Q_0)$. On regroupe tous les termes $\alpha(I, J)\varphi_j(x)\psi_j(y) + \beta(I, J)\psi_j(x)\varphi_j(y) + \gamma(I, J)\psi_j(x)\psi_j(y)$ tels que $I \in G_j$ et $J \in G_j$ et leur somme est notée $\tilde{S}(x, y)$. Alors $\tilde{S}$ appartient automatiquement à $\mathcal{D}(\mathbb{R}^2)$ et la restriction $\tilde{S}(x, y)$ de $S$ à $y \neq x$ vérifie (6.2), (6.3) et (6.4) dans $\mathbb{R}^2$ tout entier. En outre $\tilde{S}(x, y) = 0$ hors de $Q_0$.

La différence $R(x, y) = S(x, y) - \tilde{S}(x, y)$ vérifie

$$|R(x, y)| \leq \frac{C_2}{|x| + |y|} + \frac{C_2}{|x - 1| + |y - 1|}$$

et l'opérateur associé est automatiquement borné sur $L^2[0, 1]$. En outre le prolongement canonique de $R(x, y)$ hors de $Q_0$ est un noyau à support dans $(2N - 1)Q_0$ qui vérifie (6.2), (6.3) et (6.4) dans tout $\mathbb{R}^2$.

Il est alors immédiat de conclure. Si $T$ se prolonge en un opérateur linéaire continu sur $L^2[0, 1]$, alors l'opérateur $\tilde{T}$ est continu de $L^2(\mathbb{R})$ dans lui-même. Le noyau-distribution $\tilde{S}(x, y)$ de $\tilde{T}$, une fois restreint à $y \neq x$, vérifie (6.2), (6.3) et (6.4). Le théorème de David et Journé s'applique donc et la continuité en question équivaut à $\tilde{T}(1) \in \text{BMO}$ et $\tilde{T}'(1) \in \text{BMO}$. Finalement il faut comparer $\tilde{T}(1)$ à $T(1)$ d'une part, $\tilde{T}'(1)$ à $T'(1)$ d'autre part. En revenant aux décompositions des noyaux $S(x, y)$ et $\tilde{S}(x, y)$, on voit que $\tilde{T}(1)$ et $T(1)$ ne diffèrent que par deux fonctions de BMO que nous avons appelées $f_2(x)$ et $f_3(y)$ dans la Section 5.

Ainsi la continuité de l'opérateur $T$ équivaut à $T(1) \in \text{BMO}[0, 1]$ et $T'(1) \in \text{BMO}[0, 1]$.

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Spherical Means on the Heisenberg Group and a Restriction Theorem for the Symplectic Fourier Transform

S. Thangavelu
Dedicated to Prof. E. M. Stein on his 60th birthday

1. Introduction

Spherical mean value operators on a compact Riemannian manifold $M$ have been extensively studied by Sunada in a series of papers [14], [15] and [16]. He has studied the eigenvalue problem $L_r f = \alpha f$ associated with the spherical mean value operator $L_r$. The question about the eigenvalues $\alpha = 1$ and $\alpha = -1$ are related to the ergodicity and mixing properties of the geodesic random walk of step size $r$ on the manifold $M$. In a recent article [7] Pati-Shahshahani-Sitaram have investigated the eigenvalue problem in the case when $M$ is a compact symmetric space. In this case they are able to identify the eigenvalues completely in terms of the elementary spherical functions associated to $M$. They provide alternate proofs of some results of Sunada regarding the eigenvalues 1 and $-1$. Let us briefly recall their result.

Let $M = G/K$ be a compact symmetric space. Then the spherical mean value operator $L_r$ can be identified with a convolution operator $L_r f = f \ast \nu_r$ where $\nu_r$ is a certain probability measure which can be viewed as a $K$-bi-invariant measure on $G$. Let $\hat{G}_1$ denote the collection of all pairwise inequivalent, irreducible,
unitary representations of class 1 of $G$. For each $\pi \in \hat{G}_1$ there is an elementary spherical function $\phi_\pi$ associated with it. The main result of [7] can now be stated as follows.

**Theorem 0.** All the eigenvalues of the operator $L_\pi$ are of the form

$$\psi_\pi(r) = \int_G \phi_\pi(x) \, d\nu,$$

where $\pi \in \hat{G}_1$.

The aim of this paper is to study spherical mean value operators on the reduced Heisenberg group $H^n/\Gamma$. Here $H^n$ is the Heisenberg group and $\Gamma$ is the subgroup $\{(0, 2\pi k) : k \in \mathbb{Z}\}$ of $H^n$. The Heisenberg group $H^n$ is a nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$. The coordinates on $H^n$ are $(z, t)$ where $z = x + iy$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The group law is defined by

$$(z, t)(w, s) = \left( z + w, t + s + \frac{1}{2} \text{Im } z\bar{w} \right).$$

The Haar measure in $H^n$ is the Lebesgue measure $dz \, dt$. The group $H^n/\Gamma$ is a nilpotent Lie group with compact centre. A function on $H^n$ is said to be radial or rotation invariant if it is invariant under rotations in the variable $z$.

By a spherical mean value operator we mean an operator of the form $T_\mu f = f \ast \mu$ where $\mu$ is a rotation invariant compactly supported probability measure on $H^n/\Gamma$. We are able to identify all the eigenvalues of the operator $T_\mu$. For each $k = 0, 1, 2, \ldots$ and $\lambda \neq 0$ there are certain radial functions $e_\lambda^k$ on the Heisenberg group $H^n$ which can be thought of as the elementary spherical functions for the Heisenberg group. As in the case of the compact symmetric space, the eigenvalues are then given by the averages of $e_\lambda^k$ with respect to $\mu$.

**Theorem 1.** Assume that $\mu$ has no mass at the centre of $H^n/\Gamma$. Then all the eigenvalues of the operator $T_\mu$ are given by

$$\alpha_k(j) = \int_{H^n/\Gamma} e_\lambda^k(z, t) \, d\mu,$$

where $j$ is an integer. Further, any function of the form $f \ast e_{-j}^k$ satisfies

$$T_\mu(f \ast e_{-j}^k) = \alpha_k(j)(f \ast e_{-j}^k).$$

We can make more precise statements regarding the eigenvalues if we take $\mu = \mu_{r, t}$ where $\mu_{r, t}$ is the normalized Lebesgue (surface) measure on the sphere $S_{r, t} = \{(z, t) : |z| = r\}$ in $H^n/\Gamma$. Let $M_{r, t}$ stand for $T_\mu$ when $\mu = \mu_{r, t}$. The
elementary spherical functions $e^1_k(z, s)$ are radial functions of $z$ and slightly abusing the notation we write $e^1_k(r, s)$ in place of $e^1_k(z, s)$ when $|z| = r$.

**Theorem 2.**

(i) All the eigenvalues of the operator $M_{r, t}$ are given by $\alpha_k(j) = e^1_k(r, t)$.

(ii) $\alpha = 1$ and $\alpha = -1$ are not eigenvalues of the operator $M_{r, t}$ for any $r > 0$.

By writing down the Fourier series of $f * \mu_{r, t}$ we can see that it involves operators of the form $g \times \mu_r$ where $g \times \mu_r$ is the twisted convolution of $g$ with the surface measure on the sphere $|z| = r$ in $\mathbb{C}^n$. The spectral properties of the operator $T_r g = (2\pi)^{n} g \times \mu_r$ are worth studying and we have the following theorem.

**Theorem 3.**

(i) All the eigenvalues of the operator $T_r$ are given by

$$\alpha_k = \frac{k!(n-1)!}{(k+n-1)!} \phi_k(r)$$

where $\phi_k$ are the Laguerre functions of type $(n-1)$.

(ii) For each $k$ the eigenspace corresponding to the eigenvalue $\alpha_k$ is infinite dimensional; hence the operator $T_r$ is not compact.

(iii) $\alpha = 1$ and $\alpha = -1$ are not eigenvalues of $T_r$ and $\alpha = 0$ is an eigenvalue if and only if $\phi_k(r) = 0$ for some $k$.

The operators $T_r$ also arise naturally in connection with certain restriction operators $R_r$ for the symplectic Fourier transform on $\mathbb{R}^{2n}$. In Section 5 we will show that we can write

$$f(z) = (4\pi)^{-n} \omega_{2n} \int_0^\infty R_r f(z) r^{2n-1} dr$$

where $R_r$ are the restriction operators. These restriction operators $R_r$ are related to $T_r$ by $R_r f = (2\pi)^{-n} T_r (\mathbb{F} r f)$ where $\mathbb{F} r f$ is the symplectic Fourier transform of $f$. Using the above relation and the spectral properties of $T_r$ we are able to prove the following theorem regarding the mapping properties of the restriction operators $R_r$.

**Theorem 4.** Assume that $n \geq 3$. Then the following are true.

(i) $\|R_r f\|_{p'} \leq C_r \|f\|_{p'}, \quad$ for \quad $1 \leq p \leq 2$,

(ii) $\|R_r f\|_{p} \leq C_r \|f\|_{p}, \quad$ for \quad $\frac{2n}{n+1} \leq p \leq 2$, 

where $C_r$ is a constant dependent on $r$.
(iii) $|R, f|_q \leq C, \|f\|_p$, for $1 \leq p \leq \frac{2n}{n+1}$, $\frac{n-1}{n+1} < q < p'$.

To prove this theorem we need to use some mapping properties of the projection operators associated with special Hermite expansions. We will also show that the operators $R_r$ are regularising in the sense that they take $L^2(C^n)$ into $W^q(C^n)$ where $W^q(C^n)$ are the twisted Sobolev spaces to be defined in the sequel. The plan of the paper is as follows. In the next section we will define the functions $e^k$ and show that they have all the properties satisfied by the elementary spherical functions. In Section 3 we will prove Theorem 1. The spectral properties of $T_r$ will be taken up in Section 4 and finally the restriction operators will be studied in Section 5.

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2. Elementary Spherical Functions on the Heisenberg Group

Let us briefly recall the definition and properties of elementary spherical functions. Let $G$ be a semisimple, noncompact, connected Lie group with finite centre and $K$ a maximal compact subgroup. Let $C_c(K \setminus G/K)$ denote the space of continuous functions with compact support on $G$ which satisfy $f(k_1gk_2) = f(g)$ for all $k_1, k_2$ in $K$. Such functions are called spherical or $K$-bi-invariant. Then $C_c(K \setminus G/K)$ forms a commutative Banach algebra under convolution. An elementary spherical function $\phi$ is then defined to be a $K$-bi-invariant continuous function with $\phi(e) = 1$ such that $f \rightarrow f \ast \phi(e)$ defines an algebra homomorphism of $C_c(K \setminus G/K)$.

The elementary spherical functions are characterised by the following properties (see [3]):

(i) They are eigenfunctions of the convolution operator:

$$f \ast \phi = \hat{\phi}(f) \phi,$$

where

$$\hat{\phi}(f) = \int_G f(x^{-1}) \phi(x) \, dx.$$

(ii) They are eigenfunctions for a large class of left invariant differential operators on $G$. 

(iii) They satisfy
\[ \int_K \phi(xk) \, dk = \phi(x)\phi(y). \]

Let us now consider the case of the Heisenberg group \( H^n \). The role of the \( K \)-biinvariant functions will be played by the radial functions on \( H^n \). If \( L^1(\mathbb{T}^n) \) stand for the subspace of \( L^1(\mathbb{T}^n) \) containing all the radial functions then it is well known that \( L^1(\mathbb{T}^n) \) is a commutative Banach algebra under convolution (see Hulanicki-Ricci [5]). This will play the role of \( C_c(K \setminus G/K) \). On the Heisenberg group we have the following \( (2n + 1) \) left invariant vector fields \( X_j, Y_j, T \):
\[
X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.
\]
The sublaplacian on the Heisenberg group is defined by
\[
\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2).
\]

Let \( \phi_k \) be the Laguerre functions of type \( (n - 1) \) defined by the generating function identity
\[
\sum_{k=0}^{\infty} k^r \phi_k(z) = (1 - r)^{-\frac{n}{2}} e^{-\frac{(1/4)(1 + r)(1 - r)|z|^2}}.
\]

For any nonzero real number \( \lambda \) we set \( \phi_k^\lambda(z) = \phi_k(|\lambda|^{1/2} z) \) and define \( e_k^\lambda \) by
\[
e_k^\lambda(z, t) = \frac{k! (n - 1)!}{(k + n - 1)!} e^{\lambda z} \phi_k(z).
\]

It follows from the properties of the Laguerre functions (see Szego [17]) that \( e_k^\lambda(0, 0) = 1 \). We claim that these functions satisfy the following properties.

**Theorem 2.1.**

(i) For any polynomial \( p \) with constant coefficients one has
\[
p(\mathcal{L}) e_k^\lambda = p(2|\lambda|)(2k + n)) \, e_k^\lambda.
\]

(ii) For any radial function \( f \) on \( H^n \) one has
\[
f * e_k^\lambda = (2\pi)^n R_k(-\lambda, f) e_k^\lambda,
\]
where \( R_k(\lambda, f) \) is defined by the formula
\[
R_k(\lambda, f) = (2\pi)^{-\frac{n}{2}} \frac{k! (n - 1)!}{(k + n - 1)!} \int_{\mathbb{C}^n} f(z, \lambda) \phi_k^\lambda(z) \, dz,
\]
being the inverse Fourier transform

\[
\hat{f}(z, \lambda) = \int_{-\infty}^{\infty} e^{it\lambda} f(z, t) \, dt.
\]

(iii) For any \((w, s)\) in \(H^n\) with \(|w| = r\) one has the identity

\[
\int_{|w'| = 1} e^{i\lambda}((z, t) \cdot (rw', -s)) \, d\sigma(w') = e^{i\lambda}((z, t)) e^{-i\lambda}(w, s)
\]

where \(d\sigma\) is the normalized surface measure on \(|w'| = 1\).

Thus we see that the functions \(e_k^\lambda\) have all the properties satisfied by the elementary spherical functions on a semisimple Lie group. So, they can be rightly called the elementary spherical functions for the Heisenberg group. The above properties of the function \(e_k^\lambda\) are fairly well-known in the literature though not stated in the above form (see e.g. Stempak [12] and Strichartz [13]). Nevertheless, we will give a proof of the above theorem here.

To prove the theorem we need to recall several facts about the twisted convolution and the Weyl transform (see Folland [2], Mauceri [6] and Peetre [8]). The twisted convolution of two functions \(f\) and \(g\) defined on \(C^n\) is defined to be

\[
f \times g(z) = \int_{C^n} f(z - w)g(w) e^{it/2} \omega dw.
\]

The Weyl transform of a function \(f\) is the bounded operator \(W(f)\) acting on \(L^2(\mathbb{R}^n)\) given by

\[
W(f)\phi(\xi) = \int_{C^n} f(z)W(z)\phi(z) \, dz
\]

where \(\phi \in L^2(\mathbb{R}^n)\) and \(W(z)\) is the operator valued function

\[
W(z)\phi(\xi) = e^{i(z\xi + y/2)}\phi(z + y).
\]

The relation between the Weyl transform and the twisted convolution is given by \(W(f \times g) = W(f)W(g)\).

The Hermite functions \(\Phi_\alpha(x)\) play an important role in the harmonic analysis on the Heisenberg group (see Folland [2]). These are eigenfunctions of the Hermite operator \(H = (-\Delta + |x|^2)\), \(H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha\). Let \(P_k\) be the orthogonal projection of \(L^2(\mathbb{R}^n)\) onto the \(k\)th eigenspace spanned by \(\{\Phi_\alpha : |\alpha| = k\}\). We also need certain properties of the special Hermite functions. Let us define

\[
\Phi_{\alpha, \beta}(x) = (2\pi)^{-n/2} \int_{C^n} e^{i\xi x} \Phi_\alpha(\xi + \frac{y}{2}) \Phi_\beta(\xi - \frac{y}{2}) \, d\xi.
\]
Then it is well known that they form a complete orthonormal system for $L^2(\mathbb{C}^n)$ (see Strichartz [13]). Let $\pi(\alpha, \beta)$ denote the operator defined by

$$\pi(\alpha, \beta)\phi = (\phi, \Phi_\alpha)\Phi_\beta.$$  

Then one has the following proposition (see Folland [2] and Peetre [8]).

**Proposition 2.1.**

(i) $W(\Phi_{\alpha \beta}) = (2\pi)^{n/2}\pi(\alpha, \beta)$ and consequently $\Phi_{\alpha \beta} \times \Phi_\nu = 0$ if $\beta \neq \nu$ and $\Phi_{\alpha \beta} \times \Phi_\beta = (2\pi)^{n/2}\Phi_{\alpha \beta}$.

(ii) $W(\phi_k) = (2\pi)^n P_k$ and consequently $\phi_j \times \phi_k = (2\pi)^n \delta_{jk} \phi_k$ where $\delta_{jk}$ is the Kronecker $\delta$.

By abusing the notation slightly let us write $\phi_k(r)$ in place of $\phi_k(z)$ when $|z| = r$. Then the functions

$$\psi_k(r) = \left(\frac{2^{1-n}k!}{(k + n - 1)!}\right)^{1/2} \phi_k(r)$$

form a complete orthonormal system in $L^2(\mathbb{R}_+, r^{2n-1} dr)$. If $f$ is a radial function on $\mathbb{C}^n$ then we can expand $f$ in terms of $\psi_k(r)$ obtaining

$$f(z) = \sum_{k=0}^{\infty} R_k(f) \phi_k$$

where

$$R_k(f) = (2\pi)^{-n} \frac{k!(n - 1)!}{(k + n - 1)!} \int_{\mathbb{C}^n} f(z) \phi_k(z) \, dz.$$ 

This proves that when $f$ is a radial function one has $f \times \phi_k = (2\pi)^n R_k(f) \phi_k$.

Now we are ready to prove Theorem 2.1. The assertion (i) is already proved in Strichartz [13] and so we will not prove it here. For (ii) an easy calculation reveals that

$$f \ast e_k^\lambda(z, t) = \frac{k!(n - 1)!}{(k + n - 1)!} \int_{\mathbb{C}^n} f(z - w, -\lambda) \phi_k^\lambda(w) e^{-i(\lambda/2)1_m} \, dw.$$ 

It is therefore enough to show that the above integral is equal to $(2\pi)^n R_k(-\lambda, f) \phi_k^\lambda(z)$. By rescaling we can assume that $\lambda = -1$. But then we need to show that

$$\int_{\mathbb{C}^n} f(z - w, 1) \phi_k(w) e^{-i21_m} \, dw = (2\pi)^n R_k(1, f) \phi_k(z)$$

which follows from the above remark as $f$ is radial.
The proof of the assertion (iii) is similar. We have

\[ \int_{|w'|=1} e_k^\lambda (z - w, t - s - \frac{1}{2} \Im z \bar{w}) d\sigma (w') \]

\[ = \frac{k! (n - 1)!}{(k + n - 1)!} \int_{|w'|=1} \phi_k^\lambda (z - w) e^{-i(\lambda/2) \Im z \bar{w}} d\sigma (w'). \]

Again we can assume that \( \lambda = -1 \). The function

\[ F_k(z, w) = \int_{|w'|=1} \phi_k(z - w) e^{i/2 \Im z \bar{w}} d\sigma (w') \]

is a radial function of \( w \) and hence in view of (2.11)

\[ F_k(z, w) = \sum_{j=0}^\infty R_j(F_k) \phi_j(w). \]

But

\[ R_j(F_k) = (2\pi)^{-n} \frac{k! (n - 1)!}{(k + n - 1)!} \int_{C^*} F_k(z, w) \phi_j(w) dw \]

\[ = (2\pi)^{-n} \frac{k! (n - 1)!}{(k + n - 1)!} \phi_k \times \phi_j(z). \]

This proves that

\[ F_k(z, w) = \frac{k! (n - 1)!}{(k + n - 1)!} \phi_k(z) \phi_k(w). \]

We have proved that

\[ \int_{|w'|=1} e_k^\lambda (z - w, t - s - \frac{1}{2} \Im z \bar{w}) d\sigma (w') = e_k^\lambda (z, t) e_k^\lambda (w, s). \]

Hence the theorem.

We would like to end this section with the following remark. Recently Benson-Jenkins-Ratliff [1] has studied «spherical functions» on the Heisenberg group. Let \( K \) be a compact group of automorphisms of \( H^n \) such that the convolution algebra \( L^1_k \) of \( K \)-invariant functions is commutative. A bounded, continuous \( K \)-invariant function \( \varphi \) such that \( f \mapsto \int \varphi \) is an algebra homomorphism on \( L^1_k \) is called a \( K \)-spherical function. In [1] the authors have studied the \( K \)-spherical functions for various different \( K \). When \( K = U(n) \), the \( K \)-spherical functions include our \( e_k^\lambda \). (We are indebted to G. B. Folland and the referee for bringing the above work to our attention.)
3. Spherical Mean Value Operator on the Heisenberg Group

Let \( H^n \) be the \( n \) dimensional Heisenberg group defined in the previous section. Let \( \Gamma \) be the discrete subgroup \( \{ (0, 0, 2\pi k) : k \in \mathbb{Z} \} \). Then the quotient group \( H^n/\Gamma \) is called the reduced Heisenberg group. For \( g \in SO(2n, \mathbb{R}) \) we define a rotation

\[
\hat{g} : H^n/\Gamma \rightarrow H^n/\Gamma \quad \text{by} \quad \hat{g}(z, t) = (gz, t).
\]

By a radial measure we mean a measure \( \mu \) such that for every \( g \in SO(2n, \mathbb{R}) \) and every Borel set \( S \subset H^n/\Gamma \) one has \( \mu(S) = \mu(g^{-1}S) \). Let \( \mu \) be such a rotation invariant probability measure with compact support. Then the operator \( T^* \mu f = (f \ast \mu) \) is called a spherical mean value operator. In the following theorem we identify all eigenvalues of \( T^* \) as averages of the elementary spherical functions \( e^j_k \) as claimed in the introduction.

**Theorem 3.1.** Assume that \( \mu \) has no mass at the centre of \( H^n/\Gamma \). Then all the eigenvalues of the operator \( T^* \) are given by \( \alpha_k(j) \) where

\[
\alpha_k(j) = \int_{H^n/\Gamma} e^j_k(z, t) d\mu.
\]

Any function of the form \( f \ast e^{-j}_k \) is an eigenfunction corresponding to \( \alpha_k(j) \).

To prove this theorem we need to recall several results about the Fourier transform on the Heisenberg group (a good reference is Geller [4]). For each real \( \lambda \neq 0 \) we have an irreducible representation \( \pi_\lambda(z, t) \) acting on \( L^2(\mathbb{R}^n) \). It is defined by

\[
\pi_\lambda(z, t) \phi(\xi) = e^{it} e^{i(x \cdot t + \xi \cdot y)} \phi(\xi + y).
\]

The Fourier transform of a function \( f \) on \( H^n \) is the operator valued function \( \hat{f}(\lambda) \) defined by

\[
\hat{f}(\lambda) = \int_{H^n} f(z, t) \pi_\lambda(z, t) dz dt.
\]

When \( f \) is a radial function \( \hat{f}(\lambda) \) is given by the formula

\[
\hat{f}(\lambda) = (2\pi)^n \sum_{k=0}^{\infty} R_k(\lambda, f) P_k(\lambda).
\]

Here \( R_k(\lambda, f) \) are as defined in (2.5) and \( P_k(\lambda) \) are the projections of \( L^2(\mathbb{R}^n) \) onto the space spanned by \( \{|\lambda|^{n/4} \phi_k(|\lambda|^{1/2} \cdot) : |\lambda| = k \} \). In the case of \( H^n/\Gamma \), \( \pi_\lambda \) is a representation only if \( \lambda \) is an integer say \( \lambda = j \).
Let \( \chi_k \) be the function defined by

\[
\chi_k(z, t) = (2\pi)^{-n} \sum_{j=-\infty}^{\infty} e^{-itj} e^{-j^{2/2}} \phi_k^j(z) |j|^n.
\]

Then it is easy to see that \( \chi_k \in L^2(H^n) \) and \( \hat{\chi}_k(j) = e^{-j^{2/2}} P_k(j) \). We can now calculate the convolution \( \mu \ast \chi_k \).

**Lemma 3.1.** Let \( \mu \) be a rotation invariant probability measure and let \( \chi_k \) be as above. Then \( (\mu \ast \chi_k)^\wedge(j) = \alpha_k(j) \hat{\chi}_k(j) \).

**Proof.** Since \( \mu \ast \chi_k \) is a radial function we can calculate the Fourier transform using formula (3.2). A calculation shows that

\[
\int e^{it' \cdot \mu} \ast \chi_k(z, t) dt = (2\pi)^{-n} |j|^n e^{-j^{2/2}} \int e^{it' \cdot \phi_k^j(z - w)} e^{-i(j/2) \omega \cdot \overline{w}} d\mu^j(w)
\]

where \( \mu^j(w) \) is the \( j \)-th Fourier coefficient of \( \mu \) in the \( \overline{t} \) variable. It follows that

\[
R_k(j, \mu \ast \chi_k) = (2\pi)^{-n} \frac{k! (n - 1)!}{(k + n - 1)!} e^{-j^{2/2}} \int e^{it' \cdot \phi_k^j(w)} d\mu^j(w)
\]

and \( R_i(j, \mu \ast \chi_k) = 0 \) for \( i \neq k \). This shows that

\[
(\mu \ast \chi_k)^\wedge(j) = e^{-j^{2/2}} \left( \int_{H^n \cap T} e^{it' \cdot \phi_k^j(z, t)} d\mu \right) P_k(j).
\]

This completes the proof of Lemma 3.1.

We can now prove the first part of Theorem 3.1. We claim that there exists \( k \) and \( j \neq 0 \) such that \( (f \ast \chi_k)^\wedge(j) \neq 0 \). Assuming the claim for a moment we will prove the theorem. Let \( f \ast \mu = \alpha f \) for a non zero \( f \) in \( L^2(H^n \cap T) \). Then in view of the lemma \( \alpha_k(j)(f \ast \chi_k)^\wedge(j) = (f \ast \mu \ast \chi_k)^\wedge(j) = \alpha(f \ast \chi_k)^\wedge(j) \). This proves that \( \alpha = \alpha_k(j) \) as \( (f \ast \chi_k)^\wedge(j) \neq 0 \).

We will now prove the claim. If \( (f \ast \chi_k)^\wedge(j) = 0 \) for all \( k \) and \( j \neq 0 \) then calculating the Fourier coefficients of \( f \ast \chi_k \) one can see that all the Fourier coefficients of \( f \) except the zero-th one are zero and consequently

\[
f(z, t) = Af(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z, t) dt.
\]

As \( f \neq 0 \), \( Af \neq 0 \) and \( f \ast \mu = \alpha f \) becomes \( Af \ast \mu_0 = \alpha Af \) where now the convolution is on \( \mathbb{R}^{2n} \) and \( \mu_0 \) is the compactly supported measure

\[
d\mu_0(z) = \int_0^{2\pi} d\mu(z, t).
\]

But then \( Af \) has to be zero which is a contradiction.
To prove the second part of the theorem we need to recall Choquet's theorem. Let $K^n$ stand for the set of all rotation invariant probability measures on $H^n / T$. Let $\text{ext}(K^n)$ stand for the set of all extreme points of $K^n$. Then one has $\text{ext}(K^n) = E \cup \Delta_n$ where $E = \{ \mu_{r,t} \}$ and $\Delta_n = \{ \delta_t : t \in \mathbb{R} \}$. Here $\mu_{r,t}$, $r > 0$ is the normalized Lebesgue measure on the sphere $S_r = \{(z, t) : |z| = r \}$ in $H^n / T$ and $\delta_t$ are the Dirac measures. Given $\mu \in K^n$, according to Choquet's theorem there is a measure $M$ such that

$$
\mu(B) = \int_{E \cup \Delta_n} \sigma(B) \, dM(\sigma).
$$

If $\mu$ has no mass at the centre of $H^n / T$ then $M(\Delta_n) = 0$ (see Stempak [11]) and we have

$$
\mu(B) = \int_E \sigma(B) \, dM(\sigma).
$$

Let us consider $f * e_k^{-j} * \mu$. In view of (3.5) we have

$$
f * e_k^{-j} * \mu = \int_E (f * e_k^{-j} * \sigma) \, dM(\sigma).
$$

When $\sigma = \mu_{r,t}$ we can easily calculate $e_k^{-j} * \sigma$. In fact,

$$
e_k^{-j} * \mu_{r,t}(z, s) = \frac{k! (n-1)!}{(k+n-1)!} e^{i(n-1)\gamma} \int_{\mathbb{C}^n} e^{i(j/2) Im z \bar{w}} \phi_k^j(z - w) \, d\mu_{r,t}.
$$

Recall that $\mu_{r,t}$ are the normalized surface measures on the sphere $|z| = r$. Since the functions $|j|^{n/2} \phi_k(|j|^{1/2} r)$ where $\phi_k(r)$ are defined in the previous section form an orthonormal basis for $L^2(\mathbb{R}^+, r^{2n-1} \, dr)$, we can expand the radial function $G_k(r) = \int_{\mathbb{C}^n} e^{i(j/2) Im z \bar{w}} \phi_k^j(z - w) \, d\mu_{r,t}$ in terms of them. In view of the relations $\phi_j \times \phi_k = (2\pi)^3 \delta_{jk} \phi_k$ one calculates that $e_k^{-j} * \mu_{r,t} = e_k^j(w, t) e_k^{-j}(z, s)$ where $|w| = r$. This means that

$$
f * e_k^{-j} * \mu_{r,t} = e_k^j(w, t)(f * e_k^{-j}) = \mu_{r,t}(e_k^j f * e_k^{-j})
$$

where

$$
\mu_{r,t}(e_k^j) = \int e_k^j(z, s) \, d\mu_{r,t} = e_k^j(w, t), \quad |w| = r.
$$

Putting this back in (3.6) we get

$$
f * e_k^{-j} * \mu = \left( \int_E \sigma(e_k^j) \, dM(\sigma) \right)(f * e_k^{-j}) = \mu(e_k^j f * e_k^{-j}).
$$

This completes the proof of Theorem 3.1.
When \( \mu = \mu_{r,t} \), it is immediate that \( \alpha_k(j) = e^j_k(r, t) \). This proves part (i) of Theorem 2. To prove that \( \alpha = \pm 1 \) are not eigenvalues of the operator \( T_\mu = M_{r,t} \) we will prove in the next section (see Proposition 4.2) that for all \( r > 0 \) one has

\[
\frac{k!(n-1)!}{(k+n-1)!} |\phi_k(r)| < 1.
\]

This will then complete the proof of Theorem 2.

4. Spectral Properties of the Operator \( T_r f = (2\pi)^n f \times \mu_r \)

As we have seen in the introduction the study of the spherical mean value operator on the quotient group \( H^n/\Gamma \) involves operators of the form \( f \to f \times \mu_r \). These operators are interesting in their own right and we will show in the next section that they are connected to the restriction operators for the symplectic Fourier transform. We study the spectral properties of the operators \( T_r f = (2\pi)^n f \times \mu_r \), where \( \mu_r \) is the normalized surface measure on the sphere \(|z| = r\). For the operators \( T_r \) we have the following alternate description.

**Theorem 4.1.**

\[
T_r f(z) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \phi_k(r)f \times \phi_k(z).
\]

This theorem is an immediate consequence of the following Proposition in view of the relation \( W(f \times g) = W(f)W(g) \).

**Proposition 4.1.**

\[
W(\mu_r) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \phi_k(r)P_k.
\]

**Proof.** Let \( p_r(z) \) be the Poisson Kernel defined by

\[
p_r(z) = \Gamma \left( \frac{2n+1}{2} \right) \pi^{-\frac{(2n+1)/2}{2}} e^{-|z|^2} \frac{1}{(2\pi)^{n+1/2} |z|^{n+1/2}}.
\]

If \( F \) is a continuous function vanishing at infinity then we know that

\[
\int_{\mathbb{C}^n} F(z) d\mu_r = \lim_{\epsilon \to 0} \int_{\mathbb{C}^n} p_r \ast \mu_r(z) F(z) dz.
\]
Spherical Means on the Heisenberg Group and a Restriction Theorem

Given functions $\phi$ and $\psi$ in $L^2(\mathbb{R}^n)$ let us define $F(z)$ by $F(z) = (W(z)\phi, \psi)$. Since

$$(W(z)\phi, \psi) = e^{i\langle z, y \rangle / 2} \int_{\mathbb{R}^n} e^{ix \cdot \phi} \tilde{\psi}(z) \, d\xi$$

it is clear that $F(z)$ is a continuous function vanishing at infinity. Hence we have

$$(W(\mu)\phi, \psi) = \int_{\mathbb{C}^n} (W(z)\phi, \psi) \, d\mu(z)$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{C}^n} p_\epsilon * \mu_\epsilon(z) F(z) \, dz$$

$$= \lim_{\epsilon \to 0} (W(p_\epsilon * \mu_\epsilon)\phi, \psi).$$

Replacing $\phi$ by $P_k \phi$ we get

$$(W(\mu_k)P_k \phi, \psi) = \lim_{\epsilon \to 0} (W(p_\epsilon * \mu_\epsilon)P_k \phi, \psi).$$

Since $p_\epsilon * \mu_\epsilon$ is a radial function, its Weyl transform is given by

$$W(p_\epsilon * \mu_\epsilon) = (2\pi)^n \sum_{k=0}^{\infty} R_k(p_\epsilon * \mu_\epsilon)P_k$$

and consequently

$$(W(\mu_k)P_k \phi, \psi) = \lim_{\epsilon \to 0} (2\pi)^n R_k(p_\epsilon * \mu_\epsilon)(P_k \phi, \psi).$$

Let us now calculate $R_k(p_\epsilon * \mu_\epsilon)$. We have

$$\int_{\mathbb{C}^n} p_\epsilon * \mu_\epsilon(z) \phi_k(z) \, dz = \int_{\mathbb{C}^n} p_\epsilon * \phi_k(z) \, d\mu(z).$$

As $p_\epsilon * \phi_k(z) \to \phi_k(z)$ uniformly as $\epsilon \to 0$ one gets that

$$\lim_{\epsilon \to 0} R_k(p_\epsilon * \mu_\epsilon) = (2\pi)^{-n} \frac{k!(n-1)!}{(k+n-1)!} \int_{\mathbb{C}^n} \phi_k(z) \, d\mu(z).$$

This proves that

$$(W(\mu_k)P_k \phi, \psi) = \frac{k!(n-1)!}{(k+n-1)!} \phi_k(r)(P_k \phi, \psi).$$

Hence the proposition.
In view of Theorem 4.1 it is easy to see that $T_r$ is a bounded self-adjoint operator. In fact, the functions $\phi_k$ satisfy the estimate

$$|\phi_k(r)| \leq \frac{k!(n-1)!}{(k+n-1)!}$$

(see Proposition 4.2) and $f \times \phi_k$ are orthogonal projections associated to the special Hermite expansions (see [18]) and hence $T_r$ is a bounded self-adjoint operator on $L^2(C^n)$. It is also clear that all the eigenvalues of $T_r$ are given by

$$\alpha_k = \frac{k!(n-1)!}{(k+n-1)!} \phi_k(r)$$

and any function of the form $f \times \phi_k$ is an eigenfunction corresponding to the eigenvalue $\alpha_k$.

In view of the relations $W(\tilde{\Phi}_{\alpha,\beta}) = (2\pi)^{n/2} \pi(\alpha, \beta)$ and $W(\phi_k) = (2\pi)^n P_k$ one checks that $\tilde{\Phi}_{\alpha,\beta} \times \phi_k = (2\pi)^n \tilde{\Phi}_{\alpha,\beta}$ provided $|\alpha| = k$. This means that the functions $\Phi_{\alpha,\beta}$ are eigenfunctions of $T_r$ with eigenvalue $\alpha_k$. As $\Phi_{\alpha,\beta}$ are linearly independent this shows that the eigenspace corresponding to each $\alpha_k$ is infinite dimensional. Thus parts (i) and (ii) of Theorem 3 are proved. Part (iii) follows from the next proposition. The result of the proposition is not new but the novelty lies in the proof.

**Proposition 4.2.** For any $k$ and $r > 0$,

$$|\phi_k(r)| \leq \frac{(k+n-1)!}{k!(n-1)!} = \phi_k(0).$$

**Proof.** The proof is based on the following fact. If the Fourier transform of a function $f$ is positive then $|\hat{f}(x)| < \hat{f}(0)$ for $x \neq 0$. We will show that the Fourier transform of the function $L_k^n(1/2 |x|^2) e^{-1/4|x|^2}$ on $\mathbb{R}^n$ is positive. This means that

$$L_k^n \left( \frac{1}{2} |x|^2 \right) e^{-1/4|x|^2} = |\phi_k(r)| \leq \frac{(k+n-1)!}{k!(n-1)!}.$$ 

To do this we calculate the kernel $K(x,y)$ of the projection $P_k$ in different ways.

From the very definition one has the formula

$$K(x,y) = \sum_{|\alpha| = k} \Phi_{\alpha}(x) \Phi_{\alpha}(y).$$

On the other hand, as $W(\phi_k) = (2\pi)^n P_k$, $K(x,y)$ is also given by

$$K(x,y) = \int_{\mathbb{R}^n} \phi_k(\xi, y-x) e^{\xi/2 i(y-x)} d\xi$$
where we have written

\[ \phi_k(\xi, \eta) = \phi_k \left( \frac{1}{2} (|\xi|^2 + |\eta|^2) \right). \]

Therefore, setting \( x = y \) we get

\[
(2\pi)^n \sum_{|\alpha| = k} \Phi_n(x) e^{ik} L_k^{n-1} \left( \frac{1}{2} |\xi|^2 \right) e^{-1/4 |\xi|^2} \, d\xi.
\]

This proves that the Fourier transform of the Laguerre function is non-negative.

We will conclude this section with a result analogous to a theorem of Ragozin on the convolution of rotation invariant measures on \( \mathbb{R}^n \). In [9] Ragozin proved that if \( \mu \) is the surface measure on the unit sphere in \( \mathbb{R}^n, n \geq 2 \) then \( \mu \ast \mu \) is absolutely continuous with respect to the Lebesgue measure. Here we will prove a similar result for the twisted convolution. Moreover, we will identify the density explicitly.

**Proposition 4.3.** Assume that \( n \geq 2 \). Then \( \mu_r \times \mu \) is absolutely continuous with respect to the Lebesgue measure. The density is given by

\[
J(z) = (2\pi)^{-2n} \sum_{k=0}^{\infty} \left( \frac{k!}{(k + n - 1)!} \right)^2 (\phi_k(r))^2 \phi_k(z)
\]

where the series converges uniformly on every compact subset of the form \( 0 < a \leq |z| \leq b \).

**Proof.** In view of Theorem 4.1 one has

\[
f \times \mu_r(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \phi_k(r) f \times \phi_k(z).
\]

This in turn gives us

\[
f \times \mu_r \times \mu(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} \left( \frac{k!}{(k + n - 1)!} \right)^2 \phi_k(r)^2 f \times \phi_k(z).
\]

This shows that \( \mu_r \times \mu \) is given by \( J(z) \, dz \). If \( a \leq |z| \leq b \) then one has the asymptotic estimate (see Szego [17]),

\[
L_k^{n-1}(r^2) e^{-r^2/2} r^{n-1}
\]

\[
= K^{-(n-1)/2} \frac{(k+n-1)!}{k!} J_{n-1} \left( 2r \sqrt{K} \right) + O(K^{(n-1)/2} - 3/4)
\]
where $K = k + n/2$ and the bound holds uniformly in $a \leq r \leq b$. As $n \geq 2$, this shows that the series defining $J(z)$ converges uniformly on compact sets of the form $0 < a \leq |z| \leq b$.

5. A Restriction Theorem for the Symplectic Fourier Transform on $\mathbb{R}^{2n}$

The operator $T_r f = (2\pi)^d f * \mu$, is related to a restriction operator $R_r$ for the symplectic Fourier transform as we are going to see now. Before that let us briefly recall the usual restriction operators for the Fourier transform on $\mathbb{R}^n$. If we define

$$Q_r f(x) = (2\pi)^{-n} \int_{|w|=1} e^{i r w \cdot f(rw)} d\sigma(w),$$

then the Fourier inversion formula can be written as

$$f(x) = \int_0^\infty Q_r f(x) r^{n-1} dr.$$

The operators $f$ going to $Q_r f$ are called the restriction operators for the Fourier transform. It is well known that

$$\| Q_r f \|_p \leq C_r \| f \|_p, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}.$$ (5.3)

It is also known that such an estimate is not possible when $p > \frac{2n}{n+1}$. As a consequence of (5.3) one can prove the Stein-Tomas [19] restriction theorem

$$\left( \int_{|w|=1} |f(w)|^2 d\sigma \right)^{1/2} \leq C \| f \|_p, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}$$ (5.4)

which justifies the name restriction operators.

Let us now consider the symplectic Fourier transform on $\mathbb{R}^{2n}$. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the symplectic Fourier transform is defined as

$$\mathcal{F}_s f(z) = \int_{\mathbb{C}^n} f(w) e^{-i/2 \text{Im} z \cdot w} dw$$

and the inversion formula is given by

$$f(z) = (4\pi)^{-2n} \int_{\mathbb{C}^n} \mathcal{F}_s f(w) e^{-i/2 \text{Im} z \cdot w} dw.$$ We can rewrite the inversion formula as

$$f(z) = (4\pi)^{-2n} \int_{\mathbb{C}^n} \mathcal{F}_s f(z - w) e^{i/2 \text{Im} z \cdot w} dw.$$
Let \( \omega_{2n} \) be the surface area of \(|z| = 1\). We can now write

\[
f(z) = (4\pi)^{-\frac{n}{2}} \omega_{2n} \int_0^{\infty} r^{2n-1} dr \int_{|w| = r} \mathcal{F}_r f(z - w) e^{i/21 \mu w} d\mu_r
\]

where \( \mu_r \) is the normalized surface measure on \(|w| = r\). If we define \( R_r f \) by

\[
R_r f(z) = \int_{|w| = r} \mathcal{F}_r f(z - w) e^{i/21 \mu w} d\mu_r = \mathcal{F}_r f \times \mu_r
\]

then we have obtained the inversion in the form

\[
f(z) = (4\pi)^{-\frac{n}{2}} \omega_{2n} \int_0^{\infty} R_r f(z) r^{2n-1} dr.
\]

This is the analogue of (5.2) and that is the reason why we call them restriction operators. Unlike the operators \( Q_r f \), these \( R_r f \) are no longer eigenfunctions of the Laplacian.

The relation between \( T_r \) and \( R_r \) is now clear: \( R_r f = (2\pi)^{-n} T_r (\mathcal{F}_r f) \). In view of this we have the alternate formula

\[
R_r f = (2\pi)^{-n} \sum_{k=0}^{\infty} k! \frac{(n-1)!}{(k+n-1)!} \phi_k(r) \mathcal{F}_r f \times \phi_k.
\]

Using the bounds for the functions \( \phi_k \) one immediately gets

\[\| R_r f \|_2 \leq C \| \mathcal{F}_r f \|_2 \leq C \| f \|_2.\]

If we interpolate with the trivial estimate

\[\| R_r f \|_\infty = \| \mathcal{F}_r f \times \mu_r \|_\infty \leq C \| \mathcal{F}_r f \|_\infty \leq C \| f \|_1\]

we obtain the following boundedness result.

**Proposition 5.1.** For \( 1 \leq p \leq 2 \), one has

\[
\| R_r f \|_p \leq C \| f \|_p.
\]

Using the asymptotic properties of the Laguerre functions \( \phi_k(r) \) we can prove the following regularity theorem. To state the result we introduce the twisted Sobolev spaces \( W^p(C^n) \). On \( C^n \) consider the \( 2n \) vector fields

\[
Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4} \bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} z_j
\]

and the operator

\[
L = -\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j).
\]
The Hermite operator $H$ and $L$ are related by $W(Lf) = W(f)H$. Using this formula we can define $L^s$ by $W(L^sf) = W(f)L^s$. We then define the twisted Sobolev spaces by

$$\mathcal{W}^s(\mathbb{C}^n) = \{ f \in L^2(\mathbb{C}^n); L^s f \in L^2(\mathbb{C}^n) \}. \tag{5.9}$$

With this definition we can now prove the following theorem.

**Theorem 5.1.** If $s \leq (2n - 1)/4$ then $R$, maps $L^2(\mathbb{C}^n)$ continuously into $\mathcal{W}^s(\mathbb{C}^n)$.

**Proof.** Since $L^s(\phi_k) = (2k + n)^s \phi_k$ we have

$$L^s(R_f) = L^s(\mathcal{F}f \times \mu) = (2\pi)^{-n} \sum_{k=0}^{\infty} \frac{k! (n - 1)!}{(k + n - 1)!} \phi_k(r)(2k + n)^s(\mathcal{F}f \times \phi_k)$$

(where we have used the relation $L^s(f \times g) = (f \times L^s g)$). In view of the estimate (4.5) it is clear that $(2k + n)^s \frac{k! (n - 1)!}{(k + n - 1)!} \phi_k(r)$ is bounded as long as $s \leq (2n - 1)/4$ uniformly in $k$. This proves that $L^s(R_f)$ belongs to $\mathcal{W}^s(\mathbb{C}^n)$ and $|R_f|_{\mathcal{W}^s} \leq C |f|_2$.

We will now proceed to prove parts (ii) and (iii) of Theorem 4 of the introduction. We have already shown that $|R_f|_p \leq C |f|_p$ for $1 \leq p \leq 2$. The assertion (ii) that $|R_f|_p \leq C |f|_p$ for $2n/(n + 1) \leq p \leq 2$ will follow once we show that the following is true.

**Proposition 5.2.** Assume that $n \geq 3$. Then

$$|f \times \mu|_{2n/(n + 1)} \leq C |f|_{2n/(n - 1)}. \tag{5.10}$$

To see that the assertion (ii) follows from (5.10) we observe that

$$|R_f|_{2n/(n + 1)} = |\mathcal{F}f \times \mu|_{2n/(n + 1)} \leq C |\mathcal{F}f|_{2n/(n - 1)} \leq C |f|_{2n/(n + 1)}.$$

An interpolation with $|R_f|_2 \leq C |f|_2$ proves the assertion.

To prove Proposition 5.2 we need the following estimate for the projections $f \mapsto f \times \phi_k$.

**Proposition 5.3.**

$$|f \times \phi_k|_{2n/(n - 1)} \leq C |f|_{2n/(n + 1)}. \tag{5.11}$$
As the projections $f$ going to $f \times \phi_k$ are self adjoint we also have

$$\| f \times \phi_k \|_{2n/(n+1)} \leq C \| f \|_{2n/(n-1)}.$$  

Using this one immediately gets

$$\| f \times \mu_r \|_{2n/(n+1)} \leq C \sum_{k=0}^{\infty} \frac{k!}{(k+n-1)!} \| \phi_k(r) \| \| f \times \phi_k \|_{2n/(n+1)}$$

$$\leq C \left( \sum_{k=0}^{\infty} (2k+n)^{-\frac{n}{2} + \frac{1}{4}} \right) \| f \|_{2n/(n-1)}.$$

As $n \geq 3$ the series converges and this proves Proposition 5.2. So it remains to prove Proposition 5.3.

We have proved this proposition in [18]. We will briefly indicate the proof for the sake of completeness. The definition of the Laguerre polynomials $L^\alpha_k$ can be extended even for complex values of $\alpha$, $\Re \alpha > -1/2$. We then consider the functions

$$\psi^\alpha_k(z) = \frac{\Gamma(k+1)\Gamma(\alpha + 1)}{\Gamma(k + \alpha + 1)} L^\alpha_k \left( \frac{1}{2} |z|^2 \right) e^{-1/4 |z|^2}$$

and define a family of operators $G^\alpha_k f = f \times \psi^\alpha_k$. One verifies that this is an admissible analytic family of operators. By Stein's interpolation theorem [10] the estimate (5.11) will follow from the two estimates

$$\| G^\alpha_k f \|_{\infty} \leq C(1 + |r|)^{1/2} \| f \|_1,$$

$$\| G^\alpha_k f \|_2 \leq C(1 + |r|)^{\alpha} k^{-n} \| f \|_2.$$

These estimates can be proved using certain bounds for the Laguerre function $\psi^\alpha_k$. We refer to [18] for details.

We will now complete the proof of Theorem 4 by proving the assertion (iii) namely,

$$\| R, f \|_q \leq C \| f \|_p \quad \text{for} \quad 1 \leq p \leq \frac{2n}{n+1} \quad \text{where} \quad q = \frac{n-1}{n+1} p'.$$

When $p = 2n/(n+1)$, $q = p$ and we already have the inequality $\| R, f \|_{2n/(n+1)} \leq C \| f \|_{2n/(n+1)}$. Interpolating with the estimate $\| R, f \|_{\infty} \leq C \| f \|_1$ we complete the proof.

We conclude the paper with the following remarks. The estimate $\| R, f \|_q \leq C \| f \|_p$ was established in the interval $1 \leq p \leq 2n/(n+1)$. By
increasing $q$ we can extend the interval of validity. For example, by interpolating with the estimate $\| R_{x} f \|_{4n/(2n+1)} \leq C \| f \|_{4n/(2n+1)}$ we can prove

$$\| R_{x} f \|_{q} \leq C \| f \|_{p}, \quad q = \frac{2n - 1}{2n + 1} p'$$

in the range $1 \leq p \leq \frac{2n}{2n + 1}$. Similarly by decreasing the interval $1 \leq p \leq \frac{2n}{n + 1}$ we can obtain estimates valid with $q = \gamma p'$ where $\gamma < \frac{n - 1}{n + 1}$. Another remark we would like to make is regarding the assumption $n \geq 3$. It would be interesting to see if the Theorem 4 remains true for $n = 1$ and $n = 2$ also.

References


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Fonctions a support compact dans les analyses multi-résolutions

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Résumé

Nous nous intéressons aux fonctions à support compact dans une analyse multi-résolution, en particulier à celles de support minimum. Nous montrons que ces classes de fonctions sont stables par dérivation et primitivation et indiquons une méthode pour de nombreux calculs numériques.

Abstract

The main topic of this paper is the study of compactly supported functions in a multi-resolution analysis and especially of the minimally supported ones. We will show that this class of functions is stable under differentiation and integration and how to compute basic quantities with them.

1. Propriétés de base des analyses multi-résolutions

Nous rappelons dans cette section les propriétés des analyses multi-résolutions dont nous aurons besoin par la suite. La notion d'analyse multi-résolution a été introduite en 1986 par S. Mallat [10] et la plupart des propriétés que nous

Une analyse multi-résolution est une suite de sous-espaces fermés $V_j$ de $L^2(\mathbb{R})$ qui vérifie les propriétés suivantes:

1. $V_j \subset V_{j+1}$, $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ et $\bigcup_{j \in \mathbb{Z}} V_j$ est dense dans $L^2(\mathbb{R})$,
2. $f(x) \in V_j$ si et seulement si $f(2x) \in V_{j+1}$,
3. $V_0$ a une base de Riesz de la forme $g(x - k)$, $k \in \mathbb{Z}$, avec $g$ à valeurs réelles. Le plus souvent on impose à $g$ d’être à décroissance rapide à l’infini,
4. Pour tout $k \in \mathbb{N}$, $x^k g \in L^2$.

À quelles conditions une fonction $g$ vérifiant (1.4) engendre-t-elle une analyse multi-résolution? La réponse est fournie par le théorème suivant, essentiellement dû à A. Cohen [4].

**Proposition 1.** Soit $g \in L^2(\mathbb{R})$ à valeurs réelles. Alors les propriétés suivantes sont équivalentes:

(i) $g$ vérifie (1.3) et (1.4) pour une analyse multi-résolution $(V_j)_{j \in \mathbb{Z}}$;
(ii) la transformée de Fourier de $g$

$$\hat{g}(\xi) = \int g(x) e^{-ix\xi} dx,$$

s’écrit

$$\hat{g}(\xi) = \hat{g}(0) \prod_{j=1}^{\infty} m_0 \left( \frac{\xi}{2^j} \right)$$

où $\hat{g}(0) \neq 0$, $m_0$ est $C^\omega$, $2\pi$-périodique et vérifie

1. $m_0(0) = 1,$
2. $\sup_{n \in \mathbb{N}} \left\| \prod_{j=1}^{N} m_0 \left( \frac{\xi}{2^j} \right) \right\|_\infty < +\infty$,

3. il existe un compact $K$ réunion finie d’intervalles disjoints tel que

(j) $\sum_{k \in \mathbb{Z}} \chi_k(\xi + 2k\pi) = 1$ p.p.

(ji) pour tout $\xi \in K$ et $j \in \mathbb{N}^*$, $m_0 \left( \frac{\xi}{2^j} \right) \neq 0$.

**Démonstration.** La démonstration repose sur le lemme suivant:
Lemme 1. Si \( \alpha \) et \( \beta \in L^2(\mathbb{R}) \) vérifient (1.4) alors le série

\[
\sum_{k \in \mathbb{Z}} \hat{\alpha}(\xi + 2k\pi) \times \hat{\beta}(\xi + 2k\pi)
\]

converge en tout point vers une fonction \( C^\infty \).

Le lemme est immédiat, puisque d’après la formule sommatoire de Poisson on a

\[
\sum_{k \in \mathbb{Z}} \alpha(x - k) e^{i(k - x)\xi} = \sum_{k \in \mathbb{Z}} \hat{\alpha}(\xi + 2k\pi) e^{2ik\pi x}
\]

et donc

\[
\sum_{k \in \mathbb{Z}} \hat{\alpha}(\xi + 2k\pi) \hat{\beta}(\xi + 2k\pi) = \int_0^1 \left( \sum_{k \in \mathbb{Z}} \alpha(x - k) e^{i k \xi} \right) \left( \sum_{k \in \mathbb{Z}} \beta(x - k) e^{-i k \xi} \right) dx.
\]

(1.4) implique la convergence pour tout \( N \) de

\[
\int_0^1 \left( \sum_{k \in \mathbb{Z}} \sum_{|\xi| < N} |\alpha(x - k)| \right)^2 dx,
\]

d'où la convergence et la régularité de la série étudiée.

Si \( g \) vérifie (1.3) et (1.4), alors l’hypothèse que les \( g(x - k) \) forment une base de Riesz implique l’existence d’une constante \( A \geq 1 \) telle que

pour tout \( \xi \in \mathbb{R} \)

\[
\frac{1}{A} \leq \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 \leq A.
\]

De plus, puisque \( g\left(\frac{x}{2}\right) \in V_{-1} \subset V_0 \), on a \( g\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} a_k g(x - k) \) avec \( (a_k) \in l^2(\mathbb{Z}) \). D'où \( \hat{g}(2\xi) = m_0(\xi) \hat{g}(\xi) \) avec

\[
m_0(\xi) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k e^{-ik\xi}
\]

2\( \pi \) périodique et localement de carré intégrable. En particulier, on a

\[
m_0(\xi) \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} \hat{g}(2\xi + 4k\pi) \hat{g}(\xi + 2k\pi)
\]

et donc, puisque \( \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2 \) ne s’annule pas, \( m_0 \) est \( C^\infty \). De plus

\[
\sum_{k \in \mathbb{Z}} |\hat{g}(4k\pi)|^2 = |m_0(0)|^2 \sum_{k \in \mathbb{Z}} |\hat{g}(2k\pi)|^2
\]
et donc $|m_0(0)| \leq 1$; d'où

$$|\hat{g}(\xi)|^2 = \prod_{j=1}^{N} \left| m_0 \left( \frac{\xi}{2^j} \right) \right|^2 \left| \hat{g} \left( \frac{\xi}{2^N} \right) \right|^2 = \left| \hat{g}(0) \right|^2 \prod_{j=1}^{\infty} \left| m_0 \left( \frac{\xi}{2^j} \right) \right|^2,$$

et donc nécessairement $\hat{g}(0) \neq 0$ et $m_0(0) = 1$.

Comme $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule jamais, il existe autour de tout point $\xi$ de $[-\pi, \pi]$ un intervalle ouvert $I_{\xi}$ et un nombre entier $k(\xi)$ tels que $\hat{g}$ ne s'annule pas sur $I_{\xi} + 2k(\xi)\pi$. Par compacité de $[-\pi, \pi]$, on peut extraire de ces $I_{\xi}$ un recouvrement fini de $[-\pi, \pi]$. La construction de $K$ est alors immédiate. De plus, sur $K$, $\inf_{\xi \in K} |\hat{g}(\xi)|$ est positif, d'où

$$\prod_{j=0}^{N-1} m_0 \left( \frac{\xi}{2^j} \right) = \frac{\hat{g}(2^N \xi)}{\hat{g}(\xi)}$$

est borné indépendamment de $N$ sur $K$ d'où sur $\mathbb{R}$ tout entier.

Réciproquement supposons que $g \in L^2$ et

$$\hat{g}(\xi) = \hat{g}(0) \prod_{j=1}^{\infty} m_0 \left( \frac{\xi}{2^j} \right)$$

où $m_0$ vérifie (2.1) à (2.3). Alors $\hat{g}$ est $C^\infty$ et toutes ses dérivées appartiennent à $L^2$: on a en effet

$$\hat{g}^{(N+1)}(\xi) = \sum_{\alpha=1}^{N+1} \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \frac{m_0 \left( \frac{\xi}{2^j} \right)}{2^{j\alpha}} C_{N+1}^\alpha \frac{1}{2^{(N+1-\alpha)}} \hat{g}^{(N+1-\alpha)} \left( \frac{\xi}{2^j} \right)$$

d'où

$$\|\hat{g}^{(N+1)}\|_2 \leq C_N \left( \sum_{j=1}^{\infty} \frac{1}{2^{j(N+1/2)}} \right)^{N+1} \left( \sum_{\alpha=1}^{\infty} \|\hat{g}^{(N+1-\alpha)}\|_2 \right).$$

On en conclut que $g$ vérifie (1.4); en particulier $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ est borné. De plus, $\sum |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule pas, car sur $K$ $|\hat{g}(\xi)|$ est minoré par un nombre positif. Les $g(x - k)$ sont alors une base de Riesz d'un sous-espace fermé $V_0$ de $L^2(\mathbb{R})$. On définit $V_j$ par

$$f(x) \in V_j \text{ si et seulement si } f \left( \frac{x}{2^j} \right) \in V_0.$$

Puisque $\hat{g}(2\xi) = m_0(\xi)\hat{g}(\xi)$, on a facilement que $V_j \subset V_{j+1}$. De plus la série $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ converge uniformément sur tout compact (série de fonctions positives continues et de somme continue) et donc il existe $A > 0$ tel que pour tout $f \in V_0$,

$$\int_{|\xi| > A} |\hat{f}(\xi)|^2 \, d\xi \leq \frac{1}{2} |f|_2^2.$$
Si \( f \in \bigcap_{j \in \mathbb{Z}} V_j \) alors

\[
\text{pour tout } j \in \mathbb{Z}, \quad \int_{|\xi| > \lambda 2^j} |\hat{f}(\xi)|^2 \, d\xi < \frac{1}{2} |\mathcal{F}|_2^2
\]

d'où \( f = 0 \) (en faisant tendre \( j \) vers \( -\infty \)). Par ailleurs si \( f \in L^2 \) est à support compact, alors

\[
f = \lim_{j \to +\infty} \left( \sum_{k \neq 0} \hat{f}(\xi + 2^j 2k\pi) \right) \hat{g}\left(\frac{\xi}{2^j}\right) \hat{g}(0)^{-1}
\]

dans \( L^2 \) et donc \( \bigcup_j V_j \) est dense dans \( L^2 \): en effet \( \hat{f}(\xi)\hat{g}(\xi/2^j)\hat{g}(0)^{-1} \) tend vers \( \hat{f}(\xi) \) et, pour \( j \) assez grand,

\[
\left\| \left( \sum_{k \neq 0} \hat{f}(\xi + 2^j 2k\pi) \right) \hat{g}\left(\frac{\xi}{2^j}\right) \right\|_2^2 = \int |\hat{f}(\xi)|^2 \sum_{k \neq 0} \left| \hat{g}\left(\frac{\xi}{2^j} + 2k\pi\right) \right|^2 \, d\xi,
\]

tend vers

\[
\left( \int |\hat{f}(\xi)|^2 \, d\xi \right) \left( \sum_{k \neq 0} |\hat{g}(2k\pi)|^2 \right).
\]

Or on a nécessairement pour \( k \neq 0 \)

\[
\hat{g}(2k\pi) = \hat{g}(2^N 2k\pi) = \lim_{N \to +\infty} \hat{g}(2^N 2k\pi) = 0.
\]

La proposition est donc démontrée.

**Corollaire 1.** *Si \( g \) vérifie (1.3) et (1.4) pour une analyse multi-résolution, alors*

(3.1) \( \hat{g}(0) \neq 0 \) et \( \hat{g}(2k\pi) = 0 \) pour \( k \in \mathbb{Z}^* \),

(3.2) \( m_0(0) = 1 \) et \( m_0(\pi) = 0 \),

(3.3) \( \sum g(x - k) = \hat{g}(0) \).

Il reste à vérifier que \( m_0(\pi) = 0 \). Or, d’après (3.1), on a

\[
0 = \sum_{k \in \mathbb{Z}} \left| \hat{g}(2\pi + 4k\pi) \right|^2 = |m_0(\pi)|^2 \sum_{k \in \mathbb{Z}} \left| \hat{g}(\pi + 2k\pi) \right|^2;
\]

comme \( \sum_{k \in \mathbb{Z}} \left| \hat{g}(\xi + 2k\pi) \right|^2 \) ne s’annule jamais, \( m_0(\pi) = 0 \). Quant à (3.3) cela vient de la formule sommatoire de Poisson

\[
\sum_{k \in \mathbb{Z}} g(x - k) = \sum_{k \in \mathbb{Z}} \hat{g}(2k\pi) e^{2ik\pi x}.
\]
Remarques.
(i) Il est clair que dans (2.3) on peut supposer que \( 0 \in \text{Int} \, K \), puisque \( m_0(0) = 1 \) et que donc \( m_0 \) ne s'annule pas sur un voisinage de 0. De plus, si \( m_0 \) vérifie (2.1) à (2.3) avec 0 intérieur à \( K \), le produit infini

\[
\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0 \left( \frac{\xi}{2^j} \right)
\]

est dans \( L^2 \) si et seulement si on a

\[
\sup_{N} \left\| \prod_{j=1}^{N} m_0 \left( \frac{\xi}{2^j} \right) x_k \left( \frac{\xi}{2^N} \right) \right\|_2 < \infty;
\]

cela est immédiat puisque

\[
\prod_{j=1}^{N} m_0 \left( \frac{\xi}{2^j} \right) x_k \left( \frac{\xi}{2^N} \right) = x_k \left( \frac{\xi}{2^N} \right) \frac{\hat{g}(\xi)}{\hat{g}(\xi/2^N)}.
\]

De plus on a alors

\[
\prod_{j=1}^{N} m_0 \left( \frac{\xi}{2^j} \right) x_k \left( \frac{\xi}{2^N} \right) \rightarrow \hat{g} \text{ dans } L^2.
\]

(ii) Si \( m_0 \) vérifie (2.1), (2.3) et

\[
|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1,
\]

alors si

\[
\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0 \left( \frac{\xi}{2^j} \right),
\]

g est automatiquement dans \( L^2 \) et les \( g(x - k) \) sont orthonormées. (Il suffit de vérifier que les \( \theta_n(x - k) \) sont orthonormées, où

\[
\hat{\theta}_n(\xi) = \prod_{j=1}^{N} m_0 \left( \frac{\xi}{2^j} \right) x_k \left( \frac{\xi}{2^N} \right).
\]

2. Produits infinis de polynômes trigonométriques

Si \( m_0 \) est un polynôme trigonométrique tel que

\[
m_0(0) = 1 \text{ et si } \hat{g}(\xi) = \prod_{j=1}^{\infty} m_0 \left( \frac{\xi}{2^j} \right),
\]
alors $g$ est une distribution à support compact. Si de plus $m_0$ vérifie (2.3), et $g \in L^2$, alors (2.2) est immédiat, puisque $\hat{g}$ est bornée ($g$ est intégrable) et que $|\hat{g}|$ est minorée sur $K$.

Dans le cas d'un polynôme trigonométrique, la condition (2.3) s'exprime plus algébriquement de la manière suivante:

**Proposition 2.** Si $m_0$ est un polynôme trigonométrique tel que $m_0(0) = 1$ alors la condition (2.3) est équivalente à:

(i) pour tout $\xi \in \mathbb{R}$, $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 > 0$,

(ii) il n'existe pas de $\xi_0 \in ]0, 2\pi[\text{ tel que:}$

$$m_0(2^N\xi_0 + \pi) = 0.$$

**Démonstration.** Le sens direct est immédiat. (2.3) signifie que si on note

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j})$$

alors pour $\xi \in \mathbb{R}$, il existe $k \in \mathbb{Z}$, $\hat{g}(\xi + 2k\pi) \neq 0$. Si on avait

$$m_0(\xi_0) = m_0(\xi_1 + \pi) = 0$$

pour un $\xi_1 \in \mathbb{R}$ alors on aurait $\hat{g}(2\xi_1 + 2k\pi) = 0$ quel que soit $k$. De même supposons que (ii) ne soit pas vérifié; comme $m_0$ n'a qu'un nombre fini de zéros modulo $2\pi$, on doit avoir pour deux entiers $N$ et $M$: $2^N\xi_0 - 2^M\xi_0 \in 2\pi\mathbb{Z}$; on ne peut avoir $2^M\xi_0 \in 2\pi\mathbb{Z}$ (sinon pour un $M' \geq 0$ on aurait $2^M\xi_0 \in \pi + 2\pi\mathbb{Z}$ et $m_0(2^M\xi_0 + \pi) = 1$; quitte à changer $\xi_0$ en $2^M\xi_0 \mod (2\pi)$, on peut supposer $M = 0$. On va montrer que $\hat{g}(\xi_0 + 2k\pi) = 0$ pour tout $k$. Il revient au même de considérer $\hat{g}(2^N\xi_0 + 2k\pi)$. Or

$$\hat{g}(2^N\xi_0 + 2k\pi) = \hat{g}(\xi_0 + 2k\pi \frac{2^N}{2^N}) \prod_{j=1}^{N} m_0\left(\frac{2^N\xi_0 + 2k\pi}{2^j}\right);$$

si $k$ n'est pas divisible par $2^N$ on obtient 0; si $k = 2^Nk'$, alors on a $\xi_0 = 2^N\xi_0 - 2^Nk\pi$ où $1 \leq k \leq 2^N - 2$, et on est ramené à étudier $\hat{g}(2^N\xi_0 + 2(k'-k_0)\pi)$; si $k' = 0$, $k' - k_0$ n'est pas divisible par $2^N$ et $\hat{g}(2^N\xi_0 + 2(k'-k_0)\pi) = 0$; si $k' \neq 0$, alors $|k'-k_0| \leq |k'| + 2^N - 2 < 2^N|k'| = |k|$ et ce cas ne peut donc indéfiniment se reproduire. (2.3) entraîne donc bien (i) et (ii).

Réciproquement supposons que (2.3) ne soit pas vérifié et que (i) soit vrai; on va montrer que (ii) est faux. On peut supposer que

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} m_0(\frac{\xi}{2^j}).$$
est dans $L^2$, quitte à remplacer $m_0$ par $\left(\frac{1 + e^{i\xi}}{2}\right)^M m_0$ avec $M$ suffisamment grand (en effet $\prod_{j=1}^\infty m_0 \left(\frac{\xi}{2^j}\right)$ est borné sur $[-\pi, \pi]$ par continuité sur un compact et on en déduit facilement que $|\hat{g}(\xi)| \leq C(1 + |\xi|)^\alpha$ avec
\[
\alpha = \frac{\log |m_0|}{\log 2};
\]
il suffit de prendre $M > \alpha + 1$ car on a
\[
\prod_{j=1}^\infty \left(\frac{1 + e^{i\xi/2^j}}{2}\right)^M m_0 \left(\frac{\xi}{2^j}\right) = \left(\frac{e^{i\xi} - 1}{i\xi}\right)^M \hat{g}(\xi);
\]
cette substitution n'affecte ni (i) ni (ii). Si (2.3) n'est pas vérifié, alors
\[
Q(\xi) = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2
\]
a au moins un zéro $\xi_1$; de plus
\[
Q(\xi_1) = Q\left(\frac{\xi_1}{2}\right)m_0\left(\frac{\xi_1}{2}\right) + Q\left(\frac{\xi_1}{2} + \pi\right)m_0\left(\frac{\xi_1}{2} + \pi\right);
\]
quitte à changer $\xi_1$ en $\xi_1 + 2\pi$ on peut supposer $m_0\left(\frac{\xi_1}{2}\right) \neq 0$ et donc $Q\left(\frac{\xi_1}{2}\right)
\]
$= 0$. On trouve de même $Q\left(\frac{\xi_1}{4}\right) = 0$ quitte à changer $\xi_1$ en $\xi_1 + 4\pi$, et ainsi de suite. Or $g$ est à support compact et donc
\[
Q(\xi) = \sum_{k \in \mathbb{Z}} \left(\int g(x)\overline{g}(x - k) \, dx\right) e^{-ik\xi}
\]
est un polynôme trigonométrique et donc n'a qu'un nombre fini de racines modulo $2\pi$. De sorte qu'on a
\[
Q(\xi_1) = Q\left(\frac{\xi_1}{2}\right) = \cdots = Q\left(\frac{\xi_1}{2^N}\right) = 0 \text{ et } \frac{\xi_1}{2^N} - \xi_1 \in 2\pi\mathbb{Z}.
\]
Par ailleurs $Q\left(\frac{\xi_1}{2} + \pi\right), \ldots, Q\left(\frac{\xi_1}{2^N} + \pi\right)$ sont tous non nuls; en effet supposons que
\[
Q\left(\frac{\xi_1}{2^j} + \pi\right) = 0
\]
et posons

\[ Z = e^{-i \xi_j/2^j + \eta_j}, \]

le même raisonnement que ci-dessus nous fournit \( Z^{2^M} = Z \) pour un \( M \geq 1 \)
tandis que \((-Z)^N = -Z\) d'où \( Z^{2^M + N} = Z^N = -Z \) et \( Z^{2^M - N} = (-Z)^N = Z \),

ce qui est absurde. On peut donc écrire \( Q(\xi) = |A(\xi)|^2 R(\xi) \) avec

\[ A(\xi) = \prod_{j=1}^N (e^{-i \xi_j} - e^{-i \xi_j/2^j}) \]

(où \( N \) est choisi comme le plus petit entier \( \geq 1 \) tel que \( \frac{\xi_j}{2^j} \in 2\pi \mathbb{Z} \)) et

\( R(\xi) \) est un polynôme trigonométrique tel que \( R(\frac{\xi_j}{2^j} + \pi) \neq 0 \) pour

\( 1 \leq j \leq N \). On pose alors

\[ \hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{A(\xi)} \]

et nous allons voir que \( \gamma \in L^2 \) et est à support compact. Notons \( P(\xi) \) le polynôme trigonométrique

\[ P(\xi) = \sum_{k \in \mathbb{Z}} \frac{1}{2} \int \gamma \left( \frac{x}{2} \right) \hat{\gamma}(x - k) \, dx \, e^{-i k \xi} = \sum_{k \in \mathbb{Z}} \hat{\gamma}(2\xi + 4k\pi) \hat{\gamma}(\xi + 2k\pi). \]

On a alors

\[ A(2\xi)P(\xi) = \sum_{k \in \mathbb{Z}} \hat{g}(2\xi + 4k\pi) \hat{\gamma}(\xi + 2k\pi) \]

\[ = m_0(\xi)A(\xi)R(\xi). \]

Or \( A(2\xi) = A(\xi)A(\xi + \pi) \) et les zéros de \( A(\xi + \pi) \) ne sont pas des zéros de

\( R(\xi) \). D'où

\[ m_0 \left( \frac{\xi_1}{2} + \pi \right) = \cdots = m_0 \left( \frac{\xi_1}{2^N} + \pi \right) = 0, \]

et on obtient

\[ \text{pour tout } k \in \mathbb{N}, \quad m_0(2^k \xi_1 + \pi) = 0, \]

tandis que \( \xi_1 \notin 2\pi \mathbb{Z} \) puisque \( Q(0) \geq |\hat{g}(0)|^2 = 1 \). La Proposition 2 est donc
démontrée, pourvu que \( \gamma \) soit bien dans \( L^2 \) et à support compact. C'est l'objet

\textbf{Lemme 2.} \( \text{Si } g \in L^2 \text{ est à support compact et si } \)

\[ \sum_{k \in \mathbb{Z}} |\hat{g}(\xi_0 + 2k\pi)|^2 = 0 \]
alors γ définie par
\[ \hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{e^{-i\xi} - e^{-i\xi_0}} \]
est dans \( L^2 \) et à support compact.

Le lemme est classique. D’abord \( \hat{\gamma} \in L^2 \) puisque, en notant à nouveau
\[ Q(\xi) = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2, \]
on a
\[ \sum_{k \in \mathbb{Z}} |\hat{\gamma}(\xi + 2k\pi)|^2 = \frac{Q(\xi)}{|e^{-i\xi} - e^{-i\xi_0}|^2}; \]
comme \( Q \) est \( \geq 0 \), \( \xi_0 \) est zéro au moins double et \( Q(\xi)/|e^{-i\xi} - e^{-i\xi_0}|^2 \) est un polynôme trigonométrique.

Ensuite la formule sommatoire de Poisson nous donne:
\[ \sum_{k \in \mathbb{Z}} g(x - k)e^{i(k - x)\xi_0} = \sum_{k \in \mathbb{Z}} \hat{g}(\xi_0 + 2k\pi)e^{2ik\pi\xi} = 0, \]
d’où la fonction
\[ \alpha(x) = \sum_{k = -\infty}^{-1} g(x - k)e^{i(k - 1)\xi_0} = -\sum_{k = 0}^{\infty} g(x - k)e^{i(k - 1)\xi_0} \]
est à support compact et dans \( L^2 \), or \( \alpha(x) = g(x + 1) + \alpha(x + 1)e^{-i\xi_0} \) d’où
\[ \hat{\alpha}(\xi) = \hat{g}(\xi)e^{i\xi} + \hat{\alpha}(\xi)e^{i\xi - i\xi_0} \]
et donc \( \alpha = \gamma \). Le lemme est donc démontré.

**Corollaire 2.** Si \( m_0 \) est un polynôme trigonométrique à coefficients réels tel que
\[ m_0(0) = 1 \quad \text{et} \quad \hat{g}(\xi) = \prod_{j=1}^{\infty} m_0 \left( \frac{\xi}{2^j} \right) \]
soit dans \( L^2 \), alors il existe une et une seule analyse multi-résolution \( (V_j) \) de \( L^2(\mathbb{R}) \) telle que \( g \in V_0 \).

En effet, supposons que \( m_0(\xi) \) et \( m_0(\xi + \pi) \) ait une racine commune \( \xi_0 \); alors \( -\xi_0 \) est racine puisque \( m_0 \) est à coefficients réels et on a \( m_0(\xi) = (\cos 2\xi - \cos 2\xi_0)M(\xi) \) avec \( M \) polynôme trigonométrique. De plus, si
\[ Q(\xi) = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2, \]
alors il est clair que $Q(2\xi_0) = Q(-2\xi_0) = 0$ et donc, d'après le lemme précédent, si
\[
\hat{\gamma}(\xi) = \hat{g}(\xi) \frac{1 - \cos 2\xi}{\cos \xi - \cos 2\xi_0}
\]
alors $\gamma$ est dans $L^2$ et à support compact. De plus
\[
\hat{\gamma}(2\xi) = \frac{1}{\cos 2\xi - \cos 2\xi_0} (\cos \xi - \cos 2\xi_0) m_0(\xi) \hat{\gamma}(\xi)
\]
\[= M(\xi)(\cos \xi - \cos 2\xi_0) \hat{\gamma}(\xi);\]
d'où
\[
\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} m_j \left( \frac{\xi}{2^j} \right) \quad \text{avec} \quad m_j(\xi) = M(\xi)(\cos \xi - \cos 2\xi_0).
\]
Si on appelle degré d'un polynôme trigonométrique $\sum_{k \neq N_j} a_k e^{-i\xi k}$ avec $a_{N_j} \neq 0$, $a_{N_j} \neq 0$ le nombre $N_2 - N_1$, alors il est clair que $\deg m_1 = \deg m_0 = 2$.
Ainsi, d'un nombre fini d'opérations, on peut supposer que $m_0$ et $m_0(\xi + \pi)$ n'ont pas de racines communes.
Si maintenant (ii) n'est pas vérifié, on a vu que $m_0(\xi)$ admettrait un facteur
\[
\prod_{j=0}^{N-1} (e^{-i\xi} + e^{-i\xi_0 2^j})
\]
avec $2^N \xi_0 - \xi_0 \in 2\pi \mathbb{Z}$ et $\xi_0 \notin 2\pi \mathbb{Z}$. Il admet également le facteur
\[
\prod_{j=0}^{N-1} (e^{-i\xi} + e^{i\xi_0 2^j})
\]
qui est soit confondu soit premier avec le premier facteur. De plus, on a vu que si $Q(\xi) = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ alors $Q(\xi_0) = 0$ et il en va de même pour $2^j \xi_0$, $1 \leq j \leq N - 1$. On pose alors
\[
\hat{\gamma}(\xi) = \frac{A(0)}{A(\xi)} \hat{g}(\xi)
\]
où
\[
A(\xi) = \prod_{j=0}^{N-1} (e^{-i\xi} - e^{-i\xi_0 2^j})
\]
où
\[
\prod_{j=0}^{N-1} (e^{-i\xi} - e^{-i\xi_0 2^j}) \prod_{j=0}^{N-1} (e^{-i\xi} - e^{i\xi_0 2^j})
\]
suivant que les deux facteurs sont égaux ou distincts; d’après le lemme, \( \hat{\gamma} \) est dans \( L^2 \) et à support compact; de plus on a
\[
\hat{\gamma}(2\xi) = \frac{A(\xi)m_0(\xi)}{A(2\xi)} \hat{\gamma}(\xi) = \frac{m_0(\xi)}{A(\xi + \pi)} \hat{\gamma}(\xi)
\]
où \( m_1 \) est le polynôme trigonométrique à coefficients réels \( m_0(\xi)/A(\xi + \pi) \). A nouveau \( \deg m_1 < \deg m_0 \).

Au bout d’un nombre fini d’opérations, on obtient alors que
\[
\hat{g}(\xi) = P(\xi) \prod_{j=1}^{\infty} M\left(\frac{\xi}{2^j}\right)
\]
où \( P \) et \( M \) sont deux polynômes trigonométriques à coefficients réels,
\[
\prod_{j=1}^{\infty} M\left(\frac{\xi}{2^j}\right) \in L^2,
\]
\( M(0) = 1 \) et \( M \) vérifie les points (i) et (ii) de la Proposition 2. La fonction \( g \) est donc dans l’espace \( V_0 \) de l’analyse multi-résolution engendrée par la fonction \( \gamma \) définie par
\[
\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} M\left(\frac{\xi}{2^j}\right).
\]

L’existence de l’analyse multi-résolution a été démontrée. Pour l’unicité, il suffit de remarquer que si \( g \in V_0 \) pour une analyse multi-résolution alors la fonction \( \gamma \) obtenue à la fin est encore dans \( V_0 \) et que les \( \gamma(x - k) \) forment alors une base de Riesz de \( V_0 \).

**Lemme 3.** Si \( g \in V_0 \) (où \( V_0 \) correspond à une analyse multi-résolution) est à support compact et si
\[
\sum_{k \in \mathbb{Z}} |\hat{g}(\xi_0 + 2k\pi)|^2 = 0
\]
alors la fonction \( \gamma \) définie par
\[
\hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{e^{-\xi_0} - e^{i\xi_0}}
\]
est encore dans \( V_0 \).

En effet, on a vu que
\[
\gamma = \sum_{k = -\infty}^{-1} g(x - k) e^{i(k - 1)\xi_0}
\]
et il est clair que si $\alpha \in L^2$ est à support compact

$$\langle \gamma | \alpha \rangle = \sum_{k = -\infty}^{+1} e^{i(k - 1)k_0} \langle g(x - k) | \alpha \rangle.$$ 

De plus

$$\left\| \sum_{k = -N}^{-1} g(x - k) e^{i(k - 1)k_0} \right\|_2 = \frac{1}{\sqrt{2\pi}} \left\| \frac{\hat{g}(\xi)}{1 - e^{-i2k_0 - \xi}} (1 - e^{-in(k_0 - \xi)}) \right\|_2 \leq 2 |\gamma|_2.$$ 

On en conclut que la série converge vers $\gamma$ faiblement dans $L^2$ et donc que $\gamma$ appartient à l'adhérence faible de $V_0$ et donc à $V_0$.

**Lemme 4.** Si $g \in V_0$ est à support compact et si $\sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$ ne s'annule pas alors les $g(x - k)$ forment une base de Riesz de $V_0$.

Si les $h(x - k)$ forment une base de Riesz de $V_0$, alors $\hat{g}(\xi) = M(\xi)\hat{h}(\xi)$ avec $M \in L^2_{loc} 2\pi$-périodique; d'où

$$|M(\xi)|^2 \sum_{k \in \mathbb{Z}} |\hat{h}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2$$

et donc $M$ et $1/M$ sont essentiellement bornées. Le lemme est alors immédiat.

3. **Fonctions de $V_0$ à support compact**

**Théorème 1.** Soit $(V_j)_{j \in \mathbb{Z}}$ une analyse multi-résolutions de $L^2(\mathbb{R})$. Si $V_0$ contient des fonctions à support compact non nulles, alors il existe une fonction $\gamma \in V_0$ à support compact et à valeurs réelles telle que

(4.1) Les fonctions $\gamma(x - k)$, $k \in \mathbb{Z}$, forment une base de Riesz de $V_0$,

(4.2) Toute fonction de $V_0$ à support compact s'écrit comme combinaison linéaire finie des $\gamma(x - k)$.

**Démonstration.** Si $h$ est dans $L^2$ et à support compact, on sait que

$$\sum_{k \in \mathbb{Z}} |\hat{h}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} \int h(x) \overline{h}(x - k) \, dx = e^{-i\xi}$$

est un polynôme trigonométrique. Nous l'appellerons le polynôme d'auto-corrélation de $h$ et le noterons $P_h$. Si $h$ est à valeurs réelles, $P_h$ est un polynôme en $\cos \xi$ à coefficients réels. On choisit $\gamma \in V_0$ à valeurs réelles et à support compact de sorte que $P_\gamma$ soit de degré minimal.
Alors $P_\gamma(\xi)$ ne s’annule pas. Si on avait $P_\gamma(\xi_0) = 0$, on poserait

$$\hat{g}(\xi) = \frac{\hat{\gamma}(\xi)}{e^{-i\xi} - e^{-i\xi_0}}$$

si $\xi_0 \in \pi\mathbb{Z}$ et

$$\hat{g}(\xi) = \frac{\hat{\gamma}(\xi)}{\cos \xi - \cos \xi_0}$$

 sinon. D’après les Lemmes 2 et 3, $g$ serait dans $V_0$, à valeurs réelles et de support compact et $\deg P_\gamma$ serait strictement inférieur à $P_\gamma$. On en conclut que $P_\gamma$ ne s’annule pas et donc que les $\gamma(x - k)$ forment une base de Riesz de $V_0$ (d’après le Lemme 4).

Considérons maintenant $h \in V_0$ à support compact. On peut supposer $h$ à valeurs réelles, quitte à raisonner sur $\text{Re} \ h$ et $\text{Im} \ h$. On sait que $h$ s’exprime en fonction des $\gamma(x - h)$ par: $\hat{h}(\xi) = U(\xi)\hat{\gamma}(\xi)$ avec $U$ $2\pi$-périodique et dans $L^2_{\text{loc}}$. En fait $U(\xi)$ est une fraction rationnelle en $e^{-i\xi}$ puisque

$$\sum (\hat{h}(\xi + 2k\pi)\hat{\gamma}(\xi + 2k\pi)) = U(\xi)P_\gamma(\xi),$$

d’où

$$U(\xi) = \frac{C(\xi)}{P_\gamma(\xi)}$$

où $C$ est un polynôme trigonométrique à coefficients réels. De plus on a:

$$\sum |\hat{h}(\xi + 2k\pi)|^2 = |C(\xi)|^2 \frac{1}{P_\gamma(\xi)}$$

d’où

$$|C(\xi)|^2 = P_\gamma(\xi) \sum |\hat{h}(\xi + 2k\pi)|^2.$$  

Le théorème de Riesz nous permet de trouver deux polynômes $A, B \in \mathbb{R}[X]$ avec $A(0) \neq 0$, $B(0) \neq 0$, $C(\xi) = e^{-i\xi}A(e^{-i\xi})B(e^{-i\xi})$ pour un $p \in \mathbb{Z}$ et $P_\gamma(\xi) = |A(e^{-i\xi})|^2$. On a alors

$$\hat{h} = e^{-i\xi} \frac{B(e^{-i\xi})}{A(e^{-i\xi})} \hat{\gamma}.$$  

On divise $B$ en $B = A \psi + R$. Alors si $\hat{k} = \hat{h} - e^{-i\xi}Q(e^{-i\xi})\hat{\gamma}$, $k \in V_0$ est à support compact et à valeurs réelles; de plus on a:

$$P_k(\xi) = P_k(\xi) + |Q(e^{-i\xi})|^2P_\gamma(\xi) - 2 \text{Re} \sum \hat{h}(\xi + 2l\pi)\hat{\gamma}(\xi + 2l\pi)e^{i\xi}Q(e^{-i\xi})$$

$$= |B(e^{-i\xi})|^2 + |Q(e^{-i\xi})A(e^{-i\xi})|^2 - 2 \text{Re} B(e^{-i\xi})A(e^{-i\xi}) Q(e^{-i\xi})$$

$$= |R(e^{-i\xi})|^2.$$
Comme $P_γ$ est de degré minimal, nécessairement $k = 0$ et donc $h$ s'écrit comme une combinaison linéaire finie des $γ(x - k)$.

**Corollaire 3.**

(i) On a $\hat{γ}(2ξ) = m_0(ξ)\hat{γ}(ξ)$ pour $m_0$ un polynôme trigonométrique à coefficients réels tel que $m_0(0) = 1$.

(ii) $γ$ est de support minimal.

(iii) Si $h ∈ V_o$ est de support compact, les bornes inférieure et supérieure de son support sont entières.

(iv) Si on impose que $\hat{γ}(0) = 1$ et que la borne inférieure du support de $γ$ soit 0, alors $γ$ est unique.

**Démonstration.** Comme $γ(x/2) ∈ V_{-1} ⊂ V_o$ et que $γ(x/2)$ est à support compact, $γ(x/2)$ s'exprime comme une combinaison linéaire finie des $γ(x - k)$. Le point (i) est donc démontré. Si on a

$$γ\left(\frac{x}{2}\right) = \sum_{k = N_1}^{N_2} b_k γ(x - k)$$

alors nécessairement la borne inférieure du support de $γ$ est $N_1$ et la borne supérieure est $N_2$. De même si

$$h(x) = \sum_{k = M_1}^{M_2} h_k γ(x - k)$$

alors la borne inférieure de son support est $M_1 + N_1$ et la borne supérieure $M_2 + N_2$ ; en particulier la longueur de l’enveloppe convexe de ce support est $N_2 - N_1 + M_2 - M_1$ et donc supérieure à celle du support de $γ$ (avec égalité pour les seules fonctions multiples d’un $γ(x - k)$). Les points (ii), (iii), (iv) sont alors démontrés.

**Corollaire 4.** Il existe une fonction $θ ∈ W_o$ (où $W_o$ est le complémentaire orthogonal de $V_o$ dans $V_1$) à support compact et à valeurs réelles telle que

(5.1) Les fonctions $θ(x - k)$, $k ∈ Z$ forment une base de Riesz de $W_o$ ;

(5.2) Toute fonction de $W_o$ à support compact s'écrit comme une combinaison linéaire finie des $θ(x - k)$.

θ est alors de support minimum dans $W_o$.

**Démonstration.** Si $h ∈ W_o$ est à support compact, alors $h$ peut s'écrire comme

$$h(ξ) = e^{-ipξ}R(e^{-iξ/2})\hat{γ}(ξ/2)$$

pour un polynôme $R(ξ)$ et un entier $p ∈ Z$, puisque $W_o ⊂ V_1$. Notons $Q$ le polynôme de degré minimum tel que $Q(e^{-iξ/2})\hat{γ}(ξ/2)$ soit la transformée de Fourier d’un élément (normal) de $W_o$. On peut suppo-
ser $Q$ à coefficients réels (car si $\tilde{w} = Q(e^{-i\ell/2})\tilde{\gamma}(\xi/2)$ alors $\text{Re} w$ et $\text{Im} w$ sont dans $w_0$). De même, on supposera $h$ à valeurs réelles (et donc $R \in \mathbb{R}[X]$).

On remarque d'abord que $Q(z)$ et $Q(-z)$ sont premiers entre eux, car si $Q(z) \wedge Q(-z) = A(z^2)$ alors $[Q(e^{-i\ell/2})/A(e^{-i\ell})][\tilde{\gamma}(\xi/2)]$ est encore la transformée de Fourier d'un élément de $W_0$, ce qui contredit la minimalité de $Q$. On a alors, en posant $\tilde{\theta} = Q(e^{-i\ell/2})\tilde{\gamma}(\xi/2)$.

$$\sum_{\ell \in Z} |\tilde{\theta}(\xi + 2k\pi)|^2 = |Q(e^{-i\ell/2})|P_\gamma(\xi/2) + |Q(-e^{-i\ell/2})|P_\gamma\left(\frac{\xi}{2} + \pi\right) > 0$$

ce qui entraîne que les $\theta(x - k)$ forment une base de Riesz de $W_0$ (car on sait que $W_0$ a une base de Riesz de la forme $w(x - k)$, $k \in Z$).

Maintenant si $\hat{h} = e^{-i\ell\xi}R(e^{-i\ell/2})\tilde{\gamma}(\xi/2)$ est la transformée de Fourier de $h \in V_1$, alors $h \in W_0$ si et seulement si on a

$$R(e^{-i\ell})m_0(\ell)P_\gamma(\xi) + R(e^{-i(\ell + \pi)})m_0(\ell + \pi)P_\gamma(\xi + \pi) = 0.$$ 

d'où si $m_0(\ell)P_\gamma(\xi) = e^{-2iq\xi}A(e^{-i\ell})$, $R(z)A(z) + R(-z)A(-z) = 0$. En particulier $\deg R + \deg A$ est impair. On en conclut que $\deg R$ et $\deg Q$ ont la même parité. De proche en proche, on obtient que $R(z) = B(z^2)Q(z) + C(z)$ avec $\deg C < \deg Q$, d'où $R(z) = B(z^2)Q(z)$. On a alors $\hat{h} = e^{-i\ell\xi}B(e^{-i\ell})\tilde{\theta}(\xi)$.

**Remarque.** La fonction $\theta$ décrite par le Corollaire 4 a été d'abord étudiée par C. K. Chui et J. Z. Wang [3]. Cependant les propriétés décrites dans ce corollaire sont nouvelles, ces deux auteurs n'ayant pas démontré (4.2) et donc pas (5.2) ni la minimalité du support de $\theta$. Par contre, ils donnent le calcul de $Q$: si $A(z) \wedge A(-z) = B(z^2)$ et si $A(z) = B(z^2)C(z)$ (où $A(e^{-i\ell}) = e^{2iq\xi}m_0(\ell)P_\gamma(\xi)$) alors $Q(z) = zC(-z)$.

### 4. La fonction de $V_0$ de support minimal

**Théorème 2.** Soit $G \in V_0$ non nulle, à support compact et à valeurs réelles. Alors les propriétés suivantes sont équivalentes:

1. $G$ est de support minimal.
2. Le polynôme d'auto-corrélation de $G$ est de degré minimum.
3. Tout élément de $V_0$ à support compact s'écrit comme une combinaison linéaire finie des $G(x - k)$.
4. Il existe $H \in L^2$ de support compact telle que $\langle H | G(x - k) \rangle = \delta_k$.
5. $G$ se décompose en

$$\hat{G}(\xi) = \hat{G}(0) \prod_{j=1}^\infty m_0\left(\frac{x}{2^j}\right)$$

où \( m_0 \) est un polynôme trigonométrique \( m_0(\xi) = e^{-ip\xi}P(e^{-i\xi}) \) avec \( p \in \mathbb{Z} \) et \( P \in \mathbb{R}(X) \) où \( P \) vérifie:

(i) \( P(1) = 1 \).
(ii) \( P(z) \) et \( P(-z) \) sont premiers entre eux.
(iii) \( P(z) \) n’a pas de facteurs de la forme

\[
\prod_{k=1}^{N-1} (z + z_0^k) \quad \text{avec} \quad z_0^N = z_0 \neq 1.
\]

(6.6) Les restrictions à \([0, 1]\) des fonctions

\[ G(x + k), \quad \inf \text{Supp } G \leq k < \sup \text{Supp } G, \]

sont linéairement indépendantes.

DÉMONSTRATION. Le Théorème 1 donne l’équivalence de (6.1), (6.2) et (6.3).

(6.1) ⇒ (6.5). On sait d’après le Corollaire 3 que \( \hat{G}(2\xi) = m_0(\xi)\hat{G}(\xi) \) avec \( m_0 \) un polynôme trigonométrique. De plus \( \hat{G}(0) \neq 0 \) d’après (3.1) et donc \( m_0(0) = 1 \). On peut donc écrire \( m_0(\xi) = e^{-ip\xi}P(e^{-i\xi}) \) avec \( P(1) = 1 \) et \( P(0) \neq 0 \). D’après la Proposition 2, \( P(z) \) et \( P(-z) \) n’ont pas de racines communes sur le cercle unité et \( P(z) \) n’a pas de facteurs

\[
\prod_{k=1}^{N-1} (z + z_0^k) \quad \text{avec} \quad z_0 \neq 1, \quad z_0^N = z_0.
\]

Il ne reste donc à prouver que \( P(z) \land P(-z) = 1 \). Si \( P(z) \land P(-z) = R(z^2) \), on pose

\[ \hat{h}(\xi) = \frac{\hat{G}(\xi)}{R(e^{-i\xi})}, \]

alors

\[ \hat{h}(2\xi) = e^{-ip\xi}P(e^{-i\xi}) \frac{R(e^{-i\xi})}{R(e^{-2i\xi})} R(e^{-i\xi})h(\xi) \]

et on a \( \sup \text{Supp } h - \inf \text{Supp } h = \deg P - \deg R \leq \sup \text{Supp } G - \inf \text{Supp } G \) avec égalité si et seulement si \( P(z) \land P(-z) = 1 \). Puisque \( G \) est de support minimal, on a bien \( P(z) \land P(-z) = 1 \).

(6.5) ⇒ (6.1). Quitte à translater \( G \) et \( \gamma \) (où \( \gamma \) est décrite par le Théorème 1), on peut supposer \( \inf \text{Supp } G = \inf \text{Supp } \gamma = 0 \). On a alors \( \hat{G}(2\xi) = P(e^{-i\xi})\hat{G}(\xi), \) \( \hat{\gamma}(2\xi) = P(e^{-i\xi})\hat{\gamma}(\xi) \) et enfin, par (4.2), \( \hat{G}(\xi) = Q(e^{-i\xi})\hat{\gamma}(\xi) \). Remarquons que \( Q(0) \neq 0 \) (puisque \( \inf \text{Supp } G = \inf \text{Supp } \gamma \) et que \( Q \) n’a pas de racines sur le cercle unité (puisque \( P_G(\xi) = |Q(e^{-i\xi})|^2P_\gamma(\xi) \) et que \( P_G \) ne s’annule pas,
d’après la Proposition 2). De plus, on a: \( P(z)Q(z) = Q(z^2)P_0(z) \) et donc \( \deg P = \deg P_0 + \deg Q \); comme \( P(z) \land P(-z) = 1 \), il est nécessaire que \( P_0 \) divise \( P \), d’où il existe \( A \) tel que: \( Q(z^2) = Q(z)A(z) \). Si \( z \) est une racine de \( Q \), il en va de même pour \( z^2 \), et donc \( z^4, z^8, \ldots \). Comme ni 0 ni les complexes de module 1 ne sont racines de \( Q \) et que \( Q \) doit avoir un nombre fini de zéros, alors on trouve que \( Q \) est une constante et donc \( G = C^{\infty} \).

(6.1) \(\Rightarrow\) (6.6). Il suffit de reproduire la démonstration du théorème d’Yves Meyer [12] sur la restriction à \([0, 1]\) des bases d’I. Daubechies. On va donc montrer que, pour \( j \geq 0 \), la dimension de l’espace des restrictions à \([0, 1]\) des fonctions de \( V_j \) est exactement \( 2^j + \sup Supp G - \inf Supp G - 1 \).

Quite à translater \( G \) on peut supposer \( \inf Supp G = 0 \); on pose \( sup Supp G = N \). Commençons par remarquer que si

\[
\sum_{k = -M}^{N-1} \alpha_k G(x + k)
\]

est nulle sur \([0, M + 1]\) et si \( M + 1 \geq 2 \) (\( \sup Supp \theta - \inf Supp \theta \) (où \( \theta \) est la fonction décrite par le Corollaire 4) alors les \( \alpha_k \) sont nuls; en effet

\[
f(x) = \sum \alpha_k G(x + k)
\]

est portée par \([-N + 1, 0] \land [M + 1, M + N]\); dire que \( f \in V_0 \) revient à dire que \( f \) est orthogonale à toutes les fonctions \( \theta(2^jx - k), j \geq 0, k \in \mathbb{Z} \) (car \( V_0 = \bigoplus_{j \geq 0} W_j \), mais cela est alors vrai de \( f \big|_{(-N + 1, 0)} \) et de \( f \big|_{(M + 1, M + N)} \) puisque \( \theta(2^jx - k) \) est portée par un intervalle de longueur \( 2^{-j} + \inf Supp \theta \) et \( \inf Supp \theta \). Puisque \( G \) est de support minimal dans \( V_0 \), on obtient \( f = 0 \). Par dilatation, cela donne que la dimension des restrictions à \([0, 1]\) des fonctions de \( V_j \) est exactement \( 2^j + N - 1 \) pour \( j \) assez grand \( (2^j \geq 2(\sup Supp \theta - \inf Supp \theta)) \). Pour passer de \( j \) à \( j - 1 \), on suit la démonstration de Y. Meyer, basée sur le seul propriété que \( P(z) \land P(-z) = 1 \).

(6.6) \(\Rightarrow\) (6.4). On peut supposer \( \inf Supp G = 0 \). Comme \( G \big|_{(0, 1]} \) est indépendante des \( G(x + k) \big|_{(0, 1]} \) (\( k \neq 0 \)), il existe \( H \in L^2([0, 1]) \) avec \( \langle H \big| G \rangle = 1 \) et \( \langle H \big| G(x + k) \rangle = 0 \) pour \( k \neq 0 \).

(6.4) \(\Rightarrow\) (6.1). Comme \( H \) et \( G \) sont à support compact, on en déduit immédiatement que les \( G(x - k) \) sont une base de Riesz de \( V_0 \) et que tout élément de \( V_0 \) se représente comme

\[
h = \sum_{k \in \mathbb{Z}} \langle h \big| H(x - k) \rangle G(x - k).
\]

(6.3) (et donc (6.1)) est alors immédiat.
5. Calculs fondamentaux dans une analyse multi-résolution

Nous considérons une analyse multi-résolution \((V_j)_{j \in \mathbb{Z}}\) où \(V_0\) admet une base de Riesz \(\gamma(x - k)\) avec \(\gamma\) à valeurs réelles et de support compact. La fonction \(\gamma\) est choisie de manière à ce que son support soit minimal, que \(\inf \text{Supp } \gamma = 0\) et que \(\hat{\gamma}(0) = 1\). On appellera \(\gamma\) la base normalisée de \(V_0\).

On a

\[
\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} P_0(e^{-i\xi/2^j})
\]

avec \(P_0 \in \mathbb{R}[X]\), \(P_0(1) = 1\) et \(P_0(0) \neq 0\). Si \(\hat{\gamma}(\xi) \in L^1\), alors

\[
\prod_{j=1}^{N} P_0(e^{-i\xi/2^j})X_k\left(\frac{\xi}{2^N}\right)
\]

tend vers \(\hat{\gamma}\) en norme \(L^1\) (où \(K\) est décrit dans la Proposition 1). On a alors le procédé d'approximation suivant [5]:

**Proposition 3.** On pose

\[
h_0(x) = X_{[-1/2, 1/2]}(x) \quad \text{et} \quad h_{k+1}(x) = \sum_{l=0}^{N} \alpha_l h_k(2x - l)
\]

(où

\[
P_0(z) = \frac{1}{2} \sum_{l=0}^{N} \alpha_l z^l.
\]

Si \(\hat{\gamma} \in L^1\), alors \(h_k \rightarrow \gamma\) en norme \(L^\infty\), pour \(k \rightarrow +\infty\).

En effet \(h_k\) est constante par morceaux sur des intervalles de longueur \(1/2^k\).

De plus sur les points de \(1/2^k \mathbb{Z}\), \(h_k\) coïncide avec \(\theta_k\) où

\[
\theta_k = \prod_{j=1}^{k} P_0(e^{-i\xi/2^j})X_k\left(\frac{\xi}{2^k}\right)
\]

(par récurrence sur \(k\)). Enfin \(\gamma\) est uniformément continue puisque continue et à support compact. Comme \(|\theta_k - \gamma_k| \rightarrow 0\), on a bien \(|h_k - \gamma_k| \rightarrow 0\).

Néanmoins la convergence n’est pas rapide. Prenons l’exemple d’un spline cubique:

\[
\hat{\gamma}(\xi) = \left(\frac{1 - e^{-i\xi}}{-i\xi}\right)^4.
\]
On pose
\[ \hat{G}(\xi) = \frac{\tilde{\gamma}(\xi)}{P_\gamma(\xi)} \]
de sorte que \( \langle G(x) \mid \gamma(x - k) \rangle = \delta_k \). Alors il est facile de voir que
\[ h_k\left( \frac{p}{2^k} \right) = \left\langle \gamma\left( \frac{x}{2^k} \right) \mid G(x - p) \right\rangle. \]
En particulier, on a
\[ h_k(p) = \left\langle \gamma\left( \frac{x}{2^k} + p \right) \mid G(x) \right\rangle = \gamma(p) + \frac{1}{2^k} \gamma'(p) \int xG(x) \, dx + O\left( \frac{1}{4^k} \right). \]
Or
\[ \int xG(x) \, dx = 2 \quad \text{et} \quad \gamma'(1) = \frac{1}{2}, \]
d'où
\[ |h_k(1) - \gamma(1)| \sim \frac{1}{2^k}. \]
L'erreur n'est divisée que par 2 à chaque itération. Si l'on veut calculer \( \gamma \) aux points entiers, on utilisera plutôt la remarque suivante.

**Proposition 4.** On note \( M \) la transformation linéaire définie sur \( \mathbb{R}^{N-1} \) par
\[ M\begin{pmatrix} x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{N-1} \end{pmatrix} \quad \text{où} \quad y_j = \sum_{l=0}^{N} \alpha_l x_{2^j - l}, \]
(en prolongeant x_j par 0 en dehors de \{1, \ldots, N - 1\} où
\[ P_0(z) = \frac{1}{2} \sum_{l=0}^{N} \alpha_l z^l. \]
Si \( \tilde{\gamma}(\xi) \in L^1 \), alors Ker \((M - \text{Id})\) est de dimension 1 et
\[ \begin{pmatrix} \gamma(1) \\ \vdots \\ \gamma(N-1) \end{pmatrix} \]
est déterminé uniquement par

\[
M \begin{bmatrix}
\gamma(1) \\
\vdots \\
\gamma(N-1)
\end{bmatrix} = \begin{bmatrix}
\gamma(1) \\
\vdots \\
\gamma(N-1)
\end{bmatrix}
\]

et

\[
\sum_{j=1}^{N-1} \gamma(j) = 1.
\]

**Démonstration.** Soit

\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_{N-1}
\end{bmatrix}
\]

une solution de \( MX = X \). On pose

\[
h_0(x) = \sum_{j=1}^{N-1} x_j \hat{\theta}_0(x - j)
\]

où \( \hat{\theta}_0(\xi) = x_k(\xi) \) et

\[
h_{k+1}(x) = \sum_{l=0}^{N} \alpha_l h_k(2x - l).
\]

Alors

\[
\hat{h}_{k+1}(\xi) = P_0(e^{-i\xi/2})\hat{h}_k\left(\frac{\xi}{2}\right)
\]

et donc, en posant

\[
Q(\xi) = \sum_{j=1}^{N-1} x_j e^{-i\xi j},
\]

on a

\[
\hat{h}_k(\xi) = \prod_{j=1}^{k} P_0(e^{-i\xi/2})x_j\left(\frac{\xi}{2^j}\right)Q\left(\frac{\xi}{2^j}\right),
\]

de qui tend en norme \( L^1 \) vers \( \hat{\gamma}(\xi)Q(0) \). En particulier \( h_k(p) \) tend vers \( Q(0)\gamma(p) \); or \( h_k(p) = x_p \) (par récurrence sur \( k \)) et donc
\[
\begin{bmatrix}
    x_1 \\
    \vdots \\
    x_{N-1}
\end{bmatrix}
= \begin{bmatrix}
    \sum_{j=1}^{N-1} x_j \\
    \gamma(1) \\
    \vdots \\
    \gamma(N-1)
\end{bmatrix}.
\]

**Proposition 6.** Les intégrales

\[\int x^k \gamma(x) \, dx, \quad k \in \mathbb{N},\]

se calculent récursivement par les formules:

- \[\int x^k \gamma(x) \, dx = i^k \gamma^{(k)}(0).\]
- \[\hat{\gamma}(0) = 1.\]
- \[\sum_{\alpha = 0}^{k} \mathcal{C}_k^\alpha \hat{\gamma}^{(\alpha)}(0) m_0^{(k+1-\alpha)}(0).\]

Il s'agit juste d'appliquer la formule de Leibnitz à \(\hat{\gamma}(2\xi) = m_0(\xi)\hat{\gamma}(\xi)\) avec \(m_0(\xi) = P_0(e^{-i\xi}).\)

**Proposition 7.**

(i) Si \(\gamma \in H^1\) (le \(\gamma \in L^2\)) alors il existe une analyse multi-résolution \(V_j\) de \(L^2(\mathbb{R})\) de base normalisée \(M_1\) telle que

\[\gamma'(x) = M_1(x) - M_1(x - 1).\]

(ii) Il existe de même une analyse multi-résolution \(V_j''\) de \(L^2(\mathbb{R})\) de base normalisée \(M_2\) telle que \(\gamma(x) - \gamma(x - 1) = M_2(x)\).

La Proposition 7 permet donc de dériver les fonctions de \(V_0\) et d'intégrer les fonctions de \(V_0\) d'intégrales nulles. Le fait que la dérivation dans \(V_0\) revient à appliquer un opérateur de différence finie dans un autre espace \(V_0\) a été signalé par G. Malgouyres [9].

**Démonstration.**

(i) Comme \(\gamma'\) est à support compact et que

\[\sum_{k \in \mathbb{Z}} \gamma'(x - k) = 0\]

(puisque \(\sum_{k \in \mathbb{Z}} \gamma(x - k) = \hat{\gamma}(0)\)), on a

\[\gamma'(x) = M_1(x) - M_1(x - 1) \quad \text{où} \quad M_1(x) = \sum_{k = -\infty}^{-1} \gamma'(x - k)\]
est dans $L^2$ et à support compact. De plus, on a

$$\hat{M}_1(\xi) = \frac{i\xi}{1 - e^{-i\xi}} \hat{\gamma}(\xi)$$

d'où

$$\hat{M}_1(2\xi) = \frac{2}{1 + e^{-2i\xi}} P_0(e^{-i\xi}) \hat{M}_1(\xi).$$

Comme

$$P_0(-1) = 0, \quad P_1(z) = \frac{2}{1 + z} P_0(z).$$

est un polynôme avec $P_1(1) = 1$ d'où

$$\hat{M}_1(\xi) = \prod_{j=1}^{\infty} P_1(e^{i\pi/2^j}).$$

Il est clair que $P_1$ vérifie les conditions de la Proposition 2 et donc que les $M_1(x - k)$, $k \in \mathbb{Z}$, forment une base de Riesz d'un espace $V_0'$ pour une analyse multi-résolution $(V'_j)_{j \in \mathbb{Z}}$. De plus, il est clair également que $P_1$ vérifie (6.5) et donc $M_1$ est la base normalisée de $V_0$.

(ii) Il suffit de poser

$$\hat{M}_2(\xi) = \frac{1 - e^{-i\xi}}{i\xi} \hat{\gamma}(\xi);$$

alors il est clair que $M_2 \in L^2$ et que

$$M_2(x) = \gamma(x) - \gamma(x - 1)$$

et donc $M_2$ est à support compact. De plus on a

$$\hat{M}_2(2\xi) = \frac{1 + e^{-2i\xi}}{2} P_0(e^{-i\xi}) \hat{M}_2(\xi)$$
d'où

$$\hat{M}_2(\xi) = \prod_{j=1}^{\infty} P_2(e^{-i\pi/2^j})$$

où

$$P_2(z) = \frac{1 + z}{2} P_0(z),$$

et la conclusion (voir à nouveau la Proposition 2 et (6.5)) est immédiate.

**Corollaire 5.** Si $\gamma \in H^k$ (c'est-à-dire si pour $j \in \{0, \ldots, k\}$, $\gamma^{(j)} \in L^2$) alors $\hat{\gamma}^{(j)}(2l\pi) = 0$ pour $l \neq 0$ et $0 \leq j \leq k$. En particulier, tout polynôme de degré $\leq k$ se représente comme

$$Q(x) = \sum_{l \in \mathbb{Z}} R(l) \gamma(x - l)$$

où $R$ est également un polynôme de degré $\leq k$.
En effet, si \( \gamma \in H^k \) alors \( \gamma(x) = M_1(x) - M_1(x - 1) \) où \( M_1 \in H^{k-1} \). De plus si

\[
\hat{\gamma}(\xi) = \prod_{j=1}^{\infty} P_j(e^{-i\xi/2^j}) \quad \text{alors} \quad \hat{M}_1(\xi) = \prod_{j=1}^{\infty} P_j(e^{-i\xi/2^j})
\]

où

\[
P_j(z) = \frac{2}{1 + z} P_0(z).
\]

D'où si \( \gamma \in H^k \) alors \((2/(1 + z))^k P_0(z)\) est un polynôme qui vérifie de plus que sa valeur en \(-1\) est nulle (d'après (3.2)). On a donc

\[
P_0(z) = \left(\frac{1 + z}{2}\right)^{k+1} Q_0(z);
\]

comme

\[
\hat{\gamma}(2k\pi + \xi) = P_0(e^{-i\xi/2} - i\xi)e^{i\xi/2},
\]

on montre que \( \gamma^{(j)}(2l\pi) = 0 \) par récurrence sur \( M \) tel que \( l/2^M \in 2\mathbb{Z} + 1 \).

De plus, si on note \( M_j \) la base normalisée de l'analyse multi-résolution où se trouve \( \gamma^{(j)} \), alors on a:

\[
\left(\sum a_i \gamma(x - l)\right)^{(j)} = \sum a_i (M_1(x - l) - M_1(x - 1 - l))
\]

\[
= \sum (a_i - a_{i-1}) M_1(x - l)
\]

d'où

\[
\left(\sum a_i \gamma(x - l)\right)^{(k)} = \sum \Delta^k a_i M_k(x - l) \quad \text{où} \quad \Delta(a_i) = (a_i - a_{i-1}).
\]

Si \( a_i = R(P) \) avec \( \deg R \leq k \) alors \( \Delta^k a_i \) est constante et

\[
\left(\sum a_i \gamma(x - l)\right)^{(k)} = c^k M_k(0).
\]

On obtient que \( R \to \sum R(l) \gamma(x - l) \) envoie \( \mathcal{C}_k[X] \) dans \( \mathcal{C}_k[X] \); de plus cette transformation est injective donc surjective.

**Proposition 8.** Si \( V_j \) et \( V'_j \) sont deux analyses multi-résolutions de bases normalisées \( \alpha \) et \( \beta \), alors \( \alpha \ast (\beta(-x)) \) est dans un espace \( V''_0 \) d'une analyse multi-résolution, et on peut donc calculer \( \int \alpha(x)\beta(x - k) \, dx \) à l'aide de la Proposition 4.

En effet, si

\[
\gamma(x) = \int \alpha(y)\beta(y - x) \, dy,
\]
FONCTIONS A SUPPORT COMPACT DANS LES ANALYSES MULTI-RÉSOLUTIONS 181

alors on a
\[ \hat{\gamma}(\xi) = \hat{\alpha}(\xi)\hat{\beta}(\xi) \quad \text{d'où} \quad \hat{\gamma}(2\xi) = P_0(e^{-i\xi})P_1(e^{-i\xi})\hat{\gamma}(\xi) \]
(où \( \hat{\alpha}(2\xi) = P_0(e^{-i\xi})\hat{\alpha}(\xi) \) et \( \hat{\beta}(2\xi) = P_1(e^{-i\xi})\hat{\beta}(\xi) \)). Le Corollaire 2 permet alors de conclure. (Il se peut que les \( \gamma(x - k) \) ne forment toutefois pas une base de Riesz de \( V^a_0 \).)

En combinant les Propositions (7) et (8), on voit qu'on peut calculer pour un opérateur différentiel \( \sum a_\alpha \left( \frac{d}{dx} \right)^\alpha \) à coefficients constant les quantités
\[ \left\langle \sum a_\alpha \left( \frac{d}{dx} \right)^\alpha \psi_{j,k}^{(\epsilon)} \left| \psi_{j,k}^{(\eta)} \right. \right\rangle \quad \text{où} \quad \epsilon, \eta \in \{0, 1\}, \psi^{(0)} = \varphi \text{ et } \psi^{(1)} = \psi, \varphi \text{ et } \psi \text{ étant les père et mère des ondelettes orthonormées d'I. Daubechies. Or ces calculs interviennent dans l'analyse de l'opérateur } \sum a_\alpha \left( \frac{d}{dx} \right)^\alpha \text{ par l'algorithme de Beylkin-Coifman-Rokhlin [2]. Ce genre de calculs est actuellement développé par M. Lahzami [7] pour la programmation numérique.}

6. Exemples

(i) Les fonctions splines. L'analyse multi-résolution des splines de degré \( k \) est engendrée par l'espace \( V_0 \) des splines de degré \( k \) à noeuds dans \( Z \) et de carré intégrable (c'est-à-dire que \( f \in V_0 \) si et seulement si \( f \in L^2 \cap C^{k-1} \) et pour tout \( l \in Z, f|_{[l,l+1]} \) est polynomiale de degré \( \leq k \)). La base normalisée de \( V_0 \) est la fonction \( \gamma = \chi_{[0,1]} \ast \cdots \ast \chi_{[0,1]} = \chi_{[0,1]}^{(k+1)} \).

L'échelle des fonctions splines est évidemment stable par dérivation et primitivation et la Proposition 7 ne nous apprend rien.

Remarquons que la Proposition 7(ii) est un cas particulier de la Proposition 8, puis-qu'en fait \( M_2 = \chi_{[0,1]} \ast \gamma \).

L'existence d'ondelettes splines à support compact (cf. Corollaire 2) a été signalée par Lemarié en 1987 [8] et systématiquement étudiée dans la thèse de P. Auscher [1].

(ii) Les bases d'I. Daubechies. I. Daubechies a construit des fonctions \( \varphi \) telles que: \( \text{Supp}_N \varphi \subset [0,2N - 1] \), les \( \varphi (x - k) \) sont orthonormées quand \( k \) décrit \( Z \) et \( \varphi \) est de classe \( C^{\lambda N} \) pour un \( \lambda > 0 \) [5]. Il est clair que \( \varphi \) est la base normalisée de l'espace \( V_0 \) associé (puisque (6.3) est immédiat). Par contre l'échelle de ces fonctions \( \varphi \) n'est stable ni par primitivation ni par dérivation et les calculs comme ceux développés à la suite de la Proposition 8 font intervenir d'autres analyses multi-résolution.

(iii) Les analyses multi-résolution non orthogonales de J. C. Fauveau. Pour construire des analyses multi-échelle à analyse et synthèse rapides et à fil-
tre à phase linéaire, J. C. Fauveau a été amené à introduire la notion d'analyse multi-résolution non orthogonale [6]. Sa construction repose sur deux fonctions \( \gamma_1 \) et \( \gamma_2 \) à support compact telles que:

\[
\hat{\gamma}_1(\xi) = \prod_{j=1}^{m} P_1(e^{-i\xi/2^j}), \quad \hat{\gamma}_2(\xi) = \prod_{j=1}^{m} P_2(e^{-i\xi/2^j}),
\]

\[
P_1(z)P_2(z) + P_1(-z)P_2(-z) = 1
\]

et les \( \gamma_i(x-k) \) sont la base de Riesz d'un espace \( V_{0,i} \) lié à une analyse multi-résolution. Il est alors clair que \( \gamma_i \) est la base normalisée de \( V_{0,i} \) (d'après (6.5)).

Références

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Multiplicative structure of de Branges’s spaces

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1. Introduction

L. de Branges has originated a viewpoint one of whose repercussions has been the detailed analysis of certain Hilbert spaces of holomorphic functions contained within the Hardy space $H^2$ of the unit disk. The initial study of the spaces was made by de Branges and J. Rovnyak [4] about 25 years ago. Although neglected for a while, the spaces are now attracting considerable attention because of their beautiful internal structure and their relevance to function theory [21]. Our aim in this paper is to investigate their multipliers.

The starting point is a nonconstant function $b$ in $B(H^\infty)$, the unit ball in the space $H^\infty$ of all bounded holomorphic functions in the open unit disk, $D$, of the complex plane. The de Branges space $H(b)$ consists by definition of the range of the operator $(1 - T_\phi T_\overline{b})^{1/2}$ (where, for $\phi$ in $L^\infty$ of the unit circle, $T_\phi$ denotes the Toeplitz operator on $H^2$ with symbol $\phi$). The space $H(b)$ is given the Hilbert space structure that makes the operator $(1 - T_\overline{b} T_b)^{1/2}$ a coisometry of $H^2$ onto $H(b)$. By a multiplier of $H(b)$ we mean a holomorphic function $m$ in $D$ such that $mh$ is in $H(b)$ whenever $h$ is. Since the evaluation functionals on $H(b)$ at the points of $D$ are bounded, one sees from the closed graph theorem that the multiplication operator on $H(b)$ induced by such an $m$ is bounded, from which it follows that $m$ must be in $H^\infty$ [23].

There are two extreme cases. If $\|b\|_\infty < 1$, then $H(b)$ is just a renormed version of $H^2$ and every function in $H^\infty$ is a multiplier of it. At the other extreme, if $b$ is an inner function, then $H(b)$ is an ordinary subspace of $H^2$, namely, the orthogonal complement of the Beurling invariant subspace $bH^2$. 
It is thus the typical invariant subspace of $S^*$, the adjoint of the unilateral
shift operator, $S$, on $H_2^2 ((Sf)(z) = z^2 f(z))$. In this case, $H(b)$ has no noncon-
stant multipliers. (Proof: If $b$ is an inner function and $m$ is a multiplier of
$H(b)$ then, because $S^* b$ is in $H(b)$ [21], we have, for all $f$ in $H^2$, the equality
$0 = \langle m S^* b, b f \rangle$. One easily sees that the right side equals $\langle S^* m, (1 - \overline{b(0)} b) f \rangle$. Setting $f = S^* m / (1 - \overline{b(0)} b)$, we find that $S^* m = 0$).

The spaces $H(b)$ break naturally into two classes according to whether $b$ is
or is not an extreme point of $B(H^w)$, or, what is equivalent, according to
whether the function $1 - |b|^2$ is not or is log-integrable on $\partial D$ [14]. A few
results in the latter case can be found in [18]. It is shown there, for example,
that if $b$ is not an extreme point of $B(H^w)$ then any function holomorphic
in a neighborhood of $\partial$ is a multiplier of $H(b)$, and those $b$ for which every
function in $H^w$ is a multiplier of $H(b)$ are characterized. Further progress has
recently been made by B. M. Davis and J. E. McCarthy [1] who, among other
things, have characterized the functions that are multipliers of every space
$H(b)$ with $b$ nonextreme. For the case where $b$ is an extreme point, on the
other hand, next to nothing has been known up to now beyond the negative
result for inner functions cited above. In particular, it has been an open ques-
tion whether there is any extreme point $b$ such that $H(b)$ has nonconstant
multipliers.

In this paper we shall concentrate mainly on the case where $b$ is an extreme
point but not an inner function. The main thrust of our results is that $H(b)$
has an abundance of multipliers in that case.

A space closely related to $H(b)$, called $H(\overline{b})$, arises naturally in the search
for multipliers. By definition, $H(\overline{b})$ is the range of the operator $(1 - T_b T_{\overline{b}})^{1/2}$,
with the Hilbert space structure that makes this operator a coisometry of $H^2$
onto $H(\overline{b})$. The space $H(\overline{b})$ is trivial if $b$ is an inner function, but otherwise
it is infinite dimensional. It turns out that every multiplier of $H(b)$ differs by
at most a constant from a function in $H(\overline{b})$. The culmination of our efforts
will be a proof, for the case where $b$ is an extreme point of $B(H^w)$, that the
multipliers of $H(b)$ that lie in $H(\overline{b})$ are dense in $H(\overline{b})$.

In Section 2 the place of the spaces $H(b)$ and $H(\overline{b})$ in the general scheme
of de Branges is described. A lemma concerning that scheme is established
and used to obtain information about $H(b)$ and $H(\overline{b})$. (Some of the results
here can be found in the literature, but the present proofs seem particularly
apt.).

Section 3 explains the relation between $H(b)$ and $H(\overline{b})$ and certain spaces
of Cauchy integrals. The multipliers of $H(b)$ coincide with the multipliers of
its related space of Cauchy integrals. Cauchy integrals in the unit disk have
been studied extensively beginning with V. P. Havin [12], but from a view-
point rather different from ours. In Section 4 we show how our methods pro-
vide a simple proof of a theorem of S. A. Vinogradov.
The remainder of our paper addresses mainly the case where $b$ is an extreme point of $B(H^q)$. Section 5 contains two negative results for that case that say, very roughly speaking, that nonconstant multipliers cannot behave too nicely. Section 6 pertains to decompositions of the space $H(b)$ and Section 7 to the case where $b$ is invertible. In the latter case $b$ is shown to be a multiplier of $H(b)$, and the converse is shown to hold when $b$ is an extreme point. In Section 8 the multiplication operator on $H(b)$ induced by a multiplier is discussed, for the extreme point case.

In Section 9, again for the extreme point case, we introduce two conjugations, one on $H(b)$ and another on the one-dimension extension of $H(b)$ by the constant functions. It is shown that to each multiplier $m$ of $H(b)$ there corresponds a conjugate multiplier, $m^*$. The multipliers of $H(b)$ thus form a $*$-algebra, although not a $C^*$-algebra. A certain algebra of Cauchy integrals is introduced which contains all the multipliers of $H(b)$.

Section 10 contains two needed lemmas on Cauchy integrals. They are used in Section 11 to obtain more information on the space $H(b)$ and in Section 12 to establish a criterion that, among other things, enables us to construct a set of multipliers of $H(b)$ that is dense in $H(b)$ (again, for the extreme point case).

If $u$ is an inner function, then, as is explained in Section 6, every multiplier of $H(ub)$ is a multiplier of $H(b)$. In Section 13, in the extreme point case, a criterion is obtained for a multiplier of $H(b)$ to be a multiplier of $H(ub)$. The functions that are multipliers of $H(ub)$ for every inner function $u$ are characterized. A sufficient condition on $u$ is found for $H(b)$ and $H(ub)$ to have the same multipliers.

In Section 14 we give a complete description of the multipliers of $H(b)$ for a certain class of extreme points $b$. This result is related to a well-known theorem of H. Nelson and G. Szegö. We also give an example to show that, even when $b$ is an extreme point, an inner function $u$ can exist such that not every multiplier of $H(b)$ is one of $H(ub)$. (Such an example with $b$ not an extreme point can be extracted from [18].)

The concluding Section 15 contains a short list of open questions.

Besides the notations already introduced, the following additional ones are needed.

$L^2$ denotes the $L^2$ space of normalized Lebesgue measure on $\partial D$ and $P_+$ denotes the orthogonal projection in $L^2$ with range $H^2$. The norm and inner product in $L^2$ are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$.

The norm and inner product in $H(b)$ are denoted by $\|\cdot\|_b$ and $\langle \cdot, \cdot \rangle_b$, and those in $H(b)$ by $\|\cdot\|_b$ and $\langle \cdot, \cdot \rangle_b$.

The kernel function in $H^2$ for the point $w$ of $D$ is denoted by

$$k_w(z) = (1 - wz)^{-1}.$$
The kernel functions in $H(b)$ and $H(\tilde{b})$ for $w$ are denoted by $k^b_w$, as a simple argument shows. For $k^b_w$ one has the expression $k^b_w = (1 - \bar{b}(w)b)k_w$ [4], [19].

The term «operator» will always mean «bounded operator».

The following two simple properties of $H(b)$, the first of which was mentioned earlier, can be found in [19].

1. $S^*b$ belongs to $H(b)$.
2. $b$ belongs to $H(b)$ if and only if $b$ is not an extreme point of $B(H^*)$.

Alternative proofs of some of the results below have recently been found by A. V. Lipin (private communication).

2. Relations between $H(b)$ and $H(\tilde{b})$

It is helpful to fit the spaces $H(b)$ and $H(\tilde{b})$ into the general scheme promulgated by de Branges (for example, in [2]). If $H$ and $H_1$ are Hilbert spaces and $A$ is an operator in $L(H_1, H)$, then de Branges’s space $M(A)$ consists of the range of $A$, with the Hilbert space structure that makes $A$ into a coisometry of $H_1$ onto $M(A)$. Thus, for example, if $y$ is in $H_1$ and is orthogonal to the kernel of $A$, then $\|Ay\|_{M(A)} = \|y\|_{H_1}$. If $\|A\| \leq 1$, then the space $M((1 - AA^*)^{1/2})$ is called by de Branges the complementary space of $M(A)$ and denoted by $H(A)$. Our spaces $H(b)$ and $H(\tilde{b})$, therefore, coincide with $H(T_b)$ and $H(T_{\tilde{b}})$, respectively. We shall denote $M(T_b)$ by $M(b)$.

A factorization criterion of R. G. Douglas [5] is often useful in establishing containment relations between de Branges’s spaces, and in showing a given operator maps one of these spaces into another one.

**Douglas's criterion.** Let $H$, $H_1$, and $H_2$ be Hilbert spaces and $A$ and $B$ operators in $L(H_1, H)$ and $L(H_2, H)$, respectively. Then the operator inequality $BB^* \leq AA^*$ is necessary and sufficient for the existence of a factorization $B = AR$ with $R$ in $L(H_2, H_1)$ and $\|R\| \leq 1$.

This tells us, for example, that the two spaces $M(A)$ and $M(B)$ coincide as Hilbert spaces if and only if $AA^* = BB^*$. In virtue of the operator inequality $1 - T_b^*T_b \preceq 1 - T_\tilde{b}^*T_\tilde{b}$ it tells us also that $H(\tilde{b})$ is contained in $H(b)$, with the inclusion map a contraction.

If $A$ is a contraction in $L(H_1, H)$, then $M(A)$ is an ordinary subspace of $H$ if and only if $A$ is a partial isometry, in which case $H(A)$ is the ordinary orthogonal complement of $M(A)$. In the contrary case the intersection $M(A) \cap H(A)$, which we call an overlapping space, is nontrivial (de Branges and Rovnyak [3] use the term «overlapping space» in a slightly different way). A simple lemma establishes the relation between $H(A)$, $H(A^*)$, and their overlapping spaces.
Lemma 2.1. Let \( H \) and \( H_1 \) be Hilbert spaces and \( A \) a contraction in \( L(H, H_1) \). The vector \( x \) in \( H \) belongs to \( H(A) \) if and only if \( A^*x \) belongs to \( H(A^*) \), in which case,

\[
\|x\|_{H(A)}^2 = \|x\|_H^2 + \|A^*x\|_{H(A^*)}^2.
\]

The overlapping space \( M(A^*) \cap H(A^*) \) coincides with \( A^*H(A) \).

The inclusion \( A^*H(A) \subset H(A^*) \) follows from the operation identity

\[
A^*(1 - AA^*)^{1/2} = (1 - A^*A)^{1/2}A^*.
\]

which goes back at least to a paper of P. R. Halmos [10]. Suppose \( x \) is a vector in \( H \) such that \( A^*x \) is in \( H(A^*) \), say \( A^*x = (1 - A^*A)^{1/2}y \) with \( y \) in \( H_1 \) and orthogonal to the kernel of \( (1 - A^*A)^{1/2} \). Then \( AA^*x = (1 - AA^*)^{1/2}Ay \) (by the same identity used above), from which one concludes that

\[
x = (1 - AA^*)^{1/2}((1 - AA^*)^{1/2}x + Ay),
\]

showing that \( x \) is in \( H(A) \). As \( Ay \) is easily seen to be orthogonal to the kernel of \( (1 - AA^*)^{1/2} \), we have

\[
\|x\|_{H(A)} = \|(1 - AA^*)^{1/2}x + Ay\|_H.
\]

The square of the right side equals

\[
\langle (1 - AA^*)x, x \rangle_H + \|Ay\|_H^2 + 2\text{Re}\langle (1 - A^*A)^{1/2}A^*x, y \rangle_{H_1}
\]

\[
= \|x\|_H^2 - \|A^*x\|_{H_1}^2 + \|Ay\|_H^2 + 2\langle (1 - A^*A)y, y \rangle_{H_1}
\]

\[
= \|x\|_H^2 - \|A^*x\|_{H_1}^2 + \|y\|_{H_1}^2 + \|(1 - A^*A)^{1/2}y\|_{H_1}^2
\]

\[
= \|x\|_H^2 + \|y\|_{H_1}^2
\]

\[
= \|x\|_H^2 + \|A^*x\|_{H(A^*)}^2.
\]

which gives the desired expression for \( \|x\|_{H(A)} \). This completes the proof of the lemma.

For the situation of interest in this paper, we obtain the following immediate consequences of Lemma 2.1.

Lemma 2.2. The \( H^2 \) function \( h \) belongs to \( H(b) \) if and only if \( T_b h \) is in \( H(\bar{b}) \). If \( h \) is in \( H(b) \), then

\[
\|h\|_{H^2}^2 = \|h\|_{b}^2 + \|T_b h\|_{\hat{b}}^2.
\]

Lemma 2.3. The overlapping space \( M(b) \cap H(b) \) equals \( T_b H(b) \). The operator \( T_b \) acts as a contraction from \( H(\bar{b}) \) to \( H(b) \).
Two corollaries of the last lemma are worth recording.

**Corollary 2.4.** Every multiplier of \( H(b) \) is a multiplier of \( H(\overline{b}). \)

**Corollary 2.5.** The overlapping space \( M(b) \cap H(b) \) is dense in \( H(b) \) if and only if \( b \) is an outer function.

Corollary 2.4 is immediate. To establish Corollary 2.5 it suffices to note that \( T_b^* H(\overline{b}) \) is the range of the operator \((1 - T_b^* T_b)^{1/2} T_b^*\), which is dense in \( H(\overline{b}) \) if and only if the range of \( T_b \) is dense in \( H^2 \), in other words, if and only if \( b \) is an outer function.

**Lemma 2.6.** If \( \phi \) is a function in \( H^\infty \) then the spaces \( H(b) \) and \( H(\overline{b}) \) are invariant under the Toeplitz operator \( T_{\phi^2} \), whose norm as an operator in each of them does not exceed \( \|\phi\|_\infty \).

For the proof, we can assume with no loss of generality that \( \|\phi\|_\infty = 1 \). To settle the case of \( H(\overline{b}) \) it will be enough, by Douglas's criterion, to verify the operator inequality

\[
T_{\phi^2}(1 - T_b^* T_b) T_{\phi^2} \leq 1 - T_b^* T_b.
\]

One easily sees that the difference between the right and left sides equals

\[
1 - T_{|\phi|^2} - T_{|\phi^2|^2} + T_{|\phi^3|^2},
\]

which is the Toeplitz operator with symbol \((1 - |\phi|^2)(1 - |b|^2)\), hence positive semidefinite, as desired. The case of \( H(b) \) follows immediately from the case of \( H(\overline{b}) \) in conjunction with Lemma 2.2.

### 3. Cauchy integrals

For \( \mu \) a finite complex Borel measure on \( \partial \mathbb{D} \), we let \( K_\mu \) denote its Cauchy integral, that is, the holomorphic function in \( C\backslash \partial \mathbb{D} \) defined by

\[
(K_\mu)(z) = \int_{\partial \mathbb{D}} \frac{1}{1 - e^{-\theta z}} \, d\mu(e^{i\theta}).
\]

If \( \mu \) is absolutely continuous and \( \sigma \) is its Radon-Nikodym derivative with respect to normalized Lebesgue measure, we write \( K_\sigma \) in place of \( K_\mu \). (What we are calling Cauchy integrals are often referred to as integrals of Cauchy-Stieltjes type.)

If the measure \( \mu \) is positive, we define the transformation \( K_\mu \) on \( L^2(\mu) \) by \( K_\mu q = K(q\mu) \). The function \( K_\mu q \) vanishes identically in \( \mathbb{D} \) if and only if \( q \) is orthogonal to \( H^2(\mu) \), the closure of the polynomials in \( L^2(\mu) \). We denote by
\( K^2(\mu) \) the space of all functions \( K_\mu q \) with \( q \) in \( L^2(\mu) \) and give it the Hilbert space structure that makes \( K_\mu \) an isometry of \( H^2(\mu) \) onto it. As before, if \( \mu \) is absolutely continuous with Radon-Nikodym derivative \( \sigma \), we write \( K_\sigma \) and \( K^2(\sigma) \) in place of \( K_\mu \) and \( K^2(\mu) \).

We let \( \mu_b \) denote the measure on \( \partial \mathbb{D} \) whose Poisson integral is the real part of the function \((1 + b)/(1 - b)\). For \( q \) in \( L^2(\mu_b) \), we define the function \( V_b q \) in \( \mathbb{D} \) by

\[
(V_b q)(z) = (1 - b(z))(K_{\mu_b} q)(z).
\]

A proof of the following representation for \( H(b) \) can be found in [20].

**Lemma 3.1.** The transformation \( V_b \) is an isometry of \( H^2(\mu_b) \) onto \( H(b) \). It maps the function \( k_w (|w| < 1) \) to the function \((1 - \overline{b(w)})^{-1} k^b_w \).

Thus, the problem of finding the multipliers of \( H(b) \) is the same as the problem of finding the multipliers of \( K^2(\mu_b) \). We note for future reference that the equality \( H^2(\mu_b) = L^2(\mu_b) \) holds if and only if the Radon-Nikodym derivative with respect to Lebesgue measure of the absolutely continuous component of \( \mu_b \) fails to be log-integrable [14, p. 50]. That Radon-Nikodym derivative equals \((1 - |b|^2)/(1 - b)^2 \) and so is not log-integrable if and only if \( b \) is an extreme point of \( B(H^\infty) \).

The operator on \( H^2(\mu_b) \) of multiplication by \( e^{ib} \) will be denoted by \( Z^b \).

**Lemma 3.2.** The transformation \( V_b|H^2(\mu_b) \) intertwines the operators \( Z^b_b [1 - (1 - b(0))(1 \circ 1)] \) and \( S^* \).

In the proof, we shall denote the inner product in \( L^2(\mu_b) \) by \( \langle \cdot, \cdot \rangle_{\mu_b} \). Let \( q \) be any function in \( L^2(\mu_b) \), and let \( g = K_\mu q \), so that \( V_b q = (1 - b)g \). Since \((K_{\mu_b} Z^b q)(z) = \langle Z^b \mu_b q, k_{z, \mu_b} \rangle = \langle q, Z_b k_{z, \mu_b} \rangle \), we have, for \( z \neq 0, \)

\[
(K_{\mu_b} Z^b q)(z) = \int_{\partial \mathbb{D}} \frac{e^{-\theta} q(e^{i\theta})}{1 - ze^{-i\theta}} d\mu_b(e^{i\theta}) = \frac{1}{z} \int_{\partial \mathbb{D}} \left( \frac{1}{1 - ze^{-i\theta}} - 1 \right) q(e^{i\theta}) d\mu_b(e^{i\theta}) = \frac{g(z) - g(0)}{z}.
\]

Therefore,

\[
(V_b Z^b q)(z) = (1 - b(z)) \frac{g(z) - g(0)}{z}.
\]
\[
\begin{align*}
= \frac{(1 - b(z))g(z) - (1 - b(0))g(0)}{z} + g(0) \frac{b(z) - b(0)}{z} \\
= \frac{(V_q \gamma)(z) - (V_q \gamma)(0)}{Z} + \frac{(V_q \gamma)(0)}{1 - b(0)} (S^*b)(z) \\
= (S^*V_q \gamma)(z) + (V_q \gamma, V_{b,1} >_c (S^*b)(z)).
\end{align*}
\]

In the last line we have used the equality \( V_{b,1} = (1 - b(0))^{-1}k_0 \) from Lemma 3.1. The same equality shows that, when \( q \) is the constant function 1, the function \( g \) equals \((1 - b(0)b)/(1 - b(0))(1 - b)\). Inserting these expressions into the equality above (the one that gives \( K_{p,b}Z_b^{*}q \) in terms of \( g \)) one obtains, after a few lines of calculation, the formula

\[
(K_{p,b}Z_b^{*}1)(z) = \frac{1}{(1 - b(0))(1 - b(z))} \left( \frac{b(z) - b(0)}{z} \right),
\]

implying that

\[V_{b,1}Z_b^{*} = (1 - b(0))^{-1}S^*b.\]

The expression for \( V_{b}Z_b^{*}q \) can thus be rewritten as

\[
V_{b}Z_b^{*}q = S^*V_q \gamma + (1 - b(0))(V_q \gamma, V_{b,1} >_c V_{b}Z_b^{*}1),
\]
or as

\[
S^*V_q \gamma = V_{b}Z_b^{*} [q - (1 - b(0))(q, 1 >_b 1)],
\]

which is the desired conclusion.

For \( H(b) \), the situation is simpler than for \( H(b) \). We let \( q \) denote the function \( 1 - |b|^2 \) on \( \partial D \).

**Lemma 3.3.** The transformation \( K_q \) is an isometry of \( H^2(\gamma) \) onto \( H(b) \). It maps \( k_w \) to \( k_0 \). Hence \( H(b) = K^2(\gamma)\partial D \).

We denote the inner product in \( L^2(\gamma) \) by \( \langle \cdot, \cdot \rangle \). For any points \( z \) and \( w \) of \( D \) we have

\[
\langle k_w, k_z \rangle = \langle (1 - |b|^2)k_w, k_z \rangle \\
= \langle (1 - T_b^{-1}T_b)k_w, k_z \rangle \\
= \langle k_w, k_z \rangle \\
= k^\beta_{w}(z) \\
= \langle k_w, k_z^b \rangle.
\]
But \( \langle K_\theta k_w(z) \rangle = \langle k_w, k_w \rangle_\theta \), so it follows that \( K_{\theta,} k_w = k_w^\theta \) and that \( \langle K_{\theta,} k_w, k_w \rangle_\theta = \langle k_w, k_w \rangle_\theta \). Thus \( K_{\theta,} \) maps the linear manifold in \( L^2(\mu) \) spanned by the functions \( k_w \) isometrically onto a dense linear manifold in \( H(b) \) (the one spanned by the functions \( k_w^\theta \)). One can now complete the proof by a standard limit argument.

The operator on \( H^2(\mu) \) of multiplication by \( e^{i\theta} \) will be denoted by \( Z_\theta \).

**Lemma 3.4.** The transformation \( K_{\theta,} \) intertwines the operators \( Z_\theta \) and \( S^\theta \).

This is a standard property of Cauchy integrals. It is established, except for a difference in notation, as the first step in the proof of Lemma 3.2.

**Corollary 3.5.** If \( \phi \) is a function in \( H^\sigma \), then the transformation \( K_{\theta,} \) intertwines the operators \( \phi(Z_\theta) \) and \( T_\theta \).

In fact, the case where \( \phi \) is a polynomial follows immediately from Lemma 3.4. To handle the general case one takes a sequence of polynomials that is uniformly bounded on \( \partial D \) and converges almost everywhere on \( \partial D \) to \( \phi \). The obvious limit argument yields the conclusion.

Our first theorem implies, in virtue of Corollary 2.4, that any multiplier of \( H(b) \) differs by a constant from a function in \( H(b) \). (If \( b \) is not an extreme point of \( B(H^\sigma) \), then \( H(b) \) contains the constants, so one gets the stronger conclusion that the multipliers of \( H(b) \) lie in \( H(b) \). The theorem itself is trivial in that case.)

**Theorem 3.6.** If \( b \) is not an inner function, then every multiplier of \( H(b) \) differs by a constant from a function in \( H(b) \).

As noted above, the theorem is trivial if \( b \) is not an extreme point of \( B(H^\sigma) \), so we assume it is an extreme point. Let \( m \) be a multiplier of \( H(b) \) and let \( h \) be any function in \( H(b) \) such that \( h(0) \neq 0 \). By Lemma 2.6, the functions \( ms^\bullet h \) and \( S^\bullet mh \) belong to \( H(b) \). Since \( S^\bullet mh = ms^\bullet h + h(0)S^\bullet m \), it follows that \( S^\bullet m \) is in \( H(b) \). Because \( b \) is an extreme point, the function \( \varrho \) is not log-integrable, which implies that \( H^2(\mu) = L^2(\mu) \) [14, p. 50]. Therefore, by Lemma 3.4, the operator \( S^\bullet H(b) \) is unitary, so in particular \( S^\bullet H(b) = H(b) \). The function in \( H(b) \) sent to \( S^\bullet m \) by \( S^\bullet \) differs from \( m \) by a constant.

### 4. Vinogradov's theorem

If \( \mu \) is a finite complex Borel measure on \( \partial D \) then its Cauchy integral, \( K_\mu \), as a function in \( D \), belongs to \( H^p \) for \( 0 < p < 1 \) and so has an inner-outер
factorization [7, p. 39]. The theorem of Vinogradov [24] states that if the inner function \( u \) divides the inner factor of \( K_{\mu} \), then the quotient \( K_{\mu}/u \) is a Cauchy integral; in fact, it is the Cauchy integral of a measure whose norm does not exceed that of \( \mu \). A simple and natural proof of this can be based on Lemmas 2.6 and 3.1.

For simplicity we assume \( \|\mu\| = 1 \), and we choose \( b \) so that \( |\mu| = \mu_{\mu} \). By Lemma 3.1 the function \( (1 - b)K_{\mu} \) is in \( H(b) \) and has norm at most 1. Thus, by Lemma 2.6, if \( u \) is an inner function, then

\[
T_u[(1 - b)K_{\mu}] = (1 - b)K(q\mu_{\mu}),
\]

where \( q \) is in \( H^2(\mu_{\mu}) \) and has norm at most 1. But if \( u \) divides the inner factor of \( K_{\mu} \) then it divides the inner factor of \( (1 - b)K_{\mu} \), so that

\[
T_u[(1 - b)K_{\mu}] = (1 - b)K_{\mu}/u.
\]

In that case \( K_{\mu}/u = K(q\mu_{\mu}) \), which proves Vinogradov’s theorem since the measure \( q\mu_{\mu} \) has norm at most 1.

5. Nonmultipliers

Our concern from now on will be with the case where \( b \) is an extreme point of \( B(H^\infty) \). In this section we obtain two negative results about multipliers.

It was mentioned in Section 1 that, if \( b \) is not an extreme point of \( B(H^\infty) \), then every function holomorphic in a neighborhood of \( \partial D \) is a multiplier of \( H(b) \). If \( b \) is an extreme point, exactly the opposite is true: no nonconstant multiplier can be continued analytically across all of \( \partial D \). This is a consequence of the next theorem together with Theorem 3.6 and Corollary 2.4.

**Theorem 5.1.** If \( b \) is an extreme point of \( B(H^\infty) \), then no nonzero function in \( H(b) \) can be continued analytically across all of \( \partial D \).

In fact, suppose the function \( h \) in \( H(\bar{b}) \) can be continued analytically across all of \( \partial D \). By Lemma 3.3 we can write \( h = K(q_\phi) \) with \( q \) in \( L^2(\phi) \). The function \( q_\phi \) is in \( L^2 \), being the product of the \( L^2 \) function \( q^{1/2} \) and the bounded function \( q^{1/2} \). This enables us to write \( h = P_+(q_\phi) \). (Recall that \( P_+ \) is the orthogonal projection in \( L^2 \) with range \( H^2 \).) Because \( b \) is an extreme point of \( B(H^\infty) \), the function \( \phi \) is not log-integrable, and therefore neither is \( q_\phi \), because

\[
\log|q_\phi| \leq \log^+|q^{1/2}| + \frac{1}{2} \log|q|.
\]

But the forward Fourier coefficients of \( q_\phi \) coincide with the Taylor coeffi-
Corollary 5.2. If \( b \) is an extreme point of \( B(H^\infty) \) and an outer function, then no nonzero function in \( H(b) \) can be continued analytically across all of \( \overline{D} \).

In fact, suppose the function \( h \) in \( H(b) \) can be continued analytically across all of \( \partial D \). Its Taylor coefficients then tend to 0 exponentially, and a simple estimate shows that then the forward Fourier coefficients of \( bh \) exhibit the same behavior. Hence \( T_\lambda h \) can be continued analytically across all of \( \partial D \). By Lemma 2.2, \( T_\lambda h \) is in \( H(\hat{b}) \), so it is 0 by Theorem 5.1. Since \( b \) is outer it follows that \( h = 0 \), as desired.

The noncyclic vectors of the backward shift operator, \( S^* \), have been characterized by R. G. Douglas, H. S. Shapiro, and A. L. Shields [6] as the functions in \( H^2 \) that possess pseudocontinuations to the complement of \( \overline{D} \). Our next theorem implies that, if \( b \) is an extreme point of \( B(H^\infty) \) then, just as is the case with an ordinary continuation, the possession of a pseudocontinuation disqualifies a nonconstant function from being a multiplier of \( H(b) \). (Davis and McCarthy [1] have obtained this independently.) In particular, if \( b \) is an extreme point, then no nonconstant inner function is a multiplier of \( H(b) \), a result from [15].

Theorem 5.3. If \( b \) is an extreme point of \( B(H^\infty) \), then the nonzero functions in \( H(\hat{b}) \) are cyclic vectors of \( S^* \).

This theorem is nearly disjoint from Theorem 5.1: the only functions in \( H^2 \) that possess both ordinary continuations across \( \partial D \) and pseudocontinuations to the complement of \( \overline{D} \) are the rational functions [6].

To prove the theorem, let \( h \) be a nonzero function in \( H(\hat{b}) \). As in the proof of Theorem 5.1, we have \( h = P_q(\phi_0) \) where \( q \) is a function in \( L^2(\phi) \). Also as in the proof of Theorem 5.1, the function \( q_0 \) is not log-integrable.

Let \( M \) be the invariant subspace of \( S^* \) generated by \( h \) and let \( N = M + (H^\infty)^\perp \). Then \( N \) is an invariant subspace of the adjoint of the bilateral shift operator on \( L^2 \). By the known structure of these subspaces [14, p. 111], either \( N = \chi^2_E \) with \( E \) a measurable subset of \( \partial D \) or \( N = \nu H^2 \) with \( \nu \) a unimodular function in \( L^\infty \). The latter possibility is precluded because \( N \) contains the function \( q_0 \), which fails to be log-integrable (and is not the zero function). Thus \( N \) is of the form \( \chi^2_E \), and since it contains the function \( h \), which is
nonzero almost everywhere, it must actually be all of \( L^2 \). That means \( M = H^2 \), so \( h \) is a cyclic vector of \( S^* \), as desired.

**Corollary 5.4.** If \( b \) is an extreme point of \( B(H^\omega) \) and an outer function, then the nonzero functions in \( H(b) \) are cyclic vectors of \( S^* \).

In fact, suppose \( h \) is a nonzero function in \( H(b) \). Then \( T^*_b h \) is in \( H(\bar{b}) \) by Lemma 2.2 and is nonzero because \( b \) is outer. By Theorem 5.3, then, \( T^*_b h \) is a cyclic vector of \( S^* \). But \( T^*_b h \) lies in the \( S^* \)-invariant subspace generated by \( h \), so \( h \) also is a cyclic vector of \( S^* \).

### 6. Decompositions of \( H(b) \)

Let \( u_0 \) be the inner part and \( b_0 \) the outer part of the function \( b \). Then, as de Branges and Rovnyak [4, p. 32] first pointed out, the space \( H(b) \) is the orthogonal direct sum of the two subspaces \( H(u_0) \) and \( u_0 H(b_0) \). Moreover, the inclusion map of \( H(u_0) \) into \( H(b) \) is an isometry, and \( T_{u_0} \) acts as an isometry of \( H(b_0) \) into \( H(b) \). To verify these statements, it suffices to rewrite the equality

\[
1 - T_b T^*_b = 1 - T_{u_0} T^*_{u_0} + T_{u_0} (1 - T_{b_0} T^*_{b_0}) T^*_{u_0}
\]

as

\[
1 - T_b T^*_b = AA^*, \quad \text{where} \quad A = (A_1, A_2), \quad \text{an operator from} \quad H^2 \oplus H^2 \quad \text{to} \quad H^2,
\]

with

\[
A_1 = (1 - T_{u_0} T^*_{u_0})^{1/2} \quad \text{and} \quad A_2 = T_{u_0} (1 - T_{b_0} T^*_{b_0})^{1/2}.
\]

The equality tells us that \( H(b) = M(A_1) + M(A_2) \), and this is an orthogonal direct sum, with the inclusion map of each summand into \( H(b) \) isometric, because \( \ker A = \ker A_1 \oplus \{0\} \). Since \( M(A_1) = H(u_0) \) and \( M(A_2) = u_0 H(b_0) \), the decomposition of \( H(b) \) follows. One immediate consequence of the decomposition is that every multiplier of \( H(b) \) is a multiplier of \( H(b_0) \). More generally, the same reasoning shows that if \( u \) is any inner function, then \( H(ub) \) is the orthogonal direct sum of \( H(u) \) and \( u H(b) \). Thus, every multiplier of \( H(ub) \) is a multiplier of \( H(b) \).

When \( b \) is an extreme point of \( B(H^\omega) \), there is a companion orthogonal decomposition of \( H(b) \).

**Theorem 6.1.** Let \( b \) be an extreme point of \( B(H^\omega) \). Then \( H(b) \) is the orthogonal direct sum of \( H(b_0) \) and \( b_0 H(u_0) \). The inclusion map of \( H(b_0) \) into \( H(b) \) is an isometry, and the operator \( T_{b_0} \) acts as an isometry from \( H(u_0) \) into \( H(b) \).

The situation when \( b \) is not an extreme point is completely different. In
that case, $H(b_0)$ is dense in $H(b)$ [18, p. 87].

Theorem 6.1 can be established by a slight modification of the argument in the discussion preceding it. We suppose that $b$ is not inner, since otherwise the theorem reduces to a triviality. This time we use the factorization

$$I - T_b T_b^* = AA^*,$$

where $A = (A_1 A_2)$, but with

$$A_1 = (1 - T_{b_0} T_{b_0})^{1/2} \quad \text{and} \quad A_2 = T_{b_0} (1 - T_{u_0} T_{u_0})^{1/2}.$$

We assert that $\ker A = \{0\} \oplus \ker A_2$. Clearly, once this has been verified, the previous reasoning applies. To establish the assertion, let $f_1 \oplus f_2$ belong to $\ker A$, and write $g = (1 - T_{b_0} T_{b_0})^{1/2} f_1$ and $h = (1 - T_{u_0} T_{u_0}) f_2$. Then $g$ is in $H(b_0)$, while $h$ is in $H(u_0)$, and $g = -b_0 h$, implying by Lemma 2.3 that $h$ is in $H(b_0)$ (which is the same as $H(\tilde{b})$). Since $b$ is an extreme point, Theorem 5.3 implies that $h = 0$, and hence also that $g = 0$. It follows that $f_1$ is in $\ker A_1$ and $f_1 = 0$, the latter because $\ker (1 - T_{b_0} T_{b_0})$ is trivial, $b_0$ being a nonconstant outer function. This concludes the proof of the theorem.

The next theorem clarifies the relation of $H(b)$ and $H(\tilde{b})$ in the extreme point case.

**Theorem 6.2.** Let $b$ be an extreme point of $B(H^\infty)$. Then the orthogonal complement of $H(\tilde{b})$ in $H(b)$ is $b_0 H(u_0)$. The closure of $H(\tilde{b})$ in $H(b)$ is $H(b_0)$.

Again, the situation is completely different when $b$ is not an extreme point. In that case $H(\tilde{b})$ is always dense in $H(b)$ [18, p. 87].

The second assertion in Theorem 6.2 follows immediately from the first assertion together with Theorem 6.1. It will thus be enough to establish the first assertion. Some new notations are needed.

As in Section 3, we let $Q$ denote the function $1 - |b|^2$ on $\partial \Omega$ and $\langle \cdot, \cdot \rangle_b$, the inner product in $L^2(Q)$. Let $J_Q$ denote the natural injection of $H^2(Q)$ into $L^2(Q)$. One easily verifies that $K_Q$ is the adjoint of $J_Q$ and that $K_Q J_Q \equiv 1 - T_{\tilde{b}} T_{\tilde{b}}^*$. If $h$ is a function in $H(b)$ then $T_{\tilde{b}} h$ belongs to $H(\tilde{b})$, by Lemma 2.2 and so is the image under $K_Q$ of a function in $L^2(Q)$, by Lemma 3.3. The latter function is unique (also by Lemma 3.3, since $H^2(Q) = L^2(Q)$); we denote it by $W_Q h$.

That $b_0 H(u_0)$ is contained in the orthogonal complement of $H(\tilde{b})$ in $H(b)$ is an immediate consequence of Theorem 6.1. To establish the opposite containment, let $h$ be any function in $H(b)$ that is orthogonal to $H(\tilde{b})$. Let $g$ be any function in $H(\tilde{b})$ and, using Lemma 3.3, write $g = K_Q q$ with $q$ in $L^2(Q)$. Corollary 3.5 tells us that $T_{\tilde{b}} g = K_Q (\tilde{b} q)$, showing that $W_Q g = b q$. Thus, by Lemma 2.2,

$$0 = \langle h, g \rangle_b = \langle h, g \rangle + \langle T_{\tilde{b}} h, T_{\tilde{b}} g \rangle.$$
\[
\langle h, q \rangle + \langle W_q h, W_q q \rangle
\]
\[
\langle h, K_q q \rangle + \langle W_q h, \bar{b}q \rangle
\]
\[
\langle J_q h + bW_q h, q \rangle.
\]
This equality holds for all \(q\) in \(L^2(\mathbb{Q})\), so \(J_q h + bW_q h\) is the zero function in \(L^2(\mathbb{Q})\), in other words, \(W_q h = -\bar{h}/b\) on the set where \(1 - |b|^2\) is nonzero. Multiplying the last equality by \(1 - |b|^2\), we conclude that
\[
(1 - |b|^2)W_q h = -\frac{h}{b} + \bar{b}h;
\]
in particular, the function \(h/b\) belongs to \(L^2\). Projecting both sides of the preceding equality onto \(H^2\), we obtain
\[
K_q W_q h = P_+(-\bar{h}/b) + T_{\tilde{g}} h.
\]
But \(T_{\tilde{g}} h = K_q W_q h\) by the definition of \(W_q\), so the function \(h/b\) is orthogonal to \(H^2\). However, the function \(h/b_0\) is in \(H^2\) since it is in \(L^2\) and \(b_0\) is an outer function. Since \(h/b = \bar{u}_0 h/b_0\), we conclude that \(h/b_0\) is in \(H(u_0)\), which means that \(h\) is in \(b_0 H(u_0)\), as desired.

7. Consequences of invertibility

**Theorem 7.1.** If \(b\) is an extreme point of \(B(H^\infty)\), then the following conditions are equivalent

(i) \(b\) is invertible in \(H^\infty\),
(ii) \(H(b) = H(\tilde{b})\),
(iii) \(b\) is a multiplier of \(H(b)\),
(iv) \(S^* H(b)\) is similar to a unitary operator.

An analogous result for the case where \(b\) is not an extreme point of \(B(H^\infty)\) can be found in [18]. In condition (ii), by the equality \(H(b) = H(\tilde{b})\) we mean to say that the two spaces are equal as vector spaces but not that their Hilbert space structures coincide. If they are equal as vector spaces then their norms are equivalent, by the closed graph theorem.

The equivalence of conditions (ii) and (iii) in the theorem is an immediate consequence of the equality \(M(b) \cap H(b) = bH(b)\) from Lemma 2.3. To see that (i) implies (ii), assume \(b\) is invertible and write \(H(b) = T_{\tilde{g}}^{-1} T_{\tilde{g}} H(b)\). By Lemma 2.2, \(T_{\tilde{g}} H(b) \subset H(b)\), and, by Lemma 2.6, \(T_{\tilde{g}}^{-1} H(b) \subset H(b)\), so it follows that \(H(b) = H(\tilde{b})\), as desired. This much does not involve the hypothesis that \(b\) is an extreme point.
We complete the proof by showing that (ii) implies (iv) and (iv) implies (i). Actually, the first of these implications follows immediately from Lemma 3.4, which says that the operator $S^*|H(b)$ is unitarily equivalent to the operator $Z_b^*$; the last operator is unitary when $b$ is an extreme point (since then $H^2(q) = L^2(q)$). It only remains to prove that (iv) implies (i).

Assume that $b$ is not invertible in $H^\infty$. We shall show that then $S^*|H(b)$ is not similar to a unitary operator. The noninvertibility of $b$ implies the noninvertibility of $T_b$. Hence, given $\epsilon > 0$, there is an $f$ in $H^2$ with $\|f\|_2 = 1$ and $\|T_b f\|_2 < \epsilon$. Let $h = (1 - T_b T_b^*)^{1/2} f$. Then $h$ is in $H(b)$ with $\|h\|_b \leq 1$, and

$$\|h\|_b^2 \geq \|h\|_b^2 = \langle (1 - T_b T_b^*) f, f \rangle = \|f\|_2^2 - \|T_b f\|_2^2 \geq 1 - \epsilon^2.$$ 

One consequence of the assumption that $b$ is an extreme point is (in the terminology of de Branges and Rovnyak) the identity for difference quotients:

$$\|S^* g\|_b^2 = \|g\|_b^2 - |g(0)|^2 \quad (g \in H(b))$$

[19, p. 162]. From this it follows that

$$\lim_{n \to \infty} \|S^{*n} h\|_b^2 = \|h\|_b^2 - \|h\|_b^2 \leq 1 - (1 - \epsilon^2) = \epsilon^2.$$ 

As $\epsilon$ is arbitrary, the desired conclusion, that $S^*|H(b)$ is not similar to a unitary operator, follows, and the proof of Theorem 7.1 is complete.

Later, in Section 11, we shall see that the condition that $b$ be a multiplier of $H(b)$ is equivalent to the conditions of Theorem 7.1.

**Corollary 7.2.** If $b$ is an extreme point of $B(H^\infty)$ and is invertible in $H^\infty$ then $b^{-1}$ is a multiplier of $H(b)$.

In fact, if $b$ is invertible, then Lemma 2.3 and Theorem 7.1 combine to give $b H(b) = H(b)$. (The same result holds, and the same reasoning applies, when $b$ is not an extreme point. The corollary uses only the implication (i) implies (ii) from Theorem 7.1, which, as noted in the proof, holds for non-extreme points as well.)

**Corollary 7.3.** If $b$ is not an inner function then $H(b)$ has nonconstant multipliers.

We need only to treat the case where $b$ is an extreme point. Under the assumption that $b$ is not an inner function, there is a factorization $b = b_1 b_2$,
where $b_1$ and $b_2$ are in $B(H^e)$ and $b_1$ is nonconstant and invertible in $H^e$. (For example, one can take $b_1$ to be the outer function whose modulus on $\partial D$ is the maximum of $|b|$ and $1/2$). Using the reasoning at the beginning of Section 6 we obtain the decompositions

$$H(b) = H(b_1) + b_1 H(b_2) = H(b_2) + b_2 H(b_1).$$

Thus $b_1 H(b_2) \subset H(b)$ and $b_2 H(b_1) \subset H(b)$, and

$$b_1 H(b) = b_1 H(b_2) + b_2 b_1 H(b_1).$$

By Theorem 7.1 $b_1 H(b_1) \subset H(b_1)$, and hence $b_1$ is a multiplier of $H(b)$.

As in the last section, we let $u_0$ denote the inner part and $b_0$ the outer part of $b$.

Theorem 7.4. If $b$ is an extreme point of $B(H^e)$ and $b_0$ is invertible in $H^e$, then $b_0$ and $1/b_0$ are multipliers of $H(b)$ and one has the decompositions

$$H(b) = H(u_0) + H(b_0) = b_0 H(u_0) + u_0 H(b_0).$$

The proof depends on the decompositions

$$H(b) = H(u_0) + u_0 H(b_0) = H(b_0) + b_0 H(u_0)$$

from Section 6. If $b_0$ is invertible then, as seen above, we have $b_0 H(b_0) = H(b_0)$, so that

$$b_0 H(b) = b_0 H(u_0) + u_0 b_0 H(b_0)$$

$$= b_0 H(u_0) + u_0 H(b_0)$$

$$\subset H(b),$$

and

$$b_0^{-1} H(b) = b_0^{-1} H(b_0) + H(u_0)$$

$$= H(b_0) + H(u_0)$$

$$\subset H(b),$$

Thus $b_0$ and $1/b_0$ are multipliers of $H(b)$, so the preceding inclusions must actually be equalities, and the desired decompositions of $H(b)$ follow.

8. Multiplication operators

For $m$ a multiplier of $H(b)$, we let $M_m$ denote the corresponding multiplication operator on $H(b)$. For $w$ in $D$, the kernel function $k_w^b$ is an eigenvector of $M_m^*$. 


holds.

In fact, Lemma 8.1 implies that the orthogonal complement of \( S \phi \) in \( H(0) \) is invariant under \( M^m \), so that \( S \phi \) is an eigenvector of \( M^m \). To obtain the commutation relation, consider a point \( w \) in \( D \) and the corresponding eigenvalue \( \lambda \). From the expression \( \lambda = (1 - \|w\|/h)w \), one easily obtains the commutation relation \( M^m X - X M^m = S \phi (m - \alpha) \).

\[ \phi(w) = (\overline{\phi}(w) = (\overline{(m - \alpha)}(w) S \phi \bar{w}) = (m(\overline{w}) - \alpha) \phi(w). \]

where \( \phi \) is some function in \( H(0) \) that follows that \( M^m X - X M^m = S \phi (m - \alpha) \).

**Theorem 8.2.** Let \( b \) be an extreme point of \( BL^p \), and \( m \) a multiplier of \( H(0) \). Then \( S b \) is an eigenvector of \( M^m \) if \( b \) is in \( H(0) \), and \( m \) a multiplier of \( H(0) \). Then \( S b \) is an eigenvector of \( M^m \) if \( b \) is in \( H(0) \) and only if \( \phi(b) S \phi b = 0 \).

The adjoint \( X^* \) is given by \( X^* b = b - (\phi \overline{b}) \).

**Lemma 8.1.** If \( b \) is an extreme point of \( BL^p \) and \( m \) is in \( H(0) \), then \( S b \) is an eigenvector of \( M^m \). Conversely, if \( M^m \) is an operator on \( H(0) \) such that each kernel function \( k^*_w \) is an eigenvector of \( M^m \), then \( M^m \) is a multiplicity operator. This will be convenient to denote the operator \( S(H(0)) \) by \( X \). If it is a contraction by Lemma 2.6, the adjoint \( X^* \) is given by \( X^* b = b - (\phi \overline{b}) \).

When \( b \) is an extreme point and \( m \) is a multiplier of \( H(0) \), one can draw the following conclusion.
follows from Lemma 2.3 than \( m - \alpha \) is in \( H(\tilde{b}) \), and the proof of the theorem is complete.

For \( w \) in \( D \) we define the operator \( Q_w \) on \( H^2 \) by

\[
(Q_wf)(z) = \frac{f(z) - f(w)}{z - w}.
\]

A simple argument produces the alternative expression \( Q_w = (1 - wS^*)^{-1}S^* \).
In particular, \( Q_wb = (1 - wX)^{-1}S^*b \), showing that \( Q_wb \) is in \( H(b) \).

**Corollary 8.3.** If \( b \) is an extreme point of \( B(H^\omega) \) and \( m \) is a multiplier of \( H(b) \), then each function \( Q_wb \) is an eigenvector of \( M^*_m \).

The case \( w = 0 \) is given by Theorem 8.2 so, for the proof, assume \( w \neq 0 \).
The communication relation gives

\[
(1 - wX)M^*_m - M^*_m(1 - wX) = wS^*b \otimes (m - \alpha)b.
\]

Applying both sides to \( Q_wb \), we obtain

\[
(1 - wX)M^*_mQ_wb - \bar{\alpha}S^*b = \omega(Q_wb, (m - \alpha)b)b S^*b.
\]

It follows that

\[
M^*_mQ_wb = [\bar{\alpha} + w(Q_wb, (m - \alpha)b)b]Q_wb,
\]
the desired conclusion.

The properties of \( M^*_m \) given by Theorem 8.2 characterize multiplication operators in the extreme point case.

**Theorem 8.4.** If \( b \) is an extreme point of \( B(H^\omega) \) and if \( M \) is an operator on \( H(b) \) such that \( M^*S^*b = \bar{\alpha}S^*b \) and

\[
M^*X - XM^* = S^*b \otimes \phi,
\]

then \( M = M^*_m \) for a multiplier \( m \) of \( H(b) \).

To prove this it will suffice to show that the hypotheses imply \( k^b_w \) is an eigenvector of \( M^* \) whenever \( b(w) \neq 0 \). Assuming the last condition and applying the commutation relation to \( k^b_w \), we obtain

\[
\bar{\phi}(w)S^*b = M^*(\bar{w}k^b_w - \bar{b}(w)S^*b) - XM^*k^b_w
\]

\[
= (\bar{w} - X)M^*k^b_w - \bar{\alpha}b(w)S^*b,
\]
so that
\[(\overline{w} - X)M^{*}k_{\overline{w}}^{b} = (\overline{\phi(w)} + \overline{\alpha b(w)})S^{*}b.\]

But also \((\overline{w} - X)k_{\overline{w}}^{b} = \overline{b(w)}S^{*}b\), and the operator \(\overline{w} - X\) is injective because of the assumption that \(b(w) \neq 0\) (which implies that \(k_{\overline{w}}\) is not in \(H(b)\) [19]). We can conclude that

\[
M^{*}k_{\overline{w}}^{b} = \frac{\overline{\phi(w)} + \overline{\alpha b(w)}}{\overline{b(w)}} k_{\overline{w}}^{b},
\]

and the proof is complete.

9. Conjugations

We assume throughout this section that \(b\) is an extreme point of \(B(H^{\infty})\). As we mentioned earlier, in Section 3, one consequence of this assumption is the equality \(H^{2}(\mu_{b}) = L^{2}(\mu_{b})\), which enables us to define a conjugation on \(H(b)\) by transferring via the map \(V_{b}\) a conjugation on \(L^{2}(\mu_{b})\). This conjugation and another on a space related to \(H(b)\) that we shall introduce a little later are intimately connected with the structure of the multipliers of \(H(b)\).

The conjugation on \(H(b)\) that turns out to be useful is the one that corresponds to the conjugation \(q \rightarrow e^{-\overline{\theta}q}\) on \(L^{2}(\mu_{b})\). We denote it by \(C\):

\[Ch = V_{b}(Z_{b}^{\overline{\theta}r_{b}^{-1}}h) \quad (h \in H(b)).\]

That \(C\) is a conjugation (an anti-unitary involution) is obvious.

**Lemma 9.1.** For \(w\) in \(D\), \(Ck_{w}^{b} = Q_{w}b\).

This is a straightforward calculation. By the way \(\mu_{b}\) is defined, the function \((1 + b)/(1 - b)\) differs by an imaginary constant from the Herglotz integral of \(\mu_{b}\). Using the equality \(k_{w}^{b} = (1 - b\overline{w})V_{b}k_{w}\) from Lemma 3.1, we obtain

\[
(Ck_{w}^{b})(z) = (1 - b\overline{w})(1 - b(z))
\]

\[
= \frac{(1 - b\overline{w})(1 - b(z))}{z - w} \int_{\partial D} \left[ \frac{1}{1 - e^{-\theta}z - e^{\theta}w} - \frac{1}{1 - e^{-\theta}w} \right] d\mu_{b}(e^{\theta})
\]

\[
= \frac{(1 - b\overline{w})(1 - b(z))}{2(z - w)} \int_{\partial D} \left[ \frac{e^{\theta} + z}{e^{\theta} - z} - \frac{e^{\theta} + w}{e^{\theta} - w} \right] d\mu_{b}(e^{\theta})
\]

\[
= \frac{(1 - b\overline{w})(1 - b(z))}{2(z - w)} \left[ \frac{1 + b(z)}{1 - b(z)} - \frac{1 + b(w)}{1 - b(w)} \right] d\mu_{b}(e^{\theta}).
\]
\[ \frac{b(z) - b(w)}{z - w}, \]
as desired.

The conjugation \( C \) intertwines the operator \( X = S^*H(b) \) and its adjoint.

**Lemma 9.2.** \( CXC = X^* \)

It will suffice to show that \( CXk_w^b = X^*Ck_w^b \) for all \( w \). We transform both sides with the aid of Lemma 9.1. First,

\[ CXk_w^b = C(wk_w^b - \overline{b(w)}S^*b) \]
\[ = wQ_w b - b(w)k_0^b. \]

Next, by the formula for \( X^* \) mentioned in Section 8,

\[ X^*Ck_w^b = X^*Q_w b \]
\[ = SQ_w b - \langle Q_w b, S^*b \rangle b \]
\[ = SQ_w b - \langle k_0^b, k_0^b \rangle b \]
\[ = SQ_w b - (1 - \overline{b(0)b(w)})b. \]

Now

\[ SQ_w = (S - w)S^*(1 - wS^*)^{-1} + wQ_w \]
\[ = -(1 - SS^*)(1 - wS^*)^{-1} + 1 + wQ_w. \]

The operator \((1 - SS^*)(1 - wS^*)^{-1}\) is easily seen to equal \( 1 \otimes k_w \); in fact, its adjoint applied to the \( H^2 \) function \( f \) gives

\[ (1 - \overline{wS})^{-1}(1 - SS^*)f = f(0)(1 - \overline{wS})^{-1} = f(0)k_w. \]

Thus \( SQ_w b = -b(w) + b + wQ_w b \). Inserting this into the expression above for \( X^*Ck_w^b \), we get

\[ X^*Ck_w^b = -b(w) + b + wQ_w b - (1 - \overline{b(0)b(w)})b \]
\[ = wQ_w b - b(w)(1 - \overline{b(0)b}), \]
as desired.

We now introduce our second conjugation. It will act on the space \( K^2_+(q) \), by which we mean the space of functions that are sums of functions in \( K^2(q) \) and constant functions. The functions in \( K^2(q) \), and here those in \( K^2_+(q) \), are defined in the complement of \( \partial D \), and we define them at \( \infty \) in the obvious
way (namely, those in $K^2(q)$ are assigned the value 0 at infinity). For $f$ in $K^2_+(q)$ we define its conjugate, $f_*$, by

$$f_*(z) = f(1/z).$$

Straightforward calculations show that, if $f = K_q g + c$ with $q$ in $L^2(q)$ and $c$ a constant, then $f_* = -S K_q (Z_q^* q) + \bar{c}$, and also $f_* = -K_q \bar{g} + (K_q \bar{q})(0) + \bar{c}$. The latter expression shows that $f_*$ is in $K^2_-(q)$.

We let $K^\infty(q)$ denote the space of bounded functions in $K^2_+(q)$. (Here, by bounded we mean bounded in the entire complement of $\partial D$, not merely in $D$.) It is obvious that the conjugation on $K^2_+(q)$ maps $K^\infty(q)$ into itself.

The next lemma gives a relation between our two conjugations.

**Lemma 9.3.** If $f$ is in $H(b)$, then $C[(1 - b)f] = (b - 1)Sf_*$. 

To prove this, let $\bar{q}$ be the function in $L^2(q)$ such that $f = K_q \bar{g}$. Because $(1 - |b|^2)/(1 - b)$ is the Radon-Nikodym derivative with respect to normalized Lebesgue measure of the absolutely continuous component of $\mu_\rho$, we can also write $f = K_\rho(1 - |b|^2 \bar{q})$, provided we regard $|1 - b|^2 q$ as vanishing on the singular component of $\mu_\rho$, if there is one. Thus

$$(1 - b)f = V_{\rho}(|1 - b|^2 \bar{q}),$$

and we obtain

$$C[(1 - b)f] = V_{\rho}Z_\rho^*(|1 - b|^2 \bar{q})$$

$$= (1 - b)K_\rho Z_\rho^*(|1 - b|^2 \bar{q})$$

$$= (1 - b)K_q (Z_q^* \bar{q}).$$

As mentioned above, $f_* = -S K_q (Z_q^* \bar{q})$, so that $K_q (Z_q^* \bar{q}) = -Sf_*$, and the desired equality follows.

We are now able to determine the effect of conjugation on multiplication operators.

**Theorem 9.4.** If $m$ is a multiplier of $H(b)$, then $CM_m C$ is a multiplication operator, namely, it equals $M_m$.

**Corollary 9.5.** The multipliers of $H(b)$ are in $K^\infty(q)$.

The corollary follows immediately from the theorem. To prove the theorem we note first that, because $C$ is a conjugation, $(CM_m C)^* = CM_m C$. This in conjunction with Lemma 9.1 and Corollary 8.3 implies that if $m$ is a multiplier of $H(b)$ then each of the functions $k_m^h$ is an eigenvector of $(CM_m C)^*$ and
hence that $CM_\alpha C$ is a multiplication operator. It remains to determine the corresponding multiplier. Let it be denoted by $m'$.

From the proof of Corollary 8.3 we have

$$M_\alpha^*Q_{\alpha,b} = [\alpha + w\langle Q_{\alpha,b}, (m - \alpha)b\rangle_b]Q_{\alpha,b},$$

where $\alpha$ is the eigenvalue of $S^*b$ as an eigenvector of $M_\alpha^*$. Consequently

$$CM_\alpha^*Ck_{\alpha,b} = [\alpha + \tilde{w}\langle (m - \alpha)b, Q_{\alpha,b}\rangle_b]k_{\alpha,b},$$

from which we conclude that

$$m'(z) = \alpha + z\langle Q_{\alpha,b}, (m - \alpha)b\rangle_b$$

$$= \alpha + zC[(m - \alpha)b], k_{\alpha,b}$$

$$= \alpha + zC[(m - \alpha)b](z).$$

Hence $m'(0) = \alpha$, and

$$S^*m^1 = C[(m - \alpha)b].$$

By Lemma 9.3,

$$C[(m - \alpha)(b - 1)] = (1 - b)S^*m_\alpha.$$  

(The lemma applies because $m - \alpha$ belongs to $H(\tilde{b})$, by Theorem 8.2) In view of the last two equalities, we seek an expression for $C(m - \alpha)$ in terms of $m^1$.

Let $\tilde{\beta}$ be the eigenvalue of $S^*b$ as an eigenvector of $M_\alpha^*$. Because $m$ and $m^1$ play symmetric roles, we have $\beta = \overline{m(0)}$ and $C[(m^1 - \beta)b] = S^*m$. We can rewrite the last equality as

$$X(m - \alpha) = C[(m^1 - \beta)b].$$

Since $CX = X^*C$, it follows that

$$X^*C(m - \alpha) = (m^1 - \beta)b.$$

Using the formula for $X^*$ mentioned in Section 8, we can rewrite the left side here as

$$SC(m - \alpha) - \langle C(m - \alpha), S^*b\rangle_b b = SC(m - \alpha) - \langle k_{\alpha,b}^*, m - \alpha\rangle_bb$$

$$= SC(m - \alpha) - \langle \overline{m(0)} - \alpha\rangle_bb$$

$$= SC(m - \alpha) - (\alpha - \beta)b.$$  

Applying $S^*$, we find that

$$C(m - \alpha) + (\alpha - \beta)S^*b = S^*[(m^1 - \beta)b].$$
\[ \begin{align*}
C(m - \alpha) &= bS \ast m^1 \quad \text{or} \quad (m^1(0) - \beta)S \ast b \\
&= bS \ast m^1 + (\alpha - \beta)S \ast b.
\end{align*} \]

Hence \( C(m - \alpha) = bS \ast m^1 \).

Combining the last equality with the previously obtained expressions for \( C[(m - \alpha)b] \) and \( C[(m - \alpha)(b - 1)] \), we find that \( S \ast m^1 = S \ast m_\ast \), so \( m^1 \) and \( m_\ast \) differ by at most a constant. But from the way \( m_\ast \) is defined one easily sees that \( m_\ast - \beta = m_\ast - \bar{m}(0) \) belongs to \( H(\bar{b}) \). Since \( m^1 - \beta \) also belongs to \( H(\bar{b}) \), but the constant functions do not, we must have \( m^1 = m_\ast \), and the proof of the theorem is complete.

10. Lemmas on Cauchy integrals

We need two simple facts about Cauchy integrals. For \( \mu \) a finite complex Borel measure on \( \partial \mathbb{D} \), we let \( P * \mu \) denote the Poisson integral of \( \mu \) and \( Q * \mu \) the conjugate Poisson integral of \( \mu \).

**Lemma 10.1.** If \( \mu \) is a finite complex Borel measure on \( \partial \mathbb{D} \) and \( f = K_\mu \), then, in \( \mathbb{D} \),

\[ f(z) - f(1/\bar{z}) = (P * \mu)(z) \]

\[ f(z) = \frac{1}{2} \left[ (P * \mu)(z) + i(Q * \mu)(z) + (P * \mu)(0) \right] \]

**Lemma 10.2.** If \( \mu \) is a finite complex Borel measure on \( \partial \mathbb{D} \), and if \( f \) and \( g \) are holomorphic functions in \( \mathbb{D} \) such that \( f = \bar{g} = P * \mu \), then, in \( \mathbb{D} \),

\[ f(z) = (K_\mu)(z) + \bar{g}(0) \]

\[ g(\bar{z}) = (K_\mu)(1/\bar{z}) + \bar{g}(0). \]

Lemma 10.1 is a straightforward consequence of the relation between the Cauchy kernel and the Poisson and conjugate Poisson kernels, and Lemma 10.2 follows easily from Lemma 10.1.

To illustrate the use of these lemmas we show here that, when \( b \) is an extreme point of \( B(\mathbb{H}_\ast) \), the space \( K^\infty(\mathbb{Q}) \) is closed under multiplication. Let \( f \) and \( g \) be functions in \( K^\infty(\mathbb{Q}) \). The function \( fg = \tilde{\mathbb{F}} \tilde{g}_\ast \) is then bounded and harmonic in \( \mathbb{D} \), so it is the Poisson integral of its boundary function. (The function is defined in \( \mathbb{C} \setminus \partial \mathbb{D} \), but by its boundary function we mean the interior boundary function, that is, the boundary function from \( \mathbb{D} \).) By Lemma 10.2, to prove \( fg \) is in \( K^\infty(\mathbb{Q}) \) it will suffice to prove that the interior boundary
function of $fg - f*\bar{g}*$ has the form $q_Q$ with $q$ in $L^2(Q)$. Let $q_1$ and $q_2$ be the functions in $L^2(Q)$ such that $f = f(\infty) + K_Q(q_1)$ and $g = g(\infty) + K_Q(q_2)$. Then, by Lemma 10.1, the interior boundary function of $f - f*$ is $q_1q$ and the interior boundary function of $g - \bar{g}*$ is $q_2q$. Writing

$$fg - f*\bar{g}*$ = $(f - f*)g + f*(g - \bar{g}*)$$

we see that the interior boundary function of $fg - f*\bar{g}*$ is $(q_1g + q_2f*)q$. (In the last expression, of course, $g$ and $f_*$ denote interior boundary functions.) Since $g$ and $f_*$ are bounded, the function $q_1g + q_2f_*$ is in $L^2(Q)$, the desired conclusion.

Thus, $K^\infty(Q)$ is an algebra, and by reasoning like that above one easily sees that the spectrum of a function $f$ in this algebra equals the closure of $f(C \partial D)$. In fact, that the spectrum of $f$ contains the closure of $f(C \partial D)$ is obvious, so one only needs to show that $f$ is invertible in $K^\infty(Q)$ if it is bounded away from 0 in $C \partial D$. If the latter happens, and if $q_1$ is the function in $L^2(Q)$ such that $f = f(\infty) + K_Qq_1$, then by Lemma 10.1 the interior boundary function of $f^{-1} - f_*^{-1}$ is $-q_1q/ff_*$, which is of the form $q_Q$ with $q$ in $L^2(Q)$. Lemma 10.2 thus guarantees that $f^{-1}$ is in $K^\infty(Q)$.

11. More on $H(\tilde{b})$

We return in this section to the assumption that $b$ is an extreme point of $B(H^\infty)$. The functions in $H(\tilde{b})$ are restrictions to $D$ of functions in $K^\infty(Q)$, so they have natural extensions to the exterior of $\partial D$. The next lemma states the process of extension preserves multiplication, to the extent that it can. (This fails when $b$ is not an extreme point, except in the trivial case where $q$ is constant.) For $f$ in $H(\tilde{b})$, we let $f_*$ denote the restriction to $D$ of the conjugate of the extension of $f$. (It differs by a constant from a function in $H(\tilde{b})$.)

Lemma 11.1. If the function $f$, $g$, and $fg$ belong to $H(\tilde{b})$, then $(fg)_* = f_\ast g_\ast$.

For the case where $f$ and $g$ are in $K^\infty(Q)$ this is established at the end of the preceding section. The argument for the general case is similar but slightly more elaborate.

Let $q_1$, $q_2$, and $q$ be the functions in $L^2(Q)$ such that $f = K_\tilde{Q}q_1$, $g = K_\tilde{Q}q_2$, and $fg = K_\tilde{Q}q$. By Lemma 10.1, the boundary functions of $f - f*$ and $g - g*$ are $q_1q$ and $q_2q$ respectively. The function $fg - f*\bar{g}*$ is the sum of an $H^2$ function and the conjugate of an $H^1$ function, so it is the Poisson integral of its boundary function. Writing

$$fg - f*\bar{g}*$ = $(f - f*)g + f*(g - \bar{g}*)$$

206 B. A. LOTTO AND D. SARASON
we see that its boundary function is \((gq_1 + \tilde{f}_*q_2)\tilde{q}\). (In the usual way, we are identifying functions in \(D\) with their boundary functions.) Therefore, by Lemma 10.2,

\[fg = K[(gq_1 + \tilde{f}_*q_2)\tilde{q}].\]

Hence

\[K[(gq_1 + \tilde{f}_*q_2 - q)\tilde{q}] = 0.\]

As the functions \(q_1\tilde{q}^{1/2}, q_2\tilde{q}^{1/2},\) and \(q\tilde{q}^{1/2}\) are in \(L^2\), the function \((gq_1 + \tilde{f}_*q_2 - q)\tilde{q}\) is in \(L^1\), which implies that the function \((gq_1 + \tilde{f}_*q_2 - q)\tilde{q}\) fails to be log-integrable (the reasoning can be found in Section 5). Since the Cauchy integral of the latter function vanishes so do its forward Fourier coefficients, and hence it is the zero function. Thus \(fg - \tilde{f}_*\tilde{g}_*\) is actually the Poisson integral of \(q\tilde{q}\), and we can conclude by Lemma 10.2 that \((fg)_* = f_*g_*\), as desired.

Lemma 11.1 enables us to obtain the analogue of Theorem 9.4 and its corollary for multipliers of \(H(\tilde{b})\).

**Theorem 11.2.** If \(m\) is a multiplier of \(H(\tilde{b})\) then \(m\) is in \(K^\infty(\tilde{q})\) and \(m_*\) is a multiplier of \(H(\tilde{b})\).

To prove this we use the conjugation on \(H(\tilde{b})\) that corresponds, under the transformation \(K\), to the conjugation \(q \rightarrow -Z^*_q \tilde{q}\) on \(L^2(\tilde{q})\). We shall not introduce a special notation for it because we shall not have occasion to use it again. A straightforward calculation shows that it is given by \(f \rightarrow S^*f_*\).

The important property for us is that the preceding map sends \(H(b)\) onto itself, which also follows from the unitarity of \(S^*H(\tilde{b})\) (used before in the proof of Theorem 3.6).

Let \(m\) be a multiplier of \(H(\tilde{b})\), and let \(f\) be any function in \(H(\tilde{b})\). By Theorem 3.6, \(m\) is in \(K^\infty(\tilde{q})D\), so Lemma 11.1 can be applied to give \((mf)_* = m_*f_*\). Also \(f_*(0) = 0\), so \(S^*(mf)_* = m_*S^*f_*\). In view of the remark at the end of the last paragraph we can conclude that \(m_*\) is a multiplier of \(H(\tilde{b})\).

In particular, \(m_*\) is bounded in \(D\), and thus \(m\) is in \(K^\infty(\tilde{q})\).

The next result enables us to supplement Theorem 7.1.

**Theorem 11.3.** The function \(b\) belongs to \(K^\infty(\tilde{q})D\) if and only if \(1/b\) is in \(H^2\). In that case \(b_* = 1/b\).

For the proof, suppose first that \(1/b\) is in \(H^2\). Then \(1/b\) is also in \(L^2(\tilde{q})\), and we have
\[ K_\varphi(1/b) = P_+ \left( \frac{1 - |b|^2}{b} \right) = 1/b(0) - b, \]

showing that \( 1/b(0) - b \) is in \( K^2(\varphi)|D \) and hence that \( b \) is in \( K^2_\varphi(p)|D \). Moreover, because of the way the transformation \( K_\varphi \) interacts with the conjugation on \( K^2_\varphi(q) \) (as was pointed out in Section 9), we have

\[ (1/b(0) - b)_* = -SS*K_\varphi\left(Z\varphi\left(1/b\right)\right) \]
\[ = -SS*P_+ \left( \frac{1 - |b|^2}{b} \right) \]
\[ = -SS* \left( \frac{1}{b} - b(0) \right) \]
\[ = \frac{1}{b(0)} - \frac{1}{b}, \]

which gives \( b_* = 1/b \).

Suppose, conversely, that \( b \) is in \( K^2_\varphi(q)|D \), in other words, that \( b - c \) is in \( H(b) \), where \( c \) is a constant. Then \( c \neq 0 \), since \( b \) is not in \( H(b) \). Also, because \( 1 - b(0)b = k^2 \) is in \( H(b) \), we must have \( b(0) \neq 0 \) and \( c = 1/b(0) \). Let \( q \) be the function in \( L^2(\varphi) \) that maps to \( b - 1/b(0) \) under \( K_\varphi \). Then

\[ K_\varphi(bq) = T_\varphi K_\varphi q = T_\varphi(b - 1/b(0)) \]
\[ = T_\varphi b - 1 = -P_+(1 - |b|^2) \]
\[ = K_\varphi(-1). \]

Since \( K_\varphi \) has a trivial kernel, it follows that \( q = -1/b \) (modulo the measure \( \mu \)). Therefore (1 - \( |b|^2 \))/\( |b|^2 \) is in \( L^1 \), implying that \( 1/b \) is in \( L^2 \). In addition,

\[ 1/b(0) - b = K_\varphi(1/b) = P_+ \left( \frac{1 - |b|^2}{b} \right) = P_+(1/b) - b, \]

so \( P_+(1/b) = 1/b(0) \). Therefore \( 1/b \) is in \( H^2 \), in other words, \( 1/b \) is in \( H^2 \), and the proof is complete.

**Corollary 11.4.** If \( b \) is a multiplier of \( H(b) \) then \( b \) is invertible in \( H^\infty \).

The corollary is an immediate consequence of Theorems 11.2. and 11.3.
12. Construction of multipliers

We retain the assumption that $b$ is an extreme point of $B(H^\infty)$. Our next main result, Theorem 12.2, is a criterion for a function in $K^\infty(\Omega)$ to be a multiplier of $H(b)$. The criterion enables us to show that $H(b)$ has an abundance of multipliers; in particular, the multipliers of $H(b)$ that lie in $H(\tilde{b})$ are dense in $H(\tilde{b})$.

**Lemma 12.1.** Let $m$ be a function in $K^\infty(\Omega)$ and let $q$ be the function in $L^2(\Omega)$ such that $m = m(\infty) + K_q(q)$. Let $g$ be a function in $H^2$ such that $g(0) = 0$. Then $T_gm = K(\tilde{g}q)e|D$. The function $T_gm$ is in $H(b)$ if and only if $gq$ is in $L^2(\Omega)$.

In fact, by Lemma 10.1, the interior boundary function of $m - \overline{m}_e$ is $q_e$, so

$$T_gm = P_e(\overline{g}m_e) + P_e(\overline{g}q_e).$$

The first term on the right is 0 because $g(0) = 0$, and the second term is $K(\tilde{g}q_e)|D$. This proves the first assertion in the lemma. It is obvious that $T_gm$ is in $H(\tilde{b})$ if $gq$ is in $L^2(\Omega)$, which is one direction in the second assertion. For the other direction, suppose $T_gm$ is in $H(\tilde{b})$, say $T_gm = K(\tilde{q}_1(q_1))e|D$ with $q_1$ in $L^2(\Omega)$. Then $K(\tilde{g}q - q_1)e|D = 0$. But $(\tilde{g}q - q_1)e$ is not log-integrable since it is the product of the $L^1$ function $gq^{1/2} - q_1^{1/2}$ and the function $q^{1/2}$, which is not log-integrable. It follows that $q_1 = \tilde{g}q$, and the proof is complete.

**Theorem 12.2.** Let $m$ be a function in $K^\infty(\Omega)$ and let $q$ be the function in $L^2(\Omega)$ such that $m = m(\infty) + K_q(q)$.

(i) The function $m$ is a multiplier of $H(\tilde{b})$ if and only if $fq$ is in $L^2(\Omega)$ for every $f$ in $H(\tilde{b})$.

(ii) The function $m$ is a multiplier of $H(b)$ if and only if $hq$ is in $L^2(\Omega)$ for every $h$ in $H(b)$.

To prove (i), let $f$ be any function in $H(\tilde{b})$, and let $q_1$ be the function in $L^2(\Omega)$ such that $f = K(q_1)|D$. By Lemma 11.1, if $mf$ is in $H(\tilde{b})$ then $(mf)_e = m_qf_e$. This in conjunction with Lemmas 10.1 and 10.2 implies that $mf$ is in $H(\tilde{b})$ if and only if the boundary function of $mf - \overline{m}_e f_e$ has the form $q_2 e$ with $q_2$ in $L^2(\Omega)$. On $\partial D$ we have

$$mf - \overline{m}_e f_e = (m - \overline{m}_e)f + \overline{m}_e(f - f_e)$$

$$= fq e + \overline{m}_e q_1 e.$$
Since $m_*$ is bounded the function $\overline{m}_* q_1$ is in $L^2(\nu)$. Hence $mf - \overline{m}_* q_1$ has the required form if and only if $fq$ is in $L^2(\nu)$, which proves (i).

Because of (i), in proving (ii) we can assume, without loss of generality, that $m$ is a multiplier of $H(\overline{\nu})$. Let $h$ be any function in $H(b)$. By Lemma 2.2, $mh$ is in $H(b)$ if and only if $T_\nu(mh)$ is in $H(\overline{\nu})$. We have

$$T_\nu(mh) = mT_\nu h + P_\nu((\overline{bh} - P_{\nu} \overline{bh})m)$$

$$= mT_\nu h + T_\nu m,$$

where $g = (1 - P_\nu)((\overline{bh})$. The first term on the right is in $H(\overline{\nu})$ since we have assumed that $m$ is a multiplier of $H(\overline{\nu})$. Hence $mh$ is in $H(b)$ if and only if the second term on the right, $T_\nu m$, is in $H(\overline{\nu})$. By Lemma 12.1, that happens if and only if $gq$ is in $L^2(\nu)$. The function $qT_\nu((\overline{bh})$ is in $L^2(\nu)$ by (i) (since $T_\nu h$ is in $H(\overline{\nu})$ and $m$ is a multiplier of $H(\overline{\nu})$). Hence $mh$ is in $H(b)$ if and only if $bhq$ is in $L^2(\nu)$, in other words, if and only if $|h|^2 |q|^2 \nu$ is in $L^1$. But $(1 - |h|^2)|h|^2 |q|^2 \nu$ ($= |h|^2 |q|^2 \nu^2$) is in $L^1$ since $q_0$ ($= m - \overline{m}_*)$ is bounded. Hence $mh$ is in $H(b)$ if and only if $|h|^2 |q|^2 \nu$ is in $L^1$, in other words, if and only if $hq$ is in $L^2(\nu)$. This proves (ii).

**Corollary 12.3.** If $m$ is a multiplier of $H(b)$ then the spectrum of $M_m$ is the closure of $m(C/\partial D)$.

As shown in Section 10, the closure of $m(C/\partial D)$ equals the spectrum of $m$ in the algebra $K^m(\nu)$, and this set is obviously contained in the spectrum of $M_m$. To establish the opposite containment it will suffice to show that the invertibility of $m$ in $K^m(\nu)$ implies that $1/m$ is a multiplier of $H(b)$. Assume $m$ is invertible in $K^m(\nu)$, and let $q$ and $q_1$ be the function in $L^2(\nu)$ such that $m = m(\infty) + K_\nu q$ and $1/m = 1/m(\infty) + K_\nu q_1$. Since

$$\frac{1}{m} - \frac{1}{\overline{m}_*} = \frac{\overline{m}_* - m}{m\overline{m}_*}$$

we conclude by Lemmas 10.1 and 11.1 that $q_1 = -q/m\overline{m}_*$. If $h$ is in $H(b)$ then Theorem 12.2(ii) tells us that $hq$ is in $L^2(\nu)$, and therefore so is $hq_1$, since $1/m$ and $1/m_*$ are bounded. Theorem 12.2.(ii) now implies that $1/m$ is a multiplier of $H(b)$, as desired.

**Corollary 12.4.** Let $m$ be a function in $K^m(\nu)$ and let $q$ be the function in $L^2(\nu)$ such that $m = m(\infty) + K_\nu q$. If $qq^{1/2}$ is bounded, then $m$ is a multiplier of $H(b)$.

This corollary is an immediate consequence of Theorem 12.2.

**Corollary 12.5.** If $m$ is an invertible function in $H^m$ such that $(1 - |m|^2)/\nu$ is bounded on $\partial D$, then $m$ is a multiplier of $H(b)$, and $m_* = 1/m$. 
We remark that if $b$ is not an inner function, in other words, if $\varrho$ does not vanish identically, then nonconstant functions satisfying the hypotheses of Corollary 12.5 can be constructed by standard means. (One example is the outer function with modulus $\max\{b, 1/2\}$ on $\partial D$.)

To establish Corollary 12.5 it suffices to note that the bounded harmonic function $m - 1/\overline{m}$ is the Poisson integral of its boundary function, which equals $(|m|^2 - 1)/\overline{m}$. From Lemma 10.2, it follows that $m$ is in $K^w(\varrho)$ with $m - 1/\overline{m(\varrho)} = K^w((|m|^2 - 1)/\varrho\overline{m})$ in $D$, and $m^* = 1/m$. That $m$ is a multiplier of $H(b)$ is now immediate from Corollary 12.4.

**Corollary 12.6.** If $m$ is a function in $H^\infty$ such that $|\text{Re } m(\varrho)|^{1/2}$ is bounded on $\partial D$, then $m$ is a multiplier of $H(b)$ and $m^* = -m$.

It is not completely obvious that there are nonzero functions satisfying the hypotheses of the corollary in all cases where $b$ is not an inner function. That there are will be pointed out below in connection with the proof of Corollary 12.8.

To establish Corollary 12.6, it suffices to use Lemma 10.2 in the same way as in the preceding proof and earlier ones to obtain

\[ m + \overline{m(\varrho)} = K^w\left(\frac{2\text{Re } m(\varrho)}{\varrho}\right) \]

in $D$ and $m^* = -m$. Corollary 12.4 now applies to show that $m$ is a multiplier of $H(b)$.

**Corollary 12.7.** If the outer factor, $b_0$, of $b$ is invertible in $H^\infty$, then all of the functions $k_w b$ and $Q_w b_0$ are multipliers of $H(b)$, and $(Q_w b_0)^* = S k_w b_0$ and $(k_w b_0)^* = -b_0(w) S Q_w (1/b_0)$.

To simplify the notation slightly in the proof, we shall assume that $b$ itself is invertible. This is not a genuine loss of generality, because the criterion in Corollary 12.4, upon which the proof of Corollary 12.7 is based, is insensitive to the inner factor of $b$.

Assuming then that $b$ is invertible, we note that

\[ K^w_b(Z_w k_w b) = S^* K^w_b(\overline{k_w b}) \]

\[ = S^* P_+ \left( \frac{1 - |b|^2}{b} \right) \]

\[ = -S^* P_+(b \overline{k_w b}) \]

\[ = -S^*(1 - w S^*)^{-1} b \]

\[ = -Q_w b. \]
Therefore, by the relation between the transformation $K_e$ and the conjugation on $K^2_e(\mathcal{Q})$ (noted in Section 9),

$$(Q_w b)_e = SK_e(k_w/b)$$
$$= SP_e(k_w/b - \bar{b}k_w)$$
$$= S(k_w/b - T \bar{b}k_w)$$
$$= S(1 - \bar{b}(0)b)k_w/b$$
$$= Sk_w/b.$$  

Thus $Q_w b$ is in $K^\infty(\mathcal{Q})$, and Corollary 12.4 implies that it is a multiplier of $H(b)$. We see also that $SK_w/b$ is a multiplier of $H(b)$. Since the space of multipliers is invariant under $S^*$ (by Lemma 2.6), and since $b$ is a multiplier of $H(b)$ (Theorem 7.1), it follows that $k_w$ is a multiplier of $H(b)$. To determine $(k_w^b)_e$ one verifies that $k_w = \bar{b}(w)K_e(k_w/b)$, which gives the formula

$$(k_w^b)_e = -b(w)SS^*K_e(\bar{k}_w/b).$$

The right side is easily reduced to the desired expression. The details are similar to those above and we omit them.

**Corollary 12.8.** The multipliers of $H(b)$ that lie in $H(\bar{b})$ are dense in $H(\bar{b})$.

To prove this, let $q$ be a real function in $L^\infty(\mathcal{Q})$ and let $f = K_e q$. As such functions $f$ clearly span $H(\bar{b})$, it will suffice to show that $f$ can be approximated in the norm of $H(\bar{b})$ by multipliers of $H(b)$. From Lemma 10.1 one sees that the real part of $f$ is bounded in modulus by $\|q\|_\infty$ in $C\bar{\partial}D$. For $\epsilon$ a positive number smaller than $1/\|q\|_\infty$, the functions $f/(1 + \epsilon f)$ and $f/(1 + \epsilon f)\;f$ are then in $H^p$, and we have

$$\frac{f}{1 + \epsilon f} - \frac{\tilde{f}_e}{1 + \epsilon \tilde{f}_e} = \frac{f - \tilde{f}_e}{(1 + \epsilon f)(1 + \epsilon \tilde{f}_e)}.$$  

The interior boundary function of $f - \tilde{f}_e$ is by Lemma 10.1 equal to $q_e$, so the interior boundary function of the preceding function is $q_e q$, where $q_e = q/(1 + \epsilon f)(1 + \epsilon f)\;f$. By Lemma 10.2 we conclude that the function $m_e = f/(1 + \epsilon f)$ equals $K_e(q_e)$ in $D$ and that $(m_e)_e = f/(1 + \epsilon f)\;f$. Thus $m_e$ is in $K^2(\mathcal{Q})$ and in $K^\infty(\mathcal{Q})$. It now follows immediately from Corollary 12.4 that $m_e$ is a multiplier of $H(b)$. Finally, since $q_e \rightarrow q$ in $L^2(\mathcal{Q})$ as $\epsilon \rightarrow 0$, we have $\|f - m_e\|_b \rightarrow 0$ as $\epsilon \rightarrow 0$, completing the proof.

A comment on the preceding proof: Suppose for simplicity that $\|q_e\|_\infty = 1$, and let $g = f - f(0)/2$ (which makes $g_e = -g$). The functions
are then in $H^\omega$ for $0 < \epsilon < 1$, and a simple estimate shows that $|\Re m_1| \leq |q|_g/(1 - \epsilon)^2$ on $\partial D$. The functions $m_1$ thus satisfy the hypothesis of Corollary 12.6, and the functions $m_1 - m_1(\infty)$ could have been used in the proof of Corollary 12.8 in place of the functions $m$. One can also deduce Corollary 12.8 by combining Corollary 12.6 with the following nice lemma of A. M. Gleason and H. Whitney [9, Lemma 3.1] (slightly rephrased): If $k$ is a non-negative function in $L^\omega$, then there is a sequence in $H^\omega$ whose real parts lie between 0 and $k$ on $\partial D$ and converge almost everywhere to $k$.

13. The effect of the inner factor

We continue to assume that $b$ is an extreme point of $B(H^\omega)$. As we observed earlier, the multiplication criterion in Corollary 12.4 is insensitive to the inner factor of $b$: if a function $m$ passes that test then it is a multiplier not only of $H(b)$ but of $H(ub)$ for every inner function $u$. We shall show that the condition of Corollary 12.4 characterizes multipliers of the preceding kind.

Lemma 13.1. Let $m$ be a multiplier of $H(b)$ and let $q$ be the function in $L^2(q)$ such that $m = m(\infty) + K q$. Let $u$ be an inner function. Then $m$ is a multiplier of $H(ub)$ if and only if $gq$ is in $L^2(q)$ for every $g$ in $H(u)$.

This lemma follows immediately from Theorem 12.2 and the decomposition $H(ub) = H(u) + uH(b)$ (explained in Section 6).

Corollary 13.2. If $u$ is a finite Blaschke product, then every multiplier of $H(b)$ is a multiplier of $H(ub)$.

Indeed, if $u$ is a finite Blaschke product, then the functions in $H(u)$ are bounded (in fact, they are rational functions), so the condition in Lemma 13.1 is satisfied.

Theorem 13.3. Let $m$ be a function in $K^\omega(q)$ and let $q$ be the function in $L^2(q)$ such that $m = m(\infty) + K q$. Then $m$ is a multiplier of $H(ub)$ for every inner function $u$ if and only if $q^{1/2}$ is bounded.

The «if» part is Corollary 12.4. The «only if» part is an immediate consequence of the preceding lemma and the following one.

Lemma 13.4. Let $a$ be a nonnegative essentially unbounded measurable func-
tion on $\partial D$. Then there is a function $g$ in $H^2$ that is noncyclic for $S^*$ such that $g$ is not in $L^2$.

The function $g$ that we shall produce lies in $H(u)$ for an interpolating Blaschke product $u$. We shall let $|E|$ stand for the unnormalized Lebesgue measure of the measurable subset $E$ of $\partial D$.

Since $\sigma$ is unbounded there is a sequence $\{t_n\}_n^\infty$ of positive numbers such that $t_{n+1} > 2t_n$ for all $n$ and such that each set $E_n = \{ t_n \leq \sigma \leq 2t_n \}$ has positive measure. For each $n$ let $\lambda_n$ be a point of density of $E_n$. The points $\lambda_n$ are distinct so, passing to a subsequence, we can assume they converge to a point distinct from all of them. That being the case, we can find disjoint arcs $I_1, I_2, \ldots$ such that $I_n$ has center $\lambda_n$ for each $n$. Shrinking these arcs successively, if need be, we can assume $|I_n| < |I_n|/2$ and $|I_n \cap E_n| > |I_n|/2$ for each $n$.

Let $w_n$ be the point in $D$ such that $w_n/|w_n| = \lambda_n$ and $1 - |w_n| = |I_n|/2$, and let $g_n = (1 - |w_n|^2)^{1/2} k_{w_n}$, the normalized kernel function for the point $w_n$. Since $1 - |w_n + 1| < (1 - |w_n|)/2$, the sequence $\{w_n\}_n^\infty$ is an interpolating sequence. Therefore, by a theorem of H. S. Shapiro and A. L. Shields [22], the functions $g_n$ form a Riesz basis for their span in $H^2$, that span being $H(u)$, where $u$ is the Blaschke product with zero sequence $\{w_n\}_n^\infty$.

We need to estimate the size of $g_n$ on $I_n$. For that, fix an $n$ and let $r = |w_n|$. We have

$$g_n(\lambda_n e^{i\theta}) = \frac{(1 - r^2)^{1/2}}{1 - re^{i\theta}} \frac{(1 - r \cos \theta + ir \sin \theta)}{(1 - r)^2 + 4r \sin^2(\theta/2)}.$$ 

Thus $\text{Reg}_n > 0$ on $\partial D$, and for $\lambda_n e^{i\theta}$ in $I_n$, that is, for $|\theta| < 1 - r$,

$$\text{Reg}_n(\lambda_n e^{i\theta}) \geq \frac{(1 - r^2)^{1/2}(1 - r)}{(1 - r)^2 + 4r \sin^2((1 - r)/2)} \geq \frac{(1 + r)^{1/2}(1 - r)^{3/2}}{2(1 - r)^2} \geq \frac{1}{2|I_n|^{1/2}}.$$ 

Since $t_n \to \infty$ we can find a sequence $\{c_n\}_n^\infty$ of positive numbers such that $\Sigma c_n^2 < \infty$ but $\Sigma c_n^2 t_n^2 = 2$. Let $g = \Sigma c_n g_n$. By the theorem of Shapiro and Shields mentioned above, $g$ is in $H^2$ and is not a cyclic vector of $S^*$. On $I_n$ we have
\[ |g| \geq \text{Reg} \geq \text{Reg}_n \geq \frac{c_n}{2|f_n|^{1/2}}. \]

Hence
\[
\int_{\partial} |ga|^2 \, d\theta \geq \sum_{E_n \subset E_n} |ga|^2 \, d\theta \\
\geq \sum \frac{c_n^2}{4|f_n|} t_n^2 |E_n \cap E_n| \\
\geq \frac{1}{8} \sum c_n^2 t_n^2 = \infty,
\]
which proves the lemma.

Up to now we have not given an example of a multiplier that fails to satisfy the criterion in Corollary 12.4. That will come in the next section. In the other direction, one sees from Theorem 13.3 that if \( q \) is bounded away from 0 on the set where it is nonzero, then \( H(b) \) and \( H(ub) \) have the same multipliers for all inner functions \( u \).

The next result, which identifies a class of inner functions \( u \) for which \( H(ub) \) and \( H(b) \) have the same multipliers, does not require the assumption that \( b \) is an extreme point.

**Theorem 13.5.** If \( u \) is an inner function such that \( \text{dist}(b, uH^\infty) < 1 \), then every multiplier of \( H(b) \) is a multiplier of \( H(ub) \).

We first show that the distance inequality is equivalent to the equality
\[ H(u) = (1 - T_u T_u^*)H(b), \]
or, what amounts to the same thing, to the inclusion \( H(u) \subset (1 - T_u T_u^*)H(b) \). By the criterion of Douglas we used earlier (in Section 2), the inclusion is equivalent to the operator inequality
\[ 1 - T_u T_u^* \leq c (1 - T_u T_u^*)(1 - T_b T_b^*)(1 - T_u T_u^*) \]
for some \( c \geq 1 \). The operator inequality means that
\[ \|h\|_2^2 \leq c (\|h\|_2^2 - \|T_b h\|_2^2) \]
for all \( h \) in \( H(u) \), in other words, that
\[ \|T_b h\|_2^2 \leq \frac{1}{c} \|h\|_2^2 \]
for all \( h \) in \( H(u) \), in other words, that \( \|T_b \|_{H(u)} < 1 \). Since it is known [17] that \( \|T_b H(u)\| = \text{dist}(b, uH^\infty) \), the equivalence is established.
Thus, assuming $u$ satisfies the condition in the theorem, we have $H(u) = (1 - T_uT_\beta)H(b)$. Suppose $m$ is a multiplier of $H(b)$. Then, because $H(ub) = H(u) + uH(b)$, to show $m$ is a multiplier of $H(ub)$ we need only show $mH(u) \subset H(ub)$. Let $g$ be any function in $H(u)$. Then, because $H(u) = (1 - T_uT_\beta)H(b)$, there is a function $h$ in $H(b)$ whose projection onto $H(u)$ is $g$. The difference $h - g$ is then in $H(ub)$ and in $uH^2$, so it is in $uH(b)$. Hence $m \cdot (h - g)$ is in $uH(b)$ and thus in $H(ub)$. Since also $mh$ is obviously in $H(ub)$, it follows that $mg$ is in $H(ub)$, and the theorem is established.

14. Helson-Szegö weights

For certain extreme points $b$ of $B(H^p)$, those for which the conjugation operator behaves in a decent manner relative to $\mu_b$, we are able to describe the multipliers of $H(b)$ completely. By a Helson-Szegö weight we shall mean a nonnegative function $\sigma$ on $\partial D$ that has the form $\sigma = \exp(\phi + \psi)$, where $\phi$ and $\psi$ are real functions in $L^\infty$ with $\|\psi\|_\infty < \pi/2$, and $\bar{\psi}$ denotes the conjugate function of $\psi$. The following properties hold:

1. If $\sigma$ is a Helson-Szegö weight then so is $1/\sigma$.
2. A Helson-Szegö weight is in $L^1 + \epsilon$ for sufficiently small positive numbers $\epsilon$.
3. If $\sigma$ is a Helson-Szegö weight then the conjugation operator is bounded on $L^2(\sigma)$. This property characterizes Helson-Szegö weights.

Property 1 is trivial and property 2 is a well-known result of V. I. Smirnov [7, p. 34]. Property 3 is the basic theorem of H. Helson and G. Szegö [13]. A thorough discussion of these and related matters can be found in the book [8].

If $\mu_b$ is absolutely continuous and its Radon-Nikodym derivative is a Helson-Szegö weight then, as Davis and McCarthy show [1] (on the basis of the Helson-Szegö theorem), every function in $H^p$ is a multiplier of $H(b)$. (They prove the converse also.) Such a $b$ of course is not an extreme point of $B(H^p)$. The next theorem says that an analogous result holds for extreme points whose corresponding measures are made in a simple way from Helson-Szegö weights.

Theorem 14.1. If $\mu_b$ is absolutely continuous with Radon-Nikodym derivative $\chi_E\sigma$, where $\sigma$ is a Helson-Szegö weight and $E$ is a subset of $\partial D$ of positive measure whose complement has positive measure, then the following spaces coincide:

1. The space of multipliers of $H(b)$,
2. $K^\infty(q)$.
3. $K^\infty(\chi_E)$.

A lemma is needed.

Lemma 14.2. Let $\sigma$ be a Helson-Szegö weight and let $q$ be a function in $L^2(\sigma)$. Then, in $D$, the Cauchy integral $K_\sigma q$ belongs to $H^1$, and its interior boundary function has the form $q_1 \sigma$ with $q_1$ in $L^2(\sigma)$.

To see that $K_\sigma q$ is in $H^1$, choose a positive number $\epsilon$ such that $\sigma$ is in $L^{1+\epsilon}$. Then $qa$ is the product of the $L^2$ function $q\sigma^{1/2}$ and the $L^2 + 2i$ function $\sigma^{1/2}$, so it is in $L^2(2 + \sigma^{1/2} + \sigma)$, by Hölder's inequality. By M. Riesz's theorem, the conjugate function of $qa$ lies in the same space. Hence (by Lemma 10.1), the Cauchy integral $K_\sigma q$ is in $H^2(2 + \sigma^{1/2} + \sigma)$ and a fortiori in $H^1$.

To see that the interior boundary function of $K_\sigma q$ has the required form we note that, because $qa$ is in $L^2(1/\sigma)$, the Helson-Szegö theorem implies that the interior boundary function of $K_\sigma q$, its Cauchy integral, is in $L^2(1/\sigma)$. Thus, if $q_1$ is the boundary function, then the function $q_1 = q_2 / \sigma$ is in $L^2(\sigma)$, which is the desired conclusion.

As for the theorem, we already know that every multiplier of $H(b)$ is in $K^\infty(q)$, and one easily sees that $K^\infty(q)$ is contained in $K^\infty(\chi_E)$. It only remains to show that every function in $K^\infty(\chi_E)$ is a multiplier of $H(b)$, or, equivalently, of $K^2(\chi_E \sigma)$.

The argument is similar to several we have already given. Let $f$ be a function in $K^2(\chi_E \sigma)$, say $f = K(q \chi_E \sigma)$, where $q$ is in $L^2(\chi_E \sigma)$. Let $m$ be a function in $K^\infty(\chi_E)$. By Lemma 14.2 the functions $f$ and $f_*$, in $D$, belong to $H^1$. Hence $mf$ and $m\phi f_*$ are in $H^1$, implying that the harmonic function $mf - m\phi f_*$ is the Poisson integral of its boundary function. By Lemma 10.2, to prove $mf$ is in $K^2(\chi_E \sigma)$ it will suffice to prove that the boundary function of $mf - m\phi f_*$ is of the form $q_1 \chi_E \sigma$ with $q_1$ in $L^2(\chi_E \sigma)$. For this we write, as usual,

$$mf - m\phi f_* = (m - m_\gamma_f) f + m_\gamma_f (f - f_*).$$

In the first summand on the right, the boundary function of the first factor, $m - m_\gamma_f$, is bounded and vanishes off $E$ (Lemma 10.1), while the boundary function of the second factor is in $L^2(\sigma)$, by Lemma 14.2. The boundary function of the first summand is thus of the required form. In the second summand, the boundary function of the first factor, $m_\gamma_f$, is bounded, and the boundary function of the second factor, $f - f_*$, is $q \chi_E \sigma$. The boundary function of the second summand thus also has the required form, and the proof is complete.
We are now able to given an example of an extreme point b, a multiplier m of H(b), and an inner function u, such that m is not a multiplier of H(ub).

Fix δ in (0,1), and let the function σ on ∂D be defined by σ(e^iθ) = |θ|^δ, (−π ≤ θ ≤ π). This is a Helson-Szegő weight by a result of G. H. Hardy and J. E. Littlewood [11]. One can prove that nowadays by verifying that σ satisfies B. Muckenhoupt's condition (A_2), which characterizes Helson-Szegő weights. (Details are in [8].) Let E be the right half of ∂D, and let b be the function such that \((1 + b)/(1 - b)\) is the Herglotz integral of χEσ. Theorem 14.1 applies, telling us that K^∞(χE) is the space of multipliers of H(b).

Let q_0 be a C^1 function on D that vanishes off E and is nonzero at the point 1. Since q_0 is of class C^1 its conjugate function is continuous, and this implies by Lemma 10.1. that the Cauchy integral m = Kq_0 is bounded in C\(\bar{D}\) and hence belongs to K^∞(χE). Thus m is a multiplier of H(b).

We also have m = K_q q where q = q_0/\(q_0\) - q. The function q_0e^{1/2} (q_0 - q_0, q_0e^{-1/2}) is unbounded because q_0e^{-1/2} is and \(q = |1 - b|^2χEσ \leq 4χEσ\). Hence Theorem 13.3 guarantees the existence of an inner function u such that m is not a multiplier of H(ub). The proof of Lemma 13.4 provides an explicit example of such a u, a certain interpolating Blaschke product. By using estimates similar to those in the proof of Lemma 13.4 it is not hard to show that the Blaschke product with zero sequence \(1 - 2^{-n}\) also has the required property.

15. Questions

Many questions puzzle us.

1. If b is an extreme point of B(H^∞), must every function in K^∞(q) be a multiplier of H(\(\bar{b}\))? An answer most likely will involve subtleties of the conjugation operator (although we may be overlooking something simple).

2. If b is an outer function, must H(b) and H(\(\bar{b}\)) have the same multipliers? Results in [18] show that the answer can be negative when b is not an extreme point. What about the extreme point case?

3. To understand better the multipliers of H(b), one needs examples, in addition to those given by Theorem 14.1, where they can be described completely. As a very special query: Suppose in the example in Section 14 one lets δ = 1, thus passing beyond the realm of Helson-Szegő weights. What are the multipliers of H(b) for the corresponding b?

4. Davis and McCarthy [1] prove that if μ is a finite positive Borel measure on ∂D and μ_q is its absolutely continuous component, then every multiplier of K^2(μ) is a multiplier of K^2(μ_q). In case the singular component
of \( \mu \) is a finite sum of point masses and the Radon-Nikodym derivative of \( \mu_q \) is log-integrable, they are able to specify precisely which multipliers of \( K^2(\mu_q) \) are also multipliers of \( K^2(\mu) \). Can one describe in more general cases, or perhaps even in general, how the singular component of \( \mu \) influences the space of multipliers of \( K^2(\mu) \)? Progress on this will undoubtedly lead to a better understanding of the structure of the corresponding space \( H(b) \).

5. In case \( b \) is an extreme point, the algebra \( K^\infty(\varrho) \) appears to be an interesting object of study. It becomes a Banach algebra when equipped with the norm \( \|f\| = \|f\|_\infty + \|q\|_{L^2(\varrho)} \) where \( q \) is the function in \( L^2(\varrho) \) such that \( f = f(\varrho) + K_q q \), and \( \|f\|_\infty \) stands for the supremum of \( |f| \) over \( \mathbb{C} \varrho \). As shown in Section 10, the spectrum of a function \( f \) in \( K^\infty(\varrho) \) is the closure of \( f(\mathbb{C} \varrho) \). What can one say about the maximal ideal space of \( K^\infty(\varrho) \)? Is \( \mathbb{C} \varrho \) dense in it?

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Covering Lemmas and BMO Estimates for Eigenfunctions on Riemannian Surfaces

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Abstract

The principal aim of this note is to prove a covering Lemma in $\mathbb{R}^2$. As an application of this covering lemma, we can prove the BMO estimates for eigenfunctions on two-dimensional Riemannian manifolds $(M^2, g)$. We will get the upper bound estimate for $\|u\|_{\text{BMO}}$, where $u$ is the solution to $\Delta u + \lambda u = 0$, for $\lambda > 1$ and $\Delta$ is the Laplacian on $(M^2, g)$. A covering lemma in homogeneous spaces is also obtained in this note.

1. Introduction

Let $M$ be a smooth, compact and connected Riemannian manifold without boundary. Let $\Delta$ denote the Laplacian on $M$. Assume that $u$ is the solution to $\Delta u + \lambda u = 0$, $\lambda > 1$, i.e., $u$ is an eigenfunction with eigenvalue $\lambda$.

Many authors have studied the estimates of the BMO norm and the Nodal sets of eigenfunctions, see [DF1], [DF2], [C], [B], and [CM]. In [C], Cheng proved that $u$ vanishes at most to order $c\lambda$ in the two-dimensional case. In [B], Brüning showed the lower bound for the volume of the nodal set for $C^\infty$ metrics on Riemannian surfaces. Donnelly and Fefferman, see [DF1] and [DF2], obtained the growth property, estimates of the BMO norm and bounds for the volume of the Nodal set of eigenfunctions for all $n \geq 2$. Recently, Chanillo and Muckenhoupt, see [CM], improved the results of [DF2] for $n \geq 3$. 

221
The purpose of this note is to get a better BMO estimate of eigenfunctions in the two-dimensional case. The main result is the following

**Theorem 1. (BMO estimate for log |u|)** For u, λ as above, and for ε > 0

\[ \| \log |u| \|_{BMO} \leq c \lambda^{15/8 + \epsilon} \]

where \( c = c(\alpha, M) \) is independent of \( \lambda \).

The proof of Theorem 1 is based on the following covering lemma which is of independent interest.

**Lemma 1. (Covering Lemma)** Let \( \delta > 0 \) be small enough, then given any finite collection of balls \( \{B_a\}_{a \in \mathcal{A}} \), one can select a subcollection \( B_1, \ldots, B_N \) such that

(i) \[ \bigcup_a B_a \subset \sum_{i=1}^{N} (1 + \delta) B_i \]

and

(ii) \[ \sum_{i=1}^{N} \chi_{B_i}(x) \leq c \delta^{-7/4} \log \frac{1}{\delta} \]

where \( c \) is a constant independent of \( \delta \) and the given collection of balls.

The motivation of this note is from [CM]. In [CM], a covering lemma plays an important role. Lemma 1 is an improvement of the covering lemma for the two-dimensional case in [CM] so that we can get a better estimate of the BMO norm for eigenfunctions by using Lemma 1 and adapting the proof given in [CM] for \( n \geq 3 \). We would like to point out that it is quite possible to prove a covering lemma in case \( n \geq 3 \) which is better than that in [CM] by modifying the proof of Lemma 1.

This note is organized as follows: Section 2 explains why we should use a new selection of balls in order to get a better covering lemma than that in [CM]; Section 3 is devoted to the proof of a covering lemma in homogeneous spaces which is of independent interest; Section 4 and 5 deal with the proof of Lemma 1; Section 6 is devoted to the proof of Theorem 1.

One word about notations: Throughout this note, \( C \) and \( c \) will always denote generic positive constants independent of the given balls \( \{B_a\}_{a \in \mathcal{A}} \) and \( \delta > 0 \); \( q(B) \) will denote the radius of the ball \( B \); \( B(x, r) \) will denote the ball centered at \( x \) and of radius \( r \).

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2. A Covering Lemma due to Chanillo-Muckenhoupt

In this section, we first recall a covering lemma in [CM].

Lemma 2.1 (Chanillo-Muckenhoupt) Fix $0 < \delta < 1/2$. Given any finite collection of balls $\{B_\alpha_{\alpha \in I}\}$ in $\mathbb{R}^n$, one can select a subcollection $B_1, \ldots, B_N$ such that

(i) $\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + \delta) B_i$

(ii) $\sum_{i=1}^{N} \chi_{B_i}(x) \leq 4^n \delta^{-n}$

for all $x \in \mathbb{R}^n$.

In the proof of Lemma 2.1, one first selects a ball $B_1$ with the largest radius in $\{B_\alpha_{\alpha \in I}\}$. Having selected $B_1, \ldots, B_{k-1}$, one selects $B_k$ so that

$$B_k \subset \bigcup_{i=1}^{k-1} (1 + \delta) B_i$$

(2.2)

and $B_k$ has the largest possible radius in the collection $\{B_\alpha_{\alpha \in I} \setminus \{B_i\}_{i=1}^{k-1}\}$.

Here we want to point out that (ii) of Lemma 2.1 is the best possible result which can be obtained by the above selection of balls. We show this by giving the following:

EXAMPLE: Consider a family of unit balls centered inside a cube

$$Q = \{(x, y) : 0 \leq x \leq 1/2, \ 0 \leq y \leq 1/2\} \subset \mathbb{R}^2$$

and the centers of these balls have coordinates

$$[(2k\delta, 4l\delta), 0 \leq k \leq [1/4\delta^{-1}], \ 0 \leq l \leq [1/8\delta^{-1}]]$$

where $[\cdot]$ denotes the largest integer part. We also denote the unit ball with center $(2k\delta, 4l\delta)$ by $B_{k,j}$. We now select the balls in the following order:

$$B_{0,0}, B_{1,0}, \ldots, B_{[1/4\delta^{-1}],0}, B_{0,1}, B_{1,1}, \ldots, B_{[1/4\delta^{-1}],1}, \ldots, B_{0,[1/8\delta^{-1}]}, B_{1,[1/8\delta^{-1}]}, \ldots, B_{[1/4\delta^{-1}],[1/8\delta^{-1}]}$$
Then an easy calculation shows (2.2) holds. This selection exactly follows the method used in the proof of Lemma 2.1. But

\[ \sum_{i=1}^{N} m_i(x, y) \approx \delta^{-2}, \quad \text{for} \quad 0 \leq x \leq 1/2, \quad 0 \leq y \leq 1/2. \]

Therefore the above example tells us that we need to use a new selection of balls in order to get a better result than that in Lemma 2.1.

3. A Covering Lemma in Homogeneous Spaces

In this section, we are going to prove a covering lemma in homogeneous metric spaces with a doubling Borel measure. This section is independent of the others. We will apply the technique of partitioning the radii of balls to reduce the proof of the main lemma to a certain basic case. The proof as given below is an adaptation to homogeneous spaces of the proof on \( \mathbb{R}^n \) due to S. Chanillo.

We say a pair \((X, \rho)\) is a homogeneous metric space in the sense of Coifman and Weiss, if the following hold:

(i) \( \rho : X \times X \rightarrow \mathbb{R}^+ \) satisfies the following conditions:

\[ \rho(x, y) = 0 \quad \text{if and only if} \quad x = y \]

\[ \rho(x, y) = \rho(y, x) \]

\[ \rho(x, y) \leq K[\rho(x, z) + \rho(z, y)] \]

where \( K \) is a constant independent of \( x, y, z \) and

(ii) there is a Borel measure \( \mu \) such that

\[ 0 < \mu(B(x, r)) \leq A\mu(B(x, r/2)) < +\infty, \]

where \( A \) is a constant independent of the ball \( B(x, r) \) centered at \( x \) and with radius \( r \).

An easy consequence of (3.1) is

\[ \mu(B(x, r)) \leq A^{\log_2(r/r')} + 1 \mu(B(x, r')) \]

for any \( x \in X \) and \( 0 < r' < r \).
According to [CoW], both Vitali type and Whitney type covering lemmas are true. It is well-known that Besicovitch covering lemma may not be true in homogeneous spaces as pointed out in [SW]. Sometimes a Vitali covering lemma is not good enough for applications as in [CM], but the following covering lemma could be a replacement of both Vitali and Besicovitch covering lemmas.

**Lemma 3.3.** Let $\delta > 0$ be small enough and $\{B_i\}_{i=1}^{\infty}$ be a finite collection of balls in $X$. If there exists a doubling Borel measure $\mu$ on $X$ satisfying (3.1), then one can select a subcollection $B_1, \ldots, B_N$ such that

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (K + \delta)B_i$$

$$\sum_{i=1}^{N} \chi_{B_i}(x) \leq C\delta^{-d} \log \frac{1}{\delta}$$

where $C$ depends only on $K$ and $A$, and $d = \log_2 A$.

**Proof.** Select a ball $B$ with the largest radius in $\{B_i\}$. Having selected balls $B_1, \ldots, B_{k-1}$, select $B_k$ such that

$$B_k \subset \bigcup_{i=1}^{k-1} (K + \delta)B_i$$

and $B_k$ has the largest radius out of the collection

$$\{B_\alpha\}_{\alpha \in \mathbb{I}} \setminus \{B_i\}_{i=1}^{k-1}$$

The subcollection $B_1, \ldots, B_M$ chosen by the above selection obviously satisfies

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{M} (K + \delta)B_i$$

Now we prove (3.5) in the lemma. We first fix any point $x_0 \in X$. With no loss of generality, we may assume $x_0 \in \bigcap_{i=1}^{l} B_i$. We also assume $B_l = B_l(z_l, r_l)$. By the selection of $\{B_{\alpha}\}_{\alpha \in \mathbb{I}}$, we know $r_l \leq r_{l-1} \leq \ldots \leq r_1$. We note that there exists $\sigma = \sigma(x_0)$ such that
\[(3.7) \quad \sum_{i=1}^{j} \chi_{B_i}(x_0) = \sum_{k=1}^{s} \sum_{2^k \leq r_i < 2^{k+1}r_i} \chi_{B_r}(x_0)\]

We now have the following claims.

Claim (1): \(\delta 2^\sigma \leq 2K^2\)

If not, then \(\delta r_i \geq \delta 2^\sigma r_i > 2K^2 r_i\).

We note that for \(y \in B_i = B(z_i, r_i)\),

\[
q(y, z_i) \leq K[q(y, x_0) + q(x_0, z_i)] \\
\leq K[K[q(y, z_i) + q(z_i, x_0)] + q(x_0, z_i)] \\
\leq K[2Kr_i + r_i] = 2K^2r_i + Kr_i \\
< \delta r_i + Kr_i \\
= (K + \delta)r_i
\]

Thus \(B_i \subset (K + \delta)B_i\) which is a contradiction to (3.6). Thus claim (1) holds, i.e., \(\sigma \leq C \log \frac{1}{\delta}\), where \(C\) only depends on \(K\).

Claim (2): For the subcollection \([B_i]_{j=1}^{N_k}\) of \([B_i]_{j=1}^{N_k}\) with \(2^j r_i \leq q(B_i) < 2^{j+1} r_i\), we have

\[
q(z_j, z_h) > \frac{2^j r_j \delta}{K} \quad \text{for} \quad j \neq h, 1 \leq j, h \leq N_k
\]

For simplicity, we drop the subscripts and denote \(B_j\) and \(B_h\) by \(B_j\) and \(B_h\) respectively. We also assume \(j > h\). If the claim were not true, we would have for \(y \in B_j = B(z_j, r_j)\)

\[
q(y, z_h) \leq K[q(y, z_j) + q(z_j, z_h)] \\
\leq K \left[ r_j + \frac{2^j \delta}{K} \right] \\
\leq Kr_j + 2^j r_j \delta \leq Kr_j + \delta r_j \\
= (K + \delta)r_j \\
\leq (K + \delta)r_h.
\]

This implies \((K + \delta)B_h \subset B_j\) which is again a contradiction to (3.6).

Claim (3): The balls \(\left\{ B\left(z_j, \frac{2^j r_j \delta}{2K^2}\right) \right\}_{j=1}^{N_k}\) are mutually disjoint.
We set \( R_1 = 2^k r_1 \delta / 2K^2 \). If the claim were not true, there would exist \( y \in B(z_j, R_1) \cap (z_h, R_1) \) for some \( j \neq h \). Thus
\[
\varrho(z_j, z_h) \leq K[\varrho(z_j, y) + \varrho(y, z_h)] \leq \left[ \frac{2^k r_1 \delta}{2K^2} + \frac{2^k r_1 \delta}{2K^2} \right] = \frac{2^k r_1 \delta}{K},
\]
which is a contradiction to the claim (2).

**Claim (4):** The balls \([B_j]_{j=1}^{N_k}\) are all contained in \( B(x_0, R_2) \), where \( R_2 = K2^{k+2}r_i \).

For \( y \in B(z_j, r_j) \), we have
\[
\varrho(y, x_0) \leq K[\varrho(y, z_j) + \varrho(z_j, x_0)] \leq 2Kr_j \leq 2K \cdot 2^{k+1}r_i = K2^{k+2}r_i.
\]

Thus the claim holds.

**Claim (5):** \( B(x_0, R_2) \subset B(z_j, R_3) \) for each \( 1 \leq j \leq N_k \), where \( R_3 = K(2K + 1)2^{k+1}r_i \).

In fact, for \( y \in B(x_0, R_2) \), we have
\[
\varrho(y, z_j) \leq K[\varrho(y, x_0) + \varrho(x_0, z_j)] \leq K[R_2 + r_j] \leq K[2^{k+2}r_i + 2^{k+1}r_i] = K(2K + 1)2^{k+1}r_i = R_3.
\]

This proves claim (5).

Now by claims (3) and (4), we have
\[
(3.8) \quad \sum_{j=1}^{N_k} \mu(B(z_j, R_1)) \leq \mu(B(x_0, R_2)),
\]
and by claim (5) and (3.2), we have
\[
(3.9) \quad \mu(B(x_0, R_2)) \leq \mu(B(z_j, R_3)) \leq A^{\log_2(R_2/R_1)} \mu(B(z_j, R_1)).
\]

Therefore, from (3.8) and (3.9), it is easy to see
\[
N_k \leq A^{\log_2(R_2/R_1) + 1} = A^{\log_2(4K^2(2K + 1)/\delta)}.\]
By using $A = 2^d$ and an easy calculation, we get

$$N_k \leq [8K^3(2K + 1)]^d \delta^{-d}$$

Now,

$$\sum_{i=1}^l \chi_{B_i}(x_0) = \sum_{k=0}^s \sum_{2^k \leq r \leq 2^{k+1}} \chi_{B_k}(x_0) \leq \sum_{k=0}^s N_k \leq \sum_{k=1}^s [8K^3(2K + 1)]^d \delta^{-d}$$

$$\leq C \left( \log \frac{1}{\delta} \right) \delta^{-d}$$

where $C$ only depends on $K$ and $A$. This shows (3.5) and thus completes the proof of lemma (3.3).

4. A Basic Covering Lemma

The purpose of this section is to prove a covering lemma for balls which centered in any given cube in $\mathbb{R}^2$ with sidelength $\sqrt{\delta}$. Moreover, these balls have radii whose values are close to one another.

**Lemma 4.1.** Let $\delta > 0$ be given small enough. Given any cube $\mathcal{O}$ in $\mathbb{R}^2$ with sidelength $\sqrt{\delta}$ and given any finite collection of balls $\{B_a\}_{a=1}^N$ with $r = r(B_a) \leq \delta + \delta$, for some $1 \leq r \leq 2$ and centered in this cube $\mathcal{O}$, one can select a subcollection of balls $B_1, \ldots, B_N$ such that

(i) $\bigcup_{a} B_a \subset \bigcup_{i=1}^N (1 + c\delta)B_i$,

(ii) $N \leq c\delta^{-1/4}$,

where $c$ is an absolute constant independent of $\delta$ and the given balls.

In order to prove Lemma 4.1, we need the following propositions.

**Proposition 4.1.** The sum of all exterior angles of any convex polygon is $2\pi$.

This a well-known formula in plane geometry. One may also deduce this fact from the Gauss-Bonnet formula in differential geometry.
Proposition 4.2. The perimeter of any convex plane polygon contained inside a cube \( Q \) with sidelenath \( \sqrt{\delta} \) is less than \( 2\pi \sqrt{2\delta} \).

The proof of the above Proposition 4.2 uses the Cauchy-Crofton formula in \( \mathbb{R}^2 \). It states that for a given regular plane curve \( C \) with length \( l \), the measure of the set of straight lines (counted with multiplicities) which meet \( C \) is equal to \( 2l \). A proof of this assertion may be found on page 41 in [Do]. A higher dimensional version of Cauchy-Crofton formula is proved in [F].

**Proof of Proposition 4.2.** Let \( \partial P \) be the boundary of the convex polygon \( P \) inside \( Q \), and let \( S \) be the set of straight lines which meet \( P \). Then if we denote by \( q \) the distance from the origin to the lines and by \( \theta \) the angle between the positive x-axis and the line, and assume without loss of generality,

\[
\mathcal{Q} = \{(x, y) : 0 \leq x \leq \sqrt{\delta}, 0 \leq y \leq \sqrt{\delta}\},
\]

we have

\[
2 \text{ length } (\partial P) = \iint_{\mathcal{S}} d\pi d\theta \leq \int_{0}^{2\pi} \int_{0}^{\sqrt{2\delta}} 2dqd\theta = 4\pi \sqrt{2\delta}.
\]

Thus, \( \text{length } (\partial P) \leq 2\pi \sqrt{2\delta} \).

**Remark:** After this note was prepared, Prof. B. Muckenhoupt pointed out that the proof of Proposition 4.2 can be simplified by projecting \( \partial P \) to \( \partial Q \).

Proposition 4.3. Given any \( \delta > 0 \) small enough and oriented rectangle \( R \) in \( \mathbb{R}^2 \) with sidelenath \( \sqrt{\delta} \) and \( \delta \). Let \( \{B_\alpha\}_{\alpha \in \mathbb{N}} \) be a finite collection of balls centered in \( R \) and with radii \( r \leq \rho(B_\alpha) \leq r + \delta \) for some \( 1 \leq r \leq 2 \). Then we only need to select at most two balls \( B_1 \) and \( B_2 \) such that

\[
\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{2} (1 + c\delta) B_i
\]

**Proof:** If there are no more than two balls in the given collection, then there is nothing to prove. If there are more than two centers in \( R \), then we select two balls \( B_1 \) and \( B_2 \) with centers \( O_1 = (x_{o1}, y_{o1}) \) and \( O_2 = (x_{o2}, y_{o2}) \) respectively such that one of the centers is on the extreme left, the other is on the extreme right (as shown in Figure 1). We claim

\[
\bigcup_{\alpha} B_\alpha \leq \bigcup_{i=1}^{2} (1 + c\delta) B_i
\]
We will assume with no loss of generality that both $B_1$ and $B_2$ have the smallest possible radius in our collection, that is $r$. We may also assume with no loss of generality that $y_{o_1} = y_{o_2}$, then the ball $B_1$ with radius $r$ and center $(x_{o_1}, y_{o_1})$ is contained in $(1 + 2\delta)B_1$ and thus if we prove our claim with $B_1$ replaced by $B_1'$, our proposition will be proved.

Let $P$ denote the intersection point of $(1 + c\delta)B_1$ and $(1 + c\delta)B_2$. We now show that $\text{dist}(P, \partial B) > r + 2\delta$. This will prove our claim. Using the fact that $y_{o_1} = y_{o_2}$, we see that

$$\text{dist}(P, \partial B) \geq \text{dist}(P, O_2) - \delta$$

$$\geq |(1 + c\delta)^2 r^2 - (\sqrt{\delta}/2)^2|^{1/2} - \delta.$$

By choosing $c \geq 4$, we get

$$|(1 + c\delta)^2 r^2 - \delta/4|^{1/2} - \delta > r + 2\delta$$

since $r \geq 1$. This proves our claim and the proposition.

We are also going to need the following:

**Proposition 4.4.** Let $O_1O_2O_3$ be a triangle with sidelength less than $c\sqrt{\delta}$. Suppose $[B_i]_{i=1}^3$ are three balls centered at $[O_i]_{i=1}^3$ and with $r \leq \rho_i(B) \leq r + \delta$ for some $1 \leq r \leq 2$. Then any ball $B$ with $r \leq \rho(B) \leq r + \delta$ and centered at some $O$ inside the triangle $O_1O_2O_3$ can be covered by $\bigcup_{i=1}^3 (1 + c\delta)B_i$ for some absolute constant $c$. 

---

**Figure 1**

![Figure 1](image-url)
PROOF.

As shown in Fig. 2, we extend segments $O_1O_2$, $O_2O_3$, $O_3O_1$ to lines. Then these three lines subdivide $\mathbb{R}^2$ into seven pieces $\mathcal{P}_i$. Obviously, $B \cap \mathcal{P}_1$, $B \cap \mathcal{P}_2$ and $B \cap \mathcal{P}_3$ can be covered by $(1 + c\delta)B_1$, $(1 + c\delta)B_2$ and $(1 + c\delta)B_3$ respectively. Now let $O_{12}$ denote the point $O_1O_2$ nearest to $O$ and let $B'$ be the ball centered at $O_{12}$ of radius $r + \delta$. Then $B' \cap \mathcal{P}_3 \supset B \cap \mathcal{P}_3$ by the above choice of $O_{12}$. But by Proposition 4.3, $(1 + c\delta)B_1 \cup (1 + c\delta)B_2 \supset B'$. Thus

$$B \cap \mathcal{P}_3 \subset \bigcup_{i=1}^{3} (1 + c\delta)B_i.$$

A similar argument shows that

$$B \cap \mathcal{P}_j \subset \bigcup_{i=1}^{1} (1 + c\delta)B_i \quad \text{for} \quad j = 4, 6.$$

This completes the proof of the proposition.

**Proposition 4.5.** Let $\{B_a\}_{a \in \mathbb{R}^1}$ be a finite collection of balls with $r \leq q(B_a) \leq r + \delta$ for some $1 \leq r \leq 2$. If there exists a subcollection $\{B_i\}_{i=1}^m$ of $\{B_a\}$ and another further subcollection $\{B_a\}_{a \in \mathbb{R}^1}$ of $\{B_i\}$ such that

$$\bigcup_a B_a \subset \bigcup_{i=1}^{m} (1 + c\delta)B_i,$$
\[ B_i \subset \bigcup_{k=1}^{n} (1 + c \delta)B_{i_k} \]

for each \( i \). Then

\[ \bigcup_{a} B_a \subset \bigcup_{k=1}^{n} (1 + c' \delta)B_{i_k} \]

for some \( c' > c \) independent of \( \delta \) and the given balls.

**Proof.** It suffices to prove

\[ (1 + c \delta)B_i \subset \bigcup_{k=1}^{n} (1 + c' \delta)B_{i_k} \]

for each \( 1 \leq i \leq m \). Now fix \( i \), let \( B'_i = (1 + c \delta)B_i \), \( O_i \) be the center of \( B_i \), \( \partial B_i \) be the boundary of \( B_i \). Then it is enough to show

\[ B'_i \setminus B_i \subset \bigcup_{i=1}^{n} (1 + c' \delta)B_{i_k}. \]

Let \( P \subset \partial B_i \), we denote by \( Q \) the intersection point between \( B'_i \) and the half line starting with the point \( O_i \) and passing through \( P \). Then the length of the segment \( PQ \) is

\[ (1 + c \delta) g(B_i) - g(B_i) = c \delta g(B_i) \leq 3 c \delta. \]

Assume \( P \in (1 + c \delta)B_{i_k} \). In fact, \( \text{dist}(P, O_{i_k}) \leq (1 + c \delta) g(B_{i_k}) \), and then for any \( z \in PQ \), we have

\[ \text{dist}(z, O_{i_k}) \leq \text{dist}(z, P) + \text{dist}(P, O_{i_k}) \leq 3c \delta + (1 + c \delta) g(B_{i_k}) \leq (1 + c' \delta) g(B_{i_k}). \]

The last inequality follows from \( g(B_{i_k}) \geq 1 \). Thus the claim follows. We move \( P \) along \( \partial B_i \) and note that the union of all such segments \( PQ \) cover \( B'_i \setminus B_i \), this shows that

\[ B'_i \setminus B_i \subset \bigcup_{k=1}^{n} (1 + c' \delta)B_{i_k}. \]
For the remainder of this section, all balls that we will consider have radii \( q \) such that \( r_0 \leq q \leq r_0 + \delta \) for a fixed \( r_0, 1 \leq r_0 \leq 2 \).

We now begin the proof of Lemma 4.1.

Let \( S \) be the collection of centers of the balls \( \{B_q\}_{q \in \mathbb{Z}} \). Let \( V \) be the convex hull of \( S \). Then the boundary of \( V \) must be a convex plane polygon and the vertices of \( V \) consist of centers, say, \( O_i, O_j, \ldots, O_M \), enumerated in a clockwise order.

Let us consider any three vertices of \( V \), say, \( O_i, O_j, O_k, (1 \leq i < j < k \leq M) \). We introduce the following:

**Definition.** If \( O_i O_k \leq \delta^{1/4} \), we say the triple \( O_i O_j O_k \) is of type I. Otherwise, the triple \( O_i O_j O_k \) is of type II. We further split the type II triples into two cases: If \( O_i A_{ijk} \geq \delta \), we say the triple \( O_i O_j O_k \) is of type II1, otherwise, of type II2 (see Figure 3 below).

![Figure 3](image)

Now we select the vertices \( \{O_n\} \). The balls \( \{B_n\} \) with centers at \( \{O_n\} \) will be the subcollection that shall be used to prove our lemma.

We first make the following observation:

**Proposition 4.6.** If for some \( k \geq 3 \), every triple \( O_i O_j O_k \) is of type II2 for \( 2 \leq i \leq k - 1 \), then all balls centered inside the polygon with vertices \( \{O_i\}_{i=1}^k \) can be covered by \( (1 + c\delta)B_i \cup (1 + c\delta)B_j \) for some \( 1 \leq i_1 \leq i_2 \leq k \).

**Proof.** The polygon \( W \) whose vertices are \( \{O_i\}_{i=1}^k \) is contained inside a rectangle \( R \) with sidelengths \( \sqrt{2}\delta \) and \( \delta \) and \( O_i O_k \) is a part of one side with length \( \sqrt{2}\delta \) of the rectangle \( R \). Then by applying Proposition 4.3, we see that \( \bigcup_{i=1}^k B_i \subset \bigcup_{i=1}^{m-1} (1 + c\delta)B_i \). Now any ball \( B \) centered in \( W \) has its center in a triangle formed by some three vertices of \( W \). By Proposition 4.4, \( B \subset \bigcup_{i=1}^{k-1} (1 + c\delta)B_i \), thus by Proposition 4.6, we have \( B \subset \bigcup_{i=1}^{m-1} (1 + c\delta)B_i \).
We now give the selection procedure. We start with the triple $O_1O_2O_3$.

**Case 1:** If the triple $O_1O_2O_3$ is of type I or $\Pi_1$, then we select vertices $O_1$, $O_2$ and $O_3$ and pass to the next triple, $O_4O_5O_6$.

**Case 2:** If the triple $O_1O_2O_3$ is of type $\Pi_2$, then either

(i) Every triple $O_1O_iO_M$ is of type $\Pi_2$ for $2 \leq i \leq M - 1$.

Then by Proposition 4.6, We can select $B_i$ and $B_n$ as our subcollection and the proof of Lemma 4.1 is complete in this case. Or

(ii) There exists some $k$, $3 \leq k \leq M - 1$ such that every triple $O_1O_iO_k$ is of type $\Pi_2$ for $2 \leq i \leq k - 1$, but there is some triple $O_1O_nO_{k+1}$ ($2 \leq i \leq k$) which is not of type $\Pi_2$, i.e., of type I or $\Pi_1$.

Then by Proposition 4.6 again, there are two vertices $O_i$ and $O_n$ for $1 \leq i_1, i_2 \leq k$ such that all balls centered inside the polygon with vertices $\{O_i\}_{i=1}^k$ can be covered by $(1 + c\delta)B_i \cup (1 + c\delta)B_n$. Thus, we select $O_{i_1}$, $O_{i_2}$, $O_1$, $O_n$, $O_k$ and $O_{k+1}$ in this case (We note that some overlap may occur among the above five vertices since $O_1$, $O_n$, $O_k$ may be $O_1$, $O_k$). We then pass to the next triple $O_{k+1}O_{k+2}O_{k+3}$. We continue this selection as before with $O_{k+1}$ playing the role of $O_1$. Because there are only finite vertices, this process stops when $O_1$ appears again in a new triple. We thus arrive a subcollection of vertices $O_i$, $O_n$, ..., $O_l$ and a new polygon $Z$ whose vertices are formed by $O_i$, $O_n$, ..., $O_l$.

We now claim that

(i) $Z$ is a convex polygon.

(ii) Any ball $B_\alpha$ in the original collection $\{B_\alpha\}_{\alpha \in I}$ is contained in $\bigcup_{\alpha \in I} (1 + c\delta)B_\alpha$.

By noticing that $Z$ is the intersection of the convex polygon $V$ and the half spaces formed by the lines $O_nO_{k+1}$, $1 \leq k \leq L - 1$, and the intersection of convex regions is convex, this show that $Z$ is convex.

To prove (ii), we note that each center $O_\alpha$ of the balls $B_\alpha$ is in $V$ since $V$ is the convex hull of $S$. Furthermore, recalling that $\{O_i\}$ denotes the vertices of $V$, we see that the center $O_\alpha$ must be in one of the triangles $O_1O_mO_{m+1}$ ($2 \leq m \leq M - 1$). Thus by Propositions 4.4, $B_\alpha$ is contained in $\bigcup_{\alpha \in I} (1 + c\delta)B_\alpha$. But as observed above by the selection procedure, $B_\alpha \subset \bigcup_{k=1}^L (1 + c\delta)B_\alpha$, for all $\alpha \in I$, by Proposition 4.5.

In order to complete the proof of Lemma 4.1, we need the following proposition.

**Proposition 4.7.** Suppose $\{(O_{m-1}O_{m+1}, O_{m+2})\}_{m=1}^N$ be the family of type I triples selected from $O_1$, $O_2$, ..., $O_{M+1}$, then $N \leq c\delta^{-1/4}$.
PROOF. Since $O_{in} O_{in+2} \geq \delta^{3/4}$ by definition of type I triple. It follows by considering the perimeter of the triangle $O_{in} O_{in+1} O_{in+2}$, that

$$\overline{O_{in} O_{in+1}} + \overline{O_{in+1} O_{in+2}} \geq \overline{O_{in} O_{in+2}} \geq \delta^{3/4}.$$  

Summing over $m$, we get

$$N\delta^{3/4} \leq \sum_{m=1}^{N} (\overline{O_{in} O_{in+1}} + \overline{O_{in} O_{in+2}})$$

$$\leq \text{perimeter of the polygon } Z.$$ 

Since $Z$ is convex and contained in the cube $O$, the perimeter of $Z$ is no more than $c\sqrt{\delta}$ by Proposition 4.2. Thus $N\delta^{3/4} \leq c\sqrt{\delta}$ and consequently $N \leq c\delta^{-1/4}$.

**Proposition 4.8.** Suppose $\{(O_{in}, O_{in+1}, O_{in+2})\}_{m=1}^{N}$ be the family of type II$_1$ triples selected from $O_1$, $O_2$, ..., $O_M$ above, then $N \leq c\delta^{-1/4}$.

PROOF. Given a type II$_1$ triple $(O_{in}, O_{in+1}, O_{in+2})$, one has $\overline{O_{in} O_{in+2}} \leq \delta^{3/4}$ and $\overline{O_{in+1} A_m} \equiv \delta$ by the definition of type II$_1$ triple (see Figure 4).

With no loss of generality, we consider those triples such that $\theta_m \leq \pi/4$ since the number of $m$ such that $\theta_m > \pi/4$ is no more than $\frac{2\pi}{\pi/4} = 8$ by Proposition 4.1. We will show that the exterior angle $\theta_m$ is always bounded below by $\delta^{1/4}/2$ for those $\theta_m \leq \pi/4$.

![Figure 4](image-url)
In fact, we note that for $\theta_m \leq \pi/4$, we have $\tan \theta_m \leq 2\theta_m$, thus for $\theta_m \leq \pi/4$, we get

$$\theta_m \geq \alpha_m \geq \frac{1}{2} \tan \alpha_m = \frac{1}{2} \frac{O_{m+1}A_m}{O_mA_m} \geq \frac{1}{2} \frac{O_{m+1}A_m}{O_mO_{m+2}} \geq \frac{1}{2} \frac{\delta}{\delta^{3/4}} = \frac{1}{2} \delta^{1/4}.$$  

Applying Proposition 4.1 again, $\sum_{m=1}^{N} \theta_m \leq 2\pi$, where the sum is being taken over the exterior angles which arise in type $\Pi_1$ triples. But $\theta_m \geq 1/2 \delta^{1/4}$, thus $N \leq c\delta^{-1/4}$.

Q.E.D.

Finally, we note that the selected vertices $O_i, \ldots, O_n$ are from type $I$, type $\Pi_1$, or type $\Pi_2$ triples. In the convex polygon $Z$, we consider all maximal chains, where a maximal chain is a union of successive sides of the polygon $Z$ which come from type $I$ or type $\Pi_1$ triples. Then by the selection procedure, between any two maximal chains, there are no more than five vertices which are probably from type $\Pi_2$ (see case 2 (ii) at the beginning of the selection procedure in this section). Thus the number of these type $\Pi_2$ vertices are less than $c\delta^{-1/4}$ also. Therefore, the number of all vertices $O_i, \ldots, O_n$ is less than $c\delta^{-1/4}$, and this shows (ii) of Lemma 4.1.

We end this section with the following example which shows (ii) in Lemma 4.1 is the best possible result in each cube with sidelength $\sqrt{\delta}$.

EXAMPLE. Assume that the centers of the unit balls $\{B_i\}$ are on the circle centered at the origin $O$ and of radius $\sqrt{\delta}$. Furthermore, let the arclength between any two centers be $c\delta^{3/4}$. We claim that we exactly need $N = c\delta^{-1/4}$ unit balls $B_1, \ldots, B_N$ such that

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + \delta)B_i.$$ 

We consider two balls $B_1$ and $B_2$ centered at $O_1$ and $O_2$ respectively (as shown on Figure 5).

Let $OA \perp O_1O_2$, $OF \perp AO_2$, $AE = h$, $AO_2 = AO_1 = l$, and let $\angle AOF = \theta$, thus $\angle EO_2A = \theta$. Moreover,
\[ \sin \theta = \sin(\angle EO_2 A) = \frac{AE}{AO_2} = \frac{h}{l}, \]

also

\[ \sin \theta = \sin(\angle AOF) = \frac{AF}{AO} = \frac{l/2}{\sqrt{\delta}}. \]

Thus \( \frac{l/2}{\sqrt{\delta}} = \frac{h}{l}, \ i.e., \]

\[ l^2 = 2\sqrt{\delta}h. \]

\[ \text{Figure 5} \]

But by Proposition 4.3, \((1 + \delta)B_1 \cup (1 + \delta)B_2\) can cover all unit balls \(B_o\) centered between \(O_1\) and \(O_2\) if and only if \(h \leq c\delta, \ i.e., \ l \leq c\delta^{3/4}\). We also note that the arclength of \(O_1O_2 \leq c\delta^{3/4}\) if and only if \(l \leq c\delta^{3/4}\). Hence, we exactly need \(N = c\sqrt{\delta}/\delta^{3/4} = c\delta^{-1/4}\) balls.

5. **Proof of the Main Covering Lemma**

We now prove the main covering lemma (Lemma 1) stated in the introduction. We will need the following lemmas and propositions.
Lemma 5.1. Let $\delta$ be small enough. Given any cube $Q$ in $\mathbb{R}^2$ with sidelength $2^k\sqrt{\delta}$ and let $\{B_\alpha\}_{\alpha \in I}$ be any finite collection of balls with $r \leq \varrho(B_\alpha) \leq r + 2^k\delta$ for some $2^k \leq r \leq 2^{k+1}$ and centered in $Q$, where $k$ is an integer. Then one can select a subcollection of balls $B_1, \ldots, B_N$ such that

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + c\delta)B_i,$$

$$N \leq c\delta^{-1/4}.$$

The proof of Lemma 5.1 is straightforward if we use Lemma 4.1 and the scaling property.

Lemma 5.2. Let $\delta$ be small enough. Given any cube $Q$ in $\mathbb{R}^2$ with sidelength $2^k\sqrt{\delta}$ and let $\{B_\alpha\}_{\alpha \in I}$ be any finite collection of balls with $2^k \leq \varrho(B_\alpha) \leq 2^{k+1}$. Then one can select a subcollection $B_1, \ldots, B_N$ such that

$$\bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + c\delta)B_i,$$

$$N \leq c\delta^{-3/4}.$$

PROOF. With no loss of generality, we may assume the largest ball $B$ in our collection is of radius $2^{k+1}$. Any other ball $B_\alpha$ with $\varrho(B_\alpha) \leq 2^{k+1} - 2^k\sqrt{2\delta}$ is contained in $B$. Thus the balls $B_\alpha$ with $\varrho(B_\alpha) \leq 2^{k+1} - 2^k\sqrt{2\delta}$ may be ignored. Now we partition the radii into the following intervals:

$$2^{k+1} - 2^k\sqrt{2\delta} + 2^k\delta l \leq \varrho(B_\alpha) \leq 2^{k+1} - 2^k\sqrt{2\delta} + 2^k\delta(l+1)$$

for $0 \leq l \leq \sigma$.

Thus $\sigma = \sqrt{2/\delta}$. Applying Lemma 5.1 to these balls $\{B_\alpha^l\}$ whose radii are in the interval corresponding to $l$, $0 \leq l \leq \sigma$, we can select $\{B_{i}^{l}\}_{i=1}^{N_{l}}$ such that

$$\bigcup_{\alpha} B_\alpha^l \subset \bigcup_{i=1}^{N_l} (1 + c\delta)B_i^l,$$

$$N_l \leq c\delta^{-1/4}.$$
Thus
\[ \bigcup B_\alpha \subset \bigcup_{\alpha,l} B_\alpha^l \bigcup B \subset \bigcup_{i,l} (1 + c\delta)B_i^l \bigcup B \]
and the number of balls \([B_i^l]_{i,l}\) is less than
\[ \sum_{i,l} N_i \leq \sigma \cdot c\delta^{-1/4} \leq c\delta^{-1/2} \delta^{-1/4} = c\delta^{-3/4}. \]

Therefore the subcollection \([B_i^l]\) will be the one in our lemma.

Now we can show the following:

**Lemma 5.3.** Let \(\delta\) be small enough and \([B_\alpha]_{\alpha \in \mathbb{N}}\) be a finite collection of balls with \(2^k \leq q(B_\alpha) \leq 2^{k+1}\), where \(k\) is integers. Then one can select balls \(B_1, \ldots, B_N\) such that
\[ \bigcup_{\alpha} B_\alpha \subset \bigcup_{i=1}^{N} (1 + c\delta)B_i \]
and
\[ \sum_{i=1}^{N} \chi_{B_i}(x) \leq c\delta^{-7/4} \]
for all \(x \in \mathbb{R}^2\).

**Proof.** We subdivide \(\mathbb{R}^2\) into a dyadic grid of cubes \([Q_j]_{j=1}^\infty\), whose side-lengths are all \(2^k\sqrt{\delta}\). Let \([B_i^j]\) be the subcollection of \([B_\alpha]\) with centers inside \(Q_j\). For each \([B_i^j]\) and \(Q_j\), we apply Lemma 5.2 to select \([B_i^j]_{i=1}^{N_j}\) such that
\[ \bigcup_{\alpha} B_\alpha^j \subset \bigcup_{i=1}^{N_j} (1 + c\delta)B_i^j, \]
\[ N_j \leq c\delta^{-3/4}. \]
where \(c\) is independent of \(\delta, k, j\).

Then
\[ \bigcup B_\alpha \subset \bigcup_{\alpha,j} B_\alpha^j \subset \bigcup_{j=1}^{N_j} \bigcup_{i=1}^{N_j} (1 + c\delta)B_i^j = \bigcup_{i,j} (1 + c\delta)B_i^j, \]
which shows (5.4). Now let $x_0 \in \mathbb{R}^2$ be a fixed point and let $B(x_0, 2^k + 1)$ denote the ball centered at $x_0$ and of radius $2^{k + 1}$. Since $g(B_i) = 2^{k + 1}$, $\chi_{B_i}(x_0)$ vanishes for those balls $[B_i]$ centered outside $B(x_0, 2^k + 1)$. Now the cardinality of $[j]$ such that $\bigcap_{j} \subset B(x_0, 2^k + 1)$ is no more than $c\delta^{-1}$. Thus

$$\sum_{i,j} \chi_{B_i}(x_0) = \sum_{j} \sum_{i=1}^{N_j} \chi_{B_i}(x_0) \leq \sum_{j} c\delta^{-3/4} \leq c\delta^{-1} \delta^{-3/4} = c\delta^{-7/4},$$

this shows (5.5). Since $x_0$ is arbitrary, the proof is complete.

We now prove the main covering lemma.

We first select a ball $B_1$ with the largest radius. Having selected $B_1, \ldots, B_{k-1}$ select $B_k$ such that

$$B_k \subset \bigcup_{i=1}^{k-1} (1 + \delta)B_i$$

and $B_k$ has the largest possible radius out of the balls in the collection $\{B_{\alpha}\}_{\alpha \in I} \setminus \{B_i\}_{i=1}^{k-1}$. Thus clearly, we have

$$\bigcup_{\alpha} B_{\alpha} \subset \bigcup_{i=1}^{N} (1 + \delta)B_i.$$

Now let $B_i' = (1 + \delta)B_i$ and $[B_{i,k}']$ be the subcollection of $[B_i']_{i=1}^{N}$ such that $2^{k} \leq g(B_{i,k}') \leq 2^{k + 1}$. Then by using Lemma 5.3, we can select a subcollection $[B_{i,k}']$ of $[B_{i,k}']$ such that

$$\bigcup_{i} B_{i,k}' \subset \bigcup_{j=1}^{N_k} (1 + c\delta)B_{i,k}' ,$$

$$\sum_{j=1}^{N_k} \chi_{B_{i,k}'}(x) \leq c\delta^{-7/4},$$

for all $x \in \mathbb{R}^2$, where $c$ is independent of $\delta, k$ and $k = 0, \pm 1 \pm 2, \ldots$ Thus, we have
\[
\bigcup_{\alpha} B_{\alpha} \subset \bigcup_{i=1}^{N} (1 + \delta) B_{i} = \bigcup_{i=1}^{N} B_{i} = \bigcup_{i,j} B_{i,j}^{c} \subset \bigcup_{k,j} (1 + c\delta) B_{i,j}^{c} \\
\subset \bigcup_{j,k} (1 + 2c\delta) B_{i,j}^{c}
\]

for \(c > 2\) and \(\delta\) small enough. Thus, the subcollection \(\{B_{i,j}\}\) satisfies (i) of Lemma 1.

Fix \(x_{0} \in \mathbb{R}^{2}\). There exists \(k_{0} = k(x_{0})\) and \(\sigma_{0} = \sigma(x_{0})\) such that

\[
\sum_{j,k} \chi_{B_{i,j}^{c}}(x_{0}) = \sum_{k=k_{0}}^{k_{0} + \sigma_{0}} \sum_{j=1}^{N_{k}} \chi_{B_{i,j}^{c}}(x_{0}) \\
\leq \sum_{k=k_{0}}^{k_{0} + \sigma_{0}} \left(\sum_{j=1}^{N_{k}} \chi_{B_{i,j}^{c}(\delta)}\right) \\
\leq \sum_{k=k_{0}}^{k_{0} + \sigma_{0}} c\delta^{-7/4} = c\sigma_{0}\delta^{-7/4}.
\]

We claim \(2^{\sigma_{0}} \leq 8\delta^{-1}\). For otherwise there would exist a ball \(B_{i,j,k_{0} + \sigma_{0}}\) for some \(j\) with \(x_{0} \in B_{i,j,k_{0} + \sigma_{0}}\) such that

\[
\varrho(B_{i,j,k_{0} + \sigma_{0}}) \geq 1/2 \varrho(B_{i,j,k_{0} + \sigma_{0}}) \geq 1/2 \cdot 2^{k_{0} + \sigma_{0}} \geq 2\delta^{-1} 2^{k_{0} + 1}.
\]

The first inequality above is due to the fact that

\[
\varrho(B_{i,j,k_{0} + \sigma_{0}}) = (1 + \delta) \varrho(B_{i,j,k_{0} + \sigma_{0}}) \leq 2\varrho(B_{i,j,k_{0} + \sigma_{0}}) \quad \text{for} \quad \delta < 1.
\]

Let \(x_{0} \in B_{i,j,k_{0}}\) for some \(h\) be the ball with

\[
\varrho(B_{i,j,k_{0}}) \leq \varrho(B_{i,j,k_{0}}^{c}) \leq 2^{k_{0} + 1}.
\]

Then \((1 + \delta)B_{i,j,k_{0} + \sigma_{0}} \supset B_{i,j,k_{0}}\) which is a contradiction to (5.6). Thus \(2^{\sigma_{0}} \leq 8\delta^{-1}\), and the claim follows. That is, \(\sigma_{0} \leq c \log \delta^{-1}\). Therefore,

\[
\sum_{j,k} \chi_{B_{i,j}^{c}}(x_{0}) \leq c\sigma_{0}\delta^{-7/4} \leq c \log(\delta^{-1})\delta^{-7/4}.
\]

Since \(x_{0}\) is arbitrary, we are done.
6. Proof of Theorem 1

This section is devoted to the proof of the main theorem in this note (Theorem 1). We begin with recalling the following theorem in [DF2].

**Theorem 6.1 (Donnelly-Fefferman).** Let $M$, $u$, $\lambda$ be as in the introduction. Let $B(x, \delta)$ denote the ball centered at $x$ of radius $\delta$. Then

\[
(6.2) \quad \int_{B(x, \delta(1 + \lambda^{-1/2}))} |u|^2 \leq c \int_{B(x, \delta)} |u|^2
\]

\[
(6.3) \quad \left[ \int_{B(x, \delta)} |\nabla u|^2 \right]^{1/2} \leq c \frac{\sqrt{\lambda}}{\delta} \left[ \int_{B(x, \delta)} |u|^2 \right]^{1/2}
\]

Now we start the proof of Theorem 1 by showing the following lemmas.

**Lemma 6.4.** Let $u$, $\lambda$ as above, $1 \leq q < \infty$, then $u$ satisfies the Reverse-Hölder inequality

\[
(6.5) \quad \left[ \frac{1}{|B|} \int_B |u|^q \right]^{1/q} \leq c \sqrt{\lambda} \left[ \frac{1}{|B|} \int_B |u|^2 \right]^{1/2}
\]

where $c$ depends on $q$.

**PROOF.** By the Poincaré inequality, for any ball $B$, we have

\[
(6.6) \quad \left[ \frac{1}{|B|} \int_B |u - u_B|^q \right]^{1/q} \leq c |B|^{1/2} \left[ \frac{1}{|B|} \int_B |\nabla u|^p \right]^{1/p},
\]

where $u_B = \frac{1}{|B|} \int_B u$ and $1 < p < 2$, $1/q = 1/p - 1/2$, and $c = c(p, q)$.

Applying Hölder inequality and (6.3) to the right side of (6.6), we obtain

\[
\left[ \frac{1}{|B|} \int_B |u - u_B|^q \right]^{1/q} \leq c \sqrt{\lambda} \left[ \frac{1}{|B|} \int_B |u|^2 \right]^{1/2}.
\]

By Minkowski's inequality, Lemma 6.4 follows for $2 < q < \infty$, for the case $1 < q \leq 2$, we can apply Hölder inequality again.
Our theorem will follow from the following

**Lemma 6.7.** Suppose \( w > 0, q > 2, \epsilon > 0 \) and \( 1 + \epsilon \geq q' \) where \( 1/q' = 1 - 1/q \), also assume that

\[
\int_{B(r, B(1 + \lambda^{-1/2}))} w \leq c_0 \int_{B(r, B)} w
\]

and

\[
\left( \frac{1}{|B|} \int_{B} w^q \right)^{1/q} \leq c_1 \lambda \left( \frac{1}{|B|} \int_{B} w \right)
\]

then

\[
\| \log w \|_{BMO} \leq c \lambda^{15/8 + \epsilon}
\]

where \( c = c(c_0, c_1) \).

Theorem 1 will follow if we choose \( w = |u|^2 \).

In order to prove Lemma 6.7, we need the following

**Lemma 6.9.** Let \( w, q, 0 < \epsilon < 1 \) satisfy the hypothesis of Lemma 6.7, let \( B \) be a fixed ball, \( E \subset B \), then there exist \( c_2, c_3 \) such that if

\[
|E| \geq (1 - c_2 \lambda^{-15/8 - \epsilon} (\log \lambda)^{-1})^k \int_{B} w
\]

then

\[
\int_{B} w \geq (c_3 \lambda^{-7/8} (\log \lambda)^{-1})^k \int_{E} w
\]

where \( c_2 = c(c_1), c_3 = c(c_0) \).

**Proof.** We proceed as the proof given in [CM]. The method is to use induction on \( k \). We first verify the lemma for \( k = 1 \). To do so, we claim that if \( \epsilon > 0, |E| \geq (1 - \tilde{c} \lambda^{-\epsilon}) |B| \) for some appropriate \( \tilde{c} = \tilde{c}(c_1) \), then

\[
\int_{E} w \geq 1/2 \int_{B} w.
\]

To show this, we first note that \( |B \setminus E| \leq \tilde{c} \lambda^{-\epsilon} |B| \). If we choose \( q > 2 \) such that \( \frac{1 + \epsilon}{q'} \geq 1 \), thus by (6.8),
\[
\int_{B \setminus E} w \leq \left( \int_B w^q \right)^{1/q} |B \setminus E|^{1/q'} \\
\leq c_1 \varepsilon^{1/q'} \lambda^{\frac{1 + \epsilon}{q'} + 1} \int_B w \\
\leq c_1 \varepsilon^{1/q'} \int_B w
\]

If we choose \( \varepsilon \) such that \( c_1 \varepsilon^{1/q'} < 1/2 \), then \( \int_{B \setminus E} w \leq 1/2 \int_B w \), this implies \( \int_E w > \frac{1}{2} \int_B w \). Here we want to point out the choice of \( \varepsilon \) is dependent on \( \epsilon \) since \( c_1 = c_1(q) \) and \( q \) is dependent on \( \epsilon \).

Thus if \( c_2 \leq \varepsilon \), and \( |E| \geq (1 - c_2 \lambda^{-15/8 - \epsilon} (\log \lambda)^{-1}) |B| \), then

\[
\int_E w \geq 1/2 \int_B w \geq c_3 (\lambda^{-7/8} (\log \lambda)^{-1}) \int_B w,
\]

and we are done for the case \( k = 1 \). Now we assume the statement is true for \( k - 1 \). Obviously we can assume \( |E| \leq (1 - \varepsilon \lambda^{-15/8 - \epsilon}) |B| \), otherwise, there is nothing to prove. Thus for each density point \( x \) of \( E \), we can select a ball \( B_x \subset B \) such that \( x \in B_x \), and

\[
\frac{|B_x \cap E|}{|B_x|} = 1 - \varepsilon \lambda^{-15/8 - \epsilon}.
\]

Applying Lemma 1 to the balls \( B_x \) with the choice \( \delta = \lambda^{-1/2} \), and with no loss of generality, assume \( |B_x| \) are finite, and define

\[
E_1 = \left( \bigcup_{i=1}^N (1 + \lambda^{-1/2}) B_i \right) \cap B.
\]

Then \( E_1 \subset B \), and as the proof given in [CM], we can show

\[
|E| \leq (1 - c_2 \lambda^{-15/8 - \epsilon} (\log \lambda)^{-1}) |E_1|
\]

\[
\int_E w \geq c_3 \lambda^{-7/8} (\log \lambda)^{-1} \int_{E_1} w
\]

for some \( c_2 = c(c_1), c_3 = c(c_0) \).
Now we prove Lemma 6.7. The proof will be almost the same as that given in [CM]. In order to get the precise estimate, we like to show the details. It will be enough to assume \( \frac{1}{|B|} \int_B w = 1 \). It is also sufficient to show

\[
|x \in B : w^{-1}(x) > t| \leq \frac{|B|}{t^{\alpha^{15/8-\epsilon} (\log \lambda)^{-3}}}
\]

It is equivalent to show

\[
|x \in B : w(x) < t| \leq t^{\alpha^{15/8-\epsilon} (\log \lambda)^{-3}} |B|
\]

Let us denote by \( E = \{x \in B : w(x) < t\} \). Select \( k_0 \) such that

\[
|E| = \left| 1 - c_2 \lambda^{-15/8-\epsilon} (\log \lambda)^{-1} \right| |B|
\]

Thus

\[
k_0 = c(\lambda^{15/8+\epsilon} \log \lambda) \log \left( \frac{|B|}{|E|} \right)
\]

Then by Lemma 6.9, and the normalization \( \frac{1}{|B|} \int_B w = 1 \), we have

\[
|B| = \int_B w \leq (c_1^{-1} \lambda^{7/8} \log \lambda)^{k_0} \int_E w \leq (c_1^{-1} \lambda^{7/8} \log \lambda)^{k_0} t |E|
\]

Thus

\[
\frac{|B|}{|E|} \leq t e^{k_0 \log(c_1 \lambda^{-\epsilon} \log \lambda)}
\]

\[
\leq t \left( \frac{|B|}{|E|} \right)^{(c_2 \lambda^{15/8+\epsilon} \log \lambda) \log(c_1^{-1} \lambda^{7/8} \log \lambda)}
\]

\[
\leq t \left( \frac{|B|}{|E|} \right)^{c \lambda^{15/8+\epsilon} (\log \lambda)^2}
\]
Hence

$$|E| \leq \epsilon^{c' \lambda^{-15.8+\epsilon} \left(\log \lambda\right)^{-2}} |B|,$$

where $c'$ is dependent on the constant $c$ in (6.4). Since $\epsilon$ is arbitrary, we can even have

$$|E| \leq \epsilon^{c' \lambda^{-15.8+\epsilon} \left(\log \lambda\right)^{-2}} |B|.$$

**Remark.** Since for any $0 < \epsilon < 1$, we can choose $q$ so that $1 + \epsilon \geq q'$, and note that for such $q$, Lemma 6.7 holds. Unfortunately, the constant $c$ on the right side of (6.5) depends on $q$ (and then depends on $\epsilon$), and is unbounded when $q \to \infty$. Thus we have proved Theorem 1 for any $\epsilon > 0$. However we cannot obtain the theorem for replacing $\lambda^{15.8+\epsilon}$ by $\lambda^{15.8}$ since when $\epsilon \to 0$, $c_\epsilon$ is not bounded.

**References**


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*Added in Proof:* After this paper had been accepted for publication, the author has also shown a higher dimensional covering lemma in $\mathbb{R}^n$ ($n \geq 3$) by which one can refine the $\text{BMO}$ norm estimate of $|\Delta u|$ by reducing the power of $\lambda$ to $n - 1/8$ [L].
Mean value and Harnack inequalities for a certain class of degenerate parabolic equations

José C. Fernandes

Introduction

In this paper we study the behavior of solutions of degenerate parabolic equations of the form

\[(1.1) \quad \nu(x) u_t(x, t) = \sum_{i, j=1}^n D_{ij}(a_{ij}(x, t) D_{ii} u(x, t)),\]

where the coefficients are measurable functions, and the coefficient matrix \(A = (a_{ij})\) is symmetric and satisfies

\[(1.2) \quad w_1(x) \sum_{j=1}^n \lambda_j^2(x) \xi_j^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq w_2(x) \sum_{j=1}^n \lambda_j^2(x) \xi_j^2\]

for \(\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n\) and \((x, t) \in \Omega \times (a, b), \Omega\) a bounded open set in \(\mathbb{R}^n\).

We are going to assume some conditions on the weights (non-negative functions that are locally integrable) \(\nu, w_1, w_2\) and on the functions \(\lambda_j, j = 1, ..., n\), in order to be able to derive mean value and Harnack inequalities for solutions of (1.1). The assumptions on \(\lambda_j\), which we list below, are the ones stated in [FL2].

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\[ \lambda_1 = 1, \lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1}), \quad j = 2, \ldots, n, \quad x \in \mathbb{R}^n. \]

(1.4) Let \( \Pi = \{ x \in \mathbb{R}^n : \Pi x_k = 0 \} \). Then \( \lambda_j \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \Pi) \) and \( 0 < \lambda_j(x) \leq \Lambda, \quad x \in \mathbb{R}^n \setminus \Pi, \quad j = 1, \ldots, n. \)

(1.5) \( \lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, -x_i, \ldots, x_{j-1}), \) for \( j = 2, \ldots, n \) and \( i = 1, \ldots, j - 1. \)

(1.6) There is a family of \( n(n - 1)/2 \) non-negative numbers \( q_{j,i} \) such that \( 0 \leq \lambda_i(D_x \lambda_j)(x) \leq q_{j,i} \lambda_j(x) \), for \( 2 \leq j \leq n, \quad 1 \leq i \leq j - 1 \) and all \( x \in \mathbb{R}^n \setminus \Pi. \)

Denote \( \Gamma = \Omega \times (a, b) \) and define \( H = H(\Gamma) \) to be the closure of \( \text{Lip}(\Gamma) \) under the norm
\[
\| u \|^2 = \int_\Gamma u^2(x, t) (v(x) + w_2(x)) dx dt \\
+ \int_\Gamma \int_\Gamma \int_\Gamma | \nabla_x u(x, t) |^2 dx dt + \int_\Gamma \int_\Gamma u_t^2(x, t) v(x) dx dt,
\]

where \( \nabla_x u = (\lambda_1 D_{x_1} u, \ldots, \lambda_n D_{x_n} u). \) Thus, \( H(\Gamma) \) is the collection of all \( (n + 2) \)-triples \( (u, \beta, B) \) such that there exists \( u_k \in \text{Lip}(\Gamma) \) with \( u_k \rightharpoonup u, \quad \nabla_x u_k \rightharpoonup \beta, \quad (u_k)_t \rightharpoonup B, \) the convergence being in the appropriate \( L^2 \) space. We need to verify that \( \beta \) is uniquely determined and for this it is enough to show that for every \( F \in C_0^\infty(\Gamma), \)
\[
\int_\Gamma u \nabla_x F = - \int_\Gamma \beta F.
\]

In order to prove this, note that since \( u \in H, \) there exists \( \{u_k\} \subset \text{Lip}(\Gamma) \) such that \( u_k \rightharpoonup u \) in \( H. \) Then, by (1.3),
\[
\int_\Gamma u_k \lambda_i \frac{\partial F}{\partial x_i} = - \int_\Gamma \frac{\partial}{\partial x_i} (u_k \lambda_i) F = - \int_\Gamma \lambda_i \frac{\partial u_k}{\partial x_i} F.
\]

Therefore,
\[
\int_\Gamma u_k \nabla_x F = - \int_\Gamma (\nabla_x u_k) F.
\]

By Schwarz's inequality and assuming that \( w_2^{-1} \in L^1_{\text{loc}}, \)
\[ \left| \int_{\Gamma} u_k \nabla_{\lambda} F - \int_{\Gamma} u \nabla_{\lambda} F \right| \leq \int_{\Gamma} \left| u_k - u \right| w_2^{1/2} \left| \nabla_{\lambda} F \right| w_2^{-1/2} \left( \int_{\Gamma} \left| \nabla_{\lambda} F \right|^2 w_2 \right)^{1/2} \]
\[ \leq \| u_k - u \|_{L^2_{\lambda_1}} \left( \int_{\Gamma} \left| \nabla_{\lambda} F \right|^2 w_2 \right)^{1/2} \]
\[ \leq c \| u_k - u \|_{L^2_{\lambda_1}}. \]

Hence,
\[ \int_{\Gamma} u_k \nabla_{\lambda} F \rightarrow \int_{\Gamma} u \nabla_{\lambda} F \]

and similarly we can show
\[ \int_{\Gamma} (\nabla_{\lambda} u_k) F \rightarrow \int_{\Gamma} \beta F. \]

In the same way we prove \( B \) is uniquely determined, if \( v^{-1} \in L^1_{\text{loc}} \). We also define \( H_0(\Gamma) \) to be the closure of \( \text{Lip}_0(\Gamma) \), Lipschitz functions with compact support in \( \Gamma \), under the norm defined in (1.7). It is easy to see that the bilinear form \( b \) on \( \text{Lip}(\Gamma) \cap H(\Gamma) \) defined by
\[ b(u, \phi) = \int_{\Gamma} (u_t \phi u + (A \nabla u, \nabla \phi)) dx dt \]

can be continued to all of \( H(\Gamma) \) (here and in the rest of the paper the vector \( \nabla u \) is understood to be the vector \( \left( \frac{1}{\lambda_1} \beta_1, ..., \frac{1}{\lambda_n} \beta_n \right) \) where \( \nabla_{\lambda} u = (\beta_1, ..., \beta_n) \)).

We say \( u \in H(\Gamma) \) is a solution of (1.1) if \( b(u, \phi) = 0 \) for any \( \phi \in H_0; \)
\( u \in H(\Gamma) \) is a subsolution if \( b(u, \phi) \leq 0 \) for any \( \phi \in H_0(\Gamma) \), \( \phi \) positive in the \( H \)-sense, i.e., \( \phi \) can be approximated in \( H(\Gamma) \) by positive functions with compact support in \( \Gamma \); \( u \in H(\Gamma) \) is a supersolution if \( b(u, \phi) \leq 0 \) for any \( \phi \in H_0 \), \( \phi \) positive in the \( H \)-sense.

We also define \( \tilde{H} = \tilde{H}(\Omega) \) to be the closure of \( \text{Lip}(\Omega) \) under the norm
\[ \| u \|^2 = \int_{\Gamma} u^2(x)(\nu(x) + w_2(x))dx + \int_{\Gamma} \left| \nabla_{\lambda} u(x) \right|^2 w_2(x)dx, \]

and \( \tilde{H}_0(\Omega) \) to be the closure of \( \text{Lip}_0(\Omega) \) under the norm defined above.

Next we will define a natural distance (associated with the functions \( \lambda_j \), \( j = 1, ..., n \)) and state some of its properties. This metric was first introduced by [FL1].

A vector \( \nu \in \mathbb{R}^n \) is called a \( \lambda \)-subunit vector at a point \( x \) if \( (\nu, \xi)^2 \leq \Sigma \lambda_j^2(x) \xi_j^2 \), for every \( \xi \in \mathbb{R}^n \). An absolutely continuous curve \( \gamma: [0, T] \rightarrow \mathbb{R}^n \) is called a \( \lambda \)-subunit curve if \( \dot{\gamma}(t) \) is a \( \lambda \)-subunit vector at \( \gamma(t) \) for a.e. \( t \in [0, T] \).

For any \( x, y \in \mathbb{R}^n \) we define \( d: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+ \) by
\[ d(x, y) = \inf \{ T \in \mathbb{R}^+ : \text{there exists a } \lambda \text{-subunit curve } \gamma : [0, T] \to \mathbb{R}^n \}
\]

with \( \gamma(0) = x, \gamma(T) = y \).

One can check that this is a well-defined metric. There is a quasi-metric \( \delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+ \) is called a quasi-metric if there exists \( M \geq 1 \) such that \( \delta(x, y) \leq M(\delta(x, z) + \delta(z, y)) \) for all \( x, y, z \in \mathbb{R}^n \) equivalent to \( d \), and sometimes easier to work with than \( d \) (see [FL2]). If \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \) put \( H_0(x, t) = x \) and \( H_{k+1}(x, t) = H_k(x, t) + \delta_k \cdot e_k \cdot t \) for \( k = 0, \ldots, n - 1 \), where \( [e_k] \) is the standard basis in \( \mathbb{R}^n \). Define \( \varphi_j(x^*, \cdot) = (F_j(x^*, \cdot))^{-1} \), the inverse function of \( F_j(x^*, \cdot) \), where \( F_j(x, s) = s\lambda_j(H_{j-1}(x, s)) \), for \( j = 1, \ldots, n \) and \( x^* = (|x_1|, \ldots, |x_n|) \).

We define \( \delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+ \) as

\[ \delta(x, y) = \max_{1 \leq j \leq n} \varphi_j(x^*, |x_j - y_j|). \]

Note that

(1.8) \( \delta(x, y) < s \) is equivalent to \( |x_j - y_j| < F_j(x^*, s), 1 \leq j \leq n \).

In (1.9), (1.10), (1.11) below we state some basic facts concerning \( \delta \) and \( d \) (see also [FL2]).

(1.9) There exists \( a \geq 1 \) such that for any \( x, y \in \mathbb{R}^n \),

\[ a^{-1} \leq \frac{d(x, y)}{\delta(x, y)} \leq a. \]

Consequently, \( \delta \) is a quasi-metric with \( \delta(x, y) \leq a^2 [\delta(x, y) + \delta(z, y)] \) and \( \delta(x, y) \leq a^2 \delta(y, x) \).

(1.10) For any \( x \in \mathbb{R}^n \), \( s > 0 \) and \( \theta \in [0, 1] \)

\[ \theta e_j \leq \frac{F_j(x^*, \theta s)}{F_j(x^*, s)} \leq \theta \]

where \( G_1 = 1 \) and \( G_j = 1 + \sum_{i=1}^{j-1} G_i, G_i \), for \( j = 2, \ldots, n \).

(1.11) We denote \( S(x, r) = \{ y \in \mathbb{R}^n : d(x, y) < r \} \) and \( Q(x, r) = \{ y \in \mathbb{R}^n \}
\]

\( \delta(x, y) < r \) and we will call \( S(x, r) \) a \( d \)-ball and \( Q(x, r) \) a \( \delta \)-ball. Note that there is a constant \( A > 1 \) such that \( |S(x, 2r)| \leq A |S(x, r)| \) and \( |Q(x, 2r)| \leq A |Q(x, r)| \), where \( | \cdot | \) denotes Lebesgue measure. Also, by (1.8), \( |Q(x, r)| = \Pi_{j=1}^n F_j(x^*, r). \) If \( Q = Q(x, r) \), we write \( r = r(Q) \).
In general we say that a non-negative and locally integrable function \( w(x) \) is a doubling weight \( (w \in D) \) if there exists a constant \( A > 1 \) such that \( w(2Q) \leq Aw(Q) \) for any \( \delta \)-ball \( Q \), where \( 2Q = Q(x, 2r) \), if \( Q = Q(x, r) \) and

\[
  w(Q) = \int_Q w(x) \, dx.
\]

(1.12) If \( w \in D \) then there exists \( \alpha > 0 \) such that, for all \( r > 0, \theta \in [0, 1] \), and \( x \in \mathbb{R}^n \), \( w(Q(x, \theta r)) \geq \theta^\alpha w(Q(x, r)) \).

Given \( 1 < p < \infty \), we say \( w \in A_p \) if there is a constant \( c > 0 \) such that for all \( \delta \)-balls \( Q \) in \( \mathbb{R}^n \),

\[
  \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/p-1} \, dx \right)^{p-1} \leq c.
\]

(1.13) Note that if we have the \( A_p \) condition with respect to \( \delta \), we have the same condition holding for the metric \( d \), i.e. (1.13) holds with \( Q \) replaced by \( S \) (using doubling and the equivalence between \( d \) and \( \delta \)). If \( v \) is a weight, \( w \in A_p(v) \) means an analogous inequality holds with \( dx \) and \( |Q| \) replaced by \( v(x) \, dx \) and \( v(Q) \), respectively. We use the notation \( A_\nu = \bigcup_{p > 1} A_p(v) \). The theory of weights in homogeneous spaces was studied by A. P. Calderón in [C] and frequently we refer to this paper.

If \( x, y \in \mathbb{R}^n \), we shall denote by \( H(t, x, y) = (H_1(t, x, y), \ldots, H_n(t, x, y)) \) the solution at time \( t \) of the Cauchy problem \( \dot{H}(t, x, y) = y_j \dot{x}_j (H(t, x, y)), \)

\[
  H(0, x, y) = x_j, \; j = 1, \ldots, n.
\]

Given \( \alpha = (\alpha_1, \ldots, \alpha_n), \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) with \( 0 < \epsilon_j < \alpha_j, j = 1, \ldots, n \), we denote \( \Delta'' = \{ y \in \mathbb{R}^n : \epsilon_j \leq y_j \leq \alpha_j, j = 1, \ldots, n \} \). If \( \sigma \in [-1, 1]^n \), we put \( T_\sigma y = (\sigma_1 y_1, \ldots, \sigma_n y_n) \), \( Q''(x, r) = \{ y \in Q(x, r) : \sigma_j (y_j - x_j) \geq 0, j = 1, \ldots, n \} \), and \( \Delta'_{\sigma}(\sigma) = T_\sigma(\Delta'') \).

Now we can state two results proved in [FS].

Let \( \gamma \in [0, 1] \) and \( \sigma \in [-1, 1]^n \) be fixed. Then there exists \( \epsilon, \alpha \in \mathbb{R}^n \) as above such that, for all \( r > 0 \) and \( x \in \mathbb{R}^n \),

\[
  \left| H(r, x, \Delta'_{\sigma}(\sigma)) \right| \left| Q''(x, r) \right| \geq (1 - \gamma) \left| Q''(x, r) \right|,
\]

(1.14) where \( H(r, x, \Delta'_{\sigma}(\sigma)) = \{ H(x, r, y), y \in \Delta'_{\sigma}(\sigma) \} \).

Also, there are positive constants \( c_1, c_2 \) depending only on \( \epsilon, \alpha \) and \( \sigma \) such that
(1.15) \[ c_1 \left| S(x, r) \right| \leq \prod_{j=0}^{r} \lambda_j(H(t, x, y))dt \leq c_2 \left| S(x, r) \right| \]

for each \( x \in \mathbb{R}^n, r > 0 \) and \( y \in A^n_\alpha \).

If \( q \geq 2 \), we say that Sobolev inequality holds for \( w_1, w_2, \) and for any \( u \in H_0(Q), Q \) a \( \delta \)-ball in \( \mathbb{R}^n, \)

\[ (1.16) \quad \left( \frac{1}{w_2(Q)} \int_Q \left| u \right|^q w_2 dx \right)^{1/q} \leq c_r(Q) \left( \frac{1}{w_1(Q)} \int_Q \left| \nabla u \right|^2 w_1 dx \right)^{1/2}. \]

Given \( q \geq 2 \), we say the Poincaré inequality holds for \( w_1, w_2 \) and \( \mu \) if there are constant \( c > 0 \) and \( a > 0 \) (see (1.9)) such that for any \( \delta \) ball \( Q \) and every \( u \in \mathcal{C}(\delta^2 Q) \) we have

\[ (1.17) \quad \left( \frac{1}{w_2(Q)} \int_Q \left| u - av_{\mu, Q} u \right|^q w_2 dx \right)^{1/q} \]

\[ \leq c_r(Q) \left( \frac{1}{w_1(Q)} \int_Q \left| \nabla u \right|^2 w_1 dx \right)^{1/2}, \]

where \( av_{\mu, Q} u = \frac{1}{\mu(Q)} \int_Q u d\mu \) and \( a^2 Q = Q(x, a^2 r) \) if \( Q = Q(x, r). \)

The reason that we have \( a^2 Q \) on the right side of (1.17) is that we do not have a Kohn type argument (see also [1]) for the quasi-metric \( \delta \). In the \( d \)-metric, we can state (1.17) with equal balls on both sides. For the metric \( \delta \), however, we have convenient cut-off functions (see [FL1]) that are important in order to get Caccioppoli estimates for solutions of (1.1) (see C.1), (C.2) and (C.3)). This explains the reason that we work with \( \delta \) instead of \( d \).

We can now state our main results.

**Theorem A (Harnack’s inequality).**

Suppose that:

(i) \( \nu, w_1, w_2 \in A_2, \)
(ii) the Poincaré inequality holds for \( w_1, w_2 \) and \( \nu \) with \( \mu = 1 \) and some \( q > 2, \)
(iii) \( w_2 \nu^{-1} \in A_\mu(\nu). \)

If \( u \) is a non-negative solution of (1.1) in the cylinder \( R = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta) \), then
ess sup_{R^-} u \leq c \exp [c_2 (\alpha^{-2} \beta \lambda(Q(x_0, \alpha)) + \alpha^2 \beta^{-1} (\lambda(Q(x_0, \alpha))^{-1})] \inf_{R^-} u,

where \( R^- = Q(x_0, \alpha/2) \times (t_0 - \beta/4, t_0 - \beta/4), R^+ = Q(x_0, \alpha/2) \times (t_0 + \beta/4, t_0 + \beta), \lambda(Q) = w_2(Q)/\nu(Q), \lambda(Q) = w_1(Q)/\nu(Q), \) for a \( \delta \)-ball \( Q \). Here the constants \( c_1, c_2 \) depend only on the constants which arise in (i), (ii), (iii).

We write

\[
\int_R \int_R f(x, t) m(x, t) dx dt = \int_R \int_R f(x, t) m(x, t) dx dt / \int_R \int_R m(x, t) dx dt.
\]

**Theorem B (Mean value inequality).** Assume that hypotheses (i), (ii), (iii) of Theorem A hold. Let \( 0 < p < \infty, \alpha, \beta > 0, \alpha/2 < \alpha' < \alpha, \beta/2 < \beta' < \beta \) and let \( Q = Q(x_0, \alpha), Q' = Q(x_0, \alpha') \) and \( R = Q \times (t_0 - \beta, t_0 + \beta), R^+ = Q' \times (t_0 - \beta', t_0 + \beta) \). If \( u \) is a solution of (1.1) in \( R \), then \( u \) is bounded in \( R^+ \) and

\[
\text{ess sup}_{R^+} |u|^p \leq D (\alpha^2 \beta^{-1} \lambda(Q)^{-1} + 1)^{(b/h-1}(\alpha^{-2} \beta \lambda(Q) + 1)^{b/(b-h)} \int_R \int_R |u|^p (\alpha^{-2} \beta w_2 + \nu) dx dt,
\]

where \( D \leq C^{1/(h-1)} \) if \( p \geq 2 \), and \( D \leq c^{\log(1/p)/p} C^c \) if \( 0 < p < 2 \), and \( C = c \frac{\alpha^2 + b}{\alpha^2 + b (\beta - \beta')} \). Here \( h > 1, c > 0 \) and \( b > 0 \) are constants which are independent of \( u, p, \alpha, \alpha', \beta, \beta' \).

The organization of the paper is as follows. In Section 2 we prove the following Sobolev interpolation inequality:

**Theorem D:** Let \( w_1, w_2 \) be doubling weights, \( \nu \in A_2 \) and suppose (1.17) holds with \( w_1, w_2, \mu = 1 \) and some \( q > 2 \). If \( w_2^{-1} \in A_\infty(\nu) \) then there exists \( h > 1 \) and constants \( c > 0, b > 0 \) such that for every \( \epsilon \) satisfying \( 0 < \epsilon \leq 1, \)

\[
\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx \leq c \epsilon^{-b} \left( \frac{1}{\nu(Q)} \int_{(1+\epsilon)Q} u^2 v dx \right)^{b-1}
\]

\[
\times \left( \frac{r(Q)^2}{w_1(Q)} \int_{(1+\epsilon)Q} |\nabla u|^2 w_1 dx \right) + \frac{1}{\nu(Q)} \int_{(1+\epsilon)Q} u^2 v dx
\]

for all \( u \in \widetilde{H}((1 + \epsilon)Q) \).
In section 3 we prove Theorem B. First we show, for \( p \geq 2 \), the following mean value inequality for subsolutions of (1.1):

\[
(*) \quad \text{ess sup}_{R} u^+ \leq \left( p^2 C \right)^{\frac{1}{p-1}} \left( 1 + \alpha^2 \beta^{-1} \lambda(Q)^{-1} \right)^{1/(p-1)} \int_{R} u^+ (\alpha^{-2} \beta w_2 + v) dx dt,
\]

where \( C \) is as in Theorem B and \( u_+ = \max\{u, 0\} \). This inequality is less precise than the one we will eventually obtain because of the presence of the factor \( p^2 \) on the right. In order to prove the above inequality we apply Theorem D to the function \( H_M(u(\cdot, \tau)) \) where

\[
H_M(s) = \begin{cases} 
  s^{p/2} & \text{if } s \in [0, M] \\
  M^{p/2} + \frac{D}{2} M^{p-2}(s - M) & \text{if } s \geq M \\
  0 & \text{if } s < 0,
\end{cases}
\]

and therefore \( H_M(u(\cdot, \tau)) \) is an element of \( \hat{H}(Q(x_0, \alpha)) \) for a.e. \( \tau \in (t_0 - \beta', t_0 + \beta) \). The first idea would be to apply Theorem D to the function \( u^\alpha_{+}(\cdot, \tau) \) but at this point we do not know if \( u^\alpha_{+}(\cdot, \tau) \) belongs to \( \hat{H}(Q(x_0, \alpha)) \). Hence we have to work with \( H_M(u) \), and in order to proceed with the proof of \( (*) \) we show the following Caccioppoli inequality for \( H_M(u) \).

(C.1) Let \( 2 \leq p < \infty \) and \( u \) be a subsolution of (1.1) in \( R \). Let \( w_2 \in A_2 \) and \( \alpha, \alpha', \beta, \beta' \) satisfy \( \alpha/2 < \alpha' < \alpha, \beta/2 < \beta' < \beta \). Then

\[
\text{ess sup}_{(t_0 - \beta', t_0 + \beta)} \int_{Q} H_M(u(x, \tau))^2 \nu(x) dx \\
+ \int_{R} \int_{R} \left| \nabla \chi(H_M(u)) \right|^2 w_2(x) dx dt \\
\leq c \int_{R} \int_{R} u^2 H_M(u)^2 \left( \frac{w_2}{(\alpha - \alpha')^2} + \frac{\nu}{\beta - \beta'} \right) dx dt
\]

with \( c \) independent of all parameters.

The next step is to apply \( (*) \) for \( p = 2 \) to deduce that \( u_+ \) is locally bounded. This fact allow us to apply Theorem D to the function \( u^\alpha_{+}(\cdot, \tau) \) for a.e. \( \tau \in (t_0 - \beta', t_0 + \beta) \). The Caccioppoli inequality we can deduce from (C.1)
for the function $u_{p}^{p/2}$ is not precise enough since it will have a factor $p^2$ in the right hand side (note that $uH_{B}(u) \leq pu_{p}^{p/2}/2$) and this is the term we want to eliminate from (*). But with a different test function from the one used in the proof of (C.1), namely, $\phi(x, t) = \eta^2 g(u) \chi(t, \tau_1, \tau_2)$ where

$$g(s) = \begin{cases} s^{p-1} & \text{if } s \in [0, M], \\ M^{p-2}s & \text{if } s \geq M, \\ 0 & \text{if } s < 0, \end{cases}$$

and $\eta$ is a convenient $C^\infty$ function with compact support, we can deduce the following Caccioppoli inequality for subsolutions of (1.1):

(C.2) Let $2 \leq p < \infty$ and $u$ be a subsolution of (1.1) in $R$. Let $w_2 \in A_2$ and $\alpha, \alpha', \beta, \beta'$ satisfy $\alpha/2 < \alpha' < \alpha, \beta/2 < \beta' < \beta$. Then

$$\text{ess sup}_{(t_0 - \tau, t_0 + \tau)} \int_Q u_+(x, \tau)^{p} \nu(x) dx + \int_{\partial R} \left| \nabla u_+^{p/2} \right|^2 w_1(x) dt$$

$$\leq c \int_R \left( \frac{w_2}{(\alpha - \alpha')^2} + \frac{\nu}{(\beta - \beta')} \right) dx dt,$$

with $c$ independent of all parameters.

Now following the steps of the proof of (*) using (C.2) instead of (C.1) we can prove that for $p \geq 2$

$$(**) \text{ ess sup}_{R_2} u_+^p \leq$$

$$C^{\frac{n}{2}} (\alpha_2 \beta^{-1} \lambda(Q)^{-1} + 1)^{1/(h-1)} (\alpha^{-2} \beta \Lambda(Q) + 1)^{1/(h-1)} \int_R u_+^{p} (\alpha^{-2} \beta w_2 + \nu) dx dt,$$

and Theorem B will follow from (**) and an iteration argument like the one given in Lemma 3.4 of [GW2]. Finally we conclude Section 3 by making some comments about the proof of mean value inequalities for $u^p$, when $p < 0$, where $u$ is a positive solution of (1.1). These inequalities will be necessary in the proof of Theorem A and in order to show them we need the following generalization of (C.2):
(C.3) \( \text{Let } -\infty < p < +\infty, p \neq 0, 1, u \text{ satisfy } 0 < m < u(x, t) < M < \infty \text{ in } R, w_2 \in A_2. \) \( \text{Then if } p > 1 \text{ and } u \text{ is a subsolution in } R, \) \( \text{or if } p < 0 \text{ and } u \text{ is a supersolution in } R, \)

\[
\text{ess sup}_{t \in (t_0 - \delta', t_0 + \beta')} \int_{Q'} u(x, t) p v(x) \, dx + \frac{p - 1}{p} \int_{Q'} \int_{R^2} |\nabla \chi u^{p/2}|^2 w_1(x) \, dx \, dt \leq c \int_{R} \int_{R} u^p \left( \frac{p}{p - 1} \left( \frac{w_2(x)}{(\alpha - \alpha')^2} + \frac{\nu(x)}{\beta - \beta'} \right) \right) \, dx \, dt.
\]

Moreover, if \( 0 < p < 1 \) and \( u \) is a supersolution in \( R \), then

\[
\text{ess sup}_{t \in (t_0 - \beta', t_0 + \beta')} \int_{Q'} u(x, t) p v(x) \, dx + \frac{p - 1}{p} \int_{Q'} \int_{R^2} |\nabla \chi u^{p/2}|^2 w_1 \, dx \, dt \leq c \int_{R} \int_{R} u^p \left( \frac{p}{p - 1} \left( \frac{w_2(x)}{(\alpha - \alpha')^2} + \frac{\nu(x)}{\beta - \beta'} \right) \right) \, dx \, dt.
\]

In this paper we do not present the proofs of (C.2) and (C.3) since their proofs are similar to the ones given in Section 2 of [GW2].

In Section 4, we prove

**Theorem E:** Let \( \nu \) and \( w_1 \) be weights such that there exists \( s > 1 \) with

\[
(1.18) \quad \left( \frac{r(I)}{r(B)} \right)^2 \left( \frac{1}{|I|} \int_{I} \left( \frac{\nu}{\nu(B)} \right)^s \, dx \right)^{1/s} \left( \frac{1}{|I|} \int_{I} \left( \frac{w_1}{w_1(B)} \right)^{-s} \, dx \right)^{1/s} \leq c
\]

for all \( \bar{b} \)-balls \( I, B \) with \( I \subset 2a^2B \) (as in (1.9)), where \( c \) is a constant independent of the balls. Let \( Q = Q(\xi, r) \) and \( \varphi \) be a \( C^1 \) function such that \( \varphi = 1 \) in \( Q(\xi, kr) \), \( 1/2 < k < 1, 0 \leq \varphi \leq 1, \text{ supp } \varphi \subset Q \) and

\[
\varphi(x) \varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y))
\]

for all \( x, y, t, t_0 \) with \( 0 \leq t \leq t_0 \). Then, if \( u \in \text{Lip}(Q) \),

\[
\int_{Q} |u(x) - A_Q|^2 \varphi(x) \nu(x) \, dx \leq c \frac{\nu(Q)}{w_1(Q)} r(Q)^2 \int_{Q} |\nabla \chi u(x)|^2 \varphi(x) w_1(x) \, dx,
\]

where \( A_Q = \frac{1}{\varphi(Q)} \int_{Q} u(x) \varphi(x) \, dx. \)
Finally, in Section 5, we prove Theorem A. This theorem follows as an application of Bombieri's lemma ([GW2]). In order to verify the hypotheses of Bombieri's lemma we need Theorem B and Theorem F, which we state next. We write

\[(\nu \times 1)(A) = \int \int_A \nu(x)dxdt,
\]

where \(\nu = \nu(x), x \in \mathbb{R}^n, \) and \(A \subset \mathbb{R}^{n+1} = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}.

**Theorem F:** Suppose \(\nu\) is a doubling weight, \(w_2 \in A_2, (1.18)\) holds and \(w_2^{-1} \in A_\infty(\nu)\). Let \(Q_R\) be a \(b\)-ball of radius \(R, t_0 \in (a, b)\) and \(\bar{\nu}_2 = w_2^{-1}v(Q_R)\) and \(\bar{\nu} = \nu v(Q_R)\). If \(u\) is a solution of (1.1) in \(Q_{3R} \times (a, b)\) which is bounded below by a positive constant, then there are constants \(c_1, M_2, x\) and \(V\) such that if for \(s > 0\) we define

\[
E^+ = \{(x, t) \in Q_R \times (t_0, b) : \log u < -s - M_2(b - t_0) - V\}
\]

\[
E^- = \{(x, t) \in Q_R \times (a, t_0) : \log u > s - M_2(a - t_0) - V\},
\]

then

\[
\left((\bar{\nu} + \bar{\nu}_2) \times 1\right)(E^+) \leq c_1 \left(\frac{1}{s} \frac{\nu(Q_R)}{w_1(Q_R)} \frac{R^2}{b - t_0}\right)^x (b - t_0)
\]

and

\[
\left((\bar{\nu} + \bar{\nu}_2) \times 1\right)(E^-) \leq c_1 \left(\frac{1}{s} \frac{V(Q_R)}{w_1(Q_R)} \frac{R^2}{t_0 - a}\right)^x (t_0 - a).
\]

Here \(c_1\) and \(x\) depend only on the constants in the conditions on \(\nu\) and \(w_2, M_2 = \frac{w_2(Q_R)}{R^2 v(Q_R)}, and V is a constant which depends on u.\)

In order to prove this theorem, if we follow the steps of Lemma 4.9 of [GW2], we just have to verify that a certain test function (see [FL1]) satisfies the conditions of Theorem E. This will be done in Lemma 5.4.

2. Interpolation Inequality

In this section we prove Theorem D. We start with

**Theorem 2.1.** Let \(w_1, w_2,\) and \(\mu\) be doubling weights and suppose (1.17) holds for \(w_1, w_2\) with any \(\mu,\) and for some \(q > 2.\) If \(Q = Q(x, r)\) and \(w_2 v^{-1} \in A_\infty(\nu)\) then there exist \(h > 1\) and a constant \(c > 0,\) independent of \(Q\) and \(u,\) such that
\[
\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx \\
\leq c \left( \frac{1}{\nu(Q)} \int_Q u^2 \nu dx \right)^{b-1} \left( \frac{r^2}{w_1(Q)} \int_{Q(\xi, r)} |\nabla \chi u|^{2} w_1 dx + (\nu_{\nu(Q)} |u|)^{2} \right)
\]

for all \( u \in \tilde{H}(a^2 Q) \), and \( a \) as in (1.9). Also if (1.17) is replaced by (1.16), then

\[
\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx \leq c \left( \frac{1}{\nu(Q)} \int_Q u^2 \nu dx \right)^{b-1} \left( \frac{r^2}{w_1(Q)} \int_Q |\nabla \chi u|^{2} w_1 dx \right)
\]

for all \( u \in \tilde{H}_0(Q) \).

**Proof:** The proof follows as in [GW1], Theorem 3; the only differences are that we obtain \( Q(\xi, a^2 r) \) in the second integral on the right when we apply Poincaré’s inequality and in the end we use the results of Calderón for weights in homogeneous spaces (see [C]).

**Corollary 2.2.** Let \( w_1, w_2 \) be doubling weights and suppose (1.17) holds with \( w_1, w_2, \mu = 1 \) and some \( q > 2 \). If \( w_2 \nu^{-1} \in A_{\infty}(\nu) \), then there exists \( h > 1 \) and a constant \( c > 0 \) such that

\[
\frac{1}{w_2(Q)} \int_Q |u|^{2h} w_2 dx \\
\leq c \left( \frac{1}{\nu(Q)} \int_Q u^2 \nu dx \right)^{b-1} \left( \frac{r^2}{w_1(Q)} \int_Q |\nabla \chi u|^{2} w_1 dx + \frac{1}{\nu(Q)} \int_Q u^2 \nu dx \right)
\]

for all \( u \in \tilde{H}(a^2 Q) \), \( Q = Q(\xi, r) \).

**Proof:** The conclusion of Theorem 2.1 holds for \( \mu = 1 \). But, by Schwarz’s inequality,

\[
\nu \chi |u| = \frac{1}{|Q|} \int_Q |u| \nu dx \\
= \frac{1}{|Q|} \int_Q u |\nu|^{1/2} |\nu|^{-1/2} dx \leq \frac{1}{|Q|} \left( \int_Q u^2 \nu dx \right)^{1/2} \left( \int_Q \nu \nu^{-1/2} dx \right)^1 \\
\leq \left( \frac{1}{\nu(Q)} \int_Q u^2 \nu dx \right)^{1/2},
\]
where in the last inequality we used the fact that \( v \in A_2 \).

In the next section we prove mean value inequalities. In order to be able to iterate a certain inequality as was done in [GW2] we need a refinement of the above corollary. This refinement is Theorem D and to prove it we need the following lemmas.

**Lemma 2.3.** Given \( Q = Q(\xi, s) \) and \( 0 < r < s \), there exists \( x_1, \ldots, x_{m(r,s)} \) in \( Q \), and \( k \geq 1 \) independent of \( \xi, r, s \), such that

(i) \( Q(x_j, r/k) \cap Q(x_h, r/k) = \emptyset, h \neq j \)

(ii) \( Q(\xi, s) \subset \bigcup_{j=1}^{m(r,s)} Q(x_j, r) \).

Moreover, \( m(r, s) \leq c \left( \frac{s}{r} \right)^{\nu'} \) for some constant \( \nu' \) depending only on the dimension.

**Proof:** If we apply Theorem 1.2, page 69, of [CoW] to the open covering of \( Q \) given by \( S(x, r/4a)_{x \in Q} \), there exist \( x_1, \ldots, x_{m(r,s)} \) in \( Q \) such that:

\( S(x_h, r/4a) \cap S(x_j, r/4a) = \emptyset \) if \( j \neq h \) and \( Q(\xi, s) \subset \bigcup_{j=1}^{m(r,s)} S(x_j, r/4a) \).

By (1.9), \( S(x_j, r/4a) \supset Q(x_j, r/4a^2) \) and \( S(x_j, r/4a) \subset Q(x_j, r) \). Therefore, if we choose \( k = 4a^2 \), (i) and (ii) follow. It remains to find an upper bound for \( m(r, s) \). First, we note that \( Q(x_j, r/k) \subset Q(\xi, a^2(k + 1)s/k) \). But

\[
\frac{r}{k} = \frac{2a^4(k + 1)s}{k} \frac{r}{2a^4(k + 1)s},
\]

and so by (1.10), there exists \( \nu' > 0 \), such that

\[
\left| Q\left( x_j, \frac{r}{k} \right) \right| \geq \left( \frac{r}{2a^4(k + 1)s} \right)^{\nu'} \left| Q\left( x_j, \frac{2a^4(k + 1)s}{k} \right) \right|,
\]

and since the \( Q(x_j, r/k) \) are disjoint,
\[ \left| Q\left( \xi, \frac{a^2(k+1)s}{k} \right) \right| \geq \sum_{j} \left| Q\left( x_j, \frac{r}{k} \right) \right| \geq c \left( \frac{r}{s} \right)^{\nu} \sum_{j} \left| Q\left( x_j, \frac{2a^2(k+1)s}{k} \right) \right|. \]

But,
\[ Q\left( x_j, \frac{2a^4(k+1)s}{k} \right) \supset Q\left( \xi, \frac{a^2(k+1)s}{k} \right) \]
and so
\[ \left| Q\left( \xi, \frac{a^2(k+1)s}{k} \right) \right| \geq c \left( \frac{r}{s} \right)^{\nu} m(r, s) \left| Q\left( \xi, \frac{a^2(k+1)s}{k} \right) \right|. \]

Therefore, \( m(r, s) \leq c (s/r)^{\nu}. \)

\textbf{Lemma 2.4.} \ If \( \delta(y, z) < s \) then \( F(y^*, s) \leq (2a^3)Q F(y^*, s), G_j \) as in (1.10).

\textbf{Proof:} \ Since \( Q(x, s) \subset Q(y, 2a^3s), F(y^*, s) \leq F(y, 2a^3s) \). By (1.10), it follows that
\[ F(y^*, s) \leq F(y^*, 2a^3s) \leq (2a^3)^G F(y^*, s). \]

\textbf{Lemma 2.5.} \ If \( 0 < \epsilon < 1 \) and \( \eta \in Q = Q(\xi, s) \), then \( Q(\eta, \epsilon s/(2a^3)^I) \subset Q(\xi, (1 + \epsilon)s), \) where \( \xi = \max_{j=1}^{n} G_j. \)

\textbf{Proof:} \ If \( y \in Q(\eta, \epsilon s/(2a^3)^I) \) then by (1.8), \( |y - \eta| \leq F(y^*, \epsilon s/(2a^3)^I) \) and by (1.10) and Lemma 2.4
\[ F(y, \frac{\epsilon s}{(2a^3)^I}) \leq \frac{\epsilon s}{(2a^3)^I} F(y^*, s) \leq \epsilon F(y^*, s). \]

Therefore
\[ |y - \xi| \leq |y - \eta| + |\eta - \xi| \leq \epsilon F(y^*, s) + F(y^*, s) \]
\[ = (1 + \epsilon) F(y^*, s) \]
\[ \leq F(y^*, (1 + \epsilon)s), \]
where in the last inequality we used (1.10).

\textbf{Proof of Theorem D.}

Let \( Q = Q(\xi, s) \). By Lemma 2.5, \( \delta(Q, \partial(1 + \epsilon)Q) \geq \epsilon s/(2a^3)^I \). Apply Lemma 2.3 to \( r = \frac{\epsilon s}{(2a^3)^I} \) to find \( x_1, \ldots, x_{m(r, s)} \in Q \) such that:
\[ Q(x_j, r/k) \cap Q(x_h, r/k) = \emptyset \text{ if } j \neq h, \]
\( Q(x_j, s) \subset \bigcup_{j=1}^{m(r, s)} Q(x_j, r) \) and \( m(r, s) \leq c(s/r)^{\nu}. \)
Note that, by (2.5), $Q(x_j, a^2r) = Q\left(x_j, \frac{\epsilon s}{(2a^2)^\frac{1}{2}}\right) \subset Q(\xi, (1 + \epsilon)s) = (1 + \epsilon)Q$.

Then using Corollary 2.2, doubling for $w_2$, doubling for $\nu$ and $w_1$ and the fact that $Q(x_j, 2a^2s) \supset Q(\xi, s)$ and $Q(\xi, 2a^2s) \supset Q(x_j, s)$,

$$
\int_Q |u|^{2^h} w_2 dx \leq \sum_{j=1}^{m(r, s)} \int_{Q(\xi, r)} |u|^{2^h} w_2 dx
$$

$$
\leq c \sum_{j=1}^{m(r, s)} w_2(Q(x_j, r)) \left( \frac{1}{\nu(Q(x_j, r))} \int_{Q(x_j, r)} u^2 \nu dx \right)^{h-1}
$$

$$
\cdot \left\{ \frac{r^2}{w_1(Q(x_j, r))} \int_{Q(x_j, r)} |\nabla u| \left|^{2^{h+1}} \right| w_1 dx + \frac{1}{\nu(Q(x_j, r))} \int_{Q(x_j, r)} u^2 \nu dx \right\}
$$

$$
\leq c \left( \frac{s}{r} \right)^{h-1} w_2(Q(\xi, s)) \left[ \left( \frac{r}{2\alpha^2s} \right)^{-\alpha} \frac{1}{\nu(Q(\xi, s))} \int_{(1 + \epsilon)Q} u^2 \nu dx \right]^{h-1}
$$

$$
\cdot \left\{ \frac{s^2}{w_1(Q(\xi, s))} \left( \frac{r}{2\alpha^2s} \right)^{-\alpha} \int_{(1 + \epsilon)Q} |\nabla u| \left|^{2^{h+1}} \right| w_1 dx
$$

$$
+ \left( \frac{r}{2\alpha^2s} \right)^{-\alpha} \frac{1}{\nu(Q(\xi, s))} \int_{(1 + \epsilon)Q} u^2 \nu dx \right\}.
$$

The theorem follows if we choose $b = \nu + 2\alpha$, since $s/r = c\epsilon^{-1}$.

3. **Mean value inequalities.**

In this section we prove Theorem B and some other mean value inequalities. Since the proofs are similar to the ones given by [GW2], we just point out the differences. Basically, we have to be a little more careful in the iteration argument since there is a factor $\epsilon$ in Theorem D.

We assume throughout this section that:

- (a) $w_1, w_2, \nu \in A_2$
- (b) Poincaré’s inequality, (1.17), holds for both of the pairs $w_1, w_2$ and $w_1, \nu$ with some $q > 2$ and $\mu = 1$
- (c) $w_2 \nu^{-1} \in A_w(\nu)$.

Denote $R_{r, s} = Q(x_0, r) \times (t_0 - s, t_0 + s)$ and let $R = R_{r, s}$, $R' = R_{\alpha, \sigma}$ with $r/2 < \sigma < r$ and $s/2 < \alpha < s$ and define
\[(3.2) \quad C = c \frac{r^{2+b \delta}}{(r - \theta)^{2+b}(s - \alpha)} \]

where $b$ is given by Theorem D and $c$ is a constant that may vary, but which only depends on the weights and on $h$, where $h > 1$ is the index for which Theorem D holds for both $w_2$ and $\nu$ on the left hand side.

We also write $\lambda(Q) = w_1(Q)/\nu(Q)$ and $\Lambda(Q) = w_2(Q)/\nu(Q)$. We start this section with the proof of (C.1). This estimate will be important in deducing a mean value inequality for subsolutions of (1.1).

**Proof of (C.1):** If $u \in H$ define

\[\varphi(x, t) = \eta^2(x, t) \left[ \int_{t_0}^{u(x, t)} H_M(s)^2 ds + u(x, t)H_M(u(x, t)) \right] \chi(t, \tau_1, \tau_2),\]

where $\eta \in C_0^\infty(R)$ will be specified later, $t_0 - s < t < \tau_2 < t_0 + s$ and $\chi(t, \tau_1, \tau_2)$ denotes the characteristic function of $(\tau_1, \tau_2)$. The fact that the function $\varphi$ is in $H_0$ follows as a consequence of the following result: if $f$ is a piecewise smooth function on the real line with $f' \in L^\infty(-\infty, \infty)$ and if $u \in H$, then $f \circ u \in H$. Here we use the convention that $f'(u) = 0$ if $u \in L$ where $L$ denotes the set of corner points of $f$ (see the proof follows the steps of Theorem 7.8 of [GT] and it also shows that $\nabla \lambda(f \circ u) = f'(u) \nabla \lambda u$ and $(f(u))_x = f'(u)(u)_x$). The proof of the above fact also verifies that in our case $\varphi \geq 0$ in the $H_0$-sense since $H_M(s) = 0$ for $s < 0$.

Since $u$ is a subsolution, we have

\[(3.3) \quad \int_0^s \int_{\mathbb{R}} (\lambda \nabla u \cdot \nabla \varphi + u \lambda \varphi) dxdt \leq 0.\]

Note that by another limiting argument

\[u_\eta \left[ \int_0^\infty H_M(s)^2 ds \right] = \left[ u \eta^2 \int_0^\infty H_M(s)^2 ds \right] - u(\eta^2) \left[ \int_0^\infty H_M(s)^2 ds - \eta^2 H_M(u)^2 u_x \right],\]

and then by definition of $\varphi$, for $\tau_1 < t < \tau_2$,

\[u \varphi = \left[ u \int_0^\infty H_M(s)^2 ds \right] - (\eta^2)u \left[ \int_0^\infty H_M(s)^2 ds \right],\]

and
\[ \nabla \varphi = 2\eta \nabla \eta \left[ \int_0^u H_M(s)^2 \, ds + uH_M(u)^2 \right] + \eta^2 [H_M(u)^2 \nabla u + f_M(u) \nabla u], \]

where \( f_M(s) = sH_M(s)^2 \) (note that \( \nabla (f_M(u)) = f_M(u) \nabla u \), since \( f_M \) is piecewise smooth with \( f_M \in L^\infty \)). If we substitute the two last equations in (3.3) we get, with \( Q = Q(x_0, r) \),

\[
\int_Q \int_{r_1}^{r_2} \left[ \eta^2 \int_0^u H_M(s)^2 \, ds \right] \nabla u \, dx \, dt + \int_Q \int_{r_1}^{r_2} \eta^2 H_M(u)^2 (A \nabla u, \nabla u) \, dx \, dt
\]

\[ \leq \int_Q \int_{r_1}^{r_2} (\nabla^2) H_M(s)^2 \, ds \nabla u \, dx \, dt \]

\[ - 2 \int_Q \int_{r_1}^{r_2} \eta (A \nabla u, \nabla \eta) \left[ \int_0^u H_M(s)^2 \, ds + uH_M(u)^2 \right] du \, dx \, dt \]

\[ - \frac{1}{r} \eta^2 (A \nabla u, \nabla u) f_M(u) \, dx \, dt. \]

We can drop the last term on the right since the integrand is non-negative. The second term on the right is majorized in absolute value by

\[ 4 \int_Q \int_{r_1}^{r_2} |(A \nabla u, \nabla \eta)| \eta H_M(u)^2 \, u \, dx \, dt \]

\[ = 4 \int_Q \int_{r_1}^{r_2} |AH_M(u) \nabla u, uH_M(u) \nabla \eta| \, dx \, dt \]

\[ \leq 2\epsilon \int_Q \int_{r_1}^{r_2} (A \nabla (H_M(u)), \nabla (H_M(u))) \eta^2 \, dx \, dt \]

\[ + \frac{2}{\epsilon} \int_Q \int_{r_1}^{r_2} (A \nabla \eta, \nabla \eta) (H_M(u)^2 \, dx \, dt \]

where we used the fact that \( |(Ax, y)| \leq \langle Ax, x \rangle^{1/2} \langle Ay, y \rangle^{1/2} \leq \frac{\epsilon}{2} \langle Ax, x \rangle + \frac{1}{2\epsilon} \langle Ay, y \rangle \). If we pick \( \epsilon = \frac{1}{4} \), we get
\[ (3.4) \int_Q \int_0^{\tau_1} \int_0^{\tau_2} \eta^2 \left( \int_0^u H_M(s)^2 \, ds \right) \, v \, dx \, dt \]

\[ + \frac{1}{2} \int_Q \int_0^{\tau_1} \eta^2 \langle A \nabla (H_M(u)), \nabla (H_M(u)) \rangle \, dx \, dt \]

\[ \leq \delta \int_Q \int_0^{\tau_2} \langle A \nabla \eta, \nabla \eta \rangle u^2 H_M(u)^2 \, dx \, dt + \int_Q \int_0^{\tau_1} \left( \langle \nabla \eta \rangle u \right) \int_0^u H_M(s)^2 \, ds \, v \, dx \, dt. \]

Choose \( \eta \) to be zero in a neighborhood of \( \partial Q \times (t_0 - s, t_0 + s) \cup |Q| \times (t = t_0 - s), \eta \equiv 1 \) in \( R^+ \), \( 0 \leq \eta \leq 1 \), \( |\nabla \eta| \leq c/(r - q) \), \( |\eta_{x_i}| \leq c/(s - \sigma) \) (see page 537 of [FL1]). If we pick \( \tau_1 \) so close to \( t_0 - s \) that \( \eta(x, \tau_1) = 0 \) for all \( x \in Q \), drop the second term on the left of (3.4) (which is non-negative) and use Lemma 5 of [AS] it follows that

\[ (3.5) \quad \text{ess sup}_{t_0 - \sigma \leq t \leq t_0 + \sigma} \int_Q \int_0^{\tau_2} u(x, \tau_2) \int_0^{w(x, \tau_2)} H_M(s)^2 \, ds \, v \, dx \, dt \]

\[ \leq c \int_R \int_{R^+} u^2 H_M(u)^2 \left[ \frac{w_2}{(r - q)^2 + \frac{v}{s - \sigma}} \right] \, dx \, dt. \]

If we fix \( \tau_2 \in (t_0 - \sigma, t_0 + \sigma) \) and \( \tau_1 \) as before and if we drop the first term on the left of (3.4) (which we can see is non-negative after performing the integration) we obtain

\[ (3.6) \quad \int_Q \int_0^{\tau_1} \eta^2 \langle A \nabla (H_M(u)), \nabla (H_M(u)) \rangle \, dx \, dt \]

\[ \leq c \int_R \int_{R^+} u^2 H_M(u)^2 \left[ \frac{w_2}{(r - q)^2 + \frac{v}{s - \sigma}} \right] \, dx \, dt. \]

Letting \( \tau_2 \to t_0 + s \) and using (1.2) we get

\[ (3.7) \quad \int_R \int \left| \nabla (H_M(u)) \right|^2 w_1 \, dx \, dt \leq c \int_R \int u^2 H_M(u)^2 \left[ \frac{w_2}{(r - q)^2 + \frac{v}{s - \sigma}} \right] \, dx \, dt. \]

Finally note that
\[ H_M(u)^2 = \int_0^u (H_M(s)^2)' ds = \int_0^u 2H_M(s)H_M'(s) ds \]
\[ \leq 2 \int_0^u sH_M(s)^2 ds \leq 2u \int_0^u H_M(s)^2 ds, \]

since \( H_M(s) \leq sH_M(s) \). Combining this with (3.5) and (3.7), (C.1) follows with \( \alpha, \beta, \alpha', \beta' \) taken there to be \( r, s, q, \sigma \).

**Lemma 3.8.** Let \( p \geq 2, R, R' \) be as defined above and assume (3.1) holds. If \( u \) is a subsolution of (1.1) in \( R \), then \( u_+ \) is bounded in \( R_+ = Q(x_0, \rho) \times (t_0 - \sigma, t_0 + s) \) and

\[ \text{ess sup}_{R_+} u_+^p \]
\[ \leq (p^2C)^{\frac{h}{h-1}} \left( 1 + r^2 \frac{1}{s} \frac{1}{\lambda(Q)} \right)^{\frac{1}{h-1}} \left( 1 + \frac{s}{r^2} \frac{\lambda(Q)}{\Lambda(Q)} \right)^{\frac{h}{h-1}} \int_R \int_R u_+^p \left( \frac{\rho}{r^2} \frac{w_2 + \nu}{\tau} \right) dx dt, \]

with \( C \) as in (3.2).

**PROOF:** \( H_M(u) \) is a function in \( H \) since \( u \in H \) and \( H_M \) is a \( C^1 \) function with bounded derivative. Then by Fubini’s theorem we have that \( H_M(u(\cdot, \tau)) \in \mathcal{A} \) for a.e. \( \tau \in (t_0 - \sigma, t_0 + s) \). If we apply Theorem D to the function \( F(x) = H_M(u(x, \tau)), Q = Q_0 \) and \( \epsilon > 0 \) such that \( (1 + \epsilon)Q < r \) and combine this with (C.1) we obtain

\[ \frac{1}{w_2(Q_0)} \int_{Q_0} H_M(u(x, \tau))^{2\kappa} w_2(x) dx \]
\[ \leq c \epsilon^{-\beta} \left\{ \frac{1}{\nu(Q_0)} \int_R u^2 H_M(u)^2 \left( \frac{w_2}{(r - (1 + \epsilon)Q)^2} + \frac{\nu}{s - \sigma} \right) dx dt \right\}^{\frac{h}{h-1}} \]
\[ \cdot \left\{ \frac{\rho^2}{w_1(Q_0)} \int_{F_{1+\kappa}Q_0} \| \nabla (H_M(u(x, \tau))) \|^2 w_1(x) dx \right\} \]
\[ + \frac{1}{\nu(Q_0)} \int_R u^2 H_M(u)^2 \left( \frac{w_2}{(r - (1 + \epsilon)Q)^2} + \frac{\nu}{s - \sigma} \right) dx dt \]

for a.e. \( \tau \in (t_0 - \sigma, t_0 + s) \).
Integrate with respect to \( t \) over \((t_0 - \sigma, t_0 + s)\) and apply (C.1) to get

\[
\frac{1}{w_2(Q_r)} \int_{R} H_M(u(x, t))^{2k} w_2(x)\,dx\,dt
\]

\[
\leq c \varepsilon^{-b} \left( \frac{Q_2}{w_1(Q_r)} + \frac{s + \sigma}{\nu(Q_r)} \right) \left( \int_{R} u^2 H_M(u)^{2k} \left( \frac{w_2}{(r-(1+\varepsilon)Q)^2} + \frac{\nu}{s-\sigma} \right) \,dx\,dt \right)^k.
\]

Since \((r/2) < Q < r\) and \((s/2) < \sigma < s\), by the doubling property of the weights and the definitions of \( \lambda \) and \( \Lambda \), it follows that

\[
\frac{1}{w_2(Q_r)} \int_{R} H_M(u(x, t))^{2k} w_2(x)\,dx\,dt
\]

\[
\leq c \varepsilon^{-b} \left( \frac{Q_2}{\lambda(Q_r)} + s \right) \left( \int_{R} u^2 H_M(u)^{2k} \left( \frac{w_2}{(r-(1+\varepsilon)Q)^2} + \frac{\nu}{s-\sigma} \right) \,dx\,dt \right)^k.
\]

A similar inequality holds with \( w_2 \) replaced by \( \nu \) on the left, and if we add the two inequalities, we obtain

\[
(3.9) \int_{R} H_M(u)^{2k} \left( \frac{w_2}{w_2(Q_r)} + \frac{\nu}{\nu(Q_r)} \right) \,dx\,dt
\]

\[
\leq c \varepsilon^{-b} \left( \frac{Q_2}{\lambda(Q_r)} + s \right) \left( \int_{R} u^2 H_M(u)^{2k} \left( \frac{w_2}{(r-(1+\varepsilon)Q)^2} + \frac{\nu}{s-\sigma} \right) \,dx\,dt \right)^k
\]

for any \( \varepsilon \) such that \((1 + \varepsilon)Q < r\).

Now note that

\[
\frac{w_2}{(r-(1+\varepsilon)Q)^2} + \frac{\nu}{s-\sigma} \leq \frac{r^2}{(r-(1+\varepsilon)Q)^2(s-\sigma)} \left( \frac{s}{r^2} w_2 + \nu \right),
\]

\[
\int_{R} \left( \frac{w_2}{w_2(Q_r)} + \frac{\nu}{\nu(Q_r)} \right) \,dx\,dt = s,
\]

\[
\int_{R} \left( \frac{s}{r^2} w_2 + \nu \right) \,dx\,dt = s \left( \frac{s}{r^2} w_2(Q_r) + \nu(Q_r) \right) = s \nu(Q_r) \left( \frac{s}{r^2} \Lambda(Q_r) + 1 \right),
\]

\[
\frac{sr^{-2} w_2(x) + \nu(x)}{sr^{-2} w_2(Q_r) + \nu(Q_r)} \leq \frac{w_2(x)}{w_2(Q_r)} + \frac{\nu(x)}{\nu(Q_r)}.
\]
Thus, by raising both sides of (3.9) to the power $1/h$, normalizing and using the fact that $e^{-b/h} \leq e^{-b}$, we obtain

$$
(3.10) \quad \left( \int \int_{R^2} H_M(h)^{2h} \left( \frac{S}{r^2} w_2 + \nu \right) dx dt \right)^{1/h} \\
\leq c e^{-b} \frac{r^2 S}{(r - (1 + \epsilon)q)^2(s - \sigma)} \left( 1 + \frac{S}{r^2} \lambda(Q_r) \right)^{1/h} \\
\cdot \int \int_{R^2} u_H^2 H_M(u)^{2h} \left( \frac{S}{r^2} w_2 + \nu \right) dx dt
$$

for any $\epsilon$ such that $(1 + \epsilon)q < r$. Since $u_M^{p/2} \chi_{[0 < u < M]} \leq H_M(u)$ and $uH_M(u) \leq pu^{p/2}/2$, if we let $M \to \infty$ it follows by Fatou’s lemma that

$$
(3.11) \quad \left( \int \int_{R^2} u^p_{\nu} \left( \frac{S}{r^2} w_2 + \nu \right) dx dt \right)^{1/h} \\
\leq c p^2 e^{-b} \frac{r^2 S}{(r - (1 + \epsilon)q)^2(s - \sigma)} \left( 1 + \frac{S}{r^2} \lambda(Q_r) \right)^{1/h} \\
\cdot \int \int_{R^2} u^p_{\nu} \left( \frac{S}{r^2} w_2 + \nu \right) dx dt.
$$

Now, we have to iterate (3.11). Fix $r$, $s$, $q$, $\sigma$ with $r/2 < q < r$ and $s/2 < \sigma < s$. For $k = 1, 2, \ldots$ define sequences $\{s_k\}_{k \in \mathbb{N}}$ and $\{|r_k|\}_{k \in \mathbb{N}}$ and $\{\epsilon_k\}_{k \in \mathbb{N}}$ by $s_1 = s$, $s_k - s_{k+1} = \frac{s - \sigma}{2^k}$ for $k \geq 1$, $r_1 = r$, $r_k - r_{k+1} = (r - q)/2^k$ for $k \geq 1$, and $\epsilon_k = \frac{r - q}{2^k r_k} = \frac{r_k - r_{k+1}}{r_k}$ for $k \geq 1$. Also, define $R_k = Q_k \times (t_0 - s_k, t_0 + s)$ for $k \geq 1$, where $Q_k = Q(x, r_k)$. Note that $R_1 = R$ and $\bigcap_{k=1}^{\infty} R_k = R$. Since

$$
\frac{1}{2} sr^{-2} \leq s_k r_k^{-2} \leq 4sr^{-2},
$$

if we apply (3.11) with $p$ replaced by $ph^{k-1}$, $p \geq 2$, and $r = r_k$, $q = r_{k+1}$ and $\epsilon = \epsilon_{k+1}$ (note that $(1 + \epsilon_{k+1})r_{k+1} < r_k$), we obtain
\[
\left( \int_{R_{k+1}} \; u^\pm \left( \frac{s}{r^2} w_2 + v \right) \, dx \, dt \right)^{1/h'} \\
\leq \left\{ c(p h^{k-1})^2 \; \varepsilon_{k+1}^{-b} \; \frac{r^2 s_k}{(r_k - (1 + \varepsilon_{k+1}) r_k + 1)^2 (s_k - s_{k+1})} \; \left( 1 + \frac{s}{r^2} \; \Lambda(Q_r) \right) \right\}^{1/(2^{k+1})} \cdot \left\{ \int_{R_k} \; u^{\pm h^{-1}} \left( \frac{s}{r^2} w_2 + v \right) \, dx \, dt \right\}^{1/(2^{k+1})}.
\]

But note that
\[
\varepsilon_{k+1}^{-b} \; \frac{r^2 s_k}{(r_k - (1 + \varepsilon_{k+1}) r_k + 1)^2 (s_k - s_{k+1})}
= 2^{(k+1)b} \; \frac{r_{k+1}^b}{(r - q)^b} \; \frac{r^2 s_k}{2^k} \; \left( \frac{r - q}{2^k} \right)^2 \; \left( \frac{s - q}{2^k} \right)
\leq c 2^{(3+b)k} \; \frac{r^2 + b s}{(r - q)^{2+b}(s - q)}
\leq C 2^{(3+b)k},
\]

where \( C \) is given by (3.2). Thus,

\[(3.12) \quad \left( \int_{R_{k+1}} \; u^\pm \left( \frac{s}{r^2} w_2 + v \right) \, dx \, dt \right)^{1/h'} \leq \left\{ c(p h^{k-1})^2 2^{(3+b)k} \left( 1 + \frac{s}{r^2} \; \Lambda(Q_r) \right) \left( 1 + \frac{r^2}{s} \; \frac{1}{\Lambda(Q_r)} \right)^{1/h'} \right\}^{1/(2^{k+1})} \cdot \left\{ \int_{R_k} \; u^{\pm h^{-1}} \left( \frac{s}{r^2} w_2 + v \right) \, dx \, dt \right\}^{1/(2^{k+1})}.
\]

If we iterate (3.12), we obtain
ess sup_{R^+} u^p_+

\leq \prod_{k=1}^{\infty} \left( C(p h^{k-1})^2 2^{(3+b)k} \left( 1 + \frac{s}{r^2} \Lambda(Q_r) \right) \cdot \left( 1 + \frac{r^2}{s} \frac{1}{\lambda(Q_r)} \right)^{1/k} \right)^{1/h^{k-1}} \int_R^+ u^p_+ \left( \frac{s}{r^2} w_2 + \nu \right) dx dt.

Since \sum_{k=1}^{\infty} \frac{1}{h^{k-1}} = \frac{h}{h-1} and \sum_{k=1}^{\infty} \frac{k}{h^{k-1}} = \left( \frac{h}{h-1} \right)^2 , it follows that

ess sup_{R^+} u^p_+

\leq (p^2 C)^{\frac{h}{h-1}} \left( 1 + \frac{s}{r^2} \Lambda(Q_r) \right)^{\frac{h}{h-1}} \left( 1 + \frac{r^2}{s} \frac{1}{\lambda(Q_r)} \right)^{\frac{1}{h-1}} \int_R^+ u^p_+ \left( \frac{s}{r^2} w_2 + \nu \right) dx dt,

and this proves the lemma. Note that if we apply the above result for p = 2, it follows that \( u^+ \) is bounded on \( R^+ \).

PROOF OF THEOREM B: By Lemma 3.8 we know that \( u^+ \) is bounded in \( Q_{(1+\epsilon)^2} \times (t_0 - \sigma, t_0 + s) \) for all \( \epsilon \) such that \((1 + \epsilon)Q < r \). If we define \( F(x) = u^p_+ \frac{\partial}{\partial x} (x, \tau) \) then \( F \in \tilde{H}(Q_{(1+\epsilon)^2}) \) for a.e. \( \tau \in (t_0 - \sigma, t_0 + s) \) and if we follow the proof of Lemma 3.8 using (C.2) instead of (C.1), we get (see the comments in the introduction)

ess sup_{R^+} u^p_+

\leq C^{\frac{h}{h-1}} \left( 1 + \frac{r^2}{s} \frac{1}{\lambda(Q)} \right)^{\frac{h}{h-1}} \left( 1 + \frac{s}{r^2} \Lambda(Q) \right)^{\frac{h}{h-1}} \int_R^+ u^p_+ \left( \frac{s}{r^2} w_2 + \nu \right) dx dt

for \( p \geq 2 \). For \( 0 < p < 2 \), define \( I_p \) and \( I_\omega \) as in Lemma 3.4 of [GW2]. The only difference in our case is that

\[ I_\omega(\alpha', \beta')^2 \leq c \left[ \frac{1}{(\alpha - \alpha')^2 + (\beta - \beta')^2} \right]^{\frac{h}{h-1}} I_2(\alpha, \beta)^2 \]
if $1/2 < \alpha' < \alpha < 1$ and $1/2 < \beta' < \beta < 1$. Thus, arguing as in Lemma 3.4 of [GW2] we prove that if $u$ is a solution of (1.1) and $p > 0$ then

\[(3.13) \quad \text{ess sup}_R: u^p \leq D \left(1 + \frac{r^2}{s} \frac{1}{\Lambda(Q)}\right)^{\frac{h-1}{h}} \left(1 + \frac{s}{r^2} \Lambda(Q)\right)^{\frac{h-1}{h}} \int_R u^p \left(\frac{s}{r^2} w_2 + v\right) dxdt,
\]

where $D$ is as in Theorem B.

If we apply (3.13) to both $u$ and $-u$, we obtain Theorem B of the introduction, with $\alpha, \beta, \alpha', \beta'$ taken there to be $r, s, q, \sigma$.

In order to prove Harnack’s inequality we need a mean value inequality for $u^p$ when $-\infty < p < \infty$ and $u$ is a non-negative solution.

We begin by noting that if we use (C.3) instead of (C.1) we can prove the following analogue of (3.11):

**Lemma 3.14.** Suppose (3.1) holds, $0 < m < u(x, t) \leq M < \infty$ in $R = R_{r, s}$, $r/2 < q < r$, $s/2 < \sigma < s$ and $\epsilon > 0$, $(1 + \epsilon)q < r$. Then, if $p > 1$ and $u$ is a sub/solution in $R$, or if $p < 0$ and $u$ is a supersolution in $R$,

\[
\left(\int_{R^+} \int_R u^{ph'} \left(\frac{w_2}{w_2(Q)} + \frac{v}{\nu(Q)}\right) dxdt\right)^{1/h} \leq c e^{-b} \frac{r^2 s}{(r-(1+\epsilon)q)^2(s-a)} \left(1 + \frac{p}{p-1} \frac{s}{r^2} \Lambda(Q)\right)^{\frac{h-1}{h}} \left(1 + \frac{p}{p-1} \frac{r^2}{s} \frac{1}{\lambda(Q)}\right)^{\frac{h-1}{h}} \int_R u^p \left(\frac{p}{p-1} \frac{s}{r^2} w_2 + v\right) dxdt.
\]

Moreover, if $0 < p < 1$ and $u$ is a supersolution in $R$, then

\[
\left(\int_{R^+} \int_R u^{ph} \left(\frac{w_1}{w_2(Q)} + \frac{v}{\nu(Q)}\right) dxdt\right)^{1/h} \leq c e^{-b} \frac{r^2 s}{(r-(1+\epsilon)q)^2(s-a)} \left(1 + \frac{p}{|p-1|} \frac{s}{r^2} \Lambda(Q)\right)^{\frac{h-1}{h}} \left(1 + \frac{p}{|p-1|} \frac{r^2}{s} \frac{1}{\lambda(Q)}\right)^{\frac{h-1}{h}} \int_R u^p \left(\frac{p}{|p-1|} \frac{s}{r^2} w_2 + v\right) dxdt.
\]
Both inequalities are still true if we replace the integral averages on the right by the larger integral average

\[ \int_R \int_{Q_r} u^p \left( \frac{w_2}{w_2(Q_r)} + \frac{\nu}{\nu(Q_r)} \right) \, dx \, dt. \]

**Theorem 3.15.** Assume (3.1) holds, \( r, s > 0, r/2 < q < r, s/2 < \sigma < s \). If \( u \) is a non negative solution of (1.1) in \( R \), then for \( p > 0 \)

\[ \text{ess sup}_R: u^p \leq C \left( 1 + |p| \frac{s}{r^2} \Lambda(Q_1) \right)^{\frac{h}{h-1}} \left( 1 + p \frac{r^2}{s} \frac{1}{\lambda(Q_1)} \right) \int_R \int_{Q_r} u^p \left( \frac{w_2}{w_2(Q_r)} + \frac{\nu}{\nu(Q_r)} \right) \, dx \, dt, \]

and for \( p < 0 \)

\[ \text{ess sup}_R: u^p \leq C \left( 1 + |p| \frac{s}{r^2} \Lambda(Q_1) \right)^{\frac{h}{h-1}} \cdot \left( 1 + |p| \frac{r^2}{s} \frac{1}{\lambda(Q_1)} \right)^{\frac{h}{h-1}} \int_R \int_{Q_r} u^p \left( \frac{w_2}{w_2(Q_r)} + \frac{\nu}{\nu(Q_r)} \right) \, dx \, dt, \]

where \( C \) is given by (3.2).

**Proof:** In Lemma 3.17 of [GW2] we replace (3.20) by the result given here in Lemma 3.14 and then argue as in Lemma 3.17 of [GW2].

### 4. Proof of Theorem E

We start with the following lemma.

**Lemma 4.1.** Suppose \( Q = Q(\xi, r) \) and \( \varphi \) is a \( C^1 \) function such that \( \varphi = 1 \) in \( kQ = Q(\xi, kr), 0 < k < 1, 0 \leq \varphi \leq 1 \), \( \text{supp} \varphi \subset Q \) and

\[ \varphi(x) \varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y)) \quad (4.2) \]

for all \( x, y, t, t_0 \) with \( 0 \leq t \leq t_0 \). If \( u \) is a Lipschitz function,
\[ E = \{ x \in Q(\xi, kr) : u(x) = 0 \} \text{ and } |E| \geq \beta|Q| \text{ for some } 0 < \beta < 1, \text{ then if } x \in Q, \]
\[
|u(x)| \sqrt{\varphi(x)} \leq c \int_Q |\nabla_x u(z)| \sqrt{\varphi(z)} \frac{\delta(x, z)}{|Q(x, \delta(x, z))|} dz,
\]

where \(c\) is independent of \(Q, u, x\).

**Proof:** (The general outline of this proof follows the steps of the proof of Lemma 4.3 in [FS].) If \(x \in Q = Q(\xi, r)\) then \(Q(\xi, 2r) \subset Q(x, 2a^2 r)\) and \(Q(x, r) \subset Q(\xi, 2a^2 r)\). Therefore, by doubling, \(|Q(x, r)| \approx |Q|\). Now, we note that there exists \(\sigma \in [-1, 1]^n\) such that \(|E \cap Q^\sigma(x, 2a^2 r)| \geq c_\beta |Q^\sigma(x, 2a^2 r)|\).

In fact, \(E = \bigcup_{\sigma} (Q^\sigma(x, 2a^2 r) \cap E)\) and so there exists \(\sigma\) such that
\[
|Q^\sigma(x, 2a^2 r) \cap E| \geq \beta 2^{-n} |Q| \geq c \beta 2^{-n} |Q^\sigma(x, 2a^2 r)|. \tag{4.4}
\]

We also claim that there exist \(\alpha_j, \epsilon \in \mathbb{R}^n\), independent of \(x, r, 0 < \epsilon < \alpha_j, j = 1, \ldots, n\), such that
\[
|E \cap Q^\sigma(x, 2a^2 r) \cap H(2a^2 r, x, \Delta^\sigma(\alpha))| \geq \frac{c_\beta}{2} |Q^\sigma(x, 2a^2 r)|. \tag{4.5}
\]

To prove this fact, apply (1.14) to \(\gamma = \frac{c_\beta}{2}\) and find \(\alpha_j, \epsilon \in \mathbb{R}^n, 0 < \epsilon < \alpha_j, j = 1, \ldots, n\), such that
\[
|H(2a^2 r, x, \Delta^\sigma(\alpha)) \cap Q^\sigma(x, 2a^2 r)| \geq \left(1 - \frac{c_\beta}{2}\right) |Q^\sigma(x, 2a^2 r)|. \tag{4.5.1}
\]

Then,
\[
|Q^\sigma(x, 2a^2 r)| \geq |(Q^\sigma(x, 2a^2 r) \cap E) \cup (Q^\sigma(x, 2a^2 r) \cap H(...))| =
\]
\[
|Q^\sigma(x, 2a^2 r) \cap E| + |Q^\sigma(x, 2a^2 r) \cap H(...)| - |E \cap Q^\sigma(x, 2a^2 r) \cap H(...)|
\]
\[
\geq |Q^\sigma(x, 2a^2 r)| \left(c_\beta + 1 - \frac{c_\beta}{2}\right) - |E \cap Q^\sigma(x, 2a^2 r) \cap H(...)|
\]

and therefore the claim follows.

We can assume \(x \notin E\) and define \(\Sigma = \{ y \in \Delta^\alpha(\alpha) : H(2a^2 r, x, y) \in E \}\).

Let \(K\) be a smooth function supported in \(\Delta^\alpha(\alpha), 0 \leq K \leq 1, K \equiv 1\) on \(\Delta^\alpha(\alpha)\). Suppose \(u \in \text{Lip}(Q)\). If \(y \in \Sigma\) then
\[
|u(x)| \sqrt{\varphi(x)} = |u(x) - u(H(2a^2 r, x, y))| K(y) \sqrt{\varphi(x)},
\]

and if we integrate on \(\Sigma\), we obtain
\[ |u(x)| \sqrt{\varphi(x)} |\Sigma| = \int_{\Sigma} |u(x) - u(H(2a^2r, x, y))| K(y) \sqrt{\varphi(x)} dy.\]

Now we note that \( \varphi(H(2a^2r, x, y)) = 1 \) if \( y \in \Sigma \) and using (4.2) we get \( \varphi(x) \leq \varphi(H(t, x, y)) \) for any \( 0 \leq t \leq 2a^2r \). Therefore,

\[ |u(x)| \sqrt{\varphi(x)} |\Sigma| \leq \int_{\text{supp} K} \int_{0}^{2a^2r} \frac{d}{dt} (u(H(t, x, y))) dt \sqrt{\varphi(H(t, x, y))} dy\]

\[ \leq \int_{\text{supp} K} \int_{0}^{2a^2r} \langle \nabla u(H(t, x, y)), \dot{H}(t, x, y) \rangle dt \sqrt{\varphi(H(t, x, y))} dy\]

\[ \leq \int_{0}^{2a^2r} \int_{\text{supp} K} |\nabla u(H(t, x, y))| |y| \sqrt{\varphi(H(t, x, y))} dy dt.\]

If we make change of variables \( z = H(t, x, y) \) in \( \Delta_{\gamma/2}^{\alpha}(\sigma) \), then

\[ |\det \frac{\partial z}{\partial y}(t, x, y)| = \prod_{j=1}^{n} \lambda_j(H(s, x, y)) ds.\]

For \( y \in \Delta_{\gamma/2}^{\alpha}(\sigma) \), the last product is equivalent to \( |Q(x, t)| \) by (1.15). Hence

\[ (4.6) \quad |u(x)| \sqrt{\varphi(x)} \leq \frac{c}{|\Sigma|} \int_{0}^{2a^2r} \frac{1}{|Q(x, t)|} \int_{H(t, x, \Delta_{\gamma/2}^{\alpha}(\sigma))} |\nabla u(z)| \sqrt{\varphi(z)} dz dt.\]

Note that there exists \( c > 0 \) such that \( H(t, x, \Delta_{\gamma/2}^{\alpha}(\sigma)) \subset Q(x, ct) \). In fact, if we define \( \gamma(s) = H(s/|y|, x, y) \) then

\[ \langle \dot{y}(s), \xi \rangle^2 = \left( \sum_{j=1}^{n} \lambda_j \left( H \left( \frac{s}{|y|}, x, y \right) \right) y_j \xi_j \right) \frac{1}{|y|^2} \]

\[ \leq \sum_{j=1}^{n} \lambda_j^2 \left( H \left( \frac{s}{|y|}, x, y \right) \right) \xi_j^2 \]

\[ = \sum_{j=1}^{n} \lambda_j(\gamma(s)) \xi_j^2 \]

for every \( \xi \in R^n \). So, \( \gamma \) is a \( \lambda \)-subunit curve starting from \( x \) and attaining \( H(t, x, y) \) at the time \( s = t |y| \). Therefore by (1.9),

\[ \delta(x, H(t, x, y)) \leq ad(x, H(t, x, y)) \leq at |y| \leq ct \]

where \( c = 2aa \).
Thus, from (4.6), we obtain

\[ |u(x)| \leq \frac{c}{|\Sigma|} \int_0^{2\pi r} \int_{Q(x, r)} \left| \nabla u(z) \right| \frac{1}{|Q(x, t)|} \left( \int_{Q(x, t)} |\nabla \varphi(z)| \frac{dz}{|Q(x, t)|} \right) dt dt dz \]

and, interchanging the order of integration and using the fact that \( \text{supp} \varphi \subset Q \) (the argument we are going to present next is due to Chanillo, Sawyer and Wheeden), we get

\[ |u(x)| \leq \frac{c}{|\Sigma|} \int_0^{2\pi r} \int_{Q(x, r)} \left| \nabla u(z) \right| \frac{1}{|Q(x, t)|} \int_{\partial Q(x, \tau)} dt \left( \int_{Q(x, t)} |\nabla \varphi(z)| \frac{dz}{|Q(x, t)|} \right) dz. \]  

We claim that

\[ \int_{ch}^{\infty} \frac{dt}{|Q(x, t)|} \leq c \frac{ch}{|Q(x, h)|}. \]

To prove this we note that, by (1.8),

\[ \frac{|Q(x, t)|}{t} = \Pi_{j=2}^{n} F_j(x^*, t), \]

and consequently by (1.10), there exists \( \epsilon > 0 \) such that if \( t > \tau \) then

\[ \frac{|Q(x, t)|}{t} \geq \epsilon \frac{Q(x, \tau)}{\tau}. \]

Hence,

\[ \int_{ch}^{\infty} \frac{dt}{|Q(x, t)|} \leq \int_{ch}^{\infty} \frac{dt}{|Q(x, t)|} \frac{dt}{t} \leq \int_{ch}^{\infty} \frac{h}{|Q(x, h)|} \left( \frac{h}{t} \right) dt = c \frac{h}{|Q(x, h)|}. \]

Finally, we note that \( |\Sigma| \geq c > 0 \), with \( c \) independent of \( x \), since, by the change of variables \( z = H(2a^2r, x, y) \),

\[ |\Sigma| = \int_{\Sigma} dy = \int_{H(2a^2r, x, \Sigma)} \frac{1}{|Q(x, 2a^2r)|} dz \]

\[ = \frac{|H(2a^2r, x, \Sigma)|}{|Q(x, 2a^2r)|} = \frac{|E \cap H(2a^2r, x, \Delta^*(\phi))|}{|Q(x, 2a^2r)|} \]

\[ \geq c \frac{|Q^*(x, 2a^2r)|}{Q(x, 2a^2r)} \geq c > 0. \]

The lemma follows by combining the last two last estimates with (4.7).
PROOF OF THEOREM E.

Define \( Tf(x) = \int_{\mathbb{R}^n} f(y)K(x, y)dy \), where \( K(x, y) = \frac{\delta(x, y)}{|Q(x, \delta(x, y))|} \).

Fix \( S \) a \( d \)-ball. In order to show that for a pair of weights \( \tilde{v}, \tilde{w} \) we have
\[
\|Tf\|_{L^1(S, \tilde{v})} \leq \|f\|_{L^1(S, \tilde{w})} \quad \text{(where} \quad \|f\|_{L^1(S, \tilde{w})} = \left( \int_S f^2 \tilde{w} \right)^{1/2} \text{)} \quad \text{for all} \quad f \geq 0,
\]
supp \( f \subset S \), according to [SW], we need to verify that the following conditions hold:

(a) there exists \( s > 1 \) such that
\[
\varphi(I) |I| \left( \frac{1}{|I|} \int_I \tilde{v}^s dx \right)^{\frac{1}{2s}} \left( \frac{1}{|I|} \int_I \tilde{w}^{-s} dx \right)^{\frac{1}{2s}} \leq c
\]
for all \( d \)-balls \( I \subset 2S \), where \( \varphi(I) \) is defined to be
\[
\varphi(I) = \sup \left\{ K(x, y) : x, y \in I, d(x, y) \geq \frac{1}{2} r(I) \right\};
\]

(b) there is \( \epsilon > 0 \) such that
\[
\left( \frac{|I|}{|I|} \right)^{\epsilon} \leq c \left( \frac{\varphi(I)}{\varphi(I')} \right)^{\epsilon} \left( \frac{r(I')}{r(I)} \right)^{\epsilon}
\]
for all pairs of \( d \)-balls \( I' \subset I \).

Note that it is convenient to work with \( d \) since the results of [SW] hold for pseudo-metrics (a pseudo-metric \( d \) is a quasi-metric satisfying \( d(x, y) = d(y, x) \) for all \( x, y \in \mathbb{R}^n \)).

Define \( \tilde{v} = \frac{v}{v(S)} \) and \( \tilde{w} = \frac{w_1}{w_1(S)} r(S)^2 \). Note that if \( x, y \in I \) and
\[
d(x, y) \geq \frac{1}{2} r(I), \text{ then by (1.9)}
\]
\[
K(x, y) = \frac{\delta(x, y)}{|Q(x, \delta(x, y))|} \leq \frac{2ar(I)}{Q \left( x, \frac{1}{2a} r(I) \right)} \leq c \frac{r(I)}{|Q(x, r(I))|},
\]
and since \( x \in I, \ |Q(x, r(I))| = |I| \). Therefore,

\[
\varphi(I) \leq c \frac{r(I)}{|I|}.
\]

So, the expression in (a) is bounded by

\[
\begin{align*}
&c \frac{r(I)}{|I|} |I| \left( \frac{1}{|I|} \int_I \left( \frac{\nu}{\nu(S)} \right)^s dx \right)^{\frac{1}{2s}} \left( \frac{1}{|I|} \int_I \left( \frac{w_1}{w_1(S)} r(S)^2 \right)^{-s} dx \right)^{\frac{1}{2s}} \\
\leq& c \frac{r(I)}{r(S)} \left( \frac{1}{|I|} \int_I \left( \frac{\nu}{\nu(S)} \right)^s dx \right)^{\frac{1}{2s}} \left( \frac{1}{|I|} \int_I \left( \frac{w_1}{w_1(S)} \right)^{-s} dx \right)^{\frac{1}{2s}},
\end{align*}
\]

which is equivalent to the expression in condition (1.18) (if we use doubling and (1.9)). This proves (a).

To show (b) we note that if \( x, y \in I \) and \( d(x, y) \geq \frac{1}{2} r(I) \) then

\[
K(x, y) \leq \frac{(2a)^{-1} r(I)}{|Q(x, 2ar(I))|} \geq c \frac{r(I)}{|I|}.
\]

Thus \( \varphi(I) = \frac{r(I)}{|I|} \). Then, if \( I' \subset I \), \( \frac{\varphi(I)}{\varphi(I')} = \frac{r(I)}{r(I')} \frac{|I'|}{|I|} \) and we obtain (b) with \( \epsilon = 1 \).

By doubling and (1.9), it follows that

\[
\| Tf \|_{L^\infty(Q, \nu)} \leq c \| f \|_{L^1(Q, \nu)}
\]

for all \( f \geq 0, \ \text{supp} \ f \subset Q \), where \( \tilde{\nu} = \frac{\nu}{r(Q)} \) and \( \tilde{w} = \frac{w_1}{w_1(Q)} r(Q)^2 \).

Suppose \( u \) is a Lipschitz function in \( Q \) and \( |E| = |\{ x \in Q(\xi, kr): u(x) = 0 \}| \geq \beta |Q|, \ 1/2 < k < 1 \). If we combine Lemma 4.1 and the fact that \( \| Tf \|_{L^1(Q, \nu)} \leq c \| f \|_{L^1(Q, \nu)} \) we obtain

\[
(4.8) \quad \left( \int_Q \left| \frac{1}{r(Q)} \int_Q u(x)^2 \varphi(x) \nu(x) dx \right|^{1/2} \right)^{1/2} \leq c r(Q) \left( \int_Q \left| \nabla u(x)^2 \varphi(x) w_1(x) dx \right|^{1/2} \right)^{1/2}
\]
Given \( Q \) and a general Lipschitz function \( u \), there is a number \( \mu = \mu(u, Q) \), the media of \( u \) in \( Q \), such that if \( Q^+ = \{ x \in Q : u(x) \geq \mu \} \) and \( Q^- = \{ x \in Q : u(x) \leq \mu \} \) then \( |Q^+| \geq |Q|/2 \) and \( |Q^-| \geq |Q|/2 \). Hence, \( u_1 = \max\{|u - \mu(u, kQ)|, 0\} \) and \( u_2 = \max\{|\mu(u, kQ) - u|, 0\} \) satisfy the hypothesis of Lemma (4.1) for some \( \beta \) depending on \( k \) and so if we apply (4.8) to \( u_1 \) and \( u_2 \) and add both inequalities, we get

\[
(4.9) \quad \int_Q |u(x) - \mu|^2 \varphi(x) v(x) dx \leq c r(Q)^2 \frac{\nu(Q)}{w_1(Q)} \int_Q |\nabla \mu(z)^2 \varphi(z) w_1(z) dz.
\]

Finally, it is easy to see that in (4.9) \( \mu \) can be replaced by the average \( A_Q \) of \( u \) defined in Theorem E. In fact,

\[
(4.10) \quad \int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx
\]

\[
\leq 2 \int_Q |u(x) - \mu|^2 \varphi(x) v(x) dx
\]

\[
+ 2 \int_Q |\mu - A_Q|^2 \varphi(x) v(x) dx,
\]

and

\[
\int_Q |\mu - A_Q|^2 \varphi(x) v(x) dx = (\varphi v)(Q) |\mu - A_Q|^2
\]

\[
= (\varphi v)(Q) \left| \mu - \frac{1}{\varphi(Q)} \int_Q u(x) \varphi(x) dx \right|^2
\]

\[
\leq (\varphi v)(Q) \left( \frac{1}{\varphi(Q)} \int_Q |u(x) - \mu| \varphi(x) dx \right)^2
\]

\[
\leq \frac{(\varphi v)(Q)}{(\varphi(Q))^2} \int_Q |u(x) - \mu|^2 \varphi^2(x) v(x) dx \int_Q \frac{1}{v(x)} dx,
\]

where in the last inequality we used Schwarz's inequality. Since \( v \in A_2 \) and \( 0 \leq \varphi \leq 1 \), it follows from (4.9) and (4.10) that

\[
\int_Q |u(x) - A_Q|^2 \varphi(x) v(x) dx
\]

\[
\leq c r(Q)^2 \left[ 1 + \left( \frac{|Q|}{\varphi(Q)} \right)^2 \right] \frac{\nu(Q)}{w_1(Q)} \int_Q |\nabla \mu(z)|^2 \varphi(z) w_1(z) dz.
\]
This finishes the proof of Theorem E if we note that $\varphi(Q) = |Q|$ since $1/2 \leq k \leq 1$.

The next corollary is also helpful.

**Corollary 4.11.** *Theorem E is also true with $A_Q = \frac{1}{(\varphi\nu)(Q)} \int_Q u \varphi \nu dx$.*

Just note that

$$\int_Q |\mu - A_Q|^2 \varphi \nu dx = (\varphi\nu)(Q) |\mu - A_Q|^2 \int_Q \varphi dx,$$

$$\leq (\varphi\nu)(Q) \left( \frac{1}{(\varphi\nu)(Q)} \int_Q |\mu - u| \varphi dx \right)^2,$$

$$\leq \int_Q |\mu - u|^2 \varphi dx,$$

where the last inequality follows by Schwarz's inequality.

5. **Harnack's inequality**

The proof of Theorem A follows as an application of Bombieri's lemma which we state next. For its proof see Section 5 of [GW2].

**Lemma 5.1.** Let $R(q)$ be a one parameter family of rectangles in $\mathbb{R}^{n+1}$, $R(\sigma) \subset R(q)$, $1/2 \leq \sigma \leq q \leq 1$ and let $\nu$ be a doubling measure in $\mathbb{R}^{n+1}$. Let $A, \mu, M, m, \theta$ and $x$ be positive constants such that $M \geq 1/\mu$ and suppose that $f$ is a positive measurable function defined in a neighborhood of $R(1)$ satisfying

$$\text{ess sup}_{R(q)} f^p \leq \frac{A}{(q - \sigma)^m} \int_{R(q)} f^p \nu(x) dx dt$$

for all $\sigma, q, p$, $1/2 \leq \theta \leq \sigma < q < 1$, $0 < p < M$ and

$$\nu((x, t) \in R(1): \log f > s)) \leq \left( \frac{\mu}{s} \right) x \nu(R(1))$$
for all $s > 0$. Then there is a constant $\gamma = \gamma(A, m, \chi) > 0$ such that
\[
\log(\text{ess sup}_{R(\Omega)} u) \leq \frac{\gamma}{(1 - \theta)^{2m}} \mu.
\]

Hence, in order to prove Theorem A, we need a mean value inequality (that we proved in Section 3) and a logarithm estimate which is given by Theorem F (some steps of its proof will be present in this section). The next lemma shows that the test function described on page 537 of [FL1] satisfies the conditions of Theorem E. Then, as we said before, the proof of Theorem F follows as Lemma 4.9 of [GW2].

**Lemma 5.4.** Given $Q = Q(\xi, r)$ and $0 < k < 1$, there exists $\varphi \in C^1$ such that $\varphi = 1$ in $kQ$, $0 \leq \varphi \leq 1$, $\text{supp}\varphi \subset Q$, $|\nabla_x \varphi| \leq \frac{c}{r(1 - k)}$ and $\varphi(x) \cdot \varphi(H(t_0, x, y)) \leq \varphi(H(t, x, y))$ for all $x, y, t, t_0$ with $0 \leq t \leq t_0$.

**Proof:** Consider the function $\varphi$ given by [FL1], page 537:
\[
\varphi(x) = \prod_{j=1}^n \psi \left( \frac{|x_j - \xi_j|}{F_j(\xi^*, r)} \right),
\]
where $\psi \in C^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, $\psi(t) = \psi(-t)$, $\psi \equiv 1$ on $[-k, k]$, $\psi = 0$ outside $[-1, 1]$, $|\psi'(t)| \leq 2(1 - k)^{-1}$, for all $t \in \mathbb{R}$. Here, we show that $\varphi$ satisfies the last condition since all the others are proved in [FL1], page 537.

Fix $t$, $0 < t < t_0$, $x$ and $y$. Define $z = H(t, x, y)$. Then,
\[
z_j = x_j + y_j \int_0^t \lambda_j(H(s, x, y)) ds.
\]
Suppose $z_j - \xi_j \geq 0$. If $y_j \geq 0$ then
\[
|z_j - \xi_j| \leq x_j - \xi_j + y_j \int_0^{t_0} \lambda_j(H(s, x, y)) ds = H_j(t_0, x, y) - \xi_j.
\]
On the other hand, if $y_j < 0$,
\[
|z_j - \xi_j| \leq |x_j - \xi_j|.
\]
Thus, if $z_j - \xi_j \geq 0$ then $|z_j - \xi_j| \leq |H_j(t_0, x, y) - \xi_j|$ or $|z_j - \xi_j| \leq |x_j - \xi_j|$. The same holds if $z_j - \xi_j < 0$. Since $\psi(t)$ can be chosen to be non-increasing for positive $t$, then $\varphi(z) \geq a_1 \ldots a_n$, where
\[
a_j = \psi \left( \frac{|x_j - \xi_j|}{F_j(\xi^*, r)} \right).
\]
or

\[ a_j = \psi \left( \frac{|H_j(t_0, x, y) - \xi_j|}{F_j(\xi^*, r)} \right) \psi \left( \frac{|x_j - \xi_j|}{F_j(\xi^*, r)} \right) \]

Since \(0 \leq \psi \leq 1\),

\[ a_j \geq \psi \left( \frac{|H_j(t_0, x, y) - \xi_j|}{F_j(\xi^*, r)} \right) \psi \left( \frac{|x_j - \xi_j|}{F_j(\xi^*, r)} \right) \]

for \(1 \leq j \leq n\). Therefore,

\[ \varphi(z) \geq \varphi(x, y)(H(t_0, x, y)). \]

The next three lemmas are needed in order to show that the hypothesis in Theorem A imply those in Theorems D and E.

**Lemma 5.5.** Assume that Poincaré's inequality holds for \(w_1, w_2\) with \(q = 2\) and \(\mu = 1\). Then

\[ \left( \frac{r(I)}{r(B)} \right)^2 \frac{w_2(I)}{w_2(B)} \leq c \frac{w_1(I)}{w_1(B)} \]

for any pair of \(\partial\)-balls \(I, B\), with \(I \subset 2B\).

**Proof:** Suppose \(I = Q(u_0, r(I))\) and \(B = Q(x, r(B))\) and define

\[ F(u) = \sum_{j=1}^{n} \frac{|u_j - (u_0)_j|}{F_j(u_0^*, r(I))} r(I) \varphi(u) \]

where \(\varphi\) is the function described in lemma (5.4) associated with \(I\) (as opposed to \(B\)) and \(k = 1/2\). If \(u \in I\), by (1.8)

\[ \left| \frac{\partial F}{\partial u_k}(u) \right| \leq \frac{r(I)}{F_k(u_0^*, r(I))} + \frac{\partial \varphi}{\partial u_k}(u)n r(I), \]

for \(k \in [1, \ldots n]\), and using the fact that \(\lambda_k(u) = \lambda_k(u^*) \leq \lambda_k(H(u^*, r(I)))\) if \(u \in I\) we get

\[ \left| \lambda_k(u) \frac{\partial F}{\partial u_k}(u) \right| \leq \frac{F_k(u^*, r(I))}{F_k(u_0^*, r(I))} + nr(I)\lambda_k(u) \frac{\partial \varphi}{\partial u_k}(u) \]
and by Lemma 2.4 and the fact that $|\nabla \chi_{\sigma} | \leq c/r(I)$ we have $|\nabla \chi F(u)| \leq c\chi_{I}$.

We have Poincaré's inequality for $F$, i.e.,

$$(5.6) \quad \left( \frac{1}{w_2(B)} \int_{n4^{\sigma+1}B} |F(u) - av_{n4^{\sigma+1}B} F| \, w_2(u) \, du \right)^{1/2} \leq cr(B) \left( \frac{1}{w_1(B)} \int_{n2^{\sigma+1}B} |\nabla \chi F(u)| \, w_1(u) \, du \right)^{1/2},$$

where $\eta = \max_{j=1,\ldots,n}[G_j]$. The right side of (5.6) is bounded by $cr(B) \left( \frac{w_1(I)}{w_1(B)} \right)^{1/2}$ by doubling and the fact that $|\nabla \chi F| \leq c\chi_{I}$. Now, if $u \not\in \frac{1}{4}I$ there exists $k \in \{1, \ldots, n\}$ such that

$$|u_k - (u_0)_k| \geq F_k \left( u^\#, \frac{1}{4}r(I) \right)$$

and then if $u \in \frac{1}{2}I \setminus \frac{1}{4}I$ (note that $\varphi(u) = 1$)

$$(5.7) \quad F(u) \geq \frac{F_k \left( u^\#, \frac{1}{4}r(I) \right)}{F_k(u^\#, r(I))} r(I) \geq \left( \frac{1}{4} \right)^{g_k} r(I) \geq \frac{1}{4} r(I).$$

Also, if $u \in I$, $F(u) \leq nr(I)$ and therefore

$$av_{n4^{\sigma+1}B} F \leq \frac{|I|}{|n4^{\sigma+1}B|} nr(I).$$

But, by (1.10), $F_j(x^\#, n4^{\sigma+1}r(B)) \geq 2n4^\sigma F_j(x^\#, 2r(B))$, and by (1.11),

$$|n4^{\sigma+1}B| \geq (2n4^\sigma)^n \quad \text{and} \quad 2B \geq 2n4^n \quad |2B|. $$

Hence, since $I \subset 2B$, $av_{n4^{\sigma+1}B} F \leq r(I)/2 \cdot 4^\sigma$ and then if $u \in \frac{1}{2}I \setminus \frac{1}{4}I$, (using also 5.7)),

$$|F(u) - av_{n4^{\sigma+1}B} F| \geq cr(I).$$

Therefore, the left hand side of (5.6) is larger than a constant times
\[
\left( \frac{(r(I))^2}{w_2(B)} \cdot w_2 \left( \frac{1}{2} I \setminus \frac{1}{4} I \right) \right)^{1/2} \approx cr(I) \left( \frac{w_2(I)}{w_2(B)} \right)^{1/2},
\]

where in the last inequality we used the fact that
\[
w_2 \left( \frac{1}{2} I \setminus \frac{1}{4} I \right) \approx w_2(I),
\]

which is shown in the next lemma.

**Lemma 5.8.** If \( w \) is a doubling weight then \( W(Q(u, 2s) \setminus Q(u, s)) \) is equivalent to \( w(Q(u, s)) \).

**Proof:** Choose \( \eta \in Q(u, 2s) \) such that \( \delta(u, \eta) = \frac{3s}{2} \). By Lemma 2.5,
\[
Q \left( \eta, \frac{3s}{2(2a^2)^{1/2}} \right) \subset Q \left( u, (1 + \epsilon) \frac{3s}{2} \right)
\]

for any \( 0 < \epsilon < 1 \).

Choose \( j \) such that \( \delta(u, \eta) = \varphi_j(u^*, \ | \ | \eta_j - u_j | ) \). Then, if \( y \in Q \left( \eta, \frac{3s}{2(2a^2)^{1/2}} \right) \),
\[
F_j \left( u^*, \frac{3s}{2} \right) = | \eta_j - u_j | \leq | \eta_j - y_j | + | y_j - u_j |
\]
\[
\leq F_j \left( \eta^*, \frac{3s}{2(2a^2)^{1/2}} \right) + | y_j - u_j |.
\]

By (1.10) and Lemma 2.4,
\[
F_j \left( u^*, \frac{3s}{2} \right) \leq \epsilon F_j \left( u^*, \frac{3s}{2} \right) + | y_j - u_j |.
\]

Thus,
\[
| y_j - u_j | \geq (1 - \epsilon) F_j \left( u^*, \frac{3s}{2} \right) \geq F_j \left( u^*, (1 - \epsilon) \frac{3s}{2} \right).
\]

If we choose \( \epsilon = 1/3 \) we have proved that
\[
Q \left( \eta, \frac{s}{2(2a^2)^{1/2}} \right) \subset Q(u, 2s) \setminus Q(u, s).
\]

The lemma follows by doubling.
Lemma 5.9. If \( w_1 \in A_2, \nu \in A_\infty \) and Poincaré's inequality holds for \( w_1, \nu \) with \( q = 2 \) and \( \mu = 1 \), then condition (1.21) holds.

Proof: If \( \nu \in A_\infty \) there exists \( s > 1 \) such that

\[
\left( \frac{1}{|I|} \int_I \left( \frac{\nu}{\nu(B)} \right)^s \, dx \right)^{1/s} \leq \frac{1}{|I|} \frac{\nu(I)}{\nu(B)}.
\]

So, since Poincaré's inequality holds for \( w_1, \nu \) with \( q = 2 \), by Lemma 5.5

\[
\left( \frac{r(I)}{r(B)} \right)^2 \left( \frac{1}{|I|} \int_I \left( \frac{\nu}{\nu(B)} \right)^s \, dx \right)^{1/s} \leq c \frac{1}{|I|} \frac{w_1(I)}{w_1(B)},
\]

and the above condition is equivalent to condition (1.18) since \( w_1 \in A_2 \).

Now we are ready to prove Theorem A.

Proof of Theorem A

Let \( u \) be a non-negative solution of (1.1) in the cylinder \( R_{a,t} = R_{a,t}(x_0, t_0) = Q(x_0, \alpha) \times (t_0 - \beta, t_0 + \beta) \). If we define \( T(x, t) = (x, \beta t + t_0) \) and \( \bar{u}(x, t) = u(T(x, t)) \) then \( u \) is a solution in \( R_{a,t}(x_0, 0) \) of the equation

\[
\nu(x) \bar{u}_t = \text{div}(\bar{A}(x, t) \nabla \bar{u}),
\]

where the coefficients matrix \( \bar{A} = (a_{ij}) \) are defined by \( a_{ij}(x, t) = \beta a_{ij}(x, \beta t + t_0) \) and satisfies the degeneracy condition

\[
\bar{w}_1(x) \sum_{j=1}^n \lambda_j^2(x) \xi_j^2 \leq \sum_{j=1}^n \bar{a}_{ij}(x, t) \xi_i \xi_j \leq \bar{w}_2(x) \sum_{j=1}^n \lambda_j^2(x) \xi_j^2,
\]

if we put \( \bar{w}_i = \beta w_i \), for \( i = 1, 2 \).

Suppose \( |\rho| < [\alpha^{-2} \bar{\Lambda}(Q(x_0, \alpha)) + \alpha^2 / \bar{\Lambda}(Q(x_0, \alpha))]^{-1} \), where \( \bar{\Lambda}(Q) = \bar{w}_2(Q)/\nu(Q) \), \( \bar{\lambda}(Q) = \bar{w}_1(Q)/\nu(Q) \). Write

\[
R^-(\rho) = Q \left( x_0, \frac{(q + 1)\alpha}{3} \right) \times \left( -\frac{1}{2} - \frac{\rho}{2}, -\frac{1}{2} + \frac{\rho}{2} \right),
\]

\[
R^+(\rho) = Q \left( x_0, \frac{(q + 1)\alpha}{3} \right) \times \left( \frac{1}{2} - \frac{\rho}{2}, 1 \right).
\]
If we take $1/2 < q < r \leq 1$ then the mean value inequalities in Theorem 3.15 applied to $u$ give

\begin{equation}
(5.10) \quad \text{ess sup}_{R^-(q)} \bar{u}^p \leq c \frac{1}{(r-q)^m} \int_{R^-(q)} \bar{u}^p \left( \frac{\bar{w}_2}{\bar{w}_2(Q_{a})} + \frac{\nu}{\nu(Q_{a})} \right) \, dx \, dt,
\end{equation}

for some $m > 0$, if $p > 0$, where $Q_a = Q(x_0, a)$, and

\begin{equation}
(5.11) \quad \text{ess sup}_{R^-(q)} \bar{u}^p \leq c \frac{1}{(r-q)^m} \int_{R^-(q)} \bar{u}^p \left( \frac{\bar{w}_2}{\bar{w}_2(Q_{a})} + \frac{\nu}{\nu(Q_{a})} \right) \, dx \, dt,
\end{equation}

if $p < 0$. Moreover, by Theorem B, $\bar{u}$ is locally bounded and by adding $\epsilon > 0$, we may assume by letting $\epsilon \to 0$ at the end of the proof that $\bar{u}$ is bounded below in $R_{a,1}(x_0, 0)$ by a positive constant.

Now, by Theorem F, we have

\begin{equation}
(5.12) \quad \left[ \left( \frac{\nu}{\nu(Q_a)} + \frac{\bar{w}}{\bar{w}_2(Q_a)} \right) \otimes 1 \right] (E^+)
\leq \left\{ \frac{1}{s} \frac{\nu(Q_a)}{\bar{w}_1(Q_a)} \alpha^2 \right\}^* \\
\leq c \left\{ \frac{1}{s} \left[ \alpha^{-2} \bar{\Lambda}(Q_a) + \alpha^2 \frac{1}{\lambda(Q_a)} \right] \right\}^*,
\end{equation}

and the same inequality holds for $E^-$, where $E^+, E^-$ are defined in Theorem F with $u = \bar{u}$, $R = 2/3 \alpha$, $a = -1$, $b = 1$, $t_0 = 0$, $M_2 = \bar{\Lambda}(Q_a) / \alpha^2$.

By (5.10) and (5.12), we can apply Bombieri’s lemma to the family of rectangles $R^-(q)$ with $\mu = \alpha^{-2} \bar{\Lambda}(Q_a(x_0)) + \alpha^2 / \lambda(Q_a(x_0))$, $M = 1/\mu$ and $f = e^{-M_2 + \nu(0)} \bar{u}$, obtaining

\[ \text{ess sup}_{R^{-(1/2)}} \bar{u} \leq C \exp\left[ c[\alpha^{-2} \bar{\Lambda}(Q_a) + \alpha^2 / \lambda(Q_a)] \right], \]

and this implies that

\begin{equation}
(5.13) \quad \text{ess sup}_{R^{-(1/2)}} \bar{u} \leq C \exp\left[ c[\alpha^{-2} \bar{\Lambda}(Q_a(x_0, \alpha)) + \alpha^2 / \lambda(Q_a(x_0, \alpha))] - V(0) \right].
\end{equation}

Also, by (5.11) and (5.12), we can apply Bombieri’s lemma to the family of rectangles $R^+(q)$, $f = e^{-M_2 - \nu(0)} \bar{u}^{-1}$, with $\mu, M, M_2, V(0)$ as before, and we obtain
\[
\text{ess sup}_{R^{-}(1/2)} f \leq C \exp\left[ c[\alpha^{-2} \overline{\lambda}(Q_{a}) + \alpha^{2}/\overline{\lambda}(Q_{a})]\right],
\]
which implies that
\[
(5.14) \quad e^{-v(0)} \leq C e^{c[\alpha^{-2} \overline{\lambda}(Q_{a}, \alpha) + \alpha^{2}/\overline{\lambda}(Q_{a}, \alpha)]} \text{ess inf}_{R^{-}(1/2)} \overline{u}.
\]
Combining (5.13) and (5.14) it follows that
\[
\text{ess sup}_{R^{-}(1/2)} \overline{u} \leq c_{1} e^{c[\alpha^{-2} \overline{\lambda}(Q_{a}, \alpha) + \alpha^{2}/\overline{\lambda}(Q_{a}, \alpha)]} \text{ess inf}_{R^{-}(1/2)} \overline{u}.
\]
Since, \(T(R^{-}(1/2)) = R^{-}\), \(T(R^{+}(1/2)) = R^{+}\) and \(\alpha^{-2} \overline{\lambda}(Q_{a}) + \alpha^{2}/\overline{\lambda}(Q_{a}) =\)
\(\alpha^{-2} \beta \Lambda(Q_{a}) + \alpha^{2}/\lambda(Q_{a})\), Theorem A follows.

**Remark:** Using the equivalence between \(d\) and \(\delta\) we can prove the following analogues of Theorem A and B for the metric \(d\).

**Theorem A':** Assume (i), (ii), (iii) of Theorem A. If \(u\) is a non-negative solution of (1.1) in the cylinder \(R = S(x_{0}, \alpha^{2}) \times (t_{0} - \beta, t_{0} + \beta)\), then
\[
\text{ess sup}_{R^{-}} u \leq c_{1} \exp\left[ c_{2}[\alpha^{-2} \beta \Lambda(S(x_{0}, \alpha)) + \alpha^{2}/\lambda(S(x_{0}, \alpha))^{-1}]\right] \text{ess inf}_{R^{+}} u
\]
where \(R^{-} = S(x_{0}, \alpha^{2}/2) \times (t_{0} - 3 \beta/4, t_{0} - \beta/4), R^{+} = S(x_{0}, \alpha/2) \times (t_{0} + \beta/4, t_{0} + \beta)\), \(\Lambda(S) = w_{2}(S)/\nu(S)\) and \(\lambda(S) = w_{1}(S)/\nu(S)\) for a \(d\)-ball \(S\). Here the constants \(c_{1}, c_{2}\) depend only on the constants which arise in (i), (ii), (iii).

**Theorem B':** Assume hypothesis (i), (ii), (iii) of Theorem A hold. Let \(0 < p < \infty, \alpha, \beta > 0, \alpha/2 < \alpha' < \alpha, \beta/2 < \beta' < \beta\) and let \(S(x_{0}, \alpha) = S, S(x_{0}, \alpha') = S'\) and \(R(\alpha, \beta) = S \times (t_{0} - \beta, t_{0} + \beta), R^{+}(\alpha, \beta) = S' \times (t_{0} - \beta', t_{0} + \beta)\). If \(u\) is a solution of (1.1) in \(R(\alpha^{2}, \beta)\), then \(u\) is bounded in \(R^{+}(\alpha, \beta)\) and
\[
\text{ess sup}_{R^{+}(\alpha, \beta)} |u|^{p} \leq D(\alpha^{2} \beta^{-1} \lambda(S))^{-1} + 1^{1/(h-1)}(\alpha^{-2} \beta \Lambda(S) + 1)^{1/(h-1)} \int_{R^{+}(\alpha, \beta)} \nu u^{p}(\alpha^{-2} \beta w_{2} + v) dx dt
\]
where \(D\) is as in Theorem B, and \(C = c \frac{\alpha^{2} + \beta}{(\alpha - \alpha')^{2} + \beta(\beta - \beta')}\). Here \(h > 1\), constants which are independent of \(u, p, \alpha, \alpha', \beta, \beta'\).

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Analytic Dependence of Orthogonal Polynomials

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Introduction

One way we can define and construct orthogonal polynomials is by applying the Gram-Schmidt orthogonalization process to the sequence of powers $1, x, x^2, \ldots$ More precisely, we embed these power-functions in $L^2(d\mu)$, where $d\mu$ is a probability measure living on some space (usually the closed interval $[-1, 1]$, the real line $\mathbb{R}$, or the unit circle $\mathbb{T}$), and then we orthogonalize with respect to the scalar product induced by the measure $d\mu$. In the classical theory, $d\mu$ is usually absolutely continuous with respect to the Lebesgue Measure, and given by some very explicit weight function, but recently, in [CM], R. R. Coifman and M. Murray have proposed a different analytic approach to the study of general orthogonal polynomials, based on a «perturbation theory» of the orthogonalization process.

The approach of R. R. Coifman and M. Murray allows one to study orthogonal polynomials within the framework of non-linear Fourier Analysis. As in similar studies of other non-linear problems (dependence of the Riemann mapping on the domain, shape of water waves, Navier-Stokes equation) the main result consists in showing that some relevant operator quantities are analytically dependent on a B.M.O. parameter. In this kind of analysis, the theory of weighted norm inequalities, $A_p$ weights, and the relationships between good weights and BMO, play a central role (an excellent source of reference for this is [GCRF]).

This particular problem belongs also to the context of Toda flows (see [DLT]) where a certain non-linear matrix differential equation admits closed form solutions. These solutions are a «flow» of infinite tridiagonal matrices that
can be interpreted as the three-term recursion relation attached to the orthogonal polynomials.

In [CM] the program is carried out in detail starting from two situations: polynomials on the circle $\mathbb{T}$ orthogonal with respect to Lebesgue measure, and polynomials on $[-1, 1]$ orthogonal with respect to a Jacobi weight. In this work we carry out the same program starting from a set of polynomials on $\mathbb{T}$ first introduced by Szegö, in fact used by him as a tool for the proof of his equiconvergence theorems.

We then show some new results. Using the closed subspace $VMO$, we give a better characterization of the $BMO$ region where the analyticity result holds. This can be seen as the definition of a very general set of polynomials that can be «analytically perturbed». The relationship with the theory of Toda flows is exploited to derive new identities that can be used interchangeably with the Kerzman-Stein formula, shedding a new light on the connections among the different operators involved. The flow itself is expressed in terms of conjugation by a self-adjoint projection $Q$ whose dependence from the parameter gives rise to interesting formulae. In fact, starting from Chebyshev polynomials, the first Gateaux differential of $Q$ can be expressed via Hilbert transforms and a bilinear singular integral operator first considered by Calderón (see Section 7). The $L^2$-boundedness of this operator is still an open problem.

We do not deal with numerical applications, but let us note that this theory can provide some useful computer tools. The Kerzman-Stein formula can be used to efficiently compute orthogonal polynomials relative to standard and non-standard weights. The analyticity results suggest the possibility of building fast algorithms (complexity $n \log n$) to convert an orthogonal expansion relative to a fixed set of polynomials into the expansion relative to another set of polynomials, provided they are within the «analyticity range» (see Theorem 1 in Section 6, and see also [AR] for an example of numerical conversion from Chebyshev polynomials expansions into Legendre polynomials expansions).

2. **Orthogonal Polynomials Dependent on a BMO Parameter**

Consider a «perturbed» space $L^2(u^2d\mu)$ where $u(x) = e^{b(x)}$ is a suitable function. Notice that if $b(x)$ is close to zero (in some Banach Space norm which will be chosen later), then the new measure

$$u^2(x)d\mu(x) = e^{2b(x)}d\mu(x)$$

is «close» to the original one $d\mu$.  

Now, given $f \in L^2(u^2 d\mu)$, we can develop it into a series of orthogonal polynomials (the polynomials which are orthogonal with respect to $u^2 d\mu$) and consider the partial sums $S_n f$ of this series. Notice that, for a fixed measure $u^2 d\mu$ living on a fixed space, the orthogonal polynomials are uniquely determined (up to a normalization), and so is $S_n$. Notice also that the partial sums operator $S_n$ can be thought of as an operator-valued function of $b$. In fact, we want to study the mapping

\[
(2) \quad b \mapsto S_n(b).
\]

It is hard to study (2) directly because $S_n(b)$, in spite of being a well-defined projection from $L^2(u^2 d\mu)$ into polynomials of degree equal or less than $n$, acts on different $L^2$ spaces as $b$ varies. On the other hand, we can look at

\[
(3) \quad b \mapsto V_n(b)
\]

where

\[
(4) \quad V_n(b) = u S_n(b) u^{-1} = e^b S_n(b) e^{-b}.
\]

With a little abuse of notation, in the above definition we use a function to denote the operator of multiplication by that function. It is easy to check that $V_n(b)$ always acts on $L^2(d\mu)$, that the $L^2(d\mu)$-boundedness of $V_n(b)$ is equivalent to the $L^2(u^2 d\mu)$-boundedness of $S_n(b)$, and that the two operator norms are equal.

The mapping (3) can now be seen in the context of calculus on Banach Spaces, see for example [B], where notions like continuity, differentiability, and analyticity are well defined.

We claim that the «natural» Banach Space for $b$ is $\text{BMO}$ (Bounded Mean Oscillation) and that in «many» situations, depending on the choice of the measure space we start with, the dependence (3) is in fact analytic. We will make this claim precise later, and we will prove it.

Let us notice at this point that a further simplification in the study of this problem is brought about by a remarkable formula due to Kerzman and Stein [KS]. The formula is

\[
(5) \quad V = P(I + (P - P^*))^{-1}
\]

where $V$ is a self-adjoint projection sending a Hilbert Space $H$ into a proper closed subspace $K$ and $P$ is an oblique projection (non self-adjoint) also sending $H$ into $K$.

The first step in the proof of (5) is to show that the operator $I + (P - P^*)$
is invertible, but in fact, since \((P - P^*)^* = P^* - P = -(P - P^*)\), we see that \(P - P^*\) is skew adjoint, which means that its spectrum is purely imaginary. In particular \(-1 \notin \text{spec} (P - P^*)\), which shows the invertibility of \(I + (P - P^*)\). We can now rewrite (5) as

\[ V(I + (P - P^*)) = P \]

and this is the same as

\[ P = V + VP - VP^* = V + P - VP^*, \]

which simplifies into

\[ VP^* = V. \]

Finally, we prove this last operator equality using an elementary lemma from Functional Analysis (see for example [R] p. 296) and the following chain of scalar-product equalities, which hold for any element \(h \in H\):

\[(VP^*h, h) = (P^*h, Vh) = (h, PVh) = (h, Vh) = (Vh, h). \]

Having proven (5), we now apply it to our situation by choosing

\[ V = V_n(b) = e^bS_n(b)e^{-b} \]

and

\[ P = P_n(b) = e^bS_n(0)e^{-b}. \]

Notice that (with the same abuse of notation) we have used \(e^b\) to indicate the operator of multiplication by \(e^b\). Both \(V_n(b)\) and \(P_n(b)\) are projection operators from \(L^2(d\mu)\) into the same closed subspace; \(V_n(b)\) is self-adjoint while \(P_n(b)\) is an oblique projection that satisfies

\[ P_n(b)^* = P_n(-b). \]

The Kerzman-Stein formula (5) tells us that the analytic properties of \(P_n(b)\) as an operator-valued function of \(b\) are inherited by \(V_n(b)\). This is a great simplification because it is easier to deal explicitly with the integral kernel of \(S_n(0)\) than that of \(S_n(b)\). So, once we have proven the uniform analyticity of \(P_n(b)\), we have proven it also for \(V_n(b)\). To be self-contained, let us state here the following

**Definition 1 of Uniform Analyticity for a sequence \(|P_n(b)|\) of operator-valued functions of \(b \in B\) (some Banach Space).**
The sequence \( \{P_n(b)\} \) is uniformly (real) analytic in a neighborhood of \( 0 \in B \) if and only if there exists \( \delta > 0 \) such that for every \( f \in L^2(d\mu) \) and all \( b \) with \( \|b\|_B < \delta \) we have

\[
P_n(b)f = \sum_{k=0}^{\infty} \Lambda_{nk} (b, \ldots, b, f) \quad \text{for every } n
\]

where \( \Lambda_{nk} \) is a bounded \((k+1)\)-multilinear operator sending \( B^k \times L^2(d\mu) \) into \( L^2(d\mu) \) and satisfying the estimate

\[
\|\Lambda_{nk}(b, \ldots, b, f)\|_{L^2(d\mu)} \leq c^{k+1} \|b\|_B^k \|f\|_{L^2(d\mu)}.
\]

The constant \( c \) in (9) does not depend on \( n \). Notice that \( \Lambda_{nk}(b, \ldots, b, \bullet) \) is the \( k \)th Gateaux differential of \( P_n(b) \) at 0 in the direction of \( b \).

A similar definition can be given, when \( b \) belongs to a complex Banach Space, for the uniform holomorphy of \( \{P_n(b)\} \) in a neighborhood of 0. Actually, as in the case of one complex variable, the existence of one derivative in an open domain (e.g., a neighborhood of 0) implies holomorphy in the same domain (see [B]), so we have

**Definition 2** \( \{P_n(b)\} \) is uniformly holomorphic in a neighborhood \( U \) of \( 0 \in B \) (complex Banach Space) if for every \( n \), and all \( b \in U \) we have

\[
\|P_n(b)\|_{L^2 \rightarrow L^2} \leq c \quad \text{and} \quad P_n(b) \text{ is Gateaux-differentiable.}
\]

We also have a notion of the «biggest space» in which the variable \( b \) can live and maintain the dependence \( b \rightarrow P_n(b) \) holomorphic. It is given by

**Definition 3.** \( B \) is the space of uniform holomorphy at 0 for \( \{P_n(b)\} \) if \( \{P_n(b)\} \) is uniformly holomorphic in a neighborhood of 0 and

\[
\sup \|\Lambda_{n,1}(b, \bullet)\|_{L^2 \rightarrow L^2} < \infty \text{ if and only if } b \in B
\]

We will see later that, for the particular sequence of projection in our problem, proving conditions like (10) or (11) amounts to the proof of some suitable weighted norm inequalities.

### 3. Connections with Infinite-Dimensional Toda Flows

The set-up we have outlined in the previous paragraph is intimately connected with the theory of Toda flows. This theory has been studied
independently, at first without any reference to orthogonal polynomials (see for example [DLT]).

Let us consider again the «perturbed» space \( L^2(e^{2tb}d\mu) \) where we have added the real parameter \( t \in (-\epsilon, \epsilon) \) in the exponential and where we consider the function \( b(x) \) fixed.

If we apply the Gram-Schmidt orthogonalization process to the sequence \( 1, x, x^2, \ldots \) embedded in this \( L^2 \) space, we get a sequence \( \{p_j(t)\} \) of orthogonal polynomials which depends on the parameter \( t \). These polynomials satisfy a three-term recurrence

\[
(1) \quad xp_j(t) = A_j(t)p_{j+1}(t) + B_j(t)p_j(t) + A_{j-1}(t)p_{j-1}(t)
\]

with \( j = 0, 1, \ldots \) and \( p_{-1}(t) = 0 \). Since, for each \( t \), they are a complete orthonormal system in \( L^2(e^{2tb}d\mu) \), the map \( f(x) \rightarrow xf(x) \) induces a bounded linear map \( T(t) \) on \( L^2 \) that can be represented with an infinite tri-diagonal matrix

\[
(2) \quad T(t) = \begin{bmatrix}
B_0(t) & A_0(t) & 0 & 0 & \cdots \\
A_0(t) & B_1(t) & A_1(t) & 0 & \cdots \\
0 & A_1(t) & B_2(t) & A_2(t) & \cdots \\
0 & 0 & A_2(t) & B_3(t) & A_3(t) \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

If our measure \( d\mu \) (and our polynomials) live on the closed interval \([-1, 1]\), we have the following explicit formulae for the recursion coefficients (see, e.g., Szegő [Sz])

\[
(3) \quad A_j(t) = \int_{-1}^{1} xp_j(t)p_{j+1}(t)e^{2tb(t)}d\mu(x).
\]

\[
(3') \quad B_j(t) = \int_{-1}^{1} x^2 p_j(t)e^{2tb(t)}d\mu(x).
\]

We claim that, in general, the operator \( T(t) \) is a solution of the (infinite dimensional, non-linear) Toda equation, which can be written as

\[
(4) \quad \hat{T} = [T, \hat{b}(T)]
\]

where the square brackets stand for the commutator of two operators and \( \hat{b}(T) \) is a sort of Hilbert Transform obtained in the following way: first we use
operator calculus to define the function $b$ evaluated at $T$, getting some symmetric infinite matrix

\[
(5) \quad b(T) = \begin{pmatrix}
& & & b_+ \\
& d & & \\
b_- & & & \\
& & & \\
\end{pmatrix}
\]

which we have separated in (5) into lower triangular block ($b_-$), diagonal ($d$), and upper triangular block ($b_+$), then we define

\[
(6) \quad \bar{b}(T) = \begin{pmatrix}
& & & b^T_- \\
& 0 & & \\
b_- & & & \\
& & & \\
\end{pmatrix}
\]

i.e., we put zeros on the diagonal and replace the upper triangular block ($b_+$) with the transpose of the lower triangular block ($b_-$) multiplied by $-1$.

**Proof of the claim**

First we want to show that the solution of (4) is expressed by

\[
(7) \quad T(t) = Q^T(t) T(0) Q(t)
\]

where the infinite orthogonal matrix $Q(t)$ comes from

\[
(8) \quad e^{b(T(0))} = Q(t) R(t)
\]

The right hand side of (8) is the QR decomposition of the l.h.s.; this decomposition is uniquely determined by asking that $Q$ be orthogonal and $R$ upper triangular (see [DLT]). After showing that (7) is indeed a solution of (9), we will show that the $T(t)$ so obtained coincides with the $T(t)$ in (2).

In the computations that follow, we will assume implicitly that matrices denoted by capital letters are functions of $t$, we will denote differentiation with respect to $t$ with a dot, and use the notation $T_0 = T(0)$. By differentiating both members of (7) we get

\[
(9) \quad \dot{T} = \dot{Q}^T T_0 Q + Q^T T_0 \dot{Q} = \dot{Q}^T Q T^T Q + Q^T Q T^T \dot{Q} = \dot{Q}^T Q T + T Q^T \dot{Q}.
\]

The second equality is obtained by plugging in the expression for $T(0)$ that comes from (7), the third equality holds just because $Q^T Q = I$.

We want to check that

\[
\dot{T} = [T, \bar{b}(T)] = T \bar{b}(T) - \bar{b}(T) T
\]
but, looking at the last term of (9), this amounts to showing that

\[ b(T) = Q^T \dot{Q} \]  \hspace{1cm} (10)

Notice that, by the definition of \( b(T) \), both terms of this equality must be skew-adjoint, therefore if the equality holds we must have

\[ (Q^T \dot{Q})^T = \dot{Q}^T Q = -Q^T \dot{Q}. \]

and this is true, as we can see differentiating the identity \( Q^T Q = I \).

To prove (10) we differentiate (8) and use the fact that the exponential of an operator commutes with that operator. We get

\[ b(T_0)QR = \dot{Q}R + QR \]  \hspace{1cm} (11)

and multiplying both terms by \( Q^T \) on the left and by \( R^{-1} \) on the right we get

\[ Q^T b(T_0)Q = Q^T \dot{Q} + \dot{R}R^{-1} \]

which, after noticing that \( Q^T b(T_0)Q = b(Q^T T_0 Q) = b(T) \), can be rewritten as

\[ b(T) = Q^T \dot{Q} + \dot{R}R^{-1}. \]  \hspace{1cm} (12)

Let us now go back to the block notation for \( b(T) \) that we used in (5), and show that, in that notation, our last equality (12) becomes

\[
\begin{pmatrix}
\begin{array}{cc}
\ \hline
b_+ \\
\ \hline
\end{array}
& \\
\ \hline
d \\
\ \hline
\end{pmatrix}
= \\
\begin{pmatrix}
\begin{array}{cc}
\ \hline
0 \\
\ \hline
b_- \\
\ \hline
\end{array}
& \\
\ \hline
b_+ + b_- \\
\ \hline
\end{pmatrix}
+ \\
\begin{pmatrix}
\begin{array}{cc}
\ \hline
\ \hline
b_+ \\
\ \hline
\ \hline
d \\
\ \hline
\end{array}
& \\
\ \hline
0 \\
\ \hline
\ \hline
\end{pmatrix}
\]  \hspace{1cm} (13)

In fact, we know that the first matrix on the right hand side of (12) is skew-adjoint, and we know that the second one is upper triangular (because differentiating, inverting, or multiplying triangular matrices we still get triangular matrices). This knowledge induces a chain of deductions that allows us to «fill in» with the proper blocks the two matrices on the right hand side of (13). The diagonal of the first matrix on the right hand side of (13) must be 0 and this forces the diagonal of the second matrix to be \( d \). Since this second matrix is upper triangular, we know that the lower triangular block is 0 and this forces the lower triangular block of the first matrix to be \( b_- \). By skew-adjointness, the upper triangular block of the first matrix is \(-b_+^T\) and this forces the same block in the other matrix to be \( b_+ + b_-^T \), completing the picture.
The equality (12) written in the block notation (13) shows in particular that

\[
Q^T \dot{Q} = \begin{pmatrix}
- b_+ \\
\begin{array}{c}
- b_+ \\
0
\end{array}
\end{pmatrix} = \delta(T)
\]

but this is the identity (10) we wanted to prove, so (7) is indeed a solution of (4).

Now we want to check that the solution \( T(t) \) of the Toda equation defined by (7) is the same \( T(t) \) defined in (2) as the representation in the orthogonal polynomial basis \( \{P_j(x)\} \) of the operator \( M_x : f(x) \to xf(x) \). To do that, notice that we can write

\[
e^{bt} S_n(bt) e^{-bt} = e^{bt} R^{-1} R S_n(bt) R^{-1} R e^{bt} = Q S_n(0) Q^{-1}.
\]

Here, as in the first paragraph, \( S_n(bt) \) is the partial sum projection operator for the orthogonal polynomials \( P_{j,t}(x) \in L^2(e^{2bt} d\mu) \), while \( R^{-1} \) is the upper triangular infinite matrix representing the coefficients of the «perturbed» basis of \( \{P_{j,t}(x)\} \) in terms of the «unperturbed» one \( \{P_{j,0}(x)\} \), and finally \( Q(t) \) is an orthogonal transformation defined by

\[
Q(t) = e^{bt} R^{-1}(t).
\]

Let us point out that the action of \( R \) is given by

\[
\sum_{j=0}^{\infty} c_j P_{j,0}(x) \overset{R^{-1}}{\longrightarrow} \sum_{j=0}^{\infty} c_j P_{j,t}(x) \overset{R}{\longrightarrow} \sum_{j=0}^{\infty} c_j P_{j,0}(x).
\]

while the action of \( Q \) is given by

\[
\sum_{j=0}^{\infty} c_j P_{j,0}(x) \overset{Q}{\longrightarrow} \sum_{j=0}^{\infty} c_j P_{j,t}(x) e^{bt} \overset{Q^{-1}}{\longrightarrow} \sum_{j=0}^{\infty} c_j P_{j,0}(x).
\]

These last two diagrams show that \( S_n(0) = R S_n(bt) R^{-1} \), an identity that we have implicitly used in (14). Also they show the orthogonality of \( Q \).

We observe now that the matrix of \( T(0) \), defined as in (2), represents the multiplication operator \( M_x \) in the orthonormal basis \( \{P_{j,0}\} \). We also have

\[
b(T(0)) = M_{bc(t)}
\]

and
(19) \[ e^{ib(T(0))} = e^{\mathcal{M}_b} = M_e^b = e^{ib} \]

In the last step, we have simply used again the shorthand of indicating a multiplication operator directly by the multiplicator function. We observe that (15), because of (19), give us the QR decomposition of \( e^{ib(T(0))} \), and therefore the \( Q \) appearing in (15) is the same one that appears in the solution of the Toda equation (4). Because of the action of \( Q \), we see that the \( T(t) \) in (2) is the same as in (7).

4. The Role of Weighted Norm Inequalities

According to Definition 2 in Section 2, the sequence of operators \( \{ P_n(b) \} \) is uniformly holomorphic in a neighborhood \( U \) of \( 0 \in B \) if \( P_n(b) \) is uniformly bounded in operator norm and Gateaux-differentiable for \( b \in U \).

It turns out that, for the particular sequence of projections \( P_n(b) = e^b S_n(0)e^{-b} \) which we are considering, the uniform boundedness in \( U \) implies also Gateaux-differentiability. The idea that leads to this simplification is due to Coifman, Rochberg, and Weiss [CR], and goes as follows:

We can write

(1) \[ \{ P_n(b)f \}(x) = \int e^{i(b(x) - b(y))} D_n(x, y)f(y)d\mu(y) \]

where \( D_n \) is the Dirichlet Kernel for the partial sums operator \( S_n(0) \) relative to the orthogonal polynomials on \( L^2(d\mu) \). Let us fix \( b \) and multiply it by a complex parameter \( z \). We then have

(2) \[ \left\{ \frac{d}{dz} P_n(zb)f \right\}(x) = \int (b(x) - b(y))e^{i(b(x) - b(y))} D_n(x, y)f(y)d\mu(y). \]

Notice then that

(3) \[ \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(b(x) - b(y))} e^{-i\theta} d\theta = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(b(x) - b(y)) e^{ibk}}{k!} e^{-i\theta} d\theta = b(x) - b(y) \]
Therefore we can write

\[ (4) \quad \left\{ \frac{d}{dz} P_n(zb)f \right\} = \frac{1}{2\pi} \int_0^{2\pi} \exp[i(z + e^{i\theta})(b(x) - (b(y)))D_n(x,y)f(y)d\mu(y)e^{-i\theta}d\theta \]

which, using the notation \( P_{n\theta} = P_n((z + e^{i\theta})b) \) can be written as

\[ (5) \quad \left\{ \frac{d}{dz} P_n(zb)f \right\} (x) = \frac{1}{2\pi} \int_0^{2\pi} [P_{n\theta}f] (x)e^{-i\theta}d\theta \]

We are assuming by hypothesis

\[ (6) \quad \|b\| < \delta \text{ implies } \|P_n(b)\|_{L^1 \to L^1} \leq c \]

Therefore, choosing \( |z| < 1 \) and \( \|b\| < \delta/2 \), we have

\[ (7) \quad \| \frac{d}{dz} P_n(zb) \|_{L^1 \to L^1} \leq \frac{1}{2\pi} \int_0^{2\pi} \|P_{n\theta}\|_{L^1 \to L^1} d\theta \leq c. \]

So the uniform boundedness expressed by (6) implies locally also uniform analyticity.

Notice that (6) can be seen as a weighted norm inequality for \( S_n(0) \). In fact, the assumption that for \( \|b\| < \delta \) one has

\[ \|P_n(b)f\|_{L^1(\mu_0)} \leq c\|f\|_{L^1(\mu_0)} \quad \text{for all } f \in L^1(\mu_0) \]

just by setting \( f = e^{ib}g \), is seen to be equivalent to

\[ (8) \quad \|S_n(0)g\|_{L^1(e^{ib}\mu_0)} \leq c\|g\|_{L^1(e^{ib}\mu_0)} \quad \text{for all } g \in L^1(e^{ib}\mu_0). \]

We will see that in many cases the partial sums operator looks essentially like a Hilbert Transform, while \( e^{2b} \) is a good \( A_p \) weight when \( b \) has a small BMO norm. This allows us to prove weighted norm inequalities like (8) using the results of Hunt, Muckenhoupt, and Wheeden (see [HMW] and also [CF]).

In order to identify BMO also as the space of uniform analyticity at 0, we need to prove, according to Definition 3 in §2, that

\[ (9) \quad b \in \text{BMO if and only if } \sup_n \|\Lambda_n(b, \bullet)\|_{L^1 \to L^1} < \infty. \]
Let us compute the Gateaux differential that appears in (9). We have

\[
\frac{d}{dt} P_n(tb) \bigg|_{t=0} = \frac{d}{dt} \left(e^{tb} S_n(0)e^{-tb}\right) \bigg|_{t=0} = b e^{tb} S_n(0)e^{-tb} + e^{tb} S_n(0) \left(-b e^{tb}\right) \bigg|_{t=0} = [b, S_n(0)]
\]

So condition (9) can be rewritten asking that there exist \( c_1, c_2 > 0 \) such that

\[
(10) \quad c_1 \|b\|_* \leq \sup_n \| [b, S_n(0)] \|_{L^2 - L^2} \leq c_2 \|b\|_*
\]

where we indicate with a * subscript the BMO norm.

Actually, the inequality on the right is a consequence of uniform holomorphy (which implies the boundedness of all derivatives), so we only need to prove the one on the left.

5. Perturbation Theory of a Family of Orthogonal Polynomials Introduced by Szegö

In Szegö’s classic book [Sz] on orthogonal polynomials, the starting point for the proof of his equiconvergence results is the study of a particular family of orthogonal polynomials in \( L^2(T, d\mu) \). The measure \( d\mu \) on the unit circle is given by

\[
(1) \quad d\mu(\theta) = \frac{1}{g(\theta)} \frac{d\theta}{2\pi}
\]

with \( g \) trigonometric polynomial of degree \( m \) and \( g(\theta) > 0 \) for all \( \theta \in [0, 2\pi] \). It turns out that for any fixed \( g \), even though it is not easy to find a closed formula for the first \( m \) orthogonal polynomials, there is in fact a simple expression for them when the degree is greater than or equal to \( m \). This restriction does not affect Szegö’s proofs, and will not affect ours either.

We have the following:

**Proposition 1** The sequence \( \{\phi_j(z)\} \) of complex polynomials (normalized with a strictly positive coefficient for \( z^j \)) satisfying

\[
(2) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_j(z) \overline{\phi_k(z)} \frac{1}{g(\theta)} \frac{d\theta}{2\pi} = \delta_{jk}
\]

\( (z = e^{i\theta} \text{ in the integral}) \) is given, when \( j = m, m + 1, \ldots \), by
(3) \[ \phi_j(z) = z^{-m} h^*(z) \]

where \( h(z) \) is a complex polynomial of degree \( m \) such that

(4) \[ g(\theta) = |h(z)|^2 \quad \text{when} \quad z = e^{i\theta} \]

and \( h^* \) is the reciprocal polynomial of \( h \), i.e.,

(5) \[ h^*(z) = z^m \overline{h} \left( \frac{1}{z} \right). \]

**Proof.** The existence of representation (4) for the positive polynomial \( g \) is a special case of a more general one, from \( H^p \) theory, that holds in fact for all \( f \) such that \( \log f \in L^1(\mathbb{T}) \). Notice that we can choose \( h(0) > 0 \) and \( h(z) \not\equiv 0 \) for \( |z| < 1 \).

We can rewrite (3) as

(6) \[ \phi_j(z) = z^j \overline{h}(z^{-1}). \]

Here, as before in (5), we use the convention that if \( h(z) \) is a polynomial in \( z \), \( \overline{h} \) is the polynomial obtained from \( h \) by conjugating the coefficients (not \( z \)).

Plugging (6) into (2) we can easily verify the case \( j = k \) just remembering that on the unit circle we have \( \overline{z} = z^{-1} \).

To prove the case \( j \neq k \) we actually show that

(7) \[ \frac{1}{2\pi i} \int_{-\pi}^{\pi} \phi_j(z) \overline{\varphi}(z) \frac{1}{g(\theta)} \, d\theta = 0 \quad \text{when} \quad z = e^{i\theta}. \]

for any polynomial \( \varphi \) of degree \( j - 1 \). In fact we get

\[
\frac{1}{2\pi i} \int_{\Gamma} z^j \overline{h}(z^{-1}) \frac{\overline{\varphi}(z^{-1})}{h(z)h(z^{-1})} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{z^{j-1} \overline{\varphi}(z^{-1})}{h(z)} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\Gamma} \overline{\varphi}(z) \frac{dz}{h(z)} = 0.
\]

The last equality holds because we are integrating along the unit circle \( \Gamma \) a meromorphic function with no poles inside.
Given a closed expression of the polynomials, one can give an expression of the Dirichlet kernel of the partial sums operator (we refer to [Sz], p. 292, for the algebraic details). In our case we have

\[
D_n(\theta, \varphi) = \frac{h(e^{i\theta})h(e^{i\varphi}) - e^{i(n+1)(\theta - \varphi)}h(e^{i\theta})h(e^{i\varphi})}{1 - e^{i(\theta - \varphi)}}.
\]

This expression is valid whenever \( n \geq m \), and shows explicitly the dependence of the kernel on the weight. Following our general plan we look at \( P_n(b) \) which is now given by

\[
\begin{align*}
\left( P_n(b)f \right)(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ib(\psi)} D_n(\theta, \psi) e^{-b(\psi)} \frac{f(\psi)}{g(\psi)} \, d\psi \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ib(\psi)} \frac{f(\psi)}{1 - e^{i(\theta - \psi)}} \left\{ \frac{h(e^{i\theta})}{h(e^{i\psi})} - e^{i(n+1)(\theta - \psi)} \frac{h(e^{i\theta})}{h(e^{i\varphi})} \right\} d\psi \\
&= \left( (P_n(b) - P_n(b))f \right)(\theta)
\end{align*}
\]

where

\[
\begin{align*}
\left( (P_n(b)f) \right)(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ib(\psi)} h(e^{i\theta}) \frac{f(\psi)}{1 - e^{i(\theta - \psi)}} e^{-b(\psi)} h(e^{i\varphi})^{-1} \, d\psi \\
\left( (P_n(b)f) \right)(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n+1)(\theta - \psi)} \frac{f(\psi)}{1 - e^{i(\theta - \psi)}} e^{-i(n+1)\psi - b(\psi)} h(e^{i\psi})^{-1} \, d\psi.
\end{align*}
\]

The operators in (10') and (10a) consist of a singular integral (essentially a Hilbert transform) conjugated by a multiplication operator on the left and the inverse of the same multiplication on the right. Let us look at a third operator

\[
\begin{align*}
\left( (P_{\ast}(b)f) \right)(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\text{Re}b(\theta)g(\theta)^{1/2}} \frac{f(\psi)}{1 - e^{i(\theta - \psi)}} e^{-\text{Re}b(\theta)g(\psi)^{-1/2}} \, d\psi
\end{align*}
\]

which, by the result in [HMW], is bounded for \( e^{\text{Re}b(\theta)g(\theta)^{1/2}} \in A_2 \). This last condition is satisfied when the real part of \( b(\theta) \) has BMO-norm smaller
than some suitable $\delta$. (See [GCRF], chapter 4, and notice that $g(\theta)$ is a polynomial without zeroes for every $\theta$.

The same condition on $b(\theta)$ implies uniform boundedness also for (10') and (10''), just noticing that they are both obtained from (10'') multiplying on the left by a complex function of modulus 1 and on the right by the inverse of that same function. By what we have seen in the previous paragraph, for this particular kind of operators uniform boundedness implies uniform holomorphy, so the family of projections $[P_{n}(b)]$ is uniformly holomorphic for $b$ in a neighborhood of $0 \in \text{BMO}$.

By Kerzman-Stein formula (2.5), or also using the Toda Flow identities of Section 3, the same uniform result holds for the self adjoint projections $V_{n}(b)$.

The next step of the plan consists in showing that BMO is in fact the space of uniform holomorphy of the $[P_{n}(b)]$, and, as we have seen in general, that amounts to proving the inequality

\begin{equation}
(11) \quad c_{1} \| b \| s \leq \sup_{n} \| [b, S_{n}(0)] \|_{L^{2} \rightarrow L^{2}}.
\end{equation}

To do that we define

\begin{equation}
(12) \quad f_{l}(\psi) = x_{l}(\psi) \left( 1 - e^{i(\theta - \psi)} \right) h(e^{i\psi})
\end{equation}

where $I$ is a closed subinterval of $[0, 2\pi]$.

We have

\begin{equation}
(13) \quad [[b, S_{0}(0)] f_{l}](\theta) = \frac{1}{2\pi} \int_{I} \left( b(\theta) - b(\psi) \right) \left\{ h(e^{i\theta}) - e^{i(n+1)(\theta - \psi)} \frac{h(e^{i\psi}) h(e^{i\theta})}{h(e^{i\psi})} \right\} d\psi.
\end{equation}

By Riemann-Lebesgue lemma we have

\begin{equation}
(14) \quad \lim_{n \to \infty} [[b, S_{0}(0)] f_{l}](\theta) = \frac{1}{2\pi} \int_{I} \left( b(\theta) - b(\psi) \right) h(e^{i\theta}) d\psi = \frac{|I|}{2\pi} h(e^{i\theta}) (b(\theta) - m_{I}(b))
\end{equation}

where $m_{I}(b)$ is the average of the function $b$ over $I$. By Fatou's lemma we have

\begin{equation}
(15) \quad |I| \int_{I} |b(\theta) - m_{I}(b)| d\theta \leq c \limsup_{n \to \infty} \int_{I} \left| [[b, S_{n}(0)] f_{l}](\theta) \right| \frac{d\theta}{|h(e^{i\theta})|}.
\end{equation}

Therefore (11) will hold if we can bound the right hand side of this by
\[ c \mid I \mid^2 \sup_n \| [b, S_n(0)] \| \]

where \( c \) does not depend on \( n \) and \( I \).

Let \( \alpha \) be the midpoint of \( I \); a straightforward computation allows us to rewrite (12) in the form

\[
(16) \quad f_\delta(\psi) = e^{i(\theta - \delta)/2} \sin \frac{\theta - \alpha}{2} f_1(\psi) + e^{i(\theta + \delta)/2} \cos \frac{\theta - \alpha}{2} f_2(\psi)
\]

where \( f_1 \) and \( f_2 \) do not depend on \( \theta \), and are given by

\[
(16') \quad f_1(\psi) = -2i \chi_I(\psi) h(e^{i\psi}) e^{i(\alpha - \psi)/2} \cos \frac{\alpha - \psi}{2}.
\]

\[
(16') \quad f_2(\psi) = -2i \chi_I(\psi) h(e^{i\psi}) e^{i(\alpha - \psi)/2} \sin \frac{\alpha - \psi}{2}.
\]

Using (16) and Schwarz's inequality, we get

\[
(17) \quad \int_J \left| \left[ [b, S_n(0)] f_\delta \right](\theta) \right| \frac{d\theta}{|h(e^{i\theta})|} \leq \left( \int_J \left| \sin \frac{\theta - \alpha}{2} \right|^2 d\theta \right)^{1/2} \left( \int_J \left| [b, S_n(0)] f_1 \right|^2 \frac{d\theta}{|h(e^{i\theta})|^2} \right)^{1/2} + \left( \int_J \left| \cos \frac{\theta - \alpha}{2} \right|^2 d\theta \right)^{1/2} \left( \int_J \left| [b, S_n(0)] f_2 \right|^2 \frac{d\theta}{|h(e^{i\theta})|^2} \right)^{1/2}.
\]

This last inequality holds for every \( n \), so we can take the upper limit of the left hand side. Also, we notice that the right factors in the two terms on the right hand side are, for \( k = 1, 2 \), exactly the \( L^2(d\mu) \) norms of \([b, S_n(0)] f_k\), since \( d\mu(\theta) = |h(e^{i\theta})|^{-2} d\theta \).

We have

\[
(18) \quad \lim_{n \to \infty} \sup_n \int_J \left| [b, S_n(0)] f_\delta \right| \frac{d\theta}{|h(e^{i\theta})|} \leq c_1 \mid I \mid^{3/2} A \| f_1 \|_{L^2(d\mu)} + c_2 \mid I \mid^{1/2} A \| f_2 \|_{L^2(d\mu)},
\]

where \( c_1, c_2 \) do not depend on \( n \) or \( I \) and
(19) \[ A = \sup \| [b, S_n(0)] \|_{L^2(d\mu) \to L^2(d\mu)}. \]

To complete our proof we only need to check that
\[ \| f_1 \|_{L^2(d\mu)} \leq c_1 | I |^{1/2} \quad \text{and} \quad \| f_2 \|_{L^2(d\mu)} \leq c_4 | I |^{3/2} \]
and this is easily seen to be true by our choice of \( f_1 \) and \( f_2 \).

6. Generalizations and the Role of VMO

In the previous paragraph we have proven that the family of orthogonal projections
\[ V_n(b) = e^b S_n(b) e^{-b} : L^2(d\mu) \to L^2(d\mu) \]
is uniformly holomorphic (Definition 2.2) for \( b \) belonging to a neighborhood of 0 in BMO. Also we have shown that BMO is the space of uniformly holomorphy at 0 (Definition 2.3). Notice that our \( d\mu \) is a measure on \( T \) expressed via a non-vanishing weight of the form (5.1).

In fact, as long as we satisfy the proper estimates, we could have started from a measure \( d\mu \), given by a weight with zeroes. The intuitive idea is that a BMO function can contain unbounded logarithmic spikes (and the exponential of a negative spike can be 0); here we want to make this idea more precise, getting an analyticity result that holds for a «starting set» of orthogonal polynomials more general than those of the form (5.2).

We need first to introduce an important closed Banach subspace of BMO. It is called VMO (Vanishing Mean Oscillation) and is defined by
(1) \[ \text{VMO} = \{ f \in \text{BMO} : \lim_{\alpha \to 0} M_\alpha(f) = 0 \} \]
where
(2) \[ M_\alpha(f) = \sup_{|I| \leq \alpha} \frac{1}{|I|} \int_I | f(x) - f_I | \, dx. \]
As usual \( I \) denotes an interval and \( f_I \) the average of \( f \) on \( I \). Notice that, with this notation, the usual BMO norm is given by
(3) \[ \| f \|_* = \lim_{\alpha \to \infty} M_\alpha(f) \]
This space has been studied by Sarason, and in [Sa] one can find the statement and proof of some basic properties; we are going to use, in particular, the fact that VMO is the closure, in BMO-norm, of the subspace of continuous functions.

**Lemma 1**: Given a real-valued function $\gamma \in V.M.O. (\mathbb{T})$ one can find a sequence of strictly positive trigonometric polynomials $g_j(\theta)$ such that

$$\lim_{j \to \infty} \left\| \gamma(\theta) - \log \left( \frac{1}{g_j(\theta)} \right) \right\|_\ast = 0$$

**Proof.** Continuous functions are BMO-dense in VMO and since $\left\| f \right\|_\ast \leq 2 \left\| f \right\|_\infty$ we only need to show that for any real-valued $\beta(\theta) \in C(\mathbb{T})$ one can find a sequence of strictly positive trigonometric polynomials $g_j(\theta)$ such that

$$\lim_{j \to \infty} \left\| \beta(\theta) - \log \left( \frac{1}{g_j(\theta)} \right) \right\|_\infty = 0$$

we have

$$\left| \beta(\theta) - \log \left( \frac{1}{g_j(\theta)} \right) \right| = \left| \log(g_j(\theta)e^{\beta(\theta)}) \right| = \left| \log(1 + e^{\beta(\theta)}(g_j(\theta) - e^{-\beta(\theta)})) \right|.$$

And the compactness of $\mathbb{T}$ implies the existence of two positive constants $c_1, c_2$ such that

$$c_1 \leq e^{\beta(\theta)} \leq c_2.$$ But this implies that the quantity in (5) can be made uniformly small just by uniform approximation of the positive continuous function $e^{-\beta(\theta)}$ with the positive polynomials $g_j(\theta)$.

**Lemma 2**: The radius of analycity $\delta$ for the mapping relative to the orthogonal polynomials (5.2) does not depend on the particular $g(\theta)$ contained in the weight.
PROOF. Going back to the previous paragraph, we notice that we have analyticity for those functions \( b \in \text{BMO} \) which satisfy

\[
    e^{\Re b(\theta)} \sqrt{g(\theta)} \in A_2.
\]

It is a property of \( A_2 \) weights to be invariant under multiplication by functions bounded away from 0 and infinity (see [GCRF], chapter 4). On the other hand the first factor in (7) is an \( A_2 \) weight in general only if \( \| \Re b \|_* < \delta \).

Remark. If one makes additional assumptions on the nature of \( b \), one can find functions with large BMO norm whose exponential is still in \( A_2 \). This surprising fact is well illustrated by a theorem of Garnett and Jones which says the following

\[
    A(b) = \sup \{ \lambda > 0 : e^{\lambda b} \in A_2 \} = \left( \text{dist}_{\text{BMO}} [b, L^\infty] \right)^{-1},
\]

where the symbol \( \approx \) means «same order of magnitude» and where

\[
    \text{dist}_{\text{BMO}} [b, L^\infty] = \inf \{ \| b - f \|_* : f \in L^\infty \}.
\]

In particular, the exponential of any VMO function (regardless of the BMO norm) is an \( A_2 \) weight.

We can now state our general results as follows

**Theorem 1.** The uniform analitycity of the mapping \( b \to V_n(b) \) holds for all orthogonal polynomials on \( L^2(d\mu) \) where \( d\mu \) is a measure on \( \Gamma \) of the form

\[
    d\mu(\theta) = e^{2\gamma(\theta) + \beta(\theta)} \frac{d\theta}{2\pi}
\]

with \( \gamma(\theta) \) any real-valued VMO function and \( \beta(\theta) \) a complex-valued BMO function such that

\[
    \| \Re \beta \|_* < \delta
\]

where \( \delta \) is the radius of analitycity for the Szegő polynomials (5.2).

The new radius of analitycity, if we start from the polynomials relative to the measure (10), is given by

\[
    \delta' = \delta - \| \Re \beta \|_*
\]
PROOF. Let us fix some BMO function $\beta$ satisfying (11). We know that starting from a measure of the form (5.1) we can expand $V_n(b)f$ into a series of multilinear operators (2.8) satisfying an estimate (2.9). Let us define a new «starting space» for our perturbation

$$(13)\quad L^2(d\mu) = L^2 \left( e^{2\beta(\theta)} \frac{1}{g(\theta)} \frac{d\theta}{2\pi} \right).$$

Then let us denote, according to the notations of Section 2, by $S_n$ and $V_n$ the projections relative to the $L^2$ space in (13), while we denote the same projections relative to the $\beta = 0$ situation by $S_n^0$ and $V_n^0$. We have

$$(14)\quad S_n(0) = S_n^0(\beta)$$

$$(15)\quad S_n(b) = S_n^0(b + \beta)$$

$$(16)\quad V_n(b) = e^b S_n^0(b + \beta) e^{-b}$$

where the operators in the «new» space are expressed in terms of those in the «old» space. Using these identities we can write

$$(17)\quad \| V_n(b) \|_{L^2(d\mu) \rightarrow L^2(d\mu)} = \| e^{b+\beta} S_n^0(b + \beta) e^{-(b+\beta)} \|_{L^2(g) \rightarrow L^2(g)}$$

where we denote by $L^2(g)$ the space relative to the measure $\frac{1}{g(\theta)} \frac{d\theta}{2\pi}$.

By the results of Section 5, we have uniform holomorphy for $V_n(b)$ if

$$(18)\quad \| \text{Re}(b + \beta) \| < \delta$$

and this holds for any $b$ whose real part has BMO norm less than the $\delta'$ in (12).

Finally, lemma 1 and lemma 2 imply that we can substitute $1/g(\theta)$ in the right hand side of (13) with $e^{\gamma(\theta)}$ where $\gamma \in \text{VMO}$. We remark that weights of the form (10) can have any number of zeros (only with restrictions on the rate of decay of the weight around each zero).

7. The ~ Operation on Infinite Matrices Expressed Via the Hilbert Transform and a Remarkable Bilinear Operator

Starting from our knowledge of the operator-valued function

$$(19)\quad b \rightarrow V_n(b)$$
and given the identities (3.14-3.15), \( i.e., \)

\[
V_n(b) = e^b S_n(b) e^{-b} = Q(b) S_n(0) Q(b)^{-1}
\]

where

\[
Q(b) = e^b R^{-1}(b)
\]

is an orthogonal transformation depending on \( b \in \text{BMO} \), it is a natural problem to study the mapping

(1) \( b \to Q(b). \)

The first step of this study is to compute and understand the Gateaux differential of (1) at 0. We claim that

(2) \[
\frac{d}{dt} Q(bt) \big|_{t=0} = \tilde{M}_b
\]

where \( M_b \) is the operator of multiplication by \( b \) and \( \tilde{M}_b \) is defined as in (3.5-3.6).

In fact, by the Kerzman-Stein formula (2.5) we have

\[
V_n(b) = P_n(b) (I + P_n(b) - P_n(-b))^{-1}
\]

therefore

(3) \[
\frac{d}{dt} V_n(bt) \big|_{t=0} = \frac{d}{dt} P_n(tb) \big|_{t=0} - P_n(0) \frac{d}{dt} [P_n(tb) - P_n(-tb)] \big|_{t=0}
\]

\[
= [b, S_n(0)] - S_n(0) [b, S_n(0)] + [b, S_n(0)]
\]

\[
= [I - 2S_n(0)] [b, S_n(0)]
\]

\[
= [\tilde{M}_b, S_n(0)].
\]

The last equality can be checked by writing the operators as infinite matrices, remembering that \( S_n(0) \) is the identity on the first \((n + 1) \times (n + 1)\) entries while

(4) \( b = M_b = (m_{nk}) \) implies \( \tilde{M}_b = (\sgn(n - k) m_{nk}) \)

Because of identity (3.14), we have

(5) \[
[\hat{Q}(0), S_n(0)] = [\tilde{M}_b, S_n(0)]
\]
and since this commutator equality holds for \( n = 0, 1, 2, \ldots \), and \( \hat{Q}^* = -\hat{Q} \), we can conclude that

\[
\hat{Q}(0) = \tilde{M}_b.
\]

While studying the explicit form that the operator \( \tilde{M}_b \) takes for some explicit sets of orthogonal polynomials, we discovered a remarkable formula. It is an operator identity expressing the operation \( \sim \) on infinite matrices via Hilbert Transforms on the circle, and via the bilinear operator

\[
(6) \quad B(b, f)(x) = \frac{1}{2\pi} \int b(x - \theta) f(x - 2\theta) \cot \frac{\theta}{2} d\theta.
\]

Notice that it is still an open problem to establish whether operators like (6), or their analogues on the real line, are \( L^2 \)-bounded for \( b \in L^\infty \) (or \( b \in \text{BMO} \)).

The context in which the formula arises is cosine polynomials (Chebychev polynomials after a change of variable).

Consider the \( L^2 \) space of functions of the form

\[
(7) \quad f(x) = \sum_{j=0}^{\infty} f_j \cos jx \quad (b_j \in \mathbb{C})
\]

and the multiplication operator

\[
(8) \quad [M_b f](x) = b(x) f(x)
\]

where

\[
(9) \quad b(x) = \sum_{j=0}^{\infty} b_j \cos jx \quad (b_j \in \mathbb{C}).
\]

The operator \( M_b \) can be represented in the basis \( \{\cos jx\} \) by an infinite matrix

\[
(10) \quad M_b = (m_{nk}) \quad n, k = 0, 1, 2, \ldots
\]

and we can define

\[
(11) \quad \tilde{M}_b = (-i \text{ sgn}(n - k)m_{nk})
\]

which is the usual definition, apart from the constant factor \(-i\).
Introducing a new variable $\theta$, we can write an integral expression for $\tilde{M}_b$, in fact

\[(12) \quad \tilde{M}_b = \frac{1}{2\pi} \int_1^{2\pi} U_b M_b U_{-\theta} \cot \frac{\theta}{2} d\theta \]

where

\[(13) \quad U_b f = \sum_{j=0}^{\infty} e^{-i\theta f_j} \cos jx, \]
\[(13) \quad U_{-\theta} f = \sum_{j=0}^{\infty} e^{i\theta f_j} \cos jx. \]

Now, we have

\[(14) \quad U_{-\theta} f = \sum_{j=0}^{\infty} f_j (\cos j\theta \cos jx + i \sin j\theta \cos jx)\]
\[= \sum_{j=0}^{\infty} f_j \left( \frac{\cos j(x + \theta) + \cos j(x - \theta)}{2} + \frac{\sin j(x + \theta) - \sin j(x - \theta)}{2} \right)\]
\[= \frac{1}{2} [T_{-\theta} f + T_{\theta} f + i(T_{-\theta} Hf - T_{\theta} Hf)]\]

where

\[(15) \quad [T_{\theta} f](x) = f(x - \theta)\]

and $Hf$ is the Hilbert Transform of $f$ (on the circle).

A similar computation shows that

\[(16) \quad U_b f = \frac{1}{2} [T_{-\theta} f + T_{\theta} f - i(T_{-\theta} Hf - T_{\theta} Hf)].\]

Using (14) and (16), let us compute the operator $U_b M_b U_{-\theta}$. In what follows, in order to make the notation less cumbersome, we will simply write $M$ for $M_b$. We have
\( 4U_\theta MU_{-\theta} = \)
\[
= U_\theta [M_{-\theta} + MT_\theta + i(M_{-\theta}H - MT_\theta H)]
\]
\[
= T_{-\theta}MT_{-\theta} + T_{-\theta}MT_\theta + iT_{-\theta}MT_{-\theta}H - iT_{-\theta}MT_\theta H
\]
\[
+ T_\theta MT_{-\theta} + T_\theta MT_\theta + iT_\theta MT_{-\theta}H - iT_\theta MT_\theta H
\]
\[
- iT_{-\theta}HMT_{-\theta} - iT_{-\theta}HMT_\theta + T_{-\theta}HMT_{-\theta}H - T_{-\theta}HMT_\theta H
\]
\[
+ iT_\theta HMT_{-\theta} + iT_\theta HMT_\theta - T_\theta HMT_{-\theta}H + T_\theta HMT_\theta H.
\]

If we plug these sixteen terms back into (12), we notice that the eight terms giving the real part cancel after integration against \( \cot \frac{\theta}{2} \, d\theta \). In fact, we have

\[
[(T_{-\theta}MT_{-\theta} + T_{-\theta}MT_\theta + T_\theta MT_{-\theta} + T_\theta MT_\theta)f](x) =
\]
\[
= b(x + \theta)f(x + 2\theta) + b(x + \theta)f(x)
\]
\[
+ b(x - \theta)f(x) + b(x - \theta)(x - 2\theta),
\]

and these four terms define an even function of \( \theta \), which in the integral is multiplied by the odd function \( \cot \theta/2 \).

Similarly, the other four real terms applied to \( f \) give (using the notation \( Hf = \tilde{f} \))

\[
H(b(x + \theta)f(x + 2\theta) - b(x + \theta)f(x) - b(x - \theta)f(x) + b(x - \theta)f(x - 2\theta))
\]

and these cancel, too, if we assume that the integration against \( \cot \frac{\theta}{2} \, d\theta \) can be done before the Hilbert Transform \( H \) in the \( x \) variable.

Let us now look at the other eight terms in (17); they are

\[
i(T_{-\theta}MT_{-\theta}H - T_{-\theta}MT_\theta H + T_\theta MT_{-\theta}H - T_\theta MT_\theta H
\]
\[
- T_{-\theta}HMT_{-\theta} - T_{-\theta}HMT_\theta + T_\theta HMT_{-\theta} + T_\theta HMT_\theta)
\]

and applying them to \( f(x) \), we get

\[
4 \mathrm{Im}[(U_\theta MU_{-\theta})f](x)
\]
\[
= b(x + \theta)f(x + 2\theta) - b(x + \theta)f(x) + b(x - \theta)f(x) - b(x - \theta)f(x - 2\theta)
\]
\[
+ H[(x + \theta)f(x + 2\theta) - b(x + \theta)f(x)
\]
\[
+ b(x - \theta)f(x) + b(x - \theta)f(x - 2\theta)].
\]

Plugging this expression into (12), and making the change of variable \( \theta \to -\theta \) in the four integrals that contain \( x + \theta \) or \( x + 2\theta \), we get
(22) \[
\tilde{M}f = \frac{i}{4} \left\{ 2f(x) \int b(x - \theta) d\theta - \frac{1}{2\pi} \int_0^{2\pi} b(x - \theta) f(x - 2\theta) d\theta \cot \frac{\theta}{2} d\theta \right. \\
+ \left. \left( f(x) \int b(x - \theta) d\theta \right) \right\}
\]

where we have used the notation

(23) \[
d\theta = \cot \frac{\theta}{2} \frac{d\theta}{2\pi}
\]

and \(d\theta\) lives on \([0, 2\pi]\).

The final formula is

(24) \[
[\tilde{M}f](x) = \frac{i}{2} \left\{ \tilde{f}(x) \tilde{b}(x) - \frac{1}{2\pi} \int_0^{2\pi} \tilde{b}(x - \theta) \tilde{f}(x - 2\theta) \cot \frac{\theta}{2} d\theta \\
+ \left( f \tilde{b} \right) \right\}
\]

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Harnack Inequality and Green Function for a Certain Class of Degenerate Elliptic Differential Operators

Oscar Salinas

Introduction

The main purpose of this work is to obtain a Harnack inequality and estimates for the Green function for the general class of degenerate elliptic operators described below. Let

\[ Lu = - \sum_{i,j=1}^{n} D_i(a_{ij}D_ju), \]

where \( A = [a_{ij}] \) is symmetric, measurable and satisfies the following ellipticity condition

\[ v(x) \sum_{i=1}^{n} \lambda_i^2(x) \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq w(x) \sum_{i=1}^{n} \lambda_i^2(x) \xi_i^2, \]

for every \( \xi \in \mathbb{R}^n \) and almost every \( x \) in an open bounded set \( \Omega \) of \( \mathbb{R}^n \). The functions \( \lambda_i \) are defined on \( \mathbb{R}^n \) and satisfy

\[ \lambda_1 = 1 \quad \lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1}) \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n - \Pi) \]

where

\[ \Pi = \left\{ x \in \mathbb{R}^n : \prod_{i=1}^{n} x_i = 0 \right\} \quad \text{for} \quad j = 2, \ldots, n \]
\[(0.4) \quad \lambda_j(x_1, \ldots, x_i, \ldots, x_{j-1}) = \lambda_j(x_1, \ldots, -x_i, \ldots, x_{j-1}), \quad i = 1, \ldots, j - 1\]

\[(0.5) \quad 0 < \lambda_j(x) \leq \Lambda \text{ for every } x \in \mathbb{R}^n - \Pi, \quad j = 1, \ldots, n. \text{ Moreover, there exist non-negative numbers } b_{ji} \text{ such that}\]

\[0 \leq \chi_i(D_j \lambda_j)(x) \leq b_{ji} \lambda_j(x)\]

\[\text{for } i = 1, \ldots, j - 1, \quad j = 2, \ldots, n \text{ and for every } x \in \mathbb{R}^n - \Pi.\]

A vector \((\lambda_1, \ldots, \lambda_n)\) satisfying these properties generates a distance \(d\) and a quasi-distance \(\delta\) on \(\mathbb{R}^n\) in such a way that \((\mathbb{R}^n, d)\) and \((\mathbb{R}^n, \delta)\) become spaces of homogeneous type with the Lebesgue measure (see [CG], [CW] and [C]) and, moreover, there exists a constant \(a > 1\) such that \(a^{-1} \delta < d < a \delta\) (see [FL1]). The conditions on the pair of weights \((\nu, w)\) can now be stated in terms of this geometry. Given \(\alpha \in (0, 1]\) and \(\sigma > 1\), we introduce the class \(S_{\nu, \alpha}\) as the class of pairs \((\nu, w)\) such that satisfy

\[(0.6) \quad 0 < \nu(Q), \quad w(Q) < \infty \quad \text{for every } \delta\text{-ball } Q \subset \Omega, \text{ where}\]

\[w(Q) = \int_Q w, \quad \nu(Q) = \int_Q \nu,\]

\[(0.7) \quad \text{there exists } C > 0 \text{ such that}\]

\[\left(\frac{w(Q_0 \cap Q)}{w(Q_0)}\right)^{1/2\alpha} \left(\nu^{-1}(Q_0 \cap Q) \nu(Q_0)\right)^{1/2} \leq C \left| Q_0 \right|^{1-\alpha} \left| Q \right|^{\alpha}\]

\[\text{for every } Q_0 \text{ and } Q \delta\text{-balls in } \Omega \text{ such that radius } (Q) \leq 8 \ a^2 \text{ radius } (Q_0).\]

Examples of operators satisfying the preceding conditions are the following

\[(0.8) \quad Lu = -\operatorname{div}(d(0, x) x^\beta D_i u, \ d(0, x)^{-\beta} \left| x_1 \right| \gamma D_2 u)\]

for \(x = (x_1, x_2) \in \mathbb{R}^2, \gamma > 0 \text{ and } \beta > 0. \text{ Since our results will apply when}\]

\[\alpha \in (1 - \left( \sum_{j=1}^{j-1} G_j^{-1}, 1 \right), \ G_1 = 1 \text{ and } G_j = 1 + \sum_{i=1}^{j-1} b_{ji} G_i, \ j = 2, \ldots, n, \]

we get Harnack’s inequality and estimates for Green’s function for the operator \(L\) in (0.8) when

\[1 - \frac{1}{2 + \gamma} < \frac{1}{4} \left( \frac{4 + \beta}{2 + \beta} \right)\]

We point out that our results contain as special cases those in Moser ([M]), Fabes, Kenig and Serapioni ([FKS]), Fabes, Jerison and Kenig ([FJK]),...
Chanillo and Wheeden ([ChW2]), Franchi and Lanconelli ([FL2]) and ([FL3]) and Franchi and Serapioni ([FS]).

In Section 1, we present a brief survey of results of the particular geometry introduced by Franchi and Lanconelli. Section 2 is devoted to the construction of a family of δ-balls which resembles the dyadic cubes. In Section 3, we prove Sobolev and Poincaré inequalities. Section 4 contains an analysis of the relations among our conditions on (ν, w) and those in the work of Chanillo and Wheeden. Finally, Section 5 contains the statements of the results about Harnack's inequality and estimates of Green's function.

1.

In this section we give the definitions of the natural distance d and the quasi-distance δ and state its basic properties.

Let us start introducing the notions of λ-subunit vector and λ-subunit curve: a vector γ = (γ_1, ..., γ_n) ∈ R^n is a λ-subunit vector at a point x if

$$\left(\sum_{j=1}^{n} \gamma_j \xi_j \right)^2 \leq \sum_{j=1}^{n} \lambda_j^2(x) \xi_j^2, \text{ for every } \xi \in R^n;$$

we say that \( \gamma: [0, T] \to \mathbb{R}^n \) is a λ-subunit curve if it is an absolutely continuous curve and \( \dot{\gamma}(t) \) is a λ-subunit vector at \( \gamma(t) \) for a.e. \( t \in [0, T] \).

**Definition 1.1.** For any \( x, y \in \mathbb{R}^n \) we define \( d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+ \) as

\[ d(x, y) = \inf\{T \in \mathbb{R}^n: \text{there exists a λ-subunit curve } \gamma: [0, T] \to \mathbb{R}^n, \gamma(0) = x, \gamma(T) = y \}. \]

**Remark 1.2 ([FL1], [FL3]):** \( d \) is a well defined distance. In fact our hypotheses on \( \lambda = (\lambda_1, ..., \lambda_n) \) guarantee the existence of a λ-subunit curve joining \( x \) and \( y \), for any pair of points \( x \) and \( y \).

For our purposes it is useful to introduce a quasi-distance δ, more explicitly defined and sometimes easier than \( d \) to work with.

If \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R} \) put \( H_0(x, t) = x \) and \( H_{k+1}(x, t) = H_k(x, t) + t\lambda_{k+1}(H_k(x, t))e_{k+1} \) for \( k = 0, ..., n - 1 \). Here \( \{e_k\}_{k=1}^{n} \) is the usual canonical basis in \( \mathbb{R}^n \). It is clear that the function \( s \to F_j(x, s) = s\lambda_j(H_{j-1}(x, s)) \), is strictly increasing on \( (0, \infty) \) for any \( x = (x_1, ..., x_n) \) such that \( x_k \geq 0 \), \( k = 1, ..., j - 1 \), and for \( j = 1, ..., n \). Hence it is possible to define the inverse function of \( F_j(x, \cdot) \), that is \( \phi_j(x, \cdot) = (F_j(x, \cdot))^{-1} \) for \( j = 1, ..., n \).

**Definition 1.3.** For any \( x, y \in \mathbb{R}^n \) we define \( \delta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+ \) as
\[ \delta(x, y) = \max_{j=1, \ldots, n} \phi_j(x^*, |x_j - y_j|) \]

where \( x^* = (|x_1|, \ldots, |x_n|) \).

The following two Lemmas contain the basic properties of the functions \( F_j, \phi_j, d \) and \( \delta \).

**Lemma 1.4.** Put \( G_1 = 1 \) and \( G_j = 1 + \sum_{i=1}^{j-1} G_i b_{ji} \) for \( j = 2, \ldots, n \). Then

(1.5) for every \( x \in \mathbb{R}^n, s > 0, \theta \in (0, 1) \) we have

\[
\theta^{G_1} \leq \frac{F_j(x^*, \theta s)}{F_j(x^*, s)} \leq \theta^{G_j};
\]

\[
\theta \leq \frac{\phi_j(x^*, \theta s)}{\phi_j(x^*, s)} \leq \theta^{1/G_j};
\]

(1.6) if \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) verifies \( |y_i| + F_i(y^*, \theta s) \)

\[ \leq |x_i| + K F_i(x^*, s), i = 1, \ldots, j, \text{ for some } s > 0, \theta \in (0, 1) \text{ and } K \geq 1, \]

we have

\[
\frac{F_{j+1}(y^*, \theta s)}{F_{j+1}(x^*, s)} \leq \theta K^{G_j-1}.\]

**PROOF:** For (1.5) see Proposition 4.3 of [FL2]. Let us prove (1.6) from (0.5) we get that \( \lambda_{j+1} \) is increasing in each variable on \( x \in \mathbb{R}^n : x_k \geq 0 \) \( k = 1, \ldots, j \), then

\[
F_{j+1}(y^*, \theta s) = \theta s \lambda_{j+1}(|y_1| + F_1(y^*, \theta s), \ldots, |y_j| + F_j(y^*, \theta s))
\]

\[ \leq \theta s \lambda_{j+1}(|x_1| + F_1(x^*, Ks), \ldots, |x_j| + F_j(x^*, Ks))
\]

\[ \leq \theta K^{G_j-1} F_{j+1}(x^*, s), \]

the last inequality follows from (1.5).

In the sequel we shall use the following notation for \( d \)-balls, \( \delta \)-balls and their dilations

\[
S(x, r) = \{ y \in \mathbb{R}^n : d(x, y) < r \},
\]

\[
Q(x, r) = \{ y \in \mathbb{R}^n : \delta(x, y) < r \}
\]

\[
\alpha S(x, r) = S(x, \alpha r), \alpha Q(x, r) = Q(x, \alpha r), \alpha > 0.
\]
Lemma 1.7. There exist constants $a$, $b$, $A \in (1, \infty)$, depending only on $n$ and the constants in (0.5), such that

\begin{equation}
\frac{1}{a} \leq \frac{d(x, y)}{\delta(x, y)} \leq a, \text{ for all } x, y;
\end{equation}

\begin{equation}
\frac{1}{b} \leq d(x, y) \leq b |x - y|^\eta
\end{equation}

if $|x - y| \leq 1$, where $\eta = \min_j \{1/G_j\}$;

\begin{equation}
|2S| \leq A |S|, |2Q| \leq A |Q| \text{ for any } \delta\text{-ball } S \text{ and any } \delta\text{-ball } Q.
\end{equation}

PROOF: For (1.8) see Theorems 2.6 and 2.7 in [FL1]. (1.9) and (1.10) follow immediately from the above Lemma and (1.8).

2.

Here we shall construct families of $\delta$-balls that resembles the family of dyadic cubes. Let $\tau$ be the set of all $n$-tuples $s = l_1 \ldots l_n$ with $l_i = -1$, $0$, $1$; $i = 1, \ldots, n$. For $k \in \mathbb{Z}$; $l_i = -1$, $0$, $1$; $j_i \in \mathbb{Z}$ and $i = 1, \ldots, n$, define

\begin{align*}
T^k_{l_i}: \mathbb{R}^n &\to \mathbb{R}^n; \quad T^k_{l_i}(x) = x + l_i 2^k e_i, \\
T^{k}_{l_i \ldots l_n,j_i}: \mathbb{R}^n &\to \mathbb{R}^n; \quad T^{k}_{l_i \ldots l_n,j_i}(x) = T^k_{l_i \ldots l_n}(x) + l_i F_i(T^k_{l_i \ldots l_n}(x)) + 2^k j_i = 2^k e_i,
\end{align*}

and

\begin{align*}
x_{j_i} &= (2j_i - 1) 2^k e_i, \\
x_{j_i \ldots j_n} &= x_{j_i} + \ldots + x_{j_i \ldots j_n} + (2j_i - 1) F_i(x^*_{j_i \ldots j_n}) + 2^k e_i.
\end{align*}

For $k \in \mathbb{Z}$ and $s = l_1 \ldots l_n \in \tau$ given, the family of $\delta$-balls

$$D^{k, s} = \{Q(T^{k}_{l_i \ldots l_n}(x_{j_i \ldots j_n})), 2k) : j_k \in \mathbb{Z}; \quad i = 1, \ldots, n\}$$

is an a.e. covering of $\mathbb{R}^n$.

The following Lemma states the main property of these families.

Lemma 2.2. For $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^n$ and $r > 0$, there exists $s \in \tau$ and $Q_0 \in D^{k, s}$ with $2^{k-1} < 2r \leq 2^k$ such that
(2.3) \( Q(\bar{x}, r) \subset Q_0, \)

(2.4) \( |Q_0| \leq C |Q(\bar{x}, r)|, \) where \( C \) is independent of \( \bar{x}, r, k \) and \( s. \)

**Proof.** It is obvious that there exists a center \( c_1 \) of a \( \delta \)-ball belonging to \( D^{k,n}, \) with \( s_1 = 1 \ldots 1, \) and a \( l_i \) in \([-1, 0, 1]\) such that

\[
x_i(T^k_{l_i}c_1) \in [\bar{x}_i - F_i(\bar{x}^*, r), \bar{x}_i + F_i(\bar{x}^*, r)]
\leq [x_i(T^k_{l_i}c_1) - F_1(T^k_{l_i}c_1)^*, 2^k], x_i(T^k_{l_i}c_1) + F_1(T^k_{l_i}c_1)^*, 2^k],
\]

where \( x_1(T^k_{l_i}c_1) \) denote the first component of \( T^k_{l_i}c_1. \) Now, let us suppose that we have determined \( l_2, \ldots, l_m \) and \( c_1, \) with \( Q(c_1, 2^k) \in D^{k,n}, \)

\( s_i = l_i \ldots l_{i-1}1 \ldots 1, i = 1, \ldots, m \) in such a way that

(2.5) \( x_j(c_i) = x_j(T^k_{l_{i-1}l_i}(c_i)), j = 1, \ldots, i - 1, \)

\[
x_i(T^k_{l_{i-1}l_i}c_1) \in [\bar{x}_i - F_i(\bar{x}^*, r), \bar{x}_i + F_i(\bar{x}, r)]
\subset [x_i(T^k_{l_{i-1}l_i}c_1) - F_1(T^k_{l_{i-1}l_i}(c_i))^*, 2^k], x_i(T^k_{l_{i-1}l_i}c_1) + F_1(T^k_{l_{i-1}l_i}(c_i))^*, 2^k].
\]

Then

\[
|\bar{x}_i| + F_i(\bar{x}^*, r) \leq |x_i(T^k_{l_{i-1}l_i}c_1)| + F_1(T^k_{l_{i-1}l_i}(c_i))^*, 2^k,
\]

for \( i = 1, \ldots, m. \) Now, by using Lemma 1.4, we get

\[
F_{m+1}(\bar{x}^*, r) \leq 2F_{m+1}(T^k_{l_{m-1}l_m}(c_m))^*, 2^k).
\]

From this inequality it follows immediately that there exists a center \( c_{m+1} \) of a \( \delta \)-ball in \( D^{k,s_{m+1}}, \) where \( s_{m+1} = l_1 \ldots l_m 1 \ldots 1, \) and a value of \( l_{m+1} \) such that (2.5) holds with \( i = m + 1. \) The inductive process continues to obtain (2.5) for \( i = n. \) Then, taking \( c = T^k_{l_{1 \ldots l_n}}(c_n), Q_0 = Q(c, 2^k) \) and \( s = l_1 \ldots l_n, \) we get (2.3). Finally (2.4) follows from the choice of \( k \) and the doubling property (1.10).

3.

The main results of this section are the following.

**Theorem 3.1.** Let \( \beta > 0, \) fixed, let \( Q = Q(\bar{x}, r) \) be a \( \delta \)-ball such that \( Q \subset \Omega \) and let \( (\nu, w) \in S_{\alpha,n} \) for a given \( \alpha \in \left(1 - \left(\sum_j G_j\right)^{-1}, 1\right]. \) Then, for each \( u \in C^1(\bar{Q}) \) such that verifies
(3.2) \(|x \in Q : u(x) = 0| \geq \beta |Q|\),

we get

\[
(3.3) \quad \left(\frac{1}{w(Q)} \int_Q |u|^{2^*} w \, dx\right)^{1/2a} \leq C r \left(\frac{1}{\nu(Q)} \int_Q |\nabla u|^{2} \, v \, dx\right)^{1/2},
\]

where \(C\) depends only on \(n\) and the constants in (0.5) and (0.7), and \(\nabla u = (\lambda_1 D_1 u, ..., \lambda_n D_n u)\).

**Theorem 3.4** (Sobolev inequality). Let \(Q\) and \((v, w)\) be as in the above theorem. Then, for any \(u \in C_0^0\left(\frac{1}{2} Q\right)\) we get

\[
(3.5) \quad \left(\frac{1}{w(Q)} \int_{1/2Q} |u|^{2^*} w \, dx\right)^{1/2a} \leq C r \left(\frac{1}{\nu(Q)} \int_{1/2Q} |\nabla u|^{2} \, v \, dx\right)^{1/2},
\]

where \(C\) depends only on \(n\) and the constants in (0.5) and (0.7).

**Theorem 3.6** (Poincaré inequality): Let \(Q\) and \((v, w)\) be as in Theorem 3.1. Then, for any \(u \in C^1(Q)\) we get

\[
(3.7) \quad \left(\frac{1}{w(Q)} \int_Q |u - u_Q|^{2^*} w \, dx\right)^{1/2a} \leq C r \left(\frac{1}{\nu(Q)} \int_Q |\nabla u|^{2} \, v \, dx\right)^{1/2},
\]

where \(u_Q = \frac{1}{|Q|} \int_Q u \, w \, dx\) and \(C\) depends only on \(n\) and the constants in (0.5) and (0.7).

The proof of Theorem 3.1 is based on an estimate of \(u\) in terms of certain fractional integral operators applied to \(\nabla u\) and on a norm inequality with two weights for these operators.

Let us start with the definition of these operators.

**Definition 3.8.** Let \(k \in \mathbb{Z}, s \in \tau\) and \(\mu \in (0, 1]\). We define

\[
(P_{\mu}^{k,s} f)(x) = \begin{cases} 
\frac{1}{|Q|^\mu} \int_Q |f(y)| \, dy, & \text{if } x \in Q \in \mathcal{D}^{k,s} \\
0 & \text{if } x \in \mathbb{R}^n - \bigcup_{Q \in \mathcal{D}^{k,s}} Q
\end{cases}
\]
for \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Now, with these operators, we can prove

**Lemma 3.9.** Let \( \beta > 0 \), fixed, and let \( Q = Q(\bar{x}, r) \) be a \( \delta \)-ball. If \( u \in C^1(\overline{Q}) \) verifies

\[
|x \in \overline{Q} : u(x) = 0| \geq \beta |Q|
\]

then, for each \( \mu \in \left( 1 - \left( \sum j \, G_j \right)^{-1}, 1 \right) \) there exist a sequence \( \{a_i\}_{i=1}^{\infty} \subset \mathbb{R}^+ \) depending only on \( \mu \) with \( \sum a_i < \infty \), a sequence of integers \( \{k_i\} \) depending only on \( r \) and a constant \( C \) such that

\[
|u(x)| \leq C r |Q|^{\mu - 1} \sum_{i=1}^{\infty} a_i \sum_{\xi \in \tau} (P_{\mu, \xi}^a \chi_Q |\nabla \chi u|)(x)
\]

for all \( x \in Q \).

**Proof.** By using similar techniques than those in Lemma 4.3 in [FS] and a dyadic partition, we get

\[
(3.10) \quad |u(x)| \leq C_0 \int_0^{2^2 r} \frac{1}{|S(x, t)|} \int_{S(x, C_0 t)} |\nabla \chi u(y)| \chi_Q(y) dy dt
\]

\[
\leq C_0 r \sum_{i=1}^{\infty} \left( \frac{2^r}{2^{-i}} dt \right)^{\frac{n-1}{n-2}} \int_{S(x, C_0 a r/2^{i-1})} |\nabla \chi u(y)| \chi_Q(y) dy,
\]

for all \( x \in Q \). From Lemmas 1.7 and 2.2 follows that for each \( i \) there exists \( k_i \in \mathbb{Z}, s_i \in \tau \) and \( Q_i \in D_{k_i, s_i} \) such that

\[
2^{k_i - 1} < \frac{C_0 a^3 r}{2^{i-3}} \leq 2^{k_i},
\]

\[
S\left(x, \frac{C_0 a^2 r}{2^{i-1}}\right) \subset Q_i,
\]

\[
|Q_i| \leq C \left| S\left(x, \frac{C_0 a^2 r}{2^{i-1}}\right) \right|,
\]
On the other hand, by (1.5) and (1.8), we get \(|S(x, \tau r)| \geq C r^\Sigma G \cdot S(x, r)|\) for \(t \in (0, 1]\). Then, taking \(i_0 \in \mathbb{N}\) such that \(a^2/2^{i_0 - 1} < 1 \leq a^2/2^{i_0 - 3}\), we obtain from (3.10)

\[
|u(x)| \leq C r \sum_{i=1}^{i_0} \left( \int_0^{a^2/2^{i-1}} \frac{dt}{S(x, \tau r^{1-\mu})} \right) \frac{1}{|Q_i|^\mu} \int_{Q_i} \left| \nabla_x u(y) \right| \chi_Q(y) dy
\]

\[
+ \frac{1}{|S(x, r)|^{1-\mu}} \sum_{i=i_0+1}^{\infty} \left( \int_0^{a^2/2^{i-1}} \frac{dt}{S(x, 2^{i-1})} \right) \frac{1}{|Q_i|^\mu} \int_{Q_i} \left| \nabla_x u(y) \right| \chi_Q(y) dy
\]

\[
\leq C r |S(x, r)|^{\mu-1} \sum_{i=1}^{i_0} a_i \sum_{s \in r} (P_{\mu, s}^{h, t} \left( \left| \nabla_x u \right| \nu_Q \right))(x)
\]

where \(a_i = 1\) for \(i = 1, \ldots, i_0\) and \(a_i = 2^{(i-1)(\Sigma G_i - 1)}\). Thus, since \(|Q| \approx |S(x, r)|\) we get the thesis.

\textbf{Remark 3.11.} From the proof is easy to see that \(2^k \geq 8 a^2 r\) then \(Q \subset S(\bar{x}, ar) \subset S(x, 2ar) \subset S(x, C_0 ar/2^{i-2})\) and thus

\[
P_{\mu, s}^{h, t} \left( \left| \nabla_x u \right| \nu_Q \right)(x) = \frac{1}{|Q_i|^\mu} \int_{Q_i} \left| \nabla_x u \right| dy.
\]

In the following Lemma we prove a two weight norm inequality for the operators \(P_{\mu, s}^{h, t}\). The proof is based on techniques of E. T. Sawyer (see |S|).

\textbf{Lemma 3.12.} Suppose \(1 < p \leq q < \infty\). Let \(E \subset \mathbb{R}^n\) be a bounded open set and let \((v, w)\) be a pair of non negative integrable functions defined on \(E\). Then

\[
(3.13) \quad \left( \int_E \left| P_{\mu, s}^{h, t} f \right| \alpha w dx \right)^{1/q} \leq C_0 \left( \int_E \left| f \right| \beta v dx \right)^{1/p},
\]

for all \(f \in L^p(E, v dx)\) with \(\text{supp } f \subset E\), if and only if
\[(3.14) \quad w(E \cap Q)^{1/q} \left( \frac{\nu^{1/(p-1)}(E \cap Q)}{|Q|^s} \right)^{1-1/p} \leq C_0 \]

for all \( Q \in D^{k,s} \). The constant \( C_0 \) in (3.13) and (3.14) is the same.

PROOF. For sake of simplicity \( P_\alpha \) and \( D \) denote \( P_{\alpha}^{k,s} \) and \( D^{k,s} \) respectively. Now assume that (3.13) holds. Let us first show that \( \nu^{-1/(p-1)} \chi_{E \cap Q} \in L^p(E, vdx) \) for \( Q \in D \). Suppose this is not the case, then since

\[
\int_{E \cap Q} v^{-1/(p-1)} dx = \int_{E \cap Q} v^{-p/(p-1)} v dx = \infty,
\]

we can find a \( g \) in \( L^p(E, vdx) \) such that

\[
\int_{E \cap Q} g v^{-1} v dx = \infty,
\]

which is a contradiction with (3.13) taking \( f = g \) on \( E \) for every \( Q \in D \). We get (3.14) by taking \( f = \nu^{-1/(p-1)} \chi_{Q \cap Q} \) in (3.13). Conversely assume that (3.14) holds. Then

\[
\int_E \|P f\| v dx = \sum_{Q \in D} \int_{E \cap Q} \left( \frac{1}{|Q|^s} \int_Q |f| dy \right)^q v dx \leq \sum_{Q \in D} w(E \cap Q) \left( \frac{\nu^{-1/(p-1)}(E \cap Q)}{|f|} \right)^{q/(p-1)} \left( \int_Q |f|^p v dx \right)^{q/p}.
\]

Finally, from (3.14) and the fact that,

\[
\sum_{Q \in D} \left( \int_Q |f|^p v dx \right)^{q/p} \leq \left( \sum_{Q \in D} \int_Q |f|^p v dx \right)^{q/p},
\]

for \( q \geq p \), we obtain (3.13).
PROOF OF THEOREM (3.1). From Lemma (3.9) we get

\[
\left( \frac{1}{w(Q)} \int_Q |u|^{2\sigma} \, wdx \right)^{1/2\sigma} \leq \frac{Cr}{|Q|^{1-\sigma}} \sum_{i=1}^{\infty} d_i \sum_{s \leq r} \left( \frac{1}{w(Q)} \int_Q \left| P_{\mu,s} u \text{w} \chi_Q \right|^2 \, wdx \right)^{1/2\sigma}.
\]

Now, the inequality (3.3) follows by applying (0.7), (0.8) and Lemma 3.12 for the \(k_i\) such that \(2^{k_i} \leq 8a^2 r\) and the Remark 3.11, the Schwartz inequality and (0.8) for the remainders.

PROOF OF THEOREM 3.4. We need only to apply the above theorem in the ball \(Q\), keeping in mind that, by the doubling property (1.10), it follows immediately that \(|Q - (1/2)Q| = |Q|\).

PROOF OF THEOREM 3.6. With \(Q\) and \(u\) given it is always possible to find a number \(b = b(Q, u)\) such that \(Q^+ = \{x \in Q; u(x) \geq b\}\) and \(Q^- = \{x \in Q; u(x) \leq b\}\) verifies

\[
(3.15) \quad |Q^+| \geq \frac{1}{2} |Q| \quad \text{and} \quad |Q^-| \geq \frac{1}{2} |Q|.
\]

Assume this fact, then both functions \((u - b)^+\) and \((u - b)^-\) satisfy the hypotheses of Theorem (3.1) with \(\beta = 1/2\). By that Theorem we get

\[
\frac{1}{w(Q)} \int_{Q^+} |u - b|^{2\sigma} \, wdx \leq (Cr)^{2\sigma} \left( \frac{1}{w(Q)} \int_{Q^-} |\nabla u|^2 \, wdx \right)^{\sigma}.
\]

Adding these two inequalities we have

\[
\frac{1}{w(Q)} \int_Q |u - b|^{2\sigma} \, wdx \leq 2(Cr)^{2\sigma} \left( \frac{1}{w(Q)} \int_Q |\nabla u|^2 \, wdx \right)^{\sigma}.
\]
Then, since
\[
\left( \int_Q |u - u_Q|^{2\sigma} w dx \right)^{1/2\sigma} \leq \left( \int_Q |u - b|^{2\sigma} w dx \right)^{1/2\sigma} + \left( \frac{1}{w(Q)} \int_Q |u - b| w dx \right) w(Q)^{1/2\sigma} \leq 2 \left( \int_Q |u - b|^{2\sigma} w dx \right)^{1/2\sigma},
\]
we obtain the thesis. Let us prove (3.15). Observe that the two functions
\[\phi(t) = |x \in Q: u(x) \leq t| \quad \text{and} \quad \psi(t) = |x \in Q: u(x) \geq t| \]
are respectively increasing, right-continuous and decreasing, left-continuous. Define \(b = \inf \{t: \phi(t) \geq 1/2 |Q| \}\) then by the right-continuity \(\phi(b) \geq 1/2 |Q| \). Suppose now, by contradiction, that \(\psi(b) < 1/2 |Q| \). Then by the left-continuity there is \(t < b\) such that \(\psi(t) < |Q|/2\), so that \(\phi(t) > |Q|/2\) and this contradicts the definition of \(b\). Finally (3.15) holds.

4.

In [FS], B. Franchi and R. Serapioni prove inequalities of type (3.3), (3.5) and (3.7) for the case \(v = Cw\). The assumption on the weight is that \(w \in A_2\) respect to the \(d\)-balls, i.e. \(w(S) w^{-1}(S) = |S|^2\) for all \(d\)-ball \(S\). Inequalities of the same type for the euclidean case, i.e. \(\lambda_i = 1\) for all \(i\), have been proved by S. Chanillo and R. Wheeden in [ChW1]. The hypotheses on the pair of weights in that work are the euclidean case of

(4.1) \(w \in D_\infty\) respect to \(\delta\)-balls, i.e. \(w(Q) = w(2Q)\) for every \(\delta\)-ball \(Q\)

(4.2) \(v \in A_2\) respect to \(\delta\)-balls, i.e. \(v(Q) v^{-1}(Q) = |Q|^2\) for every \(\delta\)-ball \(Q\)

(4.3) there exists \(\sigma > 1\) and \(C > 0\) such that

\[
\left( \frac{|\theta Q|}{|Q|} \right)^{(\sigma - 1)/2\sigma} \left( \frac{w(\theta Q)}{w(Q)} \right)^{1/2 \sigma} \leq C \left( \frac{\nu(\theta Q)}{\nu(Q)} \right)^{1/2}, \quad \text{for every } \delta\text{-ball } Q, \text{ and } \theta \in (0,1],
\]
We say that \((\nu, w)\) belongs to \(C_\alpha\) if \((\nu, w)\) satisfy the conditions (4.1), (4.2) and (4.3). The main purpose of this section is to find relations among the conditions of type \(A_2\) and \(C_\alpha\) and the condition \(S_{\alpha, \alpha}\). We begin with the following result.

**Lemma 4.4.** Let \((\nu, w) \in S_{\alpha, \alpha}\) for some \(\alpha \in [1 - (\Sigma_j G_j)^{-1}, 1]\) and \(\Omega = \mathbb{R}^n\). The \((\nu, w)\) verifies (4.2) and (4.3).

**Proof.** By taking \(Q_0 = Q\) in (0.7), we get \(\nu(Q_0) = (\nu^{-1}(Q_0)) \leq C \left| Q_0 \right|^2\) and thus \(\nu\) satisfy (4.2). Now assume that \(\alpha = 1 - (\Sigma_j G_j)^{-1}\) then, from (0.7), we get

\[
\left( \frac{w(\theta Q)}{w(Q)} \right)^{1/2} (\nu^{-1}(\theta Q) \nu(Q))^{1/2} \leq C \left| Q \right|^{(\Sigma_j G_j)^{-1}} \left| \theta Q \right|^{1 - (\Sigma_j G_j)^{-1}}
\]

for any \(\delta\)-ball \(Q\) and any \(\theta \in (0, 1)\). From this and Hölder inequality follows (4.3), in fact

\[
\left( \frac{\left| \theta Q \right|}{\left| Q \right|} \right)^{(\Sigma_j G_j)^{-1}} \left( \frac{w(\theta Q)}{w(Q)} \right)^{1/2} \leq C \frac{\left| \theta Q \right|}{(\nu^{-1}(\theta Q) \nu(Q))^{1/2}}
\]

\[
\leq C \left( \frac{\nu(\theta Q)}{\nu(Q)} \right)^{1/2}.
\]

Now, to complete the proof, it is sufficient to prove that if \(\alpha_1, \alpha_2 \in (0, 1]\) and \(\alpha_1 > \alpha_2\) then \(S_{\alpha_1, \alpha_1} \subset S_{\alpha_2, \alpha_2}\). Note that only is necessary to prove that (0.7) with \(\alpha = \alpha_2\) holds. This is trivial if \(w(Q_0 \cap Q)\) or \(\nu^{-1}(Q_0 \cap Q)\) is zero. Assume that both \(w(Q_0 \cap Q)\) are positive. The inequality in (0.7) with \(\alpha = \alpha_1\) is equivalent to

\[
(4.5) \left( \frac{\left| Q_0 \right|}{\left| Q \right|} \right)^{\alpha_1} \leq C \left( \frac{\left| Q_0 \right|}{(\nu^{-1}(Q_0 \cap Q) \nu(Q_0))^{1/2}} \right) \left( \frac{w(Q_0)}{w(Q_0 \cap Q)} \right)^{1/2}.
\]

On the other hand, since \(Q_0 \cap Q \neq \emptyset\) and radius \((Q) \leq 8a^2\) radius \((Q_0)\), the doubling property (1.10) allow us to write
\[
\left( \frac{|Q_0|}{|Q|} \right)^{\alpha_1} \leq C \left( \frac{|Q_0|}{|Q|} \right)^{\alpha_2}
\]

with \(C\) independent of \(Q_0\) and \(Q\). From this and (4.5) follows that \((\nu, w) \in S_{\alpha, \alpha}\).

We shall next show that condition \(C_\eta\) implies a condition \(S_{\alpha, \alpha}\). In the proof of this fact we shall use the following result.

**Lemma 4.6.** Let \((\nu, w)\) be a pair of non negative weights satisfying (4.1) and (4.3). Then there exist \(\eta \in (0, 1)\) and \(\alpha' \in (1, \alpha)\) such that the inequality (4.3) holds with \((\Sigma_j G_j)/\eta\) instead of \(\Sigma_j G_j\) and \(\alpha'\) instead of \(\alpha\).

**Proof.** Since \(w \in D_\infty\) we get

\[
w(Q) = w \left( \frac{1}{2} Q \right) + w \left( Q - \frac{1}{2} Q \right) \\
\geq (1 + C) \, w \left( \frac{1}{2} Q \right)
\]

for every \(\delta\)-ball \(Q\). By iteration we have a \(\beta \geq 1\) such that

\[
(4.7) \quad \frac{w(\theta Q)}{w(Q)} \geq C \theta^\beta \quad \text{for every } \theta \in (0, 1], \quad \text{and every } \delta\text{-ball } Q.
\]

On the other hand, from (1.5) it follows that

\[
|\theta Q| \geq \theta^{\Sigma_j G_j} |Q| \quad \text{for every } \theta \in (0, 1], \quad \text{and every } \delta\text{-ball } Q.
\]

Then, this inequality (4.7) and (4.4) allow us to obtain the inequality

\[
\left( \frac{|\theta Q|}{|Q|} \right)^{(\Sigma_j G_j)^{-1}(1-\beta)} \left( \frac{w(\theta Q)}{w(Q)} \right)^{\frac{1}{2\alpha} + \epsilon} \leq C \left( \frac{|\theta Q|}{|Q|} \right)^{(\Sigma_j G_j)^{-1}(1-\beta)} \theta^{\beta} \left( \frac{w(\theta Q)}{w(Q)} \right)^{\frac{1}{2\alpha}} \\
\leq C \left( \frac{|\theta Q|}{|Q|} \right)^{(\Sigma_j G_j)^{-1}} \left( \frac{w(\theta Q)}{w(Q)} \right)^{\frac{1}{2\alpha}} \\
\leq C \left( \frac{\nu(\theta Q)}{\nu(Q)} \right)^{1/2},
\]

where \(\nu\) is the measure associated with \(\nu\) and \(w\).
for all $\theta \in (0, 1]$, $\epsilon > 0$, and all $\delta$-ball $Q$. Finally, by taking $\epsilon$ in $(0, \min \{1/\beta, (1 - 1/\alpha)/2\sigma\})$, we get the thesis with $\eta = 1 - \epsilon \beta$ and $\sigma' = \frac{\sigma}{1 + 2\epsilon \sigma}$.

**Lemma 4.8.** Let $(\nu, w) \in C_\sigma$. Then $(\nu, w) \in S_{\sigma', \alpha}$ for some $\alpha \in ((\Sigma_j G_j)^{-1}, 1]$ and some $\sigma' \in (1, \sigma)$.

**Proof.** We only need to prove (0.7). Let $Q_0 = Q(x_1, r_1)$ and $Q = Q(x_2, r_2)$ be two $\delta$-balls such that $r_2 \leq 8\sigma^2 r_1$. If $Q_0 \cap Q = \emptyset$, there is nothing to prove.

Assume $Q_0 \cap Q \neq \emptyset$, then there exists $C_1 = C_1(\sigma) > 1$ and $C_2 = C_2(\sigma)$ such that $Q_0 \subset \tilde{Q} = (C_1 r_1 / r_2) Q \subset C_2 Q_0$. Now, let $\sigma'$ and $\eta$ be as in the above Lemma and $\theta = r_2/C_1 r_1$. Then, from (4.1), (4.2), (1.10) and Lemma 4.6 it follows that

$$
\left( \frac{w(Q_0 \cap Q)}{w(Q_0)} \right)^{1/2 \sigma'} (\nu^{-1}(Q_0 \cap Q) \nu(Q_0))^{1/2} 
\leq C \left( \frac{w(\theta \tilde{Q})}{w(\tilde{Q})} \right)^{1/2 \sigma'} (\nu^{-1}(\theta \tilde{Q}) \nu(\tilde{Q}))^{1/2} 
\leq C \left( \frac{|Q|}{|\theta \tilde{Q}|} \right)^{(\Sigma_j G_j)^{-\eta}} (\nu^{-1}(\theta \tilde{Q}) \nu(\tilde{Q}))^{1/2} 
\leq C \left| \frac{Q}{\tilde{Q}} \right|^{(\Sigma_j G_j)^{-\eta}} \left| \frac{\theta \tilde{Q}}{\tilde{Q}} \right|^{1-((\Sigma_j G_j)^{-\eta})} 
\leq C \left| \frac{Q_0}{Q} \right|^{(\Sigma_j G_j)^{-\eta}} \left| \frac{\tilde{Q}}{Q} \right|^{1-((\Sigma_j G_j)^{-\eta})}.
$$

Thus, $(\nu, w) \in S_{\sigma', \alpha}$ with $\alpha = 1 - (\Sigma_j G_j)^{-1} \eta$.

Now, by using the above result, we get

**Lemma 4.9.** Let $w \in A_2$ with respect to $d$ or $\delta$-balls. Then $(w, w) \in S_{\sigma, \alpha}$ for some $\sigma > 1$ and some $\alpha \in (1 - (\Sigma_j G_j)^{-1}, 1]$. 

PROOF. From the previous Lemma, we only need to prove that \((w, w) \in C_o\) for some \(\sigma > 1\). We know that
\[
\frac{w(\theta Q)}{w(Q)} \geq C \frac{w(\theta Q) w^{-1}(Q)}{|Q|^2} \geq C \frac{w(\theta Q) w^{-1}(\theta Q)}{|Q|^2} \geq C \left( \frac{|\theta Q|}{|Q|} \right)^2
\]
for all \(\theta \in (0, 1]\) and all \(\delta\)-ball \(Q\). Then, by taking \(\sigma > 1\) such that \(\frac{1}{\sigma} > 1 - (\Sigma_j G_j)^{-1}\), we get
\[
\left( \frac{|\theta Q|}{|Q|} \right)^{\left( \Sigma_j G_j \right)^{-1}} \left( \frac{w(\theta Q)}{w(Q)} \right)^\frac{1}{2} = \left( \frac{|\theta Q|}{|Q|} \right)^{\left( \Sigma_j G_j \right)^{-1}} \left( \frac{w(\theta Q)}{w(Q)} \right)^\frac{1}{2} + \frac{1}{2} \frac{1}{2}
\]
\[
\leq C \left( \frac{w(\theta Q)}{w(Q)} \right)^\frac{1}{2} \left( \frac{|\theta Q|}{|Q|} \right)^{\left( \Sigma_j G_j \right)^{-1} + \frac{1}{\sigma} - 1}
\]
\[
\leq C \left( \frac{w(\theta Q)}{w(Q)} \right)^\frac{1}{2}.
\]
Thus \((w, w) \in C_o\).

Let us describe some examples of pairs \((\nu, w)\) that satisfy the hypotheses (0.6) and (0.7) for any \(\lambda_1, \ldots, \lambda_n\) in the conditions (0.3) to (0.5).

EXAMPLE 4.10. In [FS], Franchi and Serapioni prove that \(d(0, x)^{\beta}\) for \(\beta \in (-n, n)\) is a weight in \(A_2\) with respect to \(d\)-balls. In particular they prove the following inequalities

\[
(4.11) \int_{S(y, r)} d(0, x)^{\beta} dx \leq |d(0, y)| + r |\beta|^\beta |S(y, r)|
\]

if \(d(0, y) \geq 2r\),

\[
\int_{S(y, r)} d(0, x)^{\beta} dx \leq C r^{\beta} |S(y, r)|,
\]

if \(d(0, y) \leq 2r\).
These facts allow us to prove that there exists values of $\beta$ in $(0, n)$ such that $\nu(x) = d(0, x)^\beta$ and $w(x) = \nu(x)^{-1}$ belong to a class $S_{\sigma, \alpha}$ for some $\sigma > 1$ and some $\alpha$ in $(1 - (\Sigma_j G_j)^{-1}, 1]$. Since both $\nu$ and $w$ belong to $A_2$ we only need to show that there exists $\sigma$ and $\alpha$ such that (0.7) holds with $S$ instead of $Q_0$ and $\theta S$ instead of $Q$ for any $\theta$ in $(0, 1]$, where $S$ is any $d$-ball. Let us now prove this fact. First, note that, by the $A_2$ condition, we get

$$
(4.12) \quad \left( \frac{w(\theta S)}{w(S)} \right)^{\frac{1}{2\sigma}} (\nu^{-1}(\theta S)\nu(S))^{1/2} \leq C \frac{(w(\theta S)w^{-1}(S))^{1/2}(1 + \frac{1}{\sigma})}{|S|^{1/\alpha}}
$$

for all $\sigma > 1$. Let $S = S(y, r)$. If $d(0, y) \geq 2r$, from (4.11) we get

$$
\frac{(w(\theta S)w^{-1}(S))^{1/2}(1 + \frac{1}{\sigma})}{|S|^{1/\alpha}} \leq 3^\beta |S|^{1/2} |\theta S|^{\frac{1}{2}(1 + \frac{1}{\sigma})}.
$$

On the other hand, if $d(0, y) \leq 2r$, from the same inequalities and Lemmas 1.4 and 1.7 it follows that

$$
\frac{(w(\theta S)w^{-1}(S))^{1/2}(1 + \frac{1}{\sigma})}{|S|^{1/\alpha}} \leq C \frac{(\theta^{-\beta} |S| |\theta S|^{1/2}(1 + \frac{1}{\sigma})}{|S|^{1/\alpha}} \leq C |S|^{1/2} |\theta S|^{\frac{1}{2}(1 + \frac{1}{\sigma})(1 - \frac{\beta}{n})}.
$$

The case $2r < d(0, y) < 2r$ follows in a similar way, so we get that $(\nu, w)$ belongs $S_{\sigma, \alpha}$ for

$$
\alpha = \frac{1}{2} \left( 1 + \frac{1}{\sigma} \right) \left( 1 - \frac{\beta}{n} \right).
$$

Then, by taking $\sigma = 1 + \beta/n$, we can choose $\beta_0 \in (0, n)$ such that $\alpha \in (1 - (\Sigma_j G_j)^{-1}, 1]$ for all $\beta \in (0, \beta_0]$.

**Example 4.13.** Let $w(x) = d(0, x)^{-\beta}$ for $\beta \in (0, n)$. From (4.11) and Lemmas 1.4 and 1.7 we get

$$
(4.14) \quad \frac{w(\theta S)}{w(S)} \leq C \left( \frac{|\theta S|}{|S|} \right)^{1-\frac{\beta}{n}} \text{ for every } d\text{-ball } S \text{ and all } \theta \in (0, 1].
$$
The for \( \nu = 1 \) we get that for any \( \sigma > 1 \) and \( \epsilon \in (0, 1) \), it holds

\[
\left( \frac{w(\theta S)}{w(S)} \right)^{\frac{1}{2\sigma}} \leq C \left( \frac{\theta S}{S} \right)^{\left(1 - \frac{\beta}{n}\right)^{\frac{1}{2\sigma}}} \left( \frac{\theta S}{\nu^{-1}(\theta S)} \right)^{\frac{1}{2}}
\]

\[
\leq C \left| S \right|^{1 - \frac{1}{2}(1 + \frac{1}{\sigma}(1 - \frac{\beta}{n}))} \left| \theta S \right|^{\frac{1}{2}(1 + \frac{1}{\sigma}(1 - \frac{\beta}{n}))}
\]

Thus, \( (\nu, w) \in S_{\alpha, \epsilon} \) for \( \alpha = \epsilon/2 + (1 - \beta/n)/2\sigma \). By taking \( \epsilon = 1 - b/n \) and \( \sigma = 1 + \beta/n \), we can choose \( \beta_0 \in (0, n) \) such that \( \alpha \in (1 - \Sigma G_j)^{-1}, 1 \) for every \( \beta \in (0, \beta_0] \).

Example 4.15. Let \( \nu(x) = d(0, x)\beta, \beta \in (0, n) \), and \( w(x) = 1 \). Then, from (4.14), we get

\[
\left( \frac{w(\theta S)}{w(S)} \right)^{\frac{1}{2\sigma}} \leq C \left| S \right|^{1 - \frac{1}{2}(1 + \frac{1}{\sigma}(1 - \frac{\beta}{n}))} \left| \theta S \right|^{\frac{1}{2}(1 + \frac{1}{\sigma}(1 - \frac{\beta}{n}))}
\]

for all \( \sigma > 1 \). Thus, by reasoning as in the preceding examples, we get that there exists \( \beta_0 \in (0, n) \) such that \( (\nu, w) \in S_{\alpha, \epsilon} \), for some \( \sigma > 1 \) and some \( \alpha \in (1 - (\Sigma G_j)^{-1}, 1] \), for all \( \beta \in (0, \beta_0] \).

5.

Let \( S \) be a \( d \)-ball such that \( 2d^2 S \subset \Omega \). For \( \phi \) and \( \psi \) in Lip (\( \tilde{S} \)) we define

\[
(5.1) \quad a_0(\phi, \psi) = \int_S \langle A \nabla \phi, \nabla \psi \rangle
\]

\[
(5.2) \quad a(\phi, \psi) = a_0(\phi, \psi) + \int_S \phi \psi w.
\]
It is easy to prove that (5.1) defines a scalar product in \( \text{Lip}_0(S) \) and that (5.2) defines a scalar product in \( \text{Lip}(S) \).

Definition 5.3. We denote with \( H_0(S) \) and \( H(S) \) the completion of \( \text{Lip}_0(S) \) and \( \text{Lip}(S) \) respect to the norms \( \| \cdot \|_0 = a_0(\cdot, \cdot)^{1/2} \) and \( \| \cdot \| = a(\cdot, \cdot)^{1/2} \), respectively.

Remark 5.4. From Sobolev inequality (Theorem 3.4) we get \( H_0(S) \subset H(S) \).

Remark 5.5. It is possible to associate a function in \( L^2(S, \omega dx) \) to each element in \( H(S) \) and define its derivative as functions in \( L^2(S, \omega dx) \).

Definition 5.6. Let \( f \) be such that \( f/\omega \in L^{2/(2\alpha-1)}(S, \omega dx) \) and let \( \psi \in H(S) \). We say that \( u \in H(S) \) is a solution of

\[
Lu = f \quad \text{in} \quad S \\
\quad u = \psi \quad \text{in} \quad \partial S
\]

if

\[
a_0(u, \phi) = \int_S u \phi \quad \text{for all} \quad \phi \in H_0(S),
\]

and \( u - \psi \in H_0(S) \).

Definition 5.7. Let \( F = (f_1, \ldots, f_n) \) be such that \( |F|/\omega \in L^2(S, \omega dx) \) and let \( \psi \in H(S) \). We say that \( u \in H(S) \) is solution of

\[
Lu = -\text{div}_\omega F \quad \text{in} \quad S \\
\quad u = \psi \quad \text{in} \quad \partial S
\]

if

\[
a_0(u, \phi) = \int_S \langle F, \nabla_\omega \phi \rangle \quad \text{for all} \quad \phi \in H_0(S),
\]

and \( u - \psi \in H_0(S) \).

Remark 5.8. We can prove, by the representation theorem for continuous linear functional on Hilbert spaces, the existence and uniqueness of solutions for the above Dirichlet problems.
Remark 5.9. The above definitions and remarks hold if we change $d$-balls by $\delta$-balls.

By using the results of Sections 3 and 4, and the technique in [ChW2], we get

**Theorem 5.10 (Harnack inequality).** Let $Q_0 = Q(\bar{x}, 4R)$ be a $\delta$-ball in $\Omega$. If $u \in H(Q_0)$ is a non-negative solution of $Lu = 0$ and $Q = (1/4)Q_0$ then

$$\sup_{\bar{Q}} u \leq \exp \left\{ C \left( \frac{w(2Q)}{w((1/2)Q)} \right)^{\gamma} \frac{w(2Q)}{v(2Q)} \right\}^{1/2} \inf_{\bar{Q}} u,$$

where $\gamma = (3\sigma^2 - 2\alpha + 1)/(\sigma - 1)$ and $C$ depends only on the constants in (0.5) and (0.7).

Let $S$ be a $d$-ball such that $2aS \subset \Omega$. For $y \in S$ and $\theta > 0$ fixed such that $Q_\theta = Q(y, \theta) \subset S$, we define the mapping

$$\psi \rightarrow \frac{1}{w(Q_\theta)} \int_{Q_\theta} \psi w, \quad \psi \in H_0(S).$$

From Sobolev inequality (3.4) follows that the above mapping is a continuous linear functional on $H_0(S)$. Then, there is a unique $G_\theta^\psi \in H_0(S)$ such that

$$a_0(G_\theta^\psi, \psi) = \frac{1}{w(Q_\theta)} \int_{Q_\theta} \psi w, \quad \text{for all} \quad \psi \in H_0(S).$$

In the next, $G_\theta^\psi = G^\theta(y, \cdot, \psi)$ will be called the «$\theta$-approximate Green function for $S$ with pole $y$». For the sake of simplicity we often will use the notation $G^\theta$.

**Lemma 5.11.** $G^\theta$ is non-negative on $S$.

**Proof.** Follows the line of the euclidean case. (See Section 3 of [ChW3].)

**Lemma 5.12.** There exists a constant $C$ such that

$$w((G_\theta^\psi > t)) \leq C \left( \frac{R^2}{v(2Q)} \right)^{\sigma} \frac{w(2Q)}{t^\alpha} \quad \text{for all} \quad y \in S, \theta > 0 \quad \text{with} \quad Q(y, \theta) \subset S$$

where $R$ is the radius of $S$ and $Q$ is the $\delta$-ball with the same centre that $S$ and radius $aR$. 

**Proof.** The technique is the same that Chanillo and Wheeden have used for the euclidean case (see Section 3 of [ChW3]) but with Sobolev inequality for our particular geometry (Theorem 3.4).

Then, with the above result we have

**Lemma 5.13.** For each \( p \in (0, \sigma) \) there exists \( C \) such that

\[
\sup_{r/2 < d(y, x) < r} G_r^p \leq C \left( \frac{w(S(y, 4r/3))}{\nu(S(y, r))} \right)^{\sigma^p \over p(\sigma - 1)} \frac{r^2}{\nu(S(y, r))} \left( \frac{w(S(y, 8a^2 r/3))}{\inf_{2 < d(y, z) < r} w(S(z, r/(4a^2)))} \right)^{1/p}
\]

for all \( q \in (0, r/4a) \) and for all \( y \) and \( r \) such that \( S(y, 3a^4 r) \subset \Omega \).

**Proof.** Let \( x \in \{ x : S(x) : r/2 < d(y, x) < 3r/4 \} \) and \( q \in (0, r/4a) \), then \( S(x, r/4) \subset S(y, r) \setminus S(y, aq) \). Note that \( G^s \) satisfies \( Lu = 0 \), then for each \( p \in (0, \sigma) \), we have

\[
(5.14) \quad \sup_{Q(x, r/4a)} G^s \leq C \left( \frac{w(Q(x, r/4a))}{\nu(Q(x, r/4a))} \right)^{\sigma^p \over p(\sigma - 1)} \frac{1}{w(Q(x, r/4a))} \int_{Q(x, r/4a)} (G^s)^p w \,^\nu \left( \frac{r^2}{\nu(S(y, 2r))} \right)^{p},
\]

where \( C \) depends only on the constants of (0.5) and (0.7) (see Lemmas 3.1 and 3.11 of [ChW2]). On the other hand, from Lemma 6.2 follows

\[
\int_{Q(x, r/4a)} (G^s)^p w \leq C w(S(y, 2a^2 r)) \left( \frac{r^2}{\nu(S(y, 2r))} \right)^p
\]

for each \( p \in (0, \sigma) \). From this and (6.4) we have

\[
\sup_{Q(x, r/4a)} G^s \leq C \left( \frac{w(Q(x, r/4a))}{\nu(Q(x, r/4a))} \right)^{\sigma^p \over p(\sigma - 1)} \frac{w(S(y, 2a^2 r))}{w(Q(x, r/4a))} \left( \frac{r^2}{\nu(S(y, 2r))} \right)^{1/p}
\]
for all \( p, \sigma \) and \( x \). Then, for each \( p \in (0, \sigma) \) the inequality

\[
(5.15) \quad \sup_{r/2 < d(y, x) < 3r/4} G^0 \leq C \left( \frac{w(S(y, r))}{\nu(S(y, r))} \right)^{\frac{d}{d(p-1)}} \frac{r^2}{\nu(S(y, r))} \inf_{r/2 < d(y, z) < 3r/4} \frac{w(S(y, 2a^2 r))}{w(S(x, r/4a^3))}
\]

holds for all \( \varphi \in (0, r/4a) \), and all \( y \) and \( r \) such that \( S(y, 3a^3 r) \subset \Omega \).

The above inequality allows us to obtain a similar one but on \( S \setminus (1/2) S \). Indeed, if \( G^0 \) denotes the \( \varphi \)-approximate Green function for \( S_0 = S(y, 4r/3) \) with pole \( y \) then, by the weak maximum principle (the proof is similar that the Lemma 2.6 of [ChW3]), we have \( G^0 \leq G^0 \) on \( S \), and from this and (5.15)

\[
\sup_{2r/3 < d(y, x) < r} G^0 \leq C \left( \frac{w(S(y, 4r/3))}{\nu(S(y, 4r/3))} \right)^{\frac{d}{d(p-1)}} \frac{r^2}{\nu(S(y, 4r/3))} \left( \inf_{2r/3 < d(y, z) < r} \frac{w(S(y, 8a^2 r/3))}{w(S(2, r/3a^3))} \right)^{1/p}
\]

Then, the thesis follows from (5.15) and the last inequality.

**Lemma 5.16.** Let \( S(x_0, R) \) be a \( d \)-ball such that \( S(x_0, 13a^4 R) \subset \Omega \). Then, for each \( p \in (0, \sigma) \), there exists a constant \( C \), independent of \( x_0 \) and \( R \) such that

\[
(5.17) \quad \sup_{r/2 < d(y, x) < r} G^0(x) \leq C \left( \inf_{0 < t < R} \frac{t^2}{\nu(S(y, t))} (F_1(y, t)) \right)^{\frac{d}{d(p-1)}} \frac{r^2}{\nu(S(y, t))} (F_2(y, t))^{1/p} \frac{dt}{t},
\]

for all \( y \in S(x_0, R/2) \), \( r \in (0, R/2) \) and \( \varphi \in (0, r/4a) \), where

\[
F_1(y, t) = \frac{w(S(y, 12a^2 t))}{\nu(S(y, t))}
\]

and

\[
F_2(y, t) = \frac{w(S(y, 12a^2 t))}{\inf_{d(y, z) < 12a^2 t} w\left(S\left(z, \frac{t}{128a^5}\right)\right)}.
\]
PROOF. First, let us consider the case \( y = x_0 \). For \( s > 0 \), let us denote with \( S_t \) to \( S(y, s) \) and with \( G^p_t \) to the approximate Green function for \( S_t \) with pole \( y \). Now, for \((t_1, t_2) \in \mathbb{R}^+ \times \mathbb{R}^+\), we define

\[
g(t_1, t_2) = \left( \frac{w(S_{t_1})}{v(S_{t_1})} \right)^{\frac{p}{p(a-1)}} \frac{t_1^2}{v(S_{t_1})} \left( \frac{w(S_{t_2})}{\inf_{S(z, t_2)} w(S(z, t_2))} \right)^{1/p}
\]

Note that this function depend on \( p \), is increasing on \( t_1 \) and decreasing on \( t_2 \). Let \( r < R/2 \) and \( m \in \mathbb{N} \) such that \((3/2)^m r \leq R < (3/2)^{m-1} r\). Then on \( S_r - S_{r/2} \), holds

\[(5.18) \quad G_r \leq G_{(3/2)^m r} = G^p_r + \sum_{j=1}^{m} \left( G_{(3/2)^m} - G_{(3/2)^m} \right).
\]

From the above lemma follows

\[(5.19) \quad \sup_{S_r - S_{r/2}} G^p_r \leq C g \left( \frac{8a^2 r}{3}, \frac{r}{4a^2} \right), \text{ for all } \xi \in \left(0, \frac{r}{4a} \right),
\]

for each \( p \in (0, a) \). By the other hand, by using a similar argument to the Lemma 2.7 of [ChW3] and the above lemma, we get

\[
\sup_{S_r} \left( G^p_{(3/2)^m} - G^p_r \right) \leq C g \left( 4a^2 s, \frac{3s}{8a^3} \right), \quad \text{for all } \xi \in \left(0, \frac{s}{4a} \right),
\]

for each \( p \in (0, a) \). Then, from this inequality, (5.19) and (5.18) follows

\[(5.20) \quad \sup_{S_r - S_{r/2}} G^p_r \leq \sum_{j=1}^{m-1} g \left( 6a^2 \left( \frac{3}{2} \right)^{j-1} r, \frac{3}{2} \left( \frac{3}{2} \right)^{j-1} \frac{r}{4a^2} \right)
\]

\[\leq C \sum_{j=1}^{m-1} \int_{(3/2)^{j-1} r}^{(3/2)^j r} g \left( 6a^2 t, \frac{t}{6a^2} \right) \frac{dt}{t}
\]

\[\leq C \int_{r}^{R} g \left( 6a^2 t, \frac{t}{6a^2} \right) \frac{dt}{t}, \quad \text{for all } \xi \in \left(0, \frac{r}{4a} \right)
\]
for each \( p \in (0, a) \). Finally, since any \( y \in S(x_0, R/2) \) verifies \( S(x_0, R) \subset S(y, 2R) \), we get \( G^\varphi \leq G^\varphi_{2R} \) on \( S(x_0, R) \) and then, from (5.20), follows the thesis

\[
\sup_{r/2 < d(y, x) < r} G^\varphi \leq C \int_r^{2R} g \left( 6a^2 t, \frac{t}{6a^2} \right) \frac{dt}{t}
\]

\[
= C \int_r^R g \left( 12a^2 t, \frac{t}{3a^2} \right) \frac{dt}{t} + \int_R^{2r} g \left( 6a^2 t, \frac{t}{6a^2} \right) \frac{dt}{t}
\]

\[
\leq C \int_r^R g \left( 12a^2 t, \frac{t}{6a^2} \right) \frac{dt}{t}, \text{ for all } \varphi \in \left( 0, \frac{r}{4a} \right) .
\]

**Corollary 5.21.** With the same hypothesis as in Lemma 5.16 for a.e. \( y \in S(x_0, R/2) \) there is a constant \( C = C(y, x_0, R, w, v) > 0 \) such that

\[
(5.22) \quad \sup_{r/2 < d(y, x) < r} G^\varphi(x) \leq C \min \left\{ \int_r^R t^2 \frac{dt}{S(y, t)}, \int_r^R \frac{1}{S(y, t)} \frac{dt}{t} \right\},
\]

for all \( r \in (0, R/2) \) and all \( \varphi \in (0, r/4a) \).

**Proof.** We set \( C_1(y) = \sup_{t \leq R} \frac{w(S(y, 12a^2 t))}{v(S(y, t))} \), \( C_2(y) = \sup_{t \leq R} \frac{S(y, t)}{v(S(y, t))} \) and

\[
C_3(y) = \sup_{t \leq R} \frac{S(y, t)}{w(S(y, t))} .
\]

Now from (5.17) follows

\[
(5.23) \quad \sup_{r/2 < d(y, x) < r} G^\varphi(x) \leq C C_1(y) \left( \frac{r}{p-1} \right)^{1/p} C_2(y) \int_r^R \frac{t^2}{S(y, t)} \frac{dt}{t},
\]

for each \( y \in S(x_0, R/2) \) and for all \( r \in (0, R/2) \) and \( \varphi \in (0, r/4a) \). On the other hand, from (1.5), (1.8) and (1.10), we get

\[
\left( \frac{t}{a^2 R} \right)^{\Sigma G_\varphi} \leq \frac{|S(y, t)|}{|S(y, R)|}, \quad 0 < t \leq R,
\]
then from Lemma 4.4, it follows that
\[
\frac{l^2}{\nu(S(y, t))} \leq CR^2 \left( \frac{w(S(y, aR))}{w(S(y, t/a))} \right)^{1/q} \frac{1}{\nu(S(y, R))}, \quad 0 < r \leq R.
\]

This inequality and (5.17) allows us to obtain
\[
\sup_{r/2 < d(y, x) < r} G^\delta_y(x) \leq \frac{1}{1 \beta^{-1} - 1} \frac{1}{C_3(y)^{1/\alpha} R^2} \left( \frac{w(S(y, aR))}{\nu(S(y, R))} \right)^{1/\alpha} \int_r^R \frac{1}{|S(y, t)|^{1/\alpha}} \frac{dt}{t}.
\]

Now, from (5.23) and the above inequality, by taking infimum on \( p \), (5.22) follows.

The next Lemma gives us an estimate of \( \nabla_x G^\delta \) in terms of \( G^\delta \).

**Lemma 5.24.** Let \( S = S(x_0, R) \) be a \( d \)-ball such that \( 2S \subset \Omega \). Then there exists a constant \( C \) such that
\[
\int_{\Omega \setminus Q(y, r)} \langle \Delta \nabla G^\delta_x, \nabla G^\delta_y \rangle \leq \frac{C}{r^2} \int_{Q(y, r) \setminus Q(y, r/2)} (G^\delta_y)^2 w,
\]
for all \( y \in (1, 2) S \), \( r \in (0, R/2a) \) and \( q \in (0, r/2) \).

**Proof.** The proof is very similar to that of Lemma 4.2 of [ChW3] by keeping in mind that there exists \( \eta \in C^\infty(\mathbb{R}^d) \) such that \( \eta = 0 \) on \( Q(y, r/2) \), \( \eta = 1 \) outside \( Q(y, r) \) and \( |\nabla \eta| \leq C/r \) (see [FL2], p. 537).

Now, the above lemma and Corollary 5.21 allows us to obtain the following result.

**Lemma 5.25.** Let \( S = S(x_0, R) \) be a \( d \)-ball such that \( 13a^d S \subset \Omega \). If \( n > 2 \), then, for a.e. \( y \in (1/2) S \) there exists a constant \( C = C(\gamma, x_0, R, w, \nu) > 0 \) such that
\[
\int_{S \setminus Q(y,r)} |\nabla_x G^\varrho_x|_{vv}^2 \leq C \int_{r/2a}^R \frac{1}{|S(y,t)|^{1/\sigma}} \frac{dt}{t},
\]

for all \( r \in (0, R/2a) \) and for \( \varrho \in (0, R/2a) \).

**Proof.** Let \( \varrho \in \left(0, \frac{r}{8a^2}\right) \), then

\[
(5.26) \quad \int_{S \setminus Q(y,r)} |\nabla_x G^\varrho_x|_{vv}^2 \leq C \frac{1}{r^2} w(Q(y, r)) \left( \sup_{r/2a < d(y, x) < ar} G^\varrho_x(x) \right)^2
\]

\[
\leq C \frac{1}{r^2} w(Q(y, r)) \left( \sum_{j = \log a^d + 1}^{j \leq \log a^d + 1} \sup_{\frac{ar}{2^j}, \varrho, \frac{ar}{2^j}, \varrho} G^\varrho_x(x) \right)^2
\]

\[
\leq C w(Q(y, r)) \left( \min_{r/2a} \frac{t^2}{|S(y, t)|} \frac{dt}{t} \right)^2 \left( \frac{1}{r/2a} \frac{dt}{|S(y, t)|^{1/\sigma}} \right)^2
\]

\[
\leq C \frac{1}{r^2} w(Q(y, r)) \left( \frac{t^2}{|S(y, t)|} \frac{dt}{t} \right) \left( \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t} \right).
\]

On the other hand, from Lemmas 1.4 and 1.7 it follows that
\[
\int_{r/2a}^{R} \frac{t^2}{|S(y, t)|} \frac{dt}{t} \leq \sum_{j=0}^{\infty} \left( \frac{2^j}{2a} \right) \frac{1}{|S(y, 2^j r/a)|} \\
\leq \frac{C}{|S(y, r)|} \sum_{j=0}^{\infty} 2^{j(2-n)} \\
= \frac{C r^2}{|S(y, r)|}.
\]

Then, from this and (5.26), we get

\[
(5.27) \quad \int_{S \setminus Q(y, r)} \left| \nabla \chi G_\psi \right|^2 \nu \leq C \frac{w(Q(y, r))}{|Q(y, r)|} \int_{r/2a}^{R} \frac{1}{|S(y, t)|^{1/\alpha}} \frac{dt}{t}.
\]

Now, if \( \varrho \in [r/8a^2, R/2a] \), by applying Sobolev inequality in \( Q = Q(x_0, aR) \) we have

\[
a_0(G_\psi, G_\psi) = \frac{1}{w(Q(y, \varrho))} \int_{Q(y, \varrho)} G_\psi w \\
\leq CR \left( \frac{w(2Q)}{w(Q(y, \varrho))} \right)^{1/2} \left( \frac{1}{w(2Q)} \int_{S} \left| \nabla \chi G_\psi \right|^2 \nu \right)^{1/2} \\
\leq \frac{1}{w(Q(y, \varrho))} a_0(G_\psi, G_\psi)^{1/2}.
\]

Then
\[ \int_S |\nabla \chi G^g_y|^2 \nu \leq a_0(G^g_y, G^g_y) \leq \frac{C}{w(Q(y, \varrho))^{1/\sigma}} \]

\[ \leq \left( \frac{|S(y, r)|}{w(Q(y, r/8a^2))} \right)^{1/\sigma} \int_{r/2a}^R \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t}. \]

Finally, from (5.27) and the above inequality, we get our thesis.

With Lemma 5.25 we can prove the following integrability property of \( |\nabla \chi G^g| \).

**Lemma 5.28.** Let \( S = S(x_0, R) \) be a d-ball such that \( 13a^4 S \subset \Omega \subset \mathbb{R}^n \) with \( n > 2 \). Then for each \( q \in (0, 2\alpha/(\alpha + 1)) \) and a.e. \( y \in (1/2) S \) there exists \( C = C(q, y, x_0, R, w, \nu) > 0 \) such that

\[ (5.29) \quad \int_S |\nabla \chi G^g|^q \leq C \quad \text{for all} \quad q \in \left(0, \frac{R}{2a}\right). \]

**Proof.** Let \( y \in \frac{1}{2} S \) and \( q \in \left(0, \frac{R}{2a}\right) \). From Lemma 5.25 we get

\[ \nu(|\nabla \chi G^g| > s) \leq \frac{1}{s^q} \int_{s^q Q(y, r)} |\nabla \chi G^g|^{2}\nu + \nu(Q(y, r)) \]

\[ \leq \frac{C}{s^q} \int_{s^q r/2a}^R \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t} + |Q(y, r)| \sup_{t \leq R} \frac{\nu(Q(y, t))}{|Q(y, t)|} \]

\[ \leq C \left( \frac{1}{s^q} \int_{s^q r/2a}^R \frac{1}{|S(y, t)|^{1/\sigma}} \frac{dt}{t} + |Q(y, r)| \right). \]
for a.e. \( y \in (1/2) S \) and for all \( r \in (0, R/2a) \) and \( s > 0 \), where \( C = C(y, S, w, v) \).

Then, by using the properties of \( d \) and \( \delta \), it follows that

\[
\nu \left( \mid \nabla_y G^y_r \mid > s \right) \leq C \left( \mid Q(y, r) \mid + \frac{1}{s^2} \sum_{j=0}^{\infty} \int_{\frac{3r}{2}}^{\frac{2r}{3}} \frac{1}{S(y, t)} \mid \frac{dt}{t} \right)
\]

\[
\leq C \left( \mid Q(y, r) \mid + \frac{1}{s^2} \mid \sum_{j=0}^{\infty} 2^{-n/j} \right)
\]

\[
\leq C \left( \mid Q(y, r) \mid + \frac{1}{s^2} \mid \frac{Q(y, r)}{1/a} \right).
\]

Now, by choosing \( r \) such that \( \mid Q(y, r) \mid = s^\frac{2a}{\sigma+1} \), holds

\[
\nu \left( \mid \nabla_y G^y_r \mid > s \right) \leq C s^{-\frac{2a}{\sigma+1}}
\]

for all \( s > \mid Q(y, R/2a) \mid^{-\frac{\sigma+1}{2a}} \) for all \( y \in (0, R/2a) \) and for a.e. \( y \in (1/2) S \). From this inequality (5.29) follows immediately.

Let \( S \) be a \( d \)-ball such that \( 2aS \subset \Omega \subset \mathbb{R}^d \) with \( n \geq 3 \) and \( (p, q) \in [1, \sigma] \times [1, 2\sigma/(\sigma + 1)) \). Let \( X_{p,q} \) be the completion of \( \text{Lip}_0(S) \) with the norm

\[
\| \phi \|_{p,q} = \left( \int_S \mid \phi \mid^p w \right)^{1/p} + \left( \int_S \mid \nabla_y \phi \mid^q v dx \right)^{1/q}.
\]

Now the above results about \( G^y_r \) and \( \nabla_y G^y_r \), allow to prove existence of the Green functions \( G_y(\cdot) = G(\cdot, y) \) for \( S \) with pole \( y \).

**Theorem 5.30.** Let \( S = S(x_0, R) \) be a \( d \)-ball such that \( 13a^4S \subset \Omega \). For a.e. \( y \in (1/2) S \) exists a sequence \( \{y_k\} \subset (0, R/2a) \) and a function \( G_y \) such that
\begin{align}
(5.31) \quad & q_k \downarrow 0 \text{ and } |G_y^0| \text{ converges weakly to } G_y \text{ on } X_{p,q} \\
& \text{for all } (p, q) \in [1, \sigma) \times \left[ 1, \frac{2\sigma}{\sigma + 1} \right]; \\
(5.32) \quad & \|G_y\|_{p,q} \lesssim C \\
& \text{for every } (p, q) \in [1, \sigma) \times \left[ 1, \frac{2\sigma}{\sigma + 1} \right], \text{ where } C \text{ is independent of } y; \\
(5.33) \quad & \text{if } (v, w) \text{ satisfies } \left( w^\frac{q_0}{q_1} \right) v \in L^1(S, dx) \text{ for some } q_0 \in \left( 1, \frac{2\sigma}{\sigma + 1} \right) \\
& \text{then} \\
& \int_S \langle A \nabla G_y, \nabla \phi \rangle = \phi(y) \\
& \text{for all } \phi \in \text{Lip}_0(S), \text{ and a.e. } y \in \frac{1}{2} S; \\
(5.34) \quad & \text{if } u \text{ is solution of } (5.6) \text{ with } \psi = 0, \text{ then} \\
& u(y) = \int_S G_y(x) f(x) dx \text{ a.e. } y \in \frac{1}{2} S; \\
(5.35) \quad & \text{if } u \text{ is solution of } (5.7) \text{ with } \psi = 0, \text{ then} \\
& u(y) = \int_S \langle \nabla \psi, G_y(x), F(x) > dx \text{ a.e. } y \in \frac{1}{2} S.
\end{align}

PROOF. Follows from the above results about \( G_y^0 \) and \( \nabla \psi G_y^0 \) with the arguments in sections 5 and 6 of [ChW3].

Now, we prove another estimate for \( G_y^0 \). Then we shall to use this and the above to prove some estimates of size for the Green function.

**Lemma 5.37.** Let \( S(x_0, R) \) be a d-ball such that \( 13d^4 S \subset \Omega \). Then exists a constant \( C \) depending only on the constants of (0.5) and (0.7) such that

\begin{align}
(5.37) \quad & \inf_{r/2 < \theta(x,y) < r} G_y^0(x) \geq C \int_{x_0}^R \frac{t^2}{w(S(y, 2at))} e^{-CF_2(x_0, 0; F_1(x, t))} \frac{dt}{t}
\end{align}
for all \( y \in S(x_0, R/2), r \in (0, R/8\alpha^2) \) and \( q \in (0, r/4\alpha^2) \), where \( F_1(y, t) \) and \( F_2(y, t) \) are the functions of (5.17) and \( \gamma = \frac{3\alpha^2 - 2\alpha + 1}{\sigma - 1} \).

**Proof.** First, let us consider the case \( y = x_0 \). Let \( r \in (0, R/4\alpha^2) \). For \( s > 0 \), let us denote with \( G^s \) the approximate Green function for \( S(y, s) \) with pole \( y \). From Lemma 5.24, we get

\[
\int_{S(y, 2r) \setminus Q(y, r)} \langle A \nabla G^s_{2\alpha r}, \nabla G^s_{2\alpha r} \rangle \leq \frac{C}{r^2} w(Q(y, r) \left( \sup_{S(y, 2\alpha r) \setminus S(y, r/2\alpha)} G^s_{2\alpha r} \right)^2
\]

On the other hand, since \( LG^s_{2\alpha r} = 0 \) on \( S(y, 2\alpha r) \setminus S(y, r/4\alpha) \), the Harnack inequality (Theorem 5.10) allow us to obtain

\[
\sup_{Q(x, r/8\alpha^2)} G^s_{2\alpha r} \leq \exp \left\{ C \left( \frac{\inf_{d(y, z) < \alpha r} w(S(y, 2\alpha r))}{\inf_{d(z, x) < ar} w(S(z, r/16\alpha^3))} \right)^\gamma \left( \frac{w(S(y, 2\alpha r))}{v(S(y, 2\alpha r))} \right)^{1/2} \right\} \inf_{Q(x, r/8\alpha^2)} G^s_{2\alpha r}
\]

for all \( q \in (0, r/4\alpha^2) \) and all \( x \in S(y, ar) \setminus S(y, r/2\alpha) \). From this inequality and (5.38) follows that

\[
\left( \inf_{S(y, ar) \setminus S(y, r/2\alpha)} G^s_{2\alpha r} \right)^2 \leq \frac{Cr^2}{w(Q(y, r))} \exp \left\{ -C \left( \frac{\inf_{d(y, z) < \alpha r} w(S(y, 2\alpha r))}{\inf_{d(z, x) < ar} w(S(z, r/16\alpha^3))} \right)^\gamma \left( \frac{w(S(y, 2\alpha r))}{v(S(y, 2\alpha r))} \right)^{1/2} \right\}
\]

\[
\int_{S(y, 2\alpha r) \setminus Q(y, r)} \langle A \nabla G^s_{2\alpha r}, \nabla G^s_{2\alpha r} \rangle.
\]

Now, by taking \( \eta \in C^0(S(y, 2\alpha r)) \) such that \( \eta = 1 \) on \( Q(y, r) \) and \( |\nabla \eta| \leq \frac{C}{r} \) and applying the ellipticity condition, we get
\begin{align*}
1 &= a_0(G^\eta_{2ar}, \eta) \\
&\leq \frac{C}{r} w(y, 2ar)^{1/2} \left( \inf_{S(y, 2ar) \setminus G^\eta_{2ar}, \nabla G^\eta_{2ar}} \langle A \nabla G^\eta_{2ar}, \nabla G^\eta_{2ar} \rangle \right)^{1/2}.
\end{align*}

Then, from (5.39), holds
\begin{align*}
\inf_{S(y, ar) \setminus S(y, r/2a)} G^\eta_{2ar} \\
&\geq \frac{C r^2}{w(S(y, 2ar))} \exp \left\{ -C \left( \inf_{d(y, z) < ar} \frac{w(S(y, 2ar))}{w(S(z, r/16a^3))} \right)^\gamma \left( \frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\}
\end{align*}
for all \( q \in (0, r/4a^2) \). Now, by applying the weak maximum principle, it follows that
\begin{align*}
(5.40) \quad G^\eta_{2ar} - G^\eta_r \\
&\geq \frac{C r^2}{w(S(y, 2ar))} \exp \left\{ -C \left( \inf_{d(y, z) < ar} \frac{w(S(y, 2ar))}{w(S(z, r/16a^3))} \right)^\gamma \left( \frac{w(S(y, 2ar))}{v(S(y, 2ar))} \right)^{1/2} \right\}
\end{align*}
a.e. in \( S(y, r) \) and for all \( q \in (0, r/4a^2) \). On the other hand, if \( m \in \mathbb{N} \) is such that \( (2a)^mr \leq R < (2a)^{m+1}r \), we get
\begin{align*}
(5.42) \quad G^\eta_R \geq G^\eta_{(2a)^m r} = G^\eta_{2ar} + \sum_{j=1}^{m-1} (G^\eta_{(2a)^j r} - G^\eta_{(2a)^j r}), \text{ a.e. in } S(y, 2ar).
\end{align*}
Then, with \( S_t = S(y, t) \) and
\begin{align*}
g(t_1, t_2) &= \frac{t_1^2}{w(S_{t_1})} \exp \left\{ -C \left( \inf_{d(y, z) < ar} \frac{w(S_{t_1})}{w(S(z, t_1/16a^3))} \right)^\gamma \left( \frac{w(S_{t_1})}{v(S_{t_1})} \right)^{1/2} \right\}, \quad t_1, t_2 \in \mathbb{R}^+.
\end{align*}
from (5.40) and (5.41) the inequality
(5.43) \[ G_k^R \geq C \sum_{j=0}^{m-1} g((2a)^j r, (2a)^{j+1} r) \]

\[ \geq C \int_{2ar}^{ar} g \left( \frac{t}{4a^2}, t \right) \frac{dt}{t} \]

\[ = C \left[ \int_{4a^2 r}^{ar} g \left( \frac{t}{4a^2}, t \right) \frac{dt}{t} + \int_{2ar}^{4a^2 r} g \left( \frac{t}{4a^2}, t \right) \frac{dt}{t} \right] \]

\[ \geq C \left[ \int_{2ar}^{ar} g \left( \frac{t}{4a^2}, t \right) \frac{dt}{t} + \frac{1}{2} \int_{2ar}^{ar} g \left( \frac{t}{2a}, 2at \right) \frac{dt}{t} \right] \]

holds for a.e. in \( S(y, ar) \setminus S(y, r/2a) \) and for all \( g \in (0, r/4a^2) \). Finally, the result for \( y \neq x_0 \) follows easily from the fact that \( G_y^a \geq G_{R/2} \) a.e. in \( S(y, R/2) \) and (5.43).

In the following we prove a result of functional analysis which shall be used to prove the size estimates.

Lemma 5.44. Let \( E \subset \mathbb{R}^n \) be a measurable set, \( h \in L_{\text{loc}}(E, dx) \) be positive a.e. in \( E \) and \( p \in (1, \infty) \). If \( \int f_k \subset L^p(E, dx) \) converges weakly to a function \( f \) and satisfies \( \sup_k f_k \leq C_0 \) a.e. in a bounded set \( F \subset E \), for some constant \( C_0 \), then \( f \leq C_0 \) a.e. in \( F \).

Proof. From the hypothesis we get

\[ \int_E (C_0 - f_k) gh dx \geq 0, \]
for every $k$ and for all $g \in L^p(E, hdx)$ such that $\text{supp } g \subset F$ and $g \geq 0$ a.e. in $F$. Then, by letting $k \to \infty$, the inequality

$$
\int_E (C_0 - f) gh dx \geq 0,
$$

holds. Now, by taking $g = \chi_{F \cap \{ f > C_0 \}}$, we get the thesis.

Finally, we are in position to prove

**Theorem 5.45.** Let $S = S(x_0, R)$ be a $d$-ball such that $13a^4S \subset \Omega$. Then for a.e. $y \in (1/2)S$. $G_y$ is non-negative a.e. in $S$ and satisfies

$$(5.46) \sup_{r/2 < d(y, x) < r} G_y(x) \leq C \int_0^R \frac{t^2}{\nu(S(y, t))} ((F_1(y, t))^{\gamma_1} F_2(y, t))^{1/p} \frac{dt}{t}$$

for each $p \in (0, \sigma)$$

$$(5.47) \inf_{r/2 < d(y, x) < r} G_y(x) \geq C \int_0^R \frac{t^2}{\nu(S(y, 2at))} e^{-CF_1(y, t)^2 F_2(y, t)} \frac{dt}{t}$$

for all $r \in (0, R/8a^2)$, where $\gamma_1 = \frac{\sigma^2}{\sigma - 1}$, $\gamma_2 = \frac{3\sigma^2 - 2\sigma + 1}{\sigma - 1}$ and $F_1$ and $F_2$ are define as in (5.17).

**PROOF.** It follow from Lemmas 5.11 and 5.36 and Theorem 5.30 by using the above lemma.

Finally, we shall prove a result about the integrability without weights of the Green function.

**Theorem 5.48.** Let $S = S(x_0, R)$ be a $d$-ball such that $17a^4S \subset \mathbb{R}^n$ with $n \geq 3$. Then $G_y \in L^p(S, dx)$ for a.e. $y \in (1/2)S$ and for all $p \in \left(1, \max \left\{ \frac{\sum G_j}{\sum G_j - 2}, 1 + \frac{n}{\sum G_j} (\sigma - 1) \right\} \right)$. 
PROOF. From (5.46) by applying a similar argument to that Corollary 5.21 follows

\[ \sup_{r/2 < d(y, x) < r} G_y(x) \leq C \min \left\{ \int_r^R \frac{t^2}{|S(y, t)|} \frac{dt}{t}, \int_r^R \frac{1}{|S(y, t)|^{1/\varrho}} \frac{dt}{t} \right\} \]

for a.e. \( y \in (1/2)S \) and all \( r \in (0, R/8a^2) \) where \( C = C(y, x_0, R, w, \nu) \). By denoting with \( G_{16a^2R} \) the Green function for \( S(y, 16a^2R) \) with pole \( y \), the weak maximum principle, (5.49) and the properties of \( d \) allow us to get

\[ \int_{S_{(16a^2R)}} (G_y)^p \ dx \leq \int_{S_{(16a^2R)}} (G_{16a^2R})^p \ dx \]

\[ \leq \sum_{j=0}^\infty \int_{S(y,2(2j+1))}^{S(y,2(2j+1))} (G_{16a^2R})^p \ dx \]

\[ \leq C \sum_{j=0}^\infty \left( \sum_{i=0}^{j} \frac{R^{2i+1}}{|S(y, t)|} \right)^p |S(y, R/2^{j+1})| \]

\[ \leq C \sum_{i=0}^\infty \left( \sum_{j=0}^{i} \frac{R^{2i+1}}{|S(y, t)|} \right)^p |S(y, R/2^{i+1})| \]

\[ \leq C \sum_{i=0}^\infty \left( \sum_{j=0}^{i} \frac{R^{2(i+1)}}{|S(y, R/2^{i+1})|} \right)^p |S(y, R/2^{i+1})| \]

\[ \leq C \sum_{i=0}^\infty \left( \frac{R^{2(i+1)}}{|S(y, R/2^{i+1})|} \right)^p \sum_{j=0}^{i} 2^{(2-i-\varrho)} |S(y, R/2^{i+1})| \]

\[ \leq C \sum_{i=0}^\infty \left( \frac{R^{2(i+1)}}{|S(y, R/2^{i+1})|} \right)^p |S(y, R/2^{i+1})| \]
\[ \leq C \left( \frac{(2R)^p}{|S(y, 2R)|^{p-1}} + \frac{R^{2p}}{|S(y, R)|^{p-1}} \sum_{j=1}^{\infty} 2^{i(\Sigma_j G_j - 2)p - \Sigma_j G_j} \right). \]

Note that the right hand side is finite if and only if \( p < (\Sigma_j G_j)/(\Sigma_j G_j - 2) \). On the other hand, by applying (5.46) again and a similar argument, we get

\[
\int_{S(y, R)} (G_j)^p \, dx \leq C \left( \frac{1}{|S(y, R)|^{p/\alpha - 1}} \left( 1 + \sum_{j=1}^{\infty} 2^{\frac{1}{\alpha(\rho - 1)\Sigma_j G_j + \rho(1-\sigma)}} \right) \right),
\]

where the right member is finite iff \( p < 1 + \frac{n}{\Sigma_j G_j} (\sigma - 1) \). Finally, from this and (5.50), we obtain the thesis.

References


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