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Semigroup Commutators under Differences

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Introduction

This is the first paper on a subject whose motivation lies in a geometric problem that I shall now explain.

Motivation. Let us consider locally in $\Omega \subset \mathbb{R}^n$ (or in a more general C^∞ manifold) a self-adjoined second order differential operator with positive characteristic (*cf.* [13] [7] [8] [11] [10]):

$$Lf = h^{-1} \sum_{i,j} (\partial_i a_{ij} \partial_j h f), \qquad f \in C_0^\infty, \tag{0.1}$$

with h, h^{-1}, $a_{ij} \in C^\infty$ and real and with the matrix $A = (a_{ij}) \geqslant 0$ non-negative. We associate then to L a canonical distance by the following «recipe»: For every $x \in \Omega$ we say that $y \in B_r(x)$ lies in the r-ball ($r > 0$) centered at x if y is the end point of an absolutely continuous path $l(t)$ ($0 \leqslant t \leqslant r$) with $l(0) = x$, $l(r) = y$ that is «sub-unit» for L. This means that the speed vector

$$\dot{l}(t) = dl\left(\frac{\partial}{\partial t}\right) = \xi$$

is almost everywhere subunit for L, *i.e.* that

$$\left(\sum \xi_i \lambda_i\right)^2 \leqslant \sum a_{ij} \lambda_i \lambda_j, \qquad \lambda_i \in \mathbb{R}.$$

The value $d(x, y) = +\infty$ is admitted here if y lies in no ball centered at x.

The above operator is formally self-adjoint in $X = L^2(h^2\,dx)$, it can therefore be closed in X and the complex powers $L^\alpha(\alpha \in \mathbb{C})$ of L can be defined. We shall say that L is subelliptic if for some positive $\epsilon > 0$ we have

$$\|\Delta^\epsilon f\|_{L^2} \leqslant C(\|Lf\|_{L^2} + \|f\|_{L^2})$$

where Δ is of course the positive Laplacian. (Alternatively L is subelliptic if the distance $d(\cdot, \cdot)$ induced by L is Hölder continuous with respect to Euclidean distance (*cf.* [11] [12]).)

Let now L_1 and L_2 be two subelliptic operators as above and let d_1, d_2 denote their two corresponding distances. For two fixed $\alpha, \beta > 0$ (it will have to be $\beta \leqslant \alpha$) let us consider the following three conditions:

(i) There exists $C > 0$ such that

$$d_1^\alpha(x, y) \leqslant C d_2^\beta(x, y), \qquad x, y \in \Omega.$$

(ii) There exists $C > 0$ such that

$$\|L_2^\beta f\|_{L^2} \leqslant C(\|L_1^\alpha f\|_{L^2} + \|f\|_{L^2}), \qquad f \in C_0^\infty.$$

(iii) For every $1 < p < +\infty$ there exists $C > 0$ such that

$$\|L_2^\beta f\|_{L^p} \leqslant C(\|L_1^\alpha f\|_{L^p} + \|f\|_{L^p}), \qquad f \in C_0^\infty.$$

As we shall see below it is reasonable to conjecture that the above three conditions are all equivalent.

The basic original approach to this conjecture is due to Hörmander [14] who (among other things) proved that essentially (*i.e.* with a loss of an ϵ!) (i) implies (ii) with $L_2 = \Delta$ (the standard Laplacian) and

$$L_1 = \sum X_j^* X_j$$

a «sum of squares» operators (sometimes called a «Hörmander» operator) where the X_j's are first order differential operators, without constant term, *i.e.* X_j are C^∞ fields and X_j^* denotes their formal adjoints (with respect to Lebesgue measure). The equivalence of (i) and (ii) was proved in [11] for $0 < \alpha, \beta \leqslant 1$ when $L_2 = \Delta$. The implication (i) implies (ii) for arbitrary L_1, L_2 was shown in [15], but for $0 < \alpha, \beta < 1/2$. The equivalence of (i), (ii) and (iii) was shown in [8] for arbitrary L_1, L_2 but $\alpha = \beta$. Finally the implication (i) implies (iii) for $L_2 = \Delta$ and arbitrary $\alpha, \beta > 0$ was shown in [3]. In this paper the conjecture will be reinforced further and the following theorem will be proved

Theorem. *Let L_1, L_2, d_1, d_2 be two differential operators with their corresponding distances as above and let us assume both L_1, L_2 are subelliptic Hör-*

mander operators. I shall further impose on L_1 the following technical condition

$$d_1(x, y) \leqslant C|x - y|^{1/2}, \qquad x, y \in \Omega,$$

for some fixed $C > 0$.

Then (i) *implies* (iii) *for arbitrary $\alpha, \beta > 0$. Conversely, if* (ii) *holds for some fixed $\alpha_0, \beta_0 > 0$ then* (i) *holds for $\alpha = \alpha_0$ and $\beta = \beta_0 - \epsilon$ and arbitrary $\epsilon > 0$.*

At this stage I do not see how to eliminate the technical condition of the 1/2-Hölder continuity of d_1, except by replacing it by other conditions that are equally unnatural (*cf.* Section 8). On the other hand, in a second instalment of this paper I do hope to eliminate the other (artificial) condition of this theorem that L_1 and L_2 are «sum of squares». The $\epsilon > 0$ in the second part of this theorem should not of course be there. At the end of this paper however, I shall give an example of two operators as above that are sums of squares for which we have: $d_2(\cdot, \cdot) \leqslant C_\epsilon d_1^{1-\epsilon}(\cdot, \cdot)$ holds for all $\epsilon > 0$ but *not* for $\epsilon = 0$. A phenomenon that *cannot* occur when $L_1 = \Delta$ and L_2 is a sum of squares.

The Commutators

The technical result that makes everything work is a theorem on commutators of pseudo-differentials that I shall now describe in «general terms». Precise statements will be given later. Let $H_\alpha = \{f\colon \Lambda^\alpha f \in L^2\}$ with

$$\Lambda = \left(I - \sum \frac{\partial^2}{\partial x_i^2}\right)^{1/2}$$

denote the standard Sobolev spaces and let $L \in OPS^2_{1,0}$ be a pseudo-differential with a positive symbol that induces a non-negative Hilbert space operator A (*e.g.* $A = L(x, D) + L^*(x, D) + cI$ or $L^\omega(x, D) + cI$ or simply in our special case

$$A = \sum X_j^* X_j,$$

a sum of squares operator). The complex powers $A^\sigma (\sigma \in \mathbb{C})$ can then be defined and the basic commutation result that we shall prove in this paper is this:

For any choice $S_j \in OPS^{n_j}_{1,0}$, $n_j \in \mathbb{R}$, $1 \leqslant j \leqslant k$ and any $m \in \mathbb{R}$, $\sigma \in \mathbb{C}$ we have

$$(0.2)\quad \|[\cdots[[A^\sigma, S_1], S_2]\cdots S_k]f\|_m \leqslant C\|A^{\sigma - k/2} f\|_{m + m_1 + \cdots + n_k}, \qquad f \in C_0^\infty$$

$[A, B] = AB - BA$ denotes as usual the commutator of two operators and $\|\cdot\|_\alpha$ denote the Sobolev norms in H_α. The above will be proved for sum of squares operators in this paper and we will present a general proof in a later work.

We shall see in Section 8 the incidence that this commutator theorem has on our previous geometric problem. The result (0.2) has however another important consequence that I shall not develop here.

Indeed we shall see in Section 3 that for $\operatorname{Re}\sigma \geqslant 0$ we have

$$\|A^\sigma f\|_m \leqslant C\|f\|_{m+2\operatorname{Re}\sigma}, \qquad f \in C_0^\infty. \tag{0.3}$$

and also that for $\operatorname{Re}\sigma \leqslant 0$ we have

$$\|A^\sigma f\|_m \leqslant C\|f\|_{m+2(1-\delta)\operatorname{Re}\sigma}, \qquad f \in C_0^\infty \tag{0.4}$$

provided that A satisfies the subellipticity condition

$$\|f\|_{1-\delta} \leqslant C\|A^{1/2} f\|_{L^2} \tag{0.5}$$

for some $0 \leqslant \delta < 1$. If we combine (0.2), (0.3), (0.4) we see that for $S_j \in S^1_{1,0}$ and $\operatorname{Re}\sigma \leqslant 0$ we have

$$\|[\cdots[A^\sigma, S_1], \ldots, S_k] f\|_m \leqslant C\|f\|_{m+k\delta+2\operatorname{Re}\sigma(1-\delta)}.$$

If we combine this with Beals' characterization of pseudo-differentials (*cf.* [16], [17], [18]) we immediately see that the subellipticity condition (0.5) for $0 \leqslant \delta \leqslant 1/2$ implies that $A^\sigma \in OPS^{2\operatorname{Re}\sigma(1-\delta)}_{\nu,\nu}$ ($\operatorname{Re}\sigma \leqslant 0$, $\delta \leqslant \nu \leqslant 1-\delta$). This is already, of course, a theorem of Beals (*cf.* [16]). The present method, however, allows us to handle more general operators than in [16]. In particular the pseudo-differentials do not have to be polyhomogeneous. Another advantage of this approach is that we get (for free, so to speak) all the L^p norms. So that in (0.2), (0.3) and (0.4) we can, for operators as in (0.1), replace the H_α norms by the L^p-Sobolev norms (*i.e.* $\Lambda^\alpha f \in L^p$) and this is quite significant since, as is well-known, the general Hörmander classes $OPS^0_{\delta,\delta}$ do not operate on L^p (*cf.* [18], [19]). At any rate, I shall say no more about that aspect of the theory here. It will be developed in a later publication.

In our commutator result (0.2) and in some of its consequences the self adjointness of the operators (Alternatively real symbols) is *not* essential. This can easily be seen from the proofs. I shall only consider self-adjoint operators here however because this is all that will be needed for the Geometric Applications. Concerning self-adjointness, it turs out to be irrelevant with respect to which particular measure $d\mu = h^2\,dx$ ($h \in C^\infty$) the differential operators are self-adjoint. For simplicity therefore I shall always take $h \equiv 1$ in (0.1).

1. The Real Interpolation

Let X be a Hilbert space (more general Banach spaces can also be considered) and let $A = I + A_1$ where $(A_1 f, f) \geqslant 0$ is a positive closed self-adjoint operator that generates the semigroup $e^{-tA} \in \mathcal{L}(X)$ with $\|e^{-tA}\| \leqslant e^{-t}$. I shall denote by

$$X_\alpha = \{f \in X:\ A^{\alpha/2} f \in X\}, \qquad \alpha \in \mathbb{R}.$$

The corresponding scale of «Sobolev spaces».

Typically $X = L^2(\mathbb{R}^n)$ $(n \geqslant 1)$ and $A_1 = a(x, D) + a^*(x, D) + cI$ for an appropriate $c > 0$ where the symbol $a(x, \xi) \in S^2_{1,0}$ is non-negative $a(x, \xi) \geqslant 0$. More general manifolds can also be considered. Weyl calculus should of course be used here and we should set $A_1 = a^w(x, D) + cI$. We shall however not do so, at least not systematically, in this paper. The fact that $\|e^{-tA}\| \leqslant e^{-t}$ implies that

$$A^{-\nu} = c \int_0^\infty t^{\nu - 1} e^{-tA}\, dt$$

is a bounded operator on X_α $(\alpha \in \mathbb{R})$ for every $\nu \in \mathbb{C}$ with $\operatorname{Re} \nu > 0$. This means that the Sobolev spaces are naturally nested: $X_\alpha \subsetneq X_\beta$ for $\alpha \geqslant \beta$. When $A_1 = \Delta$ we obtain of course the standard Sobolev spaces which will be denoted as usual by H_α $(\alpha \in \mathbb{R})$.

It will be essential in what follows to introduce the corresponding Besov spaces $\Lambda^\alpha_{2,\infty}$ $(\alpha > 0)$ (*cf.* [1]) which I shall denote for simplicity by X^∞_α. We have $f \in X^\infty_\alpha$ if

$$\sup_t t^{-\alpha/2} \|D_\rho(t) f\|_X < +\infty,$$

where we denote by $D_\rho(t) = t^\rho A^\rho \exp(-tA)$. The $\rho > 0$ in the previous definition is immaterial as long as it is large enough (in fact $\rho > \alpha/2$). I shall always assume that this is the case and drop the index ρ. Different choices of ρ as above give equivalent norms (*cf.* [2], [3]). The following two facts are vital for us

First we have $X_\alpha \subsetneq X^\infty_\alpha$ $(\alpha > 0)$. Equivalently we have

$$\|D(t) f\|_X \leqslant C t^{\alpha/2} \|f\|_\alpha$$

where here and in what follows $\|\cdot\|_\alpha$ denotes the norm in X_α.

Second if we use the real interpolation method we obtain

$$X_{\theta\alpha + (1-\theta)\beta} = [X^\infty_\alpha, X^\infty_\beta]_{\theta,2} = [X_\alpha, X_\beta]_{\theta,2} \tag{1.1}$$

provided that $\alpha, \beta > 0$, $0 < \theta < 1$. *Cf.* [1] [4] for the above facts.

Let us now consider

$$\Phi(t): C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \qquad t > 0,$$

a family of mappings from C_0^∞ to C^∞ that depends on the positive parameter ($t > 0$). Notice that I have restricted myself, for simplicity, to our basic example $X = L^2(\mathbb{R}^n)$. The modifications in the abstract setting are obvious. Typical instances of $\Phi(t)$ in our applications will be

$$\Phi(t) = T_t,$$
$$\Phi(t) = T_t - I,$$
$$\Phi(t) = T_t M - M T_t = [T_t, M],$$
$$\Phi(t) = M_1[T_t, M]$$

and also higher commutators. In the above examples M, $M_1: \mathcal{S} \to \mathcal{S}$ are continuous operators and $T_t = e^{-tB}$ is a continuous semigroup of operators that acts (*i.e.* is a continuous semigroup of bounded operators) on all (or the relevant) X_α spaces. We could for instance take $X_\alpha = H_\alpha$ the classical Sobolev spaces and take $T_t = e^{-tL}$ where L is some pseudo-differential operator as above (*cf.* [3]). In this context I shall adopt once and for all the following *convention*:

Semigroups $(T_t)_{t>0}$ as above will be substituted, when necessary, and without further notice, by the corresponding semigroups $(e^{-\lambda_0 t}T_t)_{t>0}$ for appropriate $\lambda_0 > 0$ so that the behaviour at infinity will be controlled. This, equivalently, corresponds to replacing L by $L + \lambda_0$ and, since all our considerations are local, it will have no effect in our theory.

I shall now consider the integral

$$I_\theta = \int_0^\infty t^{\theta - 1}\Phi(t)\, dt, \qquad \theta \in \mathbb{C},$$

(for the principal determination of t^θ with θ complex) and I shall immediately take advantage of my convention and assume that the above integral converges in norm at infinity. The issue will be the convergence of the above integral at 0. The discussion that follows in the rest of this section will be devoted to this problem.

We shall assume that there exist two constants $A \in \mathbb{R}$, $B > 0$ such that

$$\|\Phi(t)\|_{\alpha_0 \to \beta_0} = O(t^{A + B(\alpha_0 - \beta_0)}), \qquad t \to 0$$

for some fixed $\alpha_0, \beta_0 \in \mathbb{R}$. The notation $\|\cdot\|_{\alpha \to \beta}$ here and in what follows indicates the operator norm $X_\alpha \to X_\beta$. If we have $\operatorname{Re}\theta + A + B(\alpha_0 - \beta_0) > 0$ then the integral I_θ clearly converges at 0. We shall consider therefore a $\theta_0 \in \mathbb{C}$ for which

$$\operatorname{Re}\theta_0 + A + B(\alpha_0 - \beta_0) = 0.$$

We have quite generally for any $f \in C_0^\infty$ with $\|f\|_\alpha = 1$

$$\begin{aligned}\|t^{-\beta/2}D(t)I_\theta f\|_X &\leqslant t^{-\beta/2}\int_0^\infty \xi^{\operatorname{Re}\theta-1}\|D(t)\Phi(\xi)f\|_X\,d\xi \\ &= t^{(\gamma_1-\beta)/2}\int_0^\lambda \xi^{\operatorname{Re}\theta-1}\|\Phi(\xi)f\|_{\gamma_1}\,d\xi \\ &\quad + t^{(\gamma_2-\beta)/2}\int_\lambda^\infty \xi^{\operatorname{Re}\theta-1}\|\Phi(\xi)f\|_{\gamma_2}\,d\xi \\ &= I_t + I^t,\end{aligned}$$

where we assume that $\beta, \gamma_1, \gamma_2 > 0$. I shall now make the stronger assumption that

$$\|\Phi(t)\|_{\alpha\to\beta} = O(t^{A+B(\alpha-\beta)}), \qquad t \to 0 \tag{1.2}$$

for $\alpha = \alpha_0$ and $\beta \in [\beta_0 - \epsilon, \beta_0 + \epsilon]$ for some positive $\epsilon > 0$. With $\theta = \theta_0$, $\beta = \beta_0$ and $\gamma_1 = \beta_0 - \epsilon$, $\gamma_2 = \beta_0 + \epsilon$ in the above estimate we obtain

$$\begin{aligned}&I_t \leqslant Ct^{(\gamma_1-\beta_0)/2}\lambda^{\operatorname{Re}\theta_0+A+B(\alpha_0-\gamma_1)} \\ &I^t \leqslant Ct^{(\gamma_2+\beta_0)/2}[\lambda^{\operatorname{Re}\theta_0+A+B(\alpha_0-\gamma_2)}+1], \quad \text{for} \quad \lambda \leqslant 1, \\ &I^t \leqslant Ct^{(\gamma_2-\beta_0)/2}\lambda^{-\epsilon_0}, \qquad\qquad\qquad\quad \text{for} \quad \lambda > 1.\end{aligned}$$

Where we tacitly *assume* that

$$\xi^{\operatorname{Re}\theta_0}\|\Phi(\xi)\|_{\alpha_0\to\gamma_2} = O(\xi^{-\epsilon_0}), \qquad \xi \to \infty,$$

for some fixed $\epsilon_0 > 0$. (Our convention will take care of that in all our applications.) If we assume as we may that $\epsilon > 0$ is small enough, and choose $\lambda = t^{1/2B}$ we obtain from our hypotheses that $I_t + I^t \leqslant C$ for all $t > 0$. This can be reinterpreted by saying that I_{θ_0} induces a bounded mapping. (I shall assume here that $\alpha_0, \beta_0 > 0$)

$$I_{\theta_0} = X_{\alpha_0} \to X^\infty_{\beta_0}. \tag{1.3}$$

As a final step assume that (1.2) holds for $\alpha \in [\alpha_0 - \epsilon, \alpha_0 + \epsilon]$ and $\beta \in [\beta_0 - \epsilon, \beta_0 + \epsilon]$ for some $\epsilon > 0$. The mapping I_{θ_0} is then bounded as in (1.3) for a whole range of α_0 and β_0 around the central values. The interpolation theorem (1.1) implies therefore that

$$I_{\theta_0}\colon X_{\alpha_0} \to X_{\beta_0} \tag{1.4}$$

is a bounded mapping also.

The case $\alpha_0, \beta_0 < 0$ is handled by duality, since $X^*_\alpha = X_{-\alpha}$. We have to assume that the semigroup T^*_t satisfies analogous conditions and we work with the analogous $\Phi^*(t)$. Once we have (1.4) with $\alpha_0, \beta_0 > 0$ and with $\alpha_0, \beta_0 < 0$ we interpolate and obtain (1.4) in general.

2. First Illustrations of the Real Interpolation Method

Let $S(x,\xi) \in S^{2m}_{1,0}$ $(m > 0)$ be real and let us assume that the operator $S(x,D)$ satisfies the sub-elliptic estimate

$$\|f\|^2_{H_{m-\delta}} = \|f\|^2_{m-\delta} \leqslant C(\mathrm{Re} < Sf,f > + \|f\|^2_2) \tag{2.1}$$

for some $C > 0$ and $0 \leqslant \delta \leqslant 1$. It is easy to see then that the semigroup $e^{-tS} = T_t$ acts continuously on all the classical Sobolev spaces H_α. For $0 \leqslant \delta < 1$ the proof is banal: To say that e^{-tS} acts on H_α is equivalent to saying that $\exp(-t\Delta^\alpha S\Delta^{-\alpha})$ acts on $L^2(\mathbb{R}^n)$ but the generator $\Delta^\alpha S\Delta^{-\alpha}$ is semibounded (*i.e.* $\mathrm{Re}\,(Af,f) \geqslant -C\|f\|^2_2$) by our sub-ellipticity condition and the standard commutation relation (*cf.* [5])

$$\Delta^\alpha S\Delta^{-\alpha} - S \in OPS^{2m-2}_{1,0} \bmod (\text{Skew-adjoined operators}).$$

The above argument works even for $\delta = 1$ provided that $m = 1$. For more general $m > 1$ we can still show that S is semibounded on H_n, $n = 1, 2, \ldots$, provided that we give the following equivalent norm on H_n

$$\|f\|^*_{H_n} = (\Delta^n f,f)^{1/2} + c_1(\Delta^{n-1}f,f)^{1/2} + \cdots + c_p(\Delta^{n-p}f,f)^{1/2} + \cdots$$

for appropriate $c_1, c_2, \ldots, c_n > 0$. At any rate this is not the issue here and let me *simply assume* that e^{-tS} acts on all the H_α spaces $(\alpha \in \mathbb{R})$.

Because of the sub-ellipticity condition (2.1) we can identify S with a closed Hilbert space operator and we can then define all the powers S^σ $(\sigma \in \mathbb{C})$ of S. The semigroup e^{-tS} could *for instance* be used to define these powers (*cf.* [6]).

I shall illustrate here the results of Section 1 by showing that for every $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma > 0$ the operator S^σ can be identified with a bounded operator

$$(\lambda_0 + S)^\sigma : H_\sigma \to H_{\alpha - 2m\,\mathrm{Re}\,\sigma}, \qquad \alpha \in \mathbb{R}, \tag{2.2}$$

(for $\lambda_0 > 0$ large enough. Following my convention in Section 1, I shall drop the λ_0 from now on). It is clearly enough to prove this for $0 < \mathrm{Re}\,\sigma < 1$ where we have the formula (*cf.* [6])

$$S^\sigma = C_\sigma \int_0^\infty t^{-\sigma-1}(T_t - I)\,dt. \tag{2.3}$$

Our theory in Section 1 with $X_\alpha = H_\alpha$ implies and the result (2.2) follows at once as soon as we show that $\Phi(t) = T_t - I$ satisfies

$$\|T_t - I\|_{\alpha \to \alpha - 2m} = O(t), \qquad t \to 0, \qquad \alpha \in \mathbb{R}, \tag{2.4}$$

a fact that, by interpolation, once combined with the trivial bound

$$\|T_t - I\|_{\alpha \to \alpha} = O(1)$$

gives

$$\| T_t - I \|_{\alpha \to \alpha - 2m\lambda} = O(t^\lambda), \qquad t \to 0, \quad \alpha \in \mathbb{R}, \quad \lambda \in [0, 1].$$

But (2.4) is of course evident. Indeed

$$T_t - I = c \int_0^t S T_\xi \, d\xi$$

and we only have to observe that $S \in OPS^{2m}_{1,0}$ and that T_ξ acts on the Sobolev spaces H_α.

Negative powers can also be treated that way, in other words we can assume that $\operatorname{Re} \sigma < 0$. Here we have to impose the condition that $0 \leqslant \delta < 1$. For simplicity I shall also assume that $m \equiv 1$. (The results work in general but for our applications we shall only need the case $m = 1$.) It was then shown in [3] that

$$\| T_t \|_{\alpha \to \beta} = O(t^{(\alpha - \beta)/2(1 - \delta)}), \qquad t \to 0, \tag{2.5}$$

for all $\alpha \leqslant \beta$. The theory of Section 1 applies again (with $X_\alpha = H_\alpha$) and we deduce that

$$S^{-\sigma}\colon H_\alpha \to H_{\alpha - 2(1-\delta)\operatorname{Re}\sigma}, \qquad \alpha \in \mathbb{R}, \tag{2.6}$$

for all $\sigma \in \mathbb{C}$ with $\operatorname{Re} \sigma > 0$. This was proved in [3] with the same method.

The above methods give no information on the pure imaginary powers. In fact, we shall see in the next section that

$$S^{it}\colon H_\alpha \to H_\alpha, \qquad \alpha, t \in \mathbb{R}, \tag{2.7}$$

is a bounded operator (since S^{it} is a group in t the norm of that operator increases at most exponentially in $t \in \mathbb{R}$) as soon as the symbol $S(x, \xi) \geqslant 0$ is non-negative. This fact together with complex interpolation theory (between «integer values» $\sigma = n + it$, $n = 0, 1, \ldots$) can be used to reprove (2.2).

3. Duhamel's Formula, the Pure Imaginary Powers, and an Overview of the Ideas

I shall introduce now some of the main ideas of this paper. For simplicity I shall first consider an operator that is self-adjoint and a sum of squares of fields

$$L = \sum X_j^* X_j$$

(a Hörmander operator) but on which I impose *no* sub-ellipticity conditions whatsoever (in other words I do not assume that the successive brackets of the fields X_j span the tangent space).

I shall consider the two semigroups

$$T_t = e^{-tA}, \qquad R_t = e^{-tB}, \qquad A = (I + L), \qquad B = \Lambda^{\alpha} A \Lambda^{-\alpha},$$

where $\Lambda = (1 + \Delta)^{1/2}$. «Duhamel's formula» then is the following observation

$$\begin{aligned} R_t - T_t &= \int_0^t \frac{d}{ds}(R_s T_{t-s})\, ds \\ &= \int_0^t R_s (A - B) T_{t-s}\, ds. \end{aligned}$$

Using the standard commutation relations for pseudo-differentials we see that

$$\begin{aligned} A - B &= \sum_j (X_j E) + E \\ &= \sum_j (E X_j) + E \end{aligned}$$

where from here onwards, when confusion does not arise, I shall use the letter E to indicate an arbitrary zero order in $OPS^0_{1,0}$ pseudo-differential (that can be different from place to place). We have therefore

$$R_t - T_t = \sum_j \int_0^t R_s E X_j T_{t-s}\, ds + \int_0^t R_s E T_{t-s}\, ds.$$

In the above decomposition the last term is harmless. It has an $L^2 \to L^2$ norm (in fact an $L^p \to L^p$ norm *cf.* [3] $1 < p < +\infty$) that is $O(t)$. This is simply because both T_t *and* R_{t-s} (by Section 2) are L^2 (in fact L^p) bounded uniformly for $0 < t < 1$ and so is $E \in S^0_{1,0}$.

The terms

$$\int_0^t R_s E X_j T_{t-s}\, ds = \int_0^t R_s E X_j A^{-1/2} A^{1/2} T_{t-s}\, ds \tag{3.1}$$

are more subtle. Indeed we have:

$$\| A^{1/2} e^{-tA} \|_{L^2 \to L^2} = O(t^{-1/2}) \qquad t \to 0,$$

and clearly

$$\| X_j f \|_{L^2} \leqslant \| A^{1/2} f \|_{L^2} \qquad j = 1, 2, \ldots \tag{3.2}$$

This means that $\| X_j A^{-1/2} \|_{L^2 \to L^2} \leqslant C$. Putting all these remarks together we see that we can estimate the $\| \cdot \|_{L^2 \to L^2}$ norm of the terms in (3.1) by

$$c \int_0^t \frac{ds}{(t - s)^{1/2}} = O(t^{1/2}).$$

In fact the $\|\cdot\|_{L^p \to L^p}$ $(1 < p < +\infty)$ of these terms can also be estimated by $O(t^{1/2})$ by the same method. This however is more delicate since it depends on the $L^p \to L^p$ estimate of the singular integral transform: $X_j L^{-1/2}$ (*cf.* [7], [8], [9]). This is only known to hold for a sub-elliptic operator L. I shall not press the matter further at this stage.

Another incidental fact that we notice at this stage is that the above estimate automatically implies that

$$\|T_t - R_t\|_{\alpha \to \alpha} = O(t^{1/2})$$

(just take differences!) where $\|\cdot\|_{\alpha \to \beta}$ are operator norms between the classical Sobolev spaces H_α. Furthermore, writing

$$T_{2t} - R_{2t} = T_t(T_t - R_t) + (T_t - R_t)R_t$$

allows us to conclude that for a δ-sub-elliptic operator with $\delta < 1$ (*i.e.* (2.1) is verified for $m = 1$ and that δ) we have: (we use (2.5) which is then automatically also valued for R_t)

$$\|T_t - R_t\|_{\alpha \to \beta} = O(t^{1/2 + (\alpha - \beta)/2(1 - \delta)}), \qquad t \to 0, \tag{3.3}$$

for every $\beta \geqslant \alpha$. One can also show by an easy modification of the above argument, that for $\beta \in [\alpha - 1, \alpha]$ we have in (3.3) the bound $O(t^{1/2 + (\alpha - \beta)/2})$. These remarks are not essential however *here*, they are merely meant to illustrate the ideas involved.

Let us now recall the standard formula valid for any semigroup generator D and $\epsilon > 0$, $s \in \mathbb{R}$

$$D^{-\epsilon + is} = C(\epsilon, s) \int_0^\infty t^{\epsilon - 1 + is} e^{-tD}\, dt. \tag{3.4}$$

Let us fix $s \neq 0$ and decompose

$$\int_0^\infty t^{\epsilon - 1 + is} R_t\, dt = \int_0^\infty t^{\epsilon - 1 + is} T_t\, dt + \int_0^\infty t^{\epsilon - 1 + is}(R_t - T_t)\, dt.$$

The second integral can be made to converge absolutely and uniformly as $\epsilon \to 0$ in the norm $\|\cdot\|_{L^2 \to L^2}$. This is because of (3.3) and our standing convention of Section 1 about $t \to \infty$. The first integral expresses $A^{-\epsilon + is}$ and by the spectral theorem on the spectral decomposition of L it converges strongly to the unitary operator A^{is}. Our conclusion is that $\Lambda^\alpha A^{is} \Lambda^{-\alpha}$ is an $L^2 \to L^2$ bounded operator which is our required result. In fact, since the convention of Section 1 was used to control the infinity of the above integrals, the way we have written (2.7) is *not* correct. What is correct is

$$(\lambda_0 + L)^{is}\colon\ H_\alpha \to H_\alpha \tag{3.5}$$

is bounded for $\alpha, s \in \mathbb{R}$ for some (in fact, all) $\lambda_0 > 0$. I will however systematically ignore the addition of positive constants and continue to write a formula like (3.5) in the slightly incorrect but more convenient form (2.7). Observe finally that e^{-tA} acts holomorphically on each H_α space. Indeed, consider for some appropriate $\lambda_0 > 0$, $\exp(-\lambda_0 z - zA) = U(z)$ ($\operatorname{Re} z > 0$) so that $\|U(z)\|_{L^2 \to L^2} \leqslant C$ for $\operatorname{Re} z > 0$ and $\|U(z)\|_{\alpha \to \alpha} \leqslant C$ when z lies on the real axis. By an appropriate conformal mapping we can re-express this in the form

$$\|\Lambda^{\alpha\omega/2} U(w) \Lambda^{-(\alpha\omega/2)}\|_{L^2 \to L^2} \leqslant C$$

when $\operatorname{Re} w = 0$ and $\operatorname{Re} w = 1$ (here $0 \leqslant \operatorname{Re} w \leqslant 1$) and Phragmèn-Lindelöf does the rest (*cf.* [20]).

Alternatively, we can use the action of A^{is} on H_α to interpolate the information on $\|(tA)^z e^{-tA}\|_{\alpha \to \alpha}$ for $\alpha = r \in \mathbb{R}$, $\operatorname{Re} z = 0$ and $\alpha = 0$, $\operatorname{Re} z = 1, 2, \ldots$.

In the same spirit if we use complex interpolations between the H^p_α spaces ($1 \leqslant p \leqslant +\infty$)

$$H^p_\alpha = \{f\colon\ \Lambda^{\alpha/2} f \in L^p\}$$

we see that

$$L^{is}\colon\ H^p_\alpha \to H^p_\alpha$$

for $1 < p < +\infty$, $\alpha \in \mathbb{R}$. The other non trivial information used here is the fact that

$$(I + L)^{is}\colon\ L^p \to L^p$$

is bounded for $1 < p < +\infty$ (*cf.* [20]).

Before we move on, we shall address ourselves the question whether the above considerations generalize to more general self-adjoint differential operators

$$L = h^{-1} \sum_{i,j} (\partial_i a_{ij} h \partial_j) \tag{3.6}$$

with positive characteristic $(a_{ij}) \geqslant 0$, $h, h^{-1} \in C^\infty$ or better still for pseudo-differentials with positive symbol $\rho(x, \xi) \geqslant 0$

$$L = \rho(x, D) + \rho^*(x, D) + C \tag{3.7}$$

for appropriate positive $C > 0$.

The answer is *yes* in the most general case (3.7) for L^2 norms. Everything we said about L^p norms for ($p \neq 2$) also holds in the case (3.6). But to see all this it is difficult and totally alien with the methods of this paper. It will be presented in a second instalment of this work.

4. Commutators with $E \in S^0_{1,0}$

Let us get back now to the main theme of this paper. I shall fix from here onwards the following notation

$$A = I + L = I + \sum X_j^* X_j$$

for C^∞ fields X_j and I shall also denote as in Section 1 by

$$X_\alpha = \{f\colon A^{\alpha/2} f\}, \qquad \alpha \in \mathbb{R} \tag{4.1}$$

the corresponding Sobolev spaces. (No confusion will arise, I am sure, between the fields X_j and the spaces X_α!)

Let $E \in OPS^0_{1,0}$ be an arbitrary zero order pseudo-differential. By standard commutation we have as before

$$[A, E] = AE - EA = \sum_j (EX_j) + E$$

(for different E's in $OPS^0_{1,0}$) this means that

$$\begin{aligned} \|[A,E]f\|_{L^2} &\leqslant \sum_j \|EX_j f\|_{L^2} + \|Ef\|_{L^2} \\ &\leqslant C \sum \|X_j f\|_{L^2} + \|f\|_{L^2} \\ &\leqslant C \|A^{1/2} f\|_{L^2} \end{aligned}$$

(observe, and this will be a recurrent point in what follows, that $\|A^{1/2} f\|_{L^2} \geqslant C \|f\|_{L^2}$). Considerably more is in fact true.

We shall assume that L is sub-elliptic *i.e.* that the sucessive brackets of the fields X_j span at every point the tangent space. The use of sub-ellipticity that we shall make at this stage is that it implies the estimate

$$\|X_{i_1} X_{i_2} \cdots X_{i_k} f\|_{L^2} \leqslant C \|A^{k/2} f\|_{L^2}, \qquad f \in C_0^\infty \tag{4.2}$$

for $k = 1, 2, \ldots$. This is a deep result and unlike (3.2) (for $k = 1$) it is not just a consequence of a formal identity. There are many ways of seeing this and they all depend on singular integral theory (*cf.* [9], p. 96). My favourite way of seeing it runs like this: First prove (4.2) for a nilpotent Lie group and «constant» fields (*i.e.* elements of the Lie algebra, *cf.* [21]). Then use the parametrix and the lifting theorem of [22]. Be it as it may, we are going to take (4.2) for granted here.

Using successive commutations again and developing A^k ($k = 1, 2, \ldots$) as sums of products of fields we see that (4.2) implies (observe that $\|A^j f\|$ is an increasing function of $j \in \mathbb{R}$) that

$$\|[A^k, E]f\|_{L^2} \leqslant C\|A^{k-1/2}f\|_{L^2}$$
$$\|A^n[A^k, E]f\|_{L^2} \leqslant C\|A^{n+k-1/2}f\|_{L^2}$$

for $k, n = 1, 2, \ldots$. But, denoting from here onwards by $\|\cdot\|_\alpha = \|\cdot\|_{X_\alpha}$, this simply says that

$$\|[A^k, E]\|_{2n+(2k-1)\to 2n} \leqslant C$$

which by interpolation implies that

$$\|[A^k, E]\|_{\alpha\to\alpha-2k+1} \leqslant C \tag{4.3}$$

for all $\alpha \in \mathbb{R}$, and α large enough.

If we dualize the above statement (observe that $X_\alpha^* = X_{-\alpha}$) and then interpolate again we finally obtain that (4.3) holds for all $\alpha \in \mathbb{R}$, $k = 0, 1, 2, \ldots$. In particular we have

$$\|[A, E]\|_{\alpha+1\to\alpha} \leqslant C \tag{4.4}$$

for all $\alpha \in \mathbb{R}$. Our next task will be to prove the following more general and basic fact

$$\|[A^\sigma, E]\|_{\alpha\to\alpha-2\operatorname{Re}\sigma+1} \leqslant C \tag{4.5}$$

for *any* $\sigma \in \mathbb{C}$. (4.5) follows automatically for $\operatorname{Re}\sigma < 1$ from the theory of Section 1 as soon as we have shown the estimate

$$\|[e^{-tA}, E]\|_{\alpha\to\beta} = O(t^{1/2+(\alpha-\beta)/2}), \qquad t \to 0, \tag{4.6}$$

for $\alpha, \beta \in \mathbb{R}$ with $\beta \geqslant \alpha - 1$. Indeed it suffices to insert in the I_θ of Section 1 the two formulas (2.3), (3.4) and to observe that $[e^{-tA} - I, E] = [e^{-tA}, E]$. To prove (4.6) we first observe that

$$[E, e^{-2tA}] = e^{-tA}[E, e^{-tA}] + [E, e^{-tA}]e^{-tA}$$

$$[E, e^{-tA}] = \int_0^t \frac{d}{ds}(e^{-(t-s)A}Ee^{-sA})\,ds$$

$$= \int_0^t e^{-(t-s)A}[E, A]e^{-sA}\,ds$$

which together give

$$[E, e^{-2tA}] = \int_0^t e^{-(2t-s)A}[E, A]e^{-sA}\,ds + \int_0^t e^{-(t-s)A}[E, A]e^{-(s+t)A}\,ds. \tag{4.7}$$

(In all these formulas, and they will be many, I tend to ignore irrelevant $\pm$ signs.) We shall now «factorize» the $\|[E, e^{-2tA}]\|_{\alpha\to\beta}$ inside the integrals and estimate it by

$$\int_0^t \|e^{-(2t-s)A}\|_{\alpha-1\to\beta}\,\|[E,A]\|_{\alpha\to\alpha-1}\,\|e^{-sA}\|_{\alpha\to\alpha}\,ds$$

$$+\int_0^t \|e^{-(t-s)A}\|_{\beta\to\beta}\,\|[E,A]\|_{\beta+1\to\beta}\,\|e^{-(s+t)A}\|_{\alpha\to\beta+1}\,ds.$$

$\|[E,A]\|_{\alpha\to\alpha-1}$ is of course bounded by (4.4) and we also have $\|e^{-sA}\|_{\alpha\to\beta} = O(s^{(\alpha-\beta)/2})$ for $\beta \geqslant \alpha$, because of the holomorphic action of e^{-tA} on L^2. We see therefore that the two integrals can be estimated respectively by

$$\int_0^t (2t-s)^{(\alpha-1-\beta)/2}\,ds = O(t^{1/2+(\alpha-\beta)/2}), \qquad \beta \geqslant \alpha - 1$$

$$\int_0^t (t+s)^{(\alpha-1-\beta)/2}\,ds = O(t^{1/2+(\alpha-\beta)/2}), \qquad \beta \geqslant \alpha - 1.$$

The estimate (4.6) and the estimate (4.5) for $\operatorname{Re}\sigma < 1$ follow. To obtain (4.5) for arbitrary $\sigma \in \mathbb{C}$ we just use (4.3) and formula

$$[A^{k+\sigma}, E] = A^k[A^\sigma, E] + [A^k, E]A^\sigma$$

with $k = 1, 2, \ldots$ $\operatorname{Re}\sigma < 0$ and observe that for arbitrary $\sigma \in \mathbb{C}$, $A^\sigma: X_\alpha \to X_{\alpha - 2\operatorname{Re}\sigma}$ is an isomorphism.

The estimate (4.5) is a special case of the following more general result

$$\|[\cdots[[A^\sigma, E_1], E_2]\cdots, E_k]\|_{\alpha\to\alpha-2\operatorname{Re}\sigma+k} \leqslant C \tag{4.8}$$

for arbitrary $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{C}$, $k = 1, 2, \ldots$ where $E_1, \ldots \in OPS^0_{1,0}$ are zero order pseudo-differentials. I shall give below the proof for $k = 2$. (The general case is similar and we shall have no immediate use of anyway.)

We start with the formula (4.7) where $E = E_1$ and take the commutator with E_2 so as to obtain $[[e^{-2tA}, E_1], E_2]$. We insert the commutator with E_2 inside each integral so as to obtain six different terms. The first three are

$$I_1 = \int_0^t [e^{-(2t-s)A}, E_2][A, E_1]e^{-sA}\,ds,$$

$$I_2 = \int_0^t e^{-(2t-s)A}[[A, E_1], E_2]e^{-sA}\,ds,$$

$$I_3 = \int_0^t e^{-(2t-s)A}[A, E_1][e^{-sA}, E_2]\,ds.$$

The other three terms I_1', I_2', I_3' are identical except that $2t - s$ on the first exponential is replaced by $t - s$ and the s in the second exponential is replaced by $s + t$.

We shall first analyse the terms I_1 and I_3, we have

$$\|I_1\|_{\alpha\to\beta} \leqslant \int_0^t \|\cdot\|_{\alpha-1\to\beta}\|\cdot\|_{\alpha\to\alpha-1}\|\cdot\|_{\alpha\to\alpha} ds.$$

The factors are written of course in the same order and

$$\|\cdot\|_{\alpha-1\to\beta} = O((2t-s)^{1/2+(\alpha-1-\beta)/2})$$

by our estimate (4.6) for $\beta \geqslant \alpha - 2$. For $\beta \geqslant \alpha - 2$ we have thus $\|I_1\|_{\alpha\to\beta} = O(t^{1+(\alpha-\beta)/2})$. Similarly we factorize

$$\|I_3\|_{\alpha\to\beta} \leqslant \int_0^t \|\cdot\|_{\alpha-2\to\beta}\|\cdot\|_{\alpha-1\to\alpha-2}\|\cdot\|_{\alpha\to\alpha-1} ds.$$

The term $\|\cdot\|_{\alpha\to\alpha-1} = O(s)$ because of (4.6), $\|\cdot\|_{\alpha-1\to\alpha-2} \leqslant C$ and $\|\cdot\|_{\alpha-2\to\beta} = O((2t-s)^{(\alpha-\beta)/2-1})$ for $\beta \geqslant \alpha - 2$. We obtain thus $\|I_3\|_{\alpha\to\beta} = O(t^{1+(\alpha-\beta)/2})$ for $\beta \geqslant \alpha - 2$.

To estimate the middle term we have to phase some new phenomena that give us a glimpse of what happens in (4.8) when $k \geqslant 2$ is arbitrary. We factor again

$$\|I_2\|_{\alpha\to\beta} = \int_0^t \|\cdot\|_{\gamma\to\beta}\|\cdot\|_{\alpha\to\gamma}\|\cdot\|_{\alpha\to\alpha} ds \tag{4.9}$$

and we may as well assume that $\beta \geqslant \alpha - 2$ since this condition is imposed on us by the other two terms. We distinguish two cases:

Case (i): $\beta \geqslant \alpha$. We then set $\gamma = \alpha$ and we shall show that the middle term satisfies

$$\|[[A, E_1], E_2]\|_{\alpha\to\alpha} \leqslant C. \tag{4.10}$$

Assume this for the moment. Then the first term gives $\|\cdot\|_{\gamma\to\beta} = (2t-s)^{(\alpha-\beta)/2}$ and we obtain $\|I_2\|_{\alpha\to\beta} = O(t^{1+(\alpha-\beta)/2})$ as before.

Case (ii): $\beta \in [\alpha - 2, \alpha]$. We then set $\gamma = \alpha - 2$ and from (4.10) it follows a fortiori (since $X_\alpha \subset X_\gamma$) that the middle term in (4.9) satisfies $\|\cdot\|_{\alpha\to\gamma} \leqslant C$. The first term satisfies then $\|\cdot\|_{\gamma\to\beta} = O(2t-s)^{(\alpha-\beta)/2-1}$ so that we obtain then $\|I_2\|_{\alpha\to\beta} = O(t^{(\alpha-\beta)/2})$ which for $\beta = \alpha - 2$ gives $\|I_2\|_{\alpha\to\alpha-2} = O(t)$. But since by our previous case we have $\|I_2\|_{\alpha\to\alpha} = O(t)$ it follows that $\|I_2\|_{\alpha\to\beta} = O(t)$ for all $\beta \in [\alpha - 2, \alpha]$. We can unify all the above and simply say that

$$\|I_1 + I_2 + I_3\|_{\alpha\to\beta} = O(\max\{t, t^{1+(\alpha-\beta)/2}\})$$

for $\beta \geqslant \alpha - 2$, (which in $O(t)$ if $\beta \leqslant \alpha$).

It remains to prove (4.10) which is a special case of the following more general

$$\|[[A^k, E_1], E_2]\|_{\alpha\to\alpha-2k+2} \leqslant C \tag{4.11}$$

valid for $k = 1, 2, \ldots$.

The estimate (4.11) is proved exactly as in (4.3) provided that $\alpha = 2n$ is an even integer. Indeed (4.11) simply says then that

$$\|A^n[[A^k, E_1], E_2]\|_{L^2} \leqslant C\|A^{n+k-1}f\|_{L^2} \tag{4.12}$$

and we use pseudo-differential calculus and the basic singular integral estimate (4.2). (In the development of the commutators $[[A^k, E_1]E_2]$ *two* fields X_{i_1} and X_{i_2} are *knocked out*. This is done because either two commutators of the form $[X_{i_1}, E_1]$ and $[X_{i_2}, E_1]$ appear. These commutators then convert these fields into $S^0_{1,0}$ operators. Or only one such commutator appears but then we must have a term of the form $[E_1, E_2] \in S^{-1}_{1,0}$ that combines with some $X_{i_\alpha} \in S^1_{1,0}$ to give an element in $S^0_{1,0}$.) To finish the proof of (4.12) for arbitrary $\alpha \in \mathbb{R}$, *i.e.* we dualize and interpolate.

We are left with the other three terms I_1', I_2', I_3'. The discussion is entirely analogous and will be left for the reader. The important point is that we now have to perform the «jump» from α to β at the term $e^{-(t+s)A}$. Before we made that jump at the term $e^{-(2t-s)A}$. This is to make sure that the integrals converge at the two ends.

Putting everything together we finally obtain

$$\|[[e^{-tA}, E_1], E_2]\|_{\alpha \to \beta} = O(t^{1+(\alpha-\beta)/2}) \tag{4.13}$$

for all $\beta \geqslant \alpha$.

We are finally in a position to give the proof of (4.8) for $k = 2$. Indeed the theory of Section 1 together with (4.13) and formula (3.4) allows us to conclude that

$$\|[[A^\sigma, E_1], E_2]\|_{\alpha \to \alpha - 2\operatorname{Re}\sigma + 2} \leqslant C \tag{4.14}$$

for all $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{C}$, $\operatorname{Re}\sigma < 1$. To obtain the result for arbitrary $\sigma \in \mathbb{C}$ we develop

$$\begin{aligned}[[A^{\sigma+k}, E_1], E_2] = {} & [[A^\sigma, E_1], E_2]A^k + [A^\sigma, E_1][A^k, E_2] \\ & + [A^\sigma, E_2][A^k, E_1] + A^\sigma[[A^k, E_1], E_2]\end{aligned}$$

with $k = 1, 2, \ldots$; $\operatorname{Re}\sigma < 1$. It is then only a matter of combining (4.11), (4.5) and our previous result (4.14) and we are done.

5. The Conjugation Operation c_λ

This section runs parallel to the previous one. Here we shall consider expressions of the form

$$A - \Lambda^s A \Lambda^{-s} = \Lambda^s[A, \Lambda^{-s}] = [\Lambda^s, A]\Lambda^{-s}, \qquad s \in \mathbb{R},$$

with $\Lambda = (1 + \Delta)^{1/2}$ and the standard Laplacian Δ. I shall also denote $c_s(A) = \Lambda^s A \Lambda^{-s}$ so that we have

$$\Lambda^{s_1}[A, \Lambda^{-s}]\Lambda^{s_2} = c_\lambda(A) - c_\mu(A).$$

For $s_1 + s_2 = s$ and appropriate $\lambda, \mu \in \mathbb{R}$.

The operator A, the spaces X_α ($\alpha \in \mathbb{R}$) and the norms $\|\cdot\|_\alpha = \|\cdot\|_{X_\alpha}$ have the same meaning as before. It is also clear that with $s_1 + s_2 = s$ we have

$$\|\Lambda^{s_1}[A, \Lambda^{-s}]\Lambda^{s_2} f\|_{L^2} \leqslant C \|A^{1/2} f\|_{L^2}.$$

Similarly if we impose sub-ellipticity on A we obtain by the singular integral theorem and commutations of pseudo-differentials that

$$\|c_\lambda(A^k) - c_\mu(A^k)\|_{\alpha \to \alpha - 2k + 1} \leqslant C \tag{5.1}$$

for arbitrary $\alpha \in \mathbb{R}$, and $k = 1, 2, \ldots$.

Our first aim is to prove the analogue of (4.5)

$$\|\Lambda^{s_1}[A^\sigma, \Lambda^{-s}]\Lambda^{s_2}\|_{\alpha \to \beta} = \|c_\lambda(A^\sigma) - c_\mu(A^\sigma)\|_{\alpha \to \beta} \leqslant C \tag{5.2}$$

for $\beta = \alpha - 2 \operatorname{Re} \sigma + 1$, $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{C}$ (and $s_1 + s_2 = s$ of course). It is clearly enough (take differences) to assume $\mu = 0$ and consider $(I - c_\lambda)(A^\sigma) = A^\sigma - BA^\sigma B^{-1}$ with B some *real* power of Λ.

Just as in Section 4 to show (5.2) it is enough to prove the following estimate

$$\|(I - c_\lambda)(e^{-tA})\|_{\alpha \to \beta} = O(t^{1/2 + (\alpha - \beta)/2}), \qquad t \to 0, \tag{5.3}$$

for $\beta \geqslant \alpha - 1$. This implies in particular that

$$\|c_\lambda(e^{-tA})\|_{\alpha \to \beta} = O(t^{(\alpha - \beta)/2}), \qquad t \to 0, \tag{5.4}$$

for $\beta \geqslant \alpha$. Indeed (5.4) is a consequence of the holomorphicity of e^{-tA} on X for $\lambda = 0$ and it follows for general λ by this special case and (5.3). The estimate (5.3) by the same method implies the result (5.2) for $\operatorname{Re} \sigma < 1$.

But then if we combine this special case $\operatorname{Re} \sigma < 1$ together with the special case $\sigma = k = 1, 2, \ldots$ of (5.1) and the identity

$$(1 - c_\lambda)(A^{k+\sigma}) = A^\sigma(1 - c_\lambda)(A^k) + [(1 - c_\lambda)(A^\sigma)]c_\lambda(A^k)$$

we obtain the estimate (5.2) in general. Observe here that

$$\|c_\lambda(A^\rho)\|_{\alpha \to \alpha - 2 \operatorname{Re} \rho} \leqslant \|A^\rho\|_{\alpha \to \alpha - 2 \operatorname{Re} \rho} + \|(1 - c_\lambda)A^\rho\|_{\alpha \to \alpha - 2 \operatorname{Re} \rho} \leqslant C. \tag{5.5}$$

(Alternative pseudo-differential calculus and the sub-elliptic estimate (4.2) can be used directly to control $c_\lambda(A^k)$ for $k = 1, 2, \ldots$.)

The proof of (5.3) is done in two steps. Let $c_\lambda(e^{-tA}) = Be^{-tA}B^{-1} = R_t$. The first step consists in showing that (5.4) holds for $\beta \in [\alpha, \alpha + 1[$, $\alpha \in \mathbb{R}$. Towards that let

$$
\begin{aligned}
\phi(t) &= e^{-tA} - R_t \\
&= \int_0^t e^{-(t-s)A} B[A, B^{-1}] R_s \, ds \\
&= \int_0^t e^{-(t-s)A} B[A, B^{-1}] \phi(s) + I(t), \\
I(t) &= \int_0^t e^{-(t-s)A} B[A, B^{-1}] e^{-sA} \, ds.
\end{aligned} \tag{5.6}
$$

We shall first estimate $I(t)$,

$$
\|I(t)\|_{\alpha \to \beta} \leqslant \int_0^t \|e^{-(t-s)A}\|_{\gamma-1 \to \beta} \|B[A, B^{-1}]\|_{\gamma \to \gamma-1} \|e^{-sA}\|_{\alpha \to \gamma} \, ds
$$

we shall choose $\gamma \geqslant \alpha$, $\beta \geqslant \gamma - 1$ so that the above integral can be estimated by

$$
\int_0^t (t-s)^{(\gamma-1-\beta)/2} s^{(\alpha-\gamma)/2} \, ds = O(t^{1/2 + (\alpha-\beta)/2})
$$

as long as $\gamma - 1 - \beta > -2$, $\alpha - \gamma > -2$. All in all, for the above to work we need $\gamma \in [\alpha, \alpha + 2[\cap]\beta - 1, \beta + 1] = a$, and $a \neq \varnothing$ if and only if $\beta \in [\alpha - 1, \alpha + 3[$. In the range $\beta \in [\alpha - 1, \alpha + 3[$ we have therefore the estimate $\|I(t)\|_{\alpha \to \beta} = O(t^{1/2 + (\alpha-\beta)/2})$.

If we inject that information in the formula (5.6) and set $\psi(t) = \|\phi(t) f\|_\beta$ for some $f \in C_0^\infty$ in the unit ball of X_α, we see that

$$
\psi(t) \leqslant C \int_0^t \|e^{-(t-s)A} B[A, B^{-1}]\|_{\beta \to \beta} \psi(s) \, ds + O(t^{1/2 + (\alpha-\beta)/2}). \tag{5.7}
$$

Taking the adjoint of the operator inside $\|\cdot\|_{\beta \to \beta}$ and using (5.1) we see that it satisfies $\|\cdot\|_{\beta \to \beta} = O((t-s)^{-1/2})$. We end up thus with the following integral inequality

$$
\psi(t) \leqslant C t^{1/2 + (\alpha-\beta)/2} + C \int_0^t \psi(t-s) s^{-1/2} \, ds. \tag{5.8}
$$

Now if $f \in C_0^\infty$ $\psi(t)$ is (semi-)continuous and $\psi(0) = 0$, so if t_0 is the first value of t for which $\psi(t) \geqslant 1$ we have $1 \leqslant c t_0^{1/2 + (\alpha-\beta)/2} + c t_0^{1/2}$ and therefore $t_0 \geqslant c > 0$ if $\beta \in [\alpha - 1, \alpha + 1[$.

This simply says that $\psi(t) = O(1)$ uniformly in f. Injecting this information in (5.8) gives

$$
\psi(t) = O(t^{1/2 + (\alpha-\beta)/2} + t^{1/2}), \qquad \beta \in [\alpha - 1, \alpha + 1[. \tag{5.9}
$$

That information could be reinjected in (5.8) to obtain a more precise result on $\psi(t)$. This however is unnecessary at this stage. All we need is to combine (5.9) with the estimate $\|e^{-tA}\|_{\alpha\to\beta} = O(t^{(\alpha-\beta)/2})$ $(\beta \geqslant \alpha)$ and we deduce (5.4) in the range $\beta \in [\alpha, \alpha+1[$.

At this point we start from scratch. We shall assume (by induction) that (5.4) holds in the range $\beta \in [\alpha, \alpha+k[$ for some integer $k = 1, 2, \ldots$ and we shall prove the inductive step that then (5.4) holds in fact in the range $\beta \in [\alpha, \alpha+k+1[$. (This easily implies that (5.3) holds in the range $[\alpha-1, \alpha+k+1[$.) Once this is done the proof of both (5.3) and (5.4) is completed for their full range. Towards the proof of this inductive step we proceed as follows

We have

$$e^{-2tA} - R_{2t} = \int_0^t e^{-(2t-s)A} B[A, B^{-1}] R_s \, ds + \int_0^t e^{-(t-s)A} B[A, B^{-1}] R_{s+t} \, ds.$$

It follows that we can estimate the norm $\|e^{-tA} - R_t\|_{\alpha\to\beta}$ by the sum of the following two integrals,

$$\int_0^t \|e^{-(2t-s)A}\|_{\alpha-1\to\beta} \|B[A, B^{-1}]\|_{\alpha\to\alpha-1} \|R_s\|_{\alpha\to\alpha} \, ds,$$

$$\int_0^t \|e^{-(t-s)A}\|_{\gamma-1\to\beta} \|B[A, B^{-1}]\|_{\gamma\to\gamma-1} \|R_{s+t}\|_{\alpha\to\gamma} \, ds.$$

By a process that should by now be familiar to the reader, the first integral can be estimated by $O(t^{1/2+(\alpha-\beta)/2})$ for arbitrary $\beta \geqslant \alpha - 1$. In the second integral we shall use the inductive hypothesis and choose $\gamma \in [\alpha, \alpha+k[$ (close to $\sim \alpha + k$) we must also have $\beta \in [\gamma-1, \gamma+1[$ (so that $|\beta - \gamma + 1| < 2$ which gives convergence of the integral at the upper end!). For $\beta \in [\alpha+k, a+k+1[$ we obtain therefore the required estimate $O(t^{1/2+(\alpha-\beta)/2})$ of that integral. We conclude that

$$\|e^{-tA} - R_t\|_{\alpha\to\beta} = O(t^{1/2+(\alpha-\beta)/2})$$

for $\beta \in [\alpha+k, \alpha+k+1[$. This together with the estimate $\|e^{-tA}\|_{\alpha\to\beta} = O(t^{\alpha-\beta)/2})$ $\beta \geqslant \alpha$ that always holds completes the inductive step for (5.4). The proof is now complete because once we know (5.4) for arbitrary $\beta \geqslant \alpha$ the estimate (5.3) for arbitrary $\beta \geqslant \alpha$ the estimate (5.3) follows by the previous argument.

At this stage I shall become more brief. The double «commutator» estimate of the form

$$\|(c_{\lambda_1} - c_{\lambda_2})(c_{\lambda_3} - c_{\lambda_4}) e^{-tA}\|_{\alpha\to\beta} = O(t^{1+(\alpha-\beta)/2}) \tag{5.10}$$

at least for $\beta \geqslant \alpha$ (with the same refinements as in Section 4 for $\beta \in [\alpha-2, \alpha]$) will be proved the same way as in Section 4. If anything, things are simpler

here because the situation is more symmetric. We have for instance

$$(c_\lambda - c_\mu)(c_\nu(P)) = (c_{\lambda+\nu} - c_{\mu+\nu})(P) = c_\nu(c_\lambda - c_\mu)(P).$$

To control the «middle terms» (the analogue of I_2 of Section 4) we have to check that for $k = 0, 1, 2, \ldots$

$$\|(c_\lambda - c_\lambda)(c_\lambda - c_\lambda)(A^k)\|_{\alpha-\alpha-2k+2} \leqslant C. \tag{5.11}$$

The λ's above are of course in general different (and this a notational convention that will be adopted from here onwards). (5.11) for $\alpha = 2n$ is done again by commuting pseudo-differentials and using the singular integral estimate (4.2). We then dualize and interpolate.

Just as before the use of Section 1 of (5.11) and of formula (3.4) gives

$$\|(c_\lambda - c_\lambda)(c_\lambda - c_\lambda)(A^\sigma)\|_{\alpha\to\alpha-2\operatorname{Re}\sigma+2} \leqslant C \tag{5.12}$$

provided that $\operatorname{Re}\sigma < 0$.

We shall then develop

$$(c_\lambda - c_\lambda)(c_\lambda - c_\lambda)(A^{k+\sigma})$$

for $k = 1, 2, \ldots$ $\operatorname{Re}\sigma < 0$ and combine (5.11), (5.2) and (5.12) (for $\operatorname{Re}\sigma < 0$) to conclude that (5.12) holds in general for any $\sigma \in \mathbb{C}$.

Any multiple difference can be treated that way and we obtain

$$\|(c_\lambda - c_\lambda)^k(A^\sigma)\|_{\alpha\to\beta} = \|(c_\lambda - c_\lambda)\cdots(c_\lambda - c_\lambda)(A^\sigma)\|_{\alpha\to\beta} \leqslant C$$

for $\beta = \alpha - 2\operatorname{Re}\sigma + k$. This will not be used for our main application in this paper for any $k > 2$.

We shall finally need to control mixed expressions of the form

$$M_1 = (I - c_\lambda)[A^\sigma, E], \qquad M_2 = [(I - c_\lambda)(A^\sigma), E]$$

and of course multiple commutators with $E \in S^0_{1,0}$ and powers of the operation $(c_\lambda - c_\lambda)$ can also be considered.

It should be clear to the reader by now what the appropriate estimate is. Just for the record, let me sketch a proof of the fact that

$$\|M_1\|_{\alpha\to\beta}, \qquad \|M_2\|_{\alpha\to\beta} \leqslant C$$

for $\beta = \alpha - 2\operatorname{Re}\sigma + 2$. Observe also that

$$\|M_1 - M_2\|_{\alpha\to\beta} = \|B^{-1}[B, E]A^\sigma\|_{\alpha\to\beta} + \|[B^{-1}, E][B, A^\sigma]\|_{\alpha\to\beta}$$

(for any operator norm), where $B = \Lambda^s$. We shall see in the next section that we can control the terms on the right since they are of the form $\|S_1 A^\sigma\|$ and

$\|S_2[B, A^\alpha]\|$ for appropriate pseudo-differential operators S_i. We can therefore pass early from M_1 to M_2 and vice versa, (at least this is the case when $\beta = 0$ and, in a final analysis, as we shall see, this is all that counts). At any rate, we shall not need this passage from M_1 to M_2 here since we can estimate M_1 and M_2 separately just as easily in a symmetrical manner.

Indeed observe first that

$$c_\lambda([T_t, E]) = [c_\lambda(T_t), E_\lambda] \tag{5.13}$$

with $E \to E_\lambda = c_\lambda(E)$ a bijection on $S^0_{1,0}$. It therefore is enough to control one of the two terms (5.13). Say

$$c_\lambda([T_t, E]) = \int_0^t c_\lambda(e^{-(2t-s)A})c_\lambda([A, E])c_\lambda(e^{-sA})\,ds + \int_0^t c_\lambda(e^{-(t-s)A})c_\lambda([A, E])c_\lambda(e^{-(s+t)A})\,ds.$$

Bearing in mind the estimate (5.4) and the following estimate

$$\|c_\lambda[A, E]\|_{\alpha\to\alpha-1} = \|[c_\lambda(A), E_\lambda]\|_{\alpha\to\alpha-1} \leqslant C \tag{5.14}$$

valid for all $\alpha \in \mathbb{R}$ we obtain as before

$$\|c_\lambda[T_t, E]\|_{\alpha\to\beta}, \quad \|[c_\lambda(T_t), E]\|_{\alpha\to\beta} = O(t^{1/2+(\alpha-\beta)/2}), \quad \beta \geqslant \alpha - 1. \tag{5.15}$$

The estimate (5.14) can (probably!) be reduced to the other similar estimates that we have already examined. There is no need to do so however; a direct proof along the same lines as our previous similar proofs is preferable. Indeed (5.14) is clear by pseudo-differential calculus and the sub-elliptic estimate (4.2) (just as in the proof of (5.11)) when $\alpha = 2k + 1$ $(k = 1, 2, \ldots)$. We simply dualize and interpolate as before.

We next write (we use our convention in (5.11) for the various c_λ's)

$$[(c_\lambda - c_\lambda)e^{-2tA}, E] = \left[\int_0^t c_\lambda(e^{-(2t-s)A})(c_\lambda - c_\lambda)(A)c_\lambda(e^{-sA})\,ds, E\right] + \left[\int_0^t c_\lambda(e^{-(t-s)A})(c_\lambda - c_\lambda)(A)c_\lambda(e^{-(s+t)A})\,ds, E\right]$$

and we see from (5.15) that we can subject this expression to the same treatment as in the proof of $[[e^{-2tA}, E_1], E_2]$ or $(c_\lambda - c_\lambda)(c_\lambda - c_\lambda)e^{-2tA}$ as soon as we have the estimate

$$\|[(c_\lambda - c_\lambda)A, E]\|_{\alpha\to\alpha} \leqslant C.$$

We have of course more generally

$$\|[(c_\lambda - c_\lambda)A^k, E]\|_{\alpha\to\beta}, \|(c_\lambda - c_\lambda)[A^k, E]\|_{\alpha\to\beta} \leqslant C,$$

with $\beta = \alpha + 2k - 2$ with the same proofs as before (*i.e.* we first prove it for $\alpha = 2n$, an even integer).

The estimate

$$\|[(c_\lambda - c_\lambda)A^\sigma, E]\|_{\alpha\to\beta} \leqslant C \tag{5.16}$$

for $\beta = \alpha + 2\operatorname{Re}\sigma - 2$ follows by the same machinery. Notice that when going through this last step of that proof we must use the fact that the $\|\cdot\|_{\alpha\to\alpha-2\operatorname{Re}\rho+1}$ norm of $c_\lambda[A^\rho, E] = [c_\lambda(A^\rho), \tilde{E}]$ is bounded, for $\rho = k = 1, 2, \ldots$ or $\operatorname{Re}\rho < 0$. But this is clear from the corresponding estimate

$$\|[(I - c_\lambda)(A^\rho), E]\|_{\alpha\to\alpha-2\operatorname{Re}\rho+2} \leqslant C$$

and the already known estimate of $\|[A^\rho, E]\|_{\alpha\to\alpha-2\operatorname{Re}\rho+1}$.

The estimate for M_1 is identical and I will spare the reader the details.

6. The Classical Sobolev Spaces H_α

In this section I shall face some new aspects of the theory. First of all, I shall change the scale and consider the scale of classical Sobolev spaces

$$H_\alpha = \{f\colon\ \Lambda^\alpha f \in X\}$$

where $X = L^2(\mathbb{R}^n)$ throughout, and $\Lambda = (I + \Delta)^{1/2}$ for the standard Laplacian Δ. I shall refer to $\|\cdot\|_\alpha = \|\cdot\|_{H_\alpha}$, $\|\cdot\|_{\alpha\to\beta} = \|\cdot\|_{H_\alpha\to H_\beta}$ and it is important of course that no confusion should arise with the norms $\|\cdot\|_{X_\alpha\to X_\beta}$ considered in the previous two sections. In this section we shall analyse the mapping properties of operators of the form $P = [A^\sigma, E]$, $P = (c_\lambda - c_\lambda)(A^\sigma)\cdots$ of Section 4, Section 5 (and even more general operators) between spaces of this classical Sobolev scale *i.e.* $P\colon H_\alpha \to H_\beta$.

The strange thing is that to do so we shall use the results of the previous two sections but *only* when $\beta = 0$ *i.e.* from the previous two sections we shall only retain information of the form

$$\|[A^\sigma, E]f\|_X \leqslant C\|A^{\sigma-1/2}f\|_X$$

or more generally $\|Pf\|_X \leqslant C\|A^t f\|_X$ for an appropriate $t \in \mathbb{R}$. Our aim consists simply of generalizing these estimates to the H_α ($\alpha \in \mathbb{R}$) scale so that one of the things that we shall see, for instance, is

$$\|[A^\sigma, E]\|_{H_\alpha} \leqslant C\|A^{\sigma-1/2}f\|_{H_\alpha},$$

for *all* $\alpha \in \mathbb{R}$.

In the considerations that follows the letter B will be reserved for some power $B = \Lambda^s$ ($s \in \mathbb{R}$).

The first thing to prove is that we have $\|BA^\sigma B^{-1}f\|_X \approx \|A^\sigma f\|_X$ in the sense that

$$C^{-1}\|A^\sigma f\|_X \leqslant \|BA^\sigma B^{-1}f\|_X \leqslant C\|A^\sigma f\|_X \tag{6.1}$$

for $\sigma \in \mathbb{C}$ and some $C > 0$ independent of f.

One of the two inequalities is evident. Indeed

$$\begin{aligned}\|BA^\sigma B^{-1}f\|_X &\leqslant C\|A^\sigma f\|_X + C\|B[A^\sigma, B^{-1}]f\|_X \\ &\leqslant C(\|A^\sigma f\|_X + \|A^{\sigma-1/2}f\|_X) \\ &\leqslant C\|A^\sigma f\|_X\end{aligned}$$

by Section 5 (and the fact that $\|A^{\sigma+j}f\|_X \geqslant C\|A^\sigma f\|_X$, $j \geqslant 0$).

For the other way round we shall assume first that $\operatorname{Re}\sigma > 0$ and just as before, because of Section 5, observe that

$$\begin{aligned}\|A^\sigma f\|_X &\leqslant C[\|BA^\sigma B^{-1}f\|_X + \|A^{\sigma-1/2}f\|_X] \\ &\leqslant C\sum_{j=0}^{k-1}\|BA^{\sigma-j/2}B^{-1}\|_X + C\|A^{\sigma-k/2}f\|_X\end{aligned} \tag{6.2}$$

for any $k = 0, 1, \ldots$. If we choose a k large enough for $\operatorname{Re}(\sigma - k/2) < 0$ we have

$$\|A^{\sigma-k/2}f\|_X \leqslant C\|f\|_X \leqslant C\|BA^\sigma B^{-1}f\|_X.$$

The last inequality holds because $\operatorname{Re}\sigma > 0$ and because e^{-tA} acts on H_s. Indeed our convention (*cf.* Section 1) about the behaviour at infinity of that semigroup guarantees that

$$BA^\rho B^{-1}: X \to X$$

is bounded if $\operatorname{Re}\rho < 0$. Since the terms under the summation sign of (6.2) are decreasing in j we finally obtain that $\|A^\sigma f\|_X \approx \|B^{-1}A^\sigma B\|_X$ as long as $\operatorname{Re}\sigma > 0$. By taking inverses this of course means that the same is true for $\operatorname{Re}\sigma < 0$. The pure imaginary σ's are evident since $A^{it}(t \in \mathbb{R})$ act boundedly on H_α.

The above has several consequences. In particular, we see that the norms

$$\|f\|_{a,\sigma} = \|\Lambda^\alpha A^\sigma \Lambda^\beta f\|_X \quad \text{where} \quad \alpha + \beta = a \quad \text{is fixed} \tag{6.3}$$

are all equivalent norms for fixed $a \in \mathbb{R}$, $\sigma \in \mathbb{C}$. (To see this one applies (6.1) with $f = \Lambda^\gamma \phi$ for appropriate γ.)

The next thing to observe is that

$$\|[A^\sigma, E]f\|_m \leqslant \|A^{\sigma-1/2}f\|_m \tag{6.4}$$

($\sigma \in \mathbb{C}$, $m \in \mathbb{R}$). Indeed, perform the change of variables $f = \Lambda^{-m}\phi$ and the above inequality is simply transformed (because of (6.1)) into

$$\|c_m[A^\sigma, E]\phi\|_X \leqslant C\|A^{\sigma-1/2}\phi\|_X.$$

This is a special case of (5.16) in Section 5 (and the few lines that follow).

Let us now consider the most general inequality of this type with *one* or less commutators, this is

$$\|S^{n_1}A^\sigma S^{n_2}\|_m \leqslant \|A^\sigma f\|_p, \qquad \|S^{n_1}[A^\sigma, S^{n_2}]S^{n_2}f\|_m \leqslant C\|A^{\sigma-1/2}f\|_p$$

where $S^{n_i} \in OPS^{n_i}_{1,0}$ and where we suppose that $\sum n_i + m = p$. I shall prove the second, the first being easier. I proceed as follows: I first factor $S^{n_i} = \Lambda^{n_i}E = E\Lambda^{n_i}$, expand the $[\cdot, \cdot]$ and refactor, if necessary, to bring it to the form

$$\|E\Lambda^{\nu_1}[A^\sigma, P]\Lambda^{\nu_2}Ef\|_X \leqslant C\|A^{\sigma-1/2}f\|_p$$

with $P = E$ or Λ^ν and $\operatorname{ord} P + \nu_1 + \nu_2 = p$ (here we set $\operatorname{ord} E = 0$, $\operatorname{ord}\Lambda^\nu = \nu$). The two E's can be eliminated. The one on the left is absorbed in the $\|\cdot\|_X$, the one on the right disappears by the change of variable $Ef = \phi$ and (6.4). A final change of variable $\psi = \Lambda^{-p}\phi$ reduces everything to the case $p = 0$ (we use (6.1) for that). So we are finally left with proving

$$\|\Lambda^{\mu_1}[A^\sigma, P]\Lambda^{\mu_2}f\|_X \leqslant C\|A^{\sigma-1/2}f\|_X$$

with $\mu_1 + \mu_2 + \operatorname{ord} P = 0$. But here we are on familiar ground, for the operator on the left is in one of the following two forms:

$$(c_\lambda - c_\lambda)(A^\sigma), \qquad c_\lambda[A^\sigma, E].$$

Both of these have been dealt with in Sections 4 and 5.

Let us now pass on to second commutators. The estimate we shall need is the following

$$\|S^{n_1}[[A^\sigma, S^{n_2}], S^{n_3}]S^{n_4}f\|_m \leqslant C\|A^{\sigma-1}f\|_p$$

where $p = m + \sum n_i$, and $S^{n_i} \in OPS^{n_i}_{1,0}$. The proof is easy. Use the factorization of $S^n \in OPS^n_{1,0}$ into $S^n = \Lambda^n E = E\Lambda^n$ a number of times over and we see that the expression on the left can be estimated by

$$\|E\Lambda^{\nu_1}[A^\sigma, R]\Lambda^{\nu_2}Ef\|_X + \|E\Lambda^{\mu_1}[[A^\sigma, P], Q]\Lambda^{\mu_2}Ef\|_X \tag{6.5}$$

P, Q, $R = E$ or Λ^s for some $s \in \mathbb{R}$.

Just as before, the two E's on the above two expressions can be disregarded. The one on the left is absorbed in the norm $\|\cdot\|_X$ the one on the right disappears by the change of variables $\phi = Ef$ since we clearly have, from what has already been done, $\|A^{\sigma-1}\phi\|_p \leqslant C\|A^{\sigma-1}f\|_p$. The first term in (6.5) has already been

treated and turns out to be bounded by $\|A^{\sigma-1/2}f\|_{p-1}$. To check that it is indeed the H_{p-1} norm that we actually have here, we must consider cases. The idea, however, is simple: since only one commutator appears in this term this means that two pseudo-differentials have been «contracted» by a commutation $[S^{r_1}, S^{r_2}]$ but that is of course a pseudo-differential that belongs to $OPS^{r_1+r_2-1}_{1,0}$ and that explaining the gain of one derivative that gives $p-1$. Since, by Section 2, $A^{1/2}: H_p \to H_{p-1}$ is bounded, we see that $\|A^{\sigma-1/2}f\|_{p-1} \leqslant \|A^{\sigma-1}f\|_p$. We thus see that the first term in (6.5) is controlled.

To control the second term in (6.5) we start by the change of variable $f = \Lambda^{-p}\phi$ and a use of (6.1) that reduces the problem to showing

$$\|\Lambda^{\nu}[A^{\sigma}, P], Q]\Lambda^{\mu}f\|_X \leqslant C\|A^{\sigma-1}f\|_X$$

with P, $Q = \Lambda^{\lambda}$ or E and $\operatorname{ord} P + \operatorname{ord} Q + \nu + \mu = 0$ (here again $\operatorname{ord} E = 0$, $\operatorname{ord}\Lambda^{\nu} = \nu$). This is a familiar expressions from Section 4 and 5. The four cases $P, Q = E, \Lambda^{\lambda}$ have to be checked separately

(1) $P, Q = \Lambda^{\lambda}$ gives

$$\Lambda^{\nu}[[A^{\sigma}, P], Q]\Lambda^{\mu} = (c_{\lambda} - c_{\lambda})(c_{\lambda} - c_{\lambda})A^{\sigma}.$$

(2) $P, Q = E$ gives us a term of the form $[[c_{\lambda}(A^{\sigma}), E_1], E_2]$ which can be controlled by the three terms $[[A^{\sigma}, E_1], E_2]$, $E_2[(I - c_{\lambda})A^{\sigma}, E_1]$ and $[(I - c_{\lambda})A^{\sigma}, E_1]E_2$. The E_2 on the right of the last term is then eliminated by the change of variables $E_2 f = \phi$.

(3) $P = E$, $Q = \Lambda^{\lambda}$ gives us a term of the form $(c_{\lambda} - c_{\lambda})[A, E]$.

(4) $P = \Lambda^{\lambda}$, $Q = E$ is reduced to the previous case modulo $\Lambda^{\nu}[A^{\sigma}, S^{\lambda-1}]\Lambda^{\mu}$ with $S^{\lambda-1} \in OPS^{\lambda-1}_{1,0}$.

We shall go no further here. It is clear however that if we proceed that way we obtain the basic commutation estimate of the introduction (at least for «sum of squares» sub-elliptic operators). Things can be presented in an even more symmetrical way. We can denote by

$$Z^{\alpha}_{\beta} = \{ f: A^{\alpha/2}\Lambda^{\beta}f \in X \}$$

which is a double index scale of spaces. The order at which A and Λ are set is immaterial (*cf.* (6.3)). From the above considerations together with some additional work (*cf.* Section 11) we can establish the precise mapping properties in Z^{α}_{β} of commutators of *one* power of A with an arbitrary number of pseudo-differentials. I shall not pursue the matter further here. A typical result that can thus be obtained is

$$\|A^{t}[A^{\sigma}, S^{n}]f\|_{H_{\alpha}} \leqslant C\|A^{t+s-1/2}f\|_{H_{\alpha+n}}, \qquad S^{n} \in OPS^{n}_{1,0}.$$

7. The Interaction of Two Distinct Operators

We shall now consider L and M, two distinct differential operators, both of which I shall assume are sums of squares of fields

$$L, M = \sum X_j^* X_j. \tag{7.1}$$

I shall also assume that they are both sub-elliptic

$$\|f\|_{1-\delta} \leqslant C\|(1+L)^{1/2}\|_X,$$
$$\|f\|_{1-d} \leqslant C\|(1+M)^{1/2}\|_X$$

with $0 \leqslant d, \delta < 1$. ($X = L^2$). The spaces H_α are as before the standard Sobolev spaces of Section 6 and $\|\cdot\|_\alpha = \|\cdot\|_{H_\alpha}$ refer to these norms. The aim of this section is to prove the following basic

Proposition. *Let L, M, δ and d be as above and assume that $d + \delta \leqslant 1$. The semigroup e^{-tM} acts as a continuous semigroup on each space X_α ($\alpha \in \mathbb{R}$) induced by the operator L (cf.* (Section 4)).

In fact a slightly more general result will be proved (under the same assumption that $\delta + d \leqslant 1$). I shall prove that for all $\beta \geqslant \alpha$ and all $j \in \mathbb{R}$ we have

$$\|(I+L)^{j/2} e^{-tM} (1+L)^{-j/2}\|_{\alpha \to \beta} = O(t^{(\alpha-\beta)/2(1-d)}). \tag{7.2}$$

For $\alpha = \beta = 0$ this is the content of our proposition. Observe straight away that it suffices to prove for $j = 0, 1, \ldots$. Indeed the action of $(I+L)^{is}$ ($s \in \mathbb{R}$) on H_α implies that we then have the same result for $j \in \mathbb{C}$, $\mathrm{Re}\, j = 0, 1, \ldots$. Complex interpolation of an analytic family of operators for fixed α and β gives then (7.2) for any $j \in \mathbb{C}$, $\mathrm{Re}\, j > 0$ we then dualize and have the general result. (In the above complex interpolation there is a slight difficulty: We have to recover the factor $t^{(\alpha-\beta)/2(1-d)}$ for the interpolated operator. This is no problem here since we have the *same* factor at both ends of the interpolation interval *cf.* [4]. Things would have been different if we had different factors $C_1(t)$ and $C_2(t)$. It would not then be necessarily true that we obtain $C_1^\theta C_2^{1-\theta}$ after interpolation. At any rate, if you do not feel comfortable with complex interpolation, proceed differently: Prove (7.2) for $j = 0, 1, \ldots$ and general $\alpha, \beta \in \mathbb{R}$, $\beta \geqslant \alpha$, and then interpolate *only* when $\alpha = \beta = 0$. We still obtain our proposition.)

The fact that (7.2) holds for $j = 0$ is just the content of (2.5). We shall assume therefore that (7.2) holds for $j = 0, 1, \ldots, k$, and prove it for $k+1$. The remainder of this section will be devoted to this task.

Our first step will be to analyse the expression

$$[M,(1+L)^{(k+1)/2}] = \sum_j [X_j^* X_j,(1+L)^{(k+1)/2}]$$

$$= \sum_j X_j^*[X_j,(I+L)^{(k+1)/2}] + \sum_j [X_j^*,(1+L)^{(k+1)/2}]X_j,$$

X_j will be, of course, the fields X_j in the decomposition of M and the index j as well as the summation will be dropped from here onwards. We shall rewrite this in the following form

$$M^{1/2}\{M^{-1/2}X^*[X,(I+L)^{(k+1)/2}] + M^{-1/2}X[X^*,(I+L)^{(k+1)/2}]\} + [[X^*,(I+L)^{(k+1)/2}],X]. \tag{7.3}$$

For all the X's or X^*'s that are inside a bracket $[\cdot,\cdot]$ the only information that we shall retain is the fact that they belong to $OPS^1_{1,0}$. Then we can use the results of our previous sections to estimate all the brackets that appear in (7.3). The double bracket B_2 that appears at the end of (7.3) satisfies for instance $\|B_2 f\|_\alpha \leqslant C\|(1+L)^{(k-1)/2}f\|_{\alpha+2}$ $(\alpha \in \mathbb{R})$ and the single brackets B_1 that appear within $\{\cdot\}$ in (7.3) satisfy

$$\|B_1 f\|_\alpha \leqslant C\|(I+L)^{k/2}f\|_{\alpha+1}.$$

The two «Riesz transforms» satisfy

$$\|M^{-1/2}X^*f\|_\alpha, \qquad \|M^{-1/2}Xf\|_\alpha \leqslant C\|f\|_\alpha. \tag{7.4}$$

The X^* can in fact be ignored in (7.4) since $X - X^* \in S^0_{1,0}$. The estimate (7.4) holds in general even without sub-ellipticity, *i.e.* $d = 1$. To see this, considering the adjoint operator, it is clearly enough to show that

$$\|Xf\|_\alpha \leqslant C\|M^{1/2}f\|_\alpha \qquad (\alpha \in \mathbb{R})$$

(by our convention M is really $\lambda_0 + M$, $\lambda_0 > 0$). This last estimate is clear enough since

$$\begin{aligned}\|Xf\|_\alpha &= \|\Lambda^\alpha Xf\|_X \\ &\leqslant C(\|X\Lambda^\alpha f\|_X + \|f\|_\alpha) \\ &\leqslant C\|M^{1/2}\Lambda^\alpha f\|_X + C\|f\|_\alpha \\ &\leqslant C(\|M^{1/2}f\|_\alpha + C\|[M^{1/2},\Lambda^\alpha]f\|_0 \\ &\leqslant C\|M^{1/2}f\|_\alpha\end{aligned}$$

(since $M^{-1/2}$: $H_\alpha \to H_\alpha$ is bounded) by the results of Section 6. In fact, a closer analysis of the above proof shows that the «M. Riesz transfer theorem» works both ways on the H_α scale (even without sub-ellipticity conditions)

$$C^{-1}\|M^{1/2}f\|_\alpha \leqslant \sum_j \|X_j f\|_\alpha + \|f\|_\alpha$$
$$\leqslant C\|M^{1/2}f\|_\alpha .$$

The analogue for higher Riesz transforms also holds but it *does need* subellipticity and is harder to prove (*cf.* Section 11)

$$X_{i_1}\cdots X_{i_k}M^{-k/2}\colon H_\alpha \to H_\alpha, \qquad \alpha \in \mathbb{R}, \qquad k = 1, 2, \ldots$$

is a bounded operator.

At any rate we shall not need here these higher Riesz transforms. What we need rather is to combine all the above information and simply conclude that

$$[M, (I+L)^{(k+1)/2}] = M^{1/2}A + B \tag{7.5}$$

where

$$A(1+L)^{-k/2}\colon H_{\alpha+1} \to H_\alpha, \qquad B(1+L)^{-(k-1)/2}\colon H_{\alpha+2} \to H_\alpha \tag{7.6}$$

are bounded operators ($k = 1, 2, \ldots, \alpha \in \mathbb{R}$).

We shall now denote by

$$R_t^{(k)} = (1+L)^{k/2}e^{-tM}(1+L)^{-k/2}$$

and analyse the expression

$$[e^{-2tM}, (1+L)^{(k+1)/2}](I+L)^{-(k+1)/2} = e^{-2tM} - R_{2t}^{(k+1)} = I(t)$$

we first decompose

$$[e^{-2tM}, (I+L)^{(k+1)/2}] = e^{-tM}[e^{-tM}, (1+L)^{(k+1)/2}] + [e^{-tM}, (I+L)^{(k+1)/2}]e^{-tM}$$

and then write

$$I(t) = \int_0^t e^{-(2t-s)M}[M, (1+L)^{(k+1)/2}]e^{-sM}\,ds\,(1+L)^{-(k+1)/2}$$
$$+ \int_0^t e^{-(t-s)M}[M, (1+L)^{(k+1)/2}]e^{-(t+s)M}\,ds\,(1+L)^{-(k+1)/2}.$$

In what follows I shall assume that $k \geqslant 1$ (the case $k = 0$ is much easier but has to be treated separately). The middle term in the two integrals above will be decomposed as in (7.5) and we shall have to deal separately with four different terms $I_i(t)$, $i = 1, 2, 3, 4$. We have

$$\|I_1(t)\|_{\alpha\to\beta} \leqslant \int_0^t \|e^{-(2t-s)M}M^{1/2}\|\,\|A(I+L)^{-(k/2)}\|\,\|R_s^{(k)}\|\,ds\,\|(1+L)^{-1/2}\|$$

where we have a «cascade» of $\|\cdot\|_{s\to t}$ norms that run as follows

$$\begin{aligned}\|(I+L)^{-1/2}\|_{\alpha\to\alpha+1-\delta} &\leqslant C,\\ \|A(I+L)^{-k/2}\|_{\alpha+1-\delta\to\alpha-\delta} &\leqslant C,\\ \|R_s^{(k)}\|_{\alpha+1-\delta\to\alpha+1-\delta} &= O(1),\\ \|e^{-(2t-s)M}M^{1/2}\|_{\alpha-\delta\to\beta} &= O((2t-s)^{(\alpha-\beta-\delta)/2(1-d)-1/2}).\end{aligned}$$

The first of these estimates is a consequence of (2.6). The third estimate is of course our introduction hypothesis, the second is (7.6). To obtain the fourth, we use the holomorphicity of the action of the semigroup e^{-tM} on each H_α space that allows us to conclude together with (2.6) that

$$\|M^\lambda e^{-2tM}\|_{r\to s} = \|M^\lambda e^{-tM}\|_{s\to s}\|e^{-tM}\|_{r\to s} = O(t^{(r-s)/2(1-d)-\lambda})$$

for all $s \geqslant r$, $\lambda \geqslant 0$. If we put all the above estimates together and integrate we obtain that

$$I_1(t) = O(t^{(\alpha-\beta)/2(1-d)+(1-\delta-d)/2(1-d)}).$$

The I_3 term is quite analogous except that the $(2t-s)$ has to be replaced by $(t-s)$ and the s in $R_s^{(k)}$ has to be replaced by $t+s$ and we shall perform the jump form α to β at the first level. We have again for the same reasons

$$\begin{aligned}\|(I+L)^{-1/2}\|_{\alpha\to\alpha+1-\delta} &\leqslant C,\\ \|R_{s+t}^{(k)}\|_{\alpha+1-\delta\to\beta+1} &= O((t+s)^{(\alpha-\beta-\delta)/2(1-d)}),\\ \|A(I+L)^{-k/2}\|_{\beta+1\to\beta} &\leqslant C,\\ \|e^{-(t-s)M}M^{1/2}\|_{\beta\to\beta} &= O((t-s)^{-1/2}),\end{aligned}$$

after integration we obtain exactly the same estimate for $I_3(t)$.

The decomposition of $I_2(t)$ is different

$$\|I_2(t)\|_{\alpha\to\beta} \leqslant \int_0^t \|e^{-(2t-s)M}\|\,\|B(I+L)^{(-k+1)/2}\|\,\|R_s^{(k-1)}\|\,ds\,\|(1+L)^{-1}\|.$$

The estimates are similar and for the same reasons

$$\begin{aligned}\|(I+L)^{-1}\|_{\alpha\to\alpha+2-2\delta} &\leqslant C,\\ \|B(I+L)^{-(k-1)/2}\|_{\alpha+2-2\delta\to\alpha-2\delta} &\leqslant C,\\ \|R_s^{(k-1)}\|_{\alpha+2-2\delta\to\alpha+2-2\delta} &= O(1),\\ \|e^{-(2t-s)M}\|_{\alpha-2\delta\to\beta} &= O((2t-s)^{(\alpha-\beta-2\delta)/2(1-d)}).\end{aligned}$$

After integration we obtain the estimate $I_2(t) = O(t^{(\alpha-\beta)/2(1-d)+(1-\delta-d)/(1-d)})$. The final term $I_4(t)$ is of course like the I_2 with: $2t-s \leftrightarrows t-s$, $s \rightleftarrows t+s$ interchanged, and the jump performed at the first level

$$\|(I+L)^{-1}\|_{\alpha\to\alpha+2-2\delta} \leqslant C,$$
$$\|B(I+L)^{-(k-1)/2}\|_{\beta+2\to\beta} \leqslant C,$$
$$\|R_{s+t}^{(k-1)}\|_{\alpha+2-2\delta\to\beta+2} = O((t+s)^{(\alpha-\beta-2\delta)/2(1-d)}),$$
$$\|e^{-(t-s)M}\|_{\beta\to\beta} = O(1).$$

Integration gives the same estimate as $I_2(t)$.

We can finally complete the inductive step. Indeed, by our hypothesis: $d+\delta \leqslant 1$; $I(t) = e^{-tM} - R_t^{(k+1)}$ has just been seen to satisfy $\|I(t)\|_{\alpha\to\beta} = O(t^{(\alpha-\beta)/2(1-d)})$ for $\beta \geqslant \alpha$. But since the same estimate holds for e^{-tM} by (2.5). We obtain that $\|R_t^{(k+1)}\|_{\alpha\to\beta} = O(t^{(\alpha-\beta)/2(1-d)})$ which is our inductive step and we are done when $k \geqslant 1$.

Let us now examine the case $k = 0$. The commutator $[M, (I+L)^{1/2}]$ is then factored (with the same notations as (7.3)) as follows

$$(7.7)\quad M^{1/2}\{(M^{-1/2}X^*)[X,(1+L)^{1/2}]\} + \{[X^*,(1+L)^{1/2}](XM^{-1/2})\}M^{1/2}.$$

The terms within $\{\bullet\}$ have a bounded $\|\bullet\|_{\alpha+1\to\alpha}$ norm for reasons that have been explained many times over. The $M^{1/2}$ on the extreme left of (7.7) once put in the integral is absorbed in the $e^{-(2t-s)M}$ or $e^{-(t-s)M}$ and the integral converges. Similarly the $M^{1/2}$ on the extreme right is absorbed in the e^{-sM} or $e^{-(t+s)M}$ and again gives a convergent integral. This simply gives after the computations

$$\|e^{-tM} - R_t^{(1)}\|_{\alpha\to\beta} = O(t^{(\alpha-\beta)/2(1-d)+(1-\delta-d)/2(1-d)})$$

as before. In fact, the distinction of cases $k = 0$, $k \geqslant 1$ is artificial and with a little more work I could easily have unified the presentation.

In the above proof we have used both the sub-ellipticity of L and of M (*i.e.* that both d and $\delta \leqslant 1$). It is clear however and much easier to prove that when $\delta = 0$ then d could be 1. This is just the fact that e^{-tM} acts on the classical Sobolev spaces. Slightly more can be proved when $d = 0$. Indeed we shall show in Section 10 that if we introduce the following scalar product on X_{2n}, $n = 1, 2, \ldots$

$$\ll f, f \gg_n = \sum_{j=1}^{n} c_j \langle (I+L)^{2j} f, f \rangle_{L^2}$$

then for an appropriate choice of the $c_j \geqslant 0$ the operator $\Delta = -\sum(\partial^2/\partial x_i^2)$ (the «positive» Laplacian) (or more generally any second order elliptic operator $\tilde{\Delta}$) is semibounded bellow

$$(7.8)\qquad \mathrm{Re} \ll \tilde{\Delta} f, f \gg_n \geqslant -C \ll f, f \gg_n.$$

The reason why this is important is that it easily implies that Δ^μ is also semibounded (by subordination) and so is, therefore, every operator of the form

$L^\nu + \Delta^\mu$ ($\nu \geqslant 0$, $0 \leqslant \mu \leqslant 1$). As a result it follows that the semigroup $\exp(-t(L^\nu + \Delta^\mu))$ acts as a continuous semigroup on each of the spaces X_α ($\alpha \in \mathbb{R}$), a fact that we would be hard pushed to prove otherwise.

The final question that arises is to decide to what extent the sum of squares property of L and M is essential for the above proposition and to what extend it holds for more general operators of the form (3.6) or (3.7). This issue I will analyse in a second instalment of this paper. Let me however say that the above proof will give us for free, so to speak, a slightly larger class of operators L. We shall explain in Section 9 how we can generalize all the previous machinery to make it work for perturbations of «sums of squares» of the form:

$$\tilde{L} = \sum_j X_j^* X_j + \Lambda^\alpha = L + \Lambda^\alpha \tag{7.9}$$

where X_j are C^∞ fields as before and $0 \leqslant \alpha \leqslant 2$

$$\left(\Lambda = (I + \Delta)^{1/2} = \left(I - \sum \frac{\partial^2}{\partial x_i^2}\right)^{1/2}\right).$$

These remarks will be important for our main application in the next section.

8. Powers of Operators and Subelliptic Distances

Let L and M be two operators as in Section 7 both of them subelliptic (*i.e.* $\delta, d < 1$). We shall assume that for two positive $\alpha_0, \beta_0 > 0$ we have ($X = L^2(\mathbb{R}^n)$)

$$\|L^{\alpha_0} f\|_X \leqslant C \|(I + M)^{\beta_0}\|_X \tag{8.1}$$

(it is easy to see that this can only happen if $\alpha_0 \leqslant \beta_0$). The issue is to decide to what extent we can conclude from (8.1) that we in fact have

$$\|L^{\nu\alpha_0} f\|_X \leqslant C \|(I + M)^{\nu\beta_0}\|_X, \qquad \nu > 0 \tag{8.2}$$

or more generally

$$\|L^{\nu\alpha_0} f\|_{L^p} \leqslant C \|(I + M)^{\nu\beta_0}\|_{L^p}, \qquad \nu > 0, \quad 1 < p < +\infty. \tag{8.3}$$

A few banal remarks are in order. First of all (8.1) implies (8.2) for $0 \leqslant \nu \leqslant 1$ by general operator theory (*cf.* [23]). Secondly, if (8.2) holds for arbitrary $\nu > 0$ then (8.3) also holds for arbitrary $\nu > 0$, $1 < p < +\infty$. This is because the operator $L^{z\alpha_0}(1 + M)^{-z\beta_0}$ is $L^p \to L^p$ bounded for $\operatorname{Re} z = 0$ and if it also is $L^2 \to L^2$ bounded for $\operatorname{Re} z = \nu > 0$ arbitrary we can interpolate.

Let us denote now by

$$\Lambda_L^\alpha = \{f \in L^\infty : |f(x) - f(y)| \leqslant C d_L^\alpha(x, y)\}$$

for every $0 < \alpha \leqslant 1$ where $d_L(\cdot, \cdot)$ is the distance induced by the subelliptic operators L (*cf.* Section 0), we also have the analogous definition Λ_M^α of M. The validity of (8.3) for all $1 < p < +\infty$ and all $\nu \leqslant \nu_0$ for some fixed ν_0 has the following important consequence

$$\Lambda_M^\beta \subsetneq \Lambda_L^{\alpha-\epsilon}, \qquad \epsilon > 0 \tag{8.4}$$

which in turn (trivialy) implies the following Hölder estimate of one distance with respect to the other

$$d_M^\beta(x, y) \leqslant C_\epsilon d_L^{\alpha-\epsilon}(x, y), \qquad \epsilon > 0$$

(C_ϵ does not depend on x, y but, *a priori* at least, does depend on $\epsilon > 0$). To see how (8.4) is derived we shall have to use the «Besov spaces» $\Lambda_{p,q}^\alpha$ introduced in [15]. To make the distinction between the two operators L and M, I shall note by $\Lambda_{p,q}^\alpha(L)$ and $\Lambda_{p,q}^\beta(M)$ the corresponding spaces for L and M.

By standard arguments involving real interpolation (*cf.* [4], p. 153 where it is easy to check that (4) is also valid in our setting) we see that the hypothesis we have made implies

$$\Lambda_{p,\infty}^\beta(M) \subsetneq \Lambda_{p,\infty}^\alpha(L), \qquad 1 < p < +\infty, \qquad \frac{\alpha}{\beta} = \frac{\alpha_0}{\beta_0}$$

provided that $\beta < \nu_0 \beta_0$. On the other hand, we always have

$$\Lambda_L^\alpha \subsetneq \Lambda_{\infty,\infty}^\alpha(L) \subsetneq \Lambda_{p,\infty}^\alpha(L). \tag{8.5}$$

What is less standard and very much depends on the subellipticity of M is that for all $\beta > \epsilon > 0$ there exists $p = p_\epsilon$ large enough such that

$$\Lambda_{p,\infty}^\beta(M) \subsetneq \Lambda_{\infty,\infty}^{\beta-\epsilon}(M) \subsetneq \Lambda_M^{\beta-\epsilon}. \tag{8.6}$$

To prove the first inclusion of (8.6) one should combine the abstract «dimensional» result of [2] that give these types of inclusions for general «semigroups of finite dimension» together with the rather obvious fact (*cf.* [12]) that the semigroup e^{-tM} is «finite dimensional» in the sense of [2] provided that M is subelliptic. In other words we have

$$\|e^{-tM}\|_{L^1 \to L^\infty} = O(t^{-m}), \qquad t \to 0$$

for some finite $m > 0$.

In (8.5) and (8.6) we also use twice the fact that $\Lambda_{\infty,\infty}^\alpha(M) = \Lambda_M^\alpha$ $(0 < \alpha < 1)$ this depends on subellipticity and is proved in a manner entirely analogous to the one used for the corresponding classical spaces $\Lambda_{\infty,\infty}^\alpha = \Lambda_\alpha$ on $\mathbb{R}^n$ (*cf.* [24]). Some of the ideas of [15] have to be used in this proof but the problem is much easier than the one treated in [15]. At any rate, I shall not give the details of

these proofs at this point because they are totally alien with the spirit of this paper.

The above remarks put together complete the proofs of the required inclusions (8.5), (8.6).

At this stage I shall recall that conversely $d_M^{\beta_0} \leqslant C d_L^{\alpha_0}$ implies the estimate (8.1) provided that $0 < \alpha_0, \beta_0 < 1/2$ (*cf.* [15]). All in all we finally see that to close this circle of ideas and obtain a proof of the theorem announced in the Introduction it suffices to find a way to prove that implication (8.1) implies (8.2).

The implication (8.1) implies (8.2) is an immediate consequence of the considerations of [3] as soon as we know that the semigroup e^{-tM} acts continuously on each Sobolev space X_α of the operator L. In other words, we are in the situation of Section 7. We can therefore assert straight away that the theorem holds under the subellipticity condition $d + \delta \leqslant 1$ for the *two* operators L and M. It holds in particular if $0 \leqslant d, \delta \leqslant 1/2$.

We shall finally show how (8.1) implies (8.2) under the weaker condition $0 \leqslant d \leqslant 1/2$ and in fact under a slightly more general condition that depends on α_0, β_0. Towards that we shall assume as we may that (8.1) holds for α_0, β_0 small $\ll 1$ and let us consider the perturbed operator: $\tilde{L} = L + \Lambda^\gamma$ where $\gamma = \min\{2, \beta_0/\alpha_0\}$ it follows that

$$\|L^{1/2} f\|_X = (Lf, f) \leqslant \|\tilde{L}^{1/2} f\|_X = (Lf, f) + \|f\|_{\gamma/2}.$$

But since the semigroup $e^{-t\tilde{L}}$ generated by $\tilde{L}$ acts on the Sobolev spaces X_α of L. This follows from the fact that Λ^2 is semibounded (in the sense of (7.8)) on X_α which makes Λ^γ $0 < \gamma \leqslant 2$ automatically semibounded (*cf.* end of Section 10) we deduce that

$$\|L^{\nu\alpha_0} f\|_X \leqslant C \|\tilde{L}^{\nu\alpha_0}\|_X, \qquad \nu > 0. \tag{8.7}$$

The same semiboundedness result implies that

$$L \ll C(L^{\alpha_0} + \Lambda^{\beta_0})^{1/\alpha_0}, \qquad \Lambda^\gamma \ll C(L^{\alpha_0} + \Lambda^{\beta_0})^{1/\alpha_0} \tag{8.8}$$

since we clearly have $L^{\alpha_0} \ll L^{\alpha_0} + \Lambda^{\beta_0}$ and $\Lambda^{\beta_0} \ll L^{\alpha_0} + \Lambda^{\beta_0}$, and since L is also semibounded on H_α for the natural scalar product (*cf.* [3]). (Here $\ll$ indicates the partial order of positive Hilbert space operators: $P \gg 0$ if and only if $(Pf, f) \geqslant 0$.) By adding we obtain that

$$(L + \Lambda^\gamma) \ll C(L^{\alpha_0} + \Lambda^{\beta_0})^{1/\alpha_0}$$

and since $1/\alpha_0 > 1$ by abstract operator theory we deduce that

$$\tilde{L}^{\alpha_0} = (L + \Lambda^\gamma)^{\alpha_0} \ll C(L^{\alpha_0} + \Lambda^{\beta_0}) \ll C M^{\beta_0}. \tag{8.9}$$

(Since $\Lambda \ll M$.) It remains for us to show that (8.9) implies that

$$\|\tilde{L}^{\nu\alpha_0} f\|_X \leqslant C \|M^{\nu\beta_0} f\|_X, \qquad \nu > 0 \tag{8.10}$$

and then putting together (8.7) and (8.10) we have a proof of the implication (8.1) implies (8.2).

The estimate (8.10) will follow from (8.9) (*cf.* [3]) as soon as we know that d the subellipticity index of M and $1 - \gamma/2$ [a quantity that is (if anything) bigger than the subellipticity index of $\tilde{L}$] add up to $\leqslant 1$ *i.e.* it is enough to have

$$d \leqslant \frac{\beta_0}{2\alpha_0}$$

for the implication (8.1) implies (8.2) to work with this mehod. By the fact that $\beta_0 \geqslant \alpha_0$ it follows in particular that this is always the case when $d \leqslant 1/2$ and the proof of our theorem in the Introduction has been completed.

9. The Perturbed Operator

Let us consider the operator $\tilde{L} = L + \Lambda^{2\alpha_0}$ of (7.9) $0 < \alpha_0 \leqslant 1$ and let us define the corresponding Sobolev spaces $X_\alpha = \{f\colon \tilde{L}^{\alpha/2} f \in X\}$ we shall still denote by $0 \leqslant \delta < 1$ the subellipticity index of $\tilde{L}$ *i.e.* $\|f\|_{1-\delta} \leqslant C \|\tilde{L}^{1/2}\|_X$ and we shall also consider a second operator M as in Section 7 with subellipticity index $d < 1$. We shall make one more additional hypothesis: We shall assume that L is already subelliptic (but with no control of its subellipticity index) *i.e.* we shall assume that some finite number of brackets of the fields X_j of L span at every point the tangent space. What we shall show is that the proposition of Section 8 still holds and that e^{-tM} acts on each X_α space provided that $\delta + d \leqslant 1$.

The proof of that fact relies on one simple observation: The operator $\tilde{L}$ can be written as follows

$$\tilde{L} = \sum_{j=1}^{n} Y_j^* Y_j$$

where $Y_j \in S^1_{1,0}$ and are either one of the fields X_j of L or $Y_j = \Lambda^{\alpha_0}$. With this decomposition we see that all our theory of Sections 4, 5 and 6 (and therefore also Section 7 which gives the required result) goes through, provided that we can prove the equivalent of the singular integral estimate (4.2) in Section 4. Namely all we need to prove is the following estimate

$$\|Y_{i_1} Y_{i_2} \dots Y_{i_k} f\|_X \leqslant C \|\tilde{L}^{k/2} f\|_X. \tag{9.1}$$

To show (9.1), by induction on k, I shall generalize the problem slightly and assume that the Y's above consist of exactly $p - X_i$ fields and $q - S^{\alpha_0}_{1,0}$ operators (with $p + q = k$). I shall commute the Y's in (9.1) between themselves to bring all the X_i fields at the beginning (*never* commute two fields X_i between themselves). We obtain thus that the left hand side of (9.1) is

$$\|X_{i_1}X_{i_2}\dots X_{i_p}S^{q\alpha_0}f\|_X + \cdots \tag{9.2}$$

where the ... come from the successive commutators and are identical expressions with a *lower* k. The ... can therefore be dealt with by the induction hypothesis. As for the principal term in (9.2) by our singular integral theorem it can be estimated by $\|L^{p/2}S^{q\alpha_0}\|_X$. This gives the required inductive step by Section 6 since (L is really $I + L$)

$$\|L^{p/2}f\|_{H_{q\alpha_0}} \approx \|L^{p/2}\Lambda^{q\alpha_0}f\|_X \leqslant C\|(L + \Lambda^{2\alpha_0})^{(p+q)/2}f\|_X. \tag{9.3}$$

To see how the last estimate is obtained in (9.3) observe first that by the considerations that lead to (8.8) we have

$$\begin{aligned} &\|L^p\Lambda^s f\| \leqslant C\|(L + \Lambda^r)^p\Lambda^s f\|, \\ &\|\Lambda^{\alpha_0 q}(L + \Lambda^t)^s f\| \leqslant C\|(L + \Lambda^{2\alpha_0})^{q/2}(L + \Lambda^t)^s f\|, \qquad f \in C \end{aligned} \tag{9.4}$$

for $s, t \in \mathbb{R}$, $p, q \geqslant 0$, $0 \leqslant r \leqslant 2$, $0 \leqslant \alpha_0 \leqslant 1$. It suffices therefore to prove that

$$\|L^p\Lambda^s f\| \leqslant C\|\Lambda^s(L + \Lambda^r)^p f\|; \qquad p \geqslant 0, \quad 0 \leqslant r \leqslant 2, \quad s \in \mathbb{R}. \tag{9.5}$$

The estimate (9.5) for $p = 1$ followes from (9.4) and the commutator estimate (*cf.* Section 6)

$$\|[\Lambda^s, L]f\| \leqslant C\|L^{1/2}\Lambda^s f\| \leqslant C\|L\Lambda^s f\|^{1/2}\|\Lambda^s f\|^{1/2}, \qquad f \in C_0^\infty. \tag{9.6}$$

Indeed the right hand side of (9.6) can be estimate (Peter-Paul!) by $\epsilon\|L\Lambda^s f\| + (C/\epsilon)\|\Lambda^s f\|$ and $\epsilon\|L\Lambda^s f\|$ can be absorbed with the left hand side of (9.5) (for $p = 1$). This same type of argument gives (9.5), by induction, for $p = 1, 2, \dots$. Indeed for $p = 1, 2, \dots$ we have (use (8.8) again)

$$\begin{aligned} \|L^{p+1}\Lambda^s f\| &\leqslant C\|(L + \Lambda^r)L^p\Lambda^s f\| \\ &\leqslant C\|L^p\Lambda^s(L + \Lambda^r)f\| + C\|[L, L^p\Lambda^s]f\| + C\|[\Lambda^r, L^p\Lambda^s]f\| \\ &= I_1 + I_2 + I_3, \qquad f \in C_0^\infty. \end{aligned}$$

I_1 is estimated by the inductive step (set $g = (L + \Lambda^r)f$). By Section 6, we have

$$I_2 \leqslant C\|L^{p+1/2}\Lambda^s f\| \leqslant \epsilon\|L^{p+1}\Lambda^s f\| + C(\epsilon)\|\Lambda^s f\|, \qquad \epsilon > 0,$$

and we finish as before by Peter-Paul. Finally we have

$$I_3 \leqslant C\|L^{p-1/2}\Lambda^{r+s}f\| \leqslant C\|L^p\Lambda^{r+s}f\| \leqslant C\|\Lambda^{r+s}(L + \Lambda^r)^p f\| = C\|\Lambda^{r+s}g\|,$$

$f \in C_0^\infty$, $g = (L + \Lambda^r)^p f$, by the inductive hypothesis. But

$$\|\Lambda^{r+s} g\| \leqslant \|\Lambda^s (L + \Lambda^r) g\| + \|\Lambda^s L g\|$$

and the case $p = 1$ applied to g finishes the proof of (9.5) for p and arbitrary non-negative integer.

The final problem that we have to face is that for (9.3) we need (9.5) for $p =$ half integers (and I am indebted to the referee who pointed out that difficulty to me). In fact as we have said (9.5) holds for arbitrary $p \geqslant 0$. To see how this is done let us use the notation at the end of Section 6 and write (9.5) for *fixed* $s \in \mathbb{R}$

$$\|f\|_{Z_s^p} \leqslant \|(L + \Lambda^r)^z f\|_{Z_s^0}, \qquad f \in C_0^\infty, \quad z = p. \tag{9.7}$$

It is clear that (for fixed s) $Z_s^p = Z^p$ is a «complex interpolation scale». If we can complexify $z = p + is$ in (9.7) we will then use the analytic family of operators $\phi(z) = (L + \Lambda^r)^{-z}$ and we are though. That complexification $p \to p + is$ amounts simply to showing that $\tilde{L}^{is} = (L + \Lambda^r)^{is} = H_s \to H_s$ is bounded $(0 \leqslant r \leqslant 2)$. The proof of that fact given in Section 3, for the operator L, also works for the operator $\tilde{L}$ (indeed $\tilde{L} \in S^2_{1,0}$ and already in [3] it was shown that $e^{-t\tilde{L}}$ acts on H_α) and we are done.

There is another way (more elegant but also more subtle) to deduce (9.7) for *all* $z \in \mathbb{C}$, $\operatorname{Re} z > 0$ from the validity of the same inequality for just *one* value $z = p_0 > 0$. It suffices to apply the theory of [3] on the interpolation scale Z_s^p ($p > 0$, s is fixed. The fact that this scale is not self-dual *i.e.* $(Z_s^p)^* = Z_{-s}^{-p}$ (and not Z_s^{-p}) cause no problems). For that we must show that e^{-tL} acts on spaces Z_s^p, $p = 0, 2, \ldots$. By a moments reflexion we see that this is a consequence of Section 10 and of the fact that $c_\lambda(L)$ is semibounded (in the sense (7.8)) on the spaces X_α ($\alpha = 0, 2, 4, \ldots$) of Section 1. This last fact will be left as an exercise for the (patient) reader. (Hint: $c_\lambda(L) = L + X_j E + E$ is a «small» perturbation of L.) The proof of (9.3) is complete.

10. The Semiboundedness Property of Δ

Let L be as before [*cf.* (7.1)] and let us denote by

$$\langle f, f \rangle_n = \sum_{j=0}^{n} c_j \langle (1 + L)^{2j} f, f \rangle$$

for $c_j \geqslant 0$, $(c_n > 0)$ where $\langle \cdot, \cdot \rangle$ indicates the canonical scalar product on $X = L^2$. Thus for any choice of the c_j's is a Hilbert space scalar product on X_{2n}, $n = 0, 1, \ldots$ I shall show that (with appropriate choices of the c_j's) the operator Δ is semibounded in the sence of (7.8) on these scalar products. More

precisely, I shall show by induction on n that I can choose the c_j's so that

$$\operatorname{Re}\langle(\Delta+\lambda_0)f,f\rangle_n \geqslant k\|\Lambda(I+L)^n f\|_X^2, \qquad f\in C_0^\infty$$

for some $k=k_n>0$ and all $\lambda_0>0$ large enough.

Assume that the result holds for some $n=0,1,\ldots$. We have then

$$\langle(\Delta+\lambda_0)f,f\rangle_{n+1}=\langle(I+L)^2(\Delta+\lambda_0)f,f\rangle_n+c_0\langle(\Delta+\lambda_0)f,f\rangle \tag{10.1}$$

$$\langle(I+L)^2(\Delta+\lambda_0)f,f\rangle_n=\langle(\Delta+\lambda_0)(I+L)f,(I+L)f\rangle_n+\langle[\Delta,L]f,(1+L)f\rangle_n.$$

The first term on the right hand side of (10.1) has a real part that is, by our inductive hypothesis, $\geqslant k_n\|\Lambda(1+L)^{n+1}\|_X^2$. The second term in the right hand side of (10.1) can be estimated in modulus by

$$\sum_{j=0}^{n}|\langle(1+L)^j[\Delta,L]f,(I+L)^{j+1}f\rangle|=\sum|\langle X_1X_2\ldots X_\alpha\Lambda^2Ef,Y_1\ldots Y_\beta f\rangle|$$

where $E\in S^0_{1,0}$, $\alpha\leqslant 2j+1$, $\beta\leqslant 2j+2$ and the X's and Y's are chosen among the C_0^∞ fields that define $L\left(=\sum X^*X\right)$ as in (7.1). By commutating and taking adjoints we can therefore rewrite every $\langle\bullet,\bullet\rangle$ on the right as

$$\begin{aligned}|\langle X_1X_2\ldots X_\alpha\Lambda Ef,Y_1,\ldots,Y_\beta\Lambda Ef\rangle| &\leqslant \|X_1,\ldots,X_\alpha\Lambda Ef\|_X\|Y_1,\ldots,Y_\beta\Lambda Ef\|_X\\ &\leqslant C\|(I+L)^{\alpha/2}\Lambda Ef\|_X\|(1+L)^{\beta/2}\Lambda Ef\|_X\\ &\leqslant C\|(I+L)^{j+1/2}\Lambda Ef\|_X\|(1+L)^{j+1}\Lambda Ef\|_X\\ &\leqslant C\|(I+L)^{j+1/2}\Lambda f\|_X\|(1+L)^{j+1}\Lambda f\|_X.\end{aligned}$$

By the singular integral estimate (4.2) and the estimates that we developed in Sections 4, 5 and 6. But by the log-convexity properties of $\|(1+L)^\nu\Lambda f\|_X$ (as a function of ν, since $X=L^2$) we can finally estimate this scalar product by

$$\begin{aligned}&\leqslant\|(I+L)^{j+1}\Lambda^{1/2}f\|_X[\epsilon\|(1+L)^{(j+1)}\Lambda f\|_X+C(\epsilon)\|\Lambda f\|_X]\\ &\leqslant\epsilon\|(I+L)^{j+1}\Lambda f\|_X^2+C_1(\epsilon)\|\Lambda f\|_X\end{aligned}$$

for arbitrary $\epsilon>0$ and $C(\epsilon)$, $C(\epsilon)>0$. If we put everything together and choose the c_0 on the last term on the right of (10.1) sufficiently large we obtain the required inductive step.

For odd integers (and other values of α) we can also show directly that Δ is a semibounded operator for an appropriate scalar product of X_α. It is preferable however to proceed indirectly. Observe that A semibounded on some Hilbert space X if and only if

$$\|e^{-tA}\|_{X\to X}\leqslant e^{\lambda_0 t}$$

for some $\lambda_0>0$ and this estimate, once established for the Hilbert spaces X_{2n}, goes throught to all the other X_α ($\alpha\in\mathbb{R}$) by duality and interpolation. It is also

this estimate that, by subordination (*cf.* [6]), easily gives the corresponding result for all A^α, $0 \leqslant \alpha \leqslant 1$.

The above proof that I gave is very much in the spirit of this paper but it is very unsatisfactory because it uses the subellipticity of L in a place that it probably has nothing to do. The result is probably correct without that hypothesis. I shall give a proof below of that fact for $\alpha = 1$ or 2.

If $L = \sum X_j^* X_j$, for $\alpha = 1$, we much show that

$$\operatorname{Re}(X_k \Delta f, X_k f) + c(\Delta f, f) + c(f, f) \geqslant 0$$

for appropriate $c > 0$. But this is clear enough since $X_k^*[\Delta X_k] \in S^2_{1,0}$ mod (skew adjoint terms), which allows us to write

$$\operatorname{Re}(X_k \Delta f, X_k f) = \sum \|\Delta^{1/2} X_k f\| + (S^2 f, f)$$

for some $S^2 \in OPS^2_{1,0}$. For $\alpha = 2$ we must show that

$$\operatorname{Re}(L \Delta f, Lf) + c_1 \sum \operatorname{Re}(X_k \Delta f, X_k f) + c_2(\Delta f, f) + c_3(f, f) \geqslant 0$$

again for appropriate c's that can be chosen at will. From the case $\alpha = 1$ we conclude therefore that it suffices to choose $c_i > 0$ below so that

$$\text{(10.2)} \qquad \operatorname{Re}(L \Delta f, Lf) + c_1 \sum \|\Delta^{1/2} X_k f\|_X^2 + c_2 \|\Delta^{1/2} f\|_X^2 + c_3 \|f\|_X^2 \geqslant 0,$$

$$\operatorname{Re}(L \Delta f, Lf) = \|\Delta^{1/2} Lf\|_2^2 + \operatorname{Re}([L, \Delta] f, Lf),$$

and after a number of easy calculations we finally see that it all amounts to controlling from below the expression

$$\operatorname{Re} \sum_{i,j} (A_{ij} X_i f, X_j f)$$

where $A_{ij} = X_j[\Delta, X_i]$. This however is easy for if we denote by $Q = (A_{ij})_{i,j=1}^n$ then it is clear that

$$Q + Q^* = 2([[\Delta X_i] X_j])_{i,j} \qquad \operatorname{mod}(OPS^2_{1,0})$$

and the control from below is thus supplied by the second term of (10.2).

11. Final Remarks on the Method of the Proofs

The main functional analytic tool used to obtain the critical indices in Sections 4, 5 and 6 was the machinery of real interpolation. It turns out that there is another approach, in some ways dual to the one used, which could have been adopted instead. This approach is interesting in its own right and without going into details I will describe its main features.

The key is to prove that

$$\|(c_\lambda - c_\mu)(A^\sigma e^{-tA})\|_{\alpha\to\alpha} = O(t^{-\operatorname{Re}\sigma + 1/2}) \tag{11.1}$$

for $\lambda, \mu, \alpha \in \mathbb{R}$ and $\operatorname{Re}\sigma > 1/2$. $\|\cdot\|_{\alpha\to\beta}$ refers here and in what follows to the $H_\alpha \to H_\beta$ norm. It is clear by the observation that $\|c_\lambda(\cdot)\|_{\alpha\to\alpha} = \|c_{\lambda+\alpha}(\cdot)\|_{0\to 0}$ and by taking differences that we can assume that $\alpha = 0$ (and even $\lambda = 0$). It is also enough to assume that $\sigma = 1, 2, \ldots$ and then use the standard formulae for the powers of an operator with positive real part.

The problem thus becomes to establish

$$\left\| (I - c_\lambda)\left(\frac{d^k}{dt^k} e^{-tA}\right)\right\|_{X\to X} = O(t^{1/2-k})$$

for $k = 1, 2, \ldots$. There seems to be no automatic way of obtaining this without introducing some new idea. The idea is to consider the holomorphic function

$$\Phi(z) = (1 - c_\lambda)(e^{-zA}), \qquad |\operatorname{Arg} z| < \pi/4$$

and to start by establishing that $\|\Phi(z)\|_{X\to X}$ (or even $\|\cdot\|_{\alpha\to\alpha}$) is $O(|z|^{1/2})$ in that sector. This is done just as in Section 5 since e^{-zA} is a holomorphic semigroup. Then we use Cauchy's theorem and the required estimate for $(d^k/dz^k)\Phi(z)$ is obtained.

Having estimated (11.1) we then proceed to estimate

$$\|A^\sigma(c_\lambda - c_\mu)(e^{-tA})\|_{\alpha\to\alpha} = O(t^{-\operatorname{Re}\sigma + 1/2}). \tag{11.2}$$

This time the result clearly also holds for $\operatorname{Re}\sigma = 0$ and therefore for any $\operatorname{Re}\sigma \geqslant 0$. This for $\sigma = 1, 2, \ldots$ is deduced from (11.1) and the formula

$$(I - c_\mu)(A^k e^{-tA}) = A^k[(I - c_\mu)e^{-tA}] + [(I - c_\mu)(A^k)]c_\mu(e^{-tA}) \tag{11.3}$$

where (for $k = 1, 2, \ldots$) $(1 - c_\mu)(A^k)$ is dealt with by pseudo-differential calculus as before. To obtain (11.2) we finally have to make a rather subtle use of complex interpolation. Observe that $[E, A^\sigma e^{-tA}]$ and $A^\sigma[Ee^{-tA}]$ are obtained analogously and they are in fact simpler since the last step that uses complex interpolation is not needed. I will not give the details.

A final step in this approach to our theory consists in proving estimates of the type

$$\|(1 - c_\lambda)(X_1 \cdots X_k e^{-tA})\|_{X\to X} = O(t^{(1-k)/2}) \tag{11.4}$$

for A-sub-unit fields X_j. This, in term, allows us to show at once that the multiple Riesz transforms $R = X_1 \cdots X_k A^{-k/2}$ is an $H_\alpha \to H_\alpha$ bounded

operator. To prove the estimate (11.4) we first reduce it to the corresponding estimate of $X_1 \cdots X_k(I - c_\lambda)(e^{-tA})$ by the analogue of formula (11.3) and then we use the singular integral theorem (4.2) to reduce it to (11.2). The Riesz transforms then follow from (11.4) by the same «trick» that was used in Section 3 for the pure imaginary powers (*cf.* Section 3). I will not give the details here.

12. An Example

Some of the results of this paper will be illustrated by the following example

$$L_i = X^*X + Y^*Y + Z_i^*Z_i, \qquad i = 1, 2$$

$$X = \frac{\partial}{\partial x}, \qquad Y = x\frac{\partial}{\partial y}, \qquad Z_i = a_i(y)e^{-1/y^2}\frac{\partial}{\partial y}, \qquad (x, y \in \mathbb{R})$$

where the $0 < a_i(y) \in C^\infty(\mathbb{R}\setminus\{0\})$ are chosen to be, together with all their derivatives, $O(|y|^{-n})$ as $y \to 0$ so that the two above operators are C^∞ subelliptic Hörmander operators in $\mathbb{R}^2$ as in Section 7 with $\delta = d = 1/2$.

Then a_i, $i = 1, 2$ will be chosen so that

$$(12.1) \qquad a_1/a_2 \notin L^\infty(\mathbb{R}), \qquad a_1/a_2 \in L^p_{\text{loc}}(\mathbb{R}); \quad \text{for every} \quad 1 < p < +\infty$$

(*e.g.* $a_1(y) = -\log|y|$, $a_2(y) \equiv 1$ for small y).

From this it follows that for all $C > 0$ we have

$$(12.2) \qquad \|L_1^{1/2}f\|_2 \not\leqslant C[\|L_2^{1/2}f\|_2 + \|f\|_2], \qquad f \in C_0^\infty.$$

Here we should note that $\|L_i^{1/2}f\|_2 = \|\nabla_i f\|_2$ with

$$\nabla_i f(x) = \left(\left|\frac{\partial f}{\partial x}\right|^2 + (x^2 + a_i^2(y)e^{-2/y^2})\left|\frac{\partial f}{\partial y}\right|^2\right)^{1/2}.$$

It is an automatic consequence of (12.2) (or it can be seen by direct calculation) that the corresponding distances satisfy

$$d_2(\cdot, \cdot) \not\leqslant Cd_1(\cdot, \cdot); \quad \text{for every} \quad C > 0.$$

We shall see however that for $\Omega \subset\subset \mathbb{R}^2$ and for all $\epsilon > 0$ there exists $C > 0$ such that

$$(12.3)\ d_2(\cdot, \cdot) \leqslant C_\epsilon d_1^{1-\epsilon}(\cdot, \cdot),\ \|L_1^{1/2-\epsilon}f\|_2 \leqslant C_\epsilon[\|L_2^{1/2}f\|_2 + \|f\|_2],\ f \in C_0^\infty(\Omega).$$

The above two estimates are in fact equivalent because of our main theorem. To see the second estimate (12.3) observe that the second condition of (12.1)

and Hölder's inequality implies that for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|\nabla_1 f\|_{L^{2-\delta}} \leqslant C_\delta(\|\nabla_2 f\|_{L^2} + \|f\|_{L^2}), \qquad f \in C_0^\infty. \tag{12.4}$$

The behaviour at infinity is irrelevant in the above considerations but the reader can easily check, if he so wishes, that (12.4) holds even globally provided that L_i are correctly modified at infinity.

At any rate what counts here are the following two estimates

$$\|\nabla_1 f\|_{L^p} + \|f\|_{L^p} \approx \|L_i^{1/2} f\|_{L^p} + \|f\|_{L^p}, \qquad 1 < p < +\infty, \quad f \in C_0^\infty. \tag{12.5}$$

And also for every $\epsilon > 0$, small enough, there exists $\delta > 0$ and $C > 0$ such that

$$\|L_2^{1/2-\epsilon} f\|_{L^2} + \|f\|_{L^2} \leqslant C[\|L_2^{1/2} f\|_{L^{2-\delta}} + \|f\|_{L^{2-\delta}}], \qquad f \in C_0^\infty. \tag{12.6}$$

(12.4), (12.5) and (12.6) put together imply (12.3).

The estimate (12.5) is the M. Riesz theorem for subelliptic operators already used in sections 3 and 4, *cf.* [8], [9]. The estimate (12.6) was already used in a slightly different form in Section 8 and is a consequence of the abstract Hardy-Littlewood theory and the finite dimensionality of the semigroup e^{-tL_2}, *cf.* [2], [12].

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Hardy Space Estimates for Multilinear Operators, I

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Abstract. In this article, we study bilinear operators given by inner products of finite vectors of Calderón-Zygmund operators. We find that necessary and sufficient condition for these operators to map products of Hardy spaces into Hardy spaces is to have a certain number of moments vanishing and under these assumptions we prove a Hölder-type inequality in the H^p space context.

Introduction

Probably, the most important example of a multilinear operator is the determinant of the Jacobian of a map F: $\mathbb{R}^n \to \mathbb{R}^n$. In two dimensions the determinant of the Jacobian of a pair of functions (f, g) from $\mathbb{R}^2$ to $\mathbb{R}^2$ is the bilinear map

$$J(f,g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}$$

which has very interesting properties. A theorem of P. L. Lions and Y. Meyer, says that J maps the product of Sobolev spaces $L_1^2 \times L_1^2$ into the Hardy space H^1. This theorem has been extended for L^p Sobolev spaces by [CMLS] as follows: The determinant of the Jacobian maps $L_1^p \times L_1^q$ into H^γ as long as $1 \geqslant \gamma > 2/3$; $p, q > 1$ and $p^{-1} + q^{-1} = \gamma^{-1}$. The spaces $H^\gamma(\mathbb{R}^n)$, $\gamma \leqslant 1$ are the usual real variable Hardy spaces as defined in [S] or [SW]. The result of [CMLS] is false when $\gamma = 2/3$ and leads naturally to the following question: Why can't the Jacobian-determinant map into some H^γ space for $\gamma \leqslant 2/3$?

In these articles we prove that a more general class of bilinear operators map into H^r for arbitrarily small $r > 0$ *only when* they have a certain number of moments vanishing. The determinant of the Jacobian has always integral zero but is does not have higher moments vanishing and this is the reason it cannot map into H^γ for $\gamma \leqslant 2/3$. Other bilinear operator on $\mathbb{R}^n$ have higher moments vanishing and they map into H^r for $r < n/(n+1)$. (The index 2/3 corresponds to the case $n = 2$.)

A good example of an operator with integral and first moments vanishing in $\mathbb{R}^2$ is the determinant of the Hessian of a map (f, g): $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$H(f,g) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 g}{\partial y \partial x} - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 g}{\partial x^2}.$$

By introducing the Riesz transforms, $\widehat{R_j f}(\xi) = i\xi_j \hat{f}(\xi)/|\xi|$ the Jacobian-determinant J and Hessian-determinant H can be studied through the bilinear maps $\tilde{J}$ and $\tilde{H}$ given by

$$\tilde{J}(f,g) = (R_1 f)(R_2 g) - (R_2 f)(R_1 g)$$
$$\tilde{H}(f,g) = (R_1^2 f)(R_2^2 g) - 2(R_1 R_2 f)(R_2 R_1 g) + (R_2^2 f)(R_1^2 g)$$

which are of the form

$$\sum_{j=1}^{N} (T_j^1 f)(T_j^2 g) \tag{0.1}$$

for some Calderón-Zygmund operators $\{T_j^1\}$, $\{T_j^2\}$.

In part II of this work, we prove that $\tilde{H}$ maps $L^p \times L^q$ into H^r for $p, q > 1$, $1 \geqslant r > 1/2$ and $r^{-1} = p^{-1} + q^{-1}$. We conclude that H maps functions with Laplacean in $L^p \times L^q$ into H^r for the same p, q, r as above. The exponent $r = 1/2$ is a natural lower bound in this case since the assumptions $p, q > 1$ imply that $r = (p^{-1} + q^{-1})^{-1} > 1/2$.

The question investigated in this article is under what conditions can we have boundedness into H^r for $r \leqslant 1/2$. Since $r = (p^{-1} + q^{-1})^{-1}$ we must have $p \leqslant 1$ or $q \leqslant 1$ and obviously the L^p spaces are not a suitable starting pont. If we replace the L^p spaces with H^p for $p \leqslant 1$ however, we get boundedness into H^r for arbitrarily small r. We treat general bilinear operators of the form (0.1) and we assume that these operators have for a given $r > 0$ a required number of moments vanishing to map into H^r.

1. Preliminaries

We are given two families of tempered distributions $\{K_i^1\}_{i=1}^N$, $\{K_i^2\}_{i=1}^N$, homogeneous of degree 0 and we are assuming that:

(1) The Fourier transforms of $\{K_i^1\}$, $\{K_i^2\}$ are bounded functions.
(2) $\{K_i^1\}$, $\{K_i^2\}$ are sufficiently smooth away from the origin and

$$\left|\frac{\partial^\gamma}{\partial x^\gamma} K_i^j\right| \leqslant C|x|^{-n-|\gamma|}$$

for all $|\gamma| \leqslant M$ for some sufficiently large M, ($j = 0$ or 1).
(3) For all multi-indices α and γ of norm less than M for some large M, the partial derivatives of $\{K_i^1\}$ and $\{K_i^2\}$ satisfy:

$$\left|\frac{\partial^\gamma}{\partial y^\gamma}\frac{\partial^\alpha}{\partial x^\alpha}(K_i^j(x-y) - K_i^j(x))\right| \leqslant A\frac{|y|}{|x|^{n+|\alpha|+|\gamma|+1}}$$

for $|x| > 2|y|$, ($j = 0$ or 1).

We call T_i^1 the operator given by convolving with K_i^1 and T_i^2 the operator given with convolution with K_i^2. Theorem 12 in [FS] says that the operators $\{T_i^1\}$, $\{T_i^2\}$ map $H^p \to H^p$. The main result in this article is the H^p boundedness of the bilinear product

$$B(f, g) = \sum_i (T_i^1 f)(T_i^2 g)$$

where f, g lie in suitable Hardy spaces. We have the following theorem.

Theorem. *Suppose* $\{T_i^1\}$, $\{T_i^2\}$, $i = 1, 2, \ldots, N$ *are Calderón-Zygmund operators on* $\mathbb{R}^n$ *as above. Fix* $p, q \leqslant 1$ *and let*

$$B(f, g) = \sum_{i=1}^{N} (T_i^1 f)(T_i^2 g).$$

Assume that for some $k \geqslant 0$ *integer, for all multi-indices* $|\alpha| \leqslant k$ *and for all* f H^p*-atoms and* g H^q*-atoms the moments*

$$\int x^\alpha B(f, g)(x)\, dx = 0, \qquad |\alpha| \leqslant k.$$

Then B *can be extended to a bounded operator from* $H^p \times H^q \to H^r$ *where*

$$\frac{n}{n+k+1} < r \leqslant \frac{n}{n+k} \quad \text{and} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Remarks.

(1) Note that if f is an H^p-atom and g is an H^q-atom the integral defining the moments of $B(f, g)$ is well defined, for $(T_i f)(x) \in L^p$ for all $p > 1$ and decays like $|x|^{-[n(1/p-1)]-n-1}$ as $|x| \to \infty$. Therefore, the functions

$x^\alpha(T_i^1 f)(x)(T_i^2 g)(x)$, $|\alpha| \leqslant k$ are L^1 around 0 and decay like

$$|x|^{-[n(1/p-1)]-[n(1/q-1)]-2n-2+|\alpha|} \quad \text{as} \quad |x| \to \infty.$$

The exponent above is less than $-n$ as long as $|\alpha| \leqslant k$ and $r \leqslant n/(n+k)$. The operator $B(f,g)$ is certainly well defined when f and g are finite sums of atoms and the assumptions make sense for the range of p, q and r as above. The conclusion is that $B(f,g)$ maps $H^p \times H^q \to H^r$ as long as it has r moments vanishing.

(2) Our theorem is not vacuous if we assume that $k \geqslant n$. Otherwise, $k \leqslant n-1$ implies that $n/(n+k+1) \geqslant 1/2$ and thus $r > 1/2$ which is impossible since $r = (1/p + 1/q)^{-1} \leqslant 1/2$.

Before we begin the proof of our theorem we state and prove a lemma that will be the main tool in the treatment of all the terms that will appear in the decomposition of $B(f,g)$ except the main term.

Suppose ϕ is a smooth function and f an H^p distribution ($p \leqslant 1$). We are interested in computing the value of the constant C_ϕ in the following inequality

$$\left| \int f(x)\phi(x)\,dx \right| \leqslant C_\phi$$

where, by $\int f(x)\phi(x)\,dx$ we denote the action of the distribution f on the test function ϕ. We have the following

Lemma 1. *Let ϕ be sufficiently smooth. For any $f \in H^p$, $p \leqslant 1$ there exists an L^p function f^+ independent of ϕ with $\|f^+\|_{L^p} \leqslant C_p \|f\|_{H^p}$ such that*

$$\left| \int f(x)\phi(x)\,dx \right| \leqslant N_{x_0}(\phi) f^+(x_0), \quad \textit{all} \quad x_0 \in \mathbb{R}^n,$$

where

$$N_{x_0}(\phi) = \sum_{s=0}^{N} \int |x - x_0|^{2s} |\Delta^s \phi|(x)\,dx$$

for some N depending on n and p only.

PROOF. We call $N_{x_0}(\phi)$ the «norm» of ϕ.

To prove the lemma we use the atomic decomposition of f. Assume first that $f = a_{Q_0}$ is an atom supported in the unit cube Q_0 of $\mathbb{R}^n$. We will show that

$$\left| \int a_{Q_0}(x)\phi(x)\,dx \right| \leqslant N_{x_0}(\phi) h(x_0) \tag{1.1}$$

where

$$h(x_0) = c(3 + |x_0|)^{-2s} \quad \text{and} \quad s \in \mathbb{Z}^+$$

is chosen such that $2s > n/p$. We are assuming that a_{Q_0} has at least $2s - n$ moments vanishing.

To verify (1.1) we consider two cases

Case 1. $x_0 \in 3Q_0$ (triple of Q_0). Then

$$\left| \int a_{Q_0}(x)\phi(x)\,dx \right| \leqslant C \int |\phi|\,dx \leqslant h(x_0)N_{x_0}(\phi).$$

Case 2. $x \notin 3Q_0$.

We choose a $\zeta(x)$ in $C^\infty(\mathbb{R}^n)$ function, supported in $|x| < 1$ and we call $\zeta_{x_0}(x) = \zeta(2(x - x_0)/|x_0|)$. Then $\zeta_{x_0}(x)$ is supported in $|x - x_0| \leqslant |x_0|/2$. Note that for $s > n/2p$, $\Delta^{-s}a_{Q_0}$ is defined because a_{Q_0} has enough vanishing moments. We have

$$\begin{aligned}\int a_{Q_0}\phi\,dx &= \int (\Delta^{-s}a_{Q_0})(\Delta^s\phi)\,dx \\ &= \int \zeta_{x_0}(\delta^{-s}a_{Q_0})(\Delta^s\phi)\,dx + \int (1 - \zeta_{x_0})(\Delta^{-s}a_{Q_0})(\Delta^s\phi)(x)\,dx \\ &= I + II.\end{aligned}$$

Term *II* is the easiest to treat. First we claim that $\|\Delta^{-s}a_{Q_0}\|_{L^\infty} \leqslant C$. To check this, write

$$\Delta^{-s}a_{Q_0}(x) = C\int_{Q_0} |x - y|^{-n+2s} a_{Q_0}(y)\,dy.$$

Use the fact that $a_{Q_0}(y)$ has at least $-n + 2s > 0$ moments vanishing to subtract a suitable polynomial $\sum_{|\alpha| < -n+2s} C_\alpha |x|^{-n+2s-|\alpha|} y^\alpha$ from $|x - y|^{-n+2s}$ so that

$$\left| |x - y|^{-n+2s} - \sum_{|\alpha| < -n+2s} C_\alpha |x|^{-n+2s-|\alpha|} y^\alpha \right| \leqslant C|y|^{-n+2s}$$

and thus get

$$|\Delta^{-s}a_{Q_0}(x)| \leqslant C\int_{Q_0} |y|^{-n+2s}\,dy \leqslant C.$$

Then we estimate term *II* by

$$\begin{aligned}II &\leqslant \int_{|x-x_0| \geqslant |x_0|/2} \|\Delta^{-s}a_{Q_0}\|_{L^\infty} |\Delta^s\phi|\,dx \\ &\leqslant C|x_0|^{-2s} \int_{|x-x_0| \geqslant |x_0|/2} |x - x_0|^{2s} |\Delta^s\phi(x)|\,dx \\ &\leqslant h(x_0)N_{x_0}(\phi).\end{aligned}$$

Term I can be handled with another integration by parts.

$$= \int \Delta^s(\zeta_{x_0}\Delta^{-s}a_{Q_0})\phi\,dx$$

$$= \int \left(\sum_{j=0}^{2s-1} c_j(D_x^{2s-j}\zeta_{x_0})D_x^j(\Delta^{-s}a_{Q_0}) + \zeta_{x_0}a_{Q_0}\right)\phi\,dx,$$

where D_x^jF is a sum of derivatives of F in x of total order j. Since a_{Q_0} and ζ_{x_0} have disjoint supports their product is zero.

Finally, we claim that $D_x^j(\Delta^{-s}a_{Q_0})(x)$ decays like $|x_0|^{-j}$ as $|x_0|\to\infty$ for $|x-x_0|\leqslant|x_0|/2$. We can see this by the following argument. We have

$$\Delta^{-s}a_{Q_0}(x) = \int |x-y|^{-n+2s}a_{Q_0}(y)\,dy$$

and therefore

$$D_x^j(\Delta^{-s}a_{Q_0})(x) = \int D_x^j(|x-y|^{-n+2s})a_{Q_0}(y)\,dy.$$

Note that the function $|x-y|^{-n+2s}$ is smooth near x since $y\in Q_0$ and $|x|\sim|x_0|\notin 3Q_0$. Using the fact that a_{Q_0} has $-n+2s$ moments vanishing we can subtract the Taylor polynomial of degree $-n+2s$ of $D_x^j(|x-y|^{-n-2s})$ at x to get

$$D_x^j(\Delta^{-s}a_{Q_0}(x))$$

$$= \int \left\{D_x^j(|x-y|^{-n+2s}) - \sum_{|\alpha|\leq -n+2s} \frac{\partial^\alpha}{\partial x^\alpha}(D_x^j|x|^{-n+2s})\frac{y^\alpha}{\alpha!}\right\}a_{Q_0}(y)\,dy.$$

The expression inside the curly brackets above decays like

$$|x-y|^{-n+2s-j-(-n+2s)} \sim |x|^{-j} \quad \text{as} \quad |x|\to\infty.$$

Since $|x-x_0|\leqslant|x_0|/2$ we have $|x|\sim|x_0|$ and thus we get the required estimate

$$|D_x^j(\Delta^{-s}a_{Q_0})(x)| \leqslant C|x_0|^{-j} \quad \text{as} \quad |x_0|\to\infty.$$

Clearly, we also have $|D_x^{2s-j}\zeta_{x_0}|\leqslant C|x_0|^{-2s+j}$.

Summing on j we obtain

$$\left|\sum_{j=0}^{2s-1} c_j(\Delta 2^{s-j}\zeta_{x_0})D_x^j(\Delta^{-s}a_{Q_0})\right| \leqslant C|x_0|^{-2s}$$

and thus

$$I \leqslant C|x_0|^{-2s}\int |\phi|\,dx \leqslant \zeta(x_0)N_{x_0}(\phi).$$

Putting estimates *I* and *II* together we get the desired conclusion for Case 2. Our lemma is now proved in the case where $f = a_{Q_0}$ is an atom supported in the unit cube centered at the origin. To obtain an estimate for a general atom a_Q we use translations and dilations. Observe that the norm N_{x_0} satisfies the following properties:

$$N_{x_0}\left(\frac{1}{t^n}\phi\left(\frac{\cdot}{t}\right)\right) = N_{x_0/t}(\phi)$$

$$N_{x_0}(\phi(\cdot + y)) = N_{x_0+y}(\phi).$$

For a general cube Q, the properties above and (1.1) give

$$\left|\int a_Q(x)\phi(x)\,dx\right| \leqslant |Q|^{-1/p} N_{x_0}(\phi) h\left(\frac{x_0}{|Q|^{1/n}}\right).$$

Summing over all Q we obtain

$$\left|\int f(x)\phi(x)\,dx\right| \leqslant N_{x_0}(\phi) f^+(x_0)$$

where

$$f^+(x_0) = \sum \lambda_Q |Q|^{-1/p} h\left(\frac{x_0}{|Q|^{1/n}}\right).$$

We easily check that $\|f^+\|_{L^p} \leqslant C\|f\|_{H^p}$,

$$\begin{aligned}\left(\int f^+(x_0)^p\,dx_0\right)^{1/p} &= \left(\int \left(\sum \lambda_Q |Q|^{-1/p} h\left(\frac{x_0}{|Q|^{1/n}}\right)\right)^p dx_0\right)^{1/p} \\ &\leqslant \left(\sum \lambda_Q^p \int |Q|^{-1} h^p\left(\frac{x_0}{|Q|^{1/n}}\right) dx_0\right)^{1/p} \\ &\leqslant \|h\|_{L^p}\left(\sum \lambda_Q^p\right)^{1/p} \\ &\leqslant C_p \|f\|_{H^p}.\end{aligned}$$

Our lemma is now proved.

2. Begining of the Proof

Fix p, q, r as in the statement of the theorem. Fix a smooth compactly supported function $\phi \geqslant 0$ in $\mathbb{R}^n$ and define

$$\phi_{t,x_0}(x) = \frac{1}{t^n}\phi\left(\frac{x_0 - x}{t}\right)$$

where $x_0 \in \mathbb{R}^n$ fixed. We will show that

$$\left\| \sup_{t>0} \left| \int \phi_{t,x_0}(x) B(f,g)(x)\, dx \right| \right\|_{L^r} \leqslant C \|f\|_{H^p} \|g\|_{H^q}$$

for f, g finite sums of H^p and H^q atoms respectively.

Without loss of generality we may assume that the support of ϕ lies in $\{x\colon |x| < 1\}$. Fix a smooth cut-off $\eta(x)$ supported in $|x| < 4$ such that $\eta \equiv 1$ on $|x| < 2$. Call for simplicity $\eta_0(x) = \eta(x_0 - x)/t$, $\eta_1(x) = 1 - \eta_0(x)$. The reader should remember the dependence of η_0 and η_1 on t. We write $B(f,g) = B_1 + B_2 + B_3 + B_4$ where

$$\begin{aligned} B_1 &= B(\eta_0 f, \eta_0 g), \\ B_2 &= B(\eta_1 f, g), \\ B_3 &= B(f, \eta_1 g), \\ B_4 &= -B(\eta_1 f, \eta_1 g). \end{aligned}$$

We treat term B_4 first,

$$\begin{aligned} (2.1)\quad & -\int \phi_{t,x_0} B_4\, dx \\ &= \sum \int \phi_{t,x_0}(x)(T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0))(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))\, dx \\ &\quad + \sum_i T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x)(T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0))\, dx \\ &\quad + \sum_i T_i^1(\eta_1 f)(x_0) \int \phi_{t,x_0}(x)(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))\, dx \\ &\quad + \sum_i T_i^1(\eta_1 f)(x_0) T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x)\, dx. \end{aligned}$$

We claim that for any fixed x such that $|x - x_0| \leqslant t$ we have that,

$$\sup_{t>0} |T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0)| \leqslant C f^+(x_0)$$

where $f^+(x_0)$ is an L^p function of x_0 with $\|f^+\|_{L^p} \leqslant C\|f\|_{H^p}$.

We verify this last assertion. Fix an i and write

$$\begin{aligned} &T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0) \\ &\qquad = \int \left(1 - \eta\left(\frac{x_0 - y}{t}\right)\right)(K_i^1(x - y) - K_i^1(x_0 - y)) f(y)\, dy. \end{aligned}$$

Call

$$\Phi(y) = \left(1 - \eta\left(\frac{x_0 - y}{t}\right)\right)(K_i^1(x - y) - K_i^1(x_0 - y)).$$

Φ is a smooth function of y and by Lemma 1 we get

$$|T_i^1(\eta_1 f)(x) - T_i^1(\eta_1 f)(x_0))| \leqslant N_{x_0}(\Phi) f^+(x_0).$$

It suffices to prove that $\sup_{t>0} |N_{x_0}(\Phi)| \leqslant C$ to verify our claim.

By the basic estimates for $\{K^1{}_i\}$ we get that

$$|\Delta_y^\alpha K_i^1(x - y) - \Delta_y^\alpha K_i^1(x_0 - y)| \leqslant \frac{C|x - x_0|}{|x - y|^{n + 2|\alpha|}}.$$

Since $|x - y| \sim |x_0 - y|$ and $|x - x_0| \leqslant t$, we get

$$\sup_{t>0} |N_{x_0}(\Phi)| \leqslant C \sup_{t>0} \sum_{s=0}^{N} \int_{|y - x_0| > 2t} |y - x_0|^{2s} \frac{|x - x_0|}{|x - y|^{n + 2s + 1}}\, dy \leqslant C.$$

We denote by

$$(T_i^1)_* f(x_0) = \sup_{t>0} \left| \int \left(1 - \eta\left(\frac{x_0 - y}{t}\right)\right) K_i^1(x_0 - y) f(y)\, dy \right|$$

the smoothly truncated maximal singular integral of f. By [FS] we get that $\|(T_i^1)_* f\|_{H^p} \leqslant C\|f\|_{H^p}$. We now use (2.1) to estimate B_4 as follows

$$\begin{aligned}\left| \int \phi_{t,x_0} B_4\, dx \right| \leqslant{}& \sum_i f^+(x_0) g^+(x_0) \int \phi_{t,x_0}\, dx \\ &+ \sum_i f^+(x_0)(T_i^2)_* g(x_0) \int \phi_{t,x_0}\, dx \\ &+ \sum_i (T_i^1)_* f(x_0) g^+(x_0) \int \phi_{t,x_0}\, dx \\ &+ \sum_i (T_i^1)_* f(x_0)(T_i^2)_* g(x_0) \int \phi_{t,x_0}\, dx.\end{aligned}$$

Since the right hand side above is independent of t

$$\begin{aligned}(2.2)\quad \sup_{t>0} \left| \int \phi_{t,x_0} B_4\, dx \right| \leqslant \sum_i \{& f^+(x_0) g^+(x_0) + (T_i^1)_* f(x_0) g^+(x_0) \\ &+ f^+(x_0)(T_i^2)_* g(x_0) + (T_i^1)_* f(x_0)(T_i^2)_* g(x_0)\}.\end{aligned}$$

We raise the right hand side in (2.2) to the power r and we integrate with respect to x_0. We then apply Hölder's inequality to the right hand side with exponents p/r and q/r. We finally get that

$$\begin{aligned}\left\| \sup_{t>0} \left| \int \phi_{t,x_0} B_4 \, dx \right| \right\|_{L^r} &\leqslant \sum_i \{ \| f^+ \|_{L^p} \| g^+ \|_{L^q} + \| (T_i^1)_* f \|_{L^p} \| g^+ \|_{L^q} \\ &\quad + \| f^+ \|_{L^p} \| (T_i^2)_* g \|_{L^q} + \| (T_i^1)_* f \|_{L^p} \| (T_i^2)_* g \|_{L^q} \} \\ &\leqslant C \| f \|_{H^p} \| g \|_{H^q}\end{aligned}$$

and this is the required estimate for the term B_4. We now estimate the term B_3. We have

$$\int \phi_{t,x_0} B_3 \, dx = \int \phi_{t,x_0} B(f, \eta_1 g) \, dx = I_1 + I_2$$

where

$$I_1 = \sum_i \int \phi_{t,x_0}(x) T_i^1(f)(x) (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) \, dx,$$

$$I_2 = \sum_i T_i^2(\eta_1 g)(x_0) \int \phi_{t,x_0}(x) T_i^1(f)(x) \, dx.$$

Let us start with the term I_1. Fix an i and call

$$\Phi(x) = \phi_{t,x_0}(x)(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)).$$

Φ is a smooth function of x. We claim that $N_{x_0}(\Phi) \leqslant C g^+(x_0)$. To prove the claim, we first estimate

$$\begin{aligned}&\left| \frac{\partial^\beta}{\partial x^\beta} (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) \right| \\ &\qquad = \left| \int \eta_1(y) \left(\frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0 - y)) \right) g(y) \, dy \right| \\ &\qquad = \left| \int \Psi(y) g(y) \, dy \right| \\ &\qquad \leqslant N_{x_0}(\Psi) g^+(x_0).\end{aligned}$$

where we set

$$\Psi(y) = \eta_1(y) \left(\frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0 - y)) \right).$$

An easy calculation, using the basic estimates for $\{K_i^2\}$, shows that for $|x - x_0| \leqslant t$,

$$N_{x_0}(\Psi) \leqslant \sum_{s=0}^{N} \int_{|y-x_0|>2t} |y - x_0|^{2s} \left| \Delta^s \left(\eta_1(y) \frac{\partial^\beta}{\partial x^\beta} (K_i^2(x-y) - K_i^2(x_0 - y)) \right) \right| dy$$

$$\leqslant Ct^{-|\beta|}.$$

Therefore

$$\left| \frac{\partial^\beta}{\partial x^\beta} (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)) \right| \leqslant Ct^{-|\beta|} g^+(x_0).$$

Now,

$$N_{x_0}(\Phi) \leqslant \sum_{s=0}^{N} \int |x - x_0|^{2s} |\Delta^s(\phi_{t,x_0}(T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0)))| \, dx$$

$$\leqslant C \sum_{s=0}^{N} \int_{|x-x_0|<t} |x - x_0|^{2s} \sum_{j=0}^{2s} |D_x^{2s-j} \phi_{t,x_0}| \, |D_x^j (T_i^2(\eta_1 g)(x) - T_i^2(\eta_1 g)(x_0))| \, dx$$

where by $D_x^j F$ we denote any derivative of F in x of total order j.

The above can be dominated by

$$C \sum_{s=0}^{N} \int_{|x-x_0|<t} |x - x_0|^{2s} \sum_{j=0}^{2s} t^{-n-(2s-j)} t^{-j} \, dx \, g^+(x_0) \leqslant Cg^+(x_0).$$

This estimate finishes the proof of the claim.

An application of Lemma 1 gives that

$$|I_1| \leqslant \sum_i N_{x_0}(\Phi)(T_i^1 f)^+(x_0) \leqslant C \sum_i g^+(x_0)(T_i^1 f)^+(x_0).$$

Since the right hand side does not depend on t, the $\sup_{t>0} |I_1|$ satisfies the same estimate. Hölder's inequality will give that

$$\left\| \sup_{t>0} |I_1| \right\|_{L^r} \leqslant C \sum_i \|g^+\|_{L^q} \|(T_i^1 f)^+\|_{L^p}$$

$$\leqslant C \sum_i \|g\|_{H^q} \|T_i^1 f\|_{H^p}$$

$$\leqslant C \|f\|_{H^p} \|g\|_{H^q}.$$

We now continue with the term I_2. Term I_2 satisfies the estimate

$$|I_2| \leqslant C \sum_i (T_i^2)_* g(x_0)(T_i^1 f)^*(x_0),$$

where by $(T_i^1 f)^*$ we denote some smooth maximal function of $T_i f$ and the same argument as before will give that

$$\left\| \sup_{t>0} |I_2| \right\|_{L^r} \leqslant C \|g\|_{H^q} \|f\|_{H^p}.$$

This estimate concludes the treatment of term B_3. We deal similarly with term B_2.

3. The Main Term

We are now left with the term B_1 which is the main term of the bilinear operator B. We start by introducing some notation. For $0 \leqslant \delta \leqslant 1$, we denote by Λ_δ the Lipschitz space of all bounded functions f on $\mathbb{R}^n$ with

$$\sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n} |h|^{-\delta} |f(x+h) - f(x)| = \|f\|_{\Lambda_\delta} < +\infty.$$

For $m \in \mathbb{Z}$, $m \geqslant 1$ and $0 \leqslant \delta \leqslant 1$, we denote by Λ_δ^m the space of all bounded functions f on $\mathbb{R}^n$ whose partials of order m exist and are in Λ_δ and whose partials of order up to $m-1$ are bounded. The $\| \ \|_{\Lambda_\delta^m}$ norm of a function is defined as the sum of the $\| \ \|_{\Lambda_\delta}$ norms of its partials of order m. It is easy to see that functions in Λ_δ^m satisfy

$$\left| f(x+h) - \sum_{|\alpha|\leqslant m} \frac{\partial^\alpha f}{\partial x^\alpha}(x) \frac{h^\alpha}{\alpha!} \right| \leqslant \frac{\|f\|_{\Lambda_\delta^m}}{m!} |h|^{m+\delta} \quad \text{for all} \quad x\in\mathbb{R}^n, \quad h\in\mathbb{R}^n.$$

For $m, l \in \mathbb{Z}^+ \cup \{0\}$ and $0 \leqslant \gamma, \delta \leqslant 1$ we define the space $\Lambda_{\gamma,\delta}^{m,l}$ of all bounded functions $b(y,z)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which are in Λ_γ^m as functions of y and in Λ_δ^l as functions of z and satisfy the condition:

$$\begin{aligned} (3.1) \quad \Bigg| b(y+h, z+k) &- \sum_{|\alpha|\leqslant m} \frac{\partial^\alpha}{\partial y^\alpha} b(y, z+k) \frac{h^\alpha}{\alpha!} \\ &- \sum_{|\beta|\leqslant l} \frac{\partial^\beta}{\partial z^\beta} b(y+h, z) \frac{k^\beta}{\beta!} \\ &+ \sum_{|\alpha|\leqslant m} \sum_{|\beta|\leqslant l} \frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial^\beta}{\partial z^\beta} b(y,z) \frac{h^\alpha}{\alpha!} \frac{k^\beta}{\beta!} \Bigg| \leqslant C|h|^{m+\gamma} |k|^{l+\delta} \end{aligned}$$

for all $y, z, k, h \in \mathbb{R}^n$.

This double Lipschitz condition above is equivalent to either one of the statements below

$$(3.2) \qquad |h|^{-m-\gamma} \left(b(y+h, z) - \sum_{|\alpha|\leqslant m} \frac{\partial^\alpha}{\partial y^\alpha} b(y,z) \frac{h^\alpha}{\alpha!} \right)$$

is in Λ_δ^l as a function of z

$$|k|^{-l-\delta}\left(b(y,z+k)-\sum_{|\beta|\le l}\frac{\partial^\beta}{\partial z^\beta}b(y,z)\frac{k^\beta}{\beta!}\right) \tag{3.3}$$

is in Λ_γ^m as a function of y for all $y, z, k, h \in \mathbb{R}^n$, $|h|, |k| \leqslant C$.

We will now state and prove the main lemma needed to estimate term B_1.

Lemma 2. *Suppose that $p, q \leqslant 1$ and let $m = [n(1/p - 1)]$, $l = [n(1/q - 1)]$. Suppose that the compactly supported function $b(y, z)$ is in $\Lambda_{\gamma,\delta}^{m,l}$. For $x_0 \in \mathbb{R}^n$ let*

$$S(f,g)(x_0) = \sup_{t>0}\left|\iint f(y)g(z)\frac{1}{t^{2n}}b\left(\frac{x_0-y}{t},\frac{x_0-x}{t}\right)dy\,dz\right|.$$

Then $S(f, g)$ maps $H^p \times H^q \to L^r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ as long as

$$1 \geqslant \gamma > n\left(\frac{1}{p}-1\right) - \left[n\left(\frac{1}{p}-1\right)\right]$$

and

$$1 \geqslant \delta > n\left(\frac{1}{q}-1\right) - \left[n\left(\frac{1}{q}-1\right)\right].$$

PROOF. Let $a_Q(y)$ be an H^p-atom and $b_R(z)$ be an H^q-atom. We will first estimate $S(a_Q, b_R)(x_0)$. Let y_Q be the center of the cube Q and z_R the center of the cube R. We have the following four basic estimates:

Case 1. $x_0 \in 3Q$ and $x_0 \in 3R$, then

$$S(a_Q, b_R)(x_0) \leqslant C|Q|^{-1/p}\chi_{3Q}(x_0)|R|^{-1/q}\chi_{3R}(x_0).$$

Case 2. $x_0 \in 3Q$ and $x_0 \notin 3R$, then

$$S(a_Q, b_R)(x_0) \leqslant C|Q|^{-1/p}\chi_{3Q}(x_0)\frac{|R|^{-1/q+1+(l+\delta)/n}}{\operatorname{dist}\{x_0, R\}^{n+l+\delta}}.$$

Case 3. $x_0 \notin 3Q$ and $x_0 \in 3R$, then

$$S(a_Q, b_R)(x_0) \leqslant C\frac{|Q|^{-1/p+1+(m+\gamma)/n}}{\operatorname{dist}\{x_0, Q\}^{n+m+\gamma}}|R|^{-1/q}\chi_{3R}(x_0).$$

Case 4. $x_0 \notin 3Q$ and $x_0 \notin 3R$, then

$$S(a_Q, b_R)(x_0) \leqslant C \frac{|Q|^{-1/p+1+(m+\gamma)/n}}{\operatorname{dist}\{x_0, Q\}^{n+m+\gamma}} \frac{|R|^{-1/q+1+(l+\delta)/n}}{\operatorname{dist}\{x_0, R\}^{n+l+\delta}}.$$

We indicate how to prove the basic estimates.

In Case 1 we just use the L^∞ bounds for atoms.

In Case 2 we need the following estimate which can be easily proved by integrating (3.3) over the y-support of b.

$$\int \left| b(y, z+k) - \sum_{|\beta| \leq l} \frac{\partial^\beta b}{\partial z^\beta}(y, z) \frac{k^\beta}{\beta!} \right| dy \leqslant C|k|^{l+\delta} \quad \text{for all} \quad z, k \in \mathbb{R}^n. \tag{3.4}$$

Using the fact that b_R has moments up to order l vanishing we have

$$\begin{aligned} &\iint a_Q(y) b_R(z) \frac{1}{t^{2n}} b\left(\frac{x_0 - y}{t}, \frac{x_0 - z}{t}\right) dy\, dz \\ &\quad = \int a_Q(y) \int b_R(z) \frac{1}{t^n} \left(b\left(\frac{x_0 - y}{t}, \frac{x_0 - z}{t}\right) \right. \\ &\qquad \left. - \sum_{|\beta| \leq l} \frac{\partial^\beta b}{\partial z^\beta}\left(\frac{x_0 - y}{t}, \frac{x_0 - z_R}{t}\right) \frac{1}{\beta!} \left(\frac{z_R - z}{t}\right)^\beta \right) \\ &\qquad \times \frac{dz}{t^n} dy. \end{aligned} \tag{3.5}$$

By (3.4) the above can be estimated by

$$\begin{aligned} &C|Q|^{-1/p} \chi_{3Q}(x_0) |R|^{-1/q} \frac{1}{t^n} \int_R \left| \frac{z - z_R}{t} \right|^{l+\delta} dz \\ &\qquad \leqslant C|Q|^{-1/p} \chi_{3Q}(x_0) \frac{|R|^{-1/q+1+(l+\delta)/n}}{t^{n+l+\delta}}. \end{aligned}$$

Note that (3.5) is nonzero if

$$t \geqslant \operatorname{dist}\{(x_0, x_0), Q \times R\} \sim \operatorname{dist}\{x_0, R\},$$

since $x_0 \in 3Q$ and $x_0 \notin 3R$. The required estimate for $S(a_Q, b_R)$ in Case 2 follows. Case 3 is similar.

To get the estimate in Case 4 we use the double Lipschitz estimate for b. We have

$$\begin{aligned}
&\iint a_Q(y)b_R(z)\frac{1}{t^{2n}}b\left(\frac{x_0-y}{t},\frac{x_0-z}{t}\right)dy\,dz\\
&\quad=\iint a_Q(y)b_R(z)\frac{1}{t^{2n}}\left\{b\left(\frac{x_0-y}{t},\frac{x_0-z}{t}\right)\right.\\
&\qquad-\sum_{|\alpha|\le m}\frac{\partial^\alpha b}{\partial y^\alpha}\left(\frac{x_0-y_Q}{t},\frac{x_0-z}{t}\right)\frac{1}{\alpha!}\left(\frac{y_Q-y}{t}\right)^\alpha\\
&\qquad-\sum_{|\beta|\le l}\frac{\partial^\beta b}{\partial z^\beta}\left(\frac{x_0-y}{t},\frac{x_0-z_R}{t}\right)\frac{1}{\beta!}\left(\frac{z_R-z}{t}\right)^\beta\\
&\qquad+\sum_{|\alpha|\le m}\sum_{|\beta|\le l}\frac{\partial^\alpha\partial^\beta b}{\partial y^\alpha\partial z^\beta}\left(\frac{x_0-y_Q}{t},\frac{x_0-z_R}{t}\right)\frac{1}{\alpha!}\\
&\qquad\left.\times\left(\frac{y_Q-y}{t}\right)^\alpha\frac{1}{\beta!}\left(\frac{z_R-z}{t}\right)^\beta\right\}dy\,dz.
\end{aligned}\tag{3.6}$$

Apply (3.1) to bound the above by

$$\begin{aligned}
&C|Q|^{-1/p}|R|^{-1/q}\frac{1}{t^{2n}}\int_Q\int_R\left|\frac{y-y_Q}{t}\right|^{m+\gamma}\left|\frac{z-z_R}{t}\right|^{l+\delta}dy\,dz\\
&\qquad\le Ct^{-2n-m-l-\gamma-\delta}|Q|^{-1/p+1+(m+\gamma)/n}|R|^{-1/q+1+(l+\delta)/n}.
\end{aligned}$$

Note that (3.6) is nonzero as long as

$$t\ge\operatorname{dist}\{(x_0,x_0),Q\times R\}\sim\operatorname{dist}\{x_0,Q\}+\operatorname{dist}\{x_0,R\},$$

because $x_0\notin 3Q$ and $x_0\notin 3R$.

Since

$$\begin{aligned}
&\operatorname{dist}\{(x_0,x_0),Q\times R\}^{-2n-m-\gamma-l-\delta}\\
&\qquad\le C\operatorname{dist}\{x_0,Q\}^{-n-m-\gamma}\operatorname{dist}\{x_0,R\}^{-n-l-\delta},
\end{aligned}$$

the required estimate for (3.6) follows immediately.

We have now proved our basic estimates and we continue with the proof of the lemma.

Let $f\in H^p$, $g\in H^q$ be finite sums of atoms

$$f=\sum\lambda_Q a_Q,\qquad g=\sum\mu_R b_R$$

where $\lambda_Q>0$, $\mu_R>0$ and

$$\left(\sum\lambda_Q^p\right)^{1/p}\approx\|f\|_{H^p},\qquad\left(\sum\mu_R^q\right)^{1/q}\approx\|g\|_{H^q}.$$

Bound

$$S(f,g)(x_0) \leqslant \sum_Q \sum_R \lambda_Q \mu_R S(a_Q, b_R)(x_0)$$

by $\Sigma_1(x_0) + \Sigma_2(x_0) + \Sigma_3(x_0) + \Sigma_4(x_0)$ where

$$\Sigma_j(x_0) = \sum_Q \sum_R \lambda_Q \mu_R S(a_Q, b_R)(x_0)$$

and the sum above is taken over all Q and R related to x_0 as in case j above, $1 \leqslant j \leqslant 4$.

We will show that each Σ_j is in $L^r(dx_0)$ with L^r quasinorm bounded by $C\|f\|_{H^p}\|g\|_{H^q}$. Then we can sum on j, $1 \leqslant j \leqslant 4$. Use Hölder's inequality with exponents p/r, q/r to get

$$\left(\int \Sigma_1(x_0)^r\, dx_0\right)^{1/r}$$

$$\leqslant C\left(\int \left(\sum \lambda_Q |Q|^{-1/p} \chi_{3Q}(x_0)\right)^p dx_0\right)^{1/p} \left(\int \left(\sum \mu_R |R|^{-1/q} \chi_{3R}(x_0)\right)^q dx_0\right)^{-1/q}$$

$$\leqslant C\left(\int_{3Q} \sum \lambda_Q^p |Q|^{-1}\, dx_0\right)^{1/p} \left(\int_{3R} \sum \mu_R^q |R|^{-1}\, dx_0\right)^{1/q} \quad \text{(since } p, q \leqslant 1\text{)}$$

$$\leqslant C\left(\sum \lambda_Q^p\right)^{1/p} \left(\sum \mu_R^q\right)^{1/q} \leqslant C\|f\|_{H^p}\|g\|_{H^q}.$$

Similarly

$$\left(\int \Sigma_4(x_0)^r\, dx_0\right)^{1/r}$$

$$\leqslant C\left(\int_{\operatorname{dist}\{x_0, Q\} \leqslant C|Q|^{1/n}} \left(\sum \lambda_Q \frac{|Q|^{-1/p+1+(m+\gamma)/n}}{\operatorname{dist}\{x_0, Q\}^{n+m+\gamma}}\right)^p dx_0\right)^{1/p}$$

$$\left(\int_{\operatorname{dist}\{x_0, R\} \geqslant C|R|^{1/n}} \left(\sum \mu_R \frac{|R|^{-1/q+1+(l+\delta)/n}}{\operatorname{dist}\{x_0, R\}^{n+l+\delta}}\right)^q dx_0\right)^{1/q}$$

$$\leqslant C\left(\int_{\operatorname{dist}\{x_0, Q\} \geqslant C|Q|^{1/n}} \sum \lambda_Q^p \frac{|Q|^{-1+p+(m+\gamma)p/n}}{\operatorname{dist}\{x_0, Q\}^{(n+m+\gamma)p}} dx_0\right)^{1/p}$$

$$\left(\int_{\operatorname{dist}\{x_0, R\} \geqslant C|R|^{1/n}} \sum \mu_R^q \frac{|R|^{-1+q+(l+\delta)q/n}}{\operatorname{dist}\{x_0, R\}^{(n+l+\delta)q}} dx_0\right)^{1/q}$$

$$\leqslant C\left(\sum \lambda_Q^p\right)^{1/p} \left(\sum \mu_R^q\right)^{1/q} \leqslant C\|f\|_{H^p}\|g\|_{H^q},$$

where we used the fact that $(n + m + \gamma)p > n$ and

$$\int_{\operatorname{dist}\{x_0, Q\} \geq C|Q|^{1/n}} \operatorname{dist}\{x_0, Q\}^{-(n+m+\gamma)p} dx_0 \sim |Q|^{(n-(n+m+\gamma)p)/n} \sim |Q|^{1-p-(m+\gamma)p/n}.$$

The same way we can prove that

$$\left(\int \Sigma_2(x_0)^r dx_0\right)^{1/r} \leqslant C\|f\|_{H^p}\|g\|_{H^p}$$

and

$$\left(\int \Sigma_3(x_0)^r dx_0\right)^{1/r} \leqslant C\|f\|_{H^p}\|g\|_{H^q}$$

using a combination of the estimates above. Finally we approximate the general $f \in H^p$ and $g \in H^q$ by finite sums of atoms to finish the proof of the lemma.

We now continue the proof of our theorem by estimating the term B_1. We have

$$\int \phi_{t,x_0} B_1 \, dx = \iint f(y)g(z) \frac{1}{t^{2n}} b\left(\frac{x_0 - y}{t}, \frac{x_0 - z}{t}\right) dy \, dz$$

where we set

$$b(y, z) = \sum_i \eta(y)\eta(z) \int \phi(\sigma) K_i^1(y - \sigma) K_i^2(z - \sigma) \, d\sigma.$$

To apply Lemma 2 we need to prove that $b(y, z) \in \Lambda_{\gamma,\delta}^{m,l}$ where m, γ, l, δ as in Lemma 2.

Note that the assumption that $B(f, g)$ has moments up to order k vanishing gives that the kernel of $x^\alpha B(f, g)(x)$ is identically zero for all $|\alpha| \leqslant k$, *i.e.*

$$\sum_i \int x^\alpha K_i^1(y - x) K_i^2(z - x) \, dx = 0 \quad \text{for all} \quad y, z \in \mathbb{R}^n.$$

We can therefore write

$$b(y, z) = \eta(x)\eta(y) \sum_i \int \left(\phi(\sigma) - \sum_{|\alpha| \geq k} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y) \frac{(\sigma - y)^\alpha}{\alpha!}\right) K_i^1(y - \sigma) K_i^2(z - \sigma) \, d\sigma.$$

The fact that $b(y, z) \in \Lambda_{\gamma,\delta}^{m,l}$ where m, l, γ, δ are as in Lemma 2, will be a consequence of the following two lemmas.

Lemma 3. *Let G be a function on $\mathbb{R}^n$ of class Λ_1^s and let $a(y, \sigma) = G(y - \sigma)$.*

(a) If m, l are non-negative integers such that $m + l = s$ and $\gamma, \delta > 0$ such that $\gamma + \delta = 1$, then $a(y, \sigma)$ is of class $\Lambda_{\gamma,\delta}^{m,l}$.

(b) If m, l are integers such that $m + l = s - 1$ then $a(y, \sigma)$ is of class $\Lambda_{1,1}^{m,l}$.

Lemma 4. *Let K be a convolution Calderón-Zygmund kernel on $\mathbb{R}^n$. If $a(y, \sigma) \in \Lambda_{\gamma,\delta}^{m,l}$ then*

$$b(y, z) = \int a(y, \sigma) K(z - \sigma)\, d\sigma \in \Lambda_{\gamma,\delta}^{m,l}.$$

PROOF. We denote by $\partial^\alpha G$ the partial derivative of G of order $\alpha = (\alpha_1, \ldots, \alpha_n)$. To prove Lemma 3 (*a*), by (3.2), it suffices to show that the function

$$F(\sigma) = \frac{1}{|h|^{m+\gamma}} \left(a(y + h, \sigma) - \sum_{|\alpha| \le m} \frac{\partial^\alpha a}{\partial y^\alpha}(y, \sigma) \frac{h^\alpha}{\alpha!} \right)$$

$$= \frac{1}{|h|^{m+\gamma}} \left(g(y + h - \sigma) - \sum_{|\alpha| \le m} \partial^\alpha G(y - \sigma) \frac{h^\alpha}{\alpha!} \right)$$

is in Λ_δ^l in the variable σ uniformly in $y \in \mathbb{R}^n$. This will be a consequence of the following two observations:

(1) $F(\sigma)$ is in Λ_0^l with norm $\leqslant C|h|^{1-\gamma}$.

(2) $F(\sigma)$ is in Λ_1^l with norm $\leqslant C|h|^{-\gamma}$.

Interpolation will then give that $F(\sigma)$ is in Λ_δ^l with norm $\leqslant C$. (Recall $\gamma + \delta = 1$.)

Both observations follow from Taylor's theorem. For some $\xi_{y,h}$ between y and $y + h$ we have that

$$F(\sigma) = |h|^{-m-\gamma} \left(\sum_{|\alpha| = m} (\partial^\alpha G(\xi_{y,h} - \sigma) - \partial^\alpha G(y - \sigma)) \frac{h^\alpha}{\alpha!} \right)$$

and therefore for a fixed β with $|\beta| = l$ we have

$$\partial^\beta F(\sigma) = |h|^{-m-\gamma} \left(\sum_{|\alpha| = m} (\partial^{\alpha+\beta} G(\xi_{y,h} - \sigma) - \sigma^{\alpha+\beta} G(y - \sigma)) \frac{h^\alpha}{\alpha!} \right).$$

Since $|\alpha + \beta| = m + l = s$ and since $\partial^\sigma G$ is in Λ_1 if $|\sigma| = s$ it follows that $\partial^{\alpha+\beta} G$ is in Λ_1 and thus $(\partial^\beta F)(\sigma)$ is in $\Lambda_0 = L^\infty$, with norm

$$\leqslant C|h|^{-\gamma} \sum_{|\alpha| = s} \|\partial^\alpha G\|_{\Lambda_1} |\xi_{y,h} - y| \leqslant C|h|^{1-\gamma}.$$

Also, from the translation invariance of the Lipschitz norms it follows that $\partial^{\alpha+\beta}G(\xi_{y,h}-\sigma)$ and $\partial^{\alpha+\beta}G(y-\sigma)$ are in Λ_1 in σ, and therefore the function $\partial^\beta F(\sigma)$ is in Λ_1 in σ with norm $\leqslant C|h|^{-\gamma}$. We proved that the arbitrary partial derivative $\partial^\beta F$ of F of order l is in Λ_0 with norm $\leqslant C|h|^{1-\gamma}$ and in Λ_1 with norm $\leqslant C|h|^{-\gamma}$. This concludes the proofs of the observations. Note that both norm estimates are independent of y.

Part (b) of Lemma 3 follows by a similar argument. We need to show that the function

$$F(\sigma)=\frac{1}{|h|^{m+1}}\left(a(y+h,\sigma)-\sum_{|\alpha|\leq m}\frac{\partial^\alpha a}{\partial y^\alpha}(y,\sigma)\frac{h^\alpha}{\alpha!}\right)$$

$$=\frac{1}{|h|^{m+1}}\left(G(y+h-\sigma)-\sum_{|\alpha|\leq m}\partial^\alpha G(y-\sigma)\frac{h^\alpha}{\alpha!}\right)$$

is of class Λ_1^l in the variable σ uniformly in $y\in\mathbb{R}^n$. For some $\xi'_{y,h}$ between y and $y+h$, we have that

$$F(\sigma)=\frac{|h|^{-m-1}}{(m+1)!}\sum_{|\alpha|=m+1}\partial^\alpha G(\xi'_{y,h}-\sigma)\frac{h^\alpha}{\alpha!}.$$

Fix a multi-index β with $|\beta|=l$. Clearly

$$\partial^\beta F(\sigma)=\frac{|h|^{-m-1}}{(m+1)!}\sum_{|\alpha|=m+1}\partial^{\alpha+\beta}G(\xi'_{y,h}-\sigma)\frac{h^\alpha}{\alpha!}.$$

Since in this case $|\alpha+\beta|=m+1+l=s$ and the partials of G of order s are in Λ_1 it follows that $\partial^\beta F$ is in Λ_1 with norm independent of y. Therefore F is in Λ_1^l with norm independent of y and this concludes the proof of Lemma 3.

We now indicate how to prove Lemma 4. It is a well-known fact that convolution Calderón-Zygmund operators map the Lipschitz spaces Λ_γ^m into themselves. If $a(y,\sigma)\in\Lambda_{\gamma,\delta}^{m,l}$ then by (3.2) the function

$$|h|^{-m-y}\left\{a(y+h,\sigma)-\sum_{|\alpha|<m}\frac{\partial^\alpha a}{\partial y^\alpha}(y,\sigma)\frac{h^\alpha}{\alpha!}\right\}$$

is in Λ_δ^l in the variable σ. Convolution with K in σ will give that

$$|h|^{-m-\gamma}\left\{b(y+h,z)-\sum_{|\alpha|\leq m}\frac{\partial^\alpha b}{\partial y^\alpha}(y,z)\frac{h^\alpha}{\alpha!}\right\}\in\Lambda_\delta^l\quad\text{in}\quad z$$

and by (3.2) again we get that $b(y,z)\in\Lambda_{\gamma,\delta}^{m,l}$. This finishes the proof of Lemma 4.

To conclude the proof of our theorem, it suffices to check that the functions

$$a_i(y,\sigma)=\left\{\phi(\sigma)-\sum_{|\alpha|\le k}\frac{\partial^\alpha\phi}{\partial y^\alpha}(y)\frac{(\sigma-y)^\alpha}{\alpha!}\right\}K_i^1(y-\sigma)$$

are in $\Lambda^{m,l}_{\gamma,\delta}$. Then Lemma 4 will give that $b(y,z)$ is in $\Lambda^{m,l}_{\gamma,\delta}$ also.

We write

$$a_i(y,\sigma)=\sum_{|\alpha|=k+1}\phi^{(\alpha)}(\xi_{\sigma,y})\frac{(\sigma-y)^\alpha}{\alpha!}K_i^1(y-\sigma).$$

Fix i and α with $|\alpha|=k+1$. It suffices to show that

$$\phi^{(\alpha)}(\xi_{\sigma,y})(\sigma-y)^\alpha K_i^1(y-\sigma)\in\Lambda^{m,l}_{\gamma,\delta}.$$

Since $\phi^{(\alpha)}(\xi_{\sigma,y})$ is a smooth function of y and σ it is enough to show that

$$(\sigma-y)^\alpha K_i^1(y-\sigma)\in\Lambda^{m,l}_{\gamma,\delta}$$

for suitable m, l, γ, δ.

As a function of the variable x, the function $x^\alpha K_i^1(x)$ is smooth everywhere except possibly at zero and it behaves like $|x|^{k+1}|x|^{-n}=|x|_1^{k-n}$ as $|x|\to 0$. Therfore, $x^\alpha K_i^1(x)\in\Lambda^{k-n+1}$. We are going to apply Lemma 3 with $G(x)=x^\alpha K_i^1(x)$.

Let $m=\left[n\left(\frac{1}{p}-1\right)\right]$, $l=\left[n\left(\frac{1}{q}-1\right)\right]$. Since $\frac{n+k}{n}\leqslant\frac{1}{r}\leqslant\frac{n+k+1}{n}$ it follows that $m+l$ is either $k-n$ or $k-n-1$. If $m+l=k-n$ just pick $\gamma>n\left(\frac{1}{p}-1\right)-m$ and $\delta>n\left(\frac{1}{q}-1\right)-l$ such that $\gamma+\delta=1$ (this is possible since $1>n\left(\frac{1}{r}-2\right)-m-l\Big)$ and apply part (a) of Lemma 3 with $s=k-n$. If $m+l=k-n-1=s-1$, apply part (b) of Lemma 3.

It follows that $(\sigma-y)^\alpha K_i^1(y-\sigma)$ is in $\Lambda^{m,l}_{\gamma,\delta}$ where γ and δ satisfy the hypotheses of Lemma 1. The same is true for $a_i(y,\sigma)$ and therefore for $b(y,z)$ by Lemma 4.

4. Applications and Examples

The vanishing moments properties of the bilinear operators

$$B(f,g)=\sum_j(T_j^1f)(T_j^2g)$$

can be written in terms of relations involving multipliers. Let m_j^1 and m_j^2 be the multipliers corresponding to the Calderón-Zygmund operators T_j^1 and T_j^2. $B(f, g)$ has mean value zero if and only if $\widehat{B(f,g)}(0) = 0$, *i.e.*

$$\sum (\widehat{T_j^1 f} * \widehat{T_j^2 g})(0) = 0.$$

This is equal to

$$\sum_j \int m_j^1(-\xi)\hat{f}(-\xi)m_j^2(\xi)\hat{g}(\xi)\,d\xi = 0$$

and since f and g are arbitrary the above is equivalent to the statement

$$\sum_j m_j^1(-\xi)m_j^2(\xi) = 0 \quad \text{for all} \quad \xi \neq 0.$$

Similar reasoning shows that B has two moments zero if and only if the following identities hold

$$\sum_j m_j^1(-\xi)m_j^2(\xi) = 0 \quad \text{for all} \quad \xi \neq 0,$$

$$\sum_j m_j^1(-\xi)\left(\frac{\partial}{\partial y_i} m_j^2\right)(\xi) = 0 \quad \text{for all} \quad \xi \neq 0, \qquad i = 1, 2, \ldots, n.$$

The second identity can be replaced by

$$\sum_j \left(\frac{\partial}{\partial \xi_i} m_j^1\right)(-\xi)m_j^2(\xi) = 0 \quad \text{for all } i$$

in view of the first identity and the product rule.

Generalizing the above, we get that B has all moments of order up to and including k vanishing if and only if

$$\sum_j m_j^1(-\xi)\left(\frac{\partial^m}{\partial \xi_i^m} m_j^2\right)(\xi) = 0 \tag{4.1}$$

holds for all $\xi \neq 0$, $i = 1, 2, \ldots, n$; $m = 0, 1, \ldots, k$. The identities above give us an easy way to decide whether a bilinear operator has vanishing moments. For example, using (4.1), it is trivial to check that the bilinear operator $\tilde{J}(f, g) = R_1 f R_2 g - R_2 f R_1 g$ has integral zero. To include an example, we check that the bilinear operator $\tilde{H}(f, g) = (R_1^2 f)(R_2^2 g) - 2(R_1 R_2 f)(R_2 R_1 g) + (R_2^2 f)(R_1^2 g)$ has vanishing first moments. We calculate (4.1) when $m = 1$ and $i = 1$. Let $T_1^1 = R_1^2$, $T_1^2 = R_2^2$, $T_2^1 = -2R_1 R_2$, $T_2^2 = R_2 R_1$, $T_3^1 = R_2^2$, $T_3^2 = R_1^2$ and let $m_1^1(\xi) = -\xi_1^2/|\xi|^2$, $m_1^2(\xi) = -\xi_2^2/|\xi|^2$, $m_2^1(\xi) = 2\xi_1\xi_2/|\xi|^2$, $m_2^2(\xi) = -\xi_1\xi_2/|\xi|^2$, $m_3^1(\xi) = -\xi_2^2/|\xi|^2$, $m_3^2(\xi) = -\xi_1^2/|\xi|^2$ be the corresponding multipliers. Then

$$\sum_{j=1}^{3} m_j^1(-\xi)\frac{\partial}{\partial y_1} m_j^2(\xi) = \left(-\frac{\xi_1^2}{|\xi|^2}\right)\left(\frac{2\xi_1\xi_2^2}{|\xi|^4}\right) + \left(\frac{2\xi_1\xi_2}{|\xi|^2}\right)\left(\frac{-\xi_2^3+\xi_1^2\xi_2}{|\xi|^4}\right)$$
$$+\left(-\frac{\xi_2^2}{|\xi|^2}\right)\left(-\frac{2\xi_1\xi_2^2}{|\xi|^4}\right) = 0.$$

An example of an operator with two vanishing moments is given on $\mathbb{R}^1$ by the bilinear map $B(f,g) = fg - (Cf)(Cg) + (Sf)(Sg)$, where the operators Sf and Cf are defined on the Fourier side by

$$\widehat{Sf}(\xi) = \sin(\log|\xi|)\hat{f}(\xi)$$

and

$$\widehat{Cf}(\xi) = \cos(\log|\xi|)\hat{f}(\xi).$$

One can easily check using (4.1) that B has integral and first moments zero and hence by our theorem it maps $H^p \times H^q \to H^r$ for $p, q > 1$ and $1/2 \geqslant r = (p^{-1} + q^{-1})^{-1} > 1/3$.

Examples of bilinear operators with moments of all orders vanishing are given on $\mathbb{R}^1$ by the maps

$$D_1(f,g) = fg - (Hf)(Hg)$$
$$D_2(f,g) = f(Hg) + (Hf)g$$

where H is the usual Hilbert transform. It follows from our theorem that D_1 and D_2 map $H^p \times H^q \to H^r$ for all $p, q \leqslant 1$ and r their harmonic mean. D_1 and D_2 are the real and imaginary parts of the holomorphic function $(f + iHf)(g + iHg)$ and they can also be studied through complex analysis.

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Hardy Space Estimates for Multilinear Operators, II

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Abstract. We continue the study of multilinear operators given by products of finite vectors of Calderón-Zygmund operators. We determine the set of all $r \leqslant 1$ for which these operators map products of Lebesgue spaces $L^p(\mathbb{R}^n)$ into the Hardy spaces $H^r(\mathbb{R}^n)$. At the endpoint case $r = n/(n + m + 1)$, where m is the highest vanishing moment of the multilinear operator, we prove a weak type result.

0. Introduction

A well-known by now theorem of P. L. Lions says that the derminant of the Jacobian of a function $\mathbb{R}^n \to \mathbb{R}^n$ maps the product of Sobolev spaces $L_1^n \times \cdots \times L_1^n$ into the Hardy space H^1. Coifman, Lions, Meyer and Semmes, [CLMS], went below H^1 by showing that for $p, q > 1$, the Jacobian-determinant maps $L_1^p(\mathbb{R}^2) \times L_1^q(\mathbb{R}^2)$ into $H^r(\mathbb{R}^2)$, where $r^{-1} = p^{-1} + q^{-1}$, as long as $r > 2/3$. Their result can be generalized to give the n-dimensional version that the determinant of the Jacobian maps $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_n}(\mathbb{R}^n)$ into $H^r(\mathbb{R}^n)$, as long as the harmonic mean r of the p_j's is strictly greater than $n/(n + 1)$. In this work we prove a positive result in the endpoint case $r = n/(n + 1)$. We treat more general multilinear operators with vanishing

integral since our methods show that this is the only assumption needed. We also study the case of multilinear operators with higher moments vanishing. The number of vanishing moments is related to the lowest r for which these operators map products of Lebesgue spaces into H^r. If such an operator has all moments of order $\leqslant m$ vanishing, then it maps products of Lebesgue spaces into H^r for $r > n/(n+m+1)$. Also, a weak type estimate holds in the endpoint case $r = n/(n+m+1)$ and no boundedness result holds for $r < n/(n+m+1)$.

1. Statements of Results

Throughout this article, N and K will denote fixed integers $\geqslant 2$. We are given a matrix of sufficiently smooth convolution Calderón-Zygmund kernels $\{K_i^j\}_{i=1,j=1}^{N\quad K}$ on $\mathbb{R}^n$. We call T_i^j the associated Calderón-Zygmund operator. We denote by $L(f_1,\dots,f_K)$ the K-linear operator

$$L(f_1,\dots,f_K) = \sum_{i=1}^{N} (T_i^1 f_1)\cdots(T_i^K f_K) \tag{1.1}$$

originally defined for smooth compactly supported functions $f_1,\dots,f_K$. For $p \leqslant 1$, we denote by H^p the usual real variable Hardy space as defined in [S] or [FST], *i.e.* the set of all distributions f on $\mathbb{R}^n$ for which the maximal function $\sup_{t>0} |\phi_t * f(x)|$ is in L^p, where

$$\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t^n}\right)$$

and ϕ is smooth, nonzero and compactly supported. We also denote by $H^{p,\infty}$ the weak H^p as defined in [FRS] (or [FSO] in the case $p = 1$), *i.e.* the set of all f in $\mathbb{R}^n$ for which the maximal function $\sup_{t>0} |\phi_t * f(x)|$ is in weak L^p. The weak L^p (quasi)norm of this maximal function is by definition the $\| \ \|_{H^p,\infty}$ (quasi)norm of f.

Our first result concerns the general multilinear operators L of the type above and it presents very clearly the method that will be used in this article. Note however, that there is an unpleasant restriction about the exponents that will be lifted later.

Theorem I. *Assume that for all* $(f_1,\dots,f_K) \in (C_0^\infty)^K$, *the K-linear operator L satisfies*

$$\int L(f_1,\dots,f_K)\,dx = 0.$$

Suppose that $p_1, \ldots, p_K > 1$ *are given and let* $r = (p_1^{-1} + \cdots + p_K^{-1})^{-1}$ *be their harmonic mean. Assume that the harmonic mean of any proper subset of the* p_j*'s is greater than* 1. *Then*

(1) *If* $r > 1$, L *maps* $L^{p_1} \times \cdots \times L^{p_K} \to L^r$.
(2) *If* $1 \geqslant r > n/(n+1)$, L *maps* $L^{p_1} \times \cdots \times L^{p_K} \to H^r$.
(3) *If* $r = n/(n+1)$, L *maps* $L^{p_1} \times \cdots \times L^{p_K} \to H^{r,\infty}$.

Next, we treat the case of multilinear operators with vanishing higher moments. The significance of the number of vanishing moments is that it gives the lowest r for which such operators map into H^r. We also get rid of the assumption that the harmonic mean of any subset of the p_j's is always greater than 1. We are assuming however, that the K-linear operators L have a special form.

When $K = 2$, we consider operators L of the general form (1.1), *i.e.* inner products of two vectors of Calderón-Zygmund operators. For $K \geqslant 3$, we consider operators built inductively as follows:

We are assuming that for any j there exist

$$\Lambda_i^j = \Lambda_i^j(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_K)$$

$(K-1)$-linear operators already defined by the induction hypothesis with the same number of vanishing moments, such that

$$L(f_1, \ldots, f_K) = \sum_{i=1}^{M} T_i^j(f_j)\Lambda_i^j(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_K). \tag{1.2}$$

Condition (1.2) essetially says that the multilinear operators L look like determinants of matrices. They are built by induction starting from arbitrary bilinear operators as the ones in Theorem I (when $K = 2$) and at each stage they look like sums of products of multilinear operators of one smaller degree multiplied by a Calderón-Zygmund operator. These sums have a certain degree of symmetry because it follows from a repeated application of (1.2) that for each $j_1, \ldots, j_l$, there exist $(K-l)$-linear operators $\Lambda_i^{j_1,\ldots,j_l}$ with the same number of vanishing moments such that

$$L(f_1, \ldots, f_K) = \sum_i (T_i^{j_1} f_{j_1}) \cdots (T_i^{j_1} f_{j_l}) \Lambda_i^{j_1,\ldots,j_l} \quad \text{(remaining } f_j\text{'s)}.$$

In most applications we have in mind, the multilinear operators have this form, for example determinants of matrices.

In the case of bilinear operators, $K = 2$, there are no additional assumptions about the operators L and this is why we state and prove this case separately. Also, this case is going to serve as the first step of an inductive argument that will be used later.

Theorem IIA. *Assume that for some* m, $0 \leqslant m \leqslant n-1$ *and for all* $f, g \in C_0^\infty(\mathbb{R}^n)$ *the bilinear operator*

$$B(f,g) = \sum_{i=1}^{N} (T_i^1 f)(T_i^2 g)$$

satisfies

$$\int x^\alpha B(f,g)\,dx = 0 \quad \text{for all multi-indices } \alpha \text{ with } |\alpha| \leqslant m.$$

Suppose that $p, q > 1$ *are arbitrary and let* $r = (p^{-1} + q^{-1})^{-1}$ *be their harmonic mean. Then*

(1) *If* $r > 1$, B *maps* $L^p \times L^q \to L^r$.
(2) *If* $1 \geqslant r > n/(n+m+1)$, B *maps* $L^p \times L^q \to H^r$.
(3) *If* $r = n/(n+m+1)$, B *maps* $L^p \times L^q \to H^{r,\infty}$.

Next, we generalize Theorem IIa for K-linear operators of the form (1.2) and for these type of operators we do not have any additional assumption about the p_j's

Theorem IIB. *Asume that for some* m, $0 \leqslant m \leqslant n(K-1)-1$, *and for all* $f_j \in C_0^\infty(\mathbb{R}^n)$ *the* K*-linear operator* $L(f_1, \ldots, f_K)$ *has the form* (1.2), *where each* Λ_i^j *satisfies*

$$\int x^\alpha \Lambda_i^j\,dx = 0 \quad \text{for all multi-indices } \alpha \text{ with } |\alpha| \leqslant m.$$

Suppose that $p_1, \ldots, p_K > 1$ *are arbitrary and let*

$$r = \left(\sum_k p_k^{-1}\right)^{-1}$$

be ther harmonic mean. Then

(1) *If* $r > 1$, L *maps* $L^{p_1} \times \cdots \times L^{p_K} \to L^r$.
(2) *If* $1 \geqslant r > n/(n+m+1)$, L *maps* $L^{p_1} \times \cdots \times L^{p_K} \to H^r$.
(3) *If* $r = n/(n+m+1)$, L *maps* $L^{p_1} \times \cdots \times L^{p_K} \to H^{r,\infty}$.

Remarks.

(*a*) The assumption $m \leqslant n(K-1)-1$ is necessary in Theorem II, since otherwise $r = n/(n+m+1) < 1/K$ which would contradict that $p_j > 1$.
(*b*) The hypothesis that the harmonic mean of any subset of the p_j's is greater than 1 seems to be necessary in conclusions (2) and (3) of Theorem I. It is obviously not needed in conclusion 1) of Theorem I and it is always automatically satisfied when $r = 1$ or when $K = 2$. This condition imposes

an upper bound on the degree K of multilinearity of the K-linear operator L. For, let $p_j = p > 1$ and let $r > 1$ be the harmonic mean of the p_j's. Then $Kr = p$. The assumption on the harmonic mean of any subset of the p_j's gives $p/(K-1) > 1$. We conclude that $K < 1/(1-r)$ which is a restriction on the size of K. Note, however, that when $r = 1$ there is no upper bound on K nor any restriction about the exponents and our theorem implies for example, that any K-linear operator as above with mean value zero maps $L^{p_1} \times \cdots \times L^{p_K} \to H^1$ when $\sum p_j^{-1} = 1$.

(*c*) The vanishing integral hypothesis for L in Theorem I can be relaxed to the milder condition that *for all* f_1 smooth with compact support and *for some* $f_2, \ldots, f_K$ in the corresponding Lebesgue spaces the integrals $\int L(f_1, f_2, \ldots, f_K)\,dx$ vanish. Then conclusion (2) of Theorem I will be that the operator $g \to L(g, f_2, \ldots, f_K)$ maps L^{p_1} to H^r with norm not larger than a constant times the product of the L^{p_j} norms of the f_j's, $j = 2, \ldots, K$. Conclusion (3) of Theorem I will be similar.

2. Proof of Theorem I

We fix $p_1, \ldots, p_K > 1$ and we let r be their harmonic mean. Clearly only the case $r \leqslant 1$ is interesting because the case $r > 1$ is just Hölder's inequality together with the L^p boundedness of Calderón-Zygmund operators. Fix a smooth compactly supported function ϕ in $\mathbb{R}^n$, an $x_0 \in \mathbb{R}^n$ and define

$$\phi_{t,x_0}(x) = \frac{1}{t^n}\phi\left(\frac{x - x_0}{t}\right).$$

Without loss of generality we may assume that ϕ is supported in $|x| \leqslant 1$. We need to show that

$$\sup_{t>0}\left|\int \phi_{t,x_0} L(f_1, \ldots, f_K)\,dx\right|$$

is in L^r when $r > n/(n+1)$ and in $L^{r,\infty}$ when $r = n/(n+1)$. We also fix a smooth cut-off $\eta(x)$ such that $\eta \equiv 1$ on $|x| < 2$ and supported in $|x| < 4$. We call for simplicity

$$\eta_0(x) = \eta\left(\frac{x_0 - x}{t}\right) \quad \text{and} \quad \eta_1(x) = 1 - \eta_0(x).$$

The reader should remember the dependence of η_0, η_1 on t. We now decompose

$$L(f_1, \ldots, f_K) = L_0 + L_1 + \cdots + L_{K+1},$$

where

$$L_0 = L(\eta_0 f_1, \eta_0 f_2, \dots, \eta_0 f_K)$$

$$L_1 = \sum_{j=1}^{K} L(f_1, \dots, \eta_1 f_j, \dots, f_K)$$

$$L_2 = -\sum_{\substack{1 \le j,l \le K \\ j<l}} L(f_1, \dots, \eta_1 f_j, \dots, \eta_1 f_l, \dots, f_K)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$L_K = (-1)^{K+1} L(\eta_1 f_1, \eta_1 f_2, \dots, \eta_1 f_K)$$

In each L_u above exactly u functions among the f_j's are multiplie by η_1 and the remaining are left intact. To get this decomposition of L we expand

$$L(\eta_0 f_1, \dots, \eta_0 f_K) = L(f_1 - \eta_1 f_1, \dots, f_K - \eta_K f_K)$$

and then we solve for $L(f_1, \dots, f_K)$.

Note that for any fixed i, k and any x such that $|x - x_0| \leqslant t$ we have

$$\begin{aligned} \sup_{t>0} |T_i^k(\eta_1 f)(x) - T_i^k(\eta_1)(x_0)| &\leqslant \sup_{t>0} \left| \int (K_i^k(x-y) - K_i^k(x_0 - y))\eta_1(y) f(y)\, dy \right| \\ &\leqslant C \sup_{t>0} \int_{|y-x_0| \ge t} |x - x_0|\, |y - x_0|^{-n-1} |f(y)|\, dy \\ &\leqslant C|f|^*(x_0) \end{aligned}$$

where by $g^*(x_0)$ we denote the Hardy-Littlewood maximal function of g at the point x_0. We also use the notation $(T_i^j)_*$ for the maximal truncated operator of T_i^j. The term L_0 is the main term term of the decomposition and is treated last. We begin with term L_1. We write as

$$\sum_{j=1}^{K} L(f_1, \dots, (\eta_1 f_j)(x) - (\eta_1 f_j)(x_0), \dots, f_K) + \sum_{j=1}^{K} L(f_1, \dots, (\eta_1 f_j)(x_0), \dots, f_K).$$

We then have

$$\begin{aligned} \sup_{t>0} \left| \int \phi_{t,x_0} L_1\, dx \right| &\leqslant \sum_{j=1}^{K} \sum_{i=1}^{N} \sup_{t>0} \int |\phi_{t,x_0}| \prod_{\substack{1 \le k \le K \\ k \ne j}} |T_i^k f_k| (|T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0)| \\ &\qquad + |T_i^j(\eta_1 f_j)(x_0)|)\, dx \qquad (2.1) \\ &\leqslant C \sum_{j=1}^{K} \sum_{i=1}^{N} \left(\prod_{\substack{1 \le k \le K \\ k \ne j}} |T_i^k f_k| \right)^*(x_0) \{ |f_j^*(x_0)| + (T_i^j)_* f_j(x_0) \}. \end{aligned}$$

Define σ_j by $\sigma_j^{-1} + p_j^{-1} = r^{-1}$. By Hölder's inequality the L^r norm in x_0 of the last term in (2.1) is bounded by

$$C \sum_{j=1}^{K} \sum_{i=1}^{N} (\| |f_j|^* \|_{L^{p_j}} + \| (T_i^j f_j)^* \|_{L^{p_j}}) \left\| \left(\prod_{\substack{1 \le k \le K \\ k \ne j}} |T_i^k f_k| \right)^* \right\|_{L^{\sigma_\sigma}}$$

$$\leqslant C \sum_{j=1}^{K} \sum_{i=1}^{N} \|f_j\|_{L^{p_j}} \left\| \prod_{\substack{1 \le k \le K \\ k \ne j}} |T_i^k f_k| \right\|_{L^{\sigma_j}}$$

$$\leqslant C \sum_{j=1}^{K} \sum_{i=1}^{N} \|f_j\|_{L^{p_j}} \prod_{\substack{1 \le k \le K \\ k \ne j}} \|T_i^k f_k\|_{L^{p_k}}$$

$$\leqslant C \sum_{j=1}^{K} \|f_j\|_{L^{p_j}} \prod_{\substack{1 \le k \le K \\ k \ne j}} \|f_k\|_{L^{p_k}}$$

$$= C \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}.$$

We conclude that the L^r (quasi)norm in x_0 of $\sup_{t>0} \left| \int \phi_{t,x_0} L_1 \, dx \right|$ is bounded by $C \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}$ and that the measure of the set

$$\left\{ x_0 : \sup_{t>0} \left| \int \phi_{t,x_0} L_1 \, dx \right| > \lambda \right\}$$

is bounded by $C\lambda^{-r} \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}^r$.

Term L_2 is treated similarly. First write

$$L_2 = L_{21} + L_{22} + L_{23} + L_{24}$$

where

$$L_{21} = - \sum_{\substack{1 \le j,l \le K \\ j<l}} L(f_1, \ldots, (\eta_1 f_j)(x) - (\eta_1 f_j)(x_0), \ldots, (\eta_1 f_l)(x) - (\eta_1 f_l)(x_0), \ldots, f_K),$$

$$L_{22} = - \sum_{\substack{1 \le j,l \le K \\ j<l}} L(f_1, \ldots, (\eta_1 f_j)(x_0), \ldots, (\eta_1 f_l)(x) - (\eta_1 f_l)(x_0), \ldots, f_K),$$

$$L_{23} = - \sum_{\substack{1 \le j,l \le K \\ j<l}} L(f_1, \ldots, (\eta_1 f_j)(x) - (\eta_1 f_j)(x_0), \ldots, (\eta_1 f_l)(x_0), \ldots, f_K),$$

$$L_{24} = - \sum_{\substack{1 \le j,l \le K \\ j<l}} L(f_1, \ldots, (\eta_1 f_j)(x_0), \ldots, (\eta_1 f_l)(x_0), \ldots, f_K).$$

Same reasoning as before will show that any term L_{2u}, $u = 1, 2, 3, 4$ satisfies the following estimate

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u}\,dx\right| \leqslant C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \left(\prod_{\substack{1\le k\le K\\ k\neq j,l}} |T_i^k f_k|\right)^*(x_0)(C_j f_j)(x_0)(C_l f_l)(x_0)$$

where each $C_j f_j$ is either $|f_j|^*$ or $(T_i^j)_* f$ and therefore $\|C_j f_j\|_{L^{p_j}} \leqslant C\|f_j\|_{L^{p_j}}$. Now define σ_{jl} by $\sigma_{jl}^{-1} + p_j^{-1} + p_l^{-1} = 1$. Hölder's inequality gives that for each $u = 1, 2, 3, 4$

$$\begin{aligned}
&\left\|\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u}\,dx\right|\right\|_{L^r}\\
&\qquad\leqslant C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \left\|\left(\prod_{\substack{1\le k\le K\\ k\neq j,l}} |T_i^k f_k|\right)^*\right\|_{L^{\sigma_{jl}}} \|C_j f_j\|_{L^{p_j}} \|C_l f_l\|_{L^{p_l}}\\
&\qquad\leqslant C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \left\|\prod_{\substack{1\le k\le K\\ k\neq j,l}} |T_i^k f_k|\right\|_{L^{\sigma_{jl}}} \|f_j\|_{L^{p_j}} \|f_l\|_{L^{p_l}}\\
&\qquad\leqslant C \sum_{\substack{1\le j,l\le K\\ j<l}} \sum_{i=1}^{K} \prod_{\substack{1\le k\le K\\ k\neq j,l}} \|T_i^k f_k\|_{L^{p_k}} \|f_j\|_{L^{p_j}} \|f_l\|_{L^{p_l}}\\
&\qquad\leqslant C \sum_{\substack{1\le j,l\le K\\ j<l}} \prod_{\substack{1\le k\le K\\ k\neq j,l}} \|f_k\|_{L^{p_k}} \|f_j\|_{L^{p_j}} \|f_l\|_{L^{p_l}}\\
&\qquad\leqslant C \prod_{1\le k\le K} \|f_k\|_{L^{p_k}}.
\end{aligned}$$

We conclude that the L^r (quasi)norm in x_0 of $\sup_{t>0}\left|\int \phi_{t,x_0} L_2\,dx\right|$ is bounded by $C\sum_{k=1}^{K} \|f_k\|_{L^{p_k}}$ and that the measure of the set

$$\left\{x_0 : \sup_{t>0}\left|\int \phi_{t,x_0} L_2\,dx\right| > \lambda\right\}$$

is bounded by $C\lambda^{-r}\prod_{k=1}^{K} \|f_k\|_{L^{p_k}}^r$.

We treat the terms $L_3, L_4, \ldots, L_K$ in a similar way. In particular, we write the term L_K as a sum of 2^K terms of the form $A = L(g_1, \ldots, g_K)$ where each g_j is either $(\eta_1 f_j)(x) - (\eta_1 f_j)(x_0)$ or $(\eta_1 f_j)(x_0)$. Same reasoning as before will show that the maximal function of L_K satisfies

$$\sup_{t>0}\left|\int \phi_{t,x_0} A\,dx\right| \leqslant C \sum_{i=1}^{N} (C_i^1 f_1)(x_0)\cdots(C_i^K f_K)(x_0)$$

where each $C_i^j f_j$ is $|f_j|^* + (T_i^j)_* f_j$. Hölder's inequality gives that

$$\left\| \sup_{t>0} \left| \int \phi_{t,x_0} A \, dx \right| \right\|_{L^r}$$

is bounded by $C \prod \|f_k\|_{L^{p_k}}$. Exactly the same estimate as above holds for the maximal function of L_K and the weak type estimates follow from Chebychev's inequality.

We are now left with the term L_0. This is where we are going to use the assumption that L has mean value zero. We will show that for some $1 < s_j < p_j$ we have

(2.2) When $r > n/(n+1)$, $\displaystyle\sup_{t>0} \left| \int \phi_{t,x_0} L_0 \, dx \right| \leqslant C \prod_{k=1}^{K} ((|f_k|^{s_k})^*(x_0))^{1/s_k}$.

(2.3) When $r = n/(n+1)$, $\displaystyle\sup_{t>0} \left| \int \phi_{t,x_0} L_0 \, dx \right| \leqslant C \prod_{k=1}^{K} ((|f_k|^{p_k})^*(x_0))^{1/p_k}$.

Let us now indicate how (2.2) and (2.3) imply assertions (2) and (3) of Theorem I. To get assertion (2) observe that when $r > n/(n+1)$

$$\begin{aligned} \left\| \sup_{t>0} \left| \int \phi_{t,x_0} L_0 \, dx \right| \right\|_{L^r} &\leqslant C \left\| \prod_{k=1}^{K} ((|f_k|^{s_k})^*(x_0))^{1/s_k} \right\|_{L^r} \\ &\leqslant C \prod_{k=1}^{K} \|((|f_k|^{s_k})^*(x_0))^{1/s_k}\|_{L^{p_k}} \\ &\leqslant C \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}, \end{aligned}$$

where we used above that $p_k/s_k > 1$. We denote by $|A|$ the measure of the set A. To derive conclusion (3) of Theorem I, let

$$\epsilon_0 = \frac{\lambda}{C}, \qquad \epsilon_{K+1} = 1 \quad \text{and} \quad \epsilon_1, \ldots, \epsilon_K > 0$$

be arbitrary. It follows from (2.3) that

$$\left| \left\{ x_0 : \sup \left| \int \phi_{t,x_0} L_0 \, dx \right| > \lambda \right\} \right| \leqslant \sum_{j=1}^{K} \left| \left\{ x : (|f_j|^{p_j})^*(x_0) > \left(\frac{\epsilon_{j-1}}{\epsilon_j} \right)^{p_j} \right\} \right|.$$

By the weak type (1,1) result for the Hardy-Littlewood maximal function we get that the above is bounded by

$$C \sum_{j=1}^{K} \left(\frac{\epsilon_{j-1}}{\epsilon_j} \right)^{-p_j} \int |f_j|^{p_j} \, dx.$$

This expression minimizes in $\epsilon_1, \dots, \epsilon_K > 0$ when all the terms that appear in the sum are equal. This happens when

$$\frac{\epsilon_{j-1}}{\epsilon_j} = \frac{\|f_j\|_{L^{p_j}}^{p_j}(\lambda/C)^r}{\prod \|f_j\|_{L^{p_j}}^r} \quad \text{for all} \quad j = 2, 3, \dots, K.$$

With this choice of ϵ_j's we get the weak type estimate

$$\left|\left\{x_0 : \sup_{t>0}\left|\int \phi_{t,x_0} L_0\, dx\right| > \lambda\right\}\right| \leqslant C\lambda^{-r}\prod \|f_j\|_{L^{p_j}}^r.$$

It remains to prove (2.2) and (2.3). We denote by T^* the adjoint operator of T and by $[\phi_{t,x_0}, (T_i^1)^*]$ the commutator of ϕ_{t,x_0} and $(T_i^1)^*$. Since ϕ is a Lipschitz function of order 1, it follows that

$$\begin{aligned}|[\phi_{t,x_0}, (T_i^1)^*](f)| &\leqslant \left|\int K_i^1(x-y)(\phi_{t,x_0}(x) - \phi_{t,x_0}(y))f(y)\, dy\right| \\ &\leqslant \frac{C}{t^{n+1}}\int \frac{|f(y)|}{|x-y|^{n-1}}\, dy,\end{aligned}$$

and by the Hardy-Littlewood-Sobolev fractional integral theorem we get

$$\|[\phi_{t,x_0}, (T_i^1)^*](f)\|_{L^\sigma} \leqslant Ct^{-n-1}\|f\|_{L^r} \quad \text{when} \quad 1/r - 1/\sigma = 1/n. \tag{2.4}$$

Since $L(f_1, \dots, f_K)$ has integral zero *for all* sufficiently smooth functions f_1, the identity

$$\sum_{i=1}^N (T_i^1)^*\left(\prod_{k=2}^K T_i^k(\eta_0 f_k)\right) \equiv 0$$

justifies the third equality below. We have

$$\begin{aligned}\int \phi_{t,x_0} L_0\, dx &= \sum_{i=1}^N \int \phi_{t,x_0} T_i^1(\eta_0 f_1)\cdots T_i^K(\eta_0 f_K)\, dx \\ &= \sum_{i=1}^N \int \eta_0 f_1 (T_i^1)^*\left(\phi_{t,x_0}\prod_{k=2}^K T_i^k(\eta_0 f_k)\right) dx \\ &= \sum_{i=1}^N \int \eta_0 f_1\left((T_i^1)^*\left(\phi_{t,x_0}\prod_{k=2}^K T_i^k(\eta_0 f_k)\right)\right. \\ &\qquad \left. - \phi_{t,x_0}(T_i^1)^*\left(\prod_{k=2}^K T_i^k(\eta_0 f_k)\right)\right) dx \qquad (2.5) \\ &= \sum_{i=1}^N \int \eta_0 f_1 [\phi_{t,x_0}, (T_i^1)^*](F_i)\, dx\end{aligned}$$

where

$$F_i = \prod_{k=2}^{K} T_i^k(\eta_0 f_k).$$

Apply first Hölder's inequality with exponents p_1 and $p_1' = p_1/(p_1 - 1)$ and then (2.4) with $\sigma = p_1'$ and $\tau = s = (p_2^{-1} + \cdots + p_K^{-1})^{-1}$ the harmonic mean of $p_2, \ldots, p_K$. This is where we use the assumption that $s > 1$. We get

$$\begin{aligned}
|(2.5)| &\leqslant \|\eta_0 f_1\|_{L^{p_1}} \sum_{i=1}^{N} \|[\phi_{t,x_0}, (T_i^1)^*](F_i)\|_{L^{p_1'}} \\
&\leqslant Ct^{-n-1}\|\eta_0 f_1\|_{L^{p_1}} \sum_{i=1}^{N} \|F_i\|_{L^s} \\
&\leqslant Ct^{-n-1}\|\eta_0 f_1\|_{L^{p_1}} \sum_{i=1}^{N} \|T_i^2(\eta_0 f_2)\|_{L^{p_2}} \cdots \|T_i^K(\eta_0 f_K)\|_{L^{p_K}} \\
&\leqslant Ct^{-n-1}\|\eta_0 f_1\|_{L^{p_1}} \|\eta_0 f_2\|_{L^{p_2}} \cdots \|\eta_0 f_K\|_{L^{p_K}} \\
&\leqslant Ct^{-n-1} \prod_{j=1}^{K} ((|f_j|^{p_j})^*(x_0))^{1/p_j} t^{n/p_j} \\
&= C \prod_{j=1}^{K} ((|f_j|^{p_j})^*(x_0))^{1/p_j}
\end{aligned}$$

This establishes (2.3). To prove (2.2) observe that the assumption $r > n/(n+1)$ gives

$$s^{-1} - (p_1')^{-1} = p_2^{-1} + \cdots + p_K^{-1} - (p_1')^{-1} = r^{-1} - 1 < n^{-1}.$$

Therefore for a suitable selection of $s_j < p_j$ we can make the expression $(s_2)^{-1} + \cdots + (s_K)^{-1} - (s_1')^{-1}$ equal to n^{-1}. Then the same argument as before will give that

$$|(2.5)| \leqslant Ct^{-n-1} \prod_{j=1}^{K} ((|f_j|^{s_j})^*(x_0))^{1/s_j} t^{n/s_j} = C \prod_{j=1}^{K} ((|f_j|^{s_j})^*(x_0))^{1/s_j}.$$

The exponent of t above is zero because of the choice of the s_j's. Taking the supremum over all $t > 0$ we obtain (2.2). The proof of Theorem I is now complete.

3. Proof of Theorem IIa

Clearly, we only need to do the case $r \leqslant 1$. Fix a ϕ and η as in Theorem I and split the bilinear operator $B(f, g)$ as the sum of $B_0 + B_{11} + B_{12} + B_2$ where

$$B_0 = B(\eta_0 f, \eta_0 g),$$
$$B_{11} = B(f, \eta_1 g),$$
$$B_{12} = B(\eta_1 f, g),$$
$$B_2 = -B(\eta_1 f, \eta_1 g).$$

The arguments presented in Theorem I will give the required estimates for the terms B_{11}, B_{12} and B_2. (Note the mean value zero assumption was only used in the treatment of term L_0.) It remains to get the required estimates for term B_0 which is the main term of the decomposition .We have

$$\int \phi_{t,x_0} B_0 \, dx = \iint f(y) g(z) b_t(y, z) \, dy \, dz$$

where

$$b_t(y, z) = \eta_0(y)\eta_0(z) \int \sum_{i=1}^{N} K_i^1(x - y) K_i^2(x - z) \phi_{t,x_0} \, dx.$$

The following lemma, whose proof we postpone until the end of this section describes the behavior of $b_t(y, z)$.

Lemma 1. *$b_t(y, z)$ is a smooth function off the diagonal $y = z$ and satisfies the following estimate*

$$|b_t(y, z)| \leqslant C t^{-n-m-1} |y - z|^{m+1-n} \eta\left(\frac{x_0 - y}{t}\right) \eta\left(\frac{x_0 - z}{t}\right)$$

for $|y - z|$ small.

Assuming the lemma we estimate $\left| \int \phi_{t,x_0} B_0 \, dx \right|$ by

$$(3.1) \qquad C t^{-n-m-1} \int_{|y - x_0| \leq 2t} \int_{|z - x_0| \leq 2t} |f(y)\| g(z)\| y - z|^{-n+m+1} \, dy \, dz.$$

We denote by I_{m+1} the potential of order $m + 1$, *i.e.* convolution with the kernel $|x|^{-n+m+1}$ in $\mathbb{R}^n$. Assume first that $m + 1 < n$.

In the case $r > n/(n + m + 1)$ select $p_1 < p$ and $q_1 < q$ such that

$$\frac{1}{q_1} - \frac{1}{p_1'} = \frac{m+1}{n}.$$

This is always possible since the expression

$$\frac{1}{q} - \frac{1}{p'} = \frac{1}{r} - 1$$

is by assumption strictly less than $(m+1)/n$. Hölder's inequality together with the Hardy-Littlewood-Sobolev theorem on fractional integrals give that

$$\begin{aligned}|(3.1)| &\leqslant Ct^{-n-m-1}\|f\chi_{|y-x_0|\leq 2t}\|_{L^{p_1}}\|I_{m+1}(g\chi_{|z-x_0|\leq 2t})\|_{L^{p_1'}}\\ &\leqslant Ct^{-n-m-1}\|f\chi_{|y-x_0|\leq 2t}\|_{L^{p_1}}\|g\chi_{|z-x_0|\leq 2t}\|_{L^{q_1}}\\ &\leqslant Ct^{-n-m-1}t^{n/p_1+n/q_1}((|f|^{p_1})^*(x_0))^{1/p_1}((|g|^{q_1})^*(x_0))^{1/q_1},\end{aligned}$$

where by χ_A we denote the characteristic function of the set A. By the choice of p_1 and q_1, the exponent of t above is equal to zero and we conclude that if $r > n/(n+m+1)$,

$$\sup_{t>0}\left|\int \phi_{t,x_0}B_0\,dx\right| \leqslant C((|f|^{p_1})^*(x_0))^{1/p_1}((|g|^{q_1})^*(x_0))^{1/q_1}. \tag{3.2}$$

In the case $r = n/(n+m+1)$ simply repeat the argument above with $p = p_1$ and $q = q_1$. We get if $r = n/(n+m+1)$

$$\sup_{t>0}\left|\int \phi_{t,x_0}B_0\,dx\right| \leqslant C((|f|^{p})^*(x_0))^{1/p}((|g|^{q})^*(x_0))^{1/q}. \tag{3.3}$$

Conclusions (2) and (3) of Theorem IIa follow as in Theorem I. In fact (3.2) and (3.3) are repetitions of (2.2) and (2.3) in Section 2.

When $m+1 = n$ only the case $r > n/(n+m+1) = 1/2$ can occur. Then (3.2) follows from (3.1) directly from Hölder's inequality.

It remains to prove Lemma 1. We have that

$$b_t(y,z) = \frac{1}{t^{2n}}\,b\left(\frac{x_0-y}{t}, \frac{x_0-z}{t}\right)$$

where

$$b(y,z) = \eta(y)\eta(z)\int \sum_{i=1}^{N} K_i^1(y-\sigma)K_i^2(z-\sigma)\phi(\sigma)\,d\sigma.$$

The estimate for b_t in Lemma 1 is then equivalent to the estimate

$$|b(y,z)| \leqslant C|y-z|^{m+1-n}\eta(y)\eta(z).$$

The vanishing moments assumptions for $B(f,g)$ are equivalent to the conditions

$$\int \sum_i K_i^1(y-\sigma)K_i^2(z-\sigma)\sigma^\alpha\,d\sigma = 0 \quad \text{for all} \quad |\alpha| \leqslant m.$$

We can therefore write $b(y,z) = \eta(y)\eta(z)d(y,z)$ where

$$d(y,z) = \int \sum_i K_i^1(y-\sigma)K_i^2(z-\sigma)\left[\phi(\sigma) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)(\sigma - y)^\alpha\right] d\sigma.$$

It will suffice to show that for $|y|, |z| \leqslant 4$, we have $|d(y,z)| \leqslant |y-z|^{m+1-n}$. Fix a smooth function $\zeta(\sigma)$ on $\mathbb{R}^n$, equal to 1 on $|\sigma| \leqslant 16$ and supported in $|\sigma| \leqslant 32$. Split $d(y,z) = I_1 + I_2$ where

$$I_1 = \int \sum_i K_i^1(\sigma)K_i^2((z-y)-\sigma)\left[\phi(\sigma+y) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\sigma^\alpha\right]\zeta(\sigma)\, d\sigma$$

$$I_2 = \int \sum_i K_i^1(\sigma)K_i^2((z-y)-\sigma)\left[\phi(\sigma+y) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\sigma^\alpha\right](1-\zeta(\sigma))\, d\sigma.$$

Then $\phi(\sigma + y) = 0$ in the integral I_2 and $|\sigma - (z-y)| \sim |\sigma|$. It follows that

$$\begin{aligned}
|I_2| &\leqslant C\sum_i \int_{|\sigma|\ge 16} |\sigma|^{-n}|\sigma-(z-y)|^{-n} \sum_{|\alpha|\le m} \left|\frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\right| |\sigma|^\alpha |1-\zeta(\sigma)|\, d\sigma \\
&\leqslant C\int_{|\sigma|\ge 16} |\sigma|^{-n}|\sigma|^{-n}|\sigma|^m\, d\sigma \\
&\leqslant C \leqslant C|y-z|^{m+1-n}
\end{aligned}$$

since $|y-z|$ is small and $m+1-n \leqslant 0$.

Finally we treat the term I_1. First fix an i and a y and let

$$a_y(\sigma) = K_i^1(\sigma)\zeta(\sigma)\left(\phi(\sigma+y) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \phi}{\partial y^\alpha}(y)\sigma^\alpha\right).$$

a_y is a smooth function away from zero and has compact support. We use the mean value theorem to write

$$a_y(\sigma) = K_i^1(\sigma) \sum_{|\alpha| = m+1} \frac{\partial^\alpha \phi}{\partial y^\alpha}(\xi_{y,y+\sigma})\sigma^\alpha$$

for some $\xi_{y,y+\sigma}$ between y and $y+\sigma$. It follows that

$$a_y(\sigma) = \sum_{|\alpha| = m+1} a_{y,\alpha}(\sigma)$$

where

$$a_{y,\alpha}(\sigma) = K_i^1(\sigma)\frac{\partial^\alpha \phi}{\partial y^\alpha}(\xi_{y,y+\sigma})\sigma^\alpha.$$

Each $a_{y,\alpha}(\sigma) = |\sigma|^{m+1-n}\chi_y(\sigma)$, where $\chi_y(\sigma)$ is homogeneous of degree 0 in σ and bounded above and below on the unit sphere $|\sigma| = 1$. We have the following

Lemma 2. *Let $\psi(\sigma)$ be a compactly supported function in a neighborhood of 0 such that $\psi(\sigma) = B_0|\sigma|^l\chi(\sigma)$ as $|\sigma| \to 0$, for some $-n < l < 0$, B_0 constant and $\chi(\sigma)$ homogeneous of degree 0 in σ and bounded above and below on the unit sphere $|\sigma| = 1$. Then for any Calderón-Zygmund operator T, there is a constant C depending only on T and on χ such that the estimate $|T(\psi)(w)| \leqslant CB_0|w|^l$ holds as $|w| \to 0$.*

The following short proof of Lemma 2 was suggested to me by Peter Jones. Fix a smooth compactly supported function θ equal to 1 on half of its support so that

$$|x|^l \sim \sum_{j \geq 0} 2^{-jl}\theta(2^j x) \quad \text{as} \quad |x| \to 0.$$

Then

$$T(\psi)(w) \sim \sum_{j \geq 0} B_0 2^{-jl} T(\theta\chi)(2^j w).$$

$T(\theta\chi)(w)$ is bounded near zero and has rapid decay as $|w| \to \infty$. Therefore the terms in the sum giving $T(\psi)$ with $2^j \leqslant C/|w|$ contribute

$$\leqslant CB_0 \sum_{j \leq \log(C/|w|)} 2^{-jl} \leqslant CB_0 \leqslant CB_0|w|^l$$

while the terms with with $2^j \geqslant C/|w|$ contribute

$$C_N B_0 \sum_{j \geq \log(C/|w|)} 2^{-jl}(2^j|w|)^{-N} \leqslant CB_0|w|^l$$

as $|w| \to 0$. This finishes the proof of Lemma 2.

We conclude that $(T_i^2(a_{y,\alpha})(z - y))$ is $O(|y - z|^{m+1-n})$ as $|y - z| \to 0$. Since

$$I_1 = \sum_i \sum_{|\alpha| = m+1} (T_i^2 a_{y,\alpha})(z - y)$$

it follows that I_1 satisfies the required estimate $I_1 \leqslant C|y - z|^{m+1-n}$. The proof of Lemma 1 and hence of Theorem IIa are now complete.

5. Proof of Theorem IIb

We will now combine some ideas from Theorems I and IIa to prove Theorem IIb. Again we only need to do the case $r \leqslant 1$. Fix ϕ and η as before and split the K-linear operator $L = L_0 + L_1 + \cdots + L_{K+1}$ as in Theorem I. Already the

treatment of term L_1 presents some differences. First of all for a fixed j define s_j by $s_j^{-1} + p_j^{-1} = r^{-1}$. L_1 is the sum of K terms of the form

$$L(f_1, \dots, (\eta_1 f_j)(x) - (\eta_1 f_j)(x_0), \dots, f_K) + L(f_1, \dots, (\eta_1 f_j)(x_0), \dots, f_K).$$

By Lemma 1 in [CG], for any F in H^p and ψ sufficiently smooth we have that

$$\left| \int F\psi \, dx \right| \leqslant F^+(x_0) N_{x_0}(\psi)$$

for any x_0 where $N_{x_0}(\psi)$ is the norm of ψ as defined in [CG] and F^+ is an L^p function with $\|F^+\|_{L^p} \leqslant C\|F\|_{H^p}$. A computation after Lemma 1 shows that if $\psi(x) = T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0)$, then $\sup_{t>0} N_{x_0}(\psi) \leqslant C|f_j|^*(x_0)$. An application of this fact with $\Lambda_i^j = F$ gives that the maximal function of L_2 satisfies

$$\sup_{t>0} \left| \int \phi_{t,x_0} L_2 \, dx \right|$$

$$\text{(4.1)} \quad \leqslant \sum_{j=1}^{K} \sum_{i=1}^{M} \sup_{t>0} \left| \int \phi_{t,x_0} (T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0) + T_i^j(\eta_1 f_j)(x_0)) \Lambda_i^j \, dx \right|$$

$$\leqslant C \sum_{j=1}^{K} \sum_{i=1}^{M} |f_j|^*(x_0)(\Lambda_i^j)^+(x_0) + (T_i^j)_* f_j(x_0)(\Lambda_i^j)^*(x_0).$$

If $s_j > 1$, the argument in Theorem I applies. Suppose then that $s_j \leqslant 1$. We can assume by induction that

$$\|\Lambda_i^j\|_{H^{s_j}} = \|(\Lambda_i^j)^*\|_{L^{s_j}} \leqslant C \prod_{k \neq j} \|f_k\|_{L^p}.$$

By Hölder's inequality, the L^r norm in x_0 of the last term in (4.1) is bounded by

$$C \sum_{j=1}^{K} \sum_{i=1}^{M} \| |f_j|^* \|_{L^{p_j}} \|(\Lambda_i^j)^+\|_{L^{s_j}} + \|(T_i^j)_* f_j\|_{L^{p_j}} \|(\Lambda_i^j)^*\|_{L^{s_j}}$$

$$\leqslant C \sum_{j=1}^{K} \sum_{i=1}^{M} \|f_j\|_{L^{p_j}} \prod_{\substack{1 \leq k \leq K \\ k \neq j}} \|f_k\|_{L^{p_k}}$$

$$= C \prod_{k=1}^{K} \|f_k\|_{L^{p_k}}.$$

the term L_2 can be treated similarly. A simple computation shows that the N_{x_0} norm of the function

$$\psi = (T_i^j(\eta_1 f_j)(x) - T_i^j(\eta_1 f_j)(x_0))(T_i^l(\eta_1 f_l)(x) - T_i^l(\eta_1 f_l)(x_0))$$

satisfies

$$\sup_{t>0} N_{x_0}(\psi) \leqslant C|f_j|^*(x_0)|f_l|^*(x_0).$$

As in Theorem I, we write term L_2 as $L_2 = L_{21} + L_{22} + L_{23} + L_{24}$. Since the Λ_i^j satisfy property (1.1), for each j, l we can write

$$L = \sum_{i=1}^{R} (T_i^j f_j)(T_i^l f_l)\Lambda_i^{j,l},$$

where each $\Lambda_i^{j,l}$ is a $(K-2)$-linear operator of $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{l-1}, f_{l+1}, \ldots, f_K$. Another application of Lemma 1 in [CG] will give that the maximal function of any term L_{2u}, $u = 1, 2, 3, 4$ satisfies the following estimate

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u}\,dx\right| \tag{4.2}$$
$$\leqslant C \sum_{\substack{1\leq j,l\leq K\\ j<l}} \sum_{i=1}^{M} (C_j f_j)(x_0)(C_l f_l)(x_0)[(\Lambda_i^{j,l})^*(x_0) + (\Lambda_i^{j,l})^+(x_0)]$$

where $C_j f_j = |f_j|^* + (T_i^j)_* f$ and therefore $\|C_j f_j\|_{L^{p_j}} \leqslant C\|f_j\|_{L^{p_j}}$. Now define s_{jl} by $s_{jl}^{-1} + p_j^{-1} + p_l^{-1} = 1$. If $s_{jl} > 1$, the argument in Theorem I establishes the result. If $s_{jl} \leqslant 1$, we can assume by induction that

$$\|\Lambda_i^{j,l}\|_{H^{s_{jl}}} = \|(\Lambda_i^{j,l})^*\|_{L^{s_{jl}}} \leqslant C \prod_{k\neq j,l} \|f_k\|_{L^{p_k}}.$$

By Hölder's inequality, the L^r norm in x_0 of the right hand side in (4.2) is bounded by

$$\left\|\sup_{t>0}\left|\int \phi_{t,x_0} L_{2u}\,dx\right|\right\|_{L^r}$$
$$\leqslant C \sum_{\substack{1\leq j,l\leq K\\ j<l}} \sum_{i=1}^{M} \|C_j f_j\|_{L^{p_j}} \|C_l f_l\|_{L^{p_j}} [\|(\Lambda_i^{j,l})^*\|_{L^{s_{jl}}} + \|(\Lambda_i^{j,l})^+\|_{L^{s_{jl}}}]$$
$$\leqslant C \sum_{\substack{1\leq j,l\leq K\\ j<l}} \sum_{i=1}^{M} \|f_j\|_{L^{p_j}} \|f_l\|_{L^{p_l}} \prod_{1\leq k\leq K} \|f_k\|_{L^{p_k}}$$
$$\leqslant C \prod_{1\leq k\leq K} \|f_k\|_{L^{p_k}}.$$

We conclude that the L^r (quasi)norm of the maximal function of L_2 is bounded by a constant multiple of the product of the L^{p_k} norms of the f_k's.

Clearly, this procedure can go on for terms $L_3, \dots, L_{K-1}$. In the case of the term L_{K-1}, the operators $\Lambda_i^{j_1,\dots,j_{K-1}}$ are bilinear operators as those in Theorem IIa. Finally, as in Theorem I, the terms L_K and L_{K+1} satisfy the following estimate

$$\sup_{t>0}\left|\int \phi_{t,x_0} L_K\,dx\right| + \sup_{t>0}\left|\int \phi_{t,x_0} L_{K+1}\,dx\right| \leqslant C\sum_i (C_i^1 f_1)(x_0)\cdots(C_i^K f_K)(x_0).$$

It follows that the L^r (quasi)norms of the maximal functions of L_K and L_{K+1} are bounded by constant multiples of the product of the $\|f_k\|_{L^{p_k}}$'s.

We are now left with the main term of the decomposition, L_0. We have

$$\int \phi_{t,x_0} L_0\,dx = \int\cdots\int \prod_{k=1}^{K} f_k(y_k)\, b_t(y_1,\dots,y_K)\,dy_K\cdots dy_1 \tag{4.3}$$

where

$$b_t(y_1,\dots,y_K) = \prod_{k=1}^{K}\eta_0(y_k)\int \sum_{i=1}^{N}\prod_{j=1}^{K} K_i^j(y_j - x)\phi_{t,x_0}\,dx.$$

We first do the case $r = n/(n+m+1)$. We find $q_2 > p_2, \dots, q_K > p_K$ such that

$$\sum_{k=2}^{K}\frac{1}{p_k} - \frac{1}{q_k} = \frac{m+1}{n}.$$

Let

$$\delta_k = \frac{n}{m+1}\left(\frac{1}{p_k} - \frac{1}{q_k}\right).$$

The δ_k's are positive numbers and their sum is 1. The following lemma describes the behavior of $b_t(y_1,\dots,y_K)$.

Lemma 3. *$b_t(y_1,\dots,y_K)$ is a smooth function off the planes $y_i = y_j$ and the satisfies the estimate*

$$|b_t(y_1,\dots,y_K)| \leqslant Ct^{-n-m-1}\prod_{k=1}^{K}\eta_0(y_k)\prod_{j=2}^{K}|y_1 - y_j|^{-n+\delta_j(m+1)}.$$

Assuming the lemma, we prove the theorem as follows. Hölder's inequality with exponents $p_1^{-1} + q_2^{-1} + \cdots + q_K^{-1} = 1$ together with (4.3) give the following

$$\left|\int \phi_{t,x_0} L_0\,dx\right| \leqslant Ct^{-n-m-1}\|f_1\chi_{|x_0-y_1|\leqslant 2t}\|_{L^{p_1}}\prod_{j=2}^{K}\|I_{(m+1)\delta_j}(f_j\chi_{|x_0-y_j|\leqslant 2t})\|_{L^{q_j}}$$

where by I_k we denote convolution with $|x|^{-n+k}$ on $\mathbb{R}^n$. By the Hardy-Littlewood-Sobolev fractional integral theorem we get that

$$\begin{aligned}\sup_{t>0}\left|\int \phi_{t,x_0} L_0\,dx\right| &\leqslant C\sup_{t>0} t^{-n-m-1}\|f_1\chi_{|x_0-y_1|\leq 2t}\|_{L^{p_1}}\prod_{j=2}^{K}\|f_j\chi_{|x_0-y_j|\leq 2t}\|_{L^{p_j}}\\ &\leqslant C\sup_{t>0} t^{-n-m-1}\prod_{j=1}^{K}((|f_j|^{p_j})^*(x_0))^{1/p_j}t^{n/p_j}\\ &= C\prod_{j=1}^{K}((|f_j|^{p_j})^*(x_0))^{1/p_j},\end{aligned}$$

since we are assuming that $r = n/(n+m+1)$. This estimate is the equivalent of (2.3) in Theorem I and the required weak type result follows as in Theorem I.

In the case $r > n/(n+m+1)$, select $s_j < p_j$ such that

$$\sum_{j=1}^{K} s_j^{-1} = \frac{n}{n+m+1}.$$

By the previous result, we get that

$$\sup_{t>0}\int \phi_{t,x_0} L_0\,dx \leqslant C\sum_{j=1}^{K}((|f_j|^{s_j})^*(x_0))^{1/s_j}$$

and this estimate is the equivalent of (2.2) in Theorem I. The required result follows as before.

It remains to prove Lemma 3. Note that

$$b_t(y_1,\ldots,y_K) = \frac{1}{t^{nK}}\,b\left(\frac{x_0-y_1}{t},\ldots,\frac{x_0-y_K}{t}\right)$$

where

$$b(y_1,\ldots,y_K) = \prod_{k=1}^{K}\eta(y_k)\int\sum_{i=1}^{N}K_i^1(y_1-\sigma)\cdots K_i^K(y_K-\sigma)\phi(\sigma)\,d\sigma.$$

The estimate for b_t in Lemma 3 is equivalent to the following estimate

$$|b(y_1,\ldots,y_K)| \leqslant C\prod_{k=1}^{K}\eta(y_k)\prod_{j=2}^{K}|y_1-y_j|^{-n+\delta_j(m+1)}.$$

The vanishing moments assumptions for L are equivalent to the conditions

$$\int\sum_i\prod_j K_i^j(y_j-\sigma)\sigma^\alpha\,d\sigma = 0 \quad \text{for all} \quad |\alpha|\leqslant m.$$

We can therefore write

$$b(y_1,\dots,y_K)=\prod_k \eta(y_k)\,d(y_1,\dots,y_K)$$

where

$$d(y_1,\dots,y_K)=\int \sum_i \prod_j K_i^j(y_j-\sigma)\Big[\phi(\sigma)-\sum_{|\alpha|\le m}\frac{\partial^\alpha\phi}{\partial y_1^\alpha}(y_1)(\sigma-y_1)^\alpha\Big]\,d\sigma$$

All we need to show is that for $|y_2|,\dots,|y_K|\leqslant 4$, we have that

$$|d(y_1,\dots,y_K)|\leqslant \prod_{j=2}^K |y_1-y_j|^{-n+\delta_j(m+1)}.$$

Fix a smooth function $\zeta(\sigma)$ on $\mathbb{R}^n$, equal to 1 on $|\sigma|\leqslant 16$ and supported in $|\sigma|\leqslant 32$. Split $d(y_1,\dots,y_K)=I_1+I_2$ where

$$I_1=\int \sum_i K_i^1(\sigma)\prod_{j=2}^K K_i^j((y_j-y_1)-\sigma)\Big[\phi(\sigma+y_1)-\sum_{|\alpha|\le m}\frac{\partial^\alpha\phi}{\partial x_1^\alpha}(y_1)\sigma^\alpha\Big]\zeta(\sigma)\,d\sigma,$$

$$I_2=\int \sum_i K_i^1(\sigma)\prod_{j=2}^K K_i^j((y_j-y_1)-\sigma)\Big[\phi(\sigma+y_1)-\sum_{|\alpha|\le m}\frac{\partial^\alpha\phi}{\partial y_1^\alpha}(y_1)\sigma^\alpha\Big]$$
$$\times(1-\zeta(\sigma))\,d\sigma.$$

Then $\phi(\sigma+y_1)=0$ in the integral I_2 and $|\sigma-(y_j-y_1)|\sim|\sigma|$ for all $j\geqslant 2$. It follows that

$$|I_2|\leqslant C\sum_i\int_{|\sigma|\ge 16}|\sigma|^{-n}\prod_{j=2}^K|\sigma-(y_j-y_1)|^{-n}\sum_{|\alpha|\le m}\Big|\frac{\partial^\alpha\phi}{\partial y_1^\alpha}(y_1)\Big||\sigma|^\alpha|1-\zeta(\sigma)|\,d\sigma$$

$$\leqslant C\int_{|\sigma|\ge 16}|\sigma|^{-n}|\sigma|^{-(K-1)n}|\sigma|^m\,d\sigma$$

$$\leqslant C\leqslant C\prod_{j=2}^K|y_1-y_j|^{-n+\delta_j(m+1)}$$

since $|y_j-y_1|\leqslant 8$ and the numbers $-n+\delta_j(m+1)$ are negative.

We now treat term I_1. First fix an i and $y_1,y_2,\dots,y_K$ such that $y_j\neq y_1$ for $j\neq 1$ and let

$$a_{y_1}(\sigma)=K_i^1(\sigma)\prod_{j=3}^K K_i^j((y_j-y_1)-\sigma)\zeta(\sigma)\Big(\phi(\sigma+y_1)-\sum_{|\alpha|\le m}\frac{\partial^\alpha\phi}{\partial y_i^a}(y_1)\sigma^\alpha\Big).$$

a_{y_1} is a smooth function away from 0 except possibly at the points y_i-y_1, it has compact support and like in the proof of Theorem IIA, it is equal to a

sum of functions $a_{y_1,\alpha}$, $|\alpha| = m + 1$ where each $a_{y_1,\alpha}$ satisfies the hypotheses of Lemma 2 with constant $B_0 \sim |y_3 - y_1|^{-n} \cdots |y_K - y_1|^{-n}$.

Also note that

$$I_1 = \sum_i (T_i^2 a_{y_1})(y_2 - y_1).$$

It follows from Lemma 2 that

$$|I_1| \leqslant \sum_i |(T_i^2 a_{y_1})(y_2 - y_1)| \leqslant C|y_2 - y_1|^{-n+m+1} \prod_{j=3}^{K} |y_j - y_1|^{-n}.$$

The above is also true when y_2 is replaced by y_s, $s = 3, \ldots, K$. We get that

$$|I_1| \leqslant C|y_s - y_1|^{-n+m+1} \prod_{\substack{3 \leq j \leq K \\ j \neq s}} |y_j - y_1|^{-n} \quad \text{for} \quad s = 2, \ldots, K. \tag{4.4}$$

We raise (4.4) to the power δ_s, $s = 2, \ldots, K$ and we multiply all the resulting inequalities. We get the desired conclusion for $|I_1|$ and hence for $b(y_1, \ldots, y_K)$.

6. Examples and Final Remarks

In this section, we discuss examples of operators that satisfy the hypotheses of Theorems I and II. The determinant of the Jacobian of a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ gives rise to the following bilinear operator

$$\tilde{J}(f, g) = (R_1 f)(R_2 g) - (R_2 f)(R_1 g)$$

where R_j are the usual Riesz transforms. $\tilde{J}$ has always integral 0 but it does not have any other vanishing moments. It can be easily checked that for $p, q > 1$ with $1/p + 1/q = 3/2$, $\tilde{J}$ *never* maps $L^p \times L^q$ to $H^{2/3}$ and therefore our weak type result is sharp. In general, if a bilinear operator has all moments up to and including order m vanishing and one moment of order $m + 1$ nonzero, it does not map $L^p \times L^q$ to $H^{n/(n+m+1)}$ when $1/p + 1/q = (n + m + 1)/n$. Hence, the number of vanishing moments of the bilinear operator gives the lowest r for which the operator maps products of Lebesgue spaces into H^r. The same is true for more general multilinear operators.

We now discuss probably the most important example that satisfies the hypotheses of our Theorem II, a bilinear map that involves sums of products of derivatives of order 2 and is the analogue of the determinant of the Jacobian. The Hessian of a map $F = (f, g)$: $\mathbb{R}^2 \to \mathbb{R}^2$ is the spacial $2 \times 2 \times 2$ matrix which has on the top the 2×2 matrix:

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} \\ \dfrac{\partial^2 f}{\partial y\,\partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

and on the bottom the 2×2 matrix:

$$\begin{bmatrix} \dfrac{\partial^2 g}{\partial x^2} & \dfrac{\partial^2 g}{\partial x\,\partial y} \\ \dfrac{\partial^2 g}{\partial y\,\partial x} & \dfrac{\partial^2 g}{\partial y^2} \end{bmatrix}$$

We denote by $H(f, g)$ the determinant of the $2 \times 2 \times 2$ Hessian matrix above defined as follows

$$H(f,g) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 f}{\partial x\,\partial y}\frac{\partial^2 g}{\partial y\,\partial x} - \frac{\partial^2 f}{\partial y\,\partial x}\frac{\partial^2 g}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 g}{\partial x^2}.$$

After formally replacing the partial derivatives of F with the corresponding Riesz transforms we get the following bilinear operator

$$\tilde{H}(f,g) = (R_1^2 f)(R_2^2 g) - (R_1 R_2 f)(R_2 R_1 g)(R_2 R_1 g) - (R_2 R_1 f)(R_1 R_2 g).$$

It was shown in the last section of [CG], that $\tilde{H}$ has integral and first moments zero. By Theorem IIa it follows that $\tilde{H}$ maps $L^p \times L^q$ to H^r for all $p, q > 1$, where r is their harmonic mean. It follows that the determinant of the Hessian H, of a map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ maps pairs of functions with Laplacean in $L^p \times L^q$ into H^r for all $p, q > 1$, where r is their harmonic mean. This result generalizes the correspoding theorem about the Jacobian in the case of second order derivatives and has analogues in higher dimensions.

We now discuss generalizations of $\tilde{H}$ in $\mathbb{R}^n$. Let $F = (F_1, \ldots, F_n)$ be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Form the $n \times \cdots \times n$ matrix M by stacking the $n \times n$ matrices

$$\left(\frac{\partial^2 F_k}{\partial x_i\,\partial x_j}\right)_{i,j}, \qquad k = 1, 2, \ldots, n$$

on the top of each other. We call M the n-dimensional Hessian of F. The determinant of this matrix is defined by induction on n as the sum of its n^2 minor determinants suitably signed. After formally replacing the $\partial/\partial x_j$ derivative of F by the j-th Riesz transform, the n-dimensional determinant of this matrix gives rise to an n-linear operator $\tilde{H}^n$ with vanishing integral and first moments. When $n = 2$ the resulting bilinear operator $\tilde{H}^2$ is the operator $\tilde{H}$ defined above. When $n = 3$ the resulting trilinear operator $\tilde{H}^3(f, g, h)$ is the sum of the nine terms $(-1)^{-+j}(R_i R_j h)\tilde{H}_{ij}(f, g)$ where each $\tilde{H}_{ij}$ corresponds

to the determinant of a Hessian of a $2 \times 2 \times 2$ minor. It is therefore clear that that $\tilde{H}^3$ satisfies the hypotheses of Theorem IIb. Our result then says that that for $p, q > 1$, $\tilde{H}^3$ maps $L^p \times L^q$ into H^r when the harmonic mean r of p and $q > 3/5$. In general the operator $\tilde{H}^n$ maps $L^p \times L^q$ into H^r for $1 \geqslant r > n/(n+2)$ since it follows by induction that $\tilde{H}^n$ has integral and first moments zero. ($k = 1$.)

It is conceivable that determinants of matrices of higher order derivatives of maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ give rise to multilinear operators with higher moments vanishing but these cases are not investigated in this article. Examples of bilinear operators with moments of all orders vanishing in one dimension are

$$D_1(f, g) = fg - (Hf)(Hg)$$

and

$$D_2(f, g) = f(Hg) + (Hf)g,$$

where H is the usual Hilbert transform. D_1 and D_2 are the real and imaginary parts of holomorphic functions and their mapping properties are well understood. More generally, examples of K-linear operators with all moments vanishing are given by the real and imaginary parts of $\prod_{k=1}^{K} (f_k + iHf_k)$.

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Weak Type (1,1) Estimates of the Maximal Function for the Laguerre Semigroup in Finite Dimensions

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Abstract

We prove that, in arbitrary finite dimensions, the maximal operator for the Laguerre semigroup is of weak type (1,1). This extends Muckenhoupt's one-dimensional result.

Introduction

Let (F, ν) be some positive measure space. A semigroup $\{T^t\}_{0 \le t < \infty}$ is understood to be a family of bounded linear operators, defined simultaneously on $L^p(\nu)$, $1 \leqslant p \leqslant \infty$, satisfying measurability in t and the semigroup axioms $T^{t+s} = T^t T^s$ and $T^0 = I$ is the identity. The semigroup $\{T^t\}$ is called a «symmetric diffusion semigroup» if it has the following properties

(I)	$\|T^t f\|_p \leqslant \|f\|_p, \quad 1 \leqslant p \leqslant \infty$	(Contraction)
(II)	T^t is self-adjoint on $L^2(\nu)$	(Symmetry)
(III)	$T^t f \geqslant 0$ if $f \geqslant 0$	(Positivity)
(IV)	$T^t 1 = 1$	(Conservation)
(V)	$\lim_{t \to 0^+} T^t f = f$ in $L^2(\nu)$ for all $f \in L^2(\nu)$	(Strong continuity on L^2).

The maximal operator M associated with the semigroup $\{T^t\}$ is defined by

$$Mf(x) = \sup_{t>0} |T^t f(x)|.$$

Under the assumptions (I), (II), and (V), the maximal theorem due to E. M. Stein ([St]) states that

(*a*) M is bounded on $L^p(\nu)$, *i.e.* $\|Mf\|_p \leqslant A_p \|f\|_p$, $1 < p \leqslant \infty$.

(*b*) If $f \in L^p(\nu)$, $1 < p < \infty$, then $\lim_{t \to 0+} T^t f(x) = f(x)$ a.e.

An operator is said to be of weak type (1,1) if it maps $L^1(\nu)$ boundedly into $L^{1,\infty}(\nu)$, where $L^{1,\infty}(\nu)$ is defined by means of its quasinorm

$$\|f\|_{1,\infty} = \sup_{\beta>0} \beta \nu\{x: |f(x)| > \beta\}.$$

It seems to be unknown whether weak type (1,1) holds for M, as a substitute result in the maximal theorem when $p = 1$. However, the Hopf-Dunford-Schwartz ergodic theorem ([DS] p. 690) gives that the «maximal average operator» $\tilde{M}$, defined by

$$\tilde{M}f(x) = \sup_{s>0} \left| \frac{1}{s} \int_0^s T^t f(x)\, dt \right|,$$

is of weak type (1,1) if the semigroup $\{T^t\}$ satisfies (I).

In several concrete settings, the maximal operator M of a symmetric diffusion semigroup has been investigated and found to be of weak type (1,1). This is done by well-known methods when the semigroup is defined as convolution with an «approximate identity» and the underlying measure is translation invariant. For instance the Poisson and Gaussian kernels in $(\mathbb{R}^n, dx)$ define semigroups of that kind. On the other hand, semigroups defined in terms of «classical orthogonal polynomials» have to be handled differently. For the ultraspherical harmonics, the Hermite polynomials, and the Laguerre polynomials —each with the appropriate measure space in one dimension —it has been proved that the corresponding maximal operators are of weak type (1,1). For the ultraspherical harmonics this is a joint result by B. Muckenhoupt and E. M. Stein ([MS]), whereas it is due to Muckenhoupt ([M]) for the Hermite and Laguerre semigroups. In [Sj1], P. Sjögren proves the same result for the Hermite semigroup in higher (finite) dimensions, and the Laguerre semigroup in higher dimensions is treated in the present paper.

The setting is as follows (*cf.* [M]). Let $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d: x_i > 0 \text{ for every } i\}$ and denote by γ_α the measure on $\mathbb{R}^d_+$ given by the density

$$\gamma_\alpha(x) = \prod_{i=1}^{d} x_i^\alpha e^{-x_i}.$$

For $\alpha > -1$, the d-dimensional Laguerre polynomials

$$L_m^\alpha(x) = \prod_1^d L_{m_i}^\alpha(x_i), \qquad m = (m_1, \ldots, m_d) \in \mathbb{N}^d,$$

form a complete orthogonal system in $L^2(\gamma_\alpha)$. The definition of the m_i-th Laguerre polynomial, $L_{m_i}^\alpha$, used here will be the one used in [Sz]. For a function f in $L^1(\gamma_\alpha)$, its «Laguerre-Poisson integral» $M_r^\alpha f$ is defined for $0 < r < 1$ by

$$M_r^\alpha f(x) = \int K_r^\alpha(x, y) f(y)\, d\gamma_\alpha(y),$$

where

$$K_r^\alpha(x, y) = \prod_{i=1}^d K_r^\alpha(x_i, y_i), \qquad x, y \in \mathbb{R}_+^d,$$

$$K_r^\alpha(x_i, y_i) = \sum_{n=0}^\infty r^n L_n^\alpha(x_i) L_n^\alpha(y_i)/\Gamma(\alpha + 1)\binom{n+\alpha}{n}.$$

The kernel may also be expressed in terms of the standard Bessel function J_α,

$$K_r^\alpha(s, t) = \frac{(-rst)^{-\alpha/2}}{1 - r} \exp\left(\frac{-r(s+t)}{1-r}\right) J_\alpha\left(\frac{2\sqrt{-rst}}{1-r}\right),$$

which can be estimated to give

$$(0.1)\quad K_r^\alpha(s, t) \sim \begin{cases} (1-r)^{-\alpha-1} \exp\left(\dfrac{-r(s+t)}{1-r}\right), & 0 < 4rst < (1-r)^2, \\ \dfrac{(4rst)^{-\alpha/2-1/4}}{(1-r)^{1/2}} \exp\left(\dfrac{-r(s+t)+2\sqrt{rst}}{1-r}\right), & 4rst \geqslant (1-r)^2. \end{cases}$$

Here, and in the sequel, $f \sim g$ means $c \leqslant f/g \leqslant C$ for some positive constants c and C. In (0.1) the constants depend only on α. Thus, for r and x fixed, the function $K_r^\alpha(x, \cdot)$ is bounded. Therefore, the Laguerre-Poisson integral $M_r^\alpha f$ is well defined for each $f \in L^1(\gamma_\alpha)$, $0 < r < 1$. In particular, if a function in $L^2(\gamma_\alpha)$ has the Laguerre expansion $\Sigma a_m L_m^\alpha$, then its Laguerre-Poisson integral is the function of x and r which for fixed $r (0 < r < 1)$ has the Laguerre expansion $\Sigma r^{|m|} a_m L_m^\alpha$, $|m| = \Sigma m_i$ (see [M]).

The Laguerre semigroup, with parameter $\alpha > -1$, is defined by

$$T^t = M_{e^{-t}}^\alpha, \quad 0 < t < \infty, \quad \text{and} \quad T^0 = I.$$

This defines a symmetric diffusion semigroup, whose infinitesimal generator is

$$Lu \equiv \Sigma(x_i \partial^2 u/\partial x_i^2 + (1 + \alpha - x_i)\, \partial u/\partial x_i),$$

i.e. $u(x, t) = T^t f(x)$ satisfies $\partial u/\partial t = Lu$ and $u(x, 0) = f(x)$.

By virtue of Stein's maximal theorem, the operator

$$M^{\alpha}f(x) = \sup_{0<r<1} |M_r^{\alpha} f(x)|$$

is bounded on $L^p(\gamma_\alpha)$, $1 < p \leqslant \infty$. The following theorem is the main result in this paper.

Theorem 1. *For each finite dimension d and $\alpha > -1$ there is a constant $c = c(d, \alpha)$ such that*

$$\|M^{\alpha}f\|_{1,\infty} \leqslant c\|f\|_1 \quad \textit{for each} \quad f \in L^1(\gamma_\alpha).$$

Corollary. $\lim_{r\to 1^-} M_r^{\alpha} f(x) = f(x)$ a.e. *if* $f \in L^1(\gamma_\alpha)$.

The almost everywhere convergence is as usual a consequence of the estimate for the maximal operator. Everywhere pointwise convergence is easily verified for polynomials. Since they form a dense subset of $L^1(\gamma_\alpha)$, the proof of the Corollary is then quite standard and we omit it. The use of polynomials was suggested to the author by Fulvio Ricci.

There is a natural way to define γ_α and K_r^{α}, and hence M^{α}, for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i > -1$. For such α, Theorem 1 still holds and the proof is only slightly modified.

It is noticeable that in the proof of Theorem 1 the constant c depends heavily on the dimension. Both for the Hermite and the Laguerre semigroup, it is an open problem to obtain constants independent of the dimension.

Some conventions: c and C denote various positive constants, which usually depend on the dimension and α. In case c depends on a parameter k we indicate this by writing c_k. We abbreviate $f \leqslant cg$ by $f \lesssim g$. Usually we will not indicate the underlying space for measures, *e.g.* $d\gamma_\alpha(x)$ means the measure $(\Pi x_i^{\alpha} e^{-x_i})\,dx$ on $\{x_i > 0, \text{ all } i\}$ where the dimension will be clear from x.

1. Sketch of the proof

The flavour of the proof of Theorem 1 is not «semigroup theoretical» —the semigroup property is not used at all. On the other hand, the fact that M_r^{α} is given by a symmetric positive kernel and leave constant functions invariant (implying (I)-(IV)) is essential.

The proof is split into several lemmas. First it is proved that M^{α} is of weak type $(1, 1)$ when α is an integer or half-integer. If $2\alpha = n - 2$, for some $n = 1, 2, \ldots$, this is done by a transformation to the Hermite case in $\mathbb{R}^{dn}$ and an application of Sjögren's result. Next, for general α the supremum is restrict-

ed to the intervals $0 < r < 1/2$ and $1/2 \leqslant r < 1$, respectively. The main part is the proof that the operator

$$M_1^\alpha f(x) = \sup_{1/2 \leq r < 1} |M_r^\alpha f(x)|$$

is of weak type (1,1). For this, the course of action is as follows.

For any pair of disjoint subsets I' and I'' of $I \equiv \{1, \ldots, d\}$ let

$$A(I', I'') = \{(x, y):\ y_i \leqslant x_i/D \text{ for } i \in I', x_i/D < y_i < Dx_i \text{ for } i \in I'', \\ y_i \geqslant Dx_i \quad \text{for } i \notin I' \cup I''\},$$

where D is a large constant. The operator M_r^α can be written as a finite sum of operators, each given by the kernel $K_r^\alpha(x, y)$ restricted to one of the (pairwise disjont) regions $A(I', I'')$. Of course, it then suffices to prove weak type (1,1) for each of the corresponding maximal functions.

When $I' = I'' = \varnothing$ the corresponding maximal operator turns out to be bounded on $L^1(\gamma_\alpha)$. This implies (see Proposition 1), because of the product structure and the fact that $K_r^\alpha(x, y) \geqslant 0$, that it remains to prove weak type (1,1) for the cases $I' \cup I'' = I$. If $I' = \varnothing$, *i.e.* $x_i/D < y_i < Dx_i$ for all i, this is done by a comparison (locally) with the case $\alpha = -1/2$, which is already clear. The proof for the remaining case $I' \neq \varnothing$, $I' \cup I'' = I$ is rather long and technical. It is divided into five steps, of which we give a brief summary.

In Step 1 the kernel is estimated. Dependig on whether i belongs to I' or not, basic estimates of $K_r^\alpha(x_i, y_i)$ are obtained. These imply an appropriate division of the space into regions where additional (local) estimates are made. In this way a sum emerges, to which the summation theorem in $L^{1,\infty}$ due to E. M. Stein and N. Weiss ([SW]) is applied. The range of $1 - r$ is split into the intervals $[4^{-N-1}, 4^{-N}[$, and in Step 2 the task of estimating the terms in the sum above is reduced to prove that, for $0 < a < 1$,

$$\sum_{N=0}^{\infty} a^N \gamma_0\{x \in Q:\ U_{N,k} g(x) > a^N \beta\} \leqslant \frac{A_k}{\beta} \|g\|_{L^1(dy)}, \qquad \beta > 0. \tag{1.1}$$

Here, the $U_{N,k}$'s are operators obtained in the series of estimates, Q is a «dyadic» rectangle, and g is any non-negative function supported in Q and its surrounding rectangles. For x in Q, the size of x_i is about 4^{N_i}. Assuming without loss of generality that N_1 is the largest of the N_i's, we split the sum in (1.1) into two parts, $N \leqslant N_1$ and $N > N_1$. In Step 3, the sum over $N > N_1$ is estimated by an application of Proposition 2. This proposition gives an inequality similar to that in (1.1), where the operators are averages over subsets belonging to hierarchical partitions. The sum over $N \leqslant N_1$ is estimated term by term in Step 4, where a covering argument is crucial. The covering argument is justified in Step 5.

2. Transformation to the Hermite Case

For certain α it is possible to prove the theorem by a transformation to the corresponding operator given by Hermite expansions and then apply the result in [Sj1].

Lemma 1. *$M^\alpha: L^1(\gamma_\alpha) \to L^{1,\infty}(\gamma_\alpha)$ is bounded for $2\alpha = -1, 0, 1, \ldots$.*

For a given finite dimension n we denote by μ the measure on $\mathbb{R}^n$ given by the density $e^{-|y|^2}$. Let H_k denote the k-th Hermite polynomial (see [Sz]) and define, for $0 < r < 1$, the kernels

$$P_r(x, y) = \prod_{i=1}^{n} P_r(x_i, y_i), \qquad x, y \in \mathbb{R}^n,$$

where

$$P_r(x_i, y_i) = \sum_{k=0}^{\infty} r^k H_k(x_i) H_k(y_i) / \sqrt{\pi}\, 2^k k!.$$

Define the operators N_r, $0 < r < 1$, and the corresponding maximal operator N by

$$N_r f(x) = \int P_r(x, y) f(y)\, d\mu(y),$$

$$Nf(x) = \sup_{0<r<1} |N_r f(x)|.$$

The theorem proved in [Sj1] can then be stated as follows.

Theorem A ([Sj1]) *$N: L^1(\mu) \to L^{1,\infty}(\mu)$ is bounded.*

Now, if $2\alpha = n - 2$ ($n = 1, 2, \ldots$) let μ live on $\mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{dn}$ and define $G: \mathbb{R}^{dn} \to \mathbb{R}^d_+$ by

$$G(x^1, \ldots, x^d) = (|x^1|^2, \ldots, |x^d|^2), \qquad x^i \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^{dn}} f \circ G(x)\, d\mu(x) = c \int_{\mathbb{R}^d_+} f(y)\, d\gamma_\alpha(y) \tag{2.1}$$

for each $f \in L^1(\gamma_\alpha)$.

Moreover, expanding the function $L_k^\alpha(|x|^2)$, $x \in \mathbb{R}^n$, with respect to the orthogonal basis $\{H_{k_1}(x_1) \cdots H_{k_n}(x_n)\}_{k_i \in \mathbb{N}}$ in $L^2(\mu, \mathbb{R}^n)$, we get the formula

$$L_k^\alpha(|x|^2) = \sum_{k_1 + \ldots + k_n = k} a_{k_1, \ldots, k_n} H_{2k_1}(x_1) \cdots H_{2k_n}(x_n), \qquad x \in \mathbb{R}^n \tag{2.2}$$

for $2\alpha = n - 2$, $n = 1, 2, \ldots$.

PROOF OF LEMMA 1. Let $2\alpha = n - 2$ for some $n = 1, 2, \ldots$. Assume $f \in L^2(\gamma_\alpha)$ and let

$$f(x) = \sum_{k_i = 0, 1, \ldots} b_{k_1, \ldots, k_d} L^\alpha_{k_1}(x_1) \cdots L^\alpha_{k_d}(x_d) \equiv \sum_k b_k L^\alpha_k(x)$$

be the Laguerre expansion of f. From (2.2) we have, for $y \in \mathbb{R}^{dn}$,

$$\begin{aligned} f \circ G(y) &= \sum_k b_k L^\alpha_{k_1}(|y^1|^2) \cdots L^\alpha_{k_d}(|y^d|^2) \\ &= \sum_k b_k \prod_{i=1}^d \left(\sum_{l_1 + \cdots + l_n = k_i} a_{l_1, \ldots, l_n} H_{2l_1}(y^i_1) \cdots H_{2l_n}(y^i_n) \right). \end{aligned}$$

The product is a sum of Hermite polynomials in $\mathbb{R}^{dn}$ of degree $2|k|$. Since (see [M])

$$N_r\left(\sum_k a_k H_k \right) = \sum_k r^{|k|} a_k H_k$$

this gives

$$\begin{aligned} N_r(f \circ G)(y) &= \sum_k r^{2|k|} b_k L^\alpha_k(|y^1|^2, \ldots, |y^d|^2) \\ &= M^\alpha_{r^2} f(|y^1|^2, \ldots, |y^d|^2) \\ &= (M^\alpha_{r^2} f) \circ G(y). \end{aligned}$$

Since both $K^\alpha_r(x, \cdot)$ and $P_r(x, \cdot)$ are bounded functions (see [M]), we get

$$|N_r(f \circ G)| = |(M^\alpha_{r^2} f) \circ G| \tag{2.3}$$

for all $f \in L^1(\gamma_\alpha)$, by taking the limit for $f_j \in L^2(\gamma_\alpha)$ such that $f_j \to f$ in $L^1(\gamma_\alpha)$.

Taking the supremum over $0 < r < 1$ on both sides in (2.3) gives

$$N(f \circ G) = (M^\alpha f) \circ G, \qquad f \in L^1(\gamma_\alpha). \tag{2.4}$$

Finally, from (2.1), (2.4), and Theorem A we get

$$\begin{aligned} c\gamma_\alpha\{x \in \mathbb{R}^d_+ : M^\alpha f(x) > \beta\} &= \mu\{y \in \mathbb{R}^{dn} : (M^\alpha f) \circ G(y) > \beta\} \\ &= \mu\{y \in \mathbb{R}^{dn} : N(f \circ G)(y) > \beta\} \\ &\leqslant \frac{C}{\beta} \| f \circ G \|_{L^1(\mu)} \\ &= \frac{C}{\beta} \| f \|_{L^1(\gamma_\alpha)}, \end{aligned}$$

which proves Lemma 1.

3. The Case $1/2 \leqslant r < 1$

In this and the next section we intend to prove the following lemma —the main part of the theorem.

Lemma 2. *The operator*

$$M_1^\alpha f(x) = \sup_{1/2 \leq r < 1} |M_r^\alpha f(x)|$$

is bounded from $L^1(\gamma_\alpha)$ *into* $L^{1,\infty}(\gamma_\alpha)$ *for all* $\alpha > -1$.

We split the proof into three more lemmas, where we restrict the kernel to different areas of integration. A simple result about non-negative kernels will be used several times. Therefore, it is convenient to state it in a general form.

Let (E, μ) and (F, ν) be measure spaces, with μ and ν non-negative and σ-finite. Further, let I be a countable subset of $\mathbb{R}$. Suppose that, for $r \in I$, $A_r(x, y)$ and $B_r(x', y')$ are non-negative measurable functions on $E \times E$ and $F \times F$, respectively, such that

$$A_r(x, \cdot) \in L^\infty(\mu) \quad \text{and} \quad B_r(x', \cdot) \in L^\infty(\nu) \quad \text{for all} \quad r \in I, \quad x \in E, \quad x' \in F.$$

Then the following operators are well defined, giving rise to measurable functions, on $L^1(\mu)$, $L^1(\nu)$, and $L^1(\mu \times \nu)$, respectively

$$Af(x) = \sup_{r \in I} \left| \int A_r(x, y) f(y)\, d\mu(y) \right|, \qquad x \in E,$$

$$Bf(x') = \sup_{r \in I} \left| \int B_r(x', y') f(y')\, d\nu(y') \right|, \quad x' \in F,$$

$$A \otimes Bf(x, x') = \sup_{r \in I} \left| \int A_r(x, y) B_r(x', y') f(y, y') d\mu(y) d\nu(y') \right|, \quad (x, x') \in E \times F.$$

Proposition 1. *If* $A: L^1(\mu) \to L^{1,\infty}(\mu)$ *boundedly with norm* C_A *and* $B: L^1(\nu) \to L^1(\nu)$ *boundedly with norm* C_B *then* $A \otimes B: L^1(\mu \times \nu) \to L^{1,\infty}(\mu \times \nu)$ *is bounded with norm* $C_A C_B$.

PROOF. Since the kernels are non-negative, it is enough to consider $f \geqslant 0$. Then

$$A \otimes Bf(x, x') \leqslant \sup_{r \in I} \int A_r(x, y) \left(\sup_{r \in I} \int B_r(x', y') f(y, y')\, d\nu(y') \right) d\mu(y)$$

$$= AF_{x'}(x)$$

where

$$F_{x'}(y) = Bf_y(x'), \qquad f_y(y') = f(y, y').$$

Thus

$$\begin{aligned}(\mu \times \nu)\{(x,x'):\ A \otimes Bf(x,x') > \beta\} &\leqslant \int \mu\{x:\ AF_{x'}(x) > \beta\}\, d\nu(x') \\ &\leqslant C_A \beta^{-1} \int \|F_{x'}\|_{L^1(d\mu(y))}\, d\nu(x') \\ &= C_A \beta^{-1} \int \left(\int Bf_y(x')\, d\nu(x') \right) d\mu(y) \\ &\leqslant C_A C_B \beta^{-1} \int \|f_y\|_{L^1(d\nu(y'))}\, d\mu(y) \\ &= C_A C_B \beta^{-1} \|f\|_{L^1(\mu \times \nu)},\end{aligned}$$

which proves the proposition.

Notice that the condition «I countable» (to ensure measurability) will not cause us any problem, since the kernels we are dealing with are nice enough to enable us to restrict the supremum to *e.g.* r rational.

Now, by Proposition 1, the following three lemmas will imply Lemma 2.

Lemma 3. *For large D, the operator*

$$A^\alpha f(x) = \int \sup_{1/2 \leq r < 1} K_r^\alpha(x,y) \chi_{\{y_i \geq Dx_i,\ \text{all } i\}} |f(y)|\, d\gamma_\alpha(y)$$

is bounded on $L^1(\gamma_\alpha)$.

Lemma 4. *For any $D > 1$, the operator*

$$B^\alpha f(x) = \sup_{1/2 \leq r < 1} \left| \int K_r^\alpha(x,y) \chi_{\{x_i/D < y_i < Dx_i,\ \text{all } i\}} f(y)\, d\gamma_\alpha(y) \right|$$

maps $L^1(\gamma_\alpha)$ boundedly into $L^{1,\infty}(\gamma_\alpha)$.

Lemma 5. *For large D, the operator*

$$C^\alpha f(x) = \sup_{1/2 \leq r < 1} \left| \int K_r^\alpha(x,y) \chi_{\{y_i < Dx_i,\ \text{all } i,\ \text{some } y_i \leq x_i/D\}} f(y)\, d\gamma_\alpha(y) \right|$$

maps $L^1(\gamma_\alpha)$ boundedly into $L^{1,\infty}(\gamma_\alpha)$.

Throughout the remaining proofs, (0.1) will be used frequently. Therefore we set ($s, t \in \mathbb{R}_+$, $0 < r < 1$)

$$D_r^\alpha(s,t) = (1-r)^{-\alpha-1} \exp\left(\frac{-r(s+t)}{1-r}\right) \chi_{\{4rst < (1-r)^2\}}$$

$$E_r^\alpha(s,t) = \frac{(4rst)^{-\alpha/2-1/4}}{(1-r)^{1/2}} \exp\left(\frac{-r(s+t)+2\sqrt{rst}}{1-r}\right) \chi_{\{4rst \geq (1-r)^2\}}.$$

With these notations, (0.1) says that

$$K_r^\alpha(s,t) \sim D_r^\alpha(s,t) + E_r^\alpha(s,t) \qquad (s,t \in \mathbb{R}_+) \tag{3.1}$$

and, consequently, we have

$$K_r^\alpha(x,y) \sim \sum_{I \subseteq \{1,\dots,d\}} \Big(\prod_{i \in I} D_r^\alpha(x_i,y_i)\Big)\Big(\prod_{i \notin I} E_r^\alpha(x_i,y_i)\Big), \qquad x,y \in \mathbb{R}_+^d. \tag{3.2}$$

PROOF OF LEMMA 3. The lemma follows if we estimate $K_r^\alpha(x,y)$, when $y_i \geqslant Dx_i$, $1/2 \leqslant r < 1$, by a function $A(x,y)$ such that

$$\int A(x,y) \chi_{\{x_i \leq y_i/D \text{ all } i\}} \, d\gamma_\alpha(x) \leqslant c.$$

Of course, it is enough to do this in one dimension.

For fixed x and y in $\mathbb{R}_+$ let

$$D(r) = (1-r)^{-\alpha-1} \exp\left(\frac{-r(x+y)}{1-r}\right), \qquad 1/2 \leqslant r < 1.$$

Then $D'(r)$ equals $(\alpha + 1 - (x+y)/(1-r))$ times a positive factor. Hence, $D(r)$ takes its maximum at $1 - r = (x+y)/(1+\alpha)$ if $(x+y)/(1+\alpha) \leqslant 1/2$, which gives $D(r) \leqslant (x+y)^{-\alpha-1}$. On the other hand, $D(r)$ is decreasing if $(x+y)/(1+\alpha) > 1/2$, giving $D(r) \leqslant D(1/2) \leqslant 1$.

Thus, for all $x, y \in \mathbb{R}_+$ and $1/2 \leqslant r < 1$, we have

$$D_r^\alpha(x,y) \leqslant 1 + (x+y)^{-\alpha-1}. \tag{3.3}$$

In order to estimate $E_r^\alpha(x,y)$, we set, for fixed x and y with $y \geqslant Dx$,

$$G(r) = (1-r)^{-1/2} \exp\left(\frac{-r(x+y)+2\sqrt{rxy}}{1-r}\right) \equiv (1-r)^{-1/2} \exp f(r),$$

$1/2 \leqslant r < 1$.

Then

$$f'(r) = \frac{1}{(1-r)^2}\left(\sqrt{xy/r} - (x+y) + \sqrt{rxy}\right) = \operatorname{pos}\left(\sqrt{y/x} - \sqrt{r}\right)\left(\sqrt{x/y} - \sqrt{r}\right)$$

and

$$G'(r) = \operatorname{pos}(1/2 + (1 - r)f'(r)).$$

With D large enough, $G'(r) = 0$ implies

$$(1 - r) = 2(x + y) - 2\sqrt{xy}\,(1 + r)/\sqrt{r} \geqslant y\left(2 - 4\sqrt{2/D}\right) \geqslant y.$$

Moreover, since f is decreasing for $y \geqslant 2x$,

$$f(r) \leqslant f(1/2) = -(x + y) + 2\sqrt{2xy} \leqslant x.$$

Thus, if $G'(r) = 0$ then

$$G(r) \leqslant y^{-1/2} \exp f(r) \leqslant (xy)^{-1/4} e^x.$$

Since $G(1/2) \leqslant \exp\left(-(x + y) + 2\sqrt{2xy}\right)$ and $\lim_{r \to 1^-} G(r) = 0$, this gives

$$E_r^\alpha(x, y) \leqslant (xy)^{-\alpha/2 - 1/4} \exp\left\{-(x + y) + 2\sqrt{2xy}\right\} + (xy)^{-\alpha/2 - 1/2} e^x.$$

This and (3.3) give the desired estimate of $K_r^\alpha(x, y)$.

PROOF OF LEMMA 4. Because of (3.2), it is enough to prove the lemma with $K_r^\alpha(x, y)$ replaced by the kernel

$$\left(\prod_{i \in I} D_r^\alpha(x_i, y_i)\right)\left(\prod_{i \notin I} E_r^\alpha(x_i, y_i)\right)$$

for a fixed, but arbitrary, I. The operator

$$f \to \int \left[\prod (1 + (x_i + y_i)^{-\alpha - 1})\right] \chi_{\{x_i/D < y_i < Dx_i \text{ all } i\}} |f(y)|\, d\gamma_\alpha(y)$$

is easily seen to be bounded on $L^1(\gamma_\alpha)$ for any dimension. Thus (3.3) implies, by means of Proposition 1, that we only have to prove that the operator

$$(3.4) \qquad E^\alpha f(x) = \sup_{1/2 \leq r < 1} \left| \int \left(\prod_{i=1}^d E_r^\alpha(x_i, y_i)\right) \chi_{\{x_i/D < y_i < Dx_i \text{ all } i\}} f(y)\, d\gamma_\alpha(y) \right|$$

is bounded from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$ for any dimension d, for $D > 1$ and $\alpha > -1$. We write (3.4) as

$$E^\alpha f(x) = \sup_{1/2 \leq r < 1} |E_r^\alpha f(x)|.$$

Again, it is enough to consider $f \geqslant 0$. For $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ let

$$Q^m = \{x:\ D^{m_i} < x_i \leqslant D^{m_i + 1},\ i = 1, \ldots, d\}$$

and

$$\tilde{Q}^m = \{x\colon D^{m_i-1} < x_i \leqslant D^{m_i+2}, i = 1, \ldots, d\}.$$

Since $E_r^\alpha f(x) \sim E_r^{-1/2} f(x)$ for all x, we get, by applying Lemma 1,

$$\begin{aligned}\gamma_\alpha\{x \in Q^m\colon E^\alpha f(x) > \beta\} &\leqslant D^{(\alpha+1/2)\sum m_i}\gamma_{-1/2}\{x \in Q^m\colon E^{-1/2} f(x) > c\beta\}\\ &\leqslant D^{(\alpha+1/2)\sum m_i}\gamma_{-1/2}\{x\colon M^{-1/2}(f|_{\tilde{Q}^m})(x) > c\beta\}\\ &\leqslant \frac{D^{(\alpha+1/2)\sum m_i}}{\beta}\|f|_{\tilde{Q}^m}\|_{L^1(\gamma_{-1/2})}\\ &\sim \frac{1}{\beta}\int f|_{\tilde{Q}^m}\, d\gamma_\alpha.\end{aligned}$$

Summation over m yields that E^α is bounded from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$, which proves Lemma 4.

So far we have used the Hermite result, but to prove Lemma 5 we have to investigate the kernel in more detail. Before the proof we recall a result (slightly restated) due to E. M. Stein and N. Weiss. We also state a proposition generalizing a special case of Lemma 1 in [Sj2].

Let (E, ν) be a measure space, with ν non-negative.

Theorem B ([SW], Lemma 2.3) *Suppose g_j, $j = 1, 2, \ldots$, are non-negative measurable functions on E, satisfying*

$$\nu\{x\colon g_j(x) > \alpha\} < Ac_j/\alpha, \qquad \alpha > 0,$$

where c_j are positive numbers such that $\sum c_j < \infty$ and $\sum c_j \ln(1/c_j) < \infty$. Then

$$\nu\left\{x\colon \sum_j g_j(x) > \alpha\right\} < 2AC/\alpha, \qquad \alpha > 0, \quad \textit{for some positive number } C.$$

E is said to have a piece hierarchy if for each $N = 1, 2, \ldots$ there exists a partition of E into at most countably many disjoint subsets called N-pieces with the following properties

(i) Each N-piece is measurable with positive finite measure,
(ii) Any $(N+1)$-piece is contained in some N-piece.

Now, suppose E has a piece hierarchy. Denote the N-pieces by P_N^j, $j = 1, 2, \ldots$, and let $P_N(x)$, for $x \in E$, be the N-piece that contains x. Assume $\hat{}$ is a one-to-one map from $\{P_N^j\colon j = 1, 2, \ldots\}$ into itself for each N, such that

$$\nu(P_N^j) \sim \nu(\hat{P}_N^j) \qquad \text{(uniformly in } N, j).$$

Define the operators T_N on $L^1(\nu)$ by

$$T_N g(x) = \frac{1}{\nu(\hat{P}_N(x))} \int_{\hat{P}_N(x)} |g(y)|\, d\nu(y).$$

Proposition 2. *For any $0 < a < 1$ there is a constant $c = c(a)$ such that*

$$\sum_{N=1}^{\infty} a^N \nu\{x:\ T_N g(x) > a^N \beta\} \leqslant \frac{c}{\beta} \|g\|_{L^1(\nu)}, \qquad \beta > 0.$$

PROOF. Since $T_N g$ is constant on each P_N^j, the set $I = \{(N,j):\ T_N g > a^N\beta$ on $P_N^j\}$ is well defined and the sum equals

$$\sum_{(N,j)\in I} a^N \nu(P_N^j) \sim \sum_{(N,j)\in I} (a^N - a^{N+1}) \nu(\hat{P}_N^j). \tag{3.5}$$

Regard $[0,1] \times E$ as a measure space. Then, the last sum above is just the measure of the union of the pairwise disjoint sets

$$A_{N,j} \equiv]a^{N+1}, a^N] \times \hat{P}_N^j \subseteq [0,1] \times E, \qquad (N,j) \in I.$$

The family $\{A_{N,j}:\ (N,j) \in I\}$ is partially ordered by

$$A_{M,i} \leqslant A_{N,j} \quad \text{if and only if} \quad \hat{P}_M^i \subseteq \hat{P}_N^j \quad \text{and} \quad M \geqslant N.$$

Notice that every $A_{N,j}$ is included in a maximal set. In measuring the union of the sets $A_{N,j}$, we first measure the sets smaller (not strictly) than a given maximal set. We then sum over all maximal sets. This means that the sum to the right in (3.5) equals

$$\sum_{\substack{(N,j)\in I \\ A_{N,j}\text{maximal}}} \left(\sum_{\substack{(M,i)\in I \\ A_{M,i} \leq A_{N,j}}} (a^M - a^{M+1}) \nu(\hat{P}_M^i) \right). \tag{3.6}$$

The inner sum in (3.6) may be estimated by

$$\sum_{M \geq N} (a^M - a^{M+1}) \left(\sum_{\substack{i \\ \hat{P}_M^i \subseteq \hat{P}_N^j}} \nu(\hat{P}_M^i) \right) \leqslant a^N \nu(\hat{P}_N^j).$$

Since

$$\nu(\hat{P}_N^j) < \frac{a^{-N}}{\beta} \int_{\hat{P}_N^j} |g|\, d\nu$$

for $(N,j) \in I$, (3.6) is less than

$$\frac{1}{\beta} \sum_{\substack{(N,j)\in I \\ A_{N,j}\text{maximal}}} \int_{\hat{P}_N^j} |g|\, d\nu.$$

Since these $\hat{P}_N^j$ are pairwise disjoint, this proves Proposition 2.

4. The Proof of Lemma 5

For $n = 0, \dots, d-1$ we let χ_n denote the characteristic function of the set

$$\begin{aligned} x_i/D \leq y_i < Dx_i, &\qquad i = 1, \dots, n \\ y_i \leqslant x_i/D, &\qquad i = n+1, \dots, d. \end{aligned} \tag{4.1}$$

By symmetry, it suffices to prove the lemma if we let, for $n = 0, \dots, d-1$,

$$C^\alpha f(x) = \sup_{1/2 \leq r < 1} |C_r^\alpha f(x)|$$

with

$$C_r^\alpha f(x) = \int K_r^\alpha(x, y) \chi_n f(y)\, d\gamma_\alpha(y).$$

Moreover, for the same reason as in the proof of Lemma 4, it is enough to prove boundedness (from $L^1(\gamma_\alpha)$ into $L^{1,\infty}(\gamma_\alpha)$) for the corresponding maximal operator, still called C^α, where $K_r^\alpha(x_i, y_i)$ is replaced by $E_r^\alpha(x_i, y_i)$ for $i = 1, \dots, n$. We write

$$C_r^\alpha(x, y) \equiv \prod_{i=1}^{n} E_r^\alpha(x_i, y_i) \prod_{i=n+1}^{d} K_r^\alpha(x_i, y_i) \equiv E_r^\alpha(x', y') K_r^\alpha(x'', y'') \tag{4.2}$$

with $x = (x', x'')$ in the obvious way. The proof will be split into 5 steps.

Step 1: *Estimates*

Since, for $y_i \sim x_i$,

$$E_r^\alpha(x_i, y_i) \leqslant \frac{x_i^{-\alpha - 1/2}}{(1-r)^{1/2}} \exp\left(x_i + \frac{-\left(\sqrt{x_i} - \sqrt{ry_i}\right)^2}{1-r} \right)$$

and

$$\frac{\left(\sqrt{x_i} - \sqrt{ry_i}\right)^2}{1-r} = \frac{(y_i - x_i/r)^2}{x_i(1-r)} \frac{r^2}{\left(1 + \sqrt{ry_i/x_i}\right)^2} \geqslant c \frac{(y_i - x_i/r)^2}{x_i(1-r)}$$

we get

$$E_r^\alpha(x', y') \leqslant \left[\prod_{i=1}^{n} \frac{x_i^{-\alpha - 1/2}}{(1-r)^{1/2}} \right] \exp\left(\sum_1^n x_i \right) \exp\left(-c \sum_1^n \frac{(y_i - x_i/r)^2}{x_i(1-r)} \right). \tag{4.3}$$

For $i = n+1, \dots, d$ we have

$$r(x_i + y_i) - 2\sqrt{r x_i y_i} \geqslant x_i\left(r - 2\sqrt{ry_i/x_i}\right) \geqslant x_i/4$$

if D is large enough. Hence, if $\alpha \geqslant -1/2$ we get

$$E_r^\alpha(x_i, y_i) \leqslant (1-r)^{-\alpha-1} \exp(-x_i/4(1-r)),$$

while for $-1 < \alpha < -1/2$ the following holds

$$E_r^\alpha(x_i, y_i) \leqslant \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}} \exp(-x_i/4(1-r)).$$

Since

$$D_r^\alpha(x_i, y_i) \leqslant (1-r)^{-\alpha-1} \exp(-x_i/4(1-r)),$$

these estimates imply

$$K_r^\alpha(x'', y'') \leqslant \exp\left(-\frac{1}{4(1-r)} \sum_{n+1}^{d} x_i\right) \prod_{n+1}^{d} \left(\frac{1}{(1-r)^{\alpha+1}} + \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}} \chi_{\{-1<\alpha<-1/2\}}\right). \tag{4.4}$$

For fixed x', we divide $\mathbb{R}_+^n$ into the rectangles

$$2^{k_i-1}\sqrt{x_i(1-r)} \leqslant |y_i - x_i/r| < 2^{k_i}\sqrt{x_i(1-r)}, \qquad k_i = 0, 1, \ldots,$$

where the left hand side is interpreted as 0 for $k_i = 0$.

Further, $\mathbb{R}_+^{d-n}$ is divided into strips (with respect to x'') where

$$4^{j-1}(1-r) \leqslant \sum_{n+1}^{d} x_i < 4^j(1-r), \qquad j = 0, 1, \ldots.$$

Again, $j = 0$ means 0 to the left.

Estimating the right hand side of (4.3) and (4.4) in these rectangles and strips, respectively, gives (we assume $f \geqslant 0$)

$$\int C_r^\alpha(x, y) \chi_n f(y)\, d\gamma_\alpha(y) \leqslant \sum_{j=0}^{\infty} a_j (1-r)^{-(d-n)(\alpha+1)} \chi_{\{x'': \sum x_i < 4^j(1-r)\}} \times \sum_{k_i = 0, 1, \ldots} b_k \left(\prod_1^n \frac{x_i^{-\alpha-1/2}}{(1-r)^{1/2}}\right) \exp\left(\sum_1^n x_i\right) \int_{R_k(r,x')} F(y')\, d\gamma_\alpha(y'), \tag{4.5}$$

where

$$F(y') = \int f(y', y'')\, d\gamma_\alpha(y''),$$

$$R_k(r, x') = \{y': |y_i - x_i/r| < 2^{k_i}\sqrt{x_i(1-r)}, x_i/D < y_i < Dx_i\},$$

$$a_j = 2^{(d-n)j} \exp(-4^{j-2}),$$

and

$$b_k = \exp\left(-c\sum_1^n 4^{k_i}\right).$$

The factor $2^{(d-n)j}$ in a_j is what remains from the case $-1 < \alpha < -1/2$.

Taking the supremum over $1 - r \in [4^{-N-1}, 4^{-N}[$ (written $1 - r \sim 4^{-N}$) in (4.5) yields

$$C^\alpha f(x) \leqslant \sum_j \sum_k \sup_{N=0,1,\ldots} T_{N,k,j} f(x) \equiv \sum_j \sum_k T_{k,j} f(x), \tag{4.6}$$

where

$$T_{N,k,j} f(x) = a_j b_k 4^{N(d-n)(\alpha+1)} \chi_{\{x'': \sum x_i < 4^{j-N}\}} \times \left(\prod_1^n 2^N x_i^{-\alpha-1/2}\right) \exp\left(\sum_1^n x_i\right) \sup_{1-r\sim 4^{-N}} \int_{R_k(r,x')} F(y')\, d\gamma_\alpha(y').$$

Hence, by iterating the result in Theorem B, we get Lemma 5 if we prove

$$\gamma_\alpha\{x:\ T_{k,j} f(x) > \beta\} \leqslant c_{k,j} \frac{1}{\beta} \|f\|_{L^1(\gamma_\alpha)}, \tag{4.7}$$

with $c_{k,j} = c_{k_1}^1 \cdots c_{k_n}^n c_j^0$ such that

$$\sum_{l=0}^\infty c_l^i < \infty, \qquad \sum_{l=0}^\infty c_l^i \ln(1/c_l^i) < \infty, \qquad i = 0, \ldots, n. \tag{4.8}$$

Step 2: *Localization*

Let

$$S_{N,k} F(x') = \left(\prod_{i=1}^n 2^N x_i^{-\alpha-1/2}\right) \exp\left(\sum_1^n x_i\right) \sup_{1-r\sim 4^{-N}} \int_{R_k(r,x')} F(y')\, d\gamma_\alpha(y').$$

Then

$$\begin{aligned}
\gamma_\alpha\{x:\ T_{k,j} f(x) > \beta\} &\leqslant \sum_{N=0}^\infty \gamma_\alpha\{x:\ T_{N,k,j} f(x) > \beta\} \\
&\leqslant \sum_{N=0}^\infty \gamma_\alpha\left\{x'':\ \sum_{n+1}^d x_i < 4^{j-N}\right\} \\
&\qquad \times \gamma_\alpha\{x':\ S_{N,k} F(x') > \beta a_j^{-1} b_k^{-1} 4^{-N(d-n)(\alpha+1)}\} \\
&\leqslant \sum_{N=0}^\infty 4^{(j-N)(d-n)(\alpha+1)} \\
&\qquad \times \gamma_\alpha\{x':\ S_{N,k} F(x') > \beta a_j^{-1} b_k^{-1} 4^{-N(d-n)(\alpha+1)}\}.
\end{aligned}$$

Hence, if we prove

$$(4.9)\qquad \sum_{N=0}^{\infty} a^N \gamma_\alpha\{x' :\ S_{N,k}F(x') > a^N\beta\} \leqslant \frac{A_k}{\beta}\|F\|_{L^1(d\gamma_\alpha(y'))}, \qquad 0 < a < 1,$$

then (4.7) follows (take $a = 4^{-(d-n)(\alpha+1)}$) with

$$(4.10)\qquad c_{k,j} \sim 4^{j(d-n)(\alpha+1)}2^{(d-n)j}\exp(-4^{j-2})\exp\left(-c\sum_1^n 4^{k_i}\right)A_k.$$

Thus, with A_k good enough, (4.9) would complete the proof of the lemma. Actually we have assumed here that $n = 1, \ldots, d-1$.

In the case $n = 0$, we have

$$C^\alpha f(x) \leqslant \sum_{j=0}^{\infty} \sup_{N=0,1,\ldots} 4^{Nd(\alpha+1)}2^{dj}\exp(-4^{j-2})\chi_{\{\sum x_i < 4^{j-N}\}}\|f\|_{L^1(\gamma_\alpha)}$$

$$\equiv \sum_{j=0}^{\infty} \sup_N \tilde{T}_{N,j}f(x).$$

With

$$N_0 = \min\{N:\ 4^{Nd(\alpha+1)}2^{dj}\exp(-4^{j-2})\|f\|_{L^1(\gamma_\alpha)} > \beta\},$$

we get

$$\gamma_\alpha\left\{x:\ \sup_N \tilde{T}_{N,j}f(x) > \beta\right\} \leqslant \gamma_\alpha\left\{x:\ \sum x_i < 4^{j-N_0}\right\}$$

$$\leqslant 4^{d(j-N_0)(\alpha+1)}$$

$$\leqslant 4^{dj(\alpha+1)}2^{dj}\exp(-4^{j-2})\frac{1}{\beta}\|f\|_{L^1(\gamma_\alpha)}.$$

These constants are good enough to imply, in view of Theorem B again, that C^α is bounded.

Next, to simplify the proof of (4.9), we reduce it to a local property, which in particular enables us to get rid of the α's. From now on we drop the primes, *i.e.* $x' = x \in \mathbb{R}^n_+$. Let $\{Q^m:\ m \in \mathbb{Z}^n\}$ be the partition of $\mathbb{R}^n_+$ into rectangles defined by

$$Q^m = \{x:\ D^{m_i} < x_i \leqslant D^{m_i+1},\ i = 1, \ldots, n\}$$

and let

$$\tilde{Q}^m = \{x:\ D^{m_i-1} < x_i \leqslant D^{m_i+2},\ i = 1, \ldots, n\}.$$

In proving (4.9), we may, without loss of generality, assume that $D = 4^M$ for some $M \in \mathbb{N}$. For a given $m \in \mathbb{Z}^n$, let $N_i \in \mathbb{Z}$ be such that

$$D^{m_i+1} = 4^{N_i} \qquad i = 1, \ldots, n \tag{4.11}$$

and let, for $g \geqslant 0$,

$$U^m_{N,k} g(x) = \left(\prod_{i=1}^{n} 2^{N-N_i} \right) \exp\left(\sum_1^n x_i \right) \sup_{1-r \sim 4^{-N}} \int_{|y_i - x_i/r| < 2^{k_i + N_i - N}} g(y)\, dy. \tag{4.12}$$

Then, if $x \in Q^m$

$$S_{N,k} F(x) \leqslant U^m_{N,k}(F|_{\bar{Q}^m} \gamma_0)(x).$$

Hence, the sum in (4.9) is less than or equal to

$$\sum_{m \in \mathbb{Z}^n} \left(\prod_1^n 4^{\alpha N_i} \right) \sum_{N=0}^{\infty} a^N \gamma_0 \{ x \in Q^m \colon\ U^m_{N,k}(F|_{\bar{Q}^m} \gamma_0)(x) > c a^N \beta \}. \tag{4.13}$$

If we prove that the sum over N in (4.13) is less than a constant B_k, not depending on m, times

$$\frac{1}{\beta} \int F|_{\bar{Q}^m} d\gamma_0$$

we get (4.9) with $A_k \sim B_k$.

For a fixed, but arbitrary, $m \in \mathbb{Z}^n$, let $Q = Q^m$ and $U_{N,k} = U^m_{N,k}$.

It remains to prove that, for $0 < a < 1$ and $g = g|_{\bar{Q}} \geqslant 0$,

$$\sum_{N=0}^{\infty} a^N \gamma_0 \{ x \in Q \colon\ U_{N,k} g(x) > a^N \beta \} \leqslant \frac{A_k}{\beta} \|g\|_{L^1(dy)}, \tag{4.14}$$

with A_k good enough (see (4.10) and (4.8)). Of course, A_k may depend on a.

Since the N_i's are related to m as in (4.11), $x \in Q$ clearly implies $x_i \sim 4^{N_i}$ for all i. For symmetry reasons, we may assume that

$$N_1 \geqslant \cdots \geqslant N_k \geqslant N_{k+1} \geqslant \cdots \geqslant N_n.$$

We split the sum in (4.14) into two parts, $\sum_{N > N_1}$ and $\sum_{N \leqslant N_1}$. In Step 3 we will apply Proposition 2 to $\sum_{N > N_1}$, while in Step 4 we estimate the sum $\sum_{N \leqslant N_1}$ term by term. In Step 4 we use a covering argument, which will be justified in Step 5.

We consider $k \in \mathbb{N}^n$ fixed.

Step 3: *Proof of* (4.14) *restricted to* $N > N_1$

For $x \in Q$, let

$$R_N(x) = \{ y \in \mathbb{R}^n_+ \colon\ |y_i - x_i| \leqslant 3 \cdot 2^{k_i + N_i - N} \text{ for every } i \}.$$

Since $N \geqslant N_i$ for each i, this set contains all y with $|y_i - x_i/r| < 2^{k_i + N_i - N}$, $i = 1, \ldots, n$, if $1 - r \sim 4^{-N}$. Thus,

$$U_{N,k} g(x) \leqslant \left(\prod_1^n 2^{N - N_i} \right) \exp\left(\sum_1^n x_i \right) \int_{R_N(x)} g(y)\, dy.$$

Let

$$Q_\eta = \left\{ x \in \mathbb{R}^n_+ : \ \eta \leqslant \sum_{i=1}^n x_i < \eta + 1 \right\}, \qquad \eta = 0, 1, \ldots .$$

Suppose $x \in Q \cap Q_\eta$. Then $R_N(x)$ is included in the set

$$\tilde{Q}_\eta = \bigcup_{l=-a}^{a} Q_{\eta + l}, \qquad a = 3\Sigma 2^{k_i},$$

where $Q_\eta = Q_0$ for $\eta \leqslant 0$, so that

$$U_{N,k} g(x) \leqslant e^\eta \left(\prod_{i=1}^n 2^{k_i} \right) \frac{1}{|R_N(x)|} \int_{R_N(x)} g|_{\tilde{Q}_\eta}(y)\, dy, \qquad x \in Q \cap Q_\eta. \tag{4.15}$$

Here $|\cdot|$ denotes the volume in $\mathbb{R}^n$.

Let P_N, for $N = 0, 1, \ldots$, be the partition of $\mathbb{R}^n_+$ consisting of rectangles with sidelengths $6 \cdot 2^{k_i + N_i - N}$ $(i = 1, \ldots, n)$. Then $\{P_N\}$ defines a piece hierarchy in $\mathbb{R}^n_+$. Let $P_N(x)$ denote the rectangle in P_N which contains x, and $P^j_N(x)$, $j = (j_1, \ldots, j_n)$, $j_i = -1, 0, 1$, the rectangle in P_N obtained by moving one step to the left or right in the x_i-direction depending on whether $j_i = -1$ or $j_i = 1$. If $j_i = 0$ one does not move at all in the x_i-direction. With these notations, we have

$$R_N(x) \subseteq \bigcup_{\substack{j \\ j_i = -1, 0, 1}} P^j_N(x). \tag{4.16}$$

Let

$$U^j_N h(x) = \frac{1}{|P^j_N(x)|} \int_{P^j_N(x)} |h(y)|\, dy \qquad (j_i = -1, 0, 1).$$

For each choice of j we apply Proposition 2 to U^j_N to get, using (4.15) and (4.16),

$$\sum_{N > N_1} a^N \gamma_0 \{ x \in Q : \ U_{N,k} g(x) > a^N \beta \}$$

$$\leqslant \sum_{\eta = 0}^{\infty} \sum_{N > N_1} a^N e^{-\eta} \left| \left\{ x \in Q \cap Q_\eta : \ \sum_j U^j_N (g|_{\tilde{Q}_\eta})(x) > c a^N \beta e^{-\eta} \prod 2^{-k_i} \right\} \right|$$

$$\leqslant \sum_{\eta=0}^{\infty} \frac{1}{\beta} \left(\prod 2^{k_i}\right) \|g|_{\tilde{Q}_\eta}\|_{L^1(dy)}$$

$$\leqslant \frac{\left(\sum 2^{k_i}\right)\left(\prod 2^{k_i}\right)}{\beta} \|g\|_{L^1(dy)}.$$

This proves that the inequality in (4.14), for $\sum_{N \geqslant 0}$ replaced by $\sum_{N > N_1}$, holds with

$$A_k \sim \prod_{i=1}^{n} 2^{2k_i}. \tag{4.17}$$

Step 4: *Proof of* (4.14) *restricted to* $N \leqslant N_1$

Let N be fixed, $0 \leqslant N \leqslant N_1$, and let

$$Q(z) = \{y \in \mathbb{R}^n_+ : |y_i - z_i| < 2^{k_i + N_i - N}, i = 1, \ldots, n\}$$
$$K(z) = \operatorname{con}(Q(z))$$

where $\operatorname{con}(A)$ means the conical hull of A, *i.e.* the intersection of all cones containing A. With these notations we have,

$$U_{N,k}g(x) = \left(\prod_{i=1}^{n} 2^{N-N_i}\right) \exp\left(\sum x_i\right) \sup_{1-r\sim 4^{-N}} \int_{Q(x/r)} g(y)\,dy$$

$$\leqslant \left(\prod_{i=1}^{n} 2^{N-N_i}\right) \exp\left(\sum x_i\right) \int_{K(x)} g(y)\,dy$$

$$\equiv \tilde{U}_{N,k}g(x).$$

Let $l_i = 2 \cdot 2^{k_i + N_i - N}$ be the sidelengths of $Q(x)$.

Our method of estimating

$$\gamma_0\{x \in Q : \tilde{U}_{N,k}g(x) > \beta\} \tag{4.18}$$

depends on whether l_1 is larger or smaller (up to a constant) than the sidelength of Q in the x_1-direction. We get two cases.

Case 1: $2^{k_1} > 2^{N_1 + N}(D-1)/D^2$.

In this case we may estimate the integral in the definition of $\tilde{U}_{N,k}g(x)$ by $\|g\|_{L^1(dy)}$. Set

$$\beta' = \beta \Big/ \left(\prod_i 2^{N-N_i}\right) \|g\|_{L^1(dy)}.$$

Using the fact that the area of $Q \cap \left\{\sum_1^n x_i = t\right\}$ is $\leqslant \prod_{i=2}^{n} 4^{N_i}$ and that $N_i \leqslant N_1$ for all i, we get

$$\begin{aligned}\gamma_0\{x\in Q:\ \tilde{U}_{N,k}g(x)>\beta\} &\leqslant \gamma_0\left\{x\in Q:\ \exp\left(\sum x_i\right)>\beta'\right\}\\ &\leqslant \left(\prod_{i=2}^{n}4^{N_i}\right)\int_{\ln\beta'}^{\infty}e^{-t}\,dt\\ &\leqslant \frac{1}{\beta}\left(\prod_{i=2}^{n}4^{N_i}\right)\left(\prod_{i=1}^{n}2^{N-N_i}\right)\|g\|_{L^1(dy)}\\ &\leqslant \frac{1}{\beta}2^{(n-1)(N+N_1)}2^{N-N_1}\|g\|_{L^1(dy)}.\end{aligned}$$

Thus, in Case 1 we can estimate (4.18) by

$$\frac{c}{\beta}2^{(n-1)k_1}2^{N-N_1}\|g\|_{L^1(dy)}. \tag{4.19}$$

Case 2: $2^{k_1}\leqslant 2^{N_1+N}(D-1)/D^2$.

In Step 5 we will construct a covering of Q, consisting of pairwise disjoint convex cones K_j, $j\in\mathbb{N}$, such that

(i) If $x\in Q\cap K_j$ then $\tilde{Q}\cap K(x)\subseteq\tilde{K}_j$, where $\tilde{K}_j$ is the union of a bounded number of cones «near» K_j.

(ii) $\left|K_j\cap\left\{\sum_1^n x_i=t\right\}\right|_{n-1}\leqslant\left(\sum_1^n 2^{k_i}\right)^{n-1}\prod_2^n 2^{N_i-N}$ if $0<t\leqslant T\equiv\sum_1^n 4^{N_i}$.

Here $|\cdot|_{n-1}$ denotes $(n-1)$-dimensional area. Set

$$\beta_j=\beta\Big/\left(\prod_{i=1}^{n}2^{N-N_i}\right)\int_{\tilde{K}_j}g(y)\,dy.$$

Assuming this covering, we get

$$\begin{aligned}\gamma_0\{x\in Q:\ \tilde{U}_{N,k}g(x)>\beta\} &= \sum_j\gamma_0\{x\in Q\cap K_j:\ \tilde{U}_{N,k}g(x)>\beta\}\\ &\leqslant \sum_j\gamma_0\left\{x\in Q\cap K_j:\ \exp\left(\sum x_i\right)>\beta_j\right\}\\ &\leqslant \sum_j\int_{(T\wedge\ln\beta_j)\vee(T/D)}^{T}e^{-t}\left|K_j\cap\left\{\sum_1^n x_i=t\right\}\right|_{n-1}dt\\ &\leqslant \left(\sum_1^n 2^{k_i}\right)^{n-1}\left(\prod_2^n 2^{N_i-N}\right)\left(\prod_1^n 2^{N-N_i}\right)\frac{1}{\beta}\sum_j\int_{\tilde{K}_j}g(y)\,dy\\ &\leqslant \left(\sum_1^n 2^{k_i}\right)^{n-1}2^{N-N_1}\frac{1}{\beta}\|g\|_{L^1(dy)}.\end{aligned}$$

Combining this with the estimate in Case 1, (4.19), we get that (4.18) is less than or equal to

$$\frac{c}{\beta}\left(\sum_{i=1}^{n} 2^{k_i}\right)^{n-1} 2^{N-N_1}\|g\|_{L^1(dy)}.$$

Since this quantity is summable over $N \leqslant N_1$, the inequality in (4.14), for $\sum_{N \geq 0}$ replaced by $\sum_{N=0}^{N_1}$, follows with

$$A_k \sim \prod_{i=1}^{n} 2^{(n-1)k_i}.$$

Together with the result in Step 3 (see (4.17)), this yields (4.14), with

$$A_k \sim \prod_{i=1}^{n} 2^{k_i \max\{2, n-1\}}.$$

Obviously, A_k is good enough.

Apart from the covering, this proves the lemma.

Step 5: *The covering*

Let

$$K = \operatorname{con}(\tilde{Q}), \qquad L = K \cap \{x_1 = 4^{N_1}/D^2\}.$$

Notice that $x_i \sim 4^{N_i}$, $i = 1, \ldots, n$, for all $x \in L$.

Cover L with pairwise disjoint $(n-1)$-dimensional rectangles $\tilde{R}_j$, $j \in \mathbb{N}$, with sides parallel to the axes and of length

$$L_i = l_i\left(\sum_{j=1}^{n} 2^{k_j}\right)\Big/ 2^{k_i}, \qquad i = 2, \ldots, n.$$

Next, let

$$R_j = \tilde{R}_j \cap L, \qquad K_j = \operatorname{con}(R_j).$$

We have to verify (i) and (ii).

(i) Assume $x \in Q \cap K_j$. For arbitrary y and z in $L \cap K(x)$, the conditions $2^{k_1} \leqslant 2^{N_1+N}(D-1)/D^2$ and $N_1 \geqslant N_i$ imply that $y_i - z_i \leqslant L_i$. Hence,

$$L \cap K(x) \subseteq \bigcup_{n \in I(j)} R_n,$$

where $I(j)$ represents a bounded number of the R_n's near R_j. This clearly implies

$$\tilde{Q} \cap K(x) \subseteq \bigcup_{n \in I(j)} K_n \equiv \tilde{K}_j.$$

(ii) Take $0 < t \leqslant T$, $j \in \mathbb{N}$, and let

$$A = K_j \cap \left\{ \sum_1^n x_i = t \right\}.$$

Let A' be the orthogonal projection of A into the hyperplane $\{x_1 = 0\}$ and let R' be the smallest rectangle, with sides parallel to the axes, in $\{x_1 = 0\}$ containing A'. The sidelengths, L_i', of R' are given by

$$L_i' = \max_{x \in A} x_i - \min_{x \in A} x_i = t\left(\max_{x \in R_j} x_i \Big/ \sum_1^n x_l - \min_{x \in R_j} x_i \Big/ \sum_1^n x_l \right), \qquad (i = 2, \ldots, n).$$

Suppose the function $x_i / \sum x_l$ takes its maximum and minimum (over R_j) at a and b, respectively. Set $b_l = a_l + r_l$, $l = 1, \ldots, n$. Then

$$L_i' = t\left(a_i \sum r_l - r_i \sum a_l \right) \Big/ \sum a_l \sum b_l \leqslant a_i \sum r_l \Big/ \sum a_l - r_i$$

where we used the fact that $t \leqslant T \sim \sum b_l$.

By assumption, $N_l \leqslant N_i$ if $l > i$. Therefore,

$$a_i L_l \Big/ \sum a_l \leqslant L_l \leqslant L_i, \qquad l > i.$$

On the other hand, if $l \leqslant i$ then $N_l \geqslant N_i$ implies

$$a_i L_l \Big/ \sum a_l \leqslant a_i L_l / a_l \sim 4^{N_i - N_l} \left(\sum_{j=1}^n 2^{k_j} \right) 2^{N_l - N} \leqslant L_i, \qquad l \leqslant i.$$

Thus $L_i' \leqslant L_i$ $(i = 2, \ldots, n)$ and, finally,

$$\begin{aligned} |A|_{n-1} = \sqrt{n}\, |A'|_{n-1} &\leqslant \prod_2^n L_i' \\ &\leqslant \prod_2^n L_i \\ &= \left(\sum_1^n 2^{k_i} \right)^{n-1} \prod_2^n 2^{N_i - N}. \end{aligned}$$

The proof of Lemma 5 is complete.

5. The Case $0 < r < 1/2$

The following lemma completes the proof of Theorem 1.

Lemma 6. *The operator*

$$M_0^\alpha f(x) = \sup_{0<r<1/2} |M_r^\alpha f(x)|$$

maps $L^1(\gamma_\alpha)$ *boundedly into* $L^{1,\infty}(\gamma_\alpha)$ *for all* $\alpha > -1$.

PROOF. Since $D_r^\alpha(x_i, y_i) \leqslant c$ for $0 < r < 1/2$, we may replace $K_r^\alpha(x, y)$ (see (3.2)) by

$$E_r^\alpha(x, y) = \prod_i E_r^\alpha(x_i, y_i).$$

Let

$$E^\alpha f(x) = \sup_{0<r<1/2} \left| \int E_r^\alpha(x, y) f(y)\, d\gamma_\alpha(y) \right|.$$

We claim that $(0 < r < 1/2)$

(5.1) $E_r^\alpha(s, t) \leqslant c$ if $\alpha \leqslant -1/2$, $s < 1$ or $\alpha > -1/2$, $s < \alpha + 1/2$.

Assuming $s < 1$, we get

$$\frac{-r(s+t) + 2\sqrt{rst}}{1-r} \leqslant -\frac{\sqrt{rst}}{1-r} + \frac{-rt + 3\sqrt{rt}}{1-r} \leqslant -\sqrt{rst} + C,$$

and, if also $\alpha \leqslant -1/2$,

$$E_r^\alpha(s, t) \leqslant c(rst)^{-\alpha/2 - 1/4} \exp\left(-\sqrt{rst}\right) \leqslant C.$$

Next, assume $\alpha > -1/2$, $s < \alpha + 1/2$. The derivative $(\partial/\partial t)E_r^\alpha(s, t)$ $(t \geqslant (1-r)^2/4rs)$ equals a positive factor times

$$-rt + \sqrt{rst} - \left(\frac{\alpha}{2} + \frac{1}{4}\right)(1-r),$$

which is negative for all r, s, t under consideration. Hence,

$$E_r^\alpha(s, t) \leqslant E_r^\alpha(s, (1-r)^2/4rs) \leqslant C$$

and (5.1) holds.

Because of (5.1) and Proposition 1 we may, when measuring the level sets $\{x \in \mathbb{R}_+^d : E^\alpha f(x) > \beta\}$, neglect the region where any $x_i < 1$ if $\alpha \leqslant -1/2$ or any $x_i < \alpha + 1/2$ if $\alpha > -1/2$. Therefore, we assume $x_i \geqslant c = c(\alpha)$, $i = 1, \ldots, d$.

In the interval $2^{l_i - 1}\sqrt{x_i}/r \leqslant |y_i - x_i/r| \leqslant 2^{l_i}\sqrt{x_i}/r$ $(l_i = 0, 1, \ldots)$, a simple calculation where we use $x_i \geqslant c(\alpha)$ gives the following estimate

$$E_r^\alpha(x_i, y_i) \leqslant a_{l_i} x_i^{-\alpha - 1/2} e^{x_i},$$

with

$$a_{l_i} = 4^{l_i|\alpha+1/2|} \exp(-c2^{l_i}).$$

Thus, for $f \geqslant 0$, we get

$$E^\alpha f(x) \leqslant \sum_{l_i=0}^{\infty} \left(\prod_{i=1}^{d} a_{l_i}\right)\left(\prod_{i=1}^{d} x_i^{-\alpha-1/2} e^{x_i}\right) \sup_{0<r<1/2} \int_{|y_i-x_i/r| \leqslant 2^{l_i}\sqrt{x_i}/r} f(y)\, d\gamma_\alpha(y). \tag{5.2}$$

Let

$$Q(x) = \left\{y \in \mathbb{R}^d_+ : |y_i - x_i| \leqslant 2^{l_i}\sqrt{x_i} \text{ for all } i\right\}.$$

Then the conical hull of $Q(x)$, $\operatorname{con}(Q(x)) = K(x)$, contains the domain of integration in (5.2) for all $r > 0$. Set, for $f \geqslant 0$,

$$A_l f(x) = \left(\prod_i x_i^{-\alpha-1/2} e^{x_i}\right) \int_{K(x)} f(y)\, d\gamma_\alpha(y).$$

By Theorem B, Lemma 6 follows if we prove that, for some M,

$$\gamma_\alpha\{x \in \mathbb{R}^d_+ : A_l f(x) > \beta\} \leqslant \frac{C}{\beta}\left(\prod_{i=1}^{d} 2^{l_i}\right)^M \|f\|_{L^1(\gamma_\alpha)}. \tag{5.3}$$

For $k \in \mathbb{Z}^d$ let $Q^k = \{x \in \mathbb{R}^d_+ : 2^{k_i} \leqslant x_i < 2^{k_i+1}$, for all $i\}$. Of course, (5.3) would follow from

$$\gamma_\alpha\{x \in Q^k : A_l f(x) > \beta\} \leqslant \frac{c_{k,l}}{\beta} \|f\|_{L^1(\gamma_\alpha)}, \tag{5.4}$$

with

$$\sum_k c_{k,l} \leqslant \left(\prod_i 2^{l_i}\right)^M.$$

Depending on whether a sidelength of Q^k is greater or smaller than the corresponding sidelength of $Q(x)$, $x \in Q^k$, we will treat the coordinates in different ways. Therefore, for fixed k and l, we let

$$I' = \{i : k_i \geqslant 2(l_i+1)\}, \qquad I'' = \{1, \ldots, d\} \setminus I',$$

and write $x = (x', x'') \in \mathbb{R}^d_+$, $Q^k = Q^{k'} \times Q^{k''}$ in the obvious way. Further, with $d' = \#I'$, we define $K(x')$ and $A_l F$ quite analogously as above, for $x' \in \mathbb{R}^{d'}_+$ and functions $F \geqslant 0$ on $\mathbb{R}^{d'}_+$. Let

$$F(y') = \int f(y', y'')\, d\gamma_\alpha(y'').$$

Then

$$A_l f(x', x'') \leqslant \left(\prod_{i \in I''} x_i^{-\alpha-1/2} e^{x_i}\right) A_l F(x').$$

Notice that

$$\int_{Q^{k''}} \left(\prod_{i \in I''} x_i^{-\alpha - 1/2} e^{x_i} \right) d\gamma_\alpha(x'') \sim \prod_{i \in I''} 2^{k_i/2}. \tag{5.5}$$

To prove (5.4) we will prove that (for $d' = 1, \ldots, d$)

$$\gamma_\alpha \{ x' \in Q^{k'} \colon A_l F(x') > \beta \} \leqslant \frac{c_{k',l}}{\beta} \| F \|_{L^1(d\gamma_\alpha(y'))}. \tag{5.6}$$

According to Proposition 1, (5.4) follows from (5.5) and (5.6), with

$$c_{k,l} = c_{k',l} \prod_{i \in I''} 2^{k_i/2}. \tag{5.7}$$

(We interpret products and constants with indices over empty sets as 1.)

We will construct a covering of $Q^{k'}$ consisting of pairwise disjoint convex cones $\{K_j\}$ such that, for $c_l = \prod_{I'} 2^{l_i}$,

(i) If $x' \in Q^{k'} \cap K_j$ then $K(x') \subseteq \tilde{K}_j$, where $\tilde{K}_j$ is the union of $\sim c_l^{d'-1}$ cones «near» K_j.

(ii) $\left| K_j \cap \left\{ \sum_{I'} x_i = t \right\} \right|_{d'-1} \leqslant c_l^{d'} \prod_{i \in I' \setminus \{i_0\}} 2^{k_i/2}$ if $0 < t \leqslant \sum_{I'} 2^{k_i + 1} \equiv T$ and i_0 is given by $k_{i_0} = \max_{i \in I'} k_i$.

Suppose we have this covering. Then, with

$$\beta_j = c\beta 2^{(\alpha + 1/2) \sum_{I'} k_i} \left| \int_{\tilde{K}_j} F \, d\gamma_\alpha \right|,$$

we get

$$\begin{aligned}
\gamma_\alpha \{ x' \in Q^{k'} \colon A_l F(x') > \beta \} &\leqslant \sum_j \gamma_\alpha \left\{ x' \in Q^{k'} \cap K_j \colon \exp\left(\sum_{I'} x_i \right) > \beta_j \right\} \\
&\leqslant 2^{\alpha \sum k_i} \sum_j \int_{(T \wedge \ln \beta_j) \vee (T/2)}^{T} e^{-t} \left| K_j \cap \left\{ \sum x_i = t \right\} \right|_{d'-1} dt \\
&\leqslant 2^{\alpha \sum k_i} c_l^{d'} \left(\prod_{i \neq i_0} 2^{k_i/2} \right) \frac{1}{\beta} 2^{-(\alpha + 1/2) \sum k_i} \sum_j \int_{\tilde{K}_j} F \, d\gamma_\alpha \\
&\leqslant \frac{1}{\beta} c_l^{2d'-1} 2^{-k_{i_0}/2} \| F \|_{L^1(\gamma_\alpha)} \\
&\leqslant \frac{1}{\beta} c_l^{2d'-1} \left(\prod_{I'} 2^{-k_i/2d'} \right) \| F \|_{L^1(\gamma_\alpha)}.
\end{aligned}$$

Hence, the covering would imply (5.4), with (see (5.6) and (5.7))

$$c_{k,l} \sim \Big(\prod_{I'} 2^{l_i}\Big)^{2d'-1}\Big(\prod_{I'} 2^{-k_i/2d'}\Big)\Big(\prod_{I''} 2^{k_i/2}\Big).$$

Since

$$\sum_{k\in\mathbb{Z}^d}\Big(\prod_{k_i\geq 2(l_i+1)} 2^{-k_i/2d'}\Big)\Big(\prod_{k_i<2(l_i+1)} 2^{k_i/2}\Big) \lesssim \prod_{i=1}^{d} 2^{l_i},$$

this would complete the proof.

It remains to construct the covering. This will be very similar to the construction in the proof of Lemma 5. Therefore, after describing the cones, we just sketch the verifications of (i) and (ii).

From now on everything takes place in $\mathbb{R}^{d'}_+$, so we drop the primes on the elements. Let

$$\tilde{Q} = \bigcup_{x\in Q^{k'}} Q(x), \qquad K = \operatorname{con}(\tilde{Q}),$$

and

$$L = K \cap \{x:\ x_{i_0} = 2^{k_{i_0}} - 2^{l_{i_0}+k_{i_0}/2}\}.$$

Next, cover L with pairwise disjoint $(d'-1)$-dimensional rectangles $\tilde{R}_j$, $j\in\mathbb{N}$, with sides parallel to the axes and of length

$$L_i = 2^{l_i+k_i/2}, \qquad i\in I', \qquad i\neq i_0.$$

Let

$$R_j = \tilde{R}_j \cap L \quad \text{and} \quad K_j = \operatorname{con}(R_j).$$

(i) Suppose $x\in Q^{k'}\cap K_j$. The condition $k_i \geq 2(l_i+1)$ ensures that $y_i \sim 2^{k_i}$, $i\in I'$, for all $y\in\tilde{Q}$. This and the choice of i_0 imply that $y_i - z_i \lesssim c_l L_i$, whenever y and z are points in $K(x)\cap L$. This gives (i).

(ii) Let $A = K_j \cap \Big\{\sum x_i = t\Big\}$ and let A' be the orthogonal projection of A into $\{x_{i_0}=0\}$. Further, let R' be the smallest rectangle, with sides parallel to the axes, in $\{x_{i_0}=0\}$ containing A'. Imitating the verification of (ii) in Lemma 5, we get

$$L'_i \lesssim c_l L_i, \qquad i\in I', \qquad i\neq i_0,$$

where L'_i are the sidelengths of R'. This gives (ii).

The proof of Lemma 6 is complete.

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On the S_4-Norm of a Hankel Form

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In the paper [2] a rather general theory of Hankel forms over domains Ω in $\mathbb{C}^n$ was developed, and then applied to various special cases, in the first place, to the Fock space [1]. In the latter case also a very curious result (Theorem 7.8 of [2]) was established, which however does *not* follow from the general theory. It states, roughly speaking, that the S_4-norm of a Hankel form in Fock space is exactly equal to a suitable L_4-norm of its symbol, not only equivalent to it, as predicted by the theory. This is a phenomenon peculiar to $p = 4$; it does not hold for any other value of p, except trivially for $p = 2$, of course.

Recently, Wallstén [6] (see also a forthcoming paper by Janson, Upmeier and Wallstén [3]) has published a similar result for the S_4-norm — and even stranger — for the S_6-norm of a «big» Hankel operator on a planar domain ($n = 1$) under rather general assumptions on the underlying measure. (Recall that the study of Hankel forms is essentially equivalent to the study of «small» Hankel operators.)

This has led us to reexamine Theorem 7.8. Essentially, what we do is that we give a new proof of this theorem, which is more transparent. In particular, it gives an indication of that the result is peculiar to Fock space. For instance, it is not true for the unit disk with the Drzhrabashyan (or weighted Bergman) measure

$$d\mu_\alpha(z) = (1 - |z|^2)^\alpha \, dx\, dy \qquad (\alpha > -1).$$

[1] Or Fock-Segal-Bargmann-Fischer-... space.

We shall throughout use the terminology and notation introduced in [2], with slight modifications [2]. Therefore we will assume that the reader has some previous acquaintance with that paper — it suffices that he has browsed through it once and, in particular, had a look at the appendices (Appendix 1 gives a quick summary of the whole theory).

Acknowledgement. Our thanks are due to the authors of [6] and [3] for putting their unpublished or unwritten papers at our disposal, and to Jonathan Arazy for an illuminating conversation about symmetric domains.

1. The S_4-Norm

Let Ω be a domain in $\mathbb{C}^n$ equipped with a measure $d\mu$, subject to certain assumptions, which we need not make precise, so that the main contentions of [2] are valid. Let b be an analytic function on Ω. The Hankel form H_b with symbol b is defined by the formula

$$H_b(f, g) = \int_\Omega \bar{b} f g \, dv \qquad (f, g \in A^2(\mu)),$$

where $A^2(\mu)$ denotes the Hilbert space of square integrable analytic functions on Ω with respect to $d\mu$. Here dv is the measure «associated» with $d\mu$, *i.e.*

$$dv(z) = K(z, \bar{z})^{-1} \, d\mu(z),$$

where $K(z, \bar{w})$ is the reproducing kernel in $A^2(\mu)$. Let P denote orthogonal projection from $L^2(\mu)$ onto $A^2(\mu)$.

With the form H_b we juxtapose an integral operator $\bar{H}_b$ such that

$$H_b(Pf, Pg) = \langle \bar{H}_b f, \bar{g} \rangle_\Omega \qquad (f, g \in L^2(\mu))$$

— it is the «small» Hankel operator corresponding to H_b; in symbols: $\bar{H}_b = \bar{P} M_b P$, where M_b is the multiplication operator and $\bar{P}$ is projection onto $\overline{A^2(\mu)}$.

It is clear that the kernel of $\bar{H}_b$ is

$$\int_\Omega \overline{K(z, \bar{w})} \, \overline{b(w) \frac{dv}{d\mu}(w)} \, K(w, \bar{\zeta}) \, d\mu(w),$$

---

[2] One exception is that the reproducing kernel is denoted $K(z, \bar{w})$; the bar is for book keeping reasons to indicate that the function is conjugate analytic in the second argument.

while the corresponding adjoint kernel is [3]

$$\int_\Omega K(\zeta, \bar{w}) b(w) \frac{d\nu}{d\mu}(w) K(w, \bar{z})\, d\mu(w).$$

(Here $d\nu/d\mu$ is the Radon-Nikodym derivative of $d\nu$ with respect to $d\mu$.) It follows that the operator $\bar{H}_b^* \bar{H}_b$, which may be viewed as an operator from $A^2(\mu)$ into itself, has the kernel [4]

$$\iint_{\Omega\times\Omega} K(z, \bar{w}_1) b(w_1) \frac{d\nu}{d\mu}(w_1) K(w_2, \bar{w}_1) \overline{b(w_2)} \frac{d\nu}{d\mu}(w_2) K(w_2, \bar{\zeta})\, d\mu(w_1)\, d\mu(w_2)$$

$$= \iint_{\Omega\times\Omega} K(z, \bar{w}_1) b(w_1) K(w_2, \bar{w}_1) \overline{b(w_2)} K(w_2, \bar{z})\, d\nu(w_1)\, d\nu(w_2).$$

From this we can obtain

$$\begin{aligned} \|H_b\|_{S_2}^2 &= \|\bar{H}_b\|_{S_2}^2 \\ &= \operatorname{trace} \bar{H}_b^* \bar{H}_b \\ &= \iint_{\Omega\times\Omega} K(w_2, \bar{w}_1)^2 b(w_1) \overline{b(w_2)}\, d\nu(w_1)\, d\nu(w_2). \end{aligned}$$

In particular, if $d\mu$ satisfies assumption (V) in [2], *i.e.* if

$$L(z, \bar{w}) = \text{const} \cdot K(z, \bar{w})^2,$$

where L is the reproducing kernel in $A^2(\nu)$, it follows that

$$\|H_b\|_{S_2}^2 = \|\bar{H}_b\|_{S_2}^2 = \text{const} \cdot \|b\|_{L^2(\nu)}^2.$$

This is of course in [2] (see *e.g.* Theorem 4.7).

Next we observe that the kernel of $(\bar{H}_b^* H_b)^2$ is

$$\iiiint_{\Omega\times\Omega\times\Omega\times\Omega} K(z, \bar{w}_1) b(w_1) K(w_2, \bar{w}_1) \overline{b(w_2)} K(w_2, \bar{w}_3)$$

$$\times\, b(w_3) K(w_4, \bar{w}_3) \overline{b(w_4)} K(w_4, \bar{\zeta})\, d\nu(w_1)\, d\nu(w_2)\, d\nu(w_3)\, d\nu(w_4).$$

---

[3] If $F(z, \zeta)$ is any kernel, then the adjoint kernel is $\overline{F(\zeta, z)}$.

[4] The composition $F(z, \zeta)$ of two kernels $F_1(z, \zeta)$ and $F_2(z, \zeta)$ is defined by the formula

$$F(z, \zeta) = \int_\Omega F_1(z, t) F_2(t, \zeta)\, d\mu(t).$$

This gives

$$
\begin{aligned}
\|H_b\|_{S_4}^4 &= \|\bar{H}_b\|_{S_4}^4 \\
&= \operatorname{trace}(\bar{H}_b^* H_b)^2 \\
&= \iiiint_{\Omega\times\Omega\times\Omega\times\Omega} K(w_2,\bar{w}_1)K(w_2,\bar{w}_3)K(w_4,\bar{w}_1)K(w_4,\bar{w}_3)b(w_1)b(w_3) \\
&\qquad \times \overline{b(w_2)}\;\overline{b(w_4)}\, d\nu(w_1)\, d\nu(w_2)\, d\nu(w_3)\, d\nu(w_4).
\end{aligned} \tag{1}
$$

The S_p-norm of H_b or $\bar{H}_b$, p an even integer, apparently, is given by an analogous expression (see Section 4).

We go on examining the S_4-norm, which seems to be most rewarding case beyond $p = 2$.

We shall compare the integral (1) with

$$
\|b\|_{L_w^4}^4 = \int_\Omega |b(z)|^4 K^{-4}(z,\bar{z})\, d\lambda(z) = \int_\Omega |b(z)|^4\, d\rho(z). \tag{2}
$$

Recall that $d\lambda(z) = K(z,\bar{z})\, d\mu(z)$ is the «invariant» measure (see [2]), so that, actually, $d\rho(z) = K^{-2}(z,\bar{z})\, d\mu(z)$.

We now «polarize», so that we get instead the integrals

$$
\int_{\Omega^4} \prod_{\substack{j=1,3\\k=2,4}} K(w_k,\bar{w}_j)\cdot b_1(w_1)b_3(w_3)\overline{b_2(w_2)}\;\overline{b_4(w_4)}\cdot \prod_{j=1}^{4} d\nu(w_j) \tag{1'}
$$

respectively

$$
\int_\Omega b_1(z)b_3(z)\overline{b_2(z)}\;\overline{b_4(z)}\, d\rho(z), \tag{2'}
$$

with four different functions b_1, b_2, b_3, b_4. In operator terms this means that we are considering the operator $\tilde{H}_{b_1}^* \tilde{H}_{b_2} \tilde{H}_{b_3}^* \tilde{H}_{b_4}$.

Next we let b_1, b_2, b_3, b_4 be «atoms», that is, we take

$$
b_j(z) = L(z,\bar{z}_j) \qquad (j = 1,2,3,4),
$$

where z_1, z_2, z_3, z_4 are points of Ω. It is then clear that the question of proportionality of the forms (1) and (2) (or (1') and (2')) is equivalent to the identity

$$
\begin{aligned}
&\int_\Omega L(z_1,\bar{z})L(z_3,\bar{z})L(z,\bar{z}_2)L(z,\bar{z}_4)\, d\rho(z) \\
&\qquad = \text{const}\cdot K(z_1,\bar{z}_2)K(z_1,\bar{z}_4)K(z_3,\bar{z}_2)K(z_3,\bar{z}_4).
\end{aligned} \tag{*}
$$

2. Discussion of (*)

Some things are now obvious:

1. If hypothesis (V) or (V2) (see [2]) holds, for the measures $d\mu$ and $d\nu$, then (*) is true on the diagonal ($z_1 = z_3$, $z_2 = z_4$). It is then question of the formula

$$\int_\Omega L^2(z_1, \bar{z}) L^2(z, \bar{z}_2)\, d\rho(z) = \text{const} \cdot K^4(z_1, \bar{z}_2).$$

It is clear that both members are proportional to $L^2(z_1, \bar{z}_3)$.

2. On the other hand to verify (*) it suffices to do it on another diagonal ($z_1 = z_2$, $z_3 = z_4$). This is because both members are analytic in «odd» variables, conjugate analytic in «even» ones.

3. (*) is true in the Fock case. This we know ([2], Theorem 7.8), but here is the direct argument. In the Fock case[5] we have

$$K(z, \bar{w}) = e^{z\bar{w}}, \qquad L(z, \bar{w}) = e^{2z\bar{w}}.$$

So it is question of the formula

$$\iint_C e^{2(z_1 + z_3)\bar{z}} e^{2z\overline{(z_2 + z_4)}} e^{-4|z|^2}\, dx\, dy = \text{const} \cdot e^{(z_1 + z_3)\overline{(z_2 + z_4)}},$$

or, writing $a = z_1 + z_3$, $b = z_2 + z_4$,

$$\iint_C e^{2(a\bar{z} + \bar{b}z)} e^{-4|z|^2}\, dx\, dy = \text{const} \cdot e^{a\bar{b}}.$$

To prove this, we first consider the «real» integral ($z = x + iy$)

$$\iint_C e^{2kx + 2ly} e^{-4(x^2 + y^2)}\, dx\, dy.$$

As the variables in the integral separate, it is clear that it evaluates to

$$\text{const} \cdot e^{(k^2 + l^2)/4}.$$

Next we take $k = a + \bar{b}$, $l = i(b - \bar{a})$. Then

$$kx + ly = a\bar{z} + bz,$$

while the previous exponent becomes

$$\frac{1}{4}(k^2 + l^2) = \frac{1}{4}(a + \bar{b})^2 - \frac{1}{4}(a - \bar{b})^2 = a\bar{b}.$$

---

[5] For simplicity (*i.e.* notational simplicity) we take $n = 1$ (and the parameter $\alpha = 1$).

3. The Case of a Bounded Symmetric Domain

Now we shall indicate why (*) cannot be true for a general bounded symmetric domain.

First we continue for a while the general discussion in Section 1. By 2. in Section 2 it suffices to work on the diagonal, that is we may take $z_1 = z_2$, $z_3 = z_4$. Next, we invoke the *Berezin kernel*, which is a general object known to be useful in Hankel theory (*cf.* [1], [4]). In our case, since we have also two measures μ and ν, we have two kernels, namely [6]

$$B_\mu(z, w) = \frac{|K(z, \bar{w})|^2}{K(z, \bar{z}) \cdot K(w, \bar{w})},$$

$$B_\nu(z, w) = \frac{|L(z, \bar{w})|^2}{L(z, \bar{z}) \cdot L(w, \bar{w})}.$$

Then, dividing with $K^2(z_1, \bar{z}_1)K^2(z_3, \bar{z}_3)$, we get from (*) the equivalent formula

$$(**) \qquad \int_\Omega B_\nu(z_1, z)B_\nu(z_3, z)\, d\lambda(z) = \text{const} \cdot B_\mu(z_1, z_3).$$

Putting ourselves in the homogeneous situation, it suffices, again, to establish (**) if one of the points z_1 and z_3 is «fixed», the other «variable». (That is, if Ω is a symmetric domain, we can take *e.g.* $z_3 = 0$.) We can then also say that, *a priori*, the integral to the left in (**) must be a function of $B_\mu(z_1, z_3)$, that is, we get a relation of the form

$$\int_\Omega B_\nu(z_1, z)B_\nu(z_3, z)\, d\lambda(z) = h(B_\mu(z_1, z_3)),$$

where h thus is a function of one real variable. In other words, *the question about the validity of* (*) *or* (**) *has been reduced to the question of deciding whether this function h possibly can be a multiple of the identity.*

Let us apply these considerations to the case of the unit disk, *i.e.* $\Omega = D \subset \mathbb{C}$. Then

$$d\mu(z) = \text{const} \cdot (1 - |z|^2)^\alpha\, dx\, dy,$$

$$K(z, \bar{w}) = (1 - z\bar{w})^{-(\alpha+2)},$$

$$B_\mu(z, w) = \frac{(1 - |z|^2)^{\alpha+2}(1 - |w|^2)^{\alpha+2}}{|1 - z\bar{w}|^{2(\alpha+2)}},$$

---

[6] Since these kernels display now visible properties of analyticity, it is no point to put bars over the arguments.

and similarly for $d\nu(z)$, $L(z, \bar{w})$, $B_\nu(z, w)$, with α replaced by $\beta = 2\alpha + 2$ (if and only if $2\beta + 2 = 2(\alpha + 2)$). We have also

$$d\lambda(z) = \text{const} \cdot \frac{dx\,dy}{(1 - |z|^2)^2} \qquad \text{(Poincaré measure)}.$$

So taking $z_3 = 0$ as our «base point», we are led to consider the integral

$$\int_D \frac{(1 - |z|^2)^{\beta+2}(1 - |w|^2)^{\beta+2}}{|1 - z\bar{w}|^{2(\beta+2)}} (1 - |z|^2)^{\beta+2} \frac{dx\,dy}{(1 - |z|^2)^2}.$$

Lemma. (*cf.* Rudin [5], p. 18).

$$(\gamma + 1) \int_D \frac{(1 - |z|^2)^\gamma}{(1 - |z|^2)^{2\delta}}\, dx\,dy = \pi F(\delta, \delta; \gamma + 2; |z|^2).$$

PROOF. Use the expansion

$$(1 - z\bar{w})^{-\delta} = \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} (z\bar{w})^k \qquad \text{(with } (\delta)_k = \delta(\delta + 1) \cdots (\delta + k - 1))$$

together with the fact

$$(\gamma + 1) \int_D |z|^{2k} (1 - |z|^2)^\gamma\, dx\,dy = \frac{k!\,\pi}{(\gamma + 2)_k}.$$

In our case we have $\gamma = 2\beta + 2 = 4\alpha + 6$, while $\delta = \beta + 2$. If we write out the resulting formula for a general point z_3 we get

$$\int_D B_\nu(z_1, z) B_\nu(z_3, z) \frac{dx\,dy}{(1 - |z|^2)^2}$$

$$(\dagger) \qquad = \frac{\pi}{\gamma + 1} B_\nu(z_1, z_3) F\left(\beta + 2, \beta + 2; \gamma + 2; 1 - \frac{(1 - |z_1|^2)(1 - |z_3|^2)}{|1 - z_1\bar{z}_3|^2}\right).$$

(N.B. — B_ν, *not* B_μ!)

The hypergeometric function to the right in (†) is never a multiple of the identity. Therefore, from this formula it is seen that only in the limiting case $\alpha \to \infty$ (corresponding to Fock space) that (*) or (**) can hold. *So the claim made in Introduction is substantiated.*

Let us elaborate the above somewhat. To make the transition to the Fock case we have also to invoke the «radius of curvature» R and make $R \to \infty$ in such a way that $\alpha/R^2 \to 1$. In other words, we make the substitution $z \mapsto z/R$, so that the unit disk D gets replaced by a disk D_R of radius R, and at the same

renormalize $d\mu$ and $d\nu$ so that we get probability measures. The left hand side of (†) then becomes

$$\int_{D_R} B_\nu(z_1, z)B_\nu(z_3, z)\frac{\alpha+1}{R^2}\frac{dx\,dy}{(1-|z|^2/R^2)^2}.$$

Now

$$B_\nu(z, w) = \frac{\left(1-\frac{|z|^2}{R^2}\right)^{2(\beta+2)}\left(1-\frac{|w|^2}{R^2}\right)^{2(\beta+2)}}{\left|1-\frac{z\bar{w}}{R^2}\right|^{2(\beta+2)}},$$

which in the limit is

$$B_\nu(z, w) = e^{-2|z-w|^2},$$

leading to the integral

$$\int_{\mathbb{C}} B_\nu(z_1, z)B_\nu(z_3, z)\,dx\,dy = \int_{\mathbb{C}} e^{-2|z_1-z|^2-2|z-z_3|^2}\,dx\,dy.$$

At the same time the right hand side is

$$(3)\qquad \pi B_\nu(z_1, z_3)F\left(\beta+2, \beta+2; \gamma+2; 1-\frac{\left(1-\frac{|z_1|^2}{R^2}\right)\left(1-\frac{|z_3|^2}{R^2}\right)}{\left|1-\frac{z_1z_3}{R^2}\right|^2}\right).$$

As in the proof of (†), it is now convenient to put temporarily $z_3 = 0$ and to write $z_1 = z$. The hypergeometric function in (3) is then

$$F\left(\beta+2, \beta+2; \gamma+2; \frac{|z|^2}{R^2}\right)$$

with the series expansion

$$1 + \frac{(\beta+2)^2}{(\gamma+2)}\frac{\frac{|z|^2}{R^2}}{1!} + \frac{(\beta+2)^2(\beta+3)^2}{(\gamma+2)(\gamma+3)}\frac{\left(\frac{|z|^2}{R^2}\right)^2}{2!} + \cdots$$

It is clear that we in the limit then expect the contribution $e^{|z|^2}$ [7]. On the other hand, we get $B_\nu(z, 0) = e^{-2|z|^2}$ for the first factor in (3). These two factors

[7] Although this is quite trivial, we have not been able to find in the literature any asymptotic formula of this kind for the hypergeometric function.

combine to $B_\mu(z, 0) = e^{-|z|^2}$. Restoring the original variables z_1 and z_3, we get of course $B_\mu(z_1, z_3)$. *Thus we have effectively yet another «proof» of Theorem* 7.8!

So much for the unit disk. The case of the unit ball in $\mathbb{C}^n$ is quite parallel (see again [5], p. 18). It seems that the case of a general symmetric domain involves generalized hypergeometric functions encountered by Yan [8] in his recent thesis.

Remark. So far, we have not been able to extract any other information from a formula like (†).

4. On the S_6-Norm

Continuing the computation in Section 1, we are led to the following expressions for the S_6-norm (*cf.* formula (1) there)

$$\begin{aligned}\|H_b\|_{S_6}^6 &= \|\bar{H}_b\|_{S_6}^6 \\ &= \operatorname{trace}(\bar{H}_b^* H_b)^3 \\ &= \int_{\Omega^6} b(w_1)K(w_2, \bar{w}_1)\overline{b(w_2)}K(w_2, \bar{w}_3)b(w_3)K(w_4, \bar{w}_3)\overline{b(w_4)}K(w_4, \bar{w}_5) \\ &\qquad \times b(w_5)K(w_6, \bar{w}_5)\overline{b(w_6)}K(w_6, \bar{w}_1)\prod_{j=1}^{6} d\nu(w_j).\end{aligned}$$

(From this it is also completely clear how the general formula for the S_p-norm, p an even integer (thus the analogue of the preceding formula reads for $p = 8, 10, \ldots$), must look, but we do not bother to write it down.) This expression has to be compared with the integral

$$\int_\Omega |b(z)|^6 \, d\sigma(z),$$

where $d\sigma(z) = K^3(z, \bar{z})\, d\mu(z)$. Again we can polarize. The we obtain corresponding multilinear forms formed with six functions $b_1, \ldots, b_6$. In the next step we specialize to «atoms», thus taking $b_j = K(z, \bar{z}_j)$ $(j = 1, \ldots, 6)$. Even in the Fock case we have not been able to find any interesting relationship between the resulting expressions.

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A Picard type theorem for quasiregular mappings of $\mathbb{R}^n$ into n–manifolds with many ends

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1. Introduction.

Quasiregular mappings are defined as the quasiconformal mappings by replacing the homeomorphism requirement by continuity. More precisely, a continuous map $f : G \to \mathbb{R}^n$, where G is an open set in $\mathbb{R}^n$ and $n \geq 2$, is *quasiregular* if $f \in W^1_{n,\mathrm{loc}}(G)$ and there exists K, $1 \leq K < \infty$, such that

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e.}$$

Here $W^1_{n,\mathrm{loc}}(G)$ is the space of maps that are locally in the Sobolev space W^1_n of L^n–integrable maps with distributional first order derivatives in L^n. Furthermore, $|f'(x)|$ is the operator norm of the formal derivative of f at x defined in terms of the partial derivatives, and $J_f(x) = \det f'(x)$. With this definition a quasiregular map turns out to be differentiable a.e. The definition extends easily to the case $f : M \to N$, where M and N are oriented Riemannian n–manifolds. We say that a continuous map $f : M \to N$ is *locally quasiregular* if for each $x \in M$ there exist neighborhoods U of x and V of $f(x)$ and

charts $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$ is quasiregular in the above sense. Then $T_x f$, the differential of f at x, exists a.e., and f is called *quasiregular* if there exists K, $1 \leq K < \infty$, such that

$$|T_x f|^n \leq K J_f(x) \quad \text{a.e.}$$

In this paper we call f *K-quasiregular* if it satisfies the above conditions with K.

Quasiregular maps constitute a natural generalization into n real dimensions of the analytic functions of one complex variable. For example, a Picard type theorem was established for $n \geq 3$ in 1980 in the following form.

Theorem 1.1. [R1]. *For each $n \geq 3$ and each $K \geq 1$ there exists a positive integer $q_0(n, K)$ such that every K-quasiregular map $f : \mathbb{R}^n \to \mathbf{S}^n \setminus \{a_1, \ldots, a_q\}$, where $q \geq q_0(n, K)$ and $a_1, \ldots, a_q$ are distinct, is constant.*

Theorem 1.1 is known to be sharp for $n = 3$ in the following sense.

Theorem 1.2. [R5]. *For each positive integer p there exists a nonconstant $K(p)$-quasiregular map $f : \mathbb{R}^3 \to \mathbb{R}^3$ omitting at least p points.*

Both these results can be extended considerably to yield a defect relation [R2], [R6], [R7] together with a sharpness result [R6].

The purpose of this paper is to give an affirmative answer to a question posed by M. Gromov, namely, whether one can put any Riemannian metric on $\mathbf{S}^n \setminus \{a_1, \ldots, a_q\}$ and still get the constantness of the K–quasiregular map. We formulate our result as follows:

Theorem 1.3. *For each $n \geq 3$ and each $K \geq 1$ there exists a positive integer $q_0(n, K)$ such that the following holds. Let N be an oriented compact differentiable n-manifold and let $a_1, \ldots, a_q$, $q \geq q_0(n, K)$, be distinct points in N. Suppose $M = N \setminus \{a_1, \ldots, a_q\}$ is given any Riemannian metric $\langle , \rangle$. Then any K-quasiregular map $f : \mathbb{R}^n \to M$ is constant.*

Gromov's question was partly solved in [H1]. For the proof in [H1] a regularity condition was needed on the metric which guaranteed the existence of certain path families. Up to very recently, all existing proofs of Theorem 1.1, including the extension of it in [H1], contain

estimates in terms of moduli of path families (see [R2], [R4], [R7]). The heart of the matter has been to get a rapid growth on a measure ν in $\mathbb{R}^n$ in terms of the number of omitted points. For a Borel set E, $\nu(E)$ is the average covering number of $f \mid E$ over a fixed $(n-1)$–sphere in the target.

The main ingredient to the problem is the paper [EL] by A. Eremenko and J. Lewis where they present a purely potential theoretic proof for Theorem 1.1. They obtain this as an application of a general statement on $\mathcal{A}$–harmonic functions (see Section 2 for definitions). In [EL] there is to each $\mathcal{A}$–harmonic function w attached a measure μ, and this measure can be estimated by the growth behavior of w. In some sense such measures μ replace measures ν described above. To get a proof for Theorem 1.3 a technique developed in [H1] is used to produce certain n–harmonic functions in M with prescribed behavior near the ends of M. In addition to this we need from [EL] their Lemma 1 which relates the growth behavior of the measure μ to that of the corresponding $\mathcal{A}$–harmonic function. The rest of the proof follows more or less ideas established in [R1] and [R4]. An alternate way to get a proof for Theorem 1.3 is to use the construction in Section 3 and the main theorem (Theorem 1) from [EL]. However, for our purpose we can avoid a great deal of the complications in the proof of [EL, Theorem 1]. We therefore feel that it is justified to present a direct proof were only a part of the paper [EL] is used.

2. Quasiregular mappings and $\mathcal{A}$-harmonic functions.

Throughout this section let $G \subset \mathbb{R}^n$ be an open set and let

$$\mathcal{A} : G \times \mathbb{R}^n \to \mathbb{R}^n$$

be a mapping defined by

$$\mathcal{A}(x,h) = \big(\theta(x)h \cdot h\big)^{n/2-1}\theta(x)h \tag{2.1}$$

where $\theta : G \to \mathrm{GL}(\mathbb{R}^n, \mathbb{R}^n)$ is a Borel map with the following properties. For all $x \in G$ the linear map $\theta(x)$ is self-adjoint and there are constants $0 < \alpha \le \beta < \infty$, called *structure constants* of $\mathcal{A}$, such that

$$\alpha^{2/n}\,|h|^2 \le \theta(x)h \cdot h \le \beta^{2/n}\,|h|^2 \tag{2.2}$$

for almost every $x \in G$ and all $h \in \mathbb{R}^n$. It follows from (2.1) and (2.2) that $\mathcal{A}$ has the following additional properties for almost every $x \in G$

and all $h, k \in \mathbb{R}^n$:

$$\mathcal{A}(x,h)\cdot h \geq \alpha\,|h|^n \tag{2.3}$$

$$|\mathcal{A}(x,h)\cdot k| \leq \beta\,|h|^{n-1}\,|k| \tag{2.4}$$

$$\begin{aligned}\big(\mathcal{A}(x,h)-\mathcal{A}(x,k)\big)\cdot(h-k) &\geq \frac{\alpha}{2}\,|h-k|^2\left(|h|^{n-2}+|k|^{n-2}\right)\\ &\geq \alpha\,2^{1-n}\,|h-k|^n\,.\end{aligned} \tag{2.5}$$

For the proof of these properties we refer to [BI]. A continuous function $u \in W^1_{n,\mathrm{loc}}(G)$ is said to be *$\mathcal{A}$-harmonic* in G if it is a weak solution of the equation

$$-\mathrm{div}\mathcal{A}(x,\nabla u) = 0 \tag{2.6}$$

in G, that is,

$$\int_G \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(G)$. The equation (2.6) is the Euler–Lagrange equation of the variational integral

$$\int_G F(x,\nabla u)\,dm\,, \quad F(x,h) = n^{-1}\big(\theta(x)h\cdot h\big)^{n/2}\,.$$

Therefore $\mathcal{A}$–harmonic functions are also called F–extremals in the literature.

An upper semicontinuous function $v : G \to \mathbb{R}\cup\{-\infty\}$ is called *$\mathcal{A}$-subharmonic* in G if for all domains $D \subset\subset G$ and all functions $h \in C(\bar{D})$ which are $\mathcal{A}$–harmonic in D the condition $h \geq v$ on ∂D implies $h \geq v$ in D. If u and v are $\mathcal{A}$–subharmonic in G, then obviously $\max\{u,v\}$ and $\lambda u+\mu$, $\lambda \geq 0$, $\mu \in \mathbb{R}$, are also $\mathcal{A}$–subharmonic in G. A lower semicontinuous function $u : G \to \mathbb{R}\cup\{\infty\}$ is said to be *$\mathcal{A}$-superharmonic* if $-u$ is $\mathcal{A}$–subharmonic.

There is a close connection between $\mathcal{A}$–harmonic functions and quasiregular maps which is proved by Yu. G. Reshetnyak, see *e.g.* [Re, Theorem 11.2]. Namely, if f is a quasiregular mapping of G into another open set $G' \subset \mathbb{R}^n$ and u is $\mathcal{A}$–harmonic in G', then $u\circ f$ is $f^\#\mathcal{A}$–harmonic in G where $f^\#\mathcal{A}$, the *pullback of* $\mathcal{A}$, is of type (2.1) and depends on $\mathcal{A}$ and f. We shall make use of this important result in the following case to where it easily extends. Let G' be an

open subset of an oriented Riemannian n-manifold M. A function $u \in C(G') \cap W^1_{n,\mathrm{loc}}(G')$ is said to be *n-harmonic* in G' if

$$\int_{G'} \langle |\nabla u|^{n-2} \nabla u, \nabla \varphi \rangle \, dm = 0$$

for all $\varphi \in C_0^\infty(G')$. Here $\langle \, , \, \rangle$ is the Riemannian metric of M. Suppose that $f : G \to G'$ is a K-quasiregular map. Then $u \circ f$ is $\mathcal{A}$-harmonic in G where $\mathcal{A}$ is defined by (2.1) with

$$\theta(x) = \begin{cases} J_f^{2/n}(x) T_x f^{-1} T_x f^{-1*}, & \text{if } J_f(x) > 0, \\ \mathrm{id}, & \text{otherwise.} \end{cases}$$

Here $T_x f^{-1*} : \mathbb{R}^n \to T_{f(x)}M$ is the transpose of the linear map $T_x f^{-1}$. We can choose

$$\alpha = 1/K \qquad \text{and} \qquad \beta = K^{n-1} \tag{2.7}$$

as structure constants of $\mathcal{A}$. We can apply the invariance property to chart mappings as follows. For each $x \in M$, choose a neighborhood U of x and a 2-bilipschitz chart $\varphi : U \to B^n(0,r)$. Then φ^{-1} is quasiconformal and $u \circ \varphi^{-1}$ is $\mathcal{A}$-harmonic in $B^n(0,r)$ if u is n-harmonic in U. In this way we easily obtain some basic properties for n-harmonic functions in U such as Hölder continuity, Harnack's inequality, and Harnack's principle.

During the last few years $\mathcal{A}$-harmonic functions have been extensively studied in a more general setting than the one introduced here, see [GLM], [HK]. The study has also been extended to Riemannian n-manifolds to create a classification theory of manifolds based on the existence of $\mathcal{A}$-harmonic functions with various properties [H1], [H2], [HR]. To simplify the presentation in this paper, we consider $\mathcal{A}$-harmonic functions only in the Euclidean n-space and n-harmonic functions on Riemannian n-manifolds.

3. Construction of n-harmonic functions in M.

Let N be an oriented compact differentiable n-manifold and let $M = N \setminus \{a_1, a_2, \ldots, a_q\}$ be equipped with an arbitrary Riemannian metric $\langle \, , \, \rangle$. We suppose that the points $a_i \in N$ are distinct and write $P = \{a_1, \ldots, a_q\}$. In this paper a *condenser* in N will be a pair (G, C)

where $G \neq N$ is an open set in N and C is either a compact subset of G or a closed set in $G \cap M$ such that $C \cup (G \cap P)$ is compact. The *n-capacity* of (G, C) is defined by

$$\operatorname{cap}_n(G, C) = \inf_u \int_{G \cap M} |\nabla u|^n \, dm$$

where the infimum is taken over all $u \in C(\bar{G} \cap M) \cap W^1_{n,\mathrm{loc}}(G \cap M)$ with $u = 0$ in $\partial G \cap M$ and $u = 1$ in $C \cap M$. If the infimum is attained by some function u within the class, we call u an *n-capacity function* of (G, C). In that case u will be n-harmonic in $(G \cap M) \setminus C$. From now on we assume that $\langle \, , \, \rangle$ is given such that the deleted set $P = \{a_1, \ldots, a_q\}$ is of zero n-capacity, *i.e.* $\operatorname{cap}_n(G, P) = 0$ for all open sets $G \subset N$ containing P. Let C be a smooth $(n-1)$-submanifold of N which divides N into two domains U_1 and U_2, one of them, say U_1, containing a_1, and U_2 containing the points $a_2, \ldots a_q$. Write $U = U_1 \setminus \{a_1\} \subset M$ and $V = U_2 \setminus \{a_2, \ldots, a_q\} \subset M$. This notation will be used throughout the paper.

In this section we construct n-harmonic functions in M that in some sense correspond to functions $-\log|x - a_j|$ in the case $M = \mathbf{S}^n \setminus \{a_1, \ldots, a_q\}$ equipped with the Euclidean metric.

Lemma 3.1. *There are functions $h_j \in C(V \cup C)$, $j = 2, \ldots, q$, with the following properties:*

(3.2) *h_j is n-harmonic in V,*

(3.3) *$h_j = 0$ on C,*

(3.4) *$\sup h_j = \infty$ (near a_j), and*

(3.5) *h_j is bounded in some neighborhood of a_k, $k \neq j$.*

PROOF: Fix j and choose a decreasing sequence $C_i \subset U_2$ of compact connected sets with nonempty interiors such that $\cap_i C_i = \{a_j\}$. For each i, there exists a function $w_i \in C(V \cup C)$ which is n-harmonic in $V \setminus C_i$ with $w_i = 0$ in C and $w_i = \left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(1-n)}$ in C_i, and which minimizes the Dirichlet n-integral

$$\int_{V \setminus C_i} |\nabla u|^n \, dm$$

among all functions $u \in C(V \cup C)$ that coincide with w_i in $C \cup C_i$. For each a, $0 < a \leq \left(\mathrm{cap}_n(U_2, C_i)\right)^{1/(1-n)}$, $\min\{1, w_i/a\}$ is the n–capacity function of $(U_2, \{x : w_i(x) \geq a\})$. By [H1, 3.8],

$$\mathrm{cap}_n(U_2, \{x : w_i(x) \geq a\}) = a^{1-n}. \tag{3.6}$$

If $B \subset V$ is a compact topological n–ball, then it follows from a local Harnack inequality that for sufficiently large i

$$\max_B w_i \leq c_1 \min_B w_i$$

where c_1 is independent of i. On the other hand,

$$\min_B w_i = \mathrm{cap}_n\big(U_2, \{x : w_i(x) \geq \min_B w_i\}\big)^{1/(1-n)} \leq \mathrm{cap}_n(U_2, B)^{1/(1-n)},$$

and so

$$\max_B w_i \leq c_1 \,\mathrm{cap}_n(U_2, B)^{1/(1-n)}$$

for large i. Therefore (w_i) is a locally uniformly bounded sequence in V and it follows from the Hölder continuity estimate [GLM, 4.7] that (w_i) is equicontinuous in V. Ascoli's theorem and a standard diagonal process then give a subsequence, still denoted by w_i, which converges locally uniformly in V to a function $h \in C(V)$. Since the class of n–harmonic functions is closed under uniform convergence [HK, 3.2], the limit function h is n–harmonic in V. By a boundary estimate due to V. G. Maz'ya [M, p. 236], h is continuous in $V \cup C$ and $h = 0$ in C. Therefore conditions (3.2) and (3.3) hold.

To show that h is nonconstant, take a small topological $(n-1)$–sphere $S \subset V$ about a_j such that it separates a_j and the rest of the points a_k. Then

$$\max_S w_i \leq c_S \min_S w_i$$

for large i. Here the constant c_S depends on the choice of S. But now

$$\begin{aligned}
\min_S w_i &\geq c_S^{-1} \max_S w_i \\
&= c_S^{-1} \,\mathrm{cap}_n\big(U_2, \{x : w_i(x) \geq \max_S w_i\}\big)^{1/(1-n)} \\
&\geq c_S^{-1} \,\mathrm{cap}_n(U_2, S)^{1/(1-n)} > 0.
\end{aligned}$$

Thus h is not a constant.

Next we show that h is bounded near the points a_k, $k \neq j$. Let $S \subset V$ be as above. Suppose that h takes arbitrary large values near some point a_k, $k \neq j$. In particular, some w_i takes larger values near a_k than $m = \sup_i \max_S w_i$. Truncating w_i by m near the point a_k we obtain a function that coincides with w_i in $C \cup C_i$ but its Dirichlet n–integral over $V \setminus C_i$ is strictly smaller than that of w_i. This is a contradiction, and (3.5) holds.

Suppose that $\sup_V h = \lambda < \infty$. Then $\left(\mathrm{cap}_n(U_2, C_i)\right)^{1/(n-1)} w_i \geq h/\lambda$ in $V \setminus C_i$ for all i. This implies that

$$\mathrm{cap}_n(U_2, C_i) \geq c > 0$$

since $\left(\mathrm{cap}_n(U_2, C_i)\right)^{1/(n-1)} w_i$ is the n–capacity function of (U_2, C_i). In particular,

$$\mathrm{cap}_n(U_2, \{a_j\}) > 0$$

which is a contradiction since $\langle\, , \rangle$ is assumed to be such that P is of zero n–capacity. Hence (3.4) is true.

Similarly, we can find a decreasing sequence $K_i \subset U_1$ of compact connected sets with nonempty interiors such that $\cap_i K_i = \{a_1\}$ and a sequence of functions $u_i \in C(U \cup C)$ with the following properties

(3.7) u_i is n–harmonic in $U \setminus K_i$,

(3.8) $u_i = 0$ in C and $u_i = \left(\mathrm{cap}_n(U_1, K_i)\right)^{1/(1-n)}$ in K_i,

(3.9) $\mathrm{cap}_n\left(U_1, \{x : u_i(x) \geq a\}\right) = a^{1-n}$
for all a, $0 < a \leq \left(\mathrm{cap}_n(U_1, K_i)\right)^{1/(1-n)}$, and

(3.10) $u_i \to h_1$ locally uniformly in U.

The limit function h_1 is continuous in $U \cup C$, n–harmonic in U, $h_1 = 0$ in C, and $\sup_U h_1 = \infty$.

The main result of this section is the following lemma. The proof is similar to the proof of [H1, 4.13].

Lemma 3.11. *There exist n-harmonic functions v_j, $j = 2, \ldots, q$, in*

M and a positive constant κ such that

(3.12) $|v_j| \le \kappa$ *in* C,

(3.13) $|v_j - v_i| \le 2\kappa$ *in* U,

(3.14) $\sup_U v_j = \infty$,

(3.15) $\inf_V v_j = -\infty$,

(3.16) v_j *is bounded from below near* a_k, $k \neq 1, j$, *and*

(3.17) *if* $v_j(x) > \kappa$, *then* $x \in U$; *if* $v_j(x) < -\kappa$, *then* $x \in V$.

PROOF: Fix $j \in \{2, \dots, q\}$, and let C_i and w_i, $i = 1, 2 \dots$, be as in the construction of h_j. For each i, let $\nu = \nu_i$ be the largest integer such that

$$\nu \le \min\left\{ \left(\operatorname{cap}_n(U_2, C_i)\right)^{1/(1-n)}, \left(\operatorname{cap}_n(U_1, K_i)\right)^{1/(1-n)} \right\}.$$

Then ν_i is increasing and tends to ∞ as $i \to \infty$. Write

$$G_\nu = N \setminus \overline{\{x : w_i(x) \ge \nu\}},$$
$$F_\nu = \{x : u_i(x) \ge \nu\},$$

and

$$\gamma_\nu = \left(\operatorname{cap}_n(G_\nu, F_\nu)\right)^{1/(1-n)}.$$

Let $e_\nu \in C(M)$ be a function which is n–harmonic in $G_\nu \setminus F_\nu$,

$$e_\nu | F_\nu = \gamma_\nu/2,$$

and

$$e_\nu | \{x : w_i(x) \ge \nu\} = -\gamma_\nu/2$$

such that it minimizes the Dirichlet n-integral over $G_\nu \setminus F_\nu$ among all functions taking these values in F_ν and $\{x : w_i(x) \ge \nu\}$. First we note that $-\gamma_\nu/2 \le e_\nu \le \gamma_\nu/2$ in $G_\nu \setminus F_\nu$. Applying Harnack's inequality to $e_\nu + \gamma_\nu/2$ and to $\gamma_\nu/2 - e_\nu$ we obtain that, in fact, $-\gamma_\nu/2 < e_\nu < \gamma_\nu/2$ in $G_\nu \setminus F_\nu$. Write $M_\nu = \max_C e_\nu$ and $m_\nu = \min_C e_\nu$. The function $\gamma_\nu^{-1}(e_\nu + \gamma_\nu/2)$ is the n–capacity function of (G_ν, F_ν). Applying [H1, 3.8] to this function yields

$$M_\nu - m_\nu = \operatorname{cap}_n\left(\{x : e_\nu(x) > m_\nu\}, \{x : e_\nu(x) \ge M_\nu\}\right)^{1/(1-n)}.$$

The sets $\{x : e_\nu(x) > m_\nu\}$ and $\{x : e_\nu(x) \geq M_\nu\}$ contain continua E_1 and E_2, respectively, that join C and some point of $F_{\nu_1} \cup P \cup \{x : w_1(x) \geq \nu_1\}$. Therefore

$$\begin{aligned} &\operatorname{cap}_n\big(\{x : e_\nu(x) > m_\nu\}, \{x : e_\nu(x) \geq M_\nu\}\big) \\ &\qquad \geq \inf_{E_1, E_2} \mathrm{M}_n\big(\Delta(E_1, E_2; G_{\nu_1} \setminus F_{\nu_1})\big) > 0 \end{aligned}$$

where E_1 and E_2 are as above. Here $\mathrm{M}_n\big(\Delta(E_1, E_2; G_{\nu_1} \setminus F_{\nu_1})\big)$ is the n–modulus of all curves in $G_{\nu_1} \setminus F_{\nu_1}$ connecting E_1 and E_2. Hence

$$M_\nu - m_\nu \leq \kappa \tag{3.18}$$

where $\kappa < \infty$ is independent of ν.

Next we show that $\{x : e_\nu(x) > M_\nu\} \subset U$ and $U \subset \{x : e_\nu(x) > m_\nu\}$. Suppose there is a point $x_0 \in G_\nu \setminus (U \cup C)$ such that $e_\nu(x_0) = a > M_\nu$. Then the component A of $\{x \in M : e_\nu(x) > a\}$ whose boundary contains x_0 must be a punctured neighborhood of some a_k and A is entirely contained in $G_\nu \setminus (U \cup C)$. Replacing $e_\nu \mid A$ by a decreases the Dirichlet n–integral of e_ν which gives a contradiction. On the other hand, $e_\nu(x) \geq m_\nu$ for every $x \in U$. It follows from Harnack's inequality applied to $e_\nu - m_\nu$ in $U \setminus F_\nu$ that $e_\nu > m_\nu$ in U. Applying [H1, 3.8] to $\gamma_\nu^{-1}(e_\nu + \gamma_\nu/2)$ yields

$$\begin{aligned} \gamma_\nu/2 - M_\nu &= \operatorname{cap}_n\big(\{x : e_\nu(x) > M_\nu\}, F_\nu\big)^{1/(1-n)} \\ &\leq \operatorname{cap}_n(U_1, F_\nu)^{1/(1-n)} = \nu\,. \end{aligned}$$

Similarly,

$$\begin{aligned} \gamma_\nu/2 - m_\nu &= \operatorname{cap}_n\big(\{x : e_\nu(x) > m_\nu\}, F_\nu\big)^{1/(1-n)} \\ &\geq \operatorname{cap}_n(U_1, F_\nu)^{1/(1-n)} = \nu\,. \end{aligned}$$

We claim that $m_\nu \leq 0 \leq M_\nu$, that is, each e_ν takes the value 0 in C. Suppose that we can find ν such that, for instance, $\{x : e_\nu(x) \geq 0\} \subset U$. Then

$$\begin{aligned} \nu^{1-n} &= \operatorname{cap}_n(U, F_\nu) \\ &< \operatorname{cap}_n\big(\{x : e_\nu(x) > 0\}, F_\nu\big) \\ &= (2/\gamma_\nu)^{n-1} \\ &= \operatorname{cap}_n\big(G_\nu, \{x : e_\nu(x) \geq 0\}\big) \\ &< \operatorname{cap}_n(G_\nu, U \cup C) = \nu^{1-n}\,. \end{aligned}$$

This is a contradiction. The case $U \cup C \subset \{x : e_\nu(x) > 0\}$ can be treated similarly. Since $M_\nu - m_\nu \le \kappa$, we obtain $m_\nu \ge -\kappa$ and $M_\nu \le \kappa$.

We have proved

$$\nu - \kappa \le \nu + m_\nu \le \gamma_\nu/2 \le \nu + M_\nu \le \nu + \kappa\,.$$

Thus $u_i - \kappa \le e_\nu \le u_i + \kappa$ on $\partial(U \setminus F_\nu)$ and therefore

$$u_i - \kappa \le e_\nu \le u_i + \kappa \tag{3.19}$$

in $(U \cup C) \setminus F_\nu$, see [GLM, 4.18], [HK, 3.7]. We also have

$$-w_i - \kappa \le e_\nu \le -w_i + \kappa$$

in $\partial(G_\nu \setminus U) \cap M$. We want to show that

$$-w_i - \kappa \le e_\nu \le -w_i + \kappa \tag{3.20}$$

in $(G_\nu \setminus U) \cap M$. For large ν, the boundary of $G_\nu \setminus U$ contains some points of P. Therefore we can not use [GLM, 4.18] in this case. Suppose that there is a point $x_0 \in (G_\nu \setminus U) \cap M$ and $\varepsilon > 0$ such that $e_\nu(x_0) = -w_i(x_0) + \kappa + \varepsilon$. Let A be the open connected subset of $\{x \in M : e_\nu(x) > -w_i(x) + \kappa\}$ that contains x_0. Now e_ν and $-w_i$ are n-harmonic in A, $e_\nu = -w_i + \kappa$ in $\partial A \cap M$, and ∇e_ν and ∇w_i belong to $L^n(A)$. Since P is of zero n-capacity and $e_\nu + w_i - \kappa$ is bounded, we can find a sequence $\varphi_\ell \in C_0^\infty(A)$ such that $\|\nabla e_\nu + \nabla w_i - \nabla \varphi_\ell\|_{n,A} \to 0$ as $\ell \to \infty$. Therefore

$$\begin{aligned}\int_A \big\langle |\nabla e_\nu|^{n-2} \nabla e_\nu, \nabla e_\nu + \nabla w_i \big\rangle \, dm \\ = \int_A \big\langle |\nabla(-w_i)|^{n-2} \nabla(-w_i), \nabla e_\nu + \nabla w_i \big\rangle \, dm = 0\,.\end{aligned}$$

We conclude that $e_\nu = -w_i + \kappa$ in A since

$$2^{1-n} \int_A |\nabla e_\nu + \nabla w_i|^n \, dm$$

$$\le \int_A \big\langle |\nabla e_\nu|^{n-2} \nabla e_\nu - |\nabla(-w_i)|^{n-2} \nabla(-w_i), \nabla e_\nu + \nabla w_i \big\rangle \, dm = 0$$

Hence no such x_0 and ε can exist. The left side inequality of (3.20) can be proved similarly.

As in the proof of Lemma 3.1 we find a subsequence of (e_ν) which converges locally uniformly in M to an n–harmonic function v_j. We can choose κ in (3.18) so large that it is independent of j. Then

$$h_1 - \kappa \le v_j \le h_1 + \kappa \tag{3.21}$$

in $U \cup C$, and

$$-h_j - \kappa \le v_j \le -h_j + \kappa \tag{3.22}$$

in $V \cup C$ for all $j = 2, \dots, q$. It follows from (3.21) and (3.22) that functions v_j, $j = 2, \dots, q$, and the constant κ satisfy the conditions of the Lemma.

4. Measure attached to an $\mathcal{A}$–harmonic function.

Let $\mathcal{A}$ be of type (2.1) and let w be an $\mathcal{A}$–harmonic function in a domain $G \subset \mathbb{R}^n$. Then $w^+ = \max\{0, w\}$ is a continuous $\mathcal{A}$–subharmonic function in G which belongs to $W^1_{n,\mathrm{loc}}(G)$. By [HK, 3.14], w^+ is an *$\mathcal{A}$-subsolution* in G, *i.e.*

$$\int_G \mathcal{A}(x, \nabla w^+) \cdot \nabla \varphi \, dm \le 0$$

for all nonnegative $\varphi \in C_0^\infty(G)$, see also [GLM, 5.17]. Hence

$$\varphi \mapsto -\int_G \mathcal{A}(x, \nabla w^+) \cdot \nabla \varphi \, dm\,, \quad \varphi \in C_0^\infty(G)\,,$$

is a positive linear functional. By the Riesz representation theorem, there exists a measure μ on G such that

$$-\int_G \mathcal{A}(x, \nabla w^+) \cdot \nabla \varphi \, dm = \int_G \varphi \, d\mu\,, \quad \varphi \in C_0^\infty(G)\,. \tag{4.1}$$

In the following we denote by b_i, c_i, $i = 0, 1, \dots$, positive constants which depend only on n and the structure constants of $\mathcal{A}$.

Lemma 4.2. *Let w be $\mathcal{A}$-harmonic in G, let $\bar{B}^n(x, 2r) \subset G$, and suppose $w(z) = 0$ for some $z \in B^n(x, 7r/8)$. Then*

$$\frac{1}{c_0} M(w, x, 7r/8) \le M(-w, x, r) \le c_0 M(w, x, 8r/7)$$

where

$$M(w,x,t) = \max_{\bar{B}^n(x,t)} w\,.$$

PROOF: It is enough to prove the left hand inequality. Set $g = w + M(-w,x,r)$. Then g is nonnegative in $B^n(x,r)$ and we can apply Harnack's inequality to g in the ball $\bar{B}^n(x,7r/8)$ and get for some $c_2 > 1$ that

$$\begin{aligned} M(-w,x,r) = g(z) &\geq \frac{1}{c_2} \max_{\bar{B}^n(x,7r/8)} g \\ &= \frac{1}{c_2}\big(M(w,x,7r/8) + M(-w,x,r)\big)\,, \end{aligned}$$

and the lemma follows.

The next lemma is essentially Lemma 1 in [EL] for the special case $p = n$. For completeness we include the proof which is somewhat shorter in our case.

Lemma 4.3. *Let w be $\mathcal{A}$-harmonic in G, let $\bar{B}^n(x,2r) \subset G$, and suppose $w(z) = 0$ for some $z \in B^n(x,r/4)$. Then*

$$\frac{1}{c_1}\mu(x,r/2) \leq M(w,x,r)^{n-1} \leq c_1\mu(x,2r)\,, \tag{4.4}$$

where μ is the measure defined by (4.1) *and $\mu(x,t) = \mu\big(\bar{B}^n(x,t)\big)$.*

PROOF: To prove the left hand inequality of (4.4) let $\sigma \in C_0^\infty(G)$ be such that $0 \leq \sigma \leq 1$, spt $\sigma \subset B^n(x,3r/4)$, $\sigma \mid B^n(x,r/2) = 1$, and $|\nabla\sigma| \leq 8/r$. Then by (2.4) and Hölder's inequality,

$$\begin{aligned} \mu(x,r/2) &\leq \int_G \sigma\, d\mu = -\int_G \mathcal{A}(x,\nabla w^+)\cdot\nabla\sigma\, dm \\ &\leq \beta \int_{B^n(x,3r/4)} \left|\nabla w^+\right|^{n-1} |\nabla\sigma|\, dm \qquad (4.5) \\ &\leq c_3 \left(\int_{B^n(x,3r/4)} \left|\nabla w^+\right|^n dm\right)^{(n-1)/n}. \end{aligned}$$

By the so called standard estimate (see [GLM, 4.2]) and Lemma 4.2,

$$\begin{aligned}\int_{B^n(x,3r/4)} |\nabla w^+|^n \, dm &\le \int_{B^n(x,3r/4)} |\nabla w|^n \, dm \\ &\le c_4 \operatorname{osc}\big(w, B^n(x,7r/8)\big)^n \\ &\le c_5 \, M(w,x,r)^n \,. \end{aligned} \tag{4.6}$$

The left hand inequality of (4.4) follows from (4.5) and (4.6).

For the right hand inequality of (4.4) let h be the $\mathcal{A}$–harmonic function in $B^n(x,2r)$ such that $h - w^+ \in W^1_{n,0}\big(B^n(x,2r)\big)$. Then $0 \le w^+ \le h$. By Harnack's inequality applied to h we get

$$M(w,x,r) \le M(h,x,r) \le c_2 h(y), \quad y \in B^n(x,r). \tag{4.7}$$

Hölder continuity of w ([GLM, 4.7]) gives for $0 < \varrho \le r/4$ the estimate

$$\begin{aligned} M(w,z,\varrho) &\le \operatorname{osc}\big(w, B^n(z,\varrho)\big) \\ &\le c_5 \left(\frac{\varrho}{r}\right)^\gamma \operatorname{osc}\big(w, B^n(z,r/4)\big) \\ &\le c_5 \left(\frac{\varrho}{r}\right)^\gamma \operatorname{osc}\big(w, B^n(x,r/2)\big) \\ &\le 2c_5 \left(\frac{\varrho}{r}\right)^\gamma \max\{M(w,x,r/2), M(-w,x,r/2)\} \\ &\le c_6 \left(\frac{\varrho}{r}\right)^\gamma M(w,x,r), \end{aligned} \tag{4.8}$$

where we also used Lemma 4.2 and the fact that $w(z) = 0$. Here $\gamma \in]0,1[$ is a constant which depends only on n and the structure constants of $\mathcal{A}$. Choose $\varrho \le r/4$ maximal such that

$$c_6 \left(\frac{\varrho}{r}\right)^\gamma M(w,x,r) \le \frac{1}{2} \min_{\bar{B}^n(x,r)} h \,. \tag{4.9}$$

Let $y \in B^n(z,\varrho)$. Then inequalities (4.7)–(4.9) imply

$$\begin{aligned} \frac{1}{2c_2} h(z) \le \frac{1}{2} \min_{\bar{B}^n(x,r)} h &\le h(y) - \frac{1}{2} \min_{\bar{B}^n(x,r)} h \\ &\le h(y) - M(w,z,\varrho) \\ &\le h(y) - w^+(y) \le c_2 h(z) \,. \end{aligned} \tag{4.10}$$

Let $\varphi \in W^1_{n,0}(B^n(x,2r))$ be defined by $\varphi = \min\{h - w^+, c_2 h(z)\}$ and set $F = \{y \in B^n(x,2r) : \nabla\varphi(y) \neq 0\}$. By (4.10), Poincaré's inequality, and by (2.5) we get

$$
\begin{aligned}
h(z)^n \varrho^n &\leq c_7 \int_{B^n(z,\varrho)} \varphi^n \, dm \leq c_7 \int_{B^n(x,2r)} \varphi^n \, dm \\
&\leq c_8 r^n \int_F |\nabla\varphi|^n \, dm \\
&\leq c_9 r^n \int_F \big(\mathcal{A}(x,\nabla h) - \mathcal{A}(x,\nabla w^+)\big) \cdot \nabla\varphi \, dm \qquad (4.11) \\
&\leq -c_9 r^n \int_{B^n(x,2r)} \mathcal{A}(x,\nabla w^+) \cdot \nabla\varphi \, dm \\
&= c_9 r^n \int_{B^n(x,2r)} \varphi \, d\mu \leq c_9 c_2 r^n h(z) \mu(x,2r) .
\end{aligned}
$$

Since ϱ/r has a positive lower bound depending only on n and the structure constants of $\mathcal{A}$, we obtain the right hand inequality of (4.4) from (4.11) and (4.7).

5. Proof of Theorem 1.3.

Let $f : \mathbb{R}^n \to M$ be a nonconstant K–quasiregular mapping. If the Riemannian metric of M is given such that $\{a_1, \ldots, a_q\}$ is of positive n–capacity, it is possible to construct a positive nonconstant n–harmonic function v in $f(\mathbb{R}^n)$ using the ideas from Section 3. Then $v \circ f$ is a nonconstant positive $\mathcal{A}$-harmonic function in $\mathbb{R}^n$ which is impossible by the Harnack inequality [GLM, 4.15]. Therefore we may assume that $\{a_1, \ldots, a_q\}$ is of zero n–capacity. Let $v_2, \ldots, v_q$ be the n–harmonic functions in M constructed in Section 3 and satisfying properties (3.12)–(3.17). Let $u_j = v_j \circ f$, $j = 2, \ldots, q$. Then each u_j is an $\mathcal{A}$–harmonic function in $\mathbb{R}^n$, and the structure constants of $\mathcal{A}$ depend only on n and K. By Harnack's inequality, u_2 can not be bounded either below or above. Hence u_2 and $-u_2$ take arbitrary large values in $\mathbb{R}^n$, and it follows from (3.17) that there exists $x_0 \in \mathbb{R}^n$ where $f(x_0) \in C$. Set $w_j = u_j - u_j(x_0)$. We write $w = w_2$ and define the measure μ by (4.1).

Let λ_0 be a sufficiently large number so that obvious estimates in inequalities (5.3) and (5.5) are true. We shall choose λ_0 more precisely after (5.7). Since $M(w,x,s) \to \infty$ as $s \to \infty$, Lemma 4.3 shows that $\mu(x,s) \to \infty$ as $s \to \infty$. Take r so large that

$$
\mu(x_0, r) \geq \lambda_0 . \qquad (5.1)
$$

By Lemma 4.3,

$$\mu(x_0,r) \le c_1\, M(w,x_0,2r)^{n-1}\,, \tag{5.2}$$

and by the properties (3.13), (3.17), and Lemma 4.2,

$$\begin{aligned} M(w,x_0,2r) &\le M(w_j,x_0,2r) + 4\kappa \\ &\le 2M(w_j,x_0,2r) \\ &\le 2c_0\, M(-w_j,x_0,3r)\,, \qquad j=3,\dots,q\,. \end{aligned} \tag{5.3}$$

From (5.2) and (5.3) we get

$$\mu(x_0,r) \le b_1 M(-w_j,x_0,3r)^{n-1}, \qquad j=2,\dots,q\,. \tag{5.4}$$

For $j=2,\dots,q$, let $z_j \in \bar{B}^n(x_0,3r)$ be a point such that

$$M(-w_j,x_0,3r) = -w_j(z_j).$$

For $0<\varrho<r$, Hölder continuity, Lemma 4.2, (3.13), and (3.17) imply

$$\begin{aligned} \operatorname{osc}\bigl(w_j,B^n(z_j,\varrho)\bigr) &\le b_2\left(\frac{\varrho}{r}\right)^{\gamma} \operatorname{osc}\bigl(w_j,B^n(x_0,4r)\bigr) \\ &\le 2b_2\left(\frac{\varrho}{r}\right)^{\gamma} \max\{M(w_j,x_0,4r),M(-w_j,x_0,4r)\} \\ &\le 2b_2\, c_0\left(\frac{\varrho}{r}\right)^{\gamma} M(w_j,x_0,5r) \\ &\le 2b_2\, c_0\left(\frac{\varrho}{r}\right)^{\gamma} \bigl(M(w,x_0,5r)+4\kappa\bigr) \\ &\le 4b_2\, c_0\left(\frac{\varrho}{r}\right)^{\gamma} M(w,x_0,5r)\,, \qquad j=2,\dots,q\,. \end{aligned} \tag{5.5}$$

From (5.4), (5.5), and Lemma 4.3 we obtain

$$\max_{\bar{B}(z_j,\varrho)} w_j \le -\left(\frac{\mu(x_0,r)}{b_1}\right)^{1/(n-1)} + b_3\left(\frac{\varrho}{r}\right)^{\gamma} \mu(x_0,10r)^{1/(n-1)} \tag{5.6}$$

for $0<\varrho<r$ and $j=2,\dots,q$.

By covering the ball $\bar{B}^n(x_0,10r)$ by balls of radius $r/4$ we find a constant $d_n>1$, depending only on n, such that if $\mu(x_0,10r) > d_n\mu(x_0,r)$, then there exists $x_1 \in B^n(x_0,10r)$ such that $\mu(x_1,r/4) \ge \mu(x_0,r)$. We may assume that q is so large that $\varrho<r$ can be chosen by the formula $(q-1)\varrho^n = 2(4r)^n$. Suppose that $\mu(x_0,10r) \le d_n\mu(x_0,r)$.

From (5.6) we see that there exists an integer $q_0(n,K)$ such that if $q \geq q_0(n,K)$, then

$$\max_{\bar{B}^n(z_j,\varrho)} w_j \leq -\frac{1}{2}\left(\frac{\mu(x_0,r)}{b_1}\right)^{1/(n-1)} \leq -\frac{1}{2}(\lambda_0/b_1)^{1/(n-1)}. \tag{5.7}$$

From (5.7), (3.16), and (3.17) it follows that with a sufficient large λ_0 each set $fB^n(z_j,\varrho)$ is contained in a neighborhood of a_j such that these neighborhoods are disjoint. Hence the sets $B^n(z_j,\varrho)$ are disjoint. But the balls $B^n(z_j,\varrho)$ are all in $B^n(x_0,4r)$ which is impossible by the choice of ϱ. Therefore, if $q \geq q_0(n,K)$, then $\mu(x_0,10r) > d_n\mu(x_0,r)$, and so $\mu(x_1,r/4) \geq \mu(x_0,r) \geq \lambda_0$ for some $x_1 \in \bar{B}^n(x_0,10r)$. Since the support of μ is contained in $\{x \in \mathbb{R}^n : w(x) = 0\}$, we also have $w(z) = 0$ for some $z \in \bar{B}^n(x_1,r/4)$. We can then repeat the above by starting from the ball $B^n(x_1,r/2)$ instead of $B^n(x_0,r)$. We get a sequence $B^n(x_k,r/2^k)$ of balls converging to a point y with $\mu(x_k,r/2^k) \geq \lambda_0$. By Lemma 4.3 this means that w would be discontinuous at y, which is a contradiction. The theorem is proved.

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Conjecture de Kato sur les ouverts de $\mathbb{R}$

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Résumé. On démontre la conjecture de Kato pour les opérateurs différentiels elliptiques du deuxième ordre sur un ouvert en dimension 1 avec conditions aux limites quelconques. Le cas général se réduit à celui de l'opérateur $T = -\frac{d}{dx}a(x)\frac{d}{dx}$ sur un intervalle, où $a(x)$ est une fonction bornée et accrétive. On montre dans ce cas que le domaine de l'opérateur T est engendré par une base de Riesz constituée d'ondelettes dont le caractère oscillant compense l'action de la fonction non-régulière $a(x)$.

Abstract. We prove Kato's conjecture for second order elliptic differential operators on an open set in dimension 1 with arbitrary boundary conditions. The general case reduces to studying the operator $T = -\frac{d}{dx}a(x)\frac{d}{dx}$ on an interval, when $a(x)$ is a bounded and accretive function. We show for the latter situation that the domain of T is spanned by an unconditional basis of wavelets with cancellation properties that compensate the action of the non-regular function $a(x)$.

Introduction.

La conjecture de Kato est l'énoncé suivant : si T est un opérateur accrétif et maximal, de domaine $\mathcal{D}(T)$ dans un espace de Hilbert H, défini à partir d'une forme sesquilinéaire J, le domaine de la racine carrée de T coïncide avec le domaine de J.

Le problème a été abordé dès les années '60 par T. Kato [10], ainsi que par J.-L. Lions [13]. Il est cependant trop général puisque A. McIntosh a fourni des contre-exemples de nature abstraite [19].

C'est pourquoi l'on examine cette conjecture pour des opérateurs différentiels elliptiques (lesquels ont précisément amené Kato à formuler sa conjecture). Pour ceux du deuxième ordre

$$(*)\quad T = -\sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_i \frac{\partial}{\partial x_i} b_i(x) + \sum_i c_i(x) \frac{\partial}{\partial x_i} + d(x) + s$$

où a_{ij}, $b_i(x)$, $c_i(x)$, $d(x) \in L^\infty(\Omega)$, $\Omega \in \mathbb{R}^n$, $\mathrm{Re} \sum_{i,j} a_{ij}(x)\xi_i\bar{\xi}_j \geq |\xi|^2$ pour tout $x \in \Omega$ and $\xi \in \mathbb{C}^n$, et $s \geq s_0$ assez grand, la conjecture est toujours ouverte si $n \geq 2$. On peut consulter [20] pour un état des lieux complet sur la question. Nous nous intéressons ici à la dimension 1.

Il a fallu attendre 1981 pour que la conjecture soit démontrée par R. Coifman, A. McIntosh et Y. Meyer [5] lorsque $n = 1$, $T = -\frac{d}{dx} a(x) \frac{d}{dx}$ sur $\mathbb{R}$ et $\mathrm{Re}\, a(x) \geq 1$.

Qu'en est-il cependant de la conjecture pour le «même» opérateur lorsque $\mathbb{R}$ est remplacé par un intervalle I ou plus généralement par un ouvert Ω de $\mathbb{R}$? L'objet de ce travail est l'étude de cette question à ce jour toujours ouverte.

Dans le cas d'un intervalle, par exemple $I = (0,1)$ pour fixer les idées, T est associé à une forme sesquilinéaire J définie sur un sous-espace fermé V de $H^1(I)$ qui contient $H_0^1(I)$ par

$$J(f,g) = \int_I a(x) f'(x) \overline{g'(x)}\, dx.$$

L'espace V définit les conditions aux limites et, par abus, $-\frac{d}{dx} a(x) \frac{d}{dx}$ désigne encore l'opérateur T associé à J.

Lorsque $a(x)$ n'est pas régulière, la méthode employée dans [5] (développement de $T^{1/2}$ en une série d'opérateurs multilinéaires dont le terme général est estimé) utilise l'homogénéité par dilatation du problème sur $\mathbb{R}$ et ne semble pas s'adapter à notre situation inhomogène.

Il fallait donc trouver une approche alternative qui permette de traiter ce problème. Cette approche a été mise en place dans [3] où nous avons donné une nouvelle démonstration du résultat de [5] mentionné précédemment. Cet article en est le prolongement et nous apportons ici une réponse positive à la conjecture de Kato pour l'opérateur T introduit ci-dessus quelles que soient les conditions aux limites, mêlées ou non.

Nous démontrons en fait cette conjecture pour tous les opérateurs elliptiques du second ordre à coefficients non-réguliers de la forme $(*)$ en dimension 1. Les idées développées s'appliquent également à certains opérateurs elliptiques d'ordre supérieur. Notre démarche repose sur la caractérisation explicite du domaine de l'opérateur. La conjecture de Kato en sera une conséquence facile à partir des résultats d'interpolation de J.-L. Lions [13].

Venons-en à la méthode utilisée. La caractérisation du domaine de T est obtenue à l'aide d'une base d'ondelettes spécialement conçues. L'idée sous-jacente à l'utilisation des ondelettes est qu'elles sont des «pseudo-fonctions propres» de certains opérateurs différentiels ordinaires. Par exemple, l'image d'une ondelette par l'opérateur d/dx est une autre ondelette, normalisée en tenant compte du support fréquentiel de l'ondelette utilisée et de l'ordre de la dérivation. En d'autres termes les propriétés génériques des ondelettes sont conservées par dérivation.

Bien sûr, une base d'ondelettes ordinaires ne conviendrait pas à notre opérateur; il faut également compenser l'action de la fonction non-régulière $a(x)$. Cette compensation sera contenue dans le caractère oscillant des ondelettes que nous construisons, qui fait intervenir la mesure $a(x)^{-1}dx$.

C'est pour analyser l'opérateur de Cauchy sur une courbe lipschitzienne que les ondelettes adaptées à $a(x)^{-1}$ sur $\mathbb{R}$ ont été construites dans [21]. La même construction sert donc aussi bien à démontrer la conjecture de Kato pour $-\frac{d}{dx}a(x)\frac{d}{dx}$ [3] et l'on retrouve ainsi les liens étroits qui unissent les deux problèmes [5], [12].

L'opérateur de Cauchy n'apparaît plus lorsque nous travaillons sur un intervalle, mais la méthode des bases adaptées reste un outil efficace pour résoudre notre problème. Les difficultés nouvelles liées aux conditions aux limites seront essentiellement d'ordre technique.

Le plan de l'article est le suivant:

1. Situation et énoncé du résultat.
2. Réductions du problème.
3. Esquisse de la démonstration générale pour D^*aD sur un intervalle.
4. Analyses multirésolution avec données au bord sur un intervalle.
5. Ondelettes adaptées à $1/a(x)$.
6. Mesures de Carleson et estimations quadratiques.
7. Estimations quadratiques pour les ondelettes adaptées.
8. Fin de la démonstration pour chaque donnée au bord.
9. Commentaires et extensions.
10. Appendices.

1. Situation et énoncé du résultat.

On se donne un ouvert Ω de $\mathbb{R}$ et $a(x)$, $b(x)$, $c(x)$ et $d(x)$ quatre fonctions mesurables, à valeurs complexes et bornées sur Ω. On dénotera par a, b, c et d les opérateurs de multiplication ponctuelle associés. On supposera constamment que $a(x)$ est accrétive, ce qui veut dire

$$\text{Re } a(x) \geq 1 \qquad \text{pour presque tout } x \in \Omega. \tag{1.1}$$

Soit V un sous-espace fermé de $H^1(\Omega)$ qui contient $H^1_0(\Omega)$. On désigne par D l'opérateur $-i\frac{d}{dx}$ de domaine V. Son adjoint est $D^* = -i\frac{d}{dx}$ de domaine V^*, un (autre) sous-espace fermé de $H^1(\Omega)$ (ne pas confondre avec un quelconque dual de V).

On définit une forme sesquilinéaire sur V par

$$J(f,g) = \int_\Omega \{a\,Df\,\overline{Dg} + b\,f\,\overline{Dg} + c\,Df\,\overline{g} + d\,f\,\overline{g}\}\,dx\,. \tag{1.2}$$

Il existe $r \in \mathbb{R}$ et une constante C_r telles que pour $s \geq r$ et $f \in V$

$$\text{Re } J(f,f) + s\,\|f\|^2 \geq C_r\left(\|Df\|^2 + \|f\|^2\right)\,. \tag{1.3}$$

On convient que $\|\cdot\|$ désigne la norme hilbertienne usuelle sur $L^2(\Omega)$, les autres normes figurant avec des indices.

Nous suivons la théorie de Kato [11]. La forme $J+s$ est maximale-accrétive sur V. L'espace des fonctions $f \in V$ telles que

$$|J(f,g)| \leq C\,\|g\| \qquad \forall\, g \in V,$$

est le domaine $\mathcal{D}(T)$ d'un opérateur T tel que

$$J(f,g) = \langle Tf, g\rangle \qquad \forall\, f \in \mathcal{D}(T),\, \forall\, g \in V.$$

$\mathcal{D}(T)$ est un sous-espace dense dans $L^2(\Omega)$ et dans V pour les topologies respectives. Pour $s \geq r$, l'opérateur $T + s$ est maximal-accrétif. On notera, par abus, $T = D^*aD + D^*b + cD + d$.

Pour Ω et V fixés, on dénotera par $\mathcal{A}$ la classe formée de tous ces opérateurs et par $\mathcal{B}$ la sous-classe constituée des opérateurs D^*aD.

Si $T \in \mathcal{A}$, la racine carrée de $T + s$ est donnée par la formule

$$(T+s)^{1/2} f = \frac{2}{\pi} \int_0^{+\infty} (1 + t^2(T+s))^{-1} (T+s) f\, dt$$

qui converge absolument pour $f \in \mathcal{D}(T)$. Le domaine de $(T+s)^{1/2}$ contient celui de $T + s$, il coïncide avec l'image par $(T+s)^{-1/2}$ de $L^2(\Omega)$ et est indépendant de s. On a le résultat suivant:

Théorème 1. *Soit $T \in \mathcal{A}$ et s assez grand pour que $T + s$ soit maximal-accrétif. Alors $\mathcal{D}((T+s)^{1/2}) = V$.*

Pour $T = D^*aD$, $\Omega = \mathbb{R}$ et $V = H^1(\mathbf{R})$, c'est le théorème de R. Coifman, A. McIntosh et Y. Meyer [5]. La suite de l'article est consacrée à la preuve du Théorème 1.

2. Réductions du problème.

Ces réductions consistent à voir qu'il suffit d'établir le Théorème 1 pour les opérateurs de la classe $\mathcal{B}$ sur un intervalle.

Fixons l'ouvert Ω et l'espace V. Commençons par éliminer les termes d'ordre inférieur.

Proposition 1. *Le Théorème 1 est vrai pour tout $T \in \mathcal{A}$ s'il est vrai pour tout $T \in \mathcal{B}$.*

Rappelons tout d'abord que l'adjoint d'un opérateur maximal-accrétif est maximal-accrétif. On en tire facilement que $\mathcal{A}$ est une classe autoadjointe d'opérateurs (de même pour $\mathcal{B}$).

Soient $T = D^*aD + D^*b + cD + d \in \mathcal{A}$, $T_1 = D^*aD + D^*b \in \mathcal{A}$ et $T_2^* = D^*\bar{a}D \in \mathcal{B}$. On voit facilement que $\mathcal{D}(T) = \mathcal{D}(T_1)$ et que $\mathcal{D}(T_1^*) = \mathcal{D}(T_2^*)$. Pour s choisi assez grand, $T + s$, $T_1 + s$ et $T_2 + s$ sont tous trois maximaux-accrétifs. On prendra $s = 0$ pour faciliter la notation. L'hypothèse de la proposition dit que $\mathcal{D}(T_2^*) = V$.

On utilise alors deux résultats dûs à J. L. Lions [13].

Lemme 1. *Si T, un opérateur non-borné de domaine $\mathcal{D}(T)$ dans un espace de Hilbert H, est maximal-accrétif alors $\mathcal{D}(T^{1/2})$ est l'espace d'interpolation complexe à mi-chemin entre H et $\mathcal{D}(T)$.*

Lemme 2. *Soit ϕ une forme sesquilinéaire maximale-accrétive de domaine $\mathcal{D}(\phi)$ dense dans H (H espace de Hilbert). Soit T l'opérateur maximal-accrétif de domaine $\mathcal{D}(T) \subset \mathcal{D}(\phi)$ associé à ϕ et T^* son adjoint (qui vérifie $\mathcal{D}(T^*) \subset \mathcal{D}(\phi)$). Si $\mathcal{D}((T^*)^{1/2}) = \mathcal{D}(\phi)$ alors $\mathcal{D}(T^{1/2}) = \mathcal{D}(\phi)$.*

Appliquons ces résultats. Puisque T_1^* et T_2^* ont même domaine, il en est de même pour leur racine carrée par le Lemme 1. Celui-ci étant V par hypothèse, le Lemme 2 implique que la racine carrée de T_1 a l'espace V pour domaine. Puisque $\mathcal{D}(T) = \mathcal{D}(T_1)$ une nouvelle application du Lemme 1 montre que ce domaine est également celui de la racine carrée de T. Ceci démontre la Proposition 1.

On est donc ramené à la situation plus simple

(2.1a) $J(f,g) = \int_\Omega a\, Df\, \overline{Dg}$ sur V,

(2.1b) $\operatorname{Re} J(f,f) \geq \|Df\|^2$, $f \in V$,

(2.1c) $T = D^*aD$ est maximal-accrétif de domaine $\mathcal{D}(T) \subset V$.

Bien que (1.3) soit remplacée par la version plus faible (2.1*b*), l'opérateur T reste maximal-accrétif [11].

Proposition 2. $\mathcal{D}(T) = \{f \in V : aDf \in V^*\}$.

D'après la maximalité de T, il suffit de montrer l'inclusion dans un sens. Soit $f \in \mathcal{D}(T)$, alors $h = aDf \in L^2(\Omega)$. Par (2.1a) et la définition de $\mathcal{D}(T)$, la forme antilinéaire $g \mapsto \langle h, Dg \rangle$ définie sur V est donc continue pour la topologie de $L^2(\Omega)$. Ceci revient à dire que $h \in \mathcal{D}(D^*) = V^*$ et la proposition est démontrée.

Passons maintenant à la réduction d'un ouvert arbitraire à un intervalle. Soit Ω un ouvert et V un sous-espace de $H^1(\Omega)$ comme précédemment. Soit I une composante connexe de Ω et V_I la restriction à I des fonctions de V. La restriction à V_I de la forme J définie sur V par (2.1a) induit un opérateur maximal-accrétif T_I de domaine $\mathcal{D}(T_I) \subset V_I$. Le domaine de T se décompose alors en la somme directe hilbertienne des domaines de chaque T_I et T est la somme hilbertienne de ces opérateurs, I parcourant l'ensemble des composantes connexes de Ω. Il est alors immédiat de constater que le Théorème 1 pour T se déduit du Théorème 1 pour chaque T_I. Nous remplaçons désormais l'ouvert Ω par un intervalle I.

Modulo des changements de variables évidents, on peut dresser une liste complète de toutes les situations possibles pour I et V. Pour des raisons de démonstration, nous en distinguerons neuf dans l'ordre suivant:

(2.2a)
$$I = \mathbb{R}, \qquad V = H^1(\mathbb{R}),$$

(2.2b)
$$I = (0, +\infty),\ V = H_0^1(0, +\infty) = \{f \in H^1(0, +\infty) : f(0) = 0\},$$

(2.2c)
$$I = (0, +\infty),\ V = H^1(0, +\infty),$$

(2.2d)
$$I = (0, 1), \qquad V = H_0^1(0, 1) = \{f \in H^1(0, 1) : f(0) = f(1) = 0\},$$

(2.2e)
$$I = (0, 1), \qquad V = \{f \in H^1(0, 1) : f(0) = 0\},$$

$(2.2f)$
$$I = (0,1), \quad V = H^1(0,1),$$

$(2.2g)$
$$I = (0,1), \quad V = \{f \in H^1(0,1) : f(0) = f(1)\},$$

$(2.2h)$
$$I = (0,1), \quad V = \{f \in H^1(0,1) : f(1) = rf(0)\}, r \in \mathbb{R} \setminus \{0,1\},$$

$(2.2i)$
$$I = (0,1), \quad V = \{f \in H^1(0,1) : f(1) = zf(0)\}, z \in \mathbb{C} \setminus \mathbb{R}.$$

Rappelons que le théorème d'injection de Sobolev dit que $H^1(I) \hookrightarrow C^{1/2}(\overline{I})$. Les conditions au bord sont donc bien définies et sont à comprendre comme des limites à gauche ou à droite pour $f \in H^1(I)$.

Pour illustrer les notations du début, l'espace V^* dans la situation $(2.2i)$ est $\{f \in H^1(0,1) : f(1) = \overline{z}^{-1} f(0)\}$. Il s'agit de la condition aux limites «complémentaire» de celle de V. Elle permet d'annuler le terme tout intégré dans l'intégration par parties de $\langle Df, g\rangle$, $f \in V$, $g \in V^*$.

Fixons un couple (I, V) de la liste précédente et considérons l'opérateur $T = D^*aD$ défini par les relations (2.1). La preuve du Théorème 1 repose sur une caractérisation du domaine de T à l'aide d'une base d'ondelettes appropriée.

Théorème 2. *Avec les notations précédentes, il existe un ensemble d'indices Λ, un poids positif $\omega(\lambda)$ défini sur Λ et, pour chaque $\lambda \in \Lambda$, une fonction lipschitzienne $\tau_\lambda(x), x \in I$, appartenant à $\mathcal{D}(T)$ et vérifiant les propriétés suivantes:*

(i) *la collection des fonctions τ_λ est une base de Riesz de chacun des espaces $L^2(I)$, V et $\mathcal{D}(T)$,*

(ii) *une série $\sum \alpha_\lambda \tau_\lambda(x)$ appartient à $L^2(I)$, V et $\mathcal{D}(T)$ si et seulement si $(\alpha_\lambda) \in \ell^2(\Lambda)$, $(\alpha_\lambda) \in \ell^2(\Lambda,\omega)$ et $(\alpha_\lambda) \in \ell^2(\Lambda,\omega^2)$ respectivement.*

Rappelons que dans un espace de Hilbert H, une famille de vecteurs non nuls (e_k) est une base de Riesz (on dit aussi base inconditionnelle),

si tout vecteur $v \in H$ est la somme d'une unique série inconditionnellement convergente $\sum a_k e_k$ et si $\|v\|$ et $\left(\sum |a_k|^2 \|e_k\|^2\right)^{1/2}$ définissent deux normes équivalentes. La base est dite normalisée pour H si $0 < A \leq \|e_k\| \leq B < +\infty$ pour tout k. Dans notre cas, les bases de Riesz de plusieurs espaces, dont $L^2(I)$, seront toujours normalisées pour $L^2(I)$.

Le Théorème 1 se déduit simplement du Théorème 2 comme suit. Dans la base (τ_λ), $L^2(I)$ est caractérisé par $(\alpha_\lambda) \in \ell^2(\Lambda)$ et $\mathcal{D}(T)$ par $(\alpha_\lambda) \in \ell^2(\Lambda, \omega^2)$. En interpolant d'après le Lemme 1, $\mathcal{D}(T^{1/2})$ est donc caractérisé par $(\alpha_\lambda) \in \ell^2(\Lambda, \omega)$. Par conséquent $\mathcal{D}(T^{1/2}) = V$.

La preuve du Théorème 1 repose donc complètement sur celle du Théorème 2 qui occupera la suite de cet article (excepté la Section 9). Nous mettrons en place les outils nécessaires dans les Sections 4 à 7, l'argument final étant présenté dans la Section 8, séparément pour chacune des neuf situations décrites précédemment. A cause de différences techniques notables, il nous a semblé plus simple de le faire ainsi. Toutefois, ces arguments ont une formulation commune dont nous esquissons maintenant les traits principaux.

3. Esquisse de la démonstration du Théorème 2.

Le point de départ est la construction dans $L^2(I)$ d'une base (θ_λ) d'ondelettes dont le caractère oscillant est adapté à la mesure $1/a(x)\,dx$: les intégrales $\int_I \theta_\lambda(x) \frac{dx}{a(x)}$ sont nulles ou bien petites en un sens qu'on précisera.

Les fonctions τ_λ du Théorème 2 sont définies dans V et telles que

$$D\tau_\lambda(x) = c_\lambda \frac{1}{a(x)} \theta_\lambda(x). \tag{3.1}$$

Pour que $\tau_\lambda \in \mathcal{D}(T)$, on voit d'après la Proposition 2 qu'il faut imposer la condition

$$\theta_\lambda \in \mathcal{D}(D^*) = V^*. \tag{3.2}$$

Cela permet de poser

$$c_\lambda \sigma_\lambda(x) = D^* \theta_\lambda(x). \tag{3.3}$$

Il vient alors

$$D^*aD\tau_\lambda(x) = c_\lambda^2\ \sigma_\lambda(x). \tag{3.4}$$

La valeur du facteur c_λ^2 dépend directement de l'ordre de T. C'est ce facteur qui produit le poids ω du Théorème 2. La caractérisation de $\mathcal{D}(T)$ provient du fait que les deux familles (τ_λ) et (σ_λ) forment deux bases de Riesz de $L^2(I)$. La caractérisation de l'espace intermédiaire s'obtient grâce à (3.1).

C'est la relation (3.4) qui exprime cette idée de «presque-diagonalisation» quand les τ_λ et les σ_λ vérifient des propriétés analogues de taille, de régularité et d'oscillation.

La relation (3.1) ne définit τ_λ qu'à une constante près, mais notre choix est limité par le fait que $\tau_\lambda \in V$. C'est donc sur les propriétés de θ_λ qu'il faut s'appuyer pour obtenir les estimations nécessaires sur τ_λ. L'existence des θ_λ est, comme on le voit, au centre de l'argument et la condition (3.2) nous impose de les construire à chaque situation nouvelle (remarquons que le système (θ_λ) doit former une base de $L^2(I)$; pour passer d'un problème aux limites à un autre on ne peut donc pas lui enlever ni lui ajouter une ou plusieurs fonctions sans tuer cette propriété). La méthode de construction suit un algorithme déjà utilisé dans [21]. La clef de voute de tout cet échafaudage est l'inégalité quadratique

$$\left\|\sum \alpha_\lambda\, \theta_\lambda\right\| \leq C\left(\sum |\alpha_\lambda|^2\right)^{1/2},$$

dont la démonstration utilise la notion de mesure de Carleson.

4. Analyses multirésolution avec données au bord sur un intervalle.

Une analyse multirésolution est un algorithme général pour obtenir des bases d'ondelettes. Il a été mis en place dans [14] et [15] dans le cadre de $L^2(\mathbb{R})$. On trouve dans [16], [9] et [18] des analyses multirésolution qui fonctionnent dans le cadre plus général de $L^2(I)$. Les ondelettes obtenues dans ces trois articles diffèrent dans leurs valeurs au bord: elles sont périodiques dans [16], nulles au bord dans [9] et arbitraires au bord dans [18]. Cela résulte de ce que les données au bord imposées à l'analyse multirésolution sont différentes au départ. Nous présentons ici une tentative d'unification de ces constructions.

On retrouvera donc beaucoup de points communs avec les références mentionnées à l'instant.

Notre propos étant de prouver la conjecture de Kato, nous nous sommes restreints aux analyses multirésolution constituées de fonctions splines linéaires, essentiellement à cause de leur simplicité d'emploi. Des constructions plus générales sont données dans [1].

Faisons quelques rappels sur les splines linéaires. Un spline linéaire est une fonction continue et affine par morceaux. Le spline linéaire fondamental est la fonction triangle $\Delta(x) = \sup\{1-|x|, 0\}$, $x \in \mathbb{R}$. Tout spline linéaire est alors combinaison linéaire des translatés et dilatés de $\Delta(x)$: $\Delta((x-y)/t)$, $y \in \mathbb{R}$, $t > 0$. On désignera par $\Delta_{jk}(x)$ la fonction $2^{j/2}\Delta(2^j x - k)$, $j,k \in \mathbb{Z}$. Notons que le support de cette fonction est $[(k-1)2^{-j}, (k+1)2^{-j}]$.

On a l'inégalité

$$\frac{1}{3}\sum_{k\in\mathbb{Z}} |\alpha_k|^2 \leq \int_{\mathbb{R}} \Big|\sum_{k\in\mathbb{Z}} \alpha_k \, \Delta(x-k)\Big|^2 dx \leq \sum_{k\in\mathbb{Z}} |\alpha_k|^2 \tag{4.1}$$

pour toute suite complexe (α_k). On en déduit par un calcul élémentaire

$$\frac{1}{6}\sum_{k=m}^{n} |\alpha_k|^2 \leq \int_{m}^{n} \Big|\sum_{k=m}^{n} \alpha_k \, \Delta(x-k)\Big|^2 dx \leq \sum_{k=m}^{n} |\alpha_k|^2 \tag{4.2}$$

où m, $n \in \mathbb{Z} \cup \{-\infty, +\infty\}$. La valeur $1/6$ ne joue aucun rôle sauf celui d'être strictement positive.

Précisons maintenant les données géométriques de nos analyses multirésolution.

Rappelons que I désigne l'un des trois intervalles $\mathbb{R}$, $(0, +\infty)$ ou $(0,1)$. ∂I dénote le bord de I. Dans toute la suite, j_0 est un entier fini à préciser si $I = (0,1)$, $-\infty$ sinon.

Pour $j > j_0$, Γ_j est un ensemble formé de nombres dyadiques $k2^{-j}$, $k \in \mathbb{Z}$, avec les propriétés suivantes:

- $\Gamma_j \subset \bar{I} = (I \cup \partial I)$,
- tous les nombres dyadiques $k2^{-j}$ tels que $\operatorname{dist}(k2^{-j}, \partial I) \geq 2^{-j}$ appartiennent à Γ_j,
- si $b \in \partial I$ ou bien $b \in \Gamma_j$ pour tout j ou bien $b \notin \Gamma_j$ pour tout j.

Cela entraîne aussitôt:

- $\Gamma_j \subset \Gamma_{j+1}$ pour $j > j_0$,
- la réunion des Γ_j est dense dans $\bar{I}$.

Posons $R_j = 2^j \Gamma_j \subset \mathbb{Z}$. Si $I = \mathbb{R}$ alors $R_j = \mathbb{Z}$, si $I = (0, +\infty)$ alors $R_j = \mathbb{N}$ ou $\mathbb{N}^*$ pour tout j. Enfin, si $I = (0,1)$, R_j contient au moins tous les entiers de 1 à $2^j - 1$. Tous les R_j vérifient la même condition au bord: ils contiennent simultanément 0 ou/et 2^j ou non. Dans ce cas, le cardinal de R_j vaut $2^j + \varepsilon$ où $\varepsilon \in \{-1, 0, 1\}$ et est indépendant de j.

Pour $j > j_0$, on pose $\Lambda_j = \Gamma_{j+1} \setminus \Gamma_j$. Les éléments de Λ_j sont tous les nombres de la forme $(2k+1)2^{-j-1}$ contenus dans I. L'entier k décrit un intervalle d'entiers S_j égal à $\mathbb{Z}$ si $I = \mathbb{R}$, $\mathbb{N}$ si $I = (0, +\infty)$ et $\{0, 1, \ldots 2^j - 1\}$ si $I = (0,1)$. On pose enfin $\Lambda = \cup_{j > j_0} \Lambda_j$.

Passons maintenant à la définition de l'analyse multirésolution. Pour $j > j_0$, on désigne par V_j l'espace des splines linéaires $f(x)$ de carré intégrable et de la forme

$$f(x) = \sum_{k \in R_j} \alpha_k \Delta_{jk}(x), \quad x \in I.$$

V_j hérite de la norme hilbertienne sur $L^2(I)$. On convient que V_{j_0} est nul.

Les fonctions de V_j ont donc leurs nœuds en les points de Γ_j. Les propriétés géométriques des Γ_j se traduisent en les propriétés fonctionnelles suivantes:

- $V_j \subset V_{j+1}$ si $j > j_0$,
- la réunion des espaces V_j est dense dans $L^2(I)$ et si $j_0 = -\infty$ leur intersection est nulle.

Enfin, (4.2) implique que la restriction à I des Δ_{jk}, $k \in R_j$, est une base de Riesz normalisée de V_j avec les estimations

$$\frac{1}{6} \sum_{k \in R_j} |\alpha_k|^2 \leq \int_I \Big| \sum_{k \in R_j} \alpha_k \, \Delta_{jk}(x) \Big|^2 \, dx \leq \sum_{k \in R_j} |\alpha_k|^2.$$

On appellera analyse multirésolution de $L^2(I)$ la suite des espaces V_j . Si $I = (0,1)$ par exemple, alors V_j est de dimension finie égale au cardinal de R_j (j_0 dépend alors de la valeur du nombre ε lié aux données au bord satisfaites par Γ_j). Les données au bord pour Γ_j se

traduisent en conditions aux limites pour les fonctions de V_j : celles-ci sont, par exemple, nulles au bord si les extrémités de I n'appartiennent pas à Γ_j.

Remarquons pour finir cette section que l'espace V_j est contenu dans $H^1(I)$. Cela se vérifie à la main et est laissé au lecteur.

5. Bases adaptées à $1/a(x)$.

Dans toute cette section, (V_j) désigne l'une des analyses multirésolution définies ci-dessus. Commençons par faire une remarque évidente. Toute fonction $f \in V_j$ s'écrit

$$f(x) = \sum_{k \in R_j} 2^{-j/2} f(k2^{-j}) \Delta_{jk}(x), \quad x \in I.$$

C'est une conséquence du fait que $\Delta_{jk}(\ell 2^{-j}) = \delta_{k\ell}$ (les fonctions Δ_{jk} sont parfois appelées des «splines Lagrangiens»). Les fonctions de V_j sont donc complètement déterminées par leurs valeurs sur Γ_j. Pour $j > j_0$, on appelle U_j le sous-espace fermé de V_{j+1} constitué des fonctions qui s'annulent sur Γ_j. On a le lemme évident suivant.

Lemme 3.

1. *V_{j+1} est la somme directe de V_j avec U_j.*
2. *Les fonctions à valeurs réelles $u_\lambda(x) = \Delta_{j+1,2k+1}(x)$, $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$, forment une base de Riesz normalisée de U_j.*

Si $\langle\ ,\ \rangle$ désigne le produit scalaire hilbertien sur $L^2(I)$, pour $j > j_0$ on appelle W_j le supplémentaire orthogonal de V_j dans V_{j+1}. Alors

$$L^2(I) = \bigoplus_{j=-\infty}^{+\infty} W_j, \quad \text{si } j_0 = -\infty,$$

$$L^2(I) = \bigoplus_{j=j_0+1}^{+\infty} W_j \oplus V_{j_0+1}, \quad \text{si } j_0 \text{ est fini.}$$

Soit B la forme bilinéaire, symétrique et continue sur $L^2(I)$:

$$B(f,g) = \int_I f \frac{1}{a} g.$$

Puisque $|a(x)| \leq M = \|a\|_\infty$ et Re $a(x) \geq 1$, il vient

$$\text{Re } B(f, \bar{f}) \geq \frac{1}{M^2} \|f\|^2, \quad \forall f \in L^2(I).$$

La forme B est donc bornée et accrétive sur $L^2(I)$. La restriction de B à tout sous-espace autoadjoint de $L^2(I)$ (c'est-à-dire invariant pour la conjugaison complexe des fonctions) l'est également. Observons que les espaces V_j, U_j et W_j sont autoadjoints (voir l'Appendice 1 pour l'importance de cette remarque).

Pour $j > j_0$, soit X_j le supplémentaire orthogonal de V_j dans V_{j+1} par rapport à B, c'est-à-dire

$$X_j = \{f \in V_{j+1} : B(f, v) = 0 \quad \forall v \in V_j\}.$$

A l'intérieur des espaces V_j, W_j et X_j, on construit des bases de Riesz qui sont décrites dans le théorème suivant.

Théorème 3. *Il existe des constantes $C \geq 0$, $\xi > 0$, $\zeta > 0$ et $\gamma > 0$ telles qu'on ait les assertions suivantes:*

1. *Si $j > j_0$, il existe une base de Riesz normalisée $g_{jk}(x)$, $k \in R_j$, de V_j avec*

$$\xi \sum_{k \in R_j} |\alpha_k|^2 \leq \Big\| \sum_{k \in R_j} \alpha_k \, g_{jk}(x) \Big\|^2 \leq \zeta \sum_{k \in R_j} |\alpha_k|^2.$$

Les fonctions g_{jk} sont à valeurs complexes et sont orthogonales pour B

$$B(g_{jk}, g_{j\ell}) = \delta_{k\ell}, \quad \forall k, \ell \in R_j.$$

De plus, elles vérifient les estimations

$$|g_{jk}(x)| \leq C \, 2^{j/2} \exp\{-\gamma |2^j x - k|\} \tag{5.1a}$$

$$|g_{jk}{}'(x)| \leq C \, 2^{3j/2} \exp\{-\gamma |2^j x - k|\} \tag{5.1b}$$

pour tout $x \in I$ et $k \in R_j$.

2. *Si $j > j_0$, il existe une base orthonormée $\psi_\lambda(x)$, $\lambda \in \Lambda_j$, de W_j. Ces fonctions sont à valeurs réelles et vérifient*

$$|\psi_\lambda(x)| \leq C 2^{j/2} \exp\{-\gamma 2^j |x - \lambda|\} \tag{5.2a}$$

$$|\psi_\lambda{}'(x)| \leq C 2^{3j/2} \exp\{-\gamma 2^j |x - \lambda|\} \tag{5.2b}$$

pour tout $x \in I$ *et* $\lambda \in \Lambda_j$. *Enfin si* $\lambda \in \Lambda_j$, $\ell = 0$ *ou* 1 *et* $d = \operatorname{dist}\{\lambda, \partial I\}$

$$|\int_I (x-\lambda)^\ell \psi_\lambda| \leq C\, 2^{-j\ell} 2^{-j/2} \exp\{-\gamma\, d\, 2^j\}. \tag{5.3}$$

3: *Si* $j > j_0$, *il existe une base de Riesz normalisée* $\theta_\lambda(x)$, $\lambda \in \Lambda_j$, *de* X_j *avec*

$$\xi \sum_{\lambda\in\Lambda_j} |\alpha_\lambda|^2 \leq \Big\| \sum_{\lambda\in\Lambda_j} \alpha_\lambda \theta_\lambda \Big\|^2 \leq \zeta \sum_{\lambda\in\Lambda_j} |\alpha_\lambda|^2.$$

Ces fonctions sont à valeurs complexes, elles sont orthogonales pour B *et vérifient les estimations* (5.2). *Pour* $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$, *elles sont données par*

$$\theta_\lambda(x) = \sum_{\mu\in\Lambda_j} \beta(\lambda,\mu) \mathcal{X}_\mu(x) \tag{5.4}$$

où si $\mu = (2\ell+1)2^{-j-1}$

$$|\beta(\lambda,\mu)| \leq C \exp\{-\gamma|k-\ell|\} \tag{5.5}$$

et

$$\mathcal{X}_\mu(x) = \psi_\mu(x) - \sum_{k\in R_j} \int_I \psi_\mu \frac{1}{a}\, g_{jk} \quad g_{jk}(x). \tag{5.6}$$

Enfin, avec les notations ci-dessus on a

$$|\int_I \frac{(x-\lambda)^\ell}{a} \theta_\lambda| \leq C\, 2^{-j\ell} 2^{-j/2} \exp\{-\gamma\, d\, 2^j\}. \tag{5.7}$$

4. *Les fonctions* θ_λ, $\lambda \in \Lambda$, *sont mutuellement orthogonales pour* B.

REMARQUES. 1. Les dérivées des fonctions précédentes sont des fonctions en escalier. On convient par simplicité de les supposer continues à gauche en tout point.

2. Nous ne donnons pas ici le détail de ces constructions issues de [21] et [17], et dont les étapes essentielles sont rappelées dans l'Appendice 1.

Les W_j étant mutuellement orthogonaux, il est évident que les ψ_λ, $\lambda \in \Lambda$, forment une base orthonormée de $L^2(I)$ (en leur adjoignant, si j_0 est fini, une base orthonormée de V_{j_0+1}). En revanche, la forme B n'étant pas un produit scalaire, le fait que les θ_λ, $\lambda \in \Lambda$, forment une base de Riesz requiert un argument spécifique. On a le théorème suivant.

Théorème 4.

1. *Supposons j_0 infini. Les fonctions θ_λ, $\lambda \in \Lambda$, forment une base de Riesz normalisée de $L^2(I)$ avec les estimations*

$$\xi \sum_{\lambda\in\Lambda} |\alpha_\lambda|^2 \le \Big\| \sum_{\lambda\in\Lambda} \alpha_\lambda \theta_\lambda \Big\|^2 \le \zeta \sum_{\lambda\in\Lambda} |\alpha_\lambda|^2 \tag{5.8}$$

pour toute suite (α_λ), où $\xi > 0$ et $\zeta > 0$ sont indépendantes du choix de (α_λ). Toute fonction $f \in L^2(I)$ est représentée par la série convergente dans $L^2(I)$

$$f(x) = \sum_{\lambda\in\Lambda} \alpha_\lambda \theta_\lambda(x)$$

où

$$\alpha_\lambda = \int_I f \frac{1}{a} \theta_\lambda.$$

2. *L'énoncé analogue est vrai dans le cas où j_0 est fini, à condition d'adjoindre aux θ_λ la base $g_{j_0+1,k}$, $k \in R_{j_0+1}$, de V_{j_0+1}.*

On conviendra d'appeler estimations standard les inégalités (5.1) ou (5.2).

Les inégalités (5.3) et (5.7) seront essentielles par la suite. Elles proviennent du fait que V_j contient approximativement, et en un sens faible, les polynômes de degré 1 restreints à I.

La partie difficile du Théorème 4 est l'inégalité de droite dans (5.8). Elle nécessite l'usage des mesures de Carleson. Nous en aurons par la suite fréquemment besoin et leur consacrons la section suivante.

6. Mesures de Carleson et estimations quadratiques.

Le but de cette section est de relier les estimations quadratiques du type

$$\| \sum \alpha_\lambda m_\lambda \|^2 \leq C \sum |\alpha_\lambda|^2$$

aux mesures de Carleson. Les fonctions m_λ sont soumises à des estimations standard.

Ce lien a été établi dans [17] lorsque $I = \mathbb{R}$. Nous en présentons l'extension à un intervalle quelconque (on pourrait aussi se placer sur un ouvert en dimension supérieure). Il n'y a aura pas de surprise à savoir que les résultats de [17] se généralisent in extenso.

Rappelons que I est un intervalle et ∂I son bord. Λ est un ensemble constitué d'une réunion disjointe d'ensembles Λ_j comme précédemment et $\Lambda \subset I$. Un élément $\lambda = (2k+1)2^{-j-1} \in \Lambda$ est le milieu d'un intervalle dyadique $I(\lambda) = [k2^{-j}, (k+1)2^{-j}[$. On dira que $\lambda \subset \mu$ lorsque $I(\lambda) \subset I(\mu)$. Enfin, on note $\ell(\lambda)$ la longueur de $I(\lambda)$. Par commodité, on supposera toujours $\operatorname{dist}\{\lambda, \partial I\} \geq C\,\ell(\lambda)$ où $C > 0$ est une constante numérique (cela n'est pas nécessaire pour ce qui suit).

A partir de maintenant et jusqu'à la fin de l'article, C dénote une constante positive et γ une constante strictement positive dont les valeurs changent de ligne en ligne mais en restant indépendantes des différents variables et paramètres.

Définition 1. *Soit $m_\lambda(x)$ une suite de fonctions définies sur I. On dit que (m_λ) est une famille de vaguelettes sur I si les m_λ vérifient les estimations standard* (5.2) *et*

$$\int_I m_\lambda = 0. \tag{6.1}$$

Remarquons que nous n'aurons pas vraiment besoin de la localisation exponentielle. Une décroissance inversement polynômiale intégrable suffirait.

Lemme 5. *Soit (m_λ) une famille de vaguelettes sur I. Alors*

$$\| \sum \alpha_\lambda m_\lambda \|^2 \leq C \sum |\alpha_\lambda|^2 \tag{6.2}$$

pour toute suite (α_λ).

Signalons que par dualité (6.2) est équivalente à

$$\sum |\langle f, m_\lambda \rangle|^2 \leq C \|f\|^2 \tag{6.3}$$

pour toute $f \in L^2(I)$.

La preuve du lemme est une adaptation fidèle du cas $I = \mathbb{R}$. Nous n'en décrivons que les grandes lignes.

En développant le membre de gauche dans (6.2) on est amené à montrer que la matrice M formée des coefficients $\langle m_\lambda, m_\mu \rangle$ est bornée sur $\ell^2(\Lambda)$. Soient $\lambda = (2k+1)2^{-j-1}$, $\mu = (2\ell+1)2^{-j'-1} \in \Lambda$. Supposons que $j > j'$ ou, ce qui revient au même, $\ell(\lambda) < \ell(\mu)$. La fonction m_λ est essentiellement localisée sur $I(\lambda)$ alors que m_μ est régulière et plate sur $I(\lambda)$. Grâce à (6.1) on peut écrire

$$\langle m_\lambda, m_\mu \rangle = \int_I m_\lambda \, (\overline{m_\mu - m_\mu(\lambda)}).$$

La régularité de m_λ n'intervient pas et l'on peut maintenant prolonger m_λ par 0 en dehors de I. On est alors ramené au cas $I = \mathbb{R}$. Seules les estimations standard sont utilisées et l'on obtient

$$|\langle m_\lambda, m_\mu \rangle| \leq C \, 2^{-|j-j'|(1/2+\delta)} \exp\{-\gamma 2^{\inf\{j,j'\}} |\lambda - \mu|\} \tag{6.4}$$

où $0 < \delta < 1$.

Pour finir ces estimations impliquent que M est bornée sur $\ell^1(\Lambda, p)$ et sur $\ell^\infty(\Lambda, p^{-1})$, donc sur $\ell^2(\Lambda)$ par interpolation, où $p(\lambda) = 2^{-j/2}$ si $\lambda \in \Lambda_j$. Nous omettons les détails et renvoyons à [17].

Définition 2. *Soit* (c_λ) *une suite de nombres complexes. On dit que* (c_λ) *est une suite de Carleson s'il existe* $C \geq 0$ *telle que pour tout* $\lambda \in \Lambda$,

$$\sum_{\mu \in \Lambda, \mu \subset \lambda} |c_\mu|^2 \leq C|\lambda|. \tag{6.5}$$

La raison de cette terminologie est que la mesure

$$\nu(x,t) = \sum_{\mu \in \Lambda} |c_\mu|^2 \delta(x - \mu, t - \ell(\mu))$$

est alors une mesure de Carleson sur $\mathbb{R} \times \mathbb{R}^{+*}$ (voir, par exemple, [22]). La meilleure $\sqrt{C}$ dans (6.5) sera appelée la constante de Carleson de la suite (c_λ).

Théorème 5. *Soit h_λ, $\lambda \in \Lambda$, une suite de fonctions vérifiant les estimations standard. Alors les assertions suivantes sont équivalentes.*

(i) $\left\| \sum \alpha_\lambda h_\lambda \right\|^2 \leq C \sum |\alpha_\lambda|^2$,

(ii) $\left(\int_I h_\lambda\right)$ *est une suite de Carleson.*

ESQUISSE DE LA PREUVE. Soit $\varepsilon > 0$ assez petit. Soit $\varphi(x)$ une fonction C^∞ positive, à support dans $(-\varepsilon, \varepsilon)$ et d'intégrale égale à 1. Pour $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$, posons $\varphi^\lambda(x) = 2^j \varphi(2^j(x-\lambda))$. Remarquons alors qu'un choix convenable de ε garantit que le support de φ^λ soit contenu dans I pour tout λ. Posons ensuite $m_\lambda(x) = h_\lambda(x) - \left(\int_I h_\lambda\right) \varphi^\lambda(x)$. Alors (m_λ) est une famille de vaguelettes sur I et le Lemme 5 s'applique. En utilisant (6.3) au lieu de (6.2), on en déduit que (i) est équivalent à

$$\text{(iii)} \qquad \sum |c_\lambda|^2 |\langle f, \varphi^\lambda \rangle|^2 \leq C \|f\|^2,$$

où on a posé $c_\lambda = \int_I h_\lambda$.

Cette inégalité appliquée à la fonction indicatrice de chaque $I(\mu)$ implique (ii). Réciproquement, elle est une conséquence de la version discrète d'une célèbre inégalité due à L. Carleson:

$$\sum |a_\lambda|^2 q(\lambda)^2 \leq \int_{\mathbb{R}} \sup q(\lambda)^2 \, dx$$

pour toute suite de Carleson (a_λ) de constante 1 et toute fonction positive $q(\lambda)$. Pour chaque $x \in \mathbb{R}$ la borne sup est prise sur tous les nombres $\lambda \in \Lambda$ tels que $x \in I(\lambda)$.

Ici, $q(\lambda) = |\langle f, \varphi^\lambda \rangle|$ si bien que cette borne sup est contrôlée par $C(Mf)(x)$, la fonction maximale de Hardy et Littlewood de $f(x)$ (on a posé $f(x) = 0$ si $x \notin I$). On conclut en utilisant la continuité de M sur $L^2(\mathbb{R})$.

Citons un exemple d'une suite de Carleson. Soit E un ensemble fini de points de $\mathbb{R}$. Alors une suite (c_λ) telle que

$$(6.6) \quad |c_\lambda| \leq C\, 2^{-j/2} \exp\{-\gamma\, 2^j \operatorname{dist}\{\lambda, E\}\}, \quad \lambda = (2k+1)2^{-j-1} \in \Lambda,$$

est de Carleson. Cela se vérifie à la main. Un exemple moins immédiat est donné par le

Théorème 6. *Soit (m_λ) une famille de fonctions définies sur I et vérifiant pour tout $\lambda \in \Lambda$*

$$|m_\lambda(x)| \leq C\, 2^{j/2} \exp\{-\gamma\, 2^j |x - \lambda|\}$$

et

$$\| \sum \alpha_\lambda\, m_\lambda \|^2 \leq C \sum |\alpha_\lambda|^2.$$

Alors pour tout $b(x) \in L^\infty(I)$, $(\langle b, m_\lambda \rangle)$ est une suite de Carleson.

REMARQUE. Cet énoncé nous suffira par la suite mais n'est bien sûr pas optimal. On pourrait supposer $b \in$ «BMO(I)» à condition de préciser de quel espace BMO sur l'intervalle nous parlons puisqu'il y a plusieurs définitions possibles. Dans ce cas cependant il faudrait rajouter une hypothèse sur la valeur moyenne de chaque m_λ.

PREUVE. Soit $\lambda \in \Lambda$, $I(\lambda)$ l'intervalle associé et Q l'intervalle de même centre ayant une longueur double. On pose $b(x) = b_1(x) + b_2(x)$ où $b_1(x) = b(x)\chi_Q(x)$. On observe tout d'abord que

$$\sum_{\mu \in \Lambda, \mu \subset \lambda} |\langle b_1, m_\mu \rangle|^2 \leq \sum_{\mu \in \Lambda} |\langle b_1, m_\mu \rangle|^2 \leq C\, \|b_1\|^2 \leq C\, \|b\|_\infty^2\, |\lambda|$$

par la version duale de l'inégalité quadratique dans l'hypothèse du théorème.

Il reste à établir la même estimation pour b_2. Remarquons que $b_2(x) = 0$ pour $x \in Q$. Donc, pour $\mu \subset \lambda \subset Q$, $\langle b_2, m_\mu \rangle = \int_{I \cap Q^c} b_2\, \overline{m_\mu}$ se majore brutalement grâce aux estimations sur les m_μ par

$$C\, \|b\|_\infty\, |\mu|^{1/2} \exp\left\{ -\gamma\, \frac{\ell(\lambda)}{\ell(\mu)} \right\}.$$

En remarquant que le nombre d'intervalles μ de longueur fixée contenus dans λ est $|\lambda|/|\mu|$, la somme $\sum |\langle b_2, m_\mu \rangle|^2$ indexée par les $\mu \in \Lambda$ tels que $\mu \subset \lambda$ s'estime finalement par $C\, \|b\|_\infty^2\, |\lambda|$.

7. Estimations quadratiques pour les ondelettes adaptées.

Reprenons les θ_λ construites suivant le Théorème 3. Nous démontrons tout d'abord l'inégalité de droite dans (5.8).

Les θ_λ vérifiant des estimations standard il suffit d'après le Théorème 5 de voir que $(\int_I \theta_\lambda)$ est une suite de Carleson. D'après (5.4) et (5.5) il suffit de le montrer pour la suite $(\int_I \mathcal{X}_\lambda)$ en vertu du lemme évident suivant.

Lemme 6. *Soient $m(\lambda,\mu)$ des coefficients vérifiant $m(\lambda,\mu)=0$ si $\lambda\in\Lambda_j$ et $\mu\in\Lambda_{j'}$ avec $j\neq j'$ et $|m(\lambda,\mu)|\leq C\exp\{-\gamma 2^j|\lambda-\mu|\}$ si $j=j'$. Soient (d_λ) une suite de Carleson de constante 1 et (c_λ) la suite définie par*

$$c_\lambda=\sum m(\lambda,\mu)\,d_\mu.$$

Alors (c_λ) est une suite de Carleson dont la constante ne dépend que de C et γ.

Pour calculer ces intégrales on utilise (5.6) et il vient

$$\int_I \mathcal{X}_\lambda=\int_I \psi_\lambda-\int_I \frac{1}{a}\,m_\lambda$$

où $m_\lambda(x)=\psi_\lambda(x)v_j(x)$ avec $v_j(x)=\sum_{k\in R_j}\int_I g_{jk}\;\; g_{jk}(x)$ pour $\lambda\in\Lambda_j$.

Les estimations standard (5.1) entraînent que

$$\|v_j\|_\infty\leq C\qquad\text{et}\qquad\|v_j{}'\|_\infty\leq C\,2^j\,.$$

L'orthogonalité entre W_j et V_j (rappelons que V_j est autoadjoint!) et le théorème de convergence dominée montrent que m_λ est d'intégrale nulle.

Tout ceci implique que (m_λ) est une famille de vaguelettes sur I. La suite $(\int_I \frac{1}{a}\,m_\lambda)$ est donc une suite de Carleson en appliquant successivement le Lemme 5 et le Théorème 6.

En ce qui concerne $(\int_I \psi_\lambda)$ les inégalités (5.3) permettent d'appliquer la remarque pour les suites vérifiant (6.6). Ceci achève les vérifications.

Le membre de gauche de (5.8) s'obtient ensuite par dualité à l'aide du résultat classique suivant.

Lemme 7. *Soit H un espace de Hilbert séparable. Soient (e_k) et (f_k) deux suites de H. Si pour toute suite (α_k)*

$$\Big\|\sum\alpha_k e_k\Big\|^2\leq C_1\sum|\alpha_k|^2\,,$$
$$\Big\|\sum\alpha_k f_k\Big\|^2\leq C_2\sum|\alpha_k|^2$$

et

$$\langle e_k, f_\ell \rangle = \delta_{k\ell},$$

alors

$$\sum |\alpha_k|^2 \leq C_2 \left\| \sum \alpha_k e_k \right\|^2$$

et symétriquement pour les (f_k).

Ce lemme s'applique alors aux deux familles $(\bar{\theta}_\lambda)$ et $(\frac{1}{a}\theta_\lambda)$.

Pour terminer la preuve du Théorème 4, il faut encore établir la complétude de (θ_λ). Puisque les V_j sont denses dans $L^2(I)$ et que chaque V_j se décompose en une somme $X_{j-1} + \ldots + X_N + V_N$, cela revient à démontrer que le projecteur sur V_N dont le noyau est l'espace orthogonal à V_N dans $L^2(I)$ pour B tend fortement vers 0 si N tend vers $-\infty$. Les détails sont laissés au lecteur (si j_0 est fini d'ailleurs, il n'y a donc rien à faire).

REMARQUE. Dans [21] et [2] les auteurs avaient fait dépendre l'inégalité de droite dans (5.8) du Théorème $T(1)$ de David et Journé. Ceci marche bien sûr sur $\mathbb{R}$ mais moins bien sur un intervalle avec conditions aux limites où la théorie de David et Journé ne s'applique pas immédiatement. Il s'avère que cette inégalité ne dépend en fait que des mesures de Carleson et l'équivalence du Théorème 5 prouve que c'est une formulation qui a l'avantage de s'adapter à tous les cas de figure.

8. Fin de la démonstration pour chaque condition aux limites.

Les situations considérées ici sont (2.2*a–i*). Nous les étudions à tour de rôle.

La situation (2.2*a*) est celle de [3]. Nous ne la refaisons donc pas et nous passons directement à (2.2*b*). Ce sera d'ailleurs la situation que nous détaillerons le plus.

Condition aux limites (2.2*b*): $I = (0, +\infty)$ et $V = H^1_0(I)$.

1. Les données géométriques et l'analyse multirésolution.

Pour $j \in \mathbb{Z}$, $\Gamma_j = \{k2^{-j} : k \in R_j = \mathbb{N}\}$ donc $0 \in \Gamma_j$. V_j est l'espace des splines linéaires de $L^2(I)$ ayant leurs nœuds sur Γ_j : toute $f \in V_j$ s'écrit $f(x) = \sum_{k\in\mathbb{N}} a_k \Delta_{jk}(x)$ pour tout $x \geq 0$ avec $\sum |a_k|^2 < \infty$. Observons que $V_j \subset H^1(I) = V^*$. On construit la base (θ_λ) en suivant le Théorème 3. Ces fonctions appartiennent à V^*.

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\tau_\lambda(x) = 2^j \int_0^x \frac{1}{a(t)}\,\theta_\lambda(t)\,dt, \quad x \geq 0,$$
$$\sigma_\lambda(x) = -2^{-j}\frac{d\theta_\lambda}{dx}(x), \quad x \geq 0,$$

avec la convention que σ_λ est une fonction en escalier continue à gauche en tout point.

Les fonctions τ_λ sont celles annoncées par le Théorème 2.

3. Vérifications.

Nous allons démontrer les propriétés suivantes:

(8.1) $\left\| \sum \alpha_\lambda \tau_\lambda \right\|^2 \leq C \sum |\alpha_\lambda|^2.$

(8.2) $\left\| \sum \alpha_\lambda \sigma_\lambda \right\|^2 \leq C \sum |\alpha_\lambda|^2.$

(8.3) $\int_I \tau_\lambda \sigma_\mu = \delta_{\lambda\mu}, \quad \forall\, \lambda, \mu \in \Lambda.$

(8.4) La famille (τ_λ) est complète dans $L^2(I)$.

(8.5) $\forall\, f \in L^2(I)$,

$$f(x) = \sum \alpha_\lambda \tau_\lambda(x) \text{ où } \alpha_\lambda = \int_I f\sigma_\lambda \text{ et } \|f\| \sim \left(\sum |\alpha_\lambda|^2\right)^{1/2}.$$

(8.6) $\forall\, f \in V$,

$$f(x) = \sum \alpha_\lambda \tau_\lambda(x) \text{avec} \|f\|_1 \sim \left(\sum |\alpha_\lambda|^2 \omega(\lambda)\right)^{1/2}$$

(8.7) $\forall\, f \in \mathcal{D}(T)$,

$$f(x) = \sum \alpha_\lambda \tau_\lambda(x) \text{ avec } \|f\|_2 \sim \left(\sum |\alpha_\lambda|^2 \omega(\lambda)^2\right)^{1/2}.$$

On a noté $\|\cdot\|_1$ la norme dans $V = H_0^1(I)$, $\|\cdot\|_2$ la norme du graphe dans $\mathcal{D}(T)$ et $\omega(\lambda) \equiv 1 + 2^{2j}$ si $\lambda \in \Lambda_j$.

Pour démontrer (8.1), il suffit d'après le Lemme 5 d'établir que (τ_λ) est une famille de vaguelettes sur I. Parmi les estimations standard, (5.2b) est immédiate et nous étudions (5.2a).

Fixons un $j \geq j_0$ et $\lambda \in \Lambda_j$. On observe que

$$1 = \sum_{k\in\mathbb{N}} 2^{-j/2}\Delta_{jk}(x), \quad \forall x \geq 0. \tag{8.8}$$

La fonction constante 1 est donc une limite ponctuelle de fonctions de V_j. En appliquant le théorème de convergence dominée et la définition de l'espace X_j on obtient

$$\int_I \frac{1}{a}\,\theta_\lambda = 0.$$

(C'est donc une forme plus précise que (5.7) dans cette situation.) On a donc

$$\tau_\lambda(x) = 2^j \int_0^x \frac{1}{a(t)}\,\theta_\lambda(t)\,dt = -2^j \int_x^{+\infty} \frac{1}{a(t)}\,\theta_\lambda(t)\,dt\,,$$

ce qui permet d'obtenir facilement (5.2a) pour τ_λ à l'aide de (5.2a) pour θ_λ.

Il reste à calculer l'intégrale de τ_λ. Observons que τ_λ est nulle au bord de I. En intégrant par parties il vient

$$\int_I \tau_\lambda = -2^j \int_I x\,\frac{1}{a}\,\theta_\lambda\,.$$

Cette dernière intégrale est nulle en remarquant comme précédemment que le monôme x est limite de fonctions de V_j. Plus précisément

$$x = \sum_{k\in\mathbb{N}} k2^{-j}\Delta_{jk}(x)\,, \quad \forall x \geq 0\,. \tag{8.9}$$

Démonstration de (8.2): σ_λ est une fonction en escalier qui vérifie seulement (5.2a) et pas (5.2b). La relation (6.1) n'est pas non plus satisfaite. On ne peut donc pas appliquer le Lemme 5 aux σ_λ (c'est le prix à payer pour la simplicité des splines linéaires). On peut cependant procéder comme suit.

Commençons par introduire une famille de fonctions auxiliaires. Posons $s_{j\ell}(x) = -2^{-j-1}\Delta_{j+1,\ell}{}'(x)$ pour $x \geq 0$. On a alors $s_{j\ell}(x) = -2^{(j+1)/2}$ si $x \in [(\ell-1)2^{-j-1}, \ell 2^{-j-1}[$, $2^{(j+1)/2}$ si $x \in [\ell 2^{-j-1}, (\ell +$

$1)2^{-j-1}[$, 0 ailleurs. Soit $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$, alors $\tau_\lambda \in X_j \subset V_{j+1}$. D'après une remarque antérieure

$$\theta_\lambda(x) = \sum_{\ell \in \mathbb{N}} c_j(k,\ell)\, \Delta_{j+1,\ell}(x)\,, \quad x \geq 0\,,$$

où $c_j(k,\ell) = 2^{(-j-1)/2}\theta_\lambda(\ell\, 2^{-j-1})$, et d'après (5.2$a$) il vient $|c_j(k,\ell)| \leq C\exp\{-\gamma|2k-\ell|\}$. En dérivant l'égalité ci-dessus, on voit que σ_λ s'écrit comme une combinaison linéaire des $s_{j\ell}$. On en déduit

$$\sum_{\lambda \in \Lambda_j} \alpha_\lambda \sigma_\lambda(x) = \sum_{\ell \in \mathbb{N}} \beta(j,\ell)\, s_{j\ell}(x)$$

avec $\beta(j,\ell) = 2\sum c_j(k,\ell)\alpha_\lambda$ où, comme d'habitude, $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$.

Les estimations sur les $c_j(k,\ell)$ impliquent que la transformation à j fixé : $(\alpha_\lambda) \mapsto (\beta(j,\ell))$, est bornée de $\ell^2(\Lambda_j)$ dans $\ell^2(\mathbb{N})$ avec une norme uniformément majorée par rapport à j. On est donc ramené à démontrer

$$\Big\| \sum \beta(j,\ell) s_{j\ell} \Big\|^2 \leq C \sum |\beta(j,\ell)|^2,$$

la somme étant prise sur tous les couples $(j,\ell) \in \mathbb{Z} \times \mathbb{N}$.

Pour ce faire, appelons $a(j,\ell;j',\ell')$ le produit scalaire dans $L^2(I)$ entre $s_{j\ell}$ et $s_{j'\ell'}$. On doit donc montrer que la matrice formée par ces coefficients est bornée sur ℓ^2.

Des calculs très simples montrent que $a(j,\ell;j',\ell') = 0$ sauf quand $|\ell\, 2^{-j} - \ell'\, 2^{-j'}| = 0$ ou $2^{-\inf\{j,j'\}}$, auquel cas ce coefficient s'estime par $C2^{-|j-j'|/2}$. On vérifie alors que cette matrice est bornée sur ℓ^1 et ℓ^∞, donc sur ℓ^2 par interpolation, ce qui termine la démonstration de (8.2).

La vérification de (8.3) est une simple intégration par parties. Soient $\lambda, \mu \in \Lambda$. Puisque $\tau_\lambda(0) = 0$ et que τ_λ et θ_μ sont nulles à l'infini, il vient avec des notations évidentes

$$\delta_{\lambda\mu} = \int_0^{+\infty} \theta_\lambda \frac{1}{a}\, \theta_\mu = 2^{j'-j} \int_0^{+\infty} \tau_\lambda\, \sigma_\mu.$$

Preuve de (8.4): Soit $f \in V = H_0^1(I)$, alors $df/dx \in L^2(I)$. Donc d'après le Théorème 4

$$\frac{df}{dx}(x) = \sum_{\lambda \in \Lambda} \alpha_\lambda \theta_\lambda(x) \frac{1}{a(x)} = \sum_{\lambda \in \Lambda} 2^{-j} \alpha_\lambda \frac{d\tau_\lambda}{dx}(x).$$

Les coefficients α_λ sont donnés par

$$\alpha_\lambda = \int_0^{+\infty} \frac{df}{dx}\,\theta_\lambda = 2^j \int_0^{+\infty} f\,\sigma_\lambda \equiv 2^j \beta_\lambda$$

puisque $f \in V$ et $\theta_\lambda \in V^*$. Comme $f \in L^2(I)$, l'estimation duale de (8.2) implique que $(\beta_\lambda) \in \ell^2(\Lambda)$. La série $g(x) = \sum \beta_\lambda \tau_\lambda(x)$ appartient donc à $L^2(I)$. Elle appartient aussi à V et $f = g$ par une application du théorème du graphe fermé. Ceci établit la complétude des τ_λ.

Preuve de (8.5): C'est une simple conséquence de (8.1–4) en utilisant le lemme 7.

Preuve de (8.6): L'argument de (8.4) montre que pour $f \in V$, f s'écrit sur la base (τ_λ) avec $\|f\|_1^2 \leq C \sum |\alpha_\lambda|^2 (1 + 2^{2j})$. L'inégalité en sens inverse provient de l'inégalité réciproque à (8.1) et de l'inégalité de gauche dans (5.8).

Preuve de (8.7): Soit $f \in \mathcal{D}(T)$ où, rappelons-le, $T = D^* a D$. Comme $\mathcal{D}(T) \subset L^2(I)$ on peut appliquer (8.5) : on a $f(x) = \sum \alpha_\lambda \tau_\lambda(x)$ avec $\alpha_\lambda = \int_0^{+\infty} f\,\sigma_\lambda$. Puisque $\|f\| \sim (\sum |\alpha_\lambda|^2)^{1/2}$, il reste à démontrer que $\|Tf\| \sim (\sum |\alpha_\lambda|^2 2^{4j})^{1/2}$.

Posons $\beta_\lambda = \int_0^{+\infty} Tf\,\tau_\lambda$, alors $\|Tf\|^2 \sim \sum |\beta_\lambda|^2$. On intègre ensuite β_λ deux fois par parties, en vérifiant à chaque étape que les termes tout intégrés s'annulent : puisque $\tau_\lambda \in V$ et $a\,df/dx \in V^*$ (Proposition 2) il vient

$$\begin{aligned} \beta_\lambda &= \int_I a \frac{df}{dx} \frac{d\tau_\lambda}{dx} \\ &= 2^j \int_I \frac{df}{dx}\,\theta_\lambda\ . \end{aligned}$$

Ensuite $f \in V$ et $\theta_\lambda \in V^*$, donc

$$\beta_\lambda = -2^j \int_I f \frac{d\theta_\lambda}{dx} = 2^{2j} \int_I f\,\sigma_\lambda\ .$$

On voit donc que $\beta_\lambda = 2^{2j} \alpha_\lambda$ et l'on obtient l'équivalence désirée. (Remarquer que ces calculs utilisent implicitement le fait que V et V^* sont autoadjoints car nous n'avons pas mis de barre de conjugaison. Cette propriété reste vraie sauf pour le cas de la condition aux limites (2.2i) où la conjugaison complique légèrement la situation.)

NOTE. Pour démontrer le Théorème 2 avec des conditions aux limites différentes, on suivra la même démarche, les seules différences se situant

lors de la preuve de (8.1), et parfois de (8.4) et (8.5). Afin d'éviter de lourdes répétitions, nous ne nous occuperons plus désormais que de ces points particuliers et laisserons le lecteur compléter le reste par lui-même.

Condition aux limites (2.2*c*): $I = (0, +\infty)$ et $V = H^1(I)$.

1. Les données géométriques et l'analyse multirésolution.

Pour $j \in \mathbb{Z}$, $\Gamma_j = \{k2^{-j} : k \in R_j = \mathbb{N}^*\}$ donc $0 \notin \Gamma_j$. V_j est l'espace des splines linéaires avec leurs nœuds sur Γ_j : $f \in V_j$ s'écrit $f(x) = \sum_{k\in\mathbb{N}^*} a_k \Delta_{jk}(x)$, $x \geq 0$, avec $\sum |a_k|^2 < \infty$. Observons que les fonctions de V_j s'annulent en 0 donc $V_j \subset H_0^1(I) = V^*$. On construit la base (θ_λ) en suivant le Théorème 3. Ces fonctions appartiennent toutes à V^*.

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\tau_\lambda(x) = 2^j \int_x^{+\infty} \frac{1}{a(t)} \theta_\lambda(t)\, dt, \quad x \geq 0,$$

$$\sigma_\lambda(x) = 2^{-j} \frac{d\theta_\lambda}{dx}(x), \quad x \geq 0.$$

Les τ_λ forment la base cherchée.

3. Vérifications.

Preuve de (8.1): On montre que les τ_λ sont des vaguelettes sur I. L'inégalité (5.2*b*) étant évidente, passons à (5.2*a*).

Fixons $\lambda \in \Lambda_j$. Si $x \geq \lambda$, on majore brutalement l'intégrale qui définit $\tau_\lambda(x)$. Cela donne l'inégalité désirée grâce à (5.2*a*) pour θ_λ. Si $0 \leq x \leq \lambda$, on écrit

$$\tau_\lambda(x) = 2^j \int_0^{+\infty} \frac{1}{a(t)} \theta_\lambda(t)\, dt - 2^j \int_0^x \frac{1}{a(t)} \theta_\lambda(t)\, dt\,.$$

La deuxième intégrale fournit l'estimation souhaitée. Pour majorer la première on utilise (5.7). Ceci nous donne une majoration en

$$C\, 2^{j/2} \exp\{-\gamma\, d\, 2^j\} \leq C\, 2^{j/2} \exp\{-\gamma\, 2^j\, |x - \lambda|\}$$

puisque $d = \operatorname{dist}\{\lambda, \partial I\} = \lambda \geq |x - \lambda|$ lorsque $0 \leq x \leq \lambda$.

On démontre enfin que l'intégrale de τ_λ est nulle exactement comme précédemment en remarquant que (8.9) reste valide si la sommation est restreinte à $\mathbb{N}^*$.

Les preuves de (8.4) et (8.5) sont analogues à celles du cas précédent.

Condition aux limites (2.2*d*): $I = (0,1)$ et $V = H_0^1(I)$.

1. Les données géométriques et l'analyse multirésolution.

On pose $j_0 = -1$. Pour $j \geq 0$, $\Gamma_j = \{k2^{-j} : k \in R_j = \{0, 1, \dots, 2^j\}\}$ donc 0 et $1 \in \Gamma_j$. V_j est l'espace des splines linéaires de $L^2(I)$ ayant leurs nœuds sur Γ_j : toute $f \in V_j$ s'écrit $f(x) = \sum_{k \in R_j} a_k \Delta_{jk}(x)$ pour tout $x \in I$. Notons que $\dim V_j = 2^j + 1$ et que V_0 est engendré par les monômes 1 et x. Observons aussi que $V_j \subset H^1(I) = V^*$.

On construit les fonctions θ_λ pour $\lambda \in \Lambda_j$ et $j \geq 0$ en suivant le Théorème 3. Ces fonctions appartiennent à V^*. D'après le Théorème 4, il faut les compléter par deux fonctions de V_0 (de dimension 2), notées v_0 et v_1. On exige que v_0 soit constante et que $v_1(x) = \nu x + \kappa$ sur I de telle façon que $\int_I v_0^2 \frac{1}{a} = \int_I v_1^2 \frac{1}{a} = 1$ et $\int v_1 v_0 \frac{1}{a} = 0$.

Observons enfin que l'orthogonalité entre les X_j et V_0 pour la forme B fournit

$$\int_I \frac{1}{a} \theta_\lambda = \int_I x \frac{1}{a} \theta_\lambda = 0. \tag{8.10}$$

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\tau_\lambda(x) = 2^j \int_0^x \frac{1}{a(t)} \theta_\lambda(t)\, dt, \quad x \in [0,1],$$
$$\sigma_\lambda(x) = -2^{-j} \frac{d\theta_\lambda}{dx}(x), \quad x \in [0,1].$$

On pose également

$$t_1(x) = \int_0^x v_1(t) \frac{1}{a(t)}\, dt \quad \text{et} \quad s_1(x) = -\frac{dv_1}{dx}(x).$$

On ne considère pas l'analogue pour v_0 puisque sa dérivée est nulle et ne peut donc pas appartenir au système dual des τ_λ. Observons que par (8.10) les τ_λ sont nulles au bord de I et il en est de même de t_1. Ces fonctions appartiennent donc à V.

La base qui convient pour le Théorème 2 est alors constituée de toutes les τ_λ et de t_1.

3. Vérifications.

On montre que les τ_λ sont des vaguelettes sur I en adaptant de façon immédiate de l'argument donné pendant l'étude de (2.2*b*). Cela démontre (8.1).

Faisons une remarque à ce sujet. Pour démontrer des estimations comme (8.1), on peut bien sûr oublier un nombre fini de termes sans modifier le résultat pour la famille entière. Ici, par exemple, t_1 n'est pas prise en compte, alors que la base cherchée se compose des τ_λ et de t_1.

Passons à l'étude de la complétude (8.4). Soit $f \in L^2(I)$ telle que $\langle t_1, f\rangle = \langle \tau_\lambda, f\rangle = 0$ pour tout $\lambda \in \Lambda$ et soit F une primitive de f. On intègre alors par parties les égalités précédentes. Comme les τ_λ et t_1 sont nulles au bord, il vient $\langle v_1\, /a\,, F\rangle = \langle \theta_\lambda/a, F\rangle = 0$ pour tout λ. D'après le Théorème 4, on voit que $F = C\bar{v}_0$. Donc F est constante et $f = 0$.

Terminons en écrivant la formule de représentation qui permet de finir ces vérifications (laissées au lecteur) : pour $f \in L^2(I)$,

$$f(x) = \int_I f\, s_1 \quad t_1(x) + \sum \int_I f\, \sigma_\lambda \quad \tau_\lambda(x).$$

Condition aux limites (2.2*e*): $I = (0,1)$ et $V = \{f \in H^1(I) : f(0) = 0\}$.

1. Les données géométriques et l'analyse multirésolution.

Posons $j_0 = -1$. Pour $j \geq 0$, $\Gamma_j = \{k2^{-j} : k \in R_j = \{0, 1, \ldots, 2^j - 1\}\}$ donc $0 \in \Gamma_j$ et $1 \notin \Gamma_j$. V_j est l'espace des splines linéaires de $L^2(I)$ ayant leurs nœuds sur Γ_j : toute $f \in V_j$ s'écrit $f(x) = \sum_{k \in R_j} a_k \Delta_{jk}(x)$ pour tout $x \in I$. Notons que $\dim V_j = 2^j$ et que V_0 est engendré par le polynôme $1 - x$. On appellera $v_0(x) = \eta(1-x)$ la fonction telle que $\int_I v_0^2\, /a = 1$, η étant un nombre complexe non nul. Observons aussi que les fonctions de V_j s'annulent toutes en 1, donc $V_j \subset V^*$.

On construit les fonctions θ_λ pour $\lambda \in \Lambda_j$ et $j \geq 0$ en suivant le Théorème 3. La base d'ondelettes adaptées à $1/a$ est constituée de ces fonctions et de v_0. Notons que ces fonctions appartiennent toutes à V^*.

Remarquons enfin que pour tout λ

$$\int_I (1-x)\,\frac{1}{a}\,\theta_\lambda = 0. \tag{8.11}$$

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\tau_\lambda(x) = 2^j \int_0^x \frac{1}{a(t)}\, \theta_\lambda(t)\, dt, \quad x \in [0,1],$$
$$\sigma_\lambda(x) = -2^{-j} \frac{d\theta_\lambda}{dx}(x), \quad x \in [0,1].$$

On pose également

$$t_0(x) = \int_0^x v_0(t) \frac{1}{a(t)}\, dt \quad \text{et} \quad s_0(x) = -\frac{dv_0}{dx}(x) = \eta.$$

Les fonctions τ_λ et t_0 sont nulles en 0 et, à ce titre, appartiennent à V. Ces fonctions constituent la base cherchée.

3. Vérifications.

Pour démontrer (8.1), on commence par observer qu'en intégrant (8.11) par parties on trouve l'intégrale de τ_λ qui, par conséquent, est nulle. Il reste à vérifier les estimations standard. Il suffit pour cela de reprendre la démarche utilisée pour le cas (2.2*c*).

Pour démontrer enfin la complétude, on procède comme dans le cas précédent en choisissant cette fois pour F la primitive de f qui s'annule en 1.

Condition aux limites (2.2*f*): $I = (0,1)$ et $V = H^1(I)$.

1. Les données géométriques et l'analyse multirésolution.

On pose cette fois $j_0 = 0$. Pour $j \geq 1$, $\Gamma_j = \{k2^{-j} : k \in R_j = \{1, 2, \ldots, 2^j - 1\}\}$ donc $0, 1 \notin \Gamma_j$. V_j est l'espace des splines linéaires de $L^2(I)$ ayant leurs nœuds sur Γ_j : toute $f \in V_j$ s'écrit $f(x) = \sum_{k \in R_j} a_k \Delta_{jk}(x)$ pour tout $x \in I$. Toutes les fonctions de V_j s'annulent en 0 et 1 donc $V_j \subset V^*$.

Notons que $\dim V_j = 2^j - 1$ et que V_1 est engendré par $\Delta_{1,1}(x) = \sqrt{2}(1 - |2x - 1|)$. On choisira la constante ν afin que $v_1(x) = \nu(1 - |2x - 1|)$ vérifie $\int_I v_1^2 / a = 1$.

On construit les fonctions θ_λ pour $\lambda \in \Lambda_j$ et $j \geq 1$. Avec la fonction v_1, elles constituent la base d'ondelettes adaptées à $1/a$. Notons que ces fonctions s'annulent en 0 et 1 donc elles appartiennent toutes à V^*.

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\tau_\lambda(x) = 2^j \int_{u_\lambda}^x \frac{1}{a(t)}\, \theta_\lambda(t)\, dt, \quad x \in [0,1],$$

$$\sigma_\lambda(x) = -2^{-j} \frac{d\theta_\lambda}{dx}(x), \quad x \in [0,1].$$

On pose également

$$t_1(x) = \int_{1/2}^x v_1(t) \frac{1}{a(t)}\, dt \quad \text{et} \quad s_1(x) = -\frac{dv_1}{dx}(x).$$

Le nombre u_λ est à choisir dans [0,1] en fonction de la position de λ par rapport au bord de I. Les fonctions τ_λ et t_1 appartiennent à V et ne vérifient aucune condition particulière au bord. Cependant elles ne forment pas la base cherchée : il faut les compléter par la fonction constante 1 pour obtenir la base du Théorème 2.

3. Vérifications.

A la différence avec tous les cas précédents les τ_λ ne sont plus des vaguelettes sur I. La preuve de (8.1) requiert donc l'usage du Théorème 5 et du lemme suivant.

Lemme 8. *Choisissons $u_\lambda = 1$ si $0 < \lambda \leq 1/2$ et $u_\lambda = 0$ si $1/2 < \lambda < 1$. Alors les τ_λ vérifient les estimations standard et $(\int_I \tau_\lambda)$ est une suite de Carleson.*

PREUVE. On supposera constamment $\lambda \in \Lambda_j$ où j est assez grand (voir une remarque antérieure) avec $0 < \lambda \leq 1/2$. L'autre cas, $1/2 < \lambda < 1$, se traite de façon symétrique.

Bien sûr l'estimation (5.2*b*) est évidente. Pour voir (5.2*a*), on utilise un argument qui sera très souvent employé par la suite. Voyons-en le détail.

On écrit tout d'abord

$$\tau_\lambda(x) = -2^j \int_I \chi_{[x,1]}(t) \frac{1}{a(t)}\, \theta_\lambda(t)\, dt.$$

Pour utiliser l'orthogonalité entre V_j et X_j, l'idée est d'approcher $\chi_{[x,1]}(t)$ par une fonction de V_j. Autrement dit $\chi_{[x,1]}(t) = v_j(t) + r_j(t)$ où $v_j(t) \in V_j$ et l'erreur $r_j(t)$ est bornée (par 1) et est à support dans $[1 - 2^{-j}, 1]$ (se rappeler que dans ce cas les fonctions de V_j sont nulles

au bord), et dans un intervalle de longueur 2^{-j} et qui contient x. Par orthogonalité la contribution apportée par $v_j(t)$ sous l'intégrale est nulle et il vient

$$\tau_\lambda(x) = -2^j \int_I r_j(t) \frac{1}{a(t)} \theta_\lambda(t)\, dt\,.$$

A ce point, on majore brutalement l'intégrale en tenant compte des conditions de support et en utilisant (5.2a) pour $\theta_\lambda(t)$. On obtient une majoration en

$$C\, 2^{j/2} \exp\left\{-\gamma\, 2^j \inf\{|x-\lambda|, |1-\lambda|\}\right\} \leq C\, 2^{j/2} \exp\{-\gamma\, 2^j\, |x-\lambda|\}$$

puisque $|x-\lambda| \leq |1-\lambda|$ pour $0 \leq x \leq 1$ et $0 < \lambda \leq 1/2$. Ceci démontre l'estimation (5.2a) pour τ_λ.

Passons à l'estimation de sa moyenne. Remarquons que τ_λ s'annule en 1. En intégrant par parties, on obtient alors

$$\int_I \tau_\lambda = -2^j \int_I t \frac{1}{a(t)} \theta_\lambda(t)\, dt.$$

Pour estimer cette intégrale on procède en approchant t par une function de V_j et, cette fois, le terme correctif qui se substitue à t est supporté par $[1-2^{-j}, 1]$ seulement. On obtient donc la majoration

$$\Big|\int_I \tau_\lambda\Big| \leq C\, 2^{j/2} \exp\{-\gamma\, 2^j\, |1-\lambda|\} \leq C\, 2^{j/2} \exp\{-\gamma\, 2^{j-1}\}$$

car $0 < \lambda \leq 1/2$. On voit sans peine qu'une telle estimation est de Carleson (ne pas oublier que le nombre de $\lambda \in \Lambda_j$ est 2^j). Cela achève la preuve du lemme et la vérification de (8.1).

Preuve de la complétude: Commençons par une remarque. Les fonctions θ_λ et v_1 sont nulles au bord de I. Par suite, σ_λ et s_1 ont une moyenne nulle sur I. Cela veut dire que les constantes sont orthogonales à l'espace vectoriel qu'engendrent ces dernières fonctions. Par dualité, il convient donc d'adjoindre aux fonctions τ_λ et t_1 la fonction 1 pour obtenir un système complet. C'est effectivement le cas.

Pour le voir, on prend $f \in L^2(I)$ orthogonale aux constantes, à t_1 et aux τ_λ. On appelle F la primitive de f qui s'annule en 0. Puisque f est d'intégrale nulle, on a également $F(1) = 0$ et l'on peut intégrer par parties les autres relations d'orthogonalité. Cela implique que F est orthogonale à toutes les $\theta_\lambda\, /a$ et à $v_1\, /a$. Donc $F = 0$ et $f = 0$.

La dernière chose à vérifier est que le système précédent est bien inconditionnel. Une application du Lemme 7 fournit déjà que (t_1, τ_λ) est un système inconditionnel dans $L^2(I)$ engendrant un sous-espace E. De plus, on voit aisément que les constantes n'appartiennent pas à E. Sinon, on écrit $1 = \alpha_1 t_1 + \sum \alpha_\lambda \tau_\lambda$ et en intégrant contre s_1 et les σ_λ successivement les coefficients α_1 et α_λ sont nuls, ce qui est absurde. Donc $(1, t_1, \tau_\lambda)$ forme bien un système inconditionnel générateur de $L^2(I)$.

Enfin, par le théorème de représentation de Riesz il existe une fonction s_0, orthogonale à E, telle que l'on ait la formule de décomposition suivante:

$$f(x) = \int_I f\, s_0 \quad + \int_I f\, s_1 \quad t_1(x) + \sum \int_I f\, \sigma_\lambda \quad \tau_\lambda(x)$$

pour toute $f \in L^2(I)$. Nous laissons le lecteur compléter la suite.

Condition aux limites (2.2*g*): $I = (0,1)$ et $V = \{f \in H^1(I) : f(1) = f(0)\}$.

Il s'agit d'un problème périodique et dans ce cas on voit que $V^* = V$. Cela suggère de partir d'une base adaptée sur $\mathbb{R}$ et de la périodiser ensuite.

1. Les données géométriques et l'analyse multirésolution.

Soit $j_0 = -1$. Pour $j \geq 0$, $\Gamma_j = \{k2^{-j} : k \in R_j = \mathbb{Z}\}$. V_j est l'espace des splines linéaires de $L^2(I)$ ayant leurs nœuds sur Γ_j : toute $f \in V_j$ s'écrit $f(x) = \sum_{k\in\mathbb{Z}} a_k \Delta_{jk}(x)$ pour tout x avec $\sum |a_k|^2 < \infty$.

On prolonge $a(x)$ de façon périodique en dehors de I par $a(x+k) = a(x)$, $x \in I$ et $k \in \mathbb{Z}$. Cela définit $a(x)$ p.p. sur $\mathbb{R}$ où elle y est bornée et accrétive.

On construit ensuite une base d'ondelettes adaptées à $1/a$ sur $\mathbb{R}$. Cette base est consituée des fonctions θ_λ, $\lambda \in \Lambda_j = \{(2k+1)2^{-j-1} : k \in \mathbb{Z}\}$, $j \geq 0$, et des fonctions g_{0k}, $k \in \mathbb{Z}$, qui engendrent V_0. On considère également la base g_{jk}, $k \in \mathbb{Z}$, de V_j construites dans la partie 1 du Théorème 3.

La base adaptée a la structure suivante:

$$g_{0k}(x) = g_{00}(x-k), \quad \forall\, k \in \mathbb{Z},$$
$$\theta_\mu(x) = \theta_\lambda(x - \mu + \lambda), \quad \forall\, \lambda, \mu \in \Lambda_j\,;\, \lambda - \mu \in \mathbb{Z}.$$

On a aussi pour $j \geq 0$,

$$g_{j,k+n2^j}(x) = g_{jk}(x-n), \quad \forall\, k, n \in \mathbb{Z}.$$

Cela provient du fait que dans ce cas de figure les espaces V_j et X_j, $j \geq 0$, sont invariants sous l'action des translations entières et que les bases dont on part pour appliquer le Théorème 3 possèdent cette structure, celle-ci étant conservée par les algorithmes de la preuve de ce théorème (voir Appendice 1).

Pour $j \geq 0$, on appelle V_j^p l'espace des splines linéaires périodiques de période 1 ayant leurs nœuds sur Γ_j. On peut identifier une fonction de cet espace avec sa restriction à I, ainsi V_j^p devient-il un sous-espace de $L^2(I)$ de dimension finie 2^j. Observons que $V_j^p \subset V^*$.

Grâce à la structure invariante ci-dessus, une base de V_j^p est donnée par les fonctions suivantes:

$$g_{jk}^p(x) = \sum_{n\in\mathbb{Z}} g_{j,k+n2^j}(x), \quad k = 0, 1, \ldots, 2^j - 1.$$

Cette base est aussi orthogonale pour la forme $B^p(f,g) = \int_I fg/a$. Cela découle par périodisation des relations d'orthogonalité sur $\mathbb{R}$ pour B (définition similaire à celle de B^p où l'intégrale est prise sur $\mathbb{R}$). On remarquera enfin que V_0^p est l'espace des constantes.

Soit X_j^p l'espace orthogonal à V_j^p dans V_{j+1}^p pour B^p. Cet espace est de dimension 2^j et l'on voit que les fonctions

$$\theta_\lambda^p(x) = \sum_{\mu\in\Lambda_j,\, \mu-\lambda\in\mathbb{Z}} \theta_\mu(x), \quad \lambda \in \Lambda_j^p \equiv \Lambda_j \cap I,$$

forment une base de X_j^p orthogonale pour B^p. Les espaces X_j^p étant mutuellement orthogonaux pour B^p, les fonctions $g_{00}^p(x)$ et $\theta_\lambda^p(x)$, $\lambda \in \Lambda^p = \cup_{j\geq 0} \Lambda_j^p$ forment alors un système orthogonal pour B^p, dont on établit facilement qu'il est complet dans $L^2(I)$. En vertu du Lemme 7, il reste à établir

$$\| \sum_{\lambda\in\Lambda^p} \alpha_\lambda \theta_\lambda^p \|^2 \leq C \sum |\alpha_\lambda|^2, \tag{8.12}$$

la norme étant ici celle de $L^2(I)$.

Curieusement, (8.12) n'est pas une conséquence de l'inégalité analogue sur $\mathbb{R}$. L'approche naïve qui consiste à dérouler l'intégrale sur I pour obtenir une intégrale sur $\mathbb{R}$ ne donne rien car les θ_λ^p ne sont pas orthogonales pour le produit scalaire. Pour démontrer (8.12) on utilise plutôt une variante périodique des résultats de la Section 6.

Proposition 3. *Soit h_λ, $\lambda \in \Lambda^p$, une famille de fonctions périodiques de période 1, vérifiant pour tout $\lambda \in \Lambda_j^p$, $j \geq 0$,*

$$(8.13a) \quad |h_\lambda(x)| \leq C\, 2^{j/2} \exp\{-\gamma\, 2^j \mathrm{dist}\{x-\lambda, \mathbb{Z}\}\}, \quad x \in \mathbb{R},$$

$$(8.13b) \quad |h_\lambda{}'(x)| \leq C\, 2^{3j/2} \exp\{-\gamma 2^j \mathrm{dist}\{x-\lambda, \mathbb{Z}\}\}, \quad x \in \mathbb{R},$$

et telles que $(\int_I h_\lambda)$ soit une suite de Carleson. Alors

$$\int_I \Big| \sum_{\lambda \in \Lambda^p} \alpha_\lambda\, h_\lambda \Big|^2 \leq C \sum |\alpha_\lambda|^2.$$

Cette proposition s'applique évidemment aux θ_λ^p. Les détails sur la preuve de ce résultat sont donnés dans l'Appendice 2.

Pour plus de simplicité nous abandonnerons dorénavant l'exposant p dans la notation.

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\tau_\lambda(x) = 2^j \int_{\lambda+1/2}^x \frac{1}{a(t)} \theta_\lambda(t)\, dt, \quad x \in \mathbb{R},$$

$$\sigma_\lambda(x) = -2^{-j} \frac{d\theta_\lambda}{dx}(x), \quad x \in \mathbb{R}.$$

La fonction σ_λ est périodique de période 1. Il en est de même pour τ_λ et cela provient de la relation $\int_u^{u+1} \theta_\lambda / a = 0$ pour tout réel u. En effet, θ_λ / a est périodique et sa restriction à I est orthogonale aux constantes $(=V_0)$.

Finalement la collection des restrictions à I des fonctions τ_λ forme avec la fonction constante 1 la base que nous cherchons.

3. Vérifications.

Pour démontrer (8.1), nous appliquons la Proposition 3. L'estimation (8.13b) est évidente. Pour voir (8.13a), il suffit par périodicité de prendre $x \in [\lambda - 1/2, \lambda + 1/2]$. Sur cet intervalle on a $\mathrm{dist}\{x-\lambda, \mathbb{Z}\} = |x-\lambda|$ et l'on est ramené aux situations précédentes où les estimations n'étaient pas périodisées. On écrit alors que $\chi_{[x,\lambda+1/2]}(t)$ appartient approximativement à V_j lorsque $\lambda \in \Lambda_j$ et l'on termine comme précédemment. Le choix de $\lambda + 1/2$ pour borne de l'intégrale définissant τ_λ intervient dans cette majoration.

L'intégrale de τ_λ sur I se calcule en intégrant par parties. Il vient $\int_I \tau_\lambda = \int_I g\,\theta_\lambda\,/a$ où g est une fonction périodique et bornée que l'on considère approximativement comme une fonction de V_j. Le terme d'erreur fournit une majoration de $\int_I \tau_\lambda$ qui est une suite de Carleson. Les détails sont laissés au lecteur.

Passons à la complétude. Du fait que les θ_λ sont périodiques, l'intégrale de σ_λ est nulle sur I. Il faut donc adjoindre la fonction constante 1 aux fonctions τ_λ pour obtenir un système complet. On construit ensuite une fonction $s_0 \in L^2(I)$ et la formule de représentation devient

$$f(x) = \int_I f\,s_0 \quad + \sum \int_I f\,\sigma_\lambda \quad \tau_\lambda(x)$$

pour toute $f \in L^2(I)$. Cela se montre comme dans le cas précédent et nous laissons au lecteur le soin de le vérifier.

Condition aux limites (2.2h) : $I = (0,1)$ et $V = \{f \in H^1(I) : f(1) = rf(0)\}$, $r \in \mathbb{R} \setminus \{0,1\}$.

La condition aux limites en 0 reste dépendante de celle en 1, mais plus de façon périodique. Cela affectera les propriétés de localisation des ondelettes adaptées. Chaque θ_λ ne sera plus seulement localisée autour de λ mais aussi autour de $1-\lambda$.

1. Les données géométriques et l'analyse multirésolution.

On pose $j_0 = -1$. Pour $j \geq 0$, $\Gamma_j = \{k2^{-j} : k \in R_j = \{1, 2 \ldots, 2^j - 1\}\}$. On place 0 et 1 à part. V_j est l'espace des splines linéaires f de $L^2(I)$ ayant leurs nœuds sur Γ_j et tels que $f(1) = f(0)/r$: toute $f \in V_j$ s'écrit $f(x) = a_0\varphi_{j0}(x) + \sum_{k\in R_j} a_k \Delta_{jk}(x)$ pour tout $x \in I$ où $\varphi_{j0}(x) = r\Delta_{j0}(x) + \Delta_{j2^j}(x)$. V_j est donc un espace de dimension 2^j et $V_j \subset V^*$. L'espace V_0 est engendré par le polynôme $r + (1-r)x$. Enfin, on définit $\nu \in \mathbb{C}$ afin que $v_0(x) = \nu(r + (1-r)x)$ vérifie $\int_I v_0^2\,/a = 1$.

Une adaptation de la preuve du Théorème 3 fournit des ondelettes adaptées θ_λ, $\lambda \in \Lambda_j$, $j \geq 0$, vérifiant

$$(8.14a) \qquad |\theta_\lambda(x)| \leq C\,2^{j/2} \exp\{-\gamma\,2^j\,\delta(x,\lambda)\}, \quad x \in I,$$

$$(8.14b) \qquad |\theta_\lambda'(x)| \leq C\,2^{3j/2} \exp\{-\gamma\,2^j\,\delta(x,\lambda)\}, \quad x \in I,$$

où $\delta(x,\lambda) = \inf\{|x-\lambda|, |x-(1-\lambda)|\}$. Ce sont donc des estimations standard à deux bosses placées symétriquement par rapport au milieu de I.

L'inégalité quadratique pour les θ_λ se voit en appliquant des variantes des résultats de la Section 6 (voir Appendice 3 pour tout cela).

Finalement, la base adaptée s'obtient en complétant par v_0 le système (θ_λ). On observera que les fonctions de cette base appartiennent toutes à V^*.

2. Définition des τ_λ et σ_λ.

Pour $\lambda \in \Lambda_j$ on pose

$$\sigma_\lambda(x) = -2^{-j}\frac{d\theta_\lambda}{dx}(x), \quad x \in I,$$

$$s_0(x) = -\frac{dv_0}{dx}(x) = \nu(r-1), \quad x \in I.$$

La restriction des σ_λ à chacun des intervalles (0,1/2) et (1/2,1) vérifient des estimations standard (5.2a). En coupant au point 1/2 les intégrales sur I on montre facilement (8.2) en recopiant l'argument donné plus haut.

L'observation suivante nous indiquera comment définir les τ_λ.

Lemme 9. *Soient $\lambda \in \Lambda_j$ et τ_λ une primitive de $2^j\theta_\lambda/a$. Alors*

(i) $\tau_\lambda(1) - r\tau_\lambda(0) = (1-r)\int_I \tau_\lambda$,

(ii) $\tau_\lambda \in V$ *si et seulement si* $\int_I \tau_\lambda = 0$.

La preuve en est très simple. Pour (i) il suffit d'intégrer par parties la relation $\int_I \theta_\lambda\, v_0/a = 0$ en utilisant la définition de v_0. La condition d'appartenance à V étant précisément que le membre de gauche de (i) soit nul, on en déduit (ii). On pourra noter que l'appartenance de τ_λ à V est aussi équivalente à $\int_I \tau_\lambda\, s_0 = 0$.

La conséquence de ce lemme est que nous n'avons pas le choix pour définir τ_λ. Afin d'obtenir les estimations à deux bosses, nous introduisons une fonction auxiliaire $\tau_\lambda^\sharp$ que nous corrigerons ensuite pour qu'elle soit de moyenne nulle.

Soit $\lambda \in \Lambda_j$ fixé et tel que $0 < \lambda \leq 1/2$ (on procéderait de façon symétrique si $\lambda > 1/2$). On supposera aussi j assez grand. On pose

$$\tau_\lambda^\sharp(x) = 2^j \int_{1/2}^x \frac{1}{a(t)}\theta_\lambda(t)\,dt, \quad \text{si } 0 < \lambda \leq 1/4\ ,$$

$$\tau_\lambda^\sharp(x) = 2^j \int_0^x \frac{1}{a(t)}\theta_\lambda(t)\,dt, \quad \text{si } 1/4 < \lambda \leq 1/2.$$

L'estimation (8.14a) se vérifie à l'aide de l'argument maintes fois utilisé auparavant d'approximation dans V_j des fonctions caractéristiques d'intervalle. Il est essentiel dans ces calculs que la borne de l'intégrale qui définit $\tau_\lambda^\sharp$ soit «loin» de λ.

En ce qui concerne la moyenne de $\tau_\lambda^\sharp$ on trouve

$$\left| \int_I \tau_\lambda^\sharp \right| \leq C\, 2^{j/2} \exp\{-\gamma\, 2^j\, \delta(1,\lambda)\}$$

si $1/4 < \lambda \leq 1/2$, et

$$\int_I \tau_\lambda^\sharp = 0$$

si $0 < \lambda \leq 1/4$ et $j \geq 1$. Cette intégrale est nulle car en intégrant par parties on voit qu'elle vaut $\int_I v_1\, \theta_\lambda\, /a$ où $v_1(x) = 2^j(1 - |2x-1|) \in V_1$. En ce qui concerne la première intégrale on remarque que sous la condition $1/4 < \lambda \leq 1/2$, $\delta(1,\lambda) = \lambda \geq \delta(x,\lambda)/2$ pour tout $x \in I$. La majoration obtenue est donc meilleure que (8.14a).

Cela permet donc de définir la fonction τ_λ par : pour $x \in I$, $\tau_\lambda(x) = \tau_\lambda^\sharp(x)$ si $1/4 < \lambda \leq 1/2$ et $\tau_\lambda(x) = \tau_\lambda^\sharp(x) - \int_I \tau_\lambda^\sharp$ si $0 < \lambda \leq 1/4$.

En faisant de même si $\lambda > 1/2$, on obtient alors une famille de fonctions τ_λ sur I, qui, en vertu du Lemme 9, appartiennent toutes à V et vérifient (8.14).

Pour finir, on appelle t_0 la primitive de $v_0\, /a$ qui vérifie $1 = \nu(r-1) \int_I t_0 = \int_I s_0\, t_0$. Cette fonction appartient à V. En effet, en intégrant par parties la relation $\int_I v_0^2\, /a = 1$ on tombe sur l'égalité $1 = \nu(t_0(1) - rt_0(0)) + 1$, ce qui implique que $t_0(1) - rt_0(0) = 0$.

La famille constituée des τ_λ et de t_0 est alors la base cherchée.

3. Vérifications.

Pour démontrer (8.1) on utilise une autre variante du Théorème 5.

Proposition 4. *Soit h_λ, $\lambda \in \Lambda$, une famille de fonctions définies sur I et vérifiant les estimations standard à deux bosses* (8.14). *On suppose de plus que $(\int_0^{1/2} h_\lambda)$ et $(\int_{1/2}^1 h_\lambda)$ sont deux suites de Carleson. Alors*

$$\int_I \left| \sum \alpha_\lambda\, h_\lambda \right|^2 \leq C \sum |\alpha_\lambda|^2.$$

La preuve de ce résultat repose sur l'observation que la restriction à (0,1/2) (respectivement (1/2,1)) des h_λ vérifient les estimations standard. On peut par conséquent appliquer le Théorème 5 aux familles de fonctions formées par ces restrictions. (On notera que pour $\lambda \in \Lambda_j$, $j > 0$, $\mathrm{dist}\{\lambda, 1/2\} \geq C\,\ell(\lambda)$ ce qui permet d'utiliser directement la preuve de ce théorème. Voir Section 6.)

Pour terminer, examinons la complétude. Soit $f \in L^2(I)$ orthogonale à t_0 et aux τ_λ. Posons $\kappa = (1-r)^{-1} r \int_I f$. Si F est la primitive de f qui vaut κ en 0 alors $F(1) = r^{-1}F(0)$. On peut donc intégrer par parties les relations d'orthogonalité. Ainsi F est orthogonale à v_0 et aux θ_λ. Donc $F = 0$ et par suite $f = 0$.

Condition aux limites (2.2*i*): $I = (0,1)$ et $V = \{f \in H^1(I) : f(1) = zf(0)\}$, $z \in \mathbb{C} \setminus \mathbb{R}$.

On commence comme dans le cas précédent avec la même analyse multirésolution où r est remplacé par $\overline{z}$. Une fonction de V_j vérifie donc $f(1) = \overline{z}^{-1} f(0)$ qui est la condition d'appartenance à V^*. Cette condition n'étant pas invariante pour la conjugaison, V_j n'est pas autoadjoint. On ne peut donc plus appliquer le Théorème 3 et obtenir une base adaptée, orthogonale pour B.

Qu'à cela ne tienne, on peut cependant construire deux bases adaptées bi-orthogonales, non plus pour B mais pour $\beta(f,g) = \int_I f\,\overline{g}/a$ (comme cela est fait dans [2]). Nous en rappelons quelques détails dans l'Appendice 4.

Voici en quelques mots l'application à notre problème. Appelons (θ_λ) et $(\theta^\star_\lambda)$ les deux bases ainsi construites (il faut les compléter de manière appropriée mais nous ignorons ici ce détail). Elles appartiennent à V^*, vérifient les estimations à deux bosses et

$$\int_I \theta_\lambda \frac{1}{a}\,\overline{\theta^\star_\mu} = \delta_{\lambda\mu} \tag{8.15}$$

pour tout λ, μ.

On définit ensuite τ_λ de la même façon que précédemment et $\tau^\star_\lambda$ comme la primitive de $2^j \theta^\star_\lambda / \overline{a}$ ayant une moyenne nulle sur I. Ces fonctions appartiennent alors à V. On définit aussi σ_λ et $\sigma^\star_\lambda$ comme les dérivées correctement renormalisées de θ_λ et $\theta^\star_\lambda$ respectivement.

En intégrant (8.15) par parties de deux façons différentes on obtient

$$\int_I \tau_\lambda\,\overline{\sigma^\star_\mu} = \delta_{\lambda\mu} = \int_I \tau^\star_\lambda\,\overline{\sigma_\mu}\,.$$

Le système bi-orthogonal de (τ_λ) est donc (σ_λ^*). De plus, on voit que $\tau_\lambda \in \mathcal{D}(T)$ et que $T\tau_\lambda = D^*aD\tau_\lambda = 2^{2j}\sigma_\lambda$. Le système (σ_λ), quant à lui, a pour bi-orthogonal (τ_λ^*). Ces observations permettront au lecteur de compléter les détails.

9. Commentaires et extensions.

1. Résultats d'interpolation. J.-L. Lions montre dans [13] que si T^ρ dénote la puissance fractionnaire d'ordre $\rho \in (0,1)$ d'un opérateur accrétif T, alors $\mathcal{D}(T^\rho)$ coïncide avec l'espace d'interpolation complexe $[H, \mathcal{D}(T)]_\rho$.

En utilisant notre base d'ondelette (τ_λ) pour $T = D^*aD$ on voit donc que dans cette base $\mathcal{D}(T^\rho)$ est caractérisé par $\sum |\alpha_\lambda|^2 \omega(\lambda)^{2\rho} < +\infty$. Si $\rho < 1/2$, il s'agit aussi de l'espace $[L^2(I), V]_{2\rho}$.

Si $\rho > 1/2$ en revanche on a $\mathcal{D}(T^\rho) \subset V$, et si $f \in \mathcal{D}(T^\rho)$ alors aDf est caractérisé dans la base des θ_λ par $\sum |\beta_\lambda|^2 \omega(\lambda)^{2\rho-1} < +\infty$. Or par interpolation cette condition caractérise l'espace $[L^2(I), V^*]_{2\rho-1}$ que l'on appelle $V^*_{2\rho-1}$. Donc pour $\rho > 1/2$, $\mathcal{D}(T^\rho) \subset \{f \in V : aDf \in V^*_{2\rho-1}\}$ et l'on établit facilement l'inclusion inverse.

Dans le cas où $I = \mathbb{R}$, ces résultats d'interpolation équivalent au fait que les espaces $a^{-1}H^s$ pour $1 \geq s \geq 0$ et H^s pour $-1 \leq s \leq 0$ forment une famille d'espaces d'interpolation complexe. On a dénoté par H^s l'espace de Sobolev d'ordre s. Ce résultat avait été démontré dans [6] et on le retrouve plus aisément ici. Les limitations sur le paramètre s sont seulement liées au fait que les ondelettes adaptées que nous avons choisies sont lipschitziennes. On lève cette limitation en considérant des ondelettes adaptées plus régulières, ce qui est possible mais que nous ne discuterons pas ici. Toutes ces remarques sont valides sur un intervalle et nous laissons le lecteur imaginer les résultats d'interpolation obtenus.

2. Notre démonstration utilisant l'interpolation complexe abstraite ne fournit aucune information sur la nature de l'opérateur $(D^*aD)^{1/2}$ autre que celle d'être continu de V dans L^2. Nous publierons ultérieurement une description plus fine de cet opérateur [4].

3. Considérons le problème $D^*aDu = f$ sur (0,1) avec données de Dirichlet $u(0) = u(1) = 0$ et $f \in L^2$. On peut adapter la discussion qui suit à d'autres conditions aux limites. La collection (t_1, τ_λ) construite pour $(2.2d)$ fournit une base de Riesz de $\mathcal{D}(D^*aD)$. Or la solution u de ce problème appartient à cet espace. On peut alors

obtenir cette solution par $u(x) = \alpha_1 t_1(x) + \sum \alpha_\lambda 2^{-2j} \tau_\lambda(x)$ dès que $f(x) = \alpha_1 s_1(x) + \sum \alpha_\lambda \sigma_\lambda(x)$. En d'autres termes, on a décomposé le noyau de Green de D^*aD avec donnée de Dirichlet en une série $t_1(x)t_1(y) + \sum 2^{-2j} \tau_\lambda(x)\tau_\lambda(y)$.

4. Conjecture de Kato pour des opérateurs elliptiques d'ordre supérieur. Exemple: $T = d_x^2 a(x) d_x^2$ où $d_x = \frac{d}{dx}$. La fonction $a(x)$ est supposée accrétive et bornée. La conjecture de Kato est vraie par exemple pour les données de Dirichlet au bord ou celle de Neumann (nous ne prétendons pas avoir examiné toutes les conditions aux limites possibles).

Considérons par exemple le problème de Dirichlet pour T sur $(0, +\infty)$. Le domaine de cet opérateur est contenu dans $H_0^2 = \{f \in H^2 : f(0) = f'(0) = 0\}$. Les données au bord tiennent également compte des valeurs de la dérivée. Il convient alors de remplacer les splines linéaires par des fonctions qui fournissent une interpolation d'après les valeurs de f et f'. En s'inspirant d'une construction de [8] ou de [7], on peut travailler avec des fonctions dites d'interpolation de Hermite.

On part de l'observation qu'il existe un unique polynôme P de degré 3 sur [0,1] avec des valeurs assignées pour P et P' en 0 et 1. Il existe alors un unique couple de fonctions (φ_0, φ_1) à support dans $[-1, 1]$ de classe C^1 (en fait lipschitziennes d'ordre 2) et polynomiales de degré 3 sur $[-1, 0]$ et $[0, 1]$ telles que $\varphi_0(0) = 1$, $\varphi_0'(0) = 0$, $\varphi_1(0) = 0$ et $\varphi_1'(1) = 1$. Les fonctions de classe C^1 et polynomiales de degré 3 sur chaque intervalle $[k, k+1]$, $k \in \mathbb{Z}$, s'écrivent donc $f(x) = \sum f(k)\varphi_0(x - k) + \sum f'(k)\varphi_1(x - k)$. Le couple (φ_0, φ_1) remplace donc la fonction Δ. On obtient facilement les inégalités analogues à (4.1) et (4.2) pour ce couple (voir [1] pour plus de détails).

Il suffit ensuite de réécrire *mutatis mutandis* la construction de l'analyse multirésolution, des bases adaptées ... Nous laissons au lecteur le soin de le faire.

10. Appendices.

Appendice 1 : Bases orthogonales.

Nous rappelons ici l'algorithme de construction de bases orthogonales dans un espace de Hilbert. Cet algorithme est dû à P. G. Lemarié et Y. Meyer et nous renvoyons à [21] et [17] pour une démonstration détaillée.

Pour plus de simplicité nous nous plaçons dans le cadre concret de $L^2(I) = L^2$, muni de sa structure hilbertienne complexe canonique

$\langle f, g\rangle = \int_I f\,\overline{g}$. Dans tout cet appendice, on désigne par B une forme bilinéaire, continue sur L^2 et accrétive:

$$\mathrm{Re}\, B(f, \overline{f}) \geq \|f\|^2, \quad \forall f \in L^2.$$

Nous dirons qu'un sous-espace de L^2 est autoadjoint s'il est invariant pour la conjugaison des fonctions.

Lemme A1. *Soit E un sous-espace fermé et autoadjoint de L^2. On suppose que E est engendré par une base de Riesz normalisée $e_k(x)$, $k \in K$ (K étant un sous-ensemble quelconque de $\mathbb{Z}$), de fonctions à valeurs réelles. Il existe alors une base de Riesz normalisée, $e_k^\star(x)$, $k \in K$, de E de fonctions à valeurs complexes et orthogonales pour B:*

$$B(e_k^\star, e_\ell^\star) = \delta_{k\ell}, \quad k, \ell \in K.$$

Ces fonctions sont données pour tout $k \in K$ par

$$e_k^\star(x) = \sum_{\ell \in K} \gamma(k, \ell)\, e_\ell(x),$$

où les $\gamma(k,\ell)$ sont les coefficients de la matrice $G^{-1/2}$ lorsque G est la matrice $\big((B(e_k, e_\ell))\big)$.

REMARQUE. Grâce aux hypothèses sur les fonctions e_k et au fait que E soit autoadjoint, la matrice G est bornée et accrétive sur $\ell^2(K)$. La matrice $G^{-1/2}$ est alors définie par le calcul fonctionnel de Kato.

Une deuxième remarque est que le lemme s'applique à la forme bilinéaire particulière $B_0(f, g) = \langle f, \overline{g}\rangle$. On obtient alors une base orthonormée de E et ces fonctions sont à valeurs réelles.

On se donne désormais U, V et H trois sous-espaces fermés de L^2 avec les propriétés suivantes:

- U, V et H sont autoadjoints,
- $H = U + V$ et la somme est directe,
- il existe des fonctions $v_k(x)$, $k \in K$, à valeurs réelles et formant une base de Riesz de V,
- il existe des fonctions $u_k(x)$, $k \in K'$, à valeurs réelles et formant une base de Riesz de U.

On introduit W le supplémentaire orthogonal de V dans H et l'on désigne par Π le projecteur orthogonal de H sur V.

Lemme A2. *Soit $\varphi_k(x)$, $k \in K$, la base orthonormée de V obtenue en appliquant le Lemme* A1 *à V et à (v_k) pour la forme B_0. Ces fonctions sont à valeurs réelles et l'on a*

1.
$$\Pi(h)(x) = \sum_{\ell \in K} \langle h, \varphi_\ell \rangle \varphi_\ell(x), \quad h \in H.$$

2. *$I - \Pi$ réalise un isomorphisme de U sur W et la suite de fonctions à valeurs réelles w_k définies par*

$$w_k(x) = u_k(x) - \Pi(u_k)(x), \quad \forall\, k \in K',$$

est une base de Riesz normalisée de W.

3. *En appliquant le Lemme* A1 *aux fonctions w_k dans l'espace W pour la forme B_0, on obtient une base orthonormée $\psi_k(x)$, $k \in K'$, de W composée de fonctions à valeurs réelles.*

Pour une forme accrétive B quelconque on a l'énoncé suivant:

Lemme A3.

1. *Soit X le sous-espace fermé et non autoadjoint de H défini par*

$$X = \{\theta \in H \,:\, B(\theta, v) = 0 \quad \forall\, v \in V\}.$$

Alors

$$H = V + X$$

où la somme est directe. On appelle P le projecteur sur V de noyau X associé à cette somme directe.

2. *Soit $g_k(x)$, $k \in K$, la base de Riesz obtenue en appliquant le Lemme* A1 *aux fonctions $v_k(x)$ dans V pour la forme B. On a*

$$P(h)(x) = \sum_{k \in K} B(h, g_k) g_k(x), \quad \forall\, h \in H.$$

3. *$I - P$ est un isomorphisme de W sur X. La suite de fonctions définies par*

$$\mathcal{X}_k(x) = \psi_k(x) - P(\psi_k)(x), \quad k \in K',$$

est une base de Riesz normalisée de X.

4. *Soit* $M = \big((B(\mathcal{X}_k, \mathcal{X}_\ell))\big)$. *Alors* M *est bornée et accrétive sur* $\ell^2(K')$. *Si* $\gamma(k,\ell)$ *sont les coefficients de la matrice* $M^{-1/2}$ *alors les fonctions définies par*

$$\theta_k(x) = \sum_{\ell \in K'} \gamma(k,\ell)\mathcal{X}_\ell(x), \quad k \in K',$$

forment une base de Riesz normalisée de X *et orthogonale pour* B.

Seul le dernier point appelle une remarque. L'espace X n'étant pas autoadjoint et les fonctions $\mathcal{X}_k$ n'étant pas à valeurs réelles, le point 4 n'est donc pas immédiat. C'est là qu'on utilise le fait que les ψ_k sont à valeurs réelles et que V est autoadjoint. En effet, grâce aux formules des points 2 et 3 on voit que $\mathcal{X}_k(x) - \overline{\mathcal{X}_k(x)} \in V$. Par définition de X on en tire la relation $B(\mathcal{X}_k, \mathcal{X}_\ell) = B(\mathcal{X}_k, \overline{\mathcal{X}_\ell})$ qui permet d'utiliser l'accrétivité de B.

Une autre remarque est que le détour par W pour «passer» de U à X est rendu nécessaire pour établir les inégalités quadratiques de la Section 7.

Pour terminer ce premier appendice, le lemme suivant s'applique à toutes les matrices utilisées ici et fournit les estimations standard du Théorème 3.

Lemme A4 ([9]). *Pour* $\gamma > 0$ *soit* $\mathcal{M}_\gamma$ *l'espace vectoriel des matrices à coefficients complexes* $M = \big((m(k,\ell))\big)$, $k,\ell \in K$ (*sous-ensemble quelconque de* $\mathbb{Z}$), *telles que*

$$\|M\|_\gamma \equiv \sup\{|m(k,\ell)| \exp\{\gamma|k-\ell|\} : k,\ell \in K\} < +\infty\,.$$

Pour $\gamma > 0$ *on pose*

$\mathcal{M}^\gamma = \cup_{0<\delta<\gamma} \mathcal{M}_\delta$. *Alors*

1. $\mathcal{M}^\gamma$ *est une sous-algèbre d'opérateurs bornés sur* $\ell^2(K)$.
2. *Si* $M \in \mathcal{M}^\gamma$ *est inversible sur* $\ell^2(K)$ *alors* $M^{-1} \in \mathcal{M}^\gamma$. *Si, de plus,* M *est accrétive alors* $M^{-1/2} \in \mathcal{M}^\gamma$.

Tous ces lemmes s'appliquent à chaque triplet $(U, V, H) = (U_j, V_j, V_{j+1})$. On obtient ainsi une preuve du Théorème 3.

Appendice 2: Preuve de la Proposition 3.

On reprend les notations de la Proposition 3. Nous allons montrer comment nous ramener aux arguments de la Section 6.

Le premier cas est celui où

$$\int_I h_\lambda = 0$$

pour tout $\lambda \in \Lambda^p$. Rappelons que les fonctions h_λ sont périodiques de période 1 donc l'intégrale est prise sur n'importe quel intervalle de longueur 1. Pour démontrer

$$\int_I \Big| \sum_{\lambda \in \Lambda^p} \alpha_\lambda h_\lambda \Big|^2 \leq C \sum |\alpha_\lambda|^2,$$

on commence par estimer les coefficients $\langle h_\lambda, h_\mu \rangle$. Soient $\lambda = (2k+1)2^{-j-1} \in \Lambda^p_j$, $\mu = (2k'+1)2^{-j'-1} \in \Lambda^p_{j'}$ avec $j > j'$. Grâce à la périodicité et à l'hypothèse ci-dessus, on peut écrire

$$\langle h_\lambda, h_\mu \rangle = \int_{\lambda-1/2}^{\lambda+1/2} h_\lambda \overline{h_\mu} = \int_{\lambda-1/2}^{\lambda+1/2} h_\lambda \, (\overline{h_\mu - h_\mu(\lambda)}).$$

On remarquera que pour $x \in (\lambda - 1/2, \lambda + 1/2)$ il vient $\operatorname{dist}\{x-\lambda, \mathbb{Z}\} = |x - \lambda|$. On peut donc remplacer (8.13a) par (5.2a) pour h_λ une fois prolongée par 0 en dehors de $(\lambda - 1/2, \lambda + 1/2)$.

Ensuite l'intégrale précédente se décompose en la somme de deux intégrales I_1 et I_2 en découpant le domaine d'intégration suivant que $|x - \mu_1| = \operatorname{dist}\{x - \mu, \mathbb{Z}\}$ ou que $|x - \mu_2| = \operatorname{dist}\{x - \mu, \mathbb{Z}\}$. On a désigné par μ_1 et μ_2 deux points de $\Lambda_{j'}$ à distance entière de $\mu \in \Lambda^p_{j'}$ avec $|\lambda - \mu_1| \leq 1/2 \leq |\lambda - \mu_2| \leq 1$. (On a d'ailleurs soit $\mu = \mu_1$ soit $\mu = \mu_2$.)

Pour estimer I_1 on remplace alors $h_\mu - h_\mu(\lambda)$ par $h_{\mu_1} - h_{\mu_1}(\lambda)$ où h_{μ_1} est une fonction vérifiant (5.2) pour μ_1 et qui coïncide avec h_μ sur la zone d'intégration. On est donc ramené à la majoration du Lemme 5 et l'on obtient $|\mathrm{I}_1| \leq C 2^{-|j-j'|(1/2+\delta)} \exp\{-\gamma\, 2^{\inf\{j,j'\}} |\lambda - \mu_1|\} \leq C 2^{-|j-j'|(1/2+\delta)} \exp\{-\gamma\, 2^{\inf\{j,j'\}} \operatorname{dist}\{\lambda - \mu, \mathbb{Z}\}\}$ où $0 < \delta < 1$. On procède de même pour I_2.

On conclut alors comme dans le Lemme 5 en montrant que la matrice formée par les coefficients $\langle h_\lambda, h_\mu \rangle$ est bornée sur $\ell^2(\Lambda)$ par interpolation.

Le deuxième cas est celui où la suite $(\int_I h_\lambda)$ est de Carleson. Il n'y a rien à changer à l'argument du Théorème 5 qu'il suffit de recopier mot à mot.

Appendice 3: Estimations pour les ondelettes à deux bosses.

Nous nous proposons de vérifier que la construction de la base des ondelettes adaptées avec estimations à deux bosses reste valide dans le cas de la condition aux limites (2.2h).

Tout d'abord les Lemmes A1–3 de l'Appendice 1 sont de nature purement hilbertienne et s'appliquent sans aucun changement à chaque triplet (U_j, V_j, V_{j+1}).

On observe ensuite que la fonction φ_{j_0} est localisée autour de 0 et de 1. Au cours des différentes étapes d'orthogonalisation et de projection cette double localisation va se propager à toutes les fonctions des différents systèmes obtenus. Cette propagation est contrôlée en vertu de l'extension ci-dessous du Lemme A4.

Introduisons auparavant les notations suivantes: soit K un sous-ensemble symétrique ($k \in K$ implique $-k \in K$) de $\mathbb{Z}$. Posons $\delta(k,\ell) = \inf\{|k-\ell|, |k+\ell|\}$, $k, \ell \in K$. Pour $\gamma > 0$ on appelle $\mathcal{M}^\sharp_\gamma$ l'espace vectoriel des matrices à coefficients complexes $M = \big((m(k,\ell))\big)$, $k, \ell \in K$, telles que

$$\|M\|^\sharp_\gamma \equiv \sup\{|m(k,\ell)| \exp\{\gamma\delta(k,\ell)\} \,:\, k,\ell \in K\} < +\infty\,.$$

On appelle enfin $\mathcal{M}^{\gamma\sharp} = \cup_{0<\delta<\gamma} \mathcal{M}^\sharp_\delta$.

Lemme A5.

1. *$\mathcal{M}^{\gamma\sharp}$ est une sous-algèbre d'opérateurs bornés sur $\ell^2(K)$.*
2. *Si $M \in \mathcal{M}^{\gamma\sharp}$ est inversible sur $\ell^2(K)$ alors $M^{-1} \in \mathcal{M}^{\gamma\sharp}$. Si, de plus, M est accrétive alors $M^{-1/2} \in \mathcal{M}^{\gamma\sharp}$.*

La preuve consiste à observer que pour démontrer le Lemme A4 il n'est pas nécessaire que δ soit une distance. Tout ce dont on a besoin est que δ satisfasse l'inégalité triangulaire, la condition exponentielle suivante: pour tout $\gamma > 0$, il existe $C = C(\gamma) < \infty$ telle que

$$\sup_k \sum_\ell \exp\{-\gamma\delta(k,\ell)\} \le C\,,$$

et la condition où les rôles de k et ℓ sont échangés.

En revenant aux θ_λ, on obtient grâce au Lemme A5 les estimations à deux bosses pour ces fonctions. Il faut encore démontrer l'inégalité quadratique. Celle-ci découle de la Proposition 4 et des formules de représentation (5.4–6) en suivant l'argument de la Section 7. La seule

chose à modifier est l'estimation (5.5) qui devient l'estimation à deux bosses

$$|\beta(\lambda,\mu)| \leq C\exp\{-\gamma\inf\{|k-\ell|,|k+\ell-2^j|\}\},$$

pour tous $\lambda = (2k+1)2^{-j-1} \in \Lambda_j$ et $\mu = (2\ell+1)2^{-j-1} \in \Lambda_j$.

Appendice 4: Bases bi-orthogonales dans le cas non-autoadjoint.

Cet appendice résume une construction de [2]. Lorsque l'analyse multirésolution (V_j) n'est plus autoadjointe on ne peut plus utiliser la restriction à chaque V_j d'une forme bilinéaire. Il faut donc partir d'une forme *sesquilinéaire* $\beta(f,g)$ continue sur L^2 et accrétive:

$$\text{Re}\ \beta(f,f) \geq \|f\|^2, \quad \forall f \in L^2.$$

Si maintenant l'on essaie de reprendre le Lemme A1 où B est remplacée par β, on s'aperçoit que cela n'est pas possible à cause de la sesquilinéarité de β. Il faut donc être moins exigeant et construire des bases bi-orthogonales. Enonçons par exemple une variante des lemmes A1–3.

Considérons un triplet (U, V, H) vérifiant les conditions de l'appendice 1. On n'impose plus cependant que ces espaces soient autoadjoints ni que les fonctions génératrices de U et de V soient à valeurs réelles. On appellera W le supplémentaire orthogonal pour le produit scalaire de V dans H et X le supplémentaire orthogonal pour β de V dans H:

$$\text{(A4)} \qquad X = \{\theta \in H\ :\ \beta(\theta,v) = 0 \quad \forall v \in V\}.$$

Lemme A6.

1. *Il existe une base orthonormée de V composée de fonctions $\varphi_k(x)$, $k \in K$, donnée par*

$$\varphi_k(x) = \sum_{\ell\in K} \gamma(k,\ell)v_\ell(x),$$

où les $\gamma(k,\ell)$ sont les coefficients de la matrice $M^{-1/2}$ lorsque M est la matrice définie positive $((\langle v_k, v_\ell\rangle))$.

2. *Il existe une base de Riesz normalisée, $v_k^\star(x)$, $k \in K$, de V, bi-orthogonale de la base (v_k) pour β:*

$$\beta(v_k^\star, v_\ell) = \delta_{k\ell}, \quad k, \ell \in K.$$

Ces fonctions sont données pour tout $k \in K$ par

$$v_k^\star(x) = \sum_{\ell \in K} \eta(k,\ell) v_\ell(x),$$

où les $\eta(k,\ell)$ sont les coefficients de la matrice G^{-1} lorsque

$$G = \left(\left(\beta(v_k, v_\ell)\right)\right).$$

3. *La projection orthogonale sur W de la base u_k, $k \in K'$, de U est une base de Riesz normalisée de W. Cette dernière est orthonormalisée, en suivant l'algorithme du point* 1.

4. *$H = V + X$ où la somme est directe.*

5. *La projection sur X parallèlement à V de la base (ψ_k) de W construite en* 3 *est une base de Riesz normalisée $(\mathcal{X}_k)$ de X.*

On peut compléter ce lemme en écrivant les formules de représentation similaires à celles des lemmes A1–3.

En ce qui concerne le point 4 du Lemme A3, on ne doit pas l'utiliser de la même façon. Voyons pourquoi en tentant de l'appliquer directement à la construction des bases adaptées.

Partons d'une analyse multirésolution (V_j) sur L^2. Soit $\beta(f,g) = \int_I f\,\overline{g}/a$ et pour tout j, X_j l'orthogonal pour β de V_j dans V_{j+1}. Soit $\theta \in X_j$, $\theta' \in X_{j'}$ et supposons que $j \neq j'$. Si $j > j'$, alors $X_{j'} \subset V_{j'+1}$ et donc $\beta(\theta, \theta') = 0$. En revanche il n'y a aucune raison pour que cette égalité ait lieu si $j < j'$.

Pour remédier à cela, on introduit une nouvelle famille d'espaces $(X_j^\star)$, où $X_j^\star$ est l'orthogonal de V_j dans V_{j+1} pour la forme duale de β définie par : $\beta^\star(f,g) = \overline{\beta(g,f)}$. C'est-à-dire

$$X_j^\star = \{\theta^\star \in V_{j+1} \,:\, \beta^\star(\theta^\star, v) = 0 \quad \forall\, v \in V_j\}$$

. Si maintenant $\theta \in X_j$ et $\theta^\star \in X_{j'}^\star$, avec $j \neq j'$, on voit que $\beta(\theta, \theta^\star) = 0$.

Si donc (θ_λ) est une base de X_j (j fixé) on doit construire une famille qui lui est orthogonale dans $X_j^\star$. Ainsi d'après les remarques précédentes on obtient en collectant toutes ces familles deux systèmes bi-orthogonaux pour β dans L^2.

Revenons au cadre plus abstrait ci-dessus. X est défini par (A4) et $X^\star$ par (A4) où l'on a remplacé β par sa forme duale $\beta^\star$. Il faut donc considérer la restriction de β à $X \times X^\star$. Le problème est qu'on perd ainsi l'accrétivité de β. On garde cependant une propriété plus faible mais suffisante pour notre propos.

Lemme A7.

1. *Il existe une constante $\delta > 0$ telle que*

$$\sup_{\theta^\star \in X^\star,\, \|\theta^\star\| \le 1} |\beta(\theta, \theta^\star)| \ge \delta \|\theta\| \qquad \forall\, \theta \in X$$

$$\sup_{\theta \in X,\, \|\theta\| \le 1} |\beta(\theta, \theta^\star)| \ge \delta \|\theta^\star\| \qquad \forall\, \theta^\star \in X^\star$$

2. *Soient $(\mathcal{X}_k)$ et $(\mathcal{X}_k^\star)$ des bases de Riesz de X et $X^\star$ respectivement (construites en appliquant par exemple le Lemme* A6*). Alors la matrice $G = \big((\beta(\mathcal{X}_k, \mathcal{X}_\ell^\star))\big)$ est inversible sur $\ell^2(K')$.*

3. *Les fonctions $\theta_k(x)$ définies pour $k \in K'$ par*

$$\theta_k(x) = \sum_{\ell \in K'} \gamma(k,\ell) \mathcal{X}_\ell(x)\,,$$

où $\gamma(k,\ell)$ sont les coefficients de G^{-1}, forment une base de Riesz normalisée de X telle que

$$\beta(\theta_k, \mathcal{X}_\ell^\star) = \delta_{k\ell}, \quad \forall\, k, \ell \in K'.$$

Une preuve se trouve dans [2] (Il y a dans les notations une légère différence avec cette référence).

Pour finir, on construit à l'aide de ces lemmes deux bases d'ondelettes adaptées pour la condition aux limites (2.2i). Noter que l'une sera adaptée à $1/a(x)$ et l'autre à $\overline{1/a(x)}$. On vérifie les estimations à deux bosses pour chacune de ces bases en suivant l'Appendice 3. Les détails sont laissés au lecteur.

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Criterion on L^p-Boundedness for a class of Oscillatory Singular Integrals with rough Kernels

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1. Introduction.

It was known that the following form of oscillatory singular integrals defined for smooth f with compact support had been studied by F. Ricci and E. M. Stein in [1]:

$$\text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy,$$

where $p(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, and $K(x)$ is a standard Calderón-Zygmund kernel. That means, K satisfies:

(1.1) $K(x)$ is C^1-continuous away from the origin,

(1.2) $K(x) = \Omega(x')/|x|^n$ with Ω homogeneous of degree 0 on S^{n-1},

(1.3) $$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0.$$

The following theorem is the main result in [1]:

Theorem A. *Suppose $p(x,y)$ is a real valued polynomial. If $K(x)$ satisfies* (1.1)-(1.3), *then the operator*

$$f \mapsto \text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy$$

can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself, with $1 < p < +\infty$, and the norm of this operator depends only on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$.

Evidently (1.1) shows that the kernel $K(x)$ requires certain smoothness. In this paper, we shall discuss a class of oscillatory singular integrals with rough kernel. Precisely, the kernel $K(x)$ satisfies (1.2), and

$$\Omega(x') \in L^q(S^{n-1}), \qquad \text{for some } q,\ 1 < q \le +\infty\ . \tag{1.4}$$

In this case, the kernel $K(x)$ can be very rough on S^{n-1}, and $K(x)$ is not necessary to be a standard Calderón-Zygmund kernel. Since (1.1) implies $\Omega(x') \in L^\infty(S^{n-1})$, the following result can be regarded as an improvement of Theorem A.

Theorem 1. *Suppose $p(x,y)$ is a real valued polynomial. If $K(x)$ satisfies* (1.2)-(1.4), *then the operator*

$$Tf(x) = \text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy$$

can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself, with $1 < p < +\infty$, and the norm of this operator depends only on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$.

In fact, Theorem 1 is an immediate consequence of the following stronger result.

Theorem 2. *Suppose $p(x,y)$ is a real valued polynomial, $K(x)$ satisfies* (1.2)-(1.4), *and $b(r)$ is a bounded variation function on $[0,\infty)$. If the operator*

$$f \longmapsto \text{p.v.} \int b(\mid x-y \mid) K(x-y) f(y)\, dy$$

is a bounded operator on $L^p(\mathbb{R}^n)$ to itself with $1 < p < +\infty$, then the oscillatory integral operator

$$Tf(x) = \text{p.v.} \int e^{ip(x,y)} b(|\, x - y \,|) K(x-y) f(y)\, dy$$

is a bounded operator on $L^p(\mathbb{R}^n)$ to itself, and the norm of T depends only on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$.

Let us introduce two concepts before we formulate our main result of this paper.

Definition 1. *A real valued polynomial $p(x,y)$ is called non-trivial if $p(x,y)$ does not take the form of $p_0(x) + p_1(y)$, where p_0 and p_1 are polynomials defined on $\mathbb{R}^n$.*

Definition 2. *We will say that the non-trivial polynomial $p(x,y)$ has property $\mathcal{P}$, if p satisfies*

$$p(x+h, y+h) = p(x,y) + R_0(x,h) + R_1(y,h)$$

where R_0 and R_1 are real polynomials.

The main result in this paper is

Theorem 3. *Suppose $1 < p < +\infty$. If $K(x)$ satisfies* (1.2) *and* (1.4), *then the following three facts are equivalent:*

(i) *If $p(x,y)$ is a non-trivial polynomial, then the operator*

$$Tf(x) = \text{p.v.} \int e^{ip(x,y)} K(x-y) f(y)\, dy$$

can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself.

(ii) *If $Q(x,y)$ has the property $\mathcal{P}$, then the operator*

$$Gf(x) = \text{p.v.} \int e^{iQ(x,y)} K(x-y) f(y)\, dy$$

can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ to itself.

(iii) *The truncated operator*

$$Sf(x) = \int_{|x-y|<1} K(x-y)f(y)dy$$

can be extended to be a bounded operator on $L^p(\mathbb{R}^n)$ *to itself.*

Let us give out a simple application of Theorem 3. S. Chanillo, D. S. Kurtz and G. Sampson investigated the following oscillatory integrals in [2]:

$$f \longmapsto \text{p.v.} \int_{\mathbb{R}} e^{i|x-y|^a} \frac{f(y)}{1+|x-y|}\, dy, \qquad a > 0\,. \tag{1.5}$$

Since the truncated operator

$$f \longmapsto \int_{|x-y|<1} \frac{f(y)}{1+|x-y|}\, dy$$

is a bounded operator on $L^p(\mathbb{R})$ to itself, it follows from Theorem 3 that if a is even, then the operator defined by (1.5) is a bounded operator on $L^p(\mathbb{R})$ to itself. Besides, we have the following conclusion. The operator defined by

$$f \longmapsto \int_{\mathbb{R}} e^{ip(x,y)} \frac{f(y)}{1+|x-y|}\, dy$$

is a bounded operator on $L^p(\mathbb{R})$ to itself, where $p(x,y)$ is a nontrivial polynomial. Let us make two explanations on the above conclusion. First, the above conclusion is not contained in [2]. Secondly, since the operator

$$f \longmapsto \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{1+|x-y|}\, dy$$

is not bounded on $L^p(\mathbb{R})$, the above conclusion can not be obtained from [1].

Finally, we shall get a similar result for the maximal operator corresponding to T.

Theorem 4. *Suppose* $p(x,y)$ *is a real valued polynomial. If* $K(x)$ *satisfies* (1.2)-(1.4), *then the maximal operator*

$$T_* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y)f(y)dy \right|$$

is a bounded operator on $L^p(\mathbb{R}^n)$ to itself, with $1 < p < +\infty$, and the norm of T_* depends only on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$.

2. Proof of the theorems.

First, we state several lemmas.

Lemma 1. *Suppose $\Omega(x')$ is homogeneous of degree 0 on S^{n-1}, and $\Omega(x') \in L^q(S^{n-1})$ with $1 < q \leq +\infty$. If*

$$Tf(x) = \text{p.v.} \int K(x,y) f(y)\, dy$$

is a (L^p, L^p) type operator with $1 < p < +\infty$, and $K(x,y)$ satisfies

$$|K(x,y)| \leq \frac{|\Omega[(x-y)']|}{|x-y|^n},$$

then the operators

$$T_\varepsilon f(x) = \int_{|x-y|<\varepsilon} K(x,y) f(y)\, dy$$

are (L^p, L^p) type operators, and $\|T_\varepsilon\| \leq C(\|T\| + A)$, where C is independent of T, ε and A depends only on $\Omega(x')$.

PROOF: We split f into three parts $f(y) = f_1(y) + f_2(y) + f_3(y)$ for $h \in \mathbb{R}^n$. Here

$$\begin{aligned} f_1(y) &= f(y)\chi_{\{|y-h|<\varepsilon/2\}}(y), \\ f_2(y) &= f(y)\chi_{\{\varepsilon/2\leq|y-h|<5\varepsilon/4\}}(y), \\ f_3(y) &= f(y)\chi_{\{|y-h|\geq 5\varepsilon/4\}}(y). \end{aligned}$$

When $|x-h| < \varepsilon/4$, it is easy to see $T_\varepsilon f_1(x) = Tf_1(x)$. So we have

$$\begin{aligned} \int_{|x-h|<\varepsilon/4} |T_\varepsilon f_1(x)|^p\, dx &\leq \int_{\mathbb{R}^n} |Tf_1(x)|^p\, dx \\ &\leq \|T\|^p \int_{|y-h|<\varepsilon/2} |f(y)|^p\, dy. \end{aligned} \tag{2.1}$$

If $|x-h|<\varepsilon/4, \varepsilon/2 \le |y-h| < 5\varepsilon/4$, then $\varepsilon/4 < |x-y| < 3\varepsilon/2$. So we have

$$|T_\varepsilon f_2(x)| \le \int_{\varepsilon/4<|y|\le\varepsilon} \frac{|\Omega(y')|}{|y|^n} |f_2(x-y)|\, dy.$$

By the Minkowski's inequality, it follows that

$$\begin{aligned}
&\left(\int_{|x-h|<\varepsilon/4} |T_\varepsilon f_2(x)|^p\, dx\right)^{1/p} \\
&\le \int_{\varepsilon/4<|y|\le\varepsilon} \frac{|\Omega(y')|}{|y|^n} \left(\int_{|x-h|<\varepsilon/4} |f_2(x-y)|^p\, dx\right)^{1/p} dy \\
&\le C\left(\int_{|y-h|<5\varepsilon/4} |f(y)|^p\, dy\right)^{1/p} \int_{S^{n-1}} |\Omega(y')|\, d\sigma(y') \\
&\le C\,\|\Omega\|_{L^q(S^{n-1})} \left(\int_{|y-h|<5\varepsilon/4} |f(y)|^p\, dy\right)^{1/p}.
\end{aligned} \tag{2.2}$$

If $|x-h|<\varepsilon/4$, $|y-h| \ge 5/4$, then $|x-y|>\varepsilon$. So we have

$$T_\varepsilon f_3(x) = 0. \tag{2.3}$$

From (2.1), (2.2) and (2.3) it follows that the estimate

$$\int_{|x-h|<\varepsilon/4} |T_\varepsilon f(x)|^p\, dx \le C(\|T\|+A)^p \int_{|y-h|<5\varepsilon/4} |f(y)|^p\, dy$$

holds uniformly in $h \in \mathbb{R}^n$.

The above estimates imply

$$\|T_\varepsilon f\|_p \le C(\|T\|+A)\|f\|_p.$$

Lemma 2. (Van der Corput [3]) *Suppose $\phi \in C^{(k)}[a,b]$ and $|\phi^{(k)}(t)| \ge 1$ (when $t \in (a,b)$), then we have*

$$\left|\int_a^b e^{i\lambda\phi(t)}\, dt\right| \le C\,|\lambda|^{-1/k}$$

where $\lambda \in \mathbb{R}$, and C is independent of a, b and ϕ.

Lemma 3. (See [1]) *Suppose $p(x) = \sum_{|\alpha|\le d} a_\alpha x^\alpha$ is a polynomial of degree d, and $\varepsilon < 1/d$. Then*

$$\sup_{y\in\mathbb{R}^n} \int_{|x|\le 1} |p(x-y)|^{-\varepsilon}\, dx \le A_\varepsilon \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{-\varepsilon}.$$

The bound A_ε depends on ε (and the dimension n), but not on the coefficients $\{a_\alpha\}$.

Lemma 4. (See [1]) *Suppose $p(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ is a homogeneous polynomial of degree d in $\mathbb{R}^n$, and $\varepsilon \le 1/d$. Then*

$$\int_{S^{n-1}} |p(x)|^{-\varepsilon}\, d\sigma(x) \le A_\varepsilon \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{-\varepsilon}.$$

The bound A_ε depends on ε, but not on the coefficients $\{a_\alpha\}$.

Now, let us turn to prove the theorems.

PROOF OF THEOREM 2. We shall carry out the argument by a double induction on the degrees in x and y of the polynomial p as follows. We assume the theorem is known for all polynomials which are sums of monomials degree less than k in x times monomials of any degree in y, together with monomials which are of degree k in x times monomials which are of degree less than l in y. Our inductive step will be to add to this all the monomials which have degree k in x and degree l in y.

For general $p(x, y)$, we may write

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta + R_0(x,y)$$

where $R_0(x, y)$ satisfies the above induction assumption.

For $k = 0$ and l arbitrary, the theorem is known. Let us now prove that Theorem 2 holds for arbitrary $k > 0$ and $l > 0$ by induction.

Without loss of generality, we may assume $\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| > 0$.

Case 1. $\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| = 1$.

We write

$$\begin{aligned} Tf(x) &= \int_{|x-y|\le 1} e^{ip(x,y)} b(|x-y|) K(x-y) f(y)\, dy \\ &\quad + \int_{|x-y|>1} e^{ip(x,y)} b(|x-y|) K(x-y) f(y)\, dy \\ &= T_0 f(x) + T_\infty f(x)\,. \end{aligned}$$

Take $h \in \mathbb{R}^n$, and write

$$p(x,y) = \sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}(x-h)^\alpha (y-h)^\beta + R(x,y,h)\,,$$

where the polynomial $R(x,y,h)$ satisfies the induction assumption, and the coefficients of $R(x,y,h)$ depend on h.

We have

$$\begin{aligned} |T_0 f(x)| \le & \Big| \int_{|x-y|\le 1} \exp\{i\,[R(x,y,h) + \sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}(y-h)^{\alpha+\beta}]\} \\ & \qquad \cdot b(|x-y|) K(|x-y|) f(y)\, dy \Big| \end{aligned}$$

$$\begin{aligned} + \Big| \int_{|x-y|\le 1} & \{\exp(ip(x,y)) - \exp(i[R(x,y,h) + \sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}(y-h)^{\alpha+\beta}])\} \\ & \cdot b(|x-y|) K(x-y) f(y)\, dy \Big| \end{aligned}$$

$$= |T_{01} f(x)| + |T_{02} f(x)|\,.$$

Note that $\|b\|_\infty < +\infty$, from the induction assumption and Lemma 1 we obtain that T_{01} is a (L^p, L^p) type operator, and the norm of T_{01} depends on $\|b\|_\infty$, but not on the coefficients of $p(x,y)$ and h.

When $|x-h|<1/4, |x-y|<1$, we have

$$|\exp\{ip(x,y)\}-\exp\{i\,[R(x,y,h)+\sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}(y-h)^{\alpha+\beta}]\}|$$

$$\le C\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}||x-y| \le C|x-y|.$$

Thus

$$|T_{02}f(x)| \le \int_{|x-y|\le 1} \frac{C\|b\|_\infty |\Omega[(x-y')]|}{|x-y|^{n-1}}|f(y)|\,dy$$

$$\le C\|b\|_\infty \int_{|y|\le 1} \frac{|\Omega(y')|}{|y|^{n-1}}|f(x-y)\chi_{B(h,5/4)}(x-y)|\,dy.$$

By the Minkowski's inequality, we obtain

$$\int_{|x-h|<1/4} |T_{02}f(x)|^p\,dx \le C\|b\|_\infty^p \|\Omega\|^p_{L^q(S^{n-1})} \int_{|y-h|<5/4} |f(y)|^p\,dy.$$

Thus $\quad \|T_{02}f\|_p \le C\|b\|_\infty \|\Omega\|_{L^q(S^{n-1})}\|f\|_p.$

Hence

$$\|T_0 f\|_p \le C\,\|f\|_p, \tag{2.4}$$

where C depends on $\|b\|_\infty$, but not on the coefficients of $p(x,y)$.

We write

$$T_\infty f(x) = \sum_{j-1}^{+\infty} \int_{2^{j-1}<|x-y|\le 2^j} e^{ip(x,y)} b(|x-y|)K(x-y)f(y)\,dy$$

$$= \sum_{j=1}^{+\infty} T_j f(x).$$

We have

$$T_j f(x) = \int_{2^{j-1}<|y|\le 2^j} e^{ip(x,x-y)} b(|y|)K(y)f(x-y)\,dy$$

$$= \int_{S^{n-1}} \Omega(y') \int_{2^{j-1}<r\le 2^j} e^{ip(x,x-ry')} \frac{b(r)f(x-ry')}{r}\,dr\,d\sigma(y').$$

For a fixed $y' \in S^{n-1}$, Let Y be the hyperplane through the origin orthogonal to y'. We have, for $x \in \mathbb{R}^n, x = z + sy'$, with $s \in \mathbb{R}$, $z \in Y$, and so

$$\begin{aligned}
&\int_{2^{j-i}<r\le 2^j} e^{ip(x,x-ry')} \frac{b(r)f(x-ry')}{r}\,dr \\
&\qquad = \int_{2^{j-1}<r\le 2^j} e^{ip(z+sy',z+(s-r)y')} \frac{b(r)f(z+(s-r)y')}{r}\,dr \\
&\qquad = \int_{2^{j-1}<s-t\le 2^j} e^{ip(z+sy',z+ty')} \frac{b(s-t)}{s-t} f(z+ty')\,dt \\
&\qquad = N_j[f(z+\cdot\, y')](s)
\end{aligned}$$

where N_j is a linear operator defined on $L^2(\mathbb{R})$. Denote N_j^* be its adjoint operator. Let us now consider the operator $N_j^* N_j$ with the kernel

$$\begin{aligned}
M_j(u,v) = &\int_{2^{j-1}<r-v,r-u\le 2^j} e^{i[p(z+ry',z+vy')-p(z+ry',z+uy')]} \\
&\qquad \cdot \frac{b(r-v)b(r-u)}{(r-v)(r-u)}\,dr \\
= &\int_{\substack{\frac{1}{2}<r\le 1 \\ 2^{j-1}<2^j r+v-u\le 2^j}} e^{i[p(2^j ry'+z+vy',z+vy')-p(2^j ry'+z+vy',z+uy')]} \\
&\qquad \cdot \frac{b(2^j r)b(2^j r+v-u)}{r(2^j r+v-u)}\,dr\,.
\end{aligned}$$

It is easy to see

$$|M_j(u,u)| \le \frac{C}{2^j}\,\chi_{[0,2^{j-1}]}(|v-u|). \tag{2.5}$$

Now we write $p(x,y)$ as follows

$$p(x,y) = \sum_{|\alpha|=k} x^\alpha Q_\alpha(y) + R(x,y),$$

where $R(x,y)$ is a polynomial with x-degree less than k, and $Q_\alpha(y)$ is a polynomial with degree l. So we can write

$$M_j(u,v) = \int_{\frac{1}{2}<r\le 1, 2^{j-1}<2^j r+v-u\le 2^j} e^{i(E+F)}\psi\,dr\,,$$

where

$$E = (2^j r)^k \sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')],$$

and F with r-degree less than k, and

$$\psi(r) = \frac{b(2^j r)b(2^j r + v - u)}{r(2^j r + v - u)}.$$

From Lemma 2, we have

$$\left|\int_{1/2}^{t} e^{i(E+F)}dr\right| \le C\left(2^{jk}\left|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]\right|\right)^{-\frac{1}{k}}$$

From integration by parts, we have

$$\begin{aligned}
|M_j(u,v)| &\le C\{2^{jk}\,|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]|\}^{-1/k} \\
&\qquad \cdot \{|\psi(1)| + \int_{2^{j-1}<2^j r+v-u\le 2^j}^{\frac{1}{2}<r\le 1} |d\psi(r)|\} \\
&\le C\{2^{jk}\,|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]|\}^{-1/k} \\
&\qquad \cdot \left[\frac{\|b\|_\infty^2}{2^j} + \frac{\|b\|_\infty V_0^{+\infty}(b)}{2^j}\right] \\
&\le C(b)2^{-j}\{2^{jk}|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - Q_\alpha(z+uy')]|\}^{-1/k}.
\end{aligned}$$

From (2.5) and the above inequality, we get that the estimate

$$\begin{aligned}
|M_{j(u,v)}| \le C(b)2^{-j}\{2^{jk}|\sum_{|\alpha|=k} y'^{\alpha}[Q_\alpha(z+vy') - (Q_\alpha(z+uy')]|\}^{-\delta/k} \\
\cdot \chi_{[0,2^{j-1}]}(|v-u|)
\end{aligned}$$

holds uniformly in $\delta \in (0,1]$.

Thus

$$\int |M_j(u,v)|\,dv = \int_{|v-u|<2^j} |M_j(u,v)|\,dv$$

$$\leq C(b)2^{-j}2^{-j\delta}\int_{|v-u|<2^j}|\sum_{|\alpha|=k}y'^{\alpha}[Q_\alpha(z+vy')-Q_\alpha(z+uy')]|^{-\delta/k}\,dv$$

$$\leq C(b)2^{-j\delta}\int_{|v|<1}|\sum_{|\alpha|=k}y'^{\alpha}[Q_\alpha(z+2^j(v+\frac{u}{2^j})y')-Q_\alpha(z+uy')]|^{-\delta/k}dv.$$

Now, we take $\delta\in(0,1]$ such that $\delta/k<1/l$, then from Lemma 3 it follows

$$\begin{aligned}\int|M_j(u,v)|dv &\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}2^{-jl\delta/k}\\ &\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.\end{aligned}$$

Thus

$$\|N_j^*N_j\|_{L^\infty(\mathbb{R})\to L^\infty(\mathbb{R})}\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.$$

Similarly, we have

$$\|N_j^*N_j\|_{L^1(\mathbb{R})\to L^1(\mathbb{R})}\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.$$

By the Riesz-Thorin's interpolation theorem, we obtain

$$\|N_j^*N_j\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\leq C(b)2^{-j\delta}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/k}.$$

Hence, we have

$$\|N_j\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\leq C(b)2^{-j\delta/2}|\sum_{\substack{|\alpha|=k\\|\beta|=l}}a_{\alpha\beta}y'^{\alpha+\beta}|^{-\delta/2k}.$$

Since

$$|N_jg(s)|\leq\|b\|_\infty\int_{2^{j-1}<s-t\leq 2^j}\frac{|g(t)|}{s-t}\,dt\leq C(b)\,HL(g)(s)$$

we have

$$\|N_j\|_{L^{p_0}(\mathbb{R})\to L^{p_0}(\mathbb{R})} \le C(b,p_0), \quad \text{with} \quad 1<p_0<+\infty.$$

By the Riesz-Thorin's interpolation theorem, we obtain

$$\|N_j\|_{L^p(\mathbb{R})\to L^p(\mathbb{R})} \le C(b)2^{-\theta j\delta/2}|\sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}y'^{\alpha+\beta}|^{-\theta\delta/2k} \tag{2.6}$$

where $0<\theta<1$.

From (2.6) and the Minkowski's inequality, we have

$$\begin{aligned}\|T_jf\|_p &\le \int_{S^{n-1}} |\Omega(y')|(\int_Y\int_{\mathbb{R}} |N_j[f(z+\cdot y')](s)|^p\,dsdz)^{1/p}\,d\sigma(y')\\ &\le C(b)2^{-\theta j\delta/2}\|f\|_p\int_{S^{n-1}}|\Omega(y')|\|\sum_{\substack{|az|=k\\|bz|=l}} a_{\alpha\beta}y'^{\alpha+\beta}|^{-\theta\delta/2k}\,d\sigma(y')\\ &\le C(b)2^{-\theta j\delta/2}\,\|f\|_p\,\|\Omega\|_{L^q(S^{n-1})}\\ &\qquad\cdot(\int_{S^{n-1}}|\sum_{\substack{|\alpha|=k\\|\beta|=l}} a_{\alpha\beta}y'^{\alpha+\beta}|^{-\theta\delta q'/(2k)}\,d\sigma(y'))^{1/q'}.\end{aligned}$$

We take $\delta\in(0,1]$, such that $\delta<\min\{k/l\,,2k/((k+l)\,q')\}$. Then from Lemma 4, we get

$$\|T_jf\|_p \le C(b)2^{-\theta j\delta/2}\,\|f\|_p\,. \tag{2.7}$$

Thus

$$\|T_\infty f\|_p \le C(b)\,\|f\|_p\,. \tag{2.8}$$

From (2.4) and (2.8), we obtain

$$\|Tf\|_p \le C(b)\,\|f\|_p \tag{2.9}$$

where $C(b)$ depends on the total degree of $p(x,y)$, $\|b\|_\infty$ and $V_0^\infty(b)$, but not on the coefficients of $p(x,y)$.

Case 2. $\sum_{\substack{|\alpha|=k\\|\beta|=l}} |a_{\alpha\beta}| \ne 1$

Denote $A = (\sum_{|\alpha|=k,|\beta|=l} |a_{\alpha\beta}|)^{1/(k+l)}$. We can write $p(x,y)$ as follows

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} \frac{a_{\alpha\beta}}{A^{k+l}}(Ax)^\alpha (Ay)^\beta + R_0(\frac{Ax}{A}, \frac{Ay}{A}) \stackrel{\text{def}}{=} Q(Ax, Ay).$$

Thus

$$\begin{aligned} Tf(x) &= \int e^{iQ(Ax,Ay)} b(|x-y|) K(x-y) f(y)\, dy \\ &= \int e^{iQ(Ax,y)} b(\frac{|Ax-y|}{A}) K(Ax-y) f(\frac{y}{A})\, dy. \end{aligned}$$

Since $\|b(\cdot/A)\|_\infty = \|b\|_\infty$ and $V_0^\infty(b(\cdot/A)) = V_0^\infty(b)$, from the result in Case 1, we obtain

$$\|Tf\|_p \le C\|f\|_p\,,$$

where C depends on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$. So Theorem 2 holds for any polynomial $p(x,y)$ by induction principle.

THE PROOF OF THEOREM 1. When $K(x)$ satisfies (1.2)-(1.4), from the result in [4], we know that the operator

$$f \longmapsto \text{p.v.} \int K(x-y) f(y)\, dy$$

is a (L^p, L^p) type operator with $1 < p < +\infty$. So Theorem 1 follows from Theorem 2.

THE PROOF OF THEOREM 3.
(i) *implies* (ii). This step is obvious.
(ii) *implies* (iii). Set

$$\begin{aligned} Gf(x) &= \int_{|x-y|<1} e^{iQ(x,y)} K(x-y) f(y)\, dy \\ &\quad + \int_{|x-y|\ge 1} e^{iQ(x,y)} K(x-y) f(y)\, dy \\ &= G_0 f(x) + G_\infty f(x). \end{aligned}$$

From the method similar to the proof of (2.8), we know that G_∞ is a (L^p, L^p) type operator. So G_0 is a (L^p, L^p) type operator.

We take $h \in \mathbb{R}^n$. For $|x-h| < 1$, we have

$$G_0 f(x) = G_0[f(\cdot)\chi_{B(h,2)}(\cdot)](x).$$

Thus

$$(2.10) \qquad \Big(\int_{|x-h|<1} |G_0 f(x)|^p \, dx\Big)^{1/p} \le C\Big(\int_{|y-h|<2} |f(y)|^p \, dy\Big)^{1/p}$$

where C is independent of h.

Since $Q(x,y)$ has property $\mathcal{P}$, we have

$$Q(x,y) = Q(x-h, y-h) + R_0(x,h) + R_1(y,h)$$

where R_0, R_1 are real polynomials.

It follows that

$$\begin{aligned} Sf(x) &= \int_{|x-y|<1} K(x-y) f(y) \chi_{B(h,2)}(y) \, dy \\ &= e^{-iR_0(x,h)} \int_{|x-y|<1} e^{iQ(x,y)} K(x-y) e^{-iQ(x-h,y-h)} \\ &\qquad \cdot e^{-iR_1(y,h)} f(y) \chi_{B(h,2)}(y) \, dy \, . \end{aligned}$$

Note that the Taylor's expression of $e^{-iQ(x-h,y-h)}$ is

$$\begin{aligned} e^{-iQ(x-h,y-h)} &= \sum_{m=0}^{+\infty} \frac{i^m}{m!} \Big[\sum_{\alpha,\beta} a_{\alpha\beta} (x-h)^\alpha (y-h)^\beta\Big]^m \\ &= \sum_{m=0}^{+\infty} \frac{i^m}{m!} \sum_{l} C_{m,l} b_{\alpha\beta l} (x-h)^{u(\alpha,\beta,l)} (y-h)^{v(\alpha,\beta,l)} \end{aligned}$$

where u and v are multi-index.

Thus, if we set $a = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$ and $b = (\frac{3}{2}, \cdots, \frac{3}{2}) \in \mathbb{R}^n$,

then we have

$$
\begin{aligned}
&\Big(\int_{|x-h|<1} |Sf(x)|^p\,dx\Big)^{1/p} \\
&\qquad \le \sum_{m=0}^{+\infty}\sum_{l}\frac{|C_{ml}b_{\alpha\beta l}|}{m!}\left[\int_{|x-h|<1}|(x-h)^u|^p\right. \\
&\qquad\qquad \left.|G_0[e^{-iR_1(\cdot,h)}f(\cdot)\chi_{B(h,2)}(\cdot)(\cdot-h)^v](x)|^p\,dx\right]^{1/p} \\
&\qquad \le \sum_{m=0}^{+\infty}\sum_{l}\frac{|C_{ml}b_{\alpha\beta l}|a^u}{m!}C\left[\int_{|y-h|<2}|f(y)|^p|(y-h)^v|^p\,dy)\right]^{1/p} \\
&\qquad \le \sum_{m=0}^{+\infty}\sum_{l}\frac{|C_{ml}b_{\alpha\beta l}|a^u b^v}{m!}C\left[\int_{|y-h|<2}|f(y)|^p\,dy\right]^{1/p} \\
&\qquad = C\sum_{m=0}^{+\infty}\frac{1}{m!}\Big(\sum_{\alpha,\beta}|a_{\alpha\beta}|a^\alpha b^\beta\Big)^m\left[\int_{|y-h|<2}|f(y)|^p\,dy\right]^{1/p} \\
&\qquad = C\exp\left\{\sum_{\alpha,\beta}|a_{\alpha\beta}|a^\alpha b^\beta\right\}\left[\int_{|y-h|<2}|f(y)|^p\,dy\right]^{1/p}.
\end{aligned}
$$

Thus

$$\|Sf\|_p \le C\,\|f\|_p\,.$$

(iii) *implies* (i). Set

$$b(r) = \begin{cases} 1, & r\in[0,1)\,, \\ 0, & r\in[1,+\infty)\,. \end{cases}$$

It is easy to see that $b(r)$ is a bounded variation function on $[0,+\infty)$.

Since the truncated operator

$$Sf(x) = \text{p.v.}\int b(|x-y|)K(x-y)f(y)\,dy$$

is a (L^p, L^p) type operator, from Theorem 2 we know that the operator

$$
\begin{aligned}
T_0f(x) &= \text{p.v.}\int e^{ip(x,y)}b(|x-y|)K(x-y)f(y)\,dy \\
&= \int_{|x-y|<1} e^{ip(x,y)}K(x-y)f(y)\,dy
\end{aligned}
$$

is a (L^p, L^p) type operator.

Since $p(x,y)$ is a nontrivial polynomial, by the methods similar to the proof of (2.8), we can prove that the operator

$$T_\infty f(x) = \int_{|x-y|\geq 1} e^{ip(x,y)} K(x-y) f(y)\, dy$$

is a (L^p, L^p) type operator.

Thus T is a (L^p, L^p) type operator.

THE PROOF OF THEOREM 4. We shall carry out the argument by a double induction on the degrees in x and y of the polynomial p as in the proof of Theorem 2.

As in the proof of Theorem 2, we write

$$p(x,y) = \sum_{\substack{|\alpha|=k \\ |\beta|=l}} a_{\alpha\beta} x^\alpha y^\beta + R(x,y).$$

Since our conclusion is clearly invariant under dialation, we may assume that

$$\sum_{\substack{|\alpha|=k \\ |\beta|=l}} |a_{\alpha\beta}| = 1\,.$$

If $k = 0$, we know that the conclusion holds from the result in [4]. For general $p(x,y)$, we have

$$\begin{aligned} T_* f(x) &\leq \sup_{0<\varepsilon<1} \Big| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\ &\quad + \sup_{\varepsilon\geq 1} \Big| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\ &\leq \sup_{0<\varepsilon<1} \Big| \int_{\varepsilon<|x-y|<1} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\ &\quad + \Big| \int_{|x-y|\geq 1} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\ &\quad + \sup_{\varepsilon\geq 1} \Big| \int_{|x-y|>\varepsilon} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| \\ &= T_{*0} f(x) + \Big| \int_{|x-y|\geq 1} e^{ip(x,y)} K(x-y) f(y)\, dy \Big| + T_{*\infty} f(x) \end{aligned}$$

Now, it suffices to prove that T_{*0} and $T_{*\infty}$ are (L^p, L^p) type operators.

By the method similar to proving (2.4), we can easily prove that T_{*0} is a (L^p, L^p) type operator, and the norm of T_{*0} depends on the total degree of $p(x,y)$, but on the coefficients of $p(x,y)$.

We have unique $J \in \mathbb{Z}^+$ such that $2^{J-1} \leq \varepsilon < 2^J$. Thus

$$\begin{aligned} T_{*\infty}f(x) &\leq \sup_{J\in\mathbb{Z}^+} \int_{2^{J-1}\leq|y|<2^J} \frac{|\Omega(y')|}{|y|^n}|f(x-y)|\,dy \\ &\quad + \sup_{J\in\mathbb{Z}^+} \sum_{j=J+1} \Big| \int_{2^{j-1}\leq|x-y|<2^j} e^{ip(x,y)}K(x-y)f(y)\,dy\Big| \\ &\leq \sup_{J\in\mathbb{Z}^+} \int_{2^{J-1}\leq|y|<2^J} \frac{|\Omega(y')|}{|y|^n}|f(x-y)|\,dy \\ &\quad + \sum_{j=1}^{+\infty} \Big| \int_{2^{j-1}\leq|x-y|<2^j} e^{ip(x,y)}K(x-y)f(y)\,dy\Big|. \end{aligned}$$

From the Minkowski's inequality and the method similar to proving (2.8), we get

$$\|T_{*\infty}f\|_p \leq C\,\|f\|_p\,.$$

where C depends on the total degree of $p(x,y)$, but not on the coefficients of $p(x,y)$. So we have finished the proof of the theorem.

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Analyses multi-résolutions non orthogonales, commutation entre projecteurs et derivation et ondelettes vecteurs à divergence nulle

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Résumé. La compatibilité de la notion d'analyse multi-résolution non orthogonale avec la dérivation (exprimée par la formule de commutation) permet de construire une analyse multi-résolution de $L^2(\mathbb{R}^n)^n$ adaptée à l'étude des fonctions vecteurs à divergence nulle. Cela permet en particulier la construction de bases inconditionnelles d'ondelettes vecteurs à divergence nulle et à support compact.

Abstract. The notion of non-orthogonal multi-resolution analysis and its compatibility with differentiation (as expressed by the commutation formula) lead us to the construction of a multi-resolution analysis of $L^2(\mathbb{R}^n)^n$ which is well adapted to the approximation of divergence-free vector functions. Thus, we obtain unconditional bases of compactly supported divergence-free vector wavelets.

Introduction.

G. Battle a récemment annoncé [1], [2] la construction de bases orthonormées d'ondelettes vectorielles à divergence nulle et P. Federbush a montré comment utiliser ces bases dans l'étude des solutions de l'équation de Navier-Stokes [9]. Cependant, si ces bases s'avèrent utiles théoriquement, elles ne sont guère pratiques d'un point de vue numérique en l'absence de transformations rapides associées.

Nous nous proposons ici d'aborder la représentation des fonctions vectorielles à divergence nulle sous une autre approche. Nous introduirons une analyse multi-résolution non-orthogonale de $L^2(\mathbb{R}^n)^n$ tout entier, de sorte que les projecteurs obliques associés $\vec{P}_j$ permettront l'approximation des fonctions vectorielles quelconques (localement dans $L^2(\mathbb{R}^n)^n$) et leur transformation rapide en ondelettes (obliques); cependant le choix spécifique de notre analyse multi-résolution assurera que si $\vec{\nabla} \cdot \vec{f} = 0$ alors $\vec{\nabla} \cdot \vec{P}_j(\vec{f}) = 0$, de sorte que le processus d'approximation $\vec{f} \to (\vec{P}_j(\vec{f}))_{j \in \mathbb{Z}}$ sera un processus interne pour l'espace des fonctions à divergence nulle. Cela se traduira par l'existence de bases inconditionnelles d'ondelettes vecteurs à divergence nulle (à support compact).

Notations: Pour une fonction g définie sur $\mathbb{R}^n$ on notera

$$(1) \qquad \hat{g}(\xi) = \int_{\mathbb{R}^n} g(x) e^{-ix\xi} dx \quad (\textit{transformée de Fourier}),$$

$$(2) \qquad g_{j,k}(x) = 2^{jn/2} g(2^j x - k) \quad (\textit{translatée-dilatée dyadique}).$$

1. Analyse multi-résolution en une dimension.

Nous rappelons ici les propriétés principales des analyses multi-résolutions que nous utiliserons par la suite. La notion d'analyse multi-résolution a été introduite en 1986 par S. Mallat [14] et est décrite de manière approfondie dans le livre d'Y. Meyer [15].

Definition 1. *Une analyse multi-résolution de* $L^2(\mathbb{R})$ *est une suite de sous-espaces fermés* $(V_j)_{j \in \mathbb{Z}}$ *de* $L^2(\mathbb{R})$ *telle que*

$$(3.1) \qquad V_j \subset V_{j+1}, \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \bigcup_{j \in \mathbb{Z}} V_j \quad \textit{est dense dans} \quad L^2(\mathbb{R}),$$

(3.2) $f(x) \in V_j$ *si et seulement si* $f(2x) \in V_{j+1}$, *et*

(3.3) V_0 *a une base de Riesz de la forme* $g(x-k)$, $k \in \mathbb{Z}$.

L'étude des fonctions g qui engendrent des analyses multi-résolutions a été faite par de nombreux auteurs, en particulier I. Daubechies et J. Lagarias [7] ou A. Cohen [4]. La notion suivante nous sera particulièrement utile:

Definition 2. *Une "fonction de pré-échelle" sur* $\mathbb{R}$ *sera une fonction* $g \in L^2(\mathbb{R})$ *à support compact et à valeurs réelles telle que, pour une suite finie* $(a_k)_{N_1 \leq k \leq N_2}$ *de réels et une fonction* $h \in L^2(\mathbb{R})$ *à support compact, on ait*

$$g(\frac{x}{2}) = \sum_k a_k g(x-k) \text{ p. p. } (\textit{équation à deux échelles}) \tag{4.1}$$

$$\int g(x)h(x-k)dx = \delta_{0,k} \quad \textit{pour} \quad k \in \mathbb{Z}. \tag{4.2}$$

Une fonction de pré-échelle g vérifie alors l'estimation de stabilité suivante: pour deux constantes A, B strictement positives on a

(4.3) $\forall (\lambda_k) \in \ell^2(\mathbb{Z})$,

$$A \sum \mid \lambda_k \mid^2 \leq \| \sum \lambda_k g(x-k) \|_2^2 \leq B \sum \mid \lambda_k \mid^2 .$$

Cela revient à dire que les fonctions $g(x-k)$, $k \in \mathbb{Z}$, forment une base de Riesz d'un sous-espace fermé V_0 de L^2. Le lien entre les fonctions de pré-échelle et les analyses multi-résolutions est décrite par la proposition suivante (démontrée dans [12]).

Proposition 1.
(i) *Si* g *est une fonction de pré-échelle, alors les* $g(x-k)$, $k \in \mathbb{Z}$, *forment une base de Riesz de l'espace* V_0 *d'une analyse multi-résolution* $(V_j)_{j \in \mathbb{Z}}$ *de* $L^2(\mathbb{R})$.

(ii) *Inversement, si* (V_j) *est une analyse multi-résolution telle que* V_0 *admet une base de Riesz à valeurs réelles et contient des fonctions à support compact non nulles, alors il existe une fonction de pré-échelle* g *telle que les* $g(x-k)$ *soient une base de* V_0. *De plus, si* G *est une autre fonction de pré-échelle de* V_0, *alors* $G(x) = \lambda g(x-k)$ *pour un* $\lambda \in \mathbb{R}^*$ *et un* $k \in \mathbb{Z}$.

Les fonctions de pré-échelle ont en outre de nombreuses propriétés (étudiées dans [12]) que nous rappelons ici :

Proposition 2. *Soit* g *une fonction de pré-échelle. On pose* $m_0(\xi) = \frac{1}{2}\sum_{N_1}^{N_2} a_k e^{-ik\xi}$, *où les* a_k *sont donnés par* (4.1). *Alors*

(i) *Les fonctions* m_0 *et* $\hat{g}$ *vérifient*

(5.1) $m_0(0) = 1$ *et* $m_0(\pi) = 0$,
(5.2) $\hat{g}(0) \neq 0$ *et* $\hat{g}(2k\pi) = 0$ *pour* $k \in \mathbb{Z}^*$,
(5.3) $\hat{g}(\xi) = \hat{g}(0)\prod_{j=1}^{\infty} m_0\left(\xi/2^j\right)$.

(ii) *Si* $g \in H^1$ (*c'est-à-dire si* $g' \in L^2$) *alors il existe une fonction de pré-échelle* G_1 *telle que*

(6.1) $$g'(x) = G_1(x) - G_1(x-1) \quad (\textit{formule de dérivation})$$

et le polynôme trigonométrique M_1 *associé à* G_1 *vérifie*

(6.2) $$M_1(\xi) = \frac{2}{1+e^{-i\xi}} m_0(\xi).$$

En particulier si $g \in H^p$ *alors* $\hat{g}^{(\ell)}(2k\pi) = 0$ *pour* $k \in \mathbb{Z}^*$ *et* $0 \leq \ell \leq p$ *et* $m_0^{(\ell)}(\pi) = 0$ *pour* $0 \leq \ell \leq p$.

(iii) *Il existe de même une fonction de pré-échelle* G_2 *telle que*

(7.1) $$g(x) - g(x-1) = G_2'(x) \quad (\textit{formule d' intégration})$$

et le polynôme trigonométrique M_2 *associé à* G_2 *vérifie*

(7.2) $$M_2(\xi) = \frac{1+e^{-i\xi}}{2} m_0(\xi).$$

(On pourra remarquer que (6.2) et (7.2) correspondent à la formule bien connue : $\sin x / x = \prod_{j=1}^{\infty} \cos(x/2^j)$).

Enfin, nous terminerons cette section en rappelant quelques exemples fondamentaux de fonctions de pré-échelle:

i) les fonctions ${}_N\varphi$ d'I. Daubechies [6] : ${}_N\varphi$ satisfait une équation à deux échelles ${}_N\varphi(x/2) = \sum_{k=0}^{2N-1} a_k \, {}_N\varphi(x-k)$ et les ${}_N\varphi(x-k)$, $k \in \mathbb{Z}$, forment une famille orthonormée; de plus les ${}_N\varphi$ sont règulières (de classe $C^{\alpha N}$ où α est une constante positive). Ces fonctions permettent la construction de bases d'ondelettes régulières à support compact.

ii) Les B-splines $B_k(x)$ définis par $B_1(x) = \chi_{[0,1]}(x)$ et $B_{k+1}(x) = B_k * B_1$. B_k est polynomiale par morceaux de degré $k-1$, de classe C^{k-2} et à support dans $[0,k]$.

iii) Le formalisme des analyses multi-résolution non-orthogonales de J. C. Feauveau [8] fournit également toute une classe de fonctions de pré-échelle et est décrit dans la section ci-après.

2. Analyse multi-résolution non orthogonale.

Pour définir une analyse multi-résolution non orthogonale, nous commencerons par introduire la définition suivante:

Definition 3. *Nous appellerons "fonctions d'échelle conjuguées" de $L^2(\mathbb{R})$ deux fonctions g et g^* de $L^2(\mathbb{R})$ à valeurs réelles telles que*

(8.1) $< g(x-k) \mid g^*(x-\ell) > = \delta_{k,\ell}$ *pour* $k, \ell \in \mathbb{Z}$.
(8.2) g *et* g^* *sont* C^ϵ *pour un* $\epsilon > 0$ *et à support compact.*
(8.3) $g(x/2)$ *est combinaison linéaire des* $g(x-k)$, $k \in \mathbb{Z}$.
(8.4) $g^*(x/2)$ *est combinaison linéaire des* $g^*(x-k)$, $k \in \mathbb{Z}$.
(8.5) $\hat{g}(0) = \hat{g}^*(0) = 1$.

Il est clair que (8.1) à (8.4) entraînent que g et g^* sont des fonctions de pré-échelle; en particulier, la formule sommatoire de Poisson appliquée à (8.1) et la propriété (5.2) entraînent que $\hat{g}(0)\hat{g}^*(0) = 1$ de sorte que (8.5) est une simple normalisation.

On sait que g et g^* engendrent deux analyses multi-résolution V_j

et V_j^* (Proposition 1). À l'analyse multi-résolution V_j on peut associer un processus d'approximations successives des fonctions de $L^2(\mathbb{R})$ à l'aide de projecteurs P_j. Dans le formalisme de S. Mallat [14], P_j est le projecteur orthogonal sur V_j et le processus conduit à la décomposition des fonctions de $L^2(\mathbb{R})$ sur des bases orthonormées d'ondelettes. Dans le formalisme de J. C. Feauveau [8] que nous adopterons ici, P_j est le projecteur oblique défini par

$$P_j f(x) = \sum_{k\in\mathbb{Z}} < f \mid g^*_{j,k} > g_{j,k} \ , \tag{9}$$

c'est le projecteur de L^2 sur V_j parallèlement à $(V_j^*)^\perp$. On parle alors d'analyse multi-résolution non orthogonale; le processus conduit à la décomposition de f sur une base bi-orthogonale d'ondelettes [5].

Proposition 3. *Soient g et g^* deux fonctions d'échelle conjuguées et (V_j) et (V_j^*) les analyses multi-résolution associées. Alors:*

(i) *Il existe deux polynômes trigonométriques (2π-périodiques) α et α^* tels que*

$$\hat{g}(2\xi) = \alpha(\xi)\hat{g}(\xi) \quad et \quad \hat{g}^*(2\xi) = \alpha^*(\xi)\hat{g}^*(\xi) \tag{10.1}$$

et on a alors

$$\hat{g}(\xi) = \prod_{j=1}^{\infty} \alpha\left(\frac{\xi}{2^j}\right) \quad et \quad \hat{g}^*(\xi) = \prod_{j=1}^{\infty} \alpha^*\left(\frac{\xi}{2^j}\right) \tag{10.2}$$

$$\alpha(\xi)\bar{\alpha}^*(\xi) + \alpha(\xi+\pi)\bar{\alpha}^*(\xi+\pi) = 1. \tag{10.3}$$

(ii) *$X_0 = V_1 \cap (V_0^*)^\perp$ a une base de Riesz $\gamma(x-k)$, $k \in \mathbb{Z}$, et $X_0^* = V_1^* \cap (V_0)^\perp$ a une base de Riesz $\gamma^*(x-k)$, $k \in \mathbb{Z}$, où γ et γ^* vérifient:*

$$\hat{\gamma}(2\xi) = e^{-i\xi}\bar{\alpha}^*(\xi+\pi)\hat{g}(\xi) \quad et \quad \hat{\gamma}^*(2\xi) = e^{-i\xi}\bar{\alpha}(\xi+\pi)\hat{g}^*(\xi). \tag{11}$$

(iii) *Les $\gamma_{j,k}$ et les $\gamma^*_{j,k}$ forment un système bi-orthogonal ($j,k \in \mathbb{Z}$) et les $\gamma_{j,k}$ forment une base inconditionnelle de $L^2(\mathbb{R})$*

$$\forall f \in L^2(\mathbb{R}), \quad f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} < f \mid \gamma^*_{j,k} > \gamma_{j,k} \tag{12.1}$$

$$(12.2) \qquad \frac{1}{C} \| f \|_2 \leq \left\{ \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |< f \mid \gamma^*_{j,k} >|^2 \right\}^{1/2} \leq C \| f \|_2$$

où C est une constante positive.

Les fonctions γ et γ^* seront appelées *ondelettes obliques* associées à (g, g^*) (on remarquera que $\int \gamma\, dx = \int \gamma^* dx = 0$). L'opérateur $Q_j = P_{j+1} - P_j$ est alors le projecteur de L^2 sur $X_j = V_{j+1} \cap (V_j^*)^\perp$ parallèlement à $(X_j^*)^\perp$ (où $X_j^* = V_{j+1}^* \cap V_j^\perp$) et se calcule par

$$(13) \qquad Q_j f = \sum_{k\in\mathbb{Z}} < f \mid \gamma^*_{j,k} > \gamma_{j,k} \,.$$

DÉMONSTRATION. Excepté (12.2), la proposition est élémentaire. (10.1) et (10.2) sont déjà connus (Proposition 2) et (10.3) provient directement de (8.1): la formule sommatoire de Poisson transforme (8.1) en :

$$\forall \xi \in \mathbb{R}, \qquad \sum_{k\in\mathbb{Z}} \hat{g}(\xi + 2k\pi)\bar{\hat{g}}^*(\xi + 2k\pi) = 1 \,,$$

d'où (10.3) en considérant l'égalité en 2ξ, ξ et $\xi + \pi$.

Pour établir le point ii), on remarquera que l'appartenance des $\gamma(x-k)$ à X_0 est immédiate: X_0 est stable par translation entière (puisque V_1 et V_0^* le sont), $\gamma \in V_1$ et enfin $\gamma \in (V_0^*)^\perp$ car si $u \in V_1$, $\hat{u} = U(\xi/2)\hat{g}(\xi/2)$, et si $u^* \in V_1^*$, $\hat{u}^* = U^*(\xi/2)\hat{g}^*(\xi/2)$, alors

$$< u \mid u^* > = \frac{1}{\pi} \int_0^\pi (U(\xi)\bar{U}^*(\xi) + U(\xi+\pi)\bar{U}^*(\xi+\pi))d\xi.$$

Pour $u = \gamma$ et $u^* \in V_0^*$ on a $U(\xi) = e^{-i\xi}\bar{\alpha}^*(\xi+\pi)$ et $U^*(\xi) = \alpha(\xi + \pi)V(2\xi)$ où V est 2π-périodique, et donc $U(\xi)\bar{U}^*(\xi) + U(\xi+\pi)\bar{U}^*(\xi + \pi) = 0$. Un calcul analogue donne $\gamma^*(x-k) \in X_0^*$ et $< \gamma(x-k) \mid \gamma^*(x-\ell) > = \delta_{k,\ell}$. On note Q_0 le projecteur défini par (13) ; si $f \in V_1$ on a $\hat{f}(\xi) = V(\xi/2)\hat{g}(\xi/2)$ avec V 2π-périodique et un calcul élémentaire donne

$$\widehat{P_0 f}(\xi) = \left\{ V(\frac{\xi}{2})\,\bar{\alpha}^*(\frac{\xi}{2}) + V(\frac{\xi}{2}+\pi)\,\bar{\alpha}^*(\frac{\xi}{2}+\pi) \right\} \alpha(\frac{\xi}{2})\,\hat{g}(\frac{\xi}{2})$$

et

$$\widehat{Q_0 f}(\xi) = \left\{ V(\frac{\xi}{2})\,\alpha(\frac{\xi}{2}+\pi) - V(\frac{\xi}{2}+\pi)\,\alpha(\frac{\xi}{2}) \right\} \bar{\alpha}^*(\frac{\xi}{2}+\pi)\,\hat{g}(\frac{\xi}{2})$$

(par la formule sommatoire de Poisson) et donc que $f = P_0 f + Q_0 f$; si $f \in L^2$ on a $f = f - P_1 f + P_1 f$ avec $f - P_1 f \in (V_1^*)^\perp$ d'où $P_0(f - P_1 f) = Q_0(f - P_1 f) = 0$ de sorte que $P_0 P_1 = P_0$, $Q_0 P_1 = Q_0$ et $P_1 = P_0 + Q_0$. Comme $\mathrm{Im}(P_1 - P_0) = X_0$, les $\gamma(x-k)$ forment bien une base de Riesz de X_0, et de même $\mathrm{Ker}(P_1 - P_0) = (X_0^*)^\perp$ de sorte que les $\gamma^*(x-k)$ forment une base de Riesz de X_0^*. (Pour vérifier que $\mathrm{Ker}(P_1 - P_0) = (X_0^*)^\perp$, il suffit d'écrire $f = f - P_1 f + P_0 f + Q_0 f$ et donc $\mathrm{Ker}\,(P_1 - P_0) = (V_1^*)^\perp \oplus V_0$). Enfin si $(j,k) \neq (j',k')$ alors $< \gamma_{j,k} \mid \gamma^*_{j',k'} > = 0$: si $j = j'$ cela a déjà été vu, si $j > j'$ alors $\gamma^*_{j',k'} \in X^*_{j'} \subset V^*_{j'+1} \subset V^*_j$ et $\gamma_{j,k} \in X_j \subset (V_j^*)^\perp$. (12.1) est alors immédiat: la convergence de la série double dans L^2 est assurée par le fait que $\lim_{j\to+\infty} \| f - P_j f \|_2 = 0$ et que $\lim_{j\to-\infty} \| P_j f \|_2 = 0$ (ces limites étant immédiates par équicontinuité des P_j et densité de $\bigcup_{j\in\mathbb{Z}} V_j$ et de $\bigcup_{j\in\mathbb{Z}} (V_j^*)^\perp$). L'inégalité (12.2) découlera du lemme suivant.

Lemme. *Si $\theta \in L^2$ est à support compact, de classe C^ϵ pour un $\epsilon > 0$, et si $\int \theta dx = 0$ alors il existe une constante $C(\theta)$ telle que*

$$\forall (\lambda_{j,k}) \in \ell^2(\mathbb{Z}^2), \quad \| \sum_{j,k} \lambda_{j,k}\theta_{j,k} \|_2 \leq C(\theta) \left(\sum_{j,k} | \lambda_{j,k} |^2 \right)^{1/2} .$$

Du lemme on déduit immédiatement que

$$\frac{1}{C(\gamma^*)} \left(\sum | \lambda_{j,k} |^2 \right)^{1/2} \leq \| \sum_{j,k} \lambda_{j,k}\gamma_{j,k} \|_2 \leq C(\gamma) \left(\sum_{j,k} | \lambda_{j,k} |^2 \right)^{1/2} .$$

Enfin pour démontrer le lemme il suffit de remarquer que

$$\sup_{j,k} \sum_{j',k'} |< \theta_{j,k} \mid \theta_{j',k'} >|\, 2^{(j-j')/2} < +\infty \ ;$$

en effet les hypothèses entraînent $|< \theta_{j,k} \mid \theta_{j',k'} >| \leq 2^{-|j-j'|(1/2+\epsilon)}$ et $< \theta_{j,k} \mid \theta_{j',k'} > = 0$ à j,k,j' fixés sauf pour un nombre fini d'indices

k' ($\leq M$ pour $j' \leq j$ et $\leq M2^{j'-j}$ pour $j' \geq j$ où M est une constante indépendante de j, j' et k). On a alors :

$$\| \sum_{j,k} \lambda_{j,k}\theta_{j,k} \|_2^2 \leq \left(\sum_{j,k} \sum_{j',k'} | \lambda_{j,k} |^2 | < \theta_{j,k} \mid \theta_{j',k'} > | \, 2^{(j-j')/2} \right)^{1/2}$$
$$\cdot \left(\sum_{j,k} \sum_{j',k'} | \lambda_{j',k'} |^2 | < \theta_{j,k} \mid \theta_{j',k'} > | \, 2^{(j'-j)/2} \right)^{1/2}$$

et le lemme est démontré.

3. La formule de commutation.

Les formules de dérivation (6.1) et d'intégration (7.1) sont particulièrement adaptées à l'analyse multi-résolution non orthogonale. En effet, on a le résultat suivant:

Proposition 4. *Soient g et g^* deux fonctions d'échelle conjuguées telles que g soit $C^{1+\epsilon}$ pour un $\epsilon > 0$. Alors il existe deux fonctions d'échelle conjuguées $\tilde{g}$ et $\tilde{g}^*$ telles que*

$$g'(x) = \tilde{g}(x) - \tilde{g}(x-1) \quad et \quad g^*(x+1) - g^*(x) = \tilde{g}^{*\prime}(x). \tag{14.1}$$

De plus les polynômes trigonométriques $\tilde{\alpha}$ et $\tilde{\alpha}^*$, les ondelettes obliques $\tilde{\gamma}$ et $\tilde{\gamma}^*$, et les projecteurs obliques $\tilde{P}_j$ et $\tilde{Q}_j$ associés à $\tilde{g}$ et $\tilde{g}^*$ vérifient

$$\tilde{\alpha}(\xi) = \frac{2}{1+e^{-i\xi}}\,\alpha(\xi) \quad et \quad \tilde{\alpha}^*(\xi) = \frac{1+e^{i\xi}}{2}\,\alpha^*(\xi) \tag{14.2}$$

$$\tilde{\gamma}(x) = \frac{1}{4}\,\gamma'(x) \quad et \quad \tilde{\gamma}^{*\prime}(x) = -4\,\gamma^*(x) \tag{14.3}$$

$$\frac{d}{dx} \circ P_j = \tilde{P}_j \circ \frac{d}{dx} \quad (formule\ de\ commutation) \tag{14.4}$$

$$\frac{d}{dx} \circ Q_j = \tilde{Q}_j \circ \frac{d}{dx}\,. \tag{14.5}$$

La Proposition 4 est une conséquence directe des formules de dérivation et d'intégration. Démontrons par exemple (14.4):

$$\begin{aligned}\frac{d}{dx}(P_j f) &= \frac{d}{dx}\left(\sum_k < f \mid g^*_{j,k} > g_{j,k}\right)\\ &= \sum_k < f \mid g^*_{j,k} > 2^j(\tilde{g}_{j,k} - \tilde{g}_{j,k+1})\\ &= \sum_k < f \mid 2^j(g^*_{j,k} - g^*_{j,k-1}) > \tilde{g}_{j,k}\\ &= \sum_k < f \mid -\frac{d}{dx}(\tilde{g}^*_{j,k}) > \tilde{g}_{j,k}\\ &= \sum_k < \frac{df}{dx} \mid \tilde{g}^*_{j,k} > \tilde{g}_{j,k} = \tilde{P}_j(\frac{df}{dx}).\end{aligned}$$

Cette formule de commutation exprime qu'il est équivalent de dériver l'approximation ou d'approximer la dérivée, tandis que la formule de dérivation exprime que pour dériver l'approximation il suffit de faire une différence finie. Ces deux formules sont donc importantes numérique ment.

REMARQUE. Le fait que g' (dans la formule (14.1)) s'exprime comme une différence finie $\tilde{g}(x)-\tilde{g}(x-1)$ est "classique" : c'est une conséquence des conditions de Fix et Strang [10]. Par contre, que $\tilde{g}$ soit dans ce cas précis fonction d'échelle d'une analyse multi-résolution est une observation récente de G. Malgouyres [13] et la formule de commutation est originale. Sur les rapports entre dérivation et différence finie dans les analyses multi-résolution, le lecteur pourra consulter également les travaux de Micchelli [3].

4. Analyse multi-résolution de $L^2(\mathbb{R}^n)^n$ et fonctions vectorielles à divergence nulle.

Commençons par définir les analyses multi-résolution de $L^2(\mathbb{R}^n)$. Il suffit de remplacer dans la Définition 1 $\mathbb{R}$ par $\mathbb{R}^n$ et $k \in \mathbb{Z}$ par $k \in \mathbb{Z}^n$. Une façon simple d'obtenir une analyse multi-résolution de $L^2(\mathbb{R}^n)$ est

de partir de n analyses multi-résolution de $L^2(\mathbb{R})$ $(V_j^{(\epsilon)})_{j\in\mathbb{Z}}$, $1 \leq \epsilon \leq n$, et de définir V_j comme le complété dans $L^2(\mathbb{R}^n)$ de $V_j^{(1)} \otimes \cdots \otimes V_j^{(n)}$. Si les $V_j^{(\epsilon)}$ sont associés à des fonctions d'échelle conjuguées $(g_\epsilon, g_\epsilon^*)$ alors on dispose d'un projecteur oblique P_j sur V_j défini par

$$P_j f = (P_j^{(1)} \otimes \cdots \otimes P_j^{(n)}) f = \sum_{k\in\mathbb{Z}^n} < f \mid g_{j,k}^* > g_{j,k} \,, \tag{15.1}$$

$$g^* = g_1^* \otimes \cdots \otimes g_n^* \quad \text{et} \quad g = g_1 \otimes \cdots \otimes g_n \,. \tag{15.2}$$

De même on dispose d'un système bi-orthogonal d'ondelettes obliques. Plus précisément, on note $\Omega_n = \{0,1\}^n$ et $\Omega_n^* = \Omega_n \backslash \{(0,\cdots,0)\}$. Alors la décomposition $P_{j+1}^{(\epsilon)} = P_j^{(\epsilon)} + Q_j^{(\epsilon)}$ fournit la décomposition $P_{j+1} = P_j + \sum_{\omega\in\Omega_n^*} Q_{j,\omega}$ avec $Q_{j,\omega} = A_{j,\omega_1}^{(1)} \otimes \cdots \otimes A_{j,\omega_n}^{(n)}$ où $A_{j,0}^{(\epsilon)} = P_j^{(\epsilon)}$ et $A_{j,1}^{(\epsilon)} = Q_j^{(\epsilon)}$. On a alors

$$Q_{j,\omega} f = \sum_{k\in\mathbb{Z}^n} < f \mid \gamma_{\omega,j,k}^* > \gamma_{\omega,j,k} \,, \tag{16.1}$$

$$\gamma_\omega^* = \theta_{1,\omega_1}^* \otimes \cdots \otimes \theta_{n,\omega_n}^* \quad \text{où} \quad \theta_{\epsilon,0}^* = g_\epsilon^* \quad \text{et} \quad \theta_{\epsilon,1}^* = \gamma_\epsilon^* \,, \tag{16.2}$$

$$\gamma_\omega = \theta_{1,\omega_1} \otimes \cdots \otimes \theta_{n,\omega_n} \quad \text{où} \quad \theta_{\epsilon,0} = g_\epsilon \quad \text{et} \quad \theta_{\epsilon,1} = \gamma_\epsilon \,. \tag{16.3}$$

Les $\gamma_{\omega,j,k}$, $\omega \in \Omega_n^*$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, forment alors une base inconditionnelle de $L^2(\mathbb{R}^n)$. A ces ondelettes obliques sont associées des transformations en ondelettes rapides (au sens de S. Mallat) dans la mesure où les filtres associés sont de réponse impulsionnelle finie (support compact des ondelettes) et séparables (construction de V_j par produit tensoriel).

Nous allons maintenant définir une analyse multi-résolution $(\vec{V}_j)$ de $L^2(\mathbb{R}^n)^n$ par le procédé suivant.

(i) (g_1, g_1^*) désignera un couple de fonctions d'échelle conjuguées dans $L^2(\mathbb{R})$ et on notera $V_j^{(1)}$ l'analyse multi-résolution, $P_j^{(1)}$ et $Q_j^{(1)}$ les projecteurs obliques, α_1 et α_1^* les polynômes trigonométriques et γ_1 et γ_1^* les ondelettes obliques associés.

(ii) On supposera g_1 de classe $C^{1+\epsilon}$. On notera (g_0, g_0^*) le couple de fonctions d'échelle conjuguées associé à (g_1, g_1^*) par la dérivation de g_1 et $V_j^{(0)}$, $P_j^{(0)}$, $Q_j^{(0)}$, α_0, α_0^*, γ_0, γ_0^* les espaces, projecteurs et fonctions associés.

(iii) Pour $\omega \in \Omega_n$ on notera $V_j^{(\omega)}$ l'analyse multi-résolution de $L^2(\mathbb{R}^n)$ définie par le produit tensoriel $V_j^{(\omega_1)} \otimes \cdots \otimes V_j^{(\omega_n)}$. En particulier on notera $V_j^{((0))}$ pour $V_j^{(\omega)}$ avec $\omega = (0, \cdots, 0)$ et $V_j^{((i))}$ pour $V_j^{(\omega)}$ avec $\omega = (\delta_{1,i}, \cdots, \delta_{n,i}) = (0, \cdots, 0, 1, 0, \cdots, 0)$, $(1 \leq i \leq n)$. Enfin $P_j^{((i))}$ sera la projecteur oblique sur $V_j^{((i))}$ $(0 \leq i \leq n)$.

(iv) L'analyse multi-résolution $\vec{V}_j$ est alors définie comme $\vec{V}_j = V_j^{((1))} \times \cdots \times V_j^{((n))}$. On dispose d'un projecteur oblique $\vec{P}_j$ sur $\vec{V}_j$ défini par

$$\vec{P}_j(\vec{f}) = (P_j^{((1))} f_1, \cdots, P_j^{((n))} f_n). \tag{17}$$

On a alors la propriété fondamentale suivante

Proposition 5. *Les analyses multi-résolutions* $(\vec{V}_j)$ *de* $L^2(\mathbb{R}^n)^n$ *et* $(V_j^{((0))})$ *de* $L^2(\mathbb{R}^n)$ *vérifient*

$$\vec{\nabla}(\vec{P}_j \vec{f}) = P_j^{((0))}(\vec{\nabla} \vec{f}) \quad (\textit{deuxième formule de commutation}). \tag{18}$$

La Proposition 5 est immédiate. Elle indique en particulier que les projecteurs $\vec{P}_j$ conservent la propriété de divergence nulle, c'est-à-dire que si $\vec{H} = \{\vec{f} \in L^2(\mathbb{R}^n)^n) : \vec{\nabla} \cdot \vec{f} = 0\}$ alors les espaces fermés $\vec{V}_j \cap \vec{H}$ et $\vec{P}_j(\vec{H})$ coïncident.

On introduit alors les projecteurs $\vec{Q}_{j,\omega}$ définis pour $\omega \in \Omega_n^*$ par

$$\vec{Q}_{j,\omega}(\vec{f}) = (Q_{j,\omega}^{((1))} f_1, \cdots, Q_{j,\omega}^{((n))} f_n) \tag{19}$$

de sorte que $\vec{P}_{j+1} = \vec{P}_j + \sum_{\omega \in \Omega_n^*} \vec{Q}_{j,\omega}$. On a encore

$$\vec{\nabla} \cdot \vec{Q}_{j,\omega}(\vec{f}) = Q_{j,\omega}^{((0))}(\vec{\nabla} \cdot \vec{f}) \tag{20}$$

de sorte que $\vec{H}$ se décompose sur les espaces $\vec{Q}_{j,\omega}(\vec{H})$, $j \in \mathbb{Z}$, $\omega \in \Omega_n^*$.

Nous avons donc défini un processus d'approximation sur $L^2(\mathbb{R}^n)^n$ tout entier $(\vec{P}_j)_{j\in\mathbb{Z}}$ à l'aide d'ondelettes obliques $\vec{\gamma}_{\omega,i}$ ($\omega \in \Omega_n^*$, $1 \leq i \leq n$) et $\vec{\gamma}^*_{\omega,i}$ où

$$\vec{\gamma}_{\omega,i} = (0,\cdots,0,\gamma_\omega^{((i))},0,\cdots,0), \quad \vec{\gamma}^*_{\omega,i} = (0,\cdots,0,\gamma_\omega^{*((i))},0,\cdots,0)$$

et

$$\vec{Q}_{j,\omega}(\vec{f}) = \sum_{1\leq i\leq n}\sum_{k\in\mathbb{Z}^n} < \vec{f} \mid \vec{\gamma}^*_{\omega,i,j,k} > \vec{\gamma}_{\omega,i,j,k} \,.$$

En particulier, si $\vec{f} \in \vec{H}$, $\vec{P}_j(\vec{f})$ approxime $\vec{f}$ dans $L^2(\mathbb{R}^n)^n$ quand $j \to +\infty$. Mais cette approximation se révèle interne à $\vec{H}$ grâce à la Proposition 5. Cette propriété peut également se décrire en termes d'ondelettes vecteurs à divergence nulle.

Théorème 1.

(i) *Il existe* $(n-1)(2^n-1)$ *fonctions vectorielles* $\vec{\Gamma}_{\omega,i} \in \vec{H}$ ($\omega \in \Omega_n^*$, $1 \leq i \leq n-1$) *à support compact telles que*

$$(21) \quad \begin{cases} \vec{\Gamma}_{\omega,i} \in \vec{Q}_{0,\omega}(\vec{H}) \text{ et les } \vec{\Gamma}_{\omega,i}(x-k),\ 1 \leq i \leq n-1,\ k \in \mathbb{Z}^n, \\ \text{forment une base de Riesz de } \vec{Q}_{0,\omega}(\vec{H}). \end{cases}$$

(ii) *De plus, toute fonction vectorielle* $\vec{f} \in \vec{H}$ *se décompose de manière unique en*

$$(22.1) \qquad \vec{f} = \sum_{j\in\mathbb{Z}}\sum_{\omega\in\Omega_n^*}\sum_{i=1}^{n-1}\sum_{k\in\mathbb{Z}^n} C_{\omega,i,j,k}(\vec{f})\,\vec{\Gamma}_{\omega,i,j,k}$$

et on a pour une constante $C > 0$ *indépendante de* $\vec{f}$,

(22.2)

$$\frac{1}{C} \parallel \vec{f} \parallel_{\vec{H}} \leq \left\{ \sum_{j\in\mathbb{Z}}\sum_{\omega\in\Omega_n^*}\sum_{i=1}^{n-1}\sum_{k\in\mathbb{Z}^n} \mid C_{\omega,i,j,k}(\vec{f}) \mid^2 \right\}^{1/2} \leq C \parallel \vec{f} \parallel_{\vec{H}} .$$

(iii) *Il existe* $(n-1)(2^n-1)$ *fonctions* $\vec{\theta}_{\omega,i} \in (L^2(\mathbb{R}^n))^n$ ($\omega \in \Omega_n^*$, $1 \leq i \leq n-1$) *à support compact telles que*

$$(23) \qquad \forall \vec{f} \in \vec{H},\ \forall j,\ \forall k, \quad C_{\omega,i,j,k}(\vec{f}) = < \vec{f} \mid \vec{\theta}_{\omega,i,j,k} > .$$

(iv) *Enfin si* g_1 *est de classe* $C^{p+1+\epsilon}$ *pour* $p \in \mathbb{N}$ *alors :*

$$\vec{f} \in \vec{H} \cap (H^p(\mathbb{R}^n))^n \qquad \textit{si et seulement si}$$

$$(24) \qquad \sum_{j\in\mathbb{Z}} \sum_{\omega\in\Omega_n^*} \sum_{i=1}^{n-1} \sum_{k\in\mathbb{Z}^n} (1+4^{jp}) \mid C_{\omega,i,j,k}(\vec{f}) \mid^2 < +\infty .$$

DÉMONSTRATION. Il est immédiat que toute fonction de $L^2(\mathbb{R}^n)^n$ se décompose en $\vec{f} = \sum_{j\in\mathbb{Z}} \sum_{\omega\in\Omega_n^*} \vec{Q}_{j,\omega}(\vec{f})$ et que

$$\| \vec{f} \|_{L^2(\mathbb{R}^n)^n} \approx \left(\sum_{j\in\mathbb{Z}} \sum_{\omega\in\Omega_n^*} \| \vec{Q}_{j,\omega}(\vec{f}) \|^2_{L^2(\mathbb{R}^n)^n} \right)^{1/2}$$

tandis que

$$\| \vec{f} \|_{H^p(\mathbb{R}^n)^n} \approx \left(\sum_{j\in\mathbb{Z}} \sum_{\omega\in\Omega_n^*} (1+4^{jp}) \| \vec{Q}_{j,\omega}\vec{f} \|^2_{L^2(\mathbb{R}^n)^n} \right)^{1/2} .$$

Cela se vérifie composante par composante; il s'agit alors pour l'estimation de la norme L^2 de $f \in L^2(\mathbb{R}^n)$ en fonction des normes L^2 des $Q^{((i))}_{j,\omega} f$ (i fixé) d'une simple généralisation de la Proposition 3. Pour la norme H^p, remarquons que $P_j^{((i))}$ est défini par produit tensoriel de projecteurs non-dimensionnels; la formule de commutation (14.4) et l'estimation (12.2) nous assure que, pour une certaine analyse multi-résolution $V_j^{i,p,\ell}$, on a

$$\frac{\partial^p}{\partial^p x_\ell} Q^{((i))}_{j,\omega} f = Q^{i,p,\ell}_{j,\omega} \left(\frac{\partial^p}{\partial x_\ell} f \right)$$

et donc

$$\| \frac{\partial^p}{\partial^p x_\ell} f \|_{L^2} \approx \left(\sum_{j\in\mathbb{Z}} \sum_{\omega\in\Omega_n^*} \| \frac{\partial^p}{\partial^p x_\ell} Q^{((i))}_{j,\omega} f \|_2^2 \right)^{1/2} .$$

Mais il est clair que

$$\| \frac{\partial^p}{\partial^p x_\ell} Q_{j,\omega}^{((i))} f \|_2^2 \leq C 4^{jp} \| Q_{j,\omega}^{((i))} f \|_2^2$$

d'une part et que d'autre part, d'après (14.3), si $\omega_\ell = 1$, alors

$$\| Q_{j,\omega}^{((i))} f \|_2^2 \approx 4^{jp} \| \frac{\partial^p}{\partial^p x_\ell} Q_{j,\omega}^{((i))} f \|_2^2,$$

d'où le contrôle de $\| \vec{f} \|_{H^p(\mathbb{R}^n)^n}$.

Pour démontrer le théorème, il suffit donc d'établir (21) (et (23)). Puisque $\omega \neq (0, \cdots, 0)$ il existe un indice i_0 tel que $\omega_{i_0} = 1$. Alors on a pour $i \neq i_0$:

$$Q_{0,\omega}^{((i))} f_i = \sum_{k \in \mathbb{Z}^n} < f_i \mid \gamma_{\omega,0,k}^{*((i))} > \gamma_{\omega,0,k}^{((i))},$$

où $\gamma_\omega^{((i))} = \theta_{\delta_{1,i},\omega_1} \otimes \cdots \otimes \theta_{\delta_{n,i},\omega_n}$. En particulier $\theta_{\delta_{i_0,i},\omega_{i_0}} = \theta_{0,1} = \gamma_0$. Or par construction $\gamma_0 = \frac{d}{dx}(\frac{1}{4}\gamma_1)$. On définit alors $\vec{\Gamma}_{\omega,i}$ pour $i \neq i_0$ par:

$$\vec{\Gamma}_{\omega,i} = (\Gamma_{\omega,i,1}, \cdots, \Gamma_{\omega,i,n})$$

où $\Gamma_{\omega,i,j} = 0$ si $j \notin \{i, i_0\}$, $\Gamma_{\omega,i,i} = \gamma_\omega^{((i))}$ et $\Gamma_{\omega,i,i_0} = \eta_1 \otimes \cdots \otimes \eta_n$ avec $\eta_j = \theta_{\delta_{j,i},\omega_j}$ si $j \notin \{i, i_0\}$, $\eta_{i_0} = -1/4\gamma_1$ et $\eta_i = -\frac{d}{dx}(\theta_{1,\omega_i})$ (et donc $\eta_i = g_0(x-1) - g_0(x)$ si $\omega_i = 0$ et $\eta_i = -4\gamma_0$ si $\omega_i = 1$) et on pose $\vec{\theta}_{\omega,i} = \vec{\gamma}_\omega^{*((i))}$.

Il est clair que par construction $\vec{\nabla}\vec{\Gamma}_{\omega,i} = 0$ et que $\vec{\Gamma}_{\omega,i} \in \vec{Q}_{0,\omega}\vec{H}$. De plus

$$\vec{Q}_{0,\omega}\vec{f} = \sum_{i \neq i_0} \sum_{k \in \mathbb{Z}^n} < \vec{f} \mid \vec{\theta}_{\omega,i,0,k} > \vec{\Gamma}_{\omega,i,0,k} + \vec{R}_\omega(\vec{f}).$$

$\vec{R}_\omega(\vec{f})$ a toutes ses composantes sauf la i_0-ème nulles ; de plus si $\vec{f} \in \vec{H}$ alors $\vec{R}_\omega(\vec{f}) \in \vec{H}$ mais cela entraîne $\vec{R}_\omega(\vec{f}) = \vec{0}$. Enfin les $\vec{\Gamma}_{\omega,i,0,k}$ forment une base de Riesz de $\vec{Q}_{0,\omega}\vec{H}$ de manière évidente puisqu'on dispose d'un système bi-orthogonal à support compact. Le théorème est donc démontré.

5. Etude locale des fonctions à divergence nulle.

Notre construction peut s'adapter à l'étude locale de fonctions à divergence nulle. Pour cela, nous pouvons construire une analyse multi-résolution sur le cube $[0,1]^n$ qui vérifie "la propriété de commutation", ce qui nous autorise à étudier les fonctions vectorielles à divergence nulle sur le cube.

La construction sera précisée dans [11] car elle est assez technique. On part d'une analyse multi-résolution d'I. Daubechies (avec une base orthonormale ${}_N\varphi(x-k)$ de V_0) $(V_j)_{j\in\mathbb{Z}}$ de $L^2(\mathbb{R})$. Suivant Y. Meyer, on note $V_j([0,1])$ la restriction à $[0,1]$ des fonctions de V_j [16]. De même on note $V_j^{(d)}([0,1])$ l'espace des restrictions à $[0,1]$ des dérivées d-ièmes des fonctions de V_j et $V_j^{(-d)}([0,1])$ celui de leurs primitives d-ièmes. Alors le produit scalaire de $L^2([0,1])$ introduit un crochet de dualité entre $V_j^{(d)}([0,1])$ et $V_j^{(-d)}([0,1]) \cap H_0^d([0,1])$ et on obtient alors un projecteur oblique $P_j^{(d)}$ de $L^2([0,1])$ sur $V_j^{(d)}([0,1])$ parallèlement à $\left\{V_j^{(-d)}([0,1]) \cap H_0^d([0,1])\right\}^{\perp}$. La propriété principale de $P_j^{(d)}$ est la formule suivante

$$(25)\quad \frac{d}{dx} \circ P_j^{(d)} = P_j^{(d+1)} \circ \frac{d}{dx} \quad (\textit{troisième formule de commutation}),$$

la dérivation ayant lieu dans $]0,1[$. Les bases orthonormées d'ondelettes sur l'intervalle de Y. Meyer deviennent alors des bases d'ondelettes obliques. Celles-ci peuvent être décrites assez précisément et une construction analogue à celle décrite en IV permet alors de décrire les fonctions à divergence nulle dans $]0,1[^n$.

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Uniformly Perfect Sets, Green's Function, and Fundamental Domains

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Introduction.

Suppose $\Omega \subset \bar{\mathbb{C}}$ is a domain conformally equivalent to $\mathbb{D}/\Gamma$ for some Fuchsian group Γ acting on $\mathbb{D} = \{z : |z| < 1\}$. Let $z_0 \in \Omega$ and let $G(z, z_0)$ be Green's function for Ω with pole at z_0. Define $S(z_0)$ to be the set of critical points of $G(z, z_0)$, that is

$$S(z_0) = \{z_j : \nabla G(z_j, z_0) = 0\}.$$

In Theorem 1 we characterize those domains Ω for which there exists a constant $c = c(\varepsilon)$ independent of z_0, such that

$$\sum_{z_j \in S(z_0)} G(z_j, z_0)^{1+\varepsilon} < c(\varepsilon)\,.$$

We will prove that this holds if and only if $\partial\Omega$ is a uniformly perfect set.

A set $K \subset \mathbb{C}$ is said to be uniformly perfect if there exists a constant $c > 0$ such that

$$K \cap \{z : c\, r \leq |z - a| \leq r\} \neq \emptyset \quad \text{for} \quad a \in K,\ 0 < r < \operatorname{diam}(K).$$

An equivalent definition is the following: There exists a constant $\eta > 0$ such that

$$\operatorname{cap}(K \cap \{z : |z - a| \leq r\}) \geq \eta\, r\ , \text{ for } a \in K,\ 0 < r < \operatorname{diam}(K),$$

where cap denotes logarithmic capacity, see [9].

We will also need another characterization. Let Γ be a Fuchsian group such that $\mathbb{D}/\Gamma$ is conformally equivalent to Ω, then $\partial\Omega$ is uniformly perfect if and only if there exists $\rho > 0$ such that each disk in $\mathbb{D}$ of hyperbolic radius ρ contains no two Γ-equivalent points, [9].

Examples of uniformly perfect sets are the standard Cantor sets. A Cantor set can be constructed as follows: Fix $\alpha, 0 < \alpha < 1/2$, and take from the unit square four "corner" squares of side length α. Then take from each square four "corner" squares of side length α^2. By continuing this process, at the n-th stage we get 4^n squares Q_j^n of side length α^n. The limit set $E = E(\alpha)$ is a Cantor set.

Consider now the domain whose boundary is the Cantor set E. It is shown in [1] that to each square Q_j^n can be associated a critical point z_j^n such that $G(z_j^n, \infty) \simeq \omega(\infty, Q_j^n \cap E, \bar{\mathbb{C}}\backslash E)$ and then that

$$\sum_{z_j \in S(\infty)} G(z_j, \infty) = \infty \ .$$

However, by Theorem 1, $\sum_{z_j \in S(\infty)} G(z_j, \infty)^{1+\varepsilon} < \infty$ for all $\varepsilon > 0$.

The main motivation for these results is Widom's Theorem, which states that

$$\sum_{z_j \in S(z_0)} G(z_j, z_0) < \infty \text{ if and only if } H_\lambda^\infty(\mathbb{D}) \neq \{0\} \text{ for all } \lambda \in \Gamma^*,$$

where H_λ^∞ denotes the space of bounded analytic functions in $\mathbb{D}$ satisfying $f \circ \gamma = \lambda(\gamma) f$ for all $\gamma \in \Gamma$, [12].

Also in the spirit of these results, there is a theorem by J.L. Fernández involving uniformly perfect sets. If $\Omega \cong \mathbb{D}/\Gamma$ has Geeen's function then the Poincaré series $\sum_{\gamma\in\Gamma} 1 - |\gamma(0)|$ converges. What he proves in his paper [2] is that if $\partial\Omega$ is uniformly perfect, then for some $\delta > 0$,

$$\sum_{\gamma\in\Gamma} (1 - |\gamma(0)|)^{1-\delta} < \infty \ .$$

In Theorem 2 we give a characterization of domains whose boundary is a uniformly perfect set in terms of the geometry of the fundamental domain. We will prove that $\partial\Omega$ is uniformly perfect if and only if there exists a fundamental domain whose boundary is a quasicircle. In fact, there is a fundamental domain $\mathcal{F}$ such that $\partial\mathcal{F}$ is chord arc and $\partial\mathcal{F} \cap \partial\mathbb{D}$ is also uniformly perfect.

We recall here the definition of quasicircles and chord arc curves. C is a K-quasicircle if it satisfies the three point condition $|z_3 - z_1| < c|z_2 - z_1|$ for some $c = c(K) > 0$ and any three points on C with z_3 on the arc of smaller diameter between z_1 and z_2, [6]. Equivalent definitions can be given in terms of quasiconformal mappings.

Suppose now γ is a locally rectifiable Jordan curve. For any two points $z_1, z_2 \in \gamma$, let $\ell(\gamma(z_1, z_2))$ denote the length of the arc in γ from z_1 to z_2 of smaller diameter. The curve γ is chord arc if there exists a constant $M \geq 0$ such that

$$\ell(\gamma(z_1, z_2)) \leq M|z_1 - z_2| \ , \quad \text{for all } z_1, z_2 \in \gamma.$$

To prove Theorem 2 we construct a simply connected domain $\Omega_0 \subset \Omega$ satisfying certain metric properties. The preimage of Ω_0 under the covering map will be the required fundamental domain.

Pommerenke proves in [7] that if $\sum_{z_j \in S(z_0)} G(z_j, z_0) < \infty$ then

$$\text{mes}(\partial \mathcal{F}_{z_0} \cap \partial \mathbb{D}) > 0 \ ,$$

where $\mathcal{F}_{z_0}$ denotes the normal fundamental domain of Ω with base point at z_0. Another theorem by Pommerenke involving the "size" of $\partial \mathcal{F} \cap \partial \mathbb{D}$ for a particular domain is the following: If Ω is a domain for which Green's function exists and $\mathcal{F}$ denotes the Green's fundamental domain, then $\text{cap}(\partial \mathcal{F} \cap \partial \mathbb{D}) > 0$, *cf.* [8].

Finally, we would like to point out the connection between Theorem 1 and 2 with the conjecture stated by P. Jones in [5].

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1. Uniformly Perfect Sets and Green's Function.

Theorem 1. *Suppose Ω is a domain on the Riemann sphere $\bar{\mathbb{C}}$. Let $z_0 \in \Omega$ and let $G(z, z_0)$ be Green's function for Ω with pole at z_0. Denote by $\beta_{z_0}(t)$ the first Betti number of the domain $\Omega_t = \{z \in \Omega : G(z, z_0) > t\}$. Then $\partial\Omega$ is uniformly perfect if and only if there exists $\varepsilon > 0$ such that*

$$\int_0^\infty t^\varepsilon \beta_{z_0}(t) dt < c \ , \quad \textit{for all } z_0 \in \Omega \ , \tag{1.1}$$

and for some constant $c = c(\eta) > 0$ depending only on the uniformly perfect constant η. Moreover, whenever this holds, we have

$$(1.2) \qquad \sum_{\nabla G(z_j, z_0)=0} F(G(z_j, z_0)) < c(\eta, A) \quad \textit{for all } z_0 \in \Omega \ ,$$

for all increasing function F such that $F(0) = 0$ and

$$\int_0 \frac{F(t)}{t^2} \leq A < +\infty \ .$$

REMARK. Note that if Ω is a domain whose boundary $\partial\Omega$ is a uniformly perfect set, then $\text{cap}(\partial\Omega) > 0$ and therefore there exists Green's function for Ω. Moreover, it can be shown by applying Wiener's criterion that Ω is regular for the Dirichlet problem. Therefore, as a consequence of Theorem 1 we get that if (1.1) holds for some $\varepsilon > 0$, then Ω is regular and for all $\varepsilon > 0$,

$$(1+\varepsilon)\int_0^\infty t^\varepsilon \beta_{z_0}(t)dt = \sum_{\nabla G(z_j, z_0)=0} G(z_j, z_0)^{1+\varepsilon} < c(\varepsilon) \ , \ z_0 \in \Omega.$$

PROOF OF THEOREM 1: We start by proving sufficiency. Suppose $\Omega \subset \bar{\mathbb{C}}$ is a domain whose boundary $\partial\Omega$ is a uniformly perfect set, that is

$$(1.3) \ \text{cap}(\partial\Omega \cap \{z : |z-\xi| < r\}) \geq \eta r \ , \ \text{for } \xi \in \partial\Omega, \ 0 < r < \ \text{diam}(\partial\Omega),$$

for some constant $\eta > 0$.

First, we want to make an observation which will give us the uniform bound in (1.2). If φ is a Moebius transformation and K is a set satisfying (1.3), then $\varphi(K)$ satisfies (1.3) for some constant η^* depending only on η (see Proposition 1 in [9]). Thus, it is enough to prove the theorem in the case $\partial\Omega$ has diameter 1 and the pole z_0 for Green's function is ∞. We will define $G(z) = G(z, \infty)$.

We now need an estimate on harmonic measure. Let w_0 be a point in Ω such that $\delta(w_0) = \ \text{dist}\{w_0, \partial\Omega\} < 1$ and let ξ_0 be the closest point on $\partial\Omega$ to w_0. Consider the ball $B(w_0, r_0)$ of radius $r_0 = 5\delta(w_0)$ centered at w_0. We will prove

$$(1.4) \qquad \omega(\infty, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq c \ G(w_0)$$

for some $c = c(\eta)$.

Since (1.4) is invariant under conformal mappings we can assume $\delta(w_0) = 1/3$. By (1.3), there exists a probability measure μ supported on

$$E = \partial\Omega \cap \{\xi : |\xi - \xi_0| \leq \frac{1}{3}\}$$

such that the logarithmic potential

$$\mathcal{U}^\mu(z) = \int_E \log \frac{1}{|z-\xi|} d\mu(\xi)$$

satisfies

$$\mathcal{U}^\mu(z) \leq c_0(\eta) \text{ for all } z \in \bar{\mathbb{C}},$$
$$\mathcal{U}^\mu(z) \leq 0 \text{ if } z \notin B(w_0, r_0).$$

(Note that if $z \notin B(w_0, r_0)$, then $|z - \xi| \geq 1$, for all $\xi \in E$). Thus, by the maximum principle,

$$\omega(w_0, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq \frac{1}{c_0(\eta)} \mathcal{U}^\mu(w_0) \geq \frac{\log 3/2}{c_0(\eta)},$$

and by Harnack's inequality,

$$\omega(z, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq c_1(\eta) \quad \text{if } |z - w_0| = \frac{1}{6}. \tag{1.5}$$

We consider now the function

$$F(z) = \log \frac{1}{|z - w_0|} - \mathcal{U}^\mu(z).$$

$F(z)$ is harmonic in $\Omega \backslash \{w_0\}$, bounded at ∞ and $|F(z)| \leq c_2(\eta)$ on $\partial\Omega$. Writing

$$G(z, w_0) = F(z) + (G(z, w_0) - F(z))$$

and applying the maximum principle to the harmonic function in Ω, $G(z, w_0) - F(z)$, we get .

$$G(z, w_0) \leq c_2(\eta) \quad \text{if } |z - w_0| = \frac{1}{6}.$$

If we combine this result with (1.5) and we apply again the maximum principle to the functions $\omega(z, B(w_0, r_0) \cap \partial\Omega, \Omega)$ and $G(z, w_0)$ in the domain $\Omega \backslash \{z : |z - w_0| \leq 1/6\}$ we have

$$\omega(\infty, B(w_0, r_0) \cap \partial\Omega, \Omega) \geq c(\eta)\, G(\infty, w_0).$$

Since $G(\infty, w_0) = G(w_0, \infty)$, we obtain (1.4) as required.

Define now S to be the set of critical points of $G(z)$. Since S lies in the convex hull of $\partial\Omega$, *cf.* [11], which in this case is contained in $|z| \leq 1$, it is easy to see that if $z_j \in S$ then $G(z_j) \leq t_0$ for some constant $t_0 = t_0(\eta)$.

Next, consider the set of level curves $\{G(z) = t\}$, for all $t < t_0$. For each $t < t_0$ fixed, let N_t be the number of connected components of $\{G(z) = t\}$ and let C_j^t be the j-th component. Denote by K_j^t the subset of $\partial\Omega$ contained inside the curve C_j^t. To simplify the notation we set

$$\omega(E) = \omega(\infty, E, \Omega) \quad \text{for all } E \subset \partial\Omega\,.$$

We will prove that there exists a constant $c = c(\eta) > 0$ such that

$$\omega(K_j^t) > c\,t\,, \quad j = 1, \ldots, N_t, \tag{1.6}$$

but we will first show why sufficiency follows from this result.

Since $\partial\Omega$ is uniformly perfect and in particular regular we have

$$\partial\Omega = \bigcup_{j=1}^{N_t} K_j^t\,, \quad \text{for every } t < t_0\,.$$

Thus, by (1.6)

$$1 = \omega(\partial\Omega) = \sum_{j=1}^{N_t} \omega(K_j^t) > N_t\, c\, t$$

and therefore $N_t < 1/c\,t$ for all $t < t_0$.

Now let $G_t(z)$ be Green's function for the domain $\Omega_t = \{z : G(z) > t\}$ with pole at ∞. The number of critical points of $G_t(z)$ is $N_t - 1$, *cf.* [3], which is exactly the first Betti number of Ω_t, $\beta(t)$. On the other hand, note that

$$G_t(z) = G(z) - t,$$

therefore the number of critical points of $G(z)$ contained in Ω_t is equal to the number of critical points of $G_t(z)$, that is $N_t - 1$. So, for all $t < t_0$,

$$\#\{z_j : z_j \in S,\ G(z_j) > t\} < \frac{1}{ct}\ .$$

Suppose F is an increasing function such that $F(0) = 0$ and

$$\int_0^{t_0} \frac{F(x)}{x^2} dx \leq A < +\infty.$$

Then, if $t_0 = 2^{k_0}$,

$$\sum_{z_j \in S} F(G(z_j)) = \sum_{k=-k_0}^{\infty} \sum_{\substack{z_j \in S \\ 2^{-k-1} \leq G(z_j) \leq 2^{-k}}} F(G(z_j))$$

$$\leq \frac{1}{c} \sum_{k=-k_0}^{\infty} 2^{k+1} F(2^{-k}) \leq C\ ,$$

where C is a constant depending on η and A. In particular, taking $F(x) = x^{1+\varepsilon}$ we get

$$\sum_{z_j \in S} G(z_j)^{1+\varepsilon} \leq c(\eta, \varepsilon).$$

Also, as we pointed out before

$$\#\{z_j : z_j \in S, G(z_j) > t\} = \beta(t).$$

Integration by parts gives

$$c(\eta, \varepsilon) \geq \sum_{z_j \in S} G(z_j)^{1+\varepsilon} = (1+\varepsilon) \int_0^{\infty} t^{\varepsilon} \beta(t) dt, \quad \varepsilon > 0\ .$$

We now turn to the proof of (1.6). Let $t < t_0$ and let $w_0 \in \Omega$ be a point inside the curve C_j^t such that if $d = \operatorname{dist}\{C_j^t, K_j^t\}$,

$$\operatorname{dist}\{w_0, C_j^t\} = 5 \operatorname{dist}\{w_0, K_j^t\} = \frac{5d}{6}.$$

Consider the ball $B(w_0, r_0)$ of radius $r_0 = 5d/6$ centered at w_0. Then $B(w_0, r_0)$ is inside the curve C_j^t and by (1.4)

$$\omega(B(w_0, r_0) \cap \partial\Omega) \geq c\, G(w_0).$$

Thus

$$\omega(K_j^t) \geq \omega(B(w_0, r_0) \cap \partial\Omega) \geq c\, G(w_0) \geq c't. \tag{1.7}$$

To prove the last inequality we consider the point $w_1 \in \Omega$ which is at distance $d/3$ from C_j^t and $2d/3$ from K_j^t. Let $r_1 = 2d/3$ and let D be the subset of the ball $B(w_1, r_1)$ contained inside the curve C_j^t. Denote by $\tilde{D}$ the connected component of D which contains w_1. Then

$$\omega(w_1, C_j^t \cap \tilde{D}, \tilde{D}) \geq c_0$$

for some universal constant $c_0 > 0$.

By the maximum principle applied to the function $\omega(z, C_j^t \cap \tilde{D}, \tilde{D})$ and $G(z)$ on the domain $\tilde{D}$, we get $G(w_1) \geq c_0\, t$. Hence, by Harnack's principle $G(w_0) > c\, t$, for some universal constant c. Thus (1.7) holds and this ends the proof of the sufficiency.

To prove necessity suppose $\int_0^\infty t^\varepsilon \beta_{z_0}(t) < c$ for some $c > 0$ and for all $z_0 \in \Omega$. Then, there exists $M > 0$ such that $\beta_{z_0}(t) = 0$ if $t > M$, *i.e.* the domain

$$\Omega_{M, z_0} = \{z \in \Omega : G(z, z_0) > M\}$$

is simply connected. This is equivalent to saying that there is a ball of radius $r = r(M)$ which does not contain equivalent points, (to see this, write $G(\pi(\xi), z_0) = \log(1/|B(\xi)|)$ where $B(\xi)$ is the Blaschke product with zeros at $\{\gamma(0)\}_{\gamma \in \Gamma}$ and $\pi : \mathbb{D} \mapsto \Omega$ is the universal covering map with with $\pi(0) = z_0$, [7]). Therefore, Ω is a domain whose boundary is a uniformly perfect set.

2. Uniformly Perfect Sets and Fundamental Domains.

Theorem 2. *Suppose $\Omega \cong \mathbb{D}/\Gamma$ is a domain on the Riemann sphere $\bar{\mathbb{C}}$. Then $\partial\Omega$ is a uniformly perfect set if and only if there exists a fundamental domain whose boundary is a quasicircle.*

Furthermore, in this case, we can construct a fundamental domain $\mathcal{F}$, such that

(i) *$\partial\mathcal{F}$ is chord arc,*
(ii) *$\partial\mathcal{F} \cap \mathbb{T}$ is a uniformly perfect set.*

PROOF OF THEOREM 2: To prove sufficiency we construct a simply connected domain $\Omega_0 \subset \Omega$ satisfying certain metric properties. To be more precise:

Lemma 1. *Suppose $\Omega \subset \bar{\mathbb{C}}$ is an unbounded domain whose boundary K is a uniformly perfect set. Then, there are crosscuts $\{L_j\}$ on Ω such that $\Omega_0 = \Omega \backslash \{L_j\}$ is a simply connected domain and*

(2.1) *L_j is a chord arc curve, for every j,*
(2.2) $\operatorname{dist}\{L_j, L_k\} \geq c_1 \min\{\operatorname{diam}(L_j), \operatorname{diam}(L_k)\}$, *for $j \neq k$,*
(2.3) *For any $L \in \{L_j\}$, if z_1, z_2 denote the endpoints of L and $z \in L$, we have*

$$\operatorname{dist}\{z, K\} \geq c_2 \ \min\{|z - z_1|, |z - z_2|\}.$$

The chord arc constant M as well as c_1 and c_2 only depend on the uniformly perfect constant of K, η.

PROOF OF LEMMA 1: Without loss of generality we can assume $K \subset [0,1] \times [0,1]$. Let $k_0 \in \mathbb{Z}^+$ be a large number which will be fixed later. For each $n \in \mathbb{Z}^+$, let $\langle Q \rangle_n$ be the set of closed dyadic cubes of side length $\ell_n = 2^{-k_0 n}$. (A dyadic cube $Q \subset [0,1] \times [0,1]$ is a cube of the form $Q = [2^{-j}k, 2^{-j}(k+1)] \times [2^{-i}\ell, 2^{-i}(\ell+1)]$ for some $j, k, i, \ell \in \mathbb{Z}^+$).

Define

$$K_n = \{Q \in \langle Q \rangle_n : Q \cap K \neq \emptyset\}.$$

Note that $K_n \supset K_{n+1}$ and $K = \cap_1^\infty K_n$.

To prove the lemma we will use an induction argument. At each stage n, we will construct a family of curves $\langle L \rangle_n$ satisfying $\langle L \rangle_n \subset \langle L \rangle_{n+1}$ and such that

(2.4) $K_n \cup \langle L \rangle_n$ is a connected set. The endpoints of the curves in $\langle L \rangle_n$ are either vertices or middle points of edges of cubes in K_n,

(2.5) Any $L \in \langle L \rangle_n$ is a chord arc curve with chord arc constant depending on k_0,

(2.6) If $L_j, L_k \in \langle L \rangle_n$,

$$\operatorname{dist}\{L_j, L_k\} \geq c_1 \ \min\{\operatorname{diam}(L_j), \operatorname{diam}(L_k)\},$$

(2.7) If z_1, z_2 are the endpoints of $L \in \langle L \rangle_n$, for all $z \in L$, we have

$$\operatorname{dist}\{z, K_n\} \geq c_2 \ \min\{|z - z_1|, |z - z_2|\}.$$

The constants c_1, c_2 only depend on k_0.
At the limit we will get the required crosscuts $\{L_j\}$.

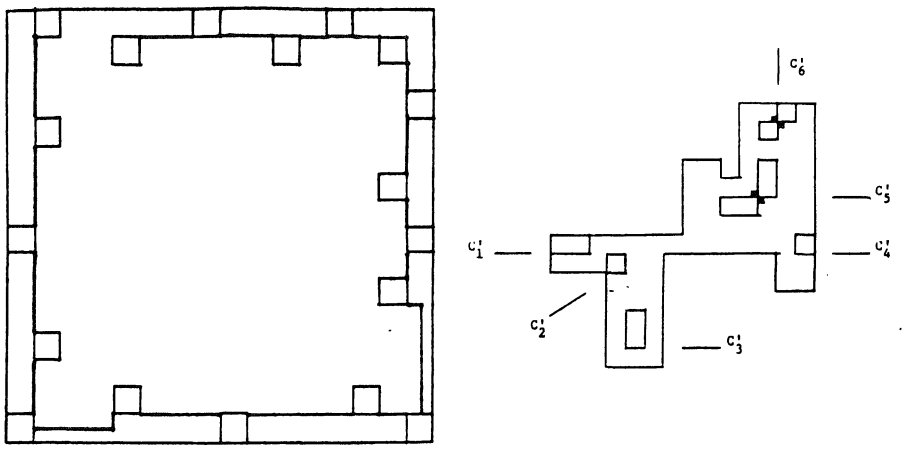


Fig. 1.

We start the proof by giving a lemma. We will consider simply connected domains P which are unions of dyadic cubes of the same side length and such that ∂P is a Jordan curve.

Lemma 2. *Fix $n_0 \in \mathbb{Z}^+$. Let P be a domain as above composed by no more than n_0 cubes of side length ℓ. Choose any collection of such cubes and call its connected components $\{C_j\}$. Then, we can construct chord arc curves $\{L_j\}$, contained in the interior of P, such that*

(i) $\{C_j\} \cup \{L_j\}$ *is a connected set and the endpoints of each curve in* $\{L_j\}$ *are either vertices or middle points of edges of cubes in* $\{C_j\}$,

(ii) *There are constants* $c_i > 0$, $i = 1, 2, 3$ *such that*

$$c_1 \ell \leq \operatorname{diam}(L_j) \leq c_2 \ell \ , \ \textit{for all } j$$
$$\operatorname{dist}(L_j, L_k) \geq c_3 \ell \ , \ \textit{for } j \neq k.$$

The chord arc constant, c_1, c_2 *and* c_3 *only depend on* n_0.

PROOF OF LEMMA 2: Suppose P is the unit cube. Choose $k \in \mathbb{Z}^+$ so that $2^{-2k} \leq n_0$. In particular, P is the union of 2^{-2k} cubes of side length 2^{-k}. If all the components $\{C_j\}$ are cubes in the grid of size 2^{-k} either touching ∂P or at distance 2^{-k} from ∂P, the statement in the lemma is obvious (Figure 1). The idea is to reduce the general case to this one by using the fact that ∂P is a chord arc curve with chord arc constant only depending on n_0.

Normalize so that $\operatorname{diam}(P) = 1$, and suppose first that P minus the collection of cubes we choose inside P is still a connected set. Let $C_1, \ldots, C_N$ be the connected components of this set of cubes. In fact, we can assume they are simply connected. In general, ∂C_j may have self intersections, so, what we do is to replace C_j by C_j', where C_j' is obtained by adding two small cubes of side length $\ell/8$ at the vertices where ∂C_j intersects itself, in case there are any. If not, let $C_j' = C_j$. Then $\partial C_j'$ is also a chord arc curve with chord arc constant only depending on n_0.

Let f be a bilipschitz map in $\mathbb{C}$ which sends P onto the unit cube Q_0, see [10]. Since we have normalized so that $\operatorname{diam} P = 1$, the bilipschitz constant of f only depends on the chord arc constant of ∂P. The images $f(\partial C_j')$, $j = 1, \ldots, N$ are chord arc and for $j \neq k$,

$$\operatorname{dist}\{f(C_j'), f(C_k')\} \geq c \operatorname{dist}\{C_j', C_k'\} \geq c\,\ell$$

for some c depending on the bilipschitz constant. Now draw disjoint neighborhoods U_j around each $f(C_j')$ with $\operatorname{dist}\{U_j, U_k\} \geq c\,\ell/2$, $j \neq k$ (Figure 2). Take a grid of small size in Q_0. Again, there exist bilipschitz maps $f_j : Q_0 \mapsto Q_0$, $j = 1, \ldots, N$, such that

(1) $f_j(f(C_j)) \subseteq S_j$ where S_j is a cube on the grid in Q_0 ,
(2) $f_j(z) = z$, for every $z \in Q_0 \backslash U_j$.

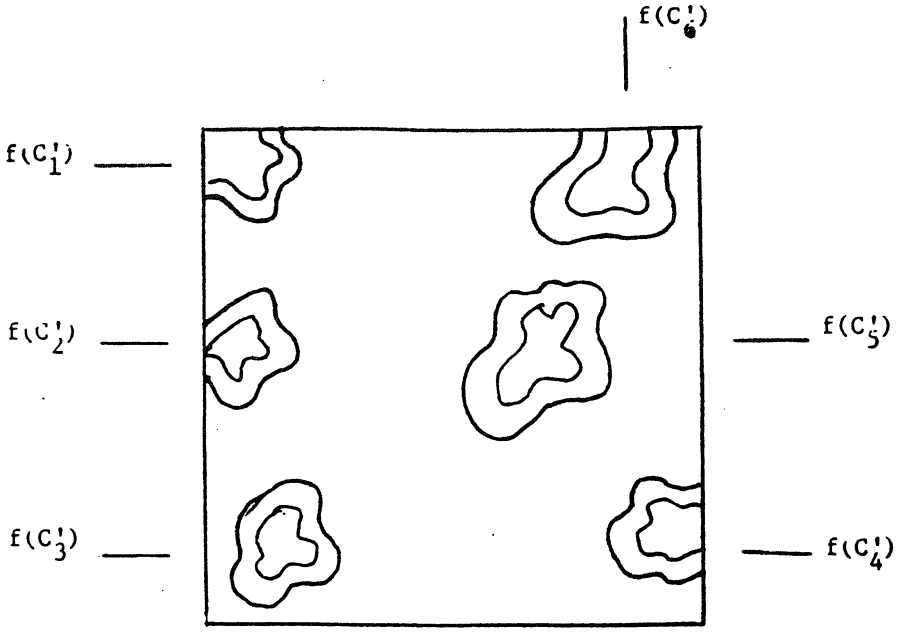


Fig. 2.

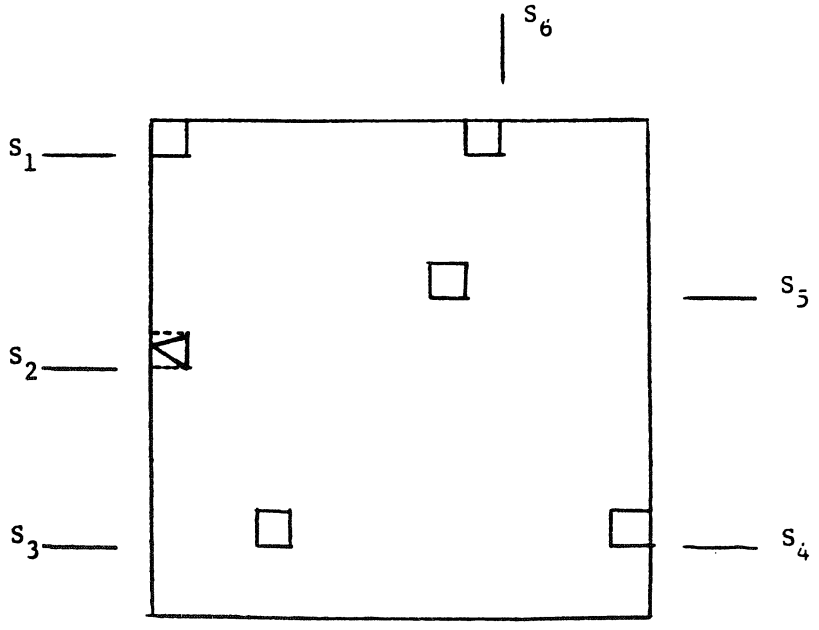


Fig. 3.

The composition $F = f_1 \circ \cdots \circ f_N \circ f$ is a bilipschitz map in $\mathbb{C}$ which sends P onto the unit cube Q_0 and each component C_j into a cube S_j of the grid (Figure 3). If the size of the grid is small enough we can find bilipschitz maps $\{g_j\}$ which will push the cubes S_j's towards the boundary reducing the general case to the one considered at the beginning of the proof. Let $G = g_N \circ \cdots \circ g_1 \circ f_N \circ \cdots \circ f_1 \circ f$. It is important to note that the size of the grid depends on $N = N(n_0)$, the chord arc constant of each $\partial(f(C_j'))$ depends as well on n_0 and therefore the bilipschitz constant of G will only depend on n_0.

Next, let $\{\gamma_j\}$ be the set of lines contained in Q_0 which connect the disjoint cubes S_j, $j = 1, \ldots, N$. Choose $\{\gamma_j\}$ so that their endpoints are either vertices or middle points of edges of the S_j's. Call these endpoints $\{v_j\}$. Without loss of generality we can assume that $F^{-1}(v_j)$ is either a vertex or a middle point of an edge of a cube of some C_j. Let $L_j = F^{-1}(\gamma_j)$. Then $\{L_j\}$ are chord arc curves satisfying (i) and (ii) in Lemma 2.

Suppose now that P minus the collection of cubes we choose inside P is a disconnected set. Then we decompose P into polygons $\{P_k\}$ and apply the argument above to the connected components $\{C_j\}$ contained inside each P_k. By adding some extra lines connecting the different P_k's, we get the required $\{L_j\}$.

We now turn to the proof of Lemma 1. The situation is the following: we approach K by a sequence K_n, $n \geq 0$, $K_n \supset K_{n+1}$ and $K = \cap_0^\infty K_n$. Each K_n is in general a disconnected union of dyadic cubes of side length ℓ_n with $\ell_{n+1}/\ell_n = 2^{-k_0}$. At each stage n, we will construct a family of chord arc curves $\langle L \rangle_n$ which connect the disjoint components of K_n and satisfy (2.4)-(2.7). By perturbing the curves slightly in $\langle L \rangle_{n-1}$ we will get that each curve in $\langle L \rangle_{n-1}$ is contained in a curve in $\langle L \rangle_n$. The limit will be the required $\{L_j\}$.

Clearly, if $n = 0$, K_0 is the unit cube Q_0, and $\langle L \rangle_0 = \emptyset$.

For $n = 1$, take the grid of size 2^{-k_0} in Q_0. K_1 is the set of cubes in the grid which contain points of K. By applying Lemma 2 to $P = Q_0$ and the collection of cubes K_1, we get the set of curves $\langle L \rangle_1$ which connect the disjoint components of K_1 and satisfy (i) and (ii) with constants only depending on k_0. Thus, $\langle L \rangle_1$ obviously satisfy (2.4)-(2.6). It is also clear by the construction of $\langle L \rangle_1$ that (2.7) holds too. We can choose $\langle L \rangle_1$ to be minimal in the following sense: $K_1 \cup \langle L \rangle_1$ is connected, but for every $L \in \langle L \rangle_1$, $K_1 \cup \langle L_1 \rangle \backslash L)$ is disconnected.

Now, given $\langle L \rangle_{n-1}$ satisfying (2.4)-(2.7), we will construct $\langle L \rangle_n$.

The set K_n is obtained by partitioning K_{n-1} into smaller cubes of

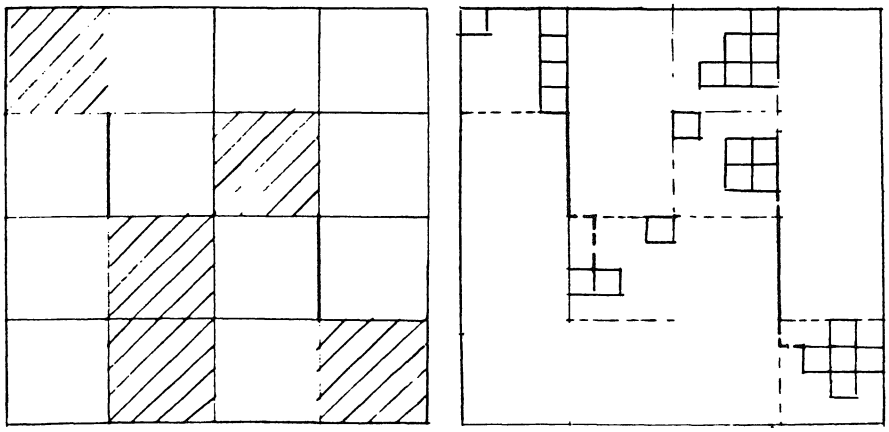


Fig. 4.

side length $2^{-k_0 n}$ and by choosing those which contain points of K. The curves in $\langle L \rangle_{n-1}$ connect the different components of K_{n-1}, thus we just have to extend these curves so they hit K_n and then apply Lemma 2 inside each component of K_{n-1} to connect the subset of K_n contained in it (Figure 4). The main problem is that as n grows, the number of cubes inside the components may get arbitrarily big and we would loose control of the constants. To avoid this, we will decompose each component of K_{n-1} into "disjoint" pieces and then apply the lemma to each of them. So, let C be a connected component of K_{n-1}. It is very easy to get a family of cubes $\{\tilde{Q}_j\}$ and connected sets $\{P_j\}$ in C such that $C = \cup P_j, P_j \subset 3\tilde{Q}_j$ and $(P_j)^\circ \cap (P_k)^\circ = \phi, j \neq k$ ($(P_j)^\circ$ denotes the interior of P_j).

For reasons which will be clear later we would like each P_j to be simply connected and satisfy:

(2.8) ∂P_j has no self-intersections,

(2.9) The concentric cube $(1 + 1/8)\tilde{Q}_j$ is contained in $(P_j)^\circ$.

To get such P_j's we only have to modify slightly the ones in the decomposition above.

If (2.8) fails for some P_j, we add two small cubes of side length $\ell_{n-1}/4$ at each vertex where ∂P_j intersects itself. If these cubes were contained in some $P_k \neq P_j$, we remove them from P_k.

If (2.9) fails, then we just add an ε-neighborhood around $\tilde{Q}_j \subset P_j$ with $\varepsilon = \ell_{n-1}/4$. Note that since $(2\tilde{Q}_j \cap C) \subset P$, the ε-neighborhood does not intersect any other $P_k \neq P_j$.

Thus, by adding some cubes we can assume that the P_j's in the decomposition above are simply connected and satisfy (2.8) and (2.9). So, we will replace any curve L in $\langle L\rangle_{n-1}$ which hits C by a curve $\tilde{L}$ which travels along L and stops when L hits $\cup P_j$ for the first time. For convenience of notation, we will set L equal to $\tilde{L}$ and we will continue referring to them as curves in $\langle L\rangle_{n-1}$.

Next, fix P_j and look at all k's such that $\partial P_k \cap \partial P_j \neq \emptyset$. For each such k, choose a point $z_{jk} \in \partial P_j \cap \partial P_k$ which is not an endpoint of any curve in $\langle L\rangle_{n-1}$. We define the set V_j as follows:

A point $z \in \partial P_j$ is in V_j if $z = z_{jk}$ for some z_{jk} as above or z is the endpoint of a curve $L \subset \langle L\rangle_{n-1}$ which hits ∂P_j. In this last case if the point z is also in ∂P_k for some $k \neq j$, we include it either in V_j or V_k. The whole point of this is to extend L, whether we do it inside P_j or P_k it does not matter. The points z_{jk}'s will be the link between the different polygons P_j's. Note that all the points in V_j can be chosen so that they are vertices or middle points of edges of cubes.

We will drop the "j's" in the notation until we need them again.

Let n_0 be the number of points in the set $V \subset \partial P$ and consider the cube $\tilde{Q}$ attached to P. We have constructed P so that P° contains the concentric cube $Q' = (1 + 1/8)\tilde{Q}$, *cf.* (2.9). The following remark is an immediate consequence of the definition of uniformly perfect sets.

REMARK. Suppose K is a (η) uniformly perfect set. Consider a cube Q with $Q \cap K \neq \emptyset$. Let ℓ be the side length of Q and let Q' denote the concentric cube of side length $(1+1/8)\ell$. Then, given any $N \in \mathbb{Z}^+$, there exists $k = k(\eta, N) \in \mathbb{Z}^+$ such that the grid of size $2^{-k}\ell$ in Q' contains N cubes $Q_1, \ldots, Q_N$ satisfying

(i) $Q_j \cap K \neq \emptyset$, $j = 1, \ldots, N$,

(ii) $\operatorname{dist}\{Q_j, Q_k\} \geq 10\, 2^{-k}\ell$, $j \neq k$.

Thus, there is $k_0 = k_0(\eta)$ such that the grid of size $\ell_n = 2^{-k_0}\ell_{n-1}$ inside $Q' = (1+1/8)\tilde{Q}$ contains at least n_0 cubes satisfying (i) and (ii). Call F to this family of cubes and note that they are cubes of the n-th

generation K_n. Note also that n_0 is bounded by a universal constant, so let us fix k_0 so that (i) and (ii) holds for this constant. Next, we will build curves $\{L_k\}, k = 1, \ldots, n_0$ connecting each point $z \in V$ to one of the cubes in F. These curves will be polygonal lines inside P with

$$\begin{aligned} &\text{(i)} \quad \operatorname{diam}(L_k) \leq c_0\, \ell_n \text{ for some universal constant } c_0 > 0, \\ &\text{(ii)} \quad \operatorname{dist}\{L_j, L_k\} \geq 10\, \ell_n\ . \end{aligned} \tag{2.10}$$

To do this, choose z_1, Q_1 such that

$$\operatorname{dist}\{z_1, Q_1\} \leq \operatorname{dist}\{z_j, Q_k\} \quad \text{for all } z_j \in V, Q_k \in F.$$

By distance here we mean the length of the shortest polygonal line which joins z_1 to Q_1 and it is contained in the grid of size ℓ_n in P. Let L_1 be such a line.

Draw now a tube T_1 running parallel to $L_1 \cup Q_1$ of radius $9\,\ell_n$. Note that T_1 does not contain any cube in F except Q_1. Moreover, since $F \subset P^\circ$ if the size of the grid is small enough (and we can assume it is), T_1 does not disconnect P, so we can apply the same construction to $P \backslash T_1$ which has $(n_0 - 1)$ points on its boundary and $(n_0 - 1)$ cubes of the family F in its interior. Choose z_2, Q_2 as before, but now the distance is measured with respect to the grid in $P \backslash T_1$. Get L_2 and draw the tube T_2. By continuing this process, we get polygonal lines $\{L_k\}, k = 1, \ldots, n_0$ such that each L_k joins V_k to Q_k inside P and (2.10) holds.

We point out again that the size of the grid which allows us to perform this last construction only depends on n_0. Now, replace L_k by Γ_k, where Γ_k is obtained by following L_k until it first hits a cube of K_n. Obviously Γ_k also satisfies (2.10). For convenience of notation call L_k to Γ_k. Next, we repeat this construction inside each component of K_{n-1}. The polygonal lines we get either extend the curves in $\langle L \rangle_{n-1}$ so that the sets $\{(K_n \cap C) : C \text{ is a component of } K_{n-1}\}$ remain connected or they connect the sets $\{P_j\}$ inside each component C. Now all that is left is to join the components of K_n inside each of the P_j's. To do this, fix P and consider the lines $\{L_k\}$ contained in P. For each k, we draw an auxiliary line $\tilde{L}_k$ running parallel to L_k at distance ℓ_n and with endpoints on the same cube as L_k. Call the thin tube in between T_k. Apply now Lemma 2 to $P' = P \backslash \cup\, T_k$ and to the collection of cubes $K_n \cap P'$. Get new lines $\{L'_j\}$ travelling inside P' satisfying (i)-(iii) in Lemma 2. Note that these lines are at distance at least ℓ_n from each of

the L_k's. Finally, remove the extra lines $\{\tilde{L}_k\}$, and perform the same construction inside each polygon $P \subset C$ for all connected components C of K_{n-1}. The lines $\langle L\rangle_{n-1} \cup \{L_k\} \cup \{L'_k\}$ will be the lines of the n-th generation $\langle L\rangle_n$. Again, we can choose $\langle L\rangle_n$ to be minimal in the sense described before.

Before returning to the proof of Theorem 2 we need the following characterization of domains with uniformly perfect boundary. Let Ω be a domain on $\bar{\mathbb{C}}$ conformally equivalent to $\mathbb{D}/\Gamma$ and let π be the universal covering map. The Poincaré metric on Ω is defined by

$$\lambda(w)|dw| = \frac{|dz|}{1-|z|^2}\,, \qquad w = \pi(z).$$

Pommerenke shows in [9] that $\partial\Omega$ is uniformly perfect if and only if there exists a constant $c > 0$ such that

$$\frac{c}{d(w)} \leq \lambda(w) \leq \frac{1}{d(w)} \tag{2.11}$$

where $d(w) = \operatorname{dist}\{w, \partial\Omega\}$.

To prove Theorem 2 we can assume again $\operatorname{diam}(\partial\Omega) = 1$.

Consider the simply connected domain Ω_0 and the crosscuts $\{L_j\}$ constructed in Lemma 1. We start by showing that any such crosscut L is chord arc in the Poincaré metric on Ω.

Let z_1, z_2 be the endpoints of L and let $z(s)$ denote the arc length parametrization of L. Choose the point $\tilde{z} \in L$ such that

$$\ell(L(z_1, \tilde{z})) = \ell(L(\tilde{z}, z_2))\,.$$

Define L_1 to be the arc of L from z_1 to $\tilde{z}$. Let w_1, w_2 be any two points on L_1, then

$$w_i = z(s_i) \qquad \text{where} \qquad s_i = \ell(L(z_1, w_i)), \qquad i = 1, 2.$$

By (2.1) L is chord arc in the Euclidean metric, with chord arc constant $M = M(\eta)$. Thus, we have for all $z \in L_1$

$$|z - z_2| \geq \frac{1}{M}\,\ell(L(z, z_2)) \geq \frac{1}{M}\,\ell(L(z_1, z)) \geq \frac{1}{M}|z_1 - z|.$$

On the other hand, by (2.3)

$$\operatorname{dist}\{z, K\} \geq c_2 \min\{|z - z_1|, |z - z_2|\}, \quad \text{for } z \in L$$

for some $c_2 = c_2(\eta)$. Thus

$$\operatorname{dist}\{z, K\} \geq c\, \ell(L(z_1, z))$$

where $c = c(\eta)$.

By convenience of notation we will use the letter $c = c(\eta)$ to denote different constants, all of them depending only on η.

We now estimate the length of the arc of L from w_1 to w_2 in the Poincaré metric. By (2.11) and the inequalities above, we obtain

$$\begin{aligned}
\rho(L(w_1, w_2)) &\leq \int_{L(w_1,w_2)} \frac{|dz|}{\operatorname{dist}\{z, K\}} \\
&\leq c \int_{L(w_1,w_2)} \frac{|dz|}{\ell(L(z_1, z))} \\
&= c \int_{s_1}^{s_2} \frac{ds}{s} \\
&= c \log \frac{s_2}{s_1} \\
&\leq c \log \left(1 + \frac{\ell(L(w_1, w_2))}{|z_1 - w_1|}\right) \\
&\leq c \log \left(1 + \frac{|w_1 - w_2|}{|z_1 - w_1|}\right).
\end{aligned}$$

Next, let σ denote the geodesic from w_1 to w_2 in the Poincaré metric. Parametrizing σ by the arc length, $z = z(s)$, that is $s = \ell(\sigma(w_1, z))$, we get

$$\begin{aligned}
\rho(w_1, w_2) &\geq c \int_{\sigma(w_1,w_2)} \frac{|dz|}{\operatorname{dist}\{z, K\}} \geq c \int_{\sigma(w_1,w_2)} \frac{|dz|}{|z - z_1|} \\
&\geq c \int_0^{\ell(\sigma(w_1,w_2))} \frac{ds}{|z_1 - w_1| + s} \geq c \log \left(1 + \frac{|w_1 - w_2|}{|z_1 - w_1|}\right).
\end{aligned}$$

Therefore, there exists a constant $C = C(\eta)$ such that

$$\rho(L(w_1, w_2)) \leq C \rho(w_1, w_2).$$

Similar computations show that the same holds for any two points w_1, w_2 on L. Hence, L is chord arc in the Poincaré metric with chord arc constant $C = C(\eta)$.

Now, let $\pi : \mathbb{D} \mapsto \Omega$ be the universal covering map of Ω with $\pi(0) = \infty$. We can define a single valued analytic branch of π^{-1} in Ω_0. Set $\mathcal{F} = \pi^{-1}(\Omega_0)$. Then $\mathcal{F}$ is a fundamental domain. We will prove that $\partial\mathcal{F}$ is chord arc.

Since each crosscut L_j is a Jordan curve, the conformal map π^{-1} extends homeomorphically to L_j, for all j. So, if L_j^+ and L_j^- denote the two sides of L_j, we can define the set of curves $\{\gamma_j\}$ by

$$\{\gamma_j\} = \{\pi^{-1}(L_j^+)\} \cup \{\pi^{-1}(L_j^-)\}.$$

Every $\gamma \in \{\gamma_j\}$ is a curve on $\mathbb{D}$ with endpoints on $\partial\mathbb{D}$. Note that $\partial\mathcal{F}$ is the union of a set on $\partial\mathbb{D}$ (may be empty) and the collection of $\{\gamma_j\}$.

To prove $\partial\mathcal{F}$ is chord arc we will show that every γ_j is chord arc and then by using the metric properties of Ω_0 we will prove that the whole boundary $\partial\mathcal{F}$ is chord arc.

Before continuing we need to introduce some notation.

Let $\xi_1, \xi_2 \in \partial\mathbb{D}$. Consider the Möbius transformation $\Phi : \mathbb{R}^2_+ \mapsto \mathbb{D}$ such that $\Phi(0) = \xi_1$ and $\Phi(\infty) = \xi_2$. For any θ, $0 < \theta < \pi/2$, define

$$\Gamma_\theta = \Phi\{z : \theta \leq \arg z \leq \pi - \theta\}.$$

We will refer to Γ_θ as a cone in $\mathbb{D}$ with endpoints ξ_1, ξ_2.

It is a well known result that any curve $\gamma \subset \mathbb{D}$ with endpoints $\xi_1, \xi_2 \subset \partial\mathbb{D}$ which is chord arc in the hyperbolic metric with chord arc constant M satisfies

(i) $\gamma \subset \Gamma_\theta$ for some $\theta = \theta(M)$,
(ii) γ is chord arc in the Euclidean metric with chord arc constant $M' = M'(M)$ [4].

As we proved before, each L_j is chord arc in the hyperbolic metric with chord arc constant depending on η, thus the same holds for γ_j. By the result above we deduce that each γ_j is contained in a cone Γ_θ for some $\theta = \theta(\eta)$, and that γ_j is chord arc in the Euclidean metric with chord arc constant $M = M(\eta)$.

Next, we will prove

$$\rho(\gamma_j, \gamma_k) \geq c(\eta) \qquad \text{for every } j, k, \qquad j \neq k$$

for some constant $c(\eta) > 0$.

If $\gamma_j = \pi^{-1}(L_j^+)$ and $\gamma_k = \pi^{-1}(L_j^-)$, then this is just an easy consequence of the fact that Ω does not contain closed geodesics of arbitrary small length in the hyperbolic metric.

If $\gamma_j = \pi^{-1}(L_j)$ and $\gamma_k = \pi^{-1}(L_k)$, all we have to show is that $\rho(L_j, L_k) \geq c(\eta)$. For, let $w_1 \in L_j$ and $w_2 \in L_k$. Denote by σ the geodesic in the hyperbolic geometry from w_1 to w_2. Suppose that $\operatorname{diam}(L_j) \leq \operatorname{diam}(L_k)$ and that z_1 is the endpoint of L_j closest to w_1. Parametrizing σ by the arc length $z = z(s)$ we get

$$\begin{aligned}
\rho(w_1, w_2) &\geq c(\eta) \int_\sigma \frac{|dz|}{\operatorname{dist}\{z, K\}} \\
&\geq c(\eta) \int_\sigma \frac{|dz|}{|z - z_1|} \\
&\geq c(\eta) \int_0^{\ell(\sigma)} \frac{ds}{s + |z_1 - w_1|} \\
&= c(\eta)\ \log\left(1 + \frac{\ell(\sigma)}{|z_1 - w_1|}\right).
\end{aligned}$$

By (2.2),

$$\ell(\sigma) \geq \operatorname{dist}\{L_j, L_k\} \geq c_1(\eta) \operatorname{diam}(L_j)$$

and since L_j is chord arc

$$|z_1 - w_1| \leq \ell(L_j) \leq M(\eta) \operatorname{diam}(L_j).$$

Hence $\rho(L_j, L_k) \geq c'(\eta)$ as required.

Summarizing, for all j, γ_j satisfies

(2.12)
$$\begin{aligned}
&\text{(i)} && \gamma_j \subset \Gamma_\theta, && \text{for some} \quad 0 < \theta < \pi/2, \\
&\text{(ii)} && \gamma_j \quad \text{is chord arc with chord arc constant} && M, \\
&\text{(iii)} && \rho(\gamma_j, \gamma_k) \geq c, && \text{for } j, k \quad j \neq k,
\end{aligned}$$

with constants only depending on η.

It is now very easy to see that (i)-(iii) imply $\partial\mathcal{F}$ is chord arc.

To prove (ii) in Theorem 2 we need to introduce some more notation.

As we remarked before $\partial\mathcal{F} = \{\gamma_j\} \cup E$ where E is a set on $\partial\mathbb{D} = \mathbb{T}$. For every γ_j, let e_j be the arc of $\mathbb{T}$ joining the endpoints of γ_j (Figure 5).

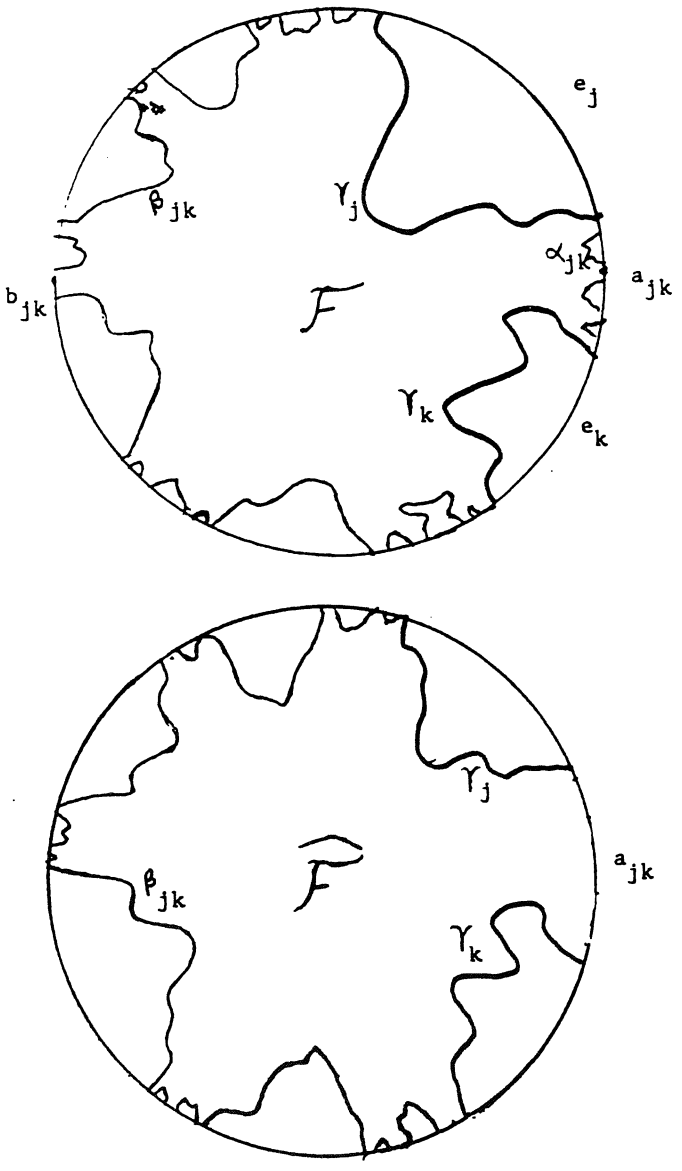


Fig. 5.

Also, for every γ_j, γ_k we denote by α_{jk} the arc of $\partial\mathcal{F}$ of smaller diameter between γ_j and γ_k and by β_{jk} the other arc. The arcs of $\mathbb{T}$ joining the endpoints of α_{jk} and β_{jk} will be denoted by a_{jk} and b_{jk} respectively. In particular,

$$\mathbb{T} = e_j \cup a_{jk} \cup e_k \cup b_{jk}.$$

The goal is to prove

$$\ell(a_{jk}) \geq c(\eta) \min\{\ell(e_j), \ell(e_k)\}, \qquad \text{for all } j,k \tag{2.13}$$

and

$$\frac{1}{2\pi}\ell(e_j) \leq 1 - \sigma(\eta), \qquad \text{for all } j, \tag{2.14}$$

for some constants $c(\eta) > 0$, $\sigma(\eta) > 0$.

If we think of $\partial\mathcal{F} \cap \mathbb{T}$ as the set obtained by removing the arcs $\{e_j\}$ successively (as we do with a Cantor set) it is clear that (2.13) and (2.14) imply $\partial\mathcal{F} \cap \mathbb{T}$ is a uniformly perfect set.

The idea is to prove

(i) There exists a point $\xi_{jk} \in \mathcal{F}$ such that

$$\omega(\xi_{jk}, \alpha_{jk}, \mathcal{F}) \geq c_0, \qquad \text{for } j,k \tag{2.15}$$

$$\omega(\xi_{jk}, \beta_{jk}, \mathcal{F}) \geq c_0, \qquad \text{for } j,k \tag{2.16}$$

for some $c_0 = c_0(\eta) > 0$.

(ii) There exists $\sigma_0 = \sigma_0(\eta) > 0$ such that

$$\omega(0, \gamma_j, \mathcal{F}) \leq 1 - \sigma_0(\eta), \qquad \text{for all } j. \tag{2.17}$$

The fact that every γ_j is a chord arc curve contained in a "cone", (2.12), will allow us to show

$$\omega(\xi_{jk}, a_{jk}, \mathbb{D}) \geq c'(\eta) \tag{2.18}$$

$$\omega(\xi_{jk}, b_{jk}, \mathbb{D}) \geq c'(\eta) \tag{2.19}$$

$$\omega(0, e_j, \mathbb{D}) \leq 1 - \sigma'(\eta) \tag{2.20}$$

for all j,k, and as an easy consequence we will get (2.13) and (2.14).

We start by proving (2.15). For, we will show that the equivalent estimate holds in the domain Ω_0. Suppose first that $\gamma_j = \pi^{-1}(L^+)$ and $\gamma_k = \pi^{-1}(L^-)$ for some crosscut L. Denote by A_1 and A_2 the two

connected components of $\partial\Omega_0 \backslash L$. Note that by the construction of Ω_0 given in Lemma 1, we have

$$\operatorname{diam}(A_i) \geq \delta \operatorname{diam}(L), \qquad i = 1, 2$$

for some $\delta = \delta(\eta) > 0$. Also note that (2.15) and (2.16) are equivalent to

$$\begin{aligned} &\omega(z, A_1, \Omega_0) \geq c_0(\eta) \\ &\omega(z, A_2, \Omega_0) \geq c_0(\eta) \end{aligned} \tag{2.21}$$

for some $z \in \Omega_0$.

Without loss of generality we can assume $\operatorname{diam}(L) = 1$. To prove (2.21) we will use an extremal length argument.

Suppose L is the unit interval. Then (2.2) and (2.3) impy that the set $\Omega_0 \backslash L$ is outside the "diamond" D_0,

$$D_0 = \{z : |\arg z| \leq \varepsilon\} \cap \{z : |\arg(z-1)| \leq \varepsilon\}$$

where $\varepsilon = \varepsilon(\eta)$.

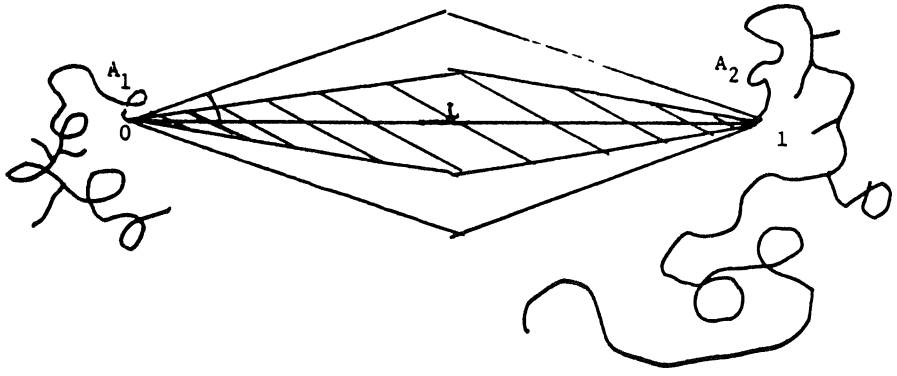


Fig. 6.

Denote by D the concentric "diamond" correspondent to the angle $\varepsilon/2$ (Figure 6). We will show

(i) For all $z \in \partial D$,

$$\omega(z, A_1 \cup A_2, \Omega) \geq c_1(\eta)$$

for some $c_1 = c_1(\eta) > 0$.

(ii) There exists a point $z_1 \in \partial D$ and a constant $c_2(\eta) > 0$ such that

$$\omega(z_1, A_1, \Omega_0) \geq c_2(\eta)$$

and similarly, there exists $z_2 \in \partial D$ such that

$$\omega(z_2, A_2, \Omega_0) \geq c_2(\eta).$$

Thus, by continuity there exists a point $z \in \partial D$, for which (2.21) holds.

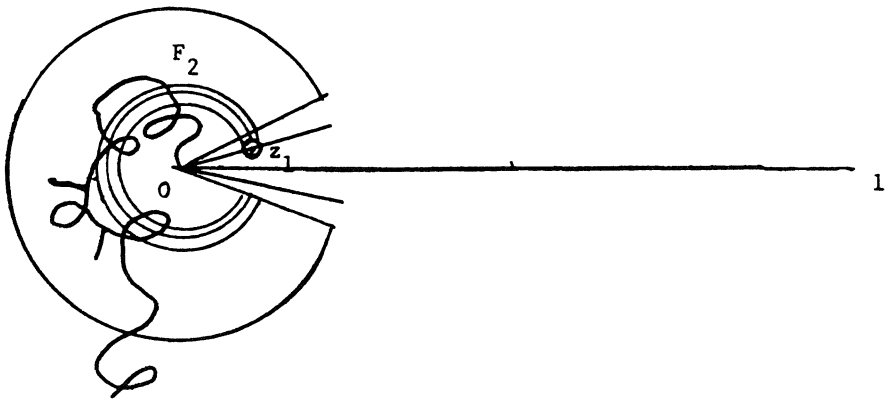


Fig. 7.

We start by proving (ii). Suppose $d = \text{dist}\{A_1, A_2\}$. In particular, A_2 is outside the ball $|z| < d$ (Figure 7). Let $r = \min\{d, \delta\}$, and let $z_1 = r/2\ e^{i\epsilon/2}$. Denote by K the ball of radius $y_1/2$ ($z_1 = x_1 + iy_1$) centered at z_1. Consider the family of curves F_1 which join K to A_1 in Ω_0, and denote by $\lambda(F_1)$ the extremal length of F_1. Then by Beurling's Theorem

$$\omega(z_1, A_1, \Omega_0) \geq c\, e^{-\pi\lambda(F_1)} \tag{2.22}$$

for some universal constant c.

Let ξ_1, ξ_2 be the two points where K intersects ∂D, and define the family of curves $F_2 = \{\gamma_r : |\xi_1| < r < |\xi_2|\}$ where $\gamma_r(\theta) = re^{i\theta}$, $\varepsilon/2 \leq \theta \leq 2\pi - \varepsilon/2$. By the choice of r, each curve F_2 contains an arc which is an element of F_1. Therefore, if $\rho(z) \geq 0$ is a measurable function admissible for $F_1, \rho \in A(F_1)$, *i.e.* $\int_\gamma \rho ds \geq 1$ for all locally rectifiable curves $\gamma \in F_1$ (ds denotes the arc length on γ), then

$$\int_{\gamma_r} \rho \, ds \geq 1 \qquad \text{for all } \gamma_r \in F_2 \, .$$

By Hölder's inequality

$$\int_{\gamma_r} \rho^2 ds \geq \frac{1}{\ell(\gamma_r)}$$

where $\ell(\gamma_r)$ denotes the length of γ_r. Thus

$$\begin{aligned} \iint \rho^2 dx \, dy &\geq \int_{|\xi_1|}^{|\xi_2|} \int_{\gamma_r} \rho^2 ds \, dr \\ &\geq \int_{|\xi_1|}^{|\xi_2|} \frac{1}{\ell(\gamma_r)} dr \geq \int_{|\xi_1|}^{|\xi_2|} \frac{1}{2\pi r} dr = \frac{1}{2\pi} \log \frac{|\xi_2|}{|\xi_1|}. \end{aligned}$$

Since $|\xi_2|/|\xi_1| = c$ where $c = c(\varepsilon)$, we get $\lambda(F_1) \leq c'(\varepsilon)$. Hence, by (2.22) we obtain

$$\omega(z_1, A_1, \Omega_0) \geq c \, e^{-\pi c'} = c_2(\varepsilon) \, .$$

The same argument gives the correspondent result for z_1. So (ii) is proved.

Note now that it is enough to show (i) for any point $z_0 = x_0 + iy_0 \in \partial D$ with $0 < x_0 < 1/2, y_0 > 0$. For, let $\tilde{A}_1 = A_1 \cap \{z : |z| \leq \delta\}$. Then by the maximum principle we get

$$\omega(z_0, A_1 \cup A_2, \Omega_0) \geq \omega(z_0, \tilde{A}_1, \bar{\mathbb{C}} \backslash \{L \cup \tilde{A}_1\}).$$

Thus if $\Omega_1 = \bar{\mathbb{C}} \backslash \{L \cup \tilde{A}_1\}$, all we have to prove is

$$\omega(z_0, \tilde{A}_1, \Omega_1) \geq c_1(\eta). \tag{2.23}$$

For, let $\xi_0 = \delta/2 \, e^{i\varepsilon_0/2}$ and let B be the ball of radius $v_0/2$, ($\xi_0 = u_0 + iv_0$) centered at ξ_0 (Figure 8).

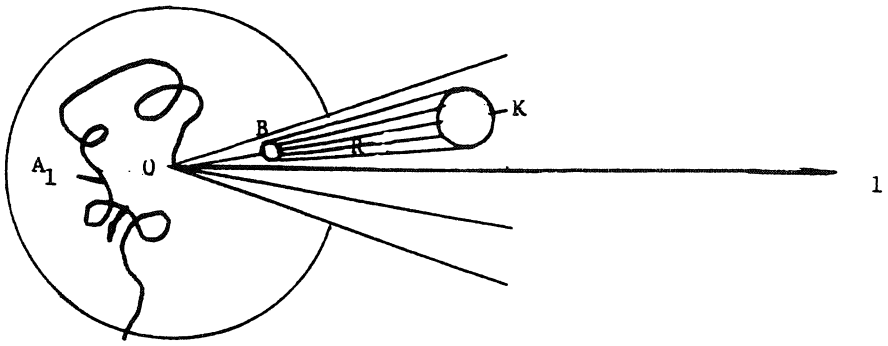


Fig. 8.

By Harnack's inequality,

$$\omega(\xi, \tilde{A}_1, \Omega_1) \geq k\,\omega(\xi_0, \tilde{A}_1, \Omega_1)$$

for all $\xi \in B$ and some universal constant k. Thus, by the maximum principle, for all $z \in \Omega_1 \backslash B$, in particular for z_0, we get

$$\omega(z, \tilde{A}_1, \Omega_1) \geq k\,\omega(z, \partial B, \Omega_1 \backslash B)\,\omega(\xi_0, \tilde{A}_1, \Omega_1). \tag{2.24}$$

Using a similar argument to the one in the proof of (ii) we get

$$\omega(\xi_0, \tilde{A}_1, \Omega_1) \geq c(\varepsilon). \tag{2.25}$$

Next, we show

$$\omega(z_0, \partial B, \Omega_1 \backslash B) \geq c(\eta). \tag{2.26}$$

For, let K be the ball of radius $y_0/2$ centered at z_0 and let F_1 be the family of curves joining K to ∂B in the domain $\Omega_1 \backslash B$.

Consider now the region R shown in Figure 8 and the rays $\{\gamma_\theta\}$ from ∂B to K with $\theta_1 \leq \theta \leq \theta_2$

$$\gamma_\theta(r) = \{re^{i\theta} : r_1 < r < r_2\}$$

where $r_2 \simeq |z_0|$ and $r_1 \simeq |\xi_0| = \delta/2$. Since $\gamma_\theta \in F$, for all $\rho \in A(F)$ we have

$$\int_{\gamma_\theta} \rho \, ds \geq 1 .$$

By Hölder's inequality, we get

$$\int_{r_1}^{r_2} \rho^2(re^{i\theta}) r \, dr \geq \frac{1}{\log \, r_2/r_1} .$$

Thus

$$\begin{aligned} \iint \rho^2 dx \, dy &\geq \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \rho^2 r \, dr \, d\theta \\ &\geq (\theta_2 - \theta_1) \frac{1}{\log \, r_2/r_1} \geq c(\varepsilon, \delta) . \end{aligned}$$

Since $\varepsilon = \varepsilon(\eta)$ and $\delta = \delta(\eta)$ we get (2.26) which with (2.24) and (2.25) give us the required estimate (2.23). This concludes the proof of (i).

So, we have proved (2.21) in the case the line L is the unit interval.

For the general case, consider the bilipschitz map f on $\mathbb{C}$ which sends the chord arc curve L of diameter 1 to the interval $[0, 1]$. Then, for all $z, w \in \mathbb{C}$,

$$\frac{1}{c(\eta)} \leq \frac{f(z) - f(w)}{|z - w|} \leq c(\eta)$$

where c only depends on the chord arc constant of L, $M = M(\eta)$.

Thus, the images of A_1 and A_2 under f are connected sets with $\operatorname{diam}(f(A_i)) \geq \delta(\eta)$, $i = 1, 2$. Furthermore, $f(A_2)$, $f(A_2)$ will stay outside a "diamond" of angle $\varepsilon = \varepsilon(\eta)$. Hence the results above apply to $f(A_1)$ and $f(A_2)$ in the domain $f(\Omega_0)$. This, together with the fact that if F is a family of curves and f is an M-bilipschitz map in $\mathbb{C}$ then

$$\frac{1}{k(M)} \lambda(F) \leq \lambda(f(F)) \leq k(M) \lambda(F) ,$$

give us (i) and (ii) in the general case.

As we remarked before, (i) and (ii) imply (2.21). So we have proved (2.15) and (2,16) whenever $\gamma_j = \pi^{-1}(L^+)$ and $\gamma_k = \pi^{-1}(L^-)$.

Suppose now $\gamma_j = \pi^{-1}(L_j^\pm)$ and $\gamma_k = \pi^{-1}(L_k^\pm)$ (whether it is $+$ or $-$ does not matter, the proof is practically the same). Assume

$\operatorname{diam}(\gamma_j) \geq \operatorname{diam}(\gamma_k)$ and $\operatorname{diam}(\gamma_k) = 1$. Denote by A_1, A_2, A_3 the three components of $\partial\Omega_0 \backslash (\gamma_j \cup \gamma_k)$, and suppose A_2 is the one which connects γ_j to γ_k. Again, we have

$$\operatorname{diam}(A_i) \geq \delta(\eta)\,, \qquad i = 1,2,3\,.$$

A similar argumet using extremal length and bilipschitz maps can be given to prove (2.15) and (2.16) in this case.

Next, we will prove (2.17). For, let $\gamma_j = \pi^{-1}(L^+)$, then (2.17) is equivalent to

$$\omega(\infty, L^+, \Omega_0) \leq 1 - \sigma_0(\eta).$$

By the maximum principle $\omega(\infty, L^+, \Omega_0) \leq \omega(\infty, L^+, \bar{\mathbb{C}} \backslash L)$. Suppose first $L = [0,1]$. Standard estimates show that for all z with $|z| = 2$,

$$\omega(z, L^-, \bar{\mathbb{C}} \backslash [0,1]) \geq c_0,$$

for some universal constant $c_0 > 0$. Thus, if $|z| = 2$, $B(z)$ is the ball of radius $1/2$ centered at z, and F_z is the family of curves joining $B(z)$ to L^-, we get $\lambda(F_z) \leq c_1$ for some universal constant $c_1 > 0$.

We now turn to the general case. Assume $\operatorname{diam}(L) = 1$, and let f be a bilipschitz map in $\mathbb{C}$ which sends L to the unit interval. Consider the curve $\Gamma = f^{-1}(|z| = 2)$. For each $\xi \in \Gamma$, set $K(\xi) = f^{-1}(B(z))$ and consider the family of curves joining $K(\xi)$ to L^-, that is $F_\xi = f^{-1}(F_z)$. Then

$$\omega(\xi, L^-, \bar{\mathbb{C}} \backslash L) \geq c\, e^{-\pi\lambda(F_\xi)} \geq c\, e^{-\pi c_2 \lambda(F_z)} \geq \sigma_0$$

where the constants c_2 and σ_0 depend on the chord arc constant of L, $M = M(\eta)$.

The next goal is to prove (2.18)-(2.20). It just says that (2.15)-(2.17) hold if we remove the arcs $\{\gamma_j\}$. This will be an easy consequence of the following result:

Let γ be a curve from 0 to 1 in $\mathbb{R}^2_+$, which is contained in a "cone" Γ_θ. The boundary of Γ_θ is the union of two arcs, C_1 and C_2. Denote by C the biggest arc of the boundary of the "cone" $\Gamma_\theta/2$ and by D the domain bounded by C and the interval $[0,1]$. Then, for all $z \in \gamma$,

$$\omega(z, I, D) \geq c_0 \tag{2.27}$$

for some constant c_0 only depending on θ.

It is easy to prove that for all $z \in C_2$

$$\omega(z, I, D) \geq c_0\,.$$

Thus, by the maximum principle, (2.27) holds.

To prove (2.18), consider the domain $\hat{\mathcal{F}} \subset \mathbb{D}$ bounded by $\gamma_j, \gamma_k, \beta_{jk}$ and a_{jk} . In particular $\hat{\mathcal{F}} \supset \mathcal{F}$, and therefore any harmonic function in $\hat{\mathcal{F}}$ is also harmonic in $\mathcal{F}$. Let $u(z) = \omega(z, a_{jk}, \hat{\mathcal{F}})$, then

$$\omega(\xi_{jk}, a_{jk}, \hat{\mathcal{F}}) = \int_{\partial \mathcal{F}} u(\xi)\, d\omega_{\xi_{jk}}(\xi).$$

By (2.15) and (2.27), we get

$$\int_{\partial \mathcal{F}} u(\xi) d\omega_{\xi_{jk}}(\xi) \geq c(\eta)\,.$$

Thus, by the maximum principle

$$\omega(\xi_{jk}, a_{jk}, \mathbb{D}) \geq c(\eta)\,.$$

Similar arguments can be given to prove (2.19) and (2.20).

Note now that (2.14) is an immediate consequence of (2.20).

To prove (2.13), let us assume that it does not hold. Then it is easy to show that (2.18) implies

$$\omega(\xi_{jk}, b_{jk}, \mathbb{D}) \underset{j,k}{\mapsto} 0$$

which contradicts (2.19). This concludes the proof of (2.13) and (2.14) and therefore the proof of sufficiency.

To prove necessity, first we need to prove a lemma on quasidisks, that is, domains bounded by quasicircles.

Lemma 3. *Suppose $\mathcal{R} \subset \mathbb{R}^2_+ = \{y \geq 0\}$ is a K-quasidisk. Let z_0 be any point in $\bar{\mathcal{R}} \backslash \mathbb{R}$ and let B be the hyperbolic ball of radius 1 centered at z_0. Then $B \cap \mathcal{R}$ contains a ball of hyperbolic radius $\varepsilon > 0$, for some ε only depending on K. ($\bar{\mathcal{R}}$ denotes the closure of $\mathcal{R}$).*

PROOF OF LEMMA 3: Suppose first $z_0 \in \partial \mathcal{R}$. Consider the K-quasiconformal map $f : \mathbb{R}^2_+ \mapsto \bar{\mathcal{R}}$ with $f(0) = z_0$. Assume $z_0 = iy_0, y_0 > 0$. Let $\tilde{B}$ be the biggest Euclidean ball centered at z_0 contained in B, and let ρ_0 be its Euclidean radius. Then, it is easy to see that $\rho_0 = (1 - 1/e)\, y_0$.

Let $B_1 \subset f^{-1}(\tilde{B})$ be the ball centered at 0 and radius $r_1 = \min_{z\in\partial\tilde{B}} |f^{-1}(z)|$. Apply the distortion theorem in [6] to $f(B_1)$ to get a ball contained in $f(B_1)$ of radius

$$\rho_1 = \min_{z\in\partial B_1} |f(z) - z_0| \geq c(K)\, \rho_0 \,.$$

Now, let $D \subset B_1$ be the ball of radius $r_1/2$ centered at $\xi = i\, r_1/2$. Then $f(D)$ is contained in $\bar{\mathcal{R}}$ and it is tangential to z_0 and to $f(\partial B_1)$. Thus, by the distortion theorem, there is a ball $B_2 \subset f(D)$ of radius ρ_2 with

$$\rho_2 \geq c(K)\, \mathrm{diam}(f(D)) \geq c_1(K)\, \rho_0 \,.$$

Let B_ε be the ball concentric to B_2 and radius $\rho_2/2$. Then $\bar{B}_\varepsilon \subset \mathcal{R} \cap B$ and since

$$\frac{\rho_2}{2} \geq \frac{c_1(K)}{2}\rho_0 = c_2(K)\, y_0 \,,$$

we get that the hyperbolic radius of $B_\varepsilon, \varepsilon$, satisfies $\varepsilon \geq c_3(K)$.

A similar argument can be given in the case z_0 lies in the interior of $\mathcal{R}$.

We now turn to the proof of the theorem.

Let $\mathcal{F}$ be a fundamental domain for the domain $\Omega \subset \bar{\mathbb{C}}$ such that $\partial\mathcal{F}$ is a K-quasicircle.

Suppose the theorem fails, that is for each $\delta > 0$, there is $\xi_0 \in \bar{\mathcal{F}}$ and an element γ in the Fuchsian group Γ such that $\rho(\xi_0, \gamma(\xi_0)) \leq \delta$. Then

$$\rho(\gamma^n(\xi_0), \xi_0) \leq n\delta \leq 1 \quad \text{if} \quad n < \frac{1}{\delta} \,. \tag{2.28}$$

Consider the ball B of hyperbolic radius 1 centered at ξ_0. By Lemma 3 there is a ball $B_\varepsilon \subset B \cap \mathcal{F}$ of hyperbolic radius ε. Since $B_\varepsilon \subset \mathcal{F}$, the balls $B_\varepsilon, \gamma(B_\varepsilon), \ldots, \gamma^n(B_\varepsilon)$ are disjoint. Furthermore, they all have hyperbolic radius ε and by (2.28) they are contained in the ball of hyperbolic radius 2 centered at ξ_0. Since this cannot happen for arbitrarily big n's, we get a contradiction. Thus, Theorem 2 is proved.

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Boundary value problems in Lipschitz cylinders for three-dimensional parabolic systems

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Abstract. We consider initial-boundary value problems for a parabolic system in a Lipschitz cylinder. When the space dimension is three, we obtain estimates for the solutions when the lateral data taken from the best possible range of L^p-spaces.

Introduction.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and for $0 < T < \infty$, let $\Omega_T = \Omega \times (0,T)$ be a Lipschitz cylinder. Consider the parabolic system

$$\frac{\partial \vec{u}}{\partial t} = \mu \Delta \vec{u} + (\lambda + \mu) \nabla (\text{div } \vec{u}) \quad \text{in } \Omega_T \tag{0.1}$$

where $\vec{u} = (u^1, u^2, u^3)$ and $\mu > 0, \lambda > -2\mu/3$ are constants. In this paper, we study the solvability of the initial-Dirichlet problem for (0.1)

$$\text{(IDP)} \qquad \begin{cases} \vec{u}|_{\Sigma_T} = \vec{g} \in L^p(\Sigma_T),\ L_1^p(\Sigma_T) \text{ or } \Lambda_0^\alpha(\Sigma_T), \\ \\ \vec{u}|_{t=0} = \vec{0}. \end{cases}$$

where $\Sigma_T = \partial\Omega \times (0,T)$ denotes the lateral boundary of Ω_T, $L_1^p(\Sigma_T)$ denotes the space of functions on Σ_T having one spatial and half of a time derivative in L^p and $\Lambda_0^\alpha(\Sigma_T)$ denotes the space of Hölder continuous functions of order α (in parabolic sense) which vanish on $\partial\Omega \times \{t = 0\}$. We also consider the initial traction problem for (0.1)

$$\text{(ITP)} \qquad \begin{cases} \lambda(\text{div } \vec{u})N + \mu\big((\nabla\vec{u}) + (\nabla\vec{u})^{tr}\big)N\big|_{\Sigma_T} = \vec{g} \in L^p(\Sigma_T), \\ \vec{u}\big|_{t=0} = \vec{0}, \end{cases}$$

where $N = (N_1, N_2, N_3)$ denotes the outward unit normal to $\partial\Omega$ and $(\cdot)^{tr}$ denotes the transpose of a matrix.

Our estimates for solutions of (0.1) will be given in terms of the parabolic maximal function denoted by $(\cdot)^*$. We will use $\partial_t^{1/2}$ to denote half of a time derivative. Throughout this paper C will denote constants depending at most on the Lipschitz character of $\partial\Omega, T, \lambda, \mu$ and p.

We now state our main results (see Section 1 below for the definition of function spaces appearing in these results).

Theorem A. (Dirichlet Problem with L^p data). *There exists $\varepsilon > 0$, such that, given any $\vec{g} \in L^p(\Sigma_T)$, $2 - \varepsilon < p \leq \infty$, there exists a unique $\vec{u}$ in Ω_T satisfying* (0.1), (IDP) *and $(\vec{u})^* \in L^p(\Sigma_T)$. Moreover, $\|(\vec{u})^*\|_{L^p(\Sigma_T)} \leq C \|\vec{g}\|_{L^p(\Sigma_T)}$.*

Theorem B. (Dirichlet Problem with Λ_0^α data). *There exists $\alpha_0 > 0$, such that, given any $\vec{g} \in \Lambda_0^\alpha(\Sigma_T)$, $0 < \alpha < \alpha_0$, there exists a unique $\vec{u}$ in Ω_T satisfying* (0.1), (IDP) *and $\vec{u} \in \Lambda_0^\alpha(\overline{\Omega} \times [0,T])$. In fact, we have $\|\vec{u}\|_{\Lambda_0^\alpha(\overline{\Omega}\times[0,T])} \leq C \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}$.*

Theorem C. (Dirichlet Problem with L_1^p data). *There exists $\varepsilon > 0$, such that, given any $\vec{g} \in L_1^p(\Sigma_T)$, $1 < p < 2 + \varepsilon$, there exists a unique $\vec{u}$ in Ω_T satisfying* (0.1), (IDP) *and $(\nabla\vec{u})^* \in L^p(\Sigma_T)$. Moreover, $\|(\nabla\vec{u})^*\|_{L^p(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^p(\Sigma_T)} \leq C \|\vec{g}\|_{L_1^p(\Sigma_T)}$.*

Theorem D. (Traction Problem with L^p data) *There exists $\varepsilon > 0$, such that, given any $\vec{g} \in L^p(\Sigma_T), 1 < p < 2 + \varepsilon$, there exists a unique $\vec{u}$ satisfying* (0.1), (ITP) *and $(\nabla\vec{u})^* \in L^p(\Sigma_T)$. In fact, we have*

$$\|(\nabla\vec{u})^*\|_{L^p(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^p(\Sigma_T)} \leq C \|\vec{g}\|_{L^p(\Sigma_T)}.$$

The results above were announced in [BS2]. The boundary value problems for (0.1) in Lipschitz cylinders were studied in [S], where the solvability of (IDP) and (ITP) was obtained for $p = 2$ on $\Omega \times (0,T)$ where Ω is any bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$.

For the system of elastostatics,

$$\mu \Delta \vec{u} + (\lambda + \mu)\nabla(\text{div } \vec{u}) = \vec{0}$$

in Lipschitz domains, the corresponding results were obtained for $p = 2$, $n \geq 3$ in [DKV2] and for p optimal, $n = 3$ in [DK2].

Our argument follows closely that for elliptic systems in three dimensions in [DK2]. In the parabolic case, the dimension of the time variable t should be counted twice. Hence, we are working in a space of (homogeneous) dimension five. This is where the difficulty lies when we try to apply the argument from [DK2]. However, since our domains are cylindrical, we are able to overcome this difficulty. This is done by using Fourier analysis in the time variable to show that we may solve the initial-Dirichlet problem with data in mixed norm L^p spaces

$$L^{p,q}(\Sigma_T) = L^q\big((0,T); L^p(\partial\Omega)\big)$$

where p is close to 2 and $q \in (1,2]$. Since the time index may be arbitarily close to one, the extra dimensions do not introduce any additional difficulties.

We remark that the techniques we use in this paper also yield some interesting new results for the heat equation in Lipschitz cylinders in all dimensions. Indeed, using the $L^p(\partial\Omega)$-valued multiplier argument in Section 2, the L^p estimates in [B] and interpolation, it is not very hard to establish the solvability of the initial-Dirichlet problem for the heat equation with boundary data in $L^{p,q}(\Sigma_T)$ for $2 \leq p < \infty$ and $p/2 < q < \infty$ and the initial-Neumann problem with boundary data in $L^{p,q}(\Sigma_T)$ for $1 < p \leq 2$ and $1 < q < p/(2-p)$. We leave the details to interested readers.

The outline of this paper is as follows. Section 1 contains notations and definitions that will be used throughout the paper. In Section 2, we study the boundary potential operators on mixed norm spaces. In Section 3, we study the initial traction problem with atomic data. Our main results are proved in Section 4.

Finally, the first author would like to thank Professor Carlos E. Kenig and the second author would like to thank Professor Andreas Seeger for several helpful conversations.

1. Notations and definitions.

We retain the notations used in the Introduction. In particular, Ω will be a bounded Lipschitz domain in $\mathbb{R}^3$, $\Omega_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. We will use P and Q for points on $\partial\Omega$, X and Y for points in Ω and t and s will be the time variables.

For v defined on Ω_T, the nontangential maximal function of v is defined by

$$(1.1) \qquad (v)^*(P,t) = \sup_{(Y,s)\in\gamma(P,t)} |v(Y,s)|$$

where $\gamma(P,t)$ is the parabolic nontangential approach region defined by

$$(1.2) \quad \gamma(P,t) = \{(Y,s):\ |Y-P| + |t-s|^{1/2} < 2\,\mathrm{dist}\{Y,\partial\Omega\}\} \cap \Omega_T\,.$$

We will use $\vec{u}$ and $\vec{f}$ to denote functions taking their values in $\mathbb{R}^3$. In the initial Dirichlet problem, the statement $\vec{u}\big|_{\Sigma_T} = \vec{g}$ is interpreted in the sense of nontangential limit

$$\lim_{\substack{(Y,s)\to(P,t)\\(Y,s)\in\gamma(P,t)}} \vec{u}(Y,s) = \vec{g}(P,t) \qquad \text{a.e. on} \quad \Sigma_T$$

with respect to the surface measure on Σ_T.

Similarly, for the initial traction problem, by

$$\lambda(\mathrm{div}\ \vec{u})N + \mu\big((\nabla\vec{u}) + (\nabla\vec{u})^{tr}\big)N\ \big|_{\Sigma_T} = \vec{g}\,,$$

we mean that

$$\lambda(\mathrm{div}\ \vec{u})\big|_{\Sigma_T} N + \mu\big((\nabla\vec{u}) + (\nabla\vec{u})^{tr}\big)\big|_{\Sigma_T} \cdot N = \vec{g}\,.$$

We say $\vec{u}$ has initial value $\vec{0}$, write $\vec{u}\big|_{t=0} = \vec{0}$, if $\vec{u}(X,t) \to \vec{0}$ uniformly on every compact subset of Ω as $t \to 0^+$.

For a smooth function f which decays rapidly at $-\infty$, we define the fractional integral of order σ by

$$(1.3) \qquad I_\sigma(f)(t) = \frac{1}{\Gamma(\sigma)} \int_{-\infty}^{t} \frac{f(s)}{(t-s)^{1-\sigma}} ds\,, \qquad 0 < \sigma \le 1\,,$$

and fractional derivative of order σ by

$$(1.4) \qquad \partial_t^\sigma(f)(t) = \frac{\partial}{\partial t} I_{1-\sigma}(f)(t)\,, \qquad 0 < \sigma < 1\,,$$

where Γ is the gamma function.

For u defined in a neighborhood of $\partial\Omega$, we define the tangential gradient of u on $\partial\Omega$ by

$$\nabla_{\tan} u = \nabla u - \langle \nabla u, N\rangle N. \tag{1.5}$$

We now give the definition of the spaces $L^{p,q}(\Sigma_T), \Lambda_0^\alpha(\Sigma_T)$, $\Lambda_0^\alpha(\overline{\Omega} \times [0,T])$, $L_1^p(\Sigma_T)$ and $L_1^{p,q}(\Sigma_T)$.

Definition 1.6. *For $1 \le p, q < \infty$, we define the mixed norm space*

$$L^{p,q}(\Sigma_T) = L^q\big((0,T), L^p(\partial\Omega)\big)$$
$$= \left\{ \vec{f}\colon \quad \|\vec{f}\|_{L^{p,q}(\Sigma_T)} = \Big(\int_0^T \Big(\int_{\partial\Omega} |\vec{f}|^p \Big)^{q/p} \Big)^{1/q} < +\infty \right\}.$$

Definition 1.7. *For $0 < \alpha < 1$, we let*

$$\|\vec{f}\|_{\Lambda^\alpha(\Sigma_T)} = \sup_{\Sigma_T} |\vec{f}| + \sup_{(P,t)\neq(Q,s)} \frac{|\vec{f}(P,t) - \vec{f}(Q,s)|}{(|P-Q| + |t-s|^{1/2})^\alpha}$$

and we define the space $\Lambda_0^\alpha(\Sigma_T)$ by

$$\Lambda_0^\alpha(\Sigma_T) = \{\vec{f} : \vec{f}(P,0) = \vec{0} \text{ for } P \in \partial\Omega \text{ and } \|\vec{f}\|_{\Lambda^\alpha(\Sigma_T)} < +\infty\}.$$

The space $\Lambda_0^\alpha(\overline{\Omega} \times [0,T])$ is defined in a similar fashion.

Definition 1.8. *For $1 < p < \infty$, $L_1^p(\partial\Omega \times \mathbb{R})$ denotes the closure of the space*

$$\{\vec{v} : \vec{v} = \vec{u}\big|_{\partial\Omega\times\mathbb{R}},\ \vec{u} \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})\}$$

with respect to the norm,

$$\|\vec{v}\|_{L_1^p(\partial\Omega\times\mathbb{R})} \equiv \|\nabla_{\tan}\vec{v}\|_{L^p(\partial\Omega\times\mathbb{R})} + \|\partial_t^{1/2}\vec{v}\|_{L^p(\partial\Omega\times\mathbb{R})} + \|\vec{v}\|_{L^p(\partial\Omega\times\mathbb{R})}.$$

We then define

$$L_1^p(\Sigma_T) = \left\{ \vec{f}\colon \ \vec{f} = \vec{g}\big|_{\Sigma_T},\ \vec{g} \in L_1^p(\partial\Omega \times \mathbb{R}) \text{ and } \vec{g}(P,t) = \vec{0} \text{ for } t < 0 \right\}.$$

The norm is given by

$$\|\vec{f}\|_{L_1^p(\Sigma_T)} = \inf \left\{ \|\vec{g}\|_{L_1^p(\partial\Omega\times\mathbb{R})}\colon\ \vec{g}\big|_{\Sigma_T} = \vec{f} \text{ and } \vec{g}(P,t) = 0 \text{ for } t < 0 \right\}.$$

We may define $L_1^{p,q}(\partial\Omega \times \mathbb{R})$ *and* $L_1^{p,q}(\Sigma_T)$ *similarly, using the norm*

$$\|\vec{v}\|_{L_1^{p,q}(\partial\Omega\times\mathbb{R})} \equiv \|\nabla_{\tan}\vec{v}\|_{L^{p,q}(\partial\Omega\times\mathbb{R})} + \|\partial_t^{1/2}\vec{v}\|_{L^{p,q}(\partial\Omega\times\mathbb{R})} + \|\vec{v}\|_{L^{p,q}(\partial\Omega\times\mathbb{R})}.$$

We close this section with the following perturbation theorem which will be very useful to us.

Theorem 1.9. *Let* M *be a measure space with a positive measure* ν. *Let* $0 < \sigma_0 < 1/2$ *and* S: $L^p(M,\nu) \to L^p(M,\nu)$ *be a bounded linear operator, with norm bounded by* B, *for* $|1/2 - 1/p| < \sigma_0$. *Also assume that* $S: L^2 \to L^2$ *in invertible and* $\|S^{-1}\|_{(2,2)} \le A$. *Then, there exist* $\delta > 0$, $C > 0$ *depending only on* σ_0, A *and* B, *such that,* S: $L^p \to L^p$ *is invertible and* $\|S^{-1}\|_{(p,p)} \le C$ *for* $|1/2 - 1/p| < \delta$.

We remark that Theorem 1.9 is a consequence of more general results of G. David and S. Semmes (unpublished). Similar results also have been given by A. P. Calderón [C]. Theorem 1.9 may be proven by applying Calderón's result to the positive operators S^*S and SS^*. See also W. Cao and Y. Sagher [CS] for a more general result.

2. Layer Potentials.

In this section we study the boundary layer potentials on mixed norm spaces. In particular, we show that the double layer potential on the lateral boundary is bounded on $L^{p,q}(\Sigma_T)$ for $1 < p, q < \infty$ and is invertible on $L^{p,q}(\Sigma_T)$ for $p \in (2-\varepsilon_0, 2]$, $q \in (1,2]$ where $\varepsilon_0 > 0$. These results are used to solve the initial-Dirichlet problem with data in $L^{p,q}(\Sigma_T)$ for $p \in (2-\varepsilon_0, 2)$ and $q \in (1,2]$.

We begin by introducing a matrix of fundamental solutions for the system (0.1), $\Gamma(X,t) = \big(\Gamma_{jk}(X,t)\big)_{3\times 3}$ where

$$\Gamma_{jk}(X,t) = \delta_{jk}\frac{e^{-|X|^2/(4\pi\mu t)}}{(4\pi\mu t)^{3/2}} + \frac{\partial^2}{\partial x_j \partial x_k}\int_{\mu t}^{(\lambda+2\mu)t} \frac{e^{-|X|^2/(4\pi s)}}{(4\pi s)^{3/2}}\,ds$$

when $t > 0$ and $\Gamma_{jk}(X,t) = 0$ for $t \le 0$. We have the following estimate

$$\Big|\frac{\partial^{|\alpha|+\alpha_0}}{\partial X^\alpha \partial t^{\alpha_0}}\,\Gamma(X,t)\Big| \le \frac{C}{(|X| + |t|^{1/2})^{3+|\alpha|+2\alpha_0}}. \tag{2.1}$$

Given $\vec{f} \in L^q(\mathbb{R}, L^p(\partial\Omega))$, $1 < p < \infty$, $1 < q < \infty$, define the single layer potential

$$\vec{u} = \mathcal{S}(\vec{f})(X,t) = \int_{-\infty}^{t} \int_{\partial\Omega} \Gamma(X-Q, t-s)\vec{f}(Q,s)dQds\,. \tag{2.2}$$

Let

$$\frac{\partial \vec{u}}{\partial \nu} = \lambda(\operatorname{div} \vec{u})N \;+\; \mu\big((\nabla\vec{u}) + (\nabla\vec{u})^{tr}\big)N \tag{2.3}$$

be the traction operator on the lateral boundary. If $\vec{u} = \mathcal{S}(\vec{f})$, then

$$\frac{\partial \vec{u}_{\pm}}{\partial \nu} = \big(\pm\frac{1}{2}I + K_\nu\big)\vec{f} \qquad \text{on } \partial\Omega \times \mathbb{R}\,,$$

where the subscripts $+$ and $-$ indicate nontangential limits taken from $\Omega \times \mathbb{R}$ and from $\overline{\Omega} \times \mathbb{R}$ respectively, I denotes the identity operator and

$$K_\nu \vec{f}(P,t) = \text{p.v.} \int_{-\infty}^{t} \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu(P)}(P-Q, t-s)\vec{f}(Q,s)dQds$$

is a singular integral (see [S]).

Lemma 2.4. *K_ν is bounded on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for any $1 < p, q < \infty$.*

PROOF. First, if $p = q$, $1 < p < \infty$, the lemma follows from the theorem of R. Coifman, A. McIntosh and Y. Meyer [CMM] on the boundedness of the Cauchy integral on Lipschitz curves and the argument of E. Fabes and N. Rivière [FR, Theorem 1.1].

Next, consider the case of $1 < q < p$. Let $B = L^p(\partial\Omega)$. We may view K_ν as a singular integral operator on B-valued functions on $\mathbb{R}^1$. Write

$$K_\nu \vec{f}(t) = \text{p.v.} \int_{-\infty}^{\infty} K_\nu(t-s)\vec{f}(s)ds\,, \quad \vec{f} \in L^q(\mathbb{R}, B)\,.$$

Using estimate (2.1), it is not difficult to see that, for $\vec{g} \in B$,

$$|K_\nu(t)\vec{g}|_B \le \frac{C}{t}\,|M_{\partial\Omega}(\vec{g})|_B$$

and

$$|\frac{d}{dt}K_\nu(t)\vec{g}|_B \leq \frac{C}{t^2}\,|M_{\partial\Omega}(\vec{g})|_B$$

where $M_{\partial\Omega}$ denotes the Hardy-Littlewood maximal function on $\partial\Omega$. Thus,

$$\|K_\nu(t)\|_{B\to B} \leq \frac{C}{t} \quad \text{and} \quad \|\frac{d}{dt}K_\nu(t)\|_{B\to B} \leq \frac{C}{t^2}\,.$$

Therefore, $K_\nu(t)$ is a standard Calderón-Zygmund kernel. Since $K_\nu : L^p(\mathbb{R},B) \to L^p(\mathbb{R},B)$ is bounded, we have $K_\nu : L^q(\mathbb{R},B) \to L^q(\mathbb{R},B)$ is bounded for $1 < q < p$, by the standard Calderón-Zygmund argument.

Finally, note that the same argument as above also shows that K_ν^*, the adjoint of K_ν, is bounded on $L^q\big(\mathbb{R},L^p(\partial\Omega)\big)$ for $1 < q \leq p$. Thus, by duality, we obtain the boundedness of K_ν on $L^q(\mathbb{R},L^p(\partial\Omega))$ when $q > p$.

For $\vec{f} \in L^q\big(\mathbb{R},L^p(\partial\Omega)\big)$, define the double layer potential

$$\begin{aligned}\vec{v}(X,t) &= \mathcal{K}(\vec{f})(X,t)\\ &= \int_{-\infty}^{t}\int_{\partial\Omega}\left\{\frac{\partial}{\partial\nu_Q}\Gamma(X-Q,t-s)\right\}^{tr}\vec{f}(Q,s)\,dQds\,.\end{aligned} \tag{2.5}$$

Then $\vec{v}_\pm = (\mp\frac{1}{2}I + \widetilde{K}_\nu)\vec{f}$ on $\partial\Omega\times\mathbb{R}$ where $\widetilde{K}_\nu = RK_\nu^*R$ and R is the reflection defined by $R(\vec{f})(P,t) = \vec{f}(P,-t)$.

Lemma 2.6. *Let $1 < p,q < \infty$ and $\vec{f} \in L^q(\mathbb{R},L^p(\partial\Omega))$.*
(i) *Let $\vec{u} = \mathcal{S}(\vec{f})$, then*

$$\|(\nabla\vec{u})^*\|_{p,q} + \|(\partial_t^{1/2}\vec{u})^*\|_{p,q} \leq C\,\|\vec{f}\|_{p,q}\,,$$

where $\|\cdot\|_{p,q}$ denotes the norm in $L^q\big(\mathbb{R},L^p(\partial\Omega)\big)$.
(ii) *Let $\vec{v} = \mathcal{K}(\vec{f})$, then*

$$\|(\vec{v})^*\|_{p,q} \leq C\,\|\vec{f}\|_{p,q}\,.$$

PROOF. Using the argument of Fabes-Rivière [FR, Theorem 1.11], we can show that for any $q_0 > 1$,

$$(\nabla\vec{u})^*(P,t) \leq C_{q_0}\Big[M_1\big(M_{\partial\Omega}(Kf)\big)(P,t) + M_1\big(M_{\partial\Omega}(|f|^{q_0})\big)^{1/q_0}(P,t)\Big]$$

where K is a singular integral operator of the same type as K_ν and M_1 is the Hardy-Littlewood maximal function on $\mathbb{R}^1$. It follows from the argument in the proof of Lemma 2.4 that K is bounded on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for $1 < p, q < \infty$. Also, note that an easy modification of the argument in [FSt] by C. Fefferman and E. M. Stein yields that M_1 is bounded on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for $1 < p, q < \infty$. The estimate for $(\nabla \vec{u})^*$ then follows easily. The estimates for $(\partial_t^{1/2} \vec{u})^*$ and $(\vec{v})^*$ follow in a similar manner. We omit the details.

Given $\vec{f} \in L^{p,q}(\Sigma_T)$, let $\vec{g}$ denote the extension of $\vec{f}$ by zero to $\partial\Omega \times \mathbb{R}$. We define $K_\nu \vec{f} \equiv K_\nu \vec{g}$. Clearly, $\pm\frac{1}{2} I + K_\nu$ is bounded on $L^{p,q}(\Sigma_T)$ for $1 < p, q < \infty$. From [S, Theorem 4.3.1], we know that $\pm\frac{1}{2} I + K_\nu$ is also invertible on $L^2(\Sigma_T)$. Using Theorem 1.9 and an $L^p(\partial\Omega)$-valued multiplier argument, we are able to extend this result.

Theorem 2.7. *There exists $\varepsilon_0 > 0$ such that $\pm\frac{1}{2} I + K_\nu$ is invertible on $L^{p,q}(\Sigma_T)$ for $p \in [2, 2 + \varepsilon_0)$ and $q \in [2, \infty)$.*

PROOF. We give the proof for $\frac{1}{2} I + K_\nu$. The invertibility of $-\frac{1}{2} I + K_\nu$ follows in the same manner.

Let $a \geq 0$, consider the parabolic system

$$\frac{\partial \vec{u}}{\partial t} + a\vec{u} = \mu \Delta \vec{u} + (\lambda + \mu) \nabla(\operatorname{div} \vec{u}). \tag{2.8}$$

Note that $\Gamma_a(X, t) = e^{-at}\Gamma(X, t)$ is a matrix of fundamental solutions for (2.8). For $\vec{f} \in L^q(\mathbb{R}, L^p(\partial\Omega))$, let

$$\mathcal{S}^a(\vec{f})(X, t) = \int_{-\infty}^{t} \int_{\partial\Omega} \Gamma_a(X - Q, t - s) \vec{f}(Q, s) dQ ds\,.$$

Clearly,

$$\frac{\partial}{\partial \nu} \mathcal{S}^a_{\pm}(\vec{f}) = \left(\pm \frac{1}{2} I + K^a_\nu\right)(\vec{f}) \quad \text{a.e. on } \partial\Omega \times \mathbb{R},$$

where $K^a_\nu(\vec{f}) = e^{-at} K_\nu e^{at}(\vec{f})$. Thus, to show $\frac{1}{2} I + K_\nu$ is invertible on $L^{p,q}(\Sigma_T)$, it suffices to show $\frac{1}{2} I + K^a_\nu$ is invertible on $L^{p,q}(\Sigma_T)$.

First, we shall show that if a is large, then $\frac{1}{2} I + K^a_\nu$ is invertible on $L^q(\mathbb{R}, L^p(\partial\Omega))$ for p near 2 and $q \in (1, \infty)$. Let $S_a = \frac{1}{2} I + K^a_\nu$ and

$$K_\nu(P, Q, t - s) = \frac{\partial \Gamma}{\partial \nu(P)}(P - Q, t - s)\,.$$

Then

$$S_a\vec{f}(P,t) = \frac{1}{2}\vec{f}(P,t) + \text{p.v.}\int_{-\infty}^{t}\int_{\partial\Omega} e^{-a(t-s)}K_\nu(P,Q,t-s)\vec{f}(Q,s)dQds\,.$$

Taking the partial Fourier transform in the t variable, we obtain

$$(S_a\vec{f})^\wedge(P,i\tau) = \frac{1}{2}(\vec{f})^\wedge(P,i\tau) + \text{p.v.}\int_{\partial\Omega}(K_\nu)^\wedge(P,Q,-a+i\tau)(\vec{f})^\wedge\,(Q,i\tau)\,dQ$$

where

$$(\vec{f})^\wedge(P,i\tau) = \int_{\mathbb{R}} e^{i\tau t}\vec{f}(P,t)dt, \qquad \tau\in\mathbb{R}$$

and

$$(K_\nu)^\wedge(P,Q,-a+i\tau) = \int_{\mathbb{R}} e^{(-a+i\tau)t}\,K_\nu(P,Q,t)dt, \quad \tau\in\mathbb{R}\,.$$

For $z\in\mathbb{C}, \operatorname{Re} z\le 0$, let $m(z): L^p(\partial\Omega)\to L^p(\partial\Omega)$ be defined by

$$m(z)(\vec{h})(P) = \frac{1}{2}\,\vec{h}(P) + \text{p.v.}\int_{\partial\Omega}(K_\nu)^\wedge(P,Q,z)\vec{h}(Q)dQ$$

for $\vec{h}\in L^p(\partial\Omega)$. Then

$$(S_a\vec{f})^\wedge(P,i\tau) = m\,(-a+i\tau)\big((\vec{f})^\wedge(\cdot,i\tau)\big)(P)\,.$$

It follows from the theorem of R. Coifman, A. McIntosh and Y. Meyer [CMM] that $m(-a+i\tau)$ is bounded on $L^p(\partial\Omega)$ for $1<p<\infty$ and $\|m(-a+i\tau)\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)}\le C_a$ with constant C_a independent of τ. Moreover, using estimate (2.1), it is not difficult to see that

$$\Big|\frac{d^\alpha}{d\tau^\alpha}m\,(-a+i\tau)\vec{h}\Big| \le \frac{C_a}{(1+|\tau|)^\alpha}\,M_{\partial\Omega}(\vec{h})$$

for integer $\alpha\ge 1$. Hence,

$$\Big\|\frac{d^\alpha}{d\tau^\alpha}m\,(-a+i\tau)\Big\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \le \frac{C_a}{(1+|\tau|)^\alpha} \tag{2.9}$$

for integer $\alpha \geq 0$. On the other hand, it follows from the Rellich identity (see [S, Lemma 4.3.13], also [BS1, Proposition 2.2]) that, if $a \geq C_0 > 0$, then

$$\|\vec{h}\|_{L^2(\partial\Omega)} \leq C \, \|m(-a+i\tau)\vec{h}\|_{L^2(\partial\Omega)} \, . \tag{2.10}$$

Now, since $m(0)$ is a Fredholm operator on $L^2(\partial\Omega)$ with index zero [DKV2, Theorem 2.7] and $m(-a+i\tau) - m(0)$ is compact on $L^2(\partial\Omega)$ by (2.1), we see that $m(-a+i\tau)$ is also a Fredholm operator with index zero. Thus, by (2.10), $m(-a+i\tau)$ is invertible on $L^2(\partial\Omega)$ and $\|m^{-1}(-a+i\tau)\|_{L^2(\partial\Omega)\to L^2(\partial\Omega)} \leq C$ for $a \geq C_0$. This implies that S_a is invertible on $L^2(\partial\Omega \times \mathbb{R})$, hence on $L^p(\partial\Omega \times \mathbb{R})$ for $|p-2| < \varepsilon_1$, by Theorem 1.9. It also follow from Theorem 1.9 that, $m(-a+i\tau)$ is invertible on $L^p(\partial\Omega)$ and $\|m^{-1}(-a+i\tau)\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \leq C_a$ for $|p-2| < \varepsilon_2$ where $C_a > 0$ and $\varepsilon_2 > 0$ are independent of τ. Moreover, by (2.9) and (2.10), we have

$$\|\frac{d^\alpha}{d\tau^\alpha} m^{-1}(-a+i\tau)\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \leq \frac{C_a}{(1+|\tau|)^\alpha} \tag{2.11}$$

for $|p-2| < \varepsilon_2$ and integer $\alpha \geq 0$.

To proceed, we let $B = L^p(\partial\Omega)$, $|p-2| < \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Since

$$(S_a^{-1}\vec{f})\hat{}\,(P, i\tau) = m^{-1}(-a+i\tau)\big((\vec{f})\hat{}\,(\cdot, i\tau)\big)(P)$$

for $\vec{f} \in L^2(\partial\Omega \times \mathbb{R})$, it follows from a standard argument and (2.11) that S_a^{-1} is associated with an $L(B)$-valued Calderón-Zygmund kernel where $L(B)$ denotes the bounded linear operators on B. Since $S_a^{-1} : L^p(\mathbb{R}, B) \to L^p(\mathbb{R}, B)$ is bounded, a standard Calderón-Zygmund argument yields that $S_a^{-1} : L^q(\mathbb{R}, B) \to L^q(\mathbb{R}, B)$ is bounded for $1 < q < p$, *i.e.* $\frac{1}{2} I + K_\nu^a : L^q\big(\mathbb{R}, L^p(\partial\Omega)\big) \to L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$ is invertible for $|p-2| < \varepsilon_0$ and $1 < q < p$. The case where $p < q < \infty$ follows by duality.

Finally, to see that $\frac{1}{2} I + K_\nu^a$ is invertible on $L^{p,q}(\Sigma_T)$ for $p \in [2, 2+\varepsilon_0)$ and $q \in [2, \infty)$, let $\vec{g} \in L^{p,q}(\Sigma_T)$, and $\vec{G}$ be the extension of $\vec{g}$ by zero to $(\partial\Omega \times \mathbb{R})$. Since $\vec{G} \in L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$, there exists $\vec{f} \in L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$ such that $(\frac{1}{2} I + K_\nu^a)\vec{f} = \vec{G}$ on $(\partial\Omega \times \mathbb{R})$. We wish to show that $\vec{f}$ vanishes when $t < 0$.

To proceed, we let $\vec{u} = \mathcal{S}^a(\vec{f})$. It follows from (2.8) and integration by parts that

$$\begin{aligned} &\int_{T_0}^t \int_{\Omega_j} a|\vec{u}|^2 \; + \frac{1}{2}\int_{\Omega_j} |\vec{u}(X,t)|^2 - |\vec{u}(X,T_0)|^2 dX \\ &\qquad + \int_{T_0}^t \int_{\Omega_j} \lambda\,|\operatorname{div}\vec{u}|^2 \; + \; \frac{1}{2}\,|\nabla\vec{u} \; + \; (\nabla\vec{u})^{tr}|^2 \\ &= \int_{T_0}^t \int_{\partial\Omega_j} \frac{\partial\vec{u}}{\partial\nu}\cdot\vec{u} \end{aligned} \tag{2.12}$$

where $T_0 < t < 0$ and $\bar{\Omega}_1 \subset \bar{\Omega}_2 \subset \cdots \subset \Omega$ is a sequence of smooth domains approximating Ω, [V]. Let $j \to \infty$ in (2.12), since $\|(\vec{u})^*\|_2 + \|(\nabla\vec{u})^*\|_2 \le C\,\|\vec{f}\|_2 \le C_{T_0}\,\|\vec{f}\|_{p,q}$, we obtain

$$\int_\Omega |\vec{u}(X,t)|^2 dX \le \int_\Omega |\vec{u}(X,T_0)|^2 dX\,.$$

Integrating in T_0, we have

$$\begin{aligned} \int_\Omega |\vec{u}(X,t)|^2 dX &\le \int_{T_0-1}^{T_0}\int_\Omega |\vec{u}(X,s)|^2 dX ds \\ &\le C\,\|(\vec{u})^*\|_{L^{p,q}\left(\partial\Omega\times(T_0-1,T_0)\right)} \\ &\le C\,\|\vec{f}\|_{L^{p,q}\left(\partial\Omega\times(-\infty,T_0)\right)} \end{aligned}$$

which goes to zero as $T_0 \to -\infty$. Hence, $\vec{u} \equiv \vec{0}$ in $\Omega\times(-\infty,0)$. Thus, $\vec{u} = \vec{0}$ a.e. on $\partial\Omega\times(-\infty,0)$. This, together with a similar argument in the complement, implies that $\vec{u} \equiv \vec{0}$ in $\overline{\Omega}^c \times (-\infty,0)$. It then follows that

$$(-\frac{1}{2}I + K_\nu)\vec{f} = \frac{\partial\vec{u}_-}{\partial\nu} = \vec{0} \quad \text{on } \partial\Omega\times(-\infty,0)\,.$$

Therefore,

$$\vec{f} = (\frac{1}{2}I + K_\nu) - (-\frac{1}{2}I + K_\nu)\vec{f} = \vec{0} \quad \text{on } \partial\Omega\times(-\infty,0)\,.$$

From Theorem 2.7 and its proof, we draw three corollaries.

Corollary 2.13. *There exists $\varepsilon_0 > 0$, such that, given any $\vec{g} \in L^{p,q}(\Sigma_T)$, $p \in [2, 2+\varepsilon_0)$, $q \in [2,\infty)$, there exists a unique $\vec{u}$ satisfying* (0.1), (ITP) *and* $\|(\nabla \vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. *Moreover, we have*

$$\|(\nabla \vec{u})^*_{L^{p,q}(\Sigma_T)} + \|(\partial_t^{1/2} \vec{u})^*\|_{L^{p,q}(\Sigma_T)} \leq C \|\vec{g}\|_{L^{p,q}(\Sigma_T)} .$$

PROOF. The existence follows from the invertibility of $\frac{1}{2} I + K_\nu$ on $L^{p,q}(\Sigma_T)$ for $p \in [2, 2+\varepsilon_0)$ and $q \in [2,\infty)$.

To see the uniqueness, let $\vec{u}$ be a solution of (0.1) in Ω_T such that, $\partial \vec{u}/\partial \nu = \vec{0}$ on Σ_T and $\|(\nabla \vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. Let $\{\Omega_j\}$ be a sequence of smooth domain approximating Ω. Since $\vec{u}$ is smooth on $\overline{\Omega}_j \times [0,T]$, by the existence and uniqueness results for the traction problem with L^2 data [S], there exists $\vec{f}_j \in L^2\big(\partial\Omega_j \times (0,T)\big)$, such that

$$\vec{u}(X,t) = \int_0^t \int_{\Omega_j} \Gamma(X-Q, t-s) \vec{f}_j(Q,s)\, dQ\, ds \tag{2.14}$$

for $(X,t) \in \Omega_j \times (0,T)$. Thus,

$$(\frac{1}{2} I + K_{\nu,j}) \vec{f}_j = \frac{\partial \vec{u}}{\partial \nu} \quad \text{on } \partial\Omega_j \times (0,T) ,$$

where $K_{\nu,j}$ is the corresponding operator on $\partial\Omega_j \times (0,T)$. Hence, by Theorem 2.7,

$$\|\vec{f}_j\|_{L^{p,q}(\partial\Omega_j \times (0,T))} \leq C \left\| \frac{\partial \vec{u}}{\partial \nu} \right\|_{L^{p,q}(\partial\Omega_j \times (0,T))} \to 0$$

as $j \to \infty$ where we used the assumption

$$\frac{\partial \vec{u}}{\partial \nu} = \vec{0} \quad \text{on } \Sigma_T \qquad \text{and} \qquad \|(\nabla \vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty .$$

It then follows from (2.14) that $\vec{u} \equiv \vec{0}$ on Ω_T and the uniqueness is proved.

Corollary 2.15. *There exists $\varepsilon_1 > 0$, such that, given any $\vec{g} \in L^{p,q}(\Sigma_T)$, $p \in (2-\varepsilon_1, 2]$ and $q \in (1,2]$, there exists a unique $\vec{u}$ satisfying* (0.1), (IDP) *and* $\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. *Moreover,*

$$\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \leq C \|\vec{g}\|_{L^{p,q}(\Sigma_T)} .$$

PROOF. Note that, restricting $\widetilde{K}_\nu$ to functions supported on Σ_T, we have $(-\frac{1}{2}I + \widetilde{K}_\nu)\vec{f} = R_T(-\frac{1}{2}I + K_\nu^*)R_T(\vec{f})$ where $\vec{f} \in L^{p,q}(\Sigma_T)$ and $R_T(\vec{f})(P,t) = \vec{f}(P, T-t)$. Hence, by Theorem 2.7 and duality, $-\frac{1}{2}I + \widetilde{K}_\nu$ is invertible on $L^{p,q}(\Sigma_T)$, $p \in (2-\varepsilon_1, 2]$, $q \in (1,2]$ for some $\varepsilon_1 > 0$. The existence follows. The uniqueness follows from an approximation argument, similar to that in the proof of Corollary 2.13.

Corollary 2.16. *There exists $\varepsilon_2 > 0$, such that, given any $\vec{g} \in L_1^{p,q}(\Sigma_T)$, $p \in [2, 2+\varepsilon_2)$, $q \in [2,\infty)$, there exists a unique $\vec{u}$ satisfying* (0.1), (IDP) *and $\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} + \|(\nabla\vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty$. Moreover,*

$$\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} + \|(\nabla\vec{u})^*\|_{L^{p,q}(\Sigma_T)} + \|(\partial_t^{1/2}\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \le C\,\|\vec{g}\|_{L_1^{p,q}(\Sigma_T)}.$$

PROOF. Clearly, the uniqueness follows from the uniqueness for the initial Dirichlet problem with $L^2(\Sigma_T)$ data. To see the existence, we shall show that $\mathcal{S}\big|_{\Sigma_T} : L^{p,q}(\Sigma_T) \to L_1^{p,q}(\Sigma_T)$ is invertible for $p \in [2, 2+\varepsilon)$, $q \in [2,\infty)$ where $\varepsilon_2 > 0$.

Let

$$V(\vec{f}) = \left(\nabla_{\tan}\mathcal{S}(\vec{f}),\ \partial_t^{1/2}\mathcal{S}(\vec{f}),\ \mathcal{S}(\vec{f})\right). \tag{2.17}$$

Then $V : L^q\big(\mathbb{R}, L^p(\partial\Omega)\big) \to \big[L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)\big]^5$ is bounded for $1 < p, q < \infty$. Taking the partial Fourier transform in the t variable in (2.17), as in the proof of Theorem 2.7, we obtain

$$(V\vec{f})^\wedge(P, i\tau) =$$
$$\Big(\nabla_{\tan}\hat{\mathcal{S}}_{i\tau}\big((\vec{f})^\wedge(\cdot,i\tau)\big)(P),\ (i\tau)^{1/2}\hat{\mathcal{S}}_{i\tau}\big((\vec{f})^\wedge(\cdot,i\tau)\big),\ \hat{\mathcal{S}}_{i\tau}\big((\cdot,i\tau)\big)(P)\Big)$$

where $\hat{\mathcal{S}}_{i\tau}$ is defined by

$$\hat{\mathcal{S}}_{i\tau}(\vec{h})(P) = \int_{\partial\Omega} (\Gamma)^\wedge(P, Q, i\tau)\vec{h}(Q)\,dQ, \quad \text{for } \vec{h} \in \quad '(\partial\Omega).$$
$$\Xi\ L^p$$

Let

$$m_{i\tau}(\vec{h})(P) = \Big(\nabla_{\tan}\hat{\mathcal{S}}_{i\tau}(\vec{h})(P),\ (i\tau)^{1/2}\hat{\mathcal{S}}_{i\tau}(\vec{h})(P),\ \hat{\mathcal{S}}_{i\tau}(\vec{h})(P)\Big).$$

Then,

$$\big(V(\vec{f})\big)^\wedge(P, i\tau) = m_{i\tau}\big((\vec{f})^\wedge(\cdot,i\tau)\big)(P$$

From [S], we know that

$$\| m_{i\tau}(\vec{h}) \|_{\left[L^p(\partial\Omega)\right]^5} \leq C \|\vec{h}\|_{L^p(\partial\Omega)}, \qquad 1 < p < \infty,$$

and

$$\|\vec{h}\|_{L^2(\partial\Omega)} \leq C \|m_{i\tau}(\vec{h})\|_{L^2(\partial\Omega)}$$

with constant C independent of τ. Moreover, using estimate (2.1), we have

$$\|\frac{d^\alpha}{d\tau^\alpha} m_{i\tau}\|_{L^p(\partial\Omega)\to L^p(\partial\Omega)} \leq \frac{C}{|\tau|^\alpha} \quad \text{for integer } \alpha \geq 0.$$

Now, consider the operator V^*V. The same argument as in the proof of Theorem 2.7, implies that there exists $\varepsilon_2 > 0$, such that, $V^*V : L^q\big(\mathbb{R}, L^p(\partial\Omega)\big) \to L^q\big(\mathbb{R}, L^p(\partial\Omega)\big)$ is invertible for $p \in (2-\varepsilon_2, 2+\varepsilon_2)$ and $1 < q < \infty$. It follows that

$$\begin{aligned}
\|\vec{f}\|_{L^q(\mathbb{R},L^p(\partial\Omega))} &\leq C \|V^*V(\vec{f})\|_{L^q(\mathbb{R},L^p(\partial\Omega))} \\
&\leq C \|V(\vec{f})\|_{\left[L^q(\mathbb{R},L^p(\partial\Omega))\right]^5} \\
&\leq C \|\mathcal{S}(\vec{f})\|_{L_1^{p,q}(\partial\Omega\times\mathbb{R})} .
\end{aligned}$$

By the definition of $L_1^{p,q}(\Sigma_T)$ norm, we have

$$\|\vec{f}\|_{L^{p,q}(\Sigma_T)} \leq C \|\mathcal{S}(\vec{f})\|_{L^{p,q}(\Sigma_T)} .$$

This implies that $\mathcal{S}\big|_{\Sigma_T}: L^{p,q}(\Sigma_T) \to L_1^{p,q}(\Sigma_T)$ is injective and has closed range. An approximation argument shows that the range is dense (see [S] for the case $p = q = 2$). Hence, we are done.

3. Main Lemma.

In this section, we will consider solutions to the initial-traction problem with atomic data on the lateral boundary. Our main result is Lemma 3.6 where we show that the nontangential maximal function of the gradient of such a solution is in L^1.

We begin with some notations. For $P \in \partial\Omega$, $t \in \mathbb{R}$ and $R > 0$ small, let

$$\begin{aligned} J_R(P) &= \{Q \in \partial\Omega : \quad |Q - P| < R\}, \\ \Delta_R(P,t) &= J_R(P) \times (t - R^2, t + R^2), \\ D_R(P) &= \{X \in \Omega : |X - P| < R\}, \\ Z_R(P,t) &= D_R(P) \times (t - R^2, t + R^2). \end{aligned}$$

For a function $\vec{u}$ on Ω_T, we define

$$(\vec{u})^*_R(P,t) = \sup_{\substack{(Y,s)\in\gamma(P,t) \\ |Y-P|\le R}} |\vec{u}(Y,s)|$$

and

$$(\vec{u})^{*R}(P,t) = \sup_{\substack{(Y,s)\in\gamma(P,t) \\ |Y-P|\ge R}} |\vec{u}(Y,s)|.$$

Clearly, $(\vec{u})^*(P,t) \le (\vec{u})^*_R(P,t) + (\vec{u})^{*R}(P,t)$.

Lemma 3.1. (Cacciopoli Inequalities) *Suppose $\vec{u}$ is a solution of* (0.1) *in $Z_{2R} = Z_{2R}(P_0,t_0)$. Assume $\partial\vec{u}/\partial\nu = \vec{0}$ on $\Delta_{2R} = \Delta_{2R}(P_0,t_0)$ and $\|(\vec{u})^*_R\|_{L^2(\Delta_{2R})} < \infty$. Then*

(i) $$\iint_{Z_{3R/2}} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2 \le \frac{C}{R^2} \iint_{Z_{2R}} |\vec{u}|^2,$$

and

(ii) $$\iint_{Z_{3R/2}} |\partial_t^{1/2}(\vec{u}\eta)|^2 \le \frac{C}{R^2} \iint_{Z_{2R}} |\vec{u}|^2$$

where $\eta = \eta(t) \in C_0^\infty(\mathbb{R})$, $\eta \equiv 1$ on $(t_0 - R^2, t_0 + R^2)$, $\operatorname{supp}\eta \subset (t_0 - 2R^2, t_0 + 2R^2)$ and $|\eta'(t)| \le C/R^2$.

PROOF. By a parabolic rescaling, we may assume that $R = 1$. We write

$$\mu\,\Delta\vec{u} + (\lambda+\mu)\nabla(\operatorname{div}\vec{u}) = a_{ij}^{kl}\,\frac{\partial^2 u^l}{\partial x_i \partial x_j}$$

where $a_{ij}^{kl} = a_{ji}^{lk}$ are constants such that

$$a_{ij}^{kl}\,\frac{\partial u^l}{\partial x_j} N_i \;=\; \Big(\frac{\partial \vec{u}}{\partial \nu}\Big)^k$$

(see [DKV2, p. 800]). Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \equiv 1$ on $D_{3/2}$, $\operatorname{supp}\psi \cap \Omega \subset D_2$ and $|\nabla\psi| \le C$. It follows from equation (0.1), the vanishing of $\partial\vec{u}/\partial\nu$ on Δ_2 and integration by parts that

$$\int_{D_2} \frac{\partial u^k}{\partial t}\,u^k(\psi\eta)^2 + \int_{D_2} a_{ij}^{kl}\,\frac{\partial u^k}{\partial x_i}\,\frac{\partial u^l}{\partial x_j}(\psi\eta)^2 = -\int_{D_2} a_{ij}^{kl}\frac{\partial u^l}{\partial x_j}\,u^k\,2\psi\,\frac{\partial\psi}{\partial x_i}\,\eta^2.$$

Integrating in t, we obtain

$$\begin{aligned}\iint_{Z_2} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2(\psi\eta)^2 &\le C\iint_{Z_2} a_{ij}^{kl}\,\frac{\partial u^k}{\partial x_i}\,\frac{\partial u^l}{\partial x_j}(\psi\eta)^2\\ &\le C\iint_{Z_2} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|\;|\vec{u}|\;|\psi\eta|\;|\nabla\psi|\;|\eta|\\ &\quad + C\iint_{Z_2} |\vec{u}|^2.\end{aligned}$$

From this, (i) follows easily.

To see part (ii), we integrate by parts and obtain

$$\begin{aligned}&\iint_{Z_2} a_{ij}^{kl}\,\frac{\partial}{\partial x_i}\,I_{1/4}(u^k\eta)\,\frac{\partial}{\partial x_j}\,\partial_t^{3/4}(u^l\eta)\,\psi^2\\ &\qquad = -\iint_{Z_2} I_{1/4}\,\Big(\frac{\partial u^l}{\partial t}\,\eta\Big)\partial_t^{3/4}(u^l\eta)\,\psi^2\\ &\qquad\quad - \iint_{Z_2} a_{ij}^{kl}\,\frac{\partial}{\partial x_i}\,(I_{1/4}(u^k\eta))\partial_t^{3/4}(u^l\eta)\,\frac{\partial}{\partial x_j}\,(\psi^2).\end{aligned}$$

Note that

$$\begin{aligned}&\iint_{Z_2} a_{ij}^{kl}\,\frac{\partial}{\partial x_i}\,I_{1/4}(u^k\eta)\,\frac{\partial}{\partial x_j}\,\partial_t^{3/4}\,(u^l\eta)\,\psi^2\\ &\qquad = \frac{1}{2}\iint_{Z_2}\frac{\partial}{\partial t}\Big(a_{ij}^{kl}\,\frac{\partial}{\partial x_i}\,I_{1/4}(u^k\eta)\,\frac{\partial}{\partial x_j}\,I_{1/4}\,(u^l\eta)\Big)\psi^2 \ge 0\end{aligned}$$

and

$$\iint_{Z_2} I_{1/4}\Big(\frac{\partial u^l}{\partial t}\eta\Big)\partial_t^{3/4}(u^l\eta)\ \psi^2 = \iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2 - \iint_{Z_2} I_{1/4}\Big(u^l\frac{\partial \eta}{\partial t}\Big)\partial_t^{3/4}(u^l\eta)\ \psi^2.$$

We have

$$\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2 \le C\Big(\iint_{Z_2} |\nabla\vec{u} + (\nabla\vec{u})^{tr}|^2\Big)^{1/2}\Big(\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2\Big)^{1/2} + C\Big(\iint_{Z_2} |\vec{u}|^2\Big)^{1/2}\Big(\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2\Big)^{1/2}.$$

Thus,

$$\iint_{Z_2} |\partial_t^{1/2}(\vec{u}\eta)|^2\psi^2 \le C\iint_{Z_2} |\partial_t^{3/4}(\vec{u}\eta)|^2\psi^2 \quad \le C\iint_{Z_2} |\vec{u}|^2.$$

Part (ii) then follows.

Lemma 3.2. *Under the same assumptions as in Lemma* 3.1, *we have*

(i) $$\iint_{Z_R} |\nabla\vec{u}|^2 \le \frac{C}{R^2}\iint_{Z_{2R}} |\vec{u}|^2$$

(ii) $$\sup_{|t-t_0|<R^2}\Big(\int_{D_R} |\vec{u}(X,t)|^{5/2}dX\Big)^{2/5} \le \frac{C}{R^{13/10}}\Big(\iint_{Z_{2R}} |\vec{u}|^2\Big)^{1/2}$$

(iii) $$\iint_{\Delta_R} (\nabla\vec{u})^*_R + \iint_{\Delta_R} \big(\partial_t^{1/2}(\vec{u}\eta)\big)^*_R \le C\,R^{1/2}\Big(\iint_{Z_{2R}} |\vec{u}|^2\Big)^{1/2}$$

where $\eta \in C_0^\infty(\mathbb{R}), \eta \equiv 1$ *on* $[t_0 - R^2, t_0 + R^2]$, supp $\eta \subset [t_0 - \frac{4}{3}R^2, t_0 + \frac{4}{3}R^2]$ *and* $|\eta'(t)| \le C/R^2$.

PROOF. By translation and rescaling, we may assume that $(P_0, t_0) = (0,0)$ and $R = 1$. Fix $\tau \in [1, 3/2]$, let $D_\tau = D_\tau(0)$ and then set

$$\vec{u}_\tau(X,t) = \int_{-4}^{t}\int_{D_\tau} \Gamma(X - Y, t - s)\vec{u}(Y,s)\eta'(s)dYds$$

and $\vec{v} = \vec{u}_\tau - \vec{u}\eta$. Then

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} = \mu\,\Delta\vec{v} + (\lambda+\mu)\nabla(\operatorname{div}\vec{v}) & \text{in } D_\tau \times (-4,4)\,, \\ \vec{v}\big|_{t=-4} = \vec{0}\,, \end{cases}$$

and $\|(\nabla\vec{u})^*\|_{L^2(\partial D_\tau \times (-4,4))} < \infty$. Hence, by the L^2-theory for the traction problem in [S], there exists $\vec{g}_\tau \in L^2(\partial D_\tau \times (-4,4))$, such that

$$\vec{v}(X,t) = \int_{-4}^{t}\int_{\partial D_\tau} \Gamma(X-Q, t-s)\vec{g}_\tau(Q,s)\,dQds$$

and

$$\|\vec{g}_\tau\|_{L^2(\partial D_\tau \times (-4,4))} \le C\,\|\frac{\partial \vec{v}}{\partial \nu}\|_{L^2(\partial D_\tau \times (-4,4))}$$

with C independent of τ.

Now, to see part (i), we use estimates (2.1) for $\Gamma(X,t)$ and obtain

$$\begin{aligned} \int_{-4}^{4}\int_{D_\tau} |\nabla\vec{v}|^2 &\le C\int_{-4}^{4}\int_{\partial D_\tau} |\vec{g}_\tau|^2 \\ &\le C\int_{-4}^{4}\int_{\partial D_\tau} \left|\frac{\partial\vec{v}}{\partial\nu}\right|^2 \\ &\le C\int_{-4}^{4}\int_{\partial D_\tau} |\nabla\vec{u}_\tau|^2 \;+\; C\int_{-4}^{4}\int_{D_2\cap\partial D_\tau} \left|\frac{\partial\vec{u}}{\partial\nu}\right|^2 \end{aligned}$$

where we used that $\partial\vec{u}/\partial\nu = \vec{0}$ on Δ_2. Therefore,

$$\begin{aligned} \int_{-1}^{1}\int_{D_1} |\nabla\vec{u}|^2 &\le C\int_{-4}^{4}\int_{D_\tau} |\nabla\vec{u}_\tau|^2 \;+\; C\int_{-4}^{4}\int_{D_\tau} |\nabla\vec{v}|^2 \\ &\le \int_{-4}^{4}\int_{D_\tau} |\vec{u}|^2 \;+\; \int_{-4}^{4}\int_{D_2\cap\partial D_\tau} \left|\frac{\partial\vec{u}}{\partial\nu}\right|^2. \end{aligned}$$

Integrating in τ, we obtain

$$\begin{aligned} \int_{-1}^{1}\int_{D_1} |\nabla\vec{u}|^2 &\le C\int_{-4}^{4}\int_{D_2} |\vec{u}|^2 \;+\; C\int_{-4}^{4}\int_{D_{3/2}} |\nabla\vec{u} \;+\; (\nabla\vec{u})^{tr}|^2 \\ &\le \int_{-4}^{4}\int_{D_2} |\vec{u}|^2 \end{aligned}$$

where we used part (i) of Lemma 3.1. Part (i) is then proved.

To see part (ii), fix $t \in [-1,1]$. Since $\vec{u} = \vec{u}_\tau - \vec{v}$ on $D_1 \times [-1,1]$, we have

$$\|\vec{u}(\cdot,t)\|_{L^{5/2}(D_1)} \le \|\vec{u}_\tau(\cdot,t)\|_{L^{5/2}(D_\tau)} + \|\vec{v}(\cdot,t)\|_{L^{5/2}(D_\tau)} .$$

Using estimates (2.1) on $\Gamma(X,t)$, Minkowski's inequality and Young's inequality, it is not hard to see that

$$\|\vec{v}(\cdot,t)\|_{L^{5/2}(D_\tau)} \le C \|\vec{g}_\tau\|_{L^2(\partial D_\tau \times (-4,4))}$$

and

$$\|\vec{u}_\tau(\cdot,t)\|_{L^{5/2}(D_\tau)} \le C \|\vec{u}\|_{L^2(D_\tau \times (-4,4))} .$$

Thus,

$$\begin{aligned} \|\vec{u}(\cdot,t)\|_{L^{5/2}(D_1)} &\le C \|\vec{g}_\tau\|_{L^2(\partial D_\tau \times (-4,4))} + C \|\vec{u}\|_{L^2(D_\tau \times (-4,4))} \\ &\le C \|\vec{u}\|_{L^2(D_2 \times [-4,4])} . \end{aligned}$$

Finally, part (iii) follows from the L^2-theory and part (i) in a similar fashion.

REMARK 3.3. The conclusions of lemmas 3.1 and 3.2 also hold if we replace the assumption $\partial\vec{u}/\partial\nu = \vec{0}$ on $\Delta_{2R}(P_0,t_0)$ by $\vec{u} = \vec{0}$ on $\Delta_{2R}(P_0, t_0)$. The proof is similar.

Lemma 3.4. *Suppose $\vec{u}$ is a solution of* (0.1) *in Ω_T, with $\|(\vec{u})^*\|_{L^2(\Sigma_T)}$ + $\|(\nabla\vec{u})^*\|_{L^2(\Sigma_T)} < \infty$. Assume that , $\vec{u}|_{t=0} = \vec{0}$ and $\partial\vec{u}/\partial\nu = \vec{a}$ where $\vec{a}$ is a parabolic atom, i.e.,* $\operatorname{supp}\vec{a} \subset \Delta_r(P_0,t_0)$ *for some $(P_0,t_0) \in \Sigma_T$, $\|\vec{a}\|_{L^2(\Sigma_T)} \le 1/r^2$ and $\iint_{\Sigma_T} \vec{a} = \vec{0}$. Then, there exists $\varepsilon > 0$, such that*

$$\|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \le C\ r^{2/p+2/q-3}$$

where $p \in (2-\varepsilon, 2]$ and $q \in (1,2]$.

PROOF. By the divergence theorem, we have, for $(X,t) \in \Omega_T$,

$$\vec{u}(X,t) = \mathcal{S}\Big(\frac{\partial\vec{u}}{\partial\nu}\Big)(X,t) - \mathcal{K}\big(\vec{u}|_{\Sigma_T}\big)(X,t) \tag{3.5}$$

where $\mathcal{S}$ and $\mathcal{K}$ are the single and double-layer potentials. Letting $X \to P \in \partial\Omega$ nontangentially in (3.5), we obtain

$$(\frac{1}{2}I + \widetilde{K}_\nu)(\vec{u}|_{\Sigma_T}) = \mathcal{S}(\frac{\partial \vec{u}}{\partial \nu})|_{\Sigma_T} = \mathcal{S}(\vec{a})|_{\Sigma_T}.$$

Recall that $\frac{1}{2}I + \widetilde{K}_\nu = R_T(\frac{1}{2}I + K_\nu)^* R_T$. Thus by Theorem 2.7 and duality, there exists $\varepsilon > 0$, such that $\frac{1}{2}I + \widetilde{K}_\nu$ is invertible on $L^{p,q}(\Sigma_T)$ for $p \in (2-\varepsilon, 2]$ and $q \in (1,2]$. This, together with Corollary 2.15, implies that

$$\begin{aligned} \|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} &\le C\,\|\vec{u}\|_{L^{p,q}(\Sigma_T)} \\ &\le C\,\|(\tfrac{1}{2}I + \widetilde{K}_\nu)\vec{u}\|_{L^{p,q}(\Sigma_T)} \\ &\le C\,\|\mathcal{S}(\vec{a})|_{\Sigma_T}\,\|_{L^{p,q}(\Sigma_T)}. \end{aligned}$$

To estimate $\|\mathcal{S}(\vec{a})|_{\Sigma_T}\,\|_{L^{p,q}(\Sigma_T)}$, first note that

$$\begin{aligned} \|\mathcal{S}(\vec{a})|_{\Sigma_T}\|_{L^{p,q}(\Delta_{10r})} &\le C\,r^{2/p+2/q-2}\|\mathcal{S}(\vec{a})|_{\Sigma_T}\|_{L^2(\Delta_{10r})} \\ &\le C\,r^{2/p+2/q-1}\|\vec{a}\|_{L^2(\Sigma_T)} \\ &\le C\,r^{2/p+2/q-3}. \end{aligned}$$

Next, using estimate (2.1) and $\iint_{\Sigma_T} \vec{a} = \vec{0}$, we get

$$\begin{aligned} &|\mathcal{S}(\vec{a})(P,t)| \\ \le &\begin{cases} \dfrac{C}{|P-P_0|^3}, & \text{if } |P-P_0| \ge 10\,r \text{ and } |t-t_0| < 100\,r^2, \\ \dfrac{C}{|t-t_0|^{3/2}}, & \text{if } |P-P_0| \le 10\,r \text{ and } |t-t_0| \ge 100\,r^2, \\ \dfrac{Cr}{\left(|P-P_0| + |t-t_0|^{1/2}\right)^4}, & \text{if } |P-P_0| \ge 10\,r \text{ and } |t-t_0| \ge 100\,r^2. \end{cases} \end{aligned}$$

An easy computation then gives the desired estimate.

We are now in a position to state and prove our main lemma which gives L^1-estimates for the nontangential maximal functions of solutions with atomic data.

Lemma 3.6. (Main Lemma) *Under the same assumptions as in Lemma* 3.4, *we have*

$$\iint_{\Sigma_T} (\nabla \vec{u})^* + \iint_{\Sigma_T} (\partial_t^{1/2} \vec{u})^* \leq C .$$

PROOF. By L^2-theory [S, Theorem 4.1.2]

$$\begin{aligned} \iint_{\Delta_{100r}(P_0,t_0)} (\nabla \vec{u})^* &\leq C\, r^2 \left(\iint_{\Delta_{100r}(P_0,t_0)} (\nabla \vec{u})^{*2} \right)^{1/2} \\ &\leq C\, r^2 \, \|\vec{a}\|_{L^2(\Sigma_T)} \leq C . \end{aligned}$$

Thus, it is easy to see that the estimate for $(\nabla \vec{u})^*$ in the lemma will follow if we can prove that

$$\iint_{\Lambda(R)} (\nabla \vec{u})^* \leq C \left(\frac{r}{R}\right)^{\alpha_0} \qquad \text{for some } \alpha_0 > 0 \tag{3.9}$$

where $\Lambda(R) = \Delta_R(P_1, t_1)$ for some $(P_1, t_1) \in \Sigma_T$, $R \geq 5r$ and $\partial \vec{u}/\partial \nu = \vec{0}$ on $\Lambda(10R)$.

First, we estimate $(\nabla \vec{u})^{*R}$ on $\Lambda(R)$. Let $(Y, s) \in \gamma(P, t)$, $|Y - P| \geq R$, $(P, t) \in \Lambda(R)$. By interior estimates,

$$|\nabla \vec{u}(Y, s)| \leq \frac{C}{R^6} \iint_{\substack{|\zeta - s| < CR^2 \\ |Z - Y| \leq CR}} |\vec{u}(Z, \zeta)| dZ d\zeta \leq \frac{C}{R^5} \iint_{\Lambda(2R)} (\vec{u})^* .$$

Thus, if $(P, t) \in \Lambda(R)$,

$$(\nabla \vec{u})^{*R}(P, t) \leq \frac{C}{R^5} \iint_{\Lambda(2R)} (\vec{u})^* .$$

Consequently,

$$\iint_{\Lambda(R)} (\nabla \vec{u})^{*R} \leq \frac{C}{R} \iint_{\Lambda(2R)} (\vec{u})^* \leq C R^{3-2/q-2/p} \|(\vec{u})^*\|_{p,q} .$$

We have applied Hölder inequality for the last inequality. Choose $p \in (2 - \varepsilon, 2)$, ε as in Lemma 3.4 and q close to 1 such that

$$3 - \frac{2}{q} - \frac{2}{p} = -\alpha_1 < 0 .$$

It then follows from Lemma 3.4 that

$$\iint_{\Lambda(R)} (\nabla \vec{u})^{*R} \leq C\left(\frac{r}{R}\right)^{\alpha_1}, \quad \alpha_1 > 0.$$

Next, we estimate $(\nabla \vec{u})^*_R$ on $\Lambda(R)$. By Lemma 3.2,

$$\iint_{\Lambda(R)} (\nabla \vec{u})^*_R \leq C\, R^{1/2} \left(\iint_{Z_{2R}(P_1,t_1)} |\vec{u}|^2 \right)^{1/2}.$$

We use Hölder's inequality, then Lemma 3.4 and Lemma 3.2 (ii) and then Hölder's inequality again to obtain

$$\begin{aligned}
\iint_{Z_{2R}(P_1,t_1)} |\vec{u}|^2 &= R^3 \int_{|t-t_1|<4R^2} \left(\frac{1}{R^3} \int_{D_{2R}(P_1)} |\vec{u}|^2 \right) \\
&\leq C\, R^3 \int_{|t-t_1|<4R^2} \left(\frac{1}{R^3} \int_{D_{2R}(P_1)} |u|^p \right)^{1/p} \\
&\qquad \left(\frac{1}{R^3} \int_{D_{2R}(P_1)} |u|^{5/2} \right)^{2/5} \\
&\leq C\, R^{1/2-2/p} \int_{|t-t_1|<4R^2} \left(\int_{J_{3R}(P_1)} (\vec{u})^{*p} \right)^{1/p} \\
&\qquad \left(\iint_{Z_{3R}(P_1,t_1)} |\vec{u}|^2 \right)^{1/2} \\
&\leq C\, R^{5/2-2/p-2/q} \, \|(\vec{u})^*\|_{L^{p,q}(\Sigma_T)} \\
&\qquad \left(\iint_{Z_{3R}(P_1,t_1)} |\vec{u}|^2 \right)^{1/2}.
\end{aligned}$$

As before, choose $p \in (2-\varepsilon, 2)$ and q close to 1 such that

$$\alpha_1 = \frac{2}{p} + \frac{2}{q} - 3 > 0.$$

Let

$$a_R = R^{1/2} \left(\iint_{Z_{2R}(P_1,t_1)} |\vec{u}|^2 \right)^{1/2}$$

and observe that

$$a_R^2 \leq C\left(\frac{r}{R}\right)^{\alpha_1} a_{2R}. \tag{3.10}$$

By Lemma 3.4,

$$a_R \le CR\|(\vec{u})^*\|_{L^2(\Sigma_T)} \le C\left(\frac{r}{R}\right)^{-1}.$$

It follows from (3.10) that

$$a_R \le C\left(\frac{r}{R}\right)^{(\alpha_1-1)/2}$$

In general, if $a_R \le C(r/R)^{\beta_n}$, then, by (3.10) we obtain the improved estimate

$$a_R \le C\left(\frac{r}{R}\right)^{(\alpha_1+\beta_n)/2}$$

Consider the sequence $\{\beta_n\}, \beta_0 = (\alpha_1 - 1)/2$, $\beta_{n+1} = (\alpha_1 + \beta_n)/2$. Then

$$\beta_{n+1} - \beta_n = \frac{\alpha_1 - \beta_n}{2} \ge \frac{\alpha_1}{2} \quad \text{if} \quad \beta_n \le 0.$$

Therefore, there exists N such that $\beta_N > \alpha_1/4$. Hence,

$$a_R \le C(\frac{r}{R})^{\alpha_1/4}$$

Let $\alpha_0 = \alpha_1/4$. The proof for (3.9) is complete.

We now give the proof that

$$\iint_{\Sigma_T} \left(\partial_t^{1/2}\vec{u}\right)^* \le C.$$

Again, we wish to show that

$$\iint_{\Lambda(R)} \left(\partial_t^{1/2}\vec{u}\right)^* \le C\left(\frac{r}{R}\right)^{\alpha_0}.$$

Note that we may only consider those $\Lambda(R) = \Delta_R(P_1, t_1)$ where $|t_1 - t_0| < CR^2$.

To proceed we let $\eta \in C_0^\infty(\mathbb{R}), \eta \equiv 1$ on $\left[t_1 - \frac{7}{6}R^2, t_1 + \frac{7}{6}R^2\right]$, $\operatorname{supp}\eta \subset \left[t_1 - \frac{4}{3}R^2, t_1 + \frac{4}{3}R^2\right]$ and $|\eta'(t)| \le C/R^2$. Then, by Lemma 3.2,

$$\iint_{\Lambda(R)} \left(\dot{\partial}_t^{1/2}(\vec{u}\eta)\right)^* \le C\,R^{1/2}\left(\iint_{Z_{2R}} |\vec{u}|^2\right)^{1/2} + \frac{C}{R}\iint_{\Lambda(2R)} (\vec{u})^*$$

$$\le C\left(\frac{r}{R}\right)^{\alpha_0}.$$

The estimate for $\left(\partial_t^{1/2}(\vec{u}(1-\eta))\right)^*$ is easy. In fact, if $(P,t) \in \Lambda(R)$,

$$\left|\left[\partial_t^{1/2}(\vec{u}(1-\eta))\right]^*(P,t)\right| \leq \frac{C}{R^3}\int_{t_0-R^2}^{t_1-R^2} |(\vec{u})^*(P,s)|ds$$

where we used $\vec{u}(Q,s) = \vec{0}$ if $s < t_0 - R^2$. Thus,

$$\begin{aligned}\iint_{\Lambda(R)} |\partial_t^{1/2}(\vec{u}(1-\eta))| &\leq \frac{C}{R}\int_{t_0-R^2}^{t_1-R^2}\int_{I_R(P_1)} (\vec{u})^* \\ &\leq C\,R^{3-2/p-2/q}\|\vec{u}\|_{L^{p,q}(\Sigma_T)} \\ &\leq C\left(\frac{r}{R}\right)^{2/p+2/q-3}\end{aligned}$$

where $2/p + 2/q - 3 > 0$. The estimate for $(\partial_t^{1/2}\vec{u})^*$ then follows. This completes the proof of the main lemma.

We omit the proof of the following lemma which may be carried out using an argument similar to that of Lemma 3.6.

Lemma 3.11. *Suppose $\vec{u}$ is a solution of* (0.1) *in Ω_T with $\|(\vec{u})^*\|_{L^2(\Sigma_T)}$ $+ \|(\nabla\vec{u})^*\|_{L^2(\Sigma_T)} < \infty$ and $\vec{u}\big|_{t=0} = \vec{0}$. Assume that* $\operatorname{supp}\vec{u}\big|_{\Sigma_T} \subset \Delta_r(P_0,t_0)$ *for some $(P_0,t_0) \in \Sigma_T$, r small and $\|\vec{u}\|_{L_1^2(\Sigma_T)} \leq 1/r^2$. Then, there exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$,*

$$\iint_{\Sigma_T}\Big((\nabla\vec{u})^*(P,t) + (\partial_t^{1/2}\vec{u})^*(P,t)\Big) \cdot\Big(r + |P-P_0| + |t-t_0|^{1/2}\Big)^{\alpha} dP\,dt \leq C\,r^{\alpha}.$$

4. Boundary Value Problems.

In this section we give the proofs for Theorem A, B, C, and D which were stated in the Introduction. We begin by introducing the matrix Green's function for (0.1) in Ω_T. Fix $X \in \Omega$, let $v^X(Y,s)$ be the matrix valued solution of (0.1) in Ω_T, which satisfies the boundary conditions

$$\begin{cases} v^X\big|_{\Sigma_T} = \Gamma(X-\cdot,\cdot)\big|_{\Sigma_T} \\ \\ v^X\big|_{s=0} = 0\,. \end{cases}$$

By the L^2-theory for the Dirichlet problem with data in $L^2_1(\Sigma_T)$, such a solution exists and satisfies $\|(v^X)^*\|_{L^2(\Sigma_T)} + \|(\nabla v^X)^*\|_{L^2(\Sigma_T)} < \infty$. Let

$$G(X,Y,t,s) = G(X,Y,t-s) = \Gamma(X-Y,t-s) - v^X(Y,t-s)$$

be the matrix Green's function.

Lemma 4.1. *Let $X \in \Omega$, $P \in \partial\Omega$ and $|X-P| < 2\,\mathrm{dist}\{X,\partial\Omega\}$. Then, there exists $\alpha_0 > 0$, such that*

$$\iint_{\Sigma_T} |\nabla_Q G(X,Q,t)|\,(r + |Q-P| + t^{1/2})^\alpha dQds \le C\,r^\alpha$$

where $r = |X-P|$, $0 < \alpha < \alpha_0$.

PROOF. We may assume that r is small. Let $X^* \in \bar{\Omega}^C$ such that $|X^*-P| = r$ and $|X^*-P| < 2\,\mathrm{dist}\{X^*,\partial\Omega\}$. Let $\tilde{v}^X(Y,s)$ be the matrix valued solution of (0.1) in Ω_T such that

$$\begin{cases} \tilde{v}^X\big|_{\Sigma_T} = \Gamma(X-\cdot,\cdot) - \Gamma(X^*-\cdot,\cdot)\big|_{\Sigma_T} \\ \tilde{v}^X\big|_{s=0} = 0 \end{cases}$$

given by L^2-theory. We claim that there exist $\alpha_0 > 0$ such that

$$\iint_{\Sigma_T} \big|\nabla_Q \tilde{v}^X(Q,s)\big|(r + |Q-P| + s^{1/2})^\alpha dQds \le C\,r^\alpha\,, \tag{4.2}$$

for $0 < \alpha < \alpha_0$.

Assume (4.2) for a moment. We give the proof of the lemma. By the uniqueness for the initial-Dirichlet problem with L^2-data, it is easy to see that

$$G(X,Y,s) = \Gamma(X-Y,s) - \Gamma(X^*-Y,s) - \tilde{v}^X(Y,s)\,.$$

By estimate (2.1) for $\Gamma(X,t)$, we have, for $|Y-P| \ge 10$,

$$\begin{aligned} &\big|\nabla_Y\big(\Gamma(X-Y,s) - \Gamma(X^*-Y,s)\big)\big| \le \frac{Cr}{(|Y-P| + s^{1/2})^5} \\ &\big|\frac{\partial}{\partial s}\big(\Gamma(X-Y,s) - \Gamma(X^*-Y,s)\big)\big| \le \frac{Cr}{(|Y-P| + s^{1/2})^6} \end{aligned} \tag{4.3}$$

The lemma then follows easily from (4.2) and (4.3).

To prove (4.2), let

$$B(Q,t) = B^X(Q,t) = \Gamma(X-Q,t) - \Gamma(X^*-Q,t), \quad (Q,t) \in \Sigma_T .$$

We may choose a partition of unity, $1 = \sum_{k\geq 0,\, j=1,2,\dots,N} \psi_{k,j}$ which satisfies

1) $\operatorname{supp} \psi_{k,j} \subset \Delta_{2^k r}(P_{k,j}, t_{k,j})$, $P_{0,j} = P$,
2) $2^k r \leq |P_{k,j} - P| + t_{k,j}^{1/2} \leq C\, 2^k r$,
3) $|\psi_{k,j}| \leq 1$ and $|\nabla \psi_{k,j}| + 2^k r |\partial \psi_{k,j}/\partial t| \leq C\,(2^k r)^{-1}$.

Then

$$B(Q,t) = \sum \psi_{k,j} B(Q,t) = \sum 2^{-k} a_{k,j}(Q,t)$$

where $a_{k,j}(Q,t) = 2^k \psi_{k,j}(Q,t) B(Q,t)$. It follows from (4.3) and an argument in [B, Proposition 1.9] that

$$\|a_{k,j}\|_{L^2_1(\Sigma_T)} \leq \frac{C}{(2^k r)^2} .$$

Now, let $\tilde{v}_{k,j}$ be the solution to the initial-Dirichlet problem for (0.1) with $\tilde{v}_{k,j}\big|_{\Sigma_T} = a_{k,j}$. Clearly, $\tilde{v}^X = \sum 2^{-k} \tilde{v}_{k,j}$. By Lemma 3.11, we have, for $0 < \alpha < \alpha_0$,

$$\iint_{\Sigma_T} |\nabla_Q \tilde{v}_{k,j}(Q,s)| \big(2^k r + |Q - P_{k,j}| + |s - t_{k,j}|^{1/2}\big)^{\alpha} \leq C \big(2^k r\big)^{\alpha} .$$

Since

$$\begin{aligned} |Q-P| + |s|^{1/2} &\leq |Q - P_{k,j}| + |P_{k,j} - P\,| + |s - t_{k,j}|^{1/2} + |t_{k,j}|^{1/2} \\ &\leq C\, 2^k r + |Q - P_{k,j}| + |s - t_{k,j}|^{1/2} \end{aligned}$$

it follows that

$$\iint_{\Sigma_T} |\nabla_Q \tilde{v}_{k,j}(Q,s)| \;\; (r + |Q-P| + s^{1/2})^{\alpha} dQds \leq C\,(2^k r)^{\alpha}.$$

Thus

$$\begin{aligned} \iint_{\Sigma_T} |\nabla_Q \tilde{v}(Q,s)| \;\; &\big(r + |Q-P| + s|^{1/2}\big)^{\alpha} dQds \\ &\leq C \sum 2^{-k} (2^k r)^{\alpha} \leq C\, r^{\alpha} . \end{aligned}$$

The claim (4.2) is proved and the proof is finished.

We are now in a position to prove Theorem A and B.

PROOF OF THEOREM A. The case $2-\varepsilon < p \leq 2$ is contained in Corollary 2.15. Consider the case $2 < p \leq \infty$. The uniqueness follows the uniqueness for L^2-solutions [S]. To see the existence, note that, by Lemma 4.1,

$$\iint_{\Sigma_T} |\nabla_Q G(X,Q,t)| dQdt \leq C .$$

Hence

$$\begin{aligned} |\vec{u}(X,t)| &= |\int_0^t \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,t-s)\vec{u}(Q,s) dQds| \\ &\leq C \|\vec{u}\|_{L^\infty(\Sigma_T)} . \end{aligned}$$

Thus, $\|\vec{u}\|_{L^\infty(\Omega_T)} \leq C\|\vec{u}\|_{L^\infty(\Sigma_T)}$. The existence then follows by interpolation.

PROOF OF THEOREM B. Given $\vec{g} \in \Lambda_0^\alpha(\Sigma_T)$, $0 < \alpha < \alpha_0$, α_0 as in Lemma 4.1. Let $\vec{u}$ be the solution of (0.1), $\vec{u}|_{\Sigma_T} = \vec{g}$ and $\vec{u}|_{t=0} = \vec{0}$. To show $\|\vec{u}\|_{\Lambda_0^\alpha(\Omega_T)} \leq C \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}$, it is enough to prove

$$|\vec{u}(X_1,t) - \vec{u}(X_2,t)| \leq C |X_1 - X_2|^\alpha \quad \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)} \tag{4.4}$$

and

$$|\vec{u}(X,t_1) - \vec{u}(X,t_2)| \leq C |t_1 - t_2|^{\alpha/2} \quad \|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)} . \tag{4.5}$$

The estimate (4.5) is easy. In fact, write

$$\begin{aligned} \vec{u}(X,t) &= \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,t-s)\vec{g}(Q,S) dQds \\ &= \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial \nu_Q} G(X,Q,s)\vec{g}(Q,t-s) dQds , \end{aligned}$$

where we have put $\vec{g}(Q,t) = \vec{0}$ for $t < 0$. From this, (4.5) follows easily.

To see (4.4), it suffices to show

$$|\nabla_X \vec{u}(X,t)| \leq \frac{C\,\|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}}{\operatorname{dist}\{X,\partial\Omega\}^{1-\alpha}}\,. \tag{4.6}$$

We write

$$\begin{aligned}\vec{u}(X,t) &= \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial\nu_Q} G(X,Q,s)\vec{u}(Q,t-s)dQds \\ &= \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial\nu_Q} G(X,Q,s)\,\big(\vec{u}(Q,t-s)-\vec{u}(P,t)\big)dQds \\ &\qquad + \int_0^T \int_{\partial\Omega} \frac{\partial}{\partial\nu_Q} G(X,Q,s)\vec{u}(P,t)dQds \\ &= \mathrm{I}(X,t) + \mathrm{II}(X,t)\,,\end{aligned}$$

where $P \in \partial\Omega$ and $|X-P| < 2\operatorname{dist}\{X,\partial\Omega\}$. By interior estimates and Lemma 4.1, we have

$$|\nabla_X I(X,t)| \leq \frac{C\,\|\vec{g}\|_{\Lambda_0^\alpha(\Sigma_T)}}{\operatorname{dist}\{X,\partial\Omega\}^{1-\alpha}}\,.$$

To estimate ∇_X II (X,t), let

$$v(X,t) = \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial\nu_Q} G(X,Q,t-s)dQds\,.$$

We wish to show that

$$|\nabla v(X,T)| \leq \frac{C}{r^{1-\alpha}} \tag{4.7}$$

where $r = \operatorname{dist}\{X,\partial\Omega\}$. Clearly, (4.7), together with the estimate for $\nabla_X I(X,t)$ yields (4.6).

To show (4.7), note that, v is a solution of (0.1) in Ω_T, $v\big|_{\Sigma_T}$ is the identity matrix and $v\big|_{t=0} = 0$. Hence, by Theorem A, $|v(X,t)| \leq C$.

Let

$$w(X,t) = \int_0^t \int_\Omega \Gamma(X-Y,t-s)v(Y,s)\eta'(s)dYds$$

where $\eta(0) = 0$, $\eta(t) = 1$ for $t > T/2$ and $|\eta'(t)| \leq C$. Then, $|\nabla w| + |\partial_t^{1/2} w| \leq C$. Let $v_1 = w - v\eta$. Then, v_1 is a solution of (0.1) in

Ω_T, $v_1\big|_{t=0} = 0$. Note that $\|v_1\|_{L_1^{p,q}(\Sigma_T)} \le C$ for $p > 2, q > 2$. Choose $p \in (2, 2+\varepsilon_2)$, ε_2 as in Corollary 2.16 and $q > 2$ large such that $2/p + 2/q < 1$. It then follows from interior estimates, Hölder's inequality and Corollary 2.16 that

$$\begin{aligned} |\nabla_X v_1(X,t)| &\le \frac{C}{r^{2/p+2/q}} \|(\nabla v_1)^*\|_{L^{p,q}(\Sigma_T)} \\ &\le \frac{C}{r^{2/p+2/q}} \|v_1\|_{L_1^{p,q}(\Sigma_T)} \\ &\le \frac{C}{r^{2/p+2/q}} . \end{aligned}$$

Thus,

$$\begin{aligned} |\nabla_X v(X,T)| &\le |\nabla_X v_1(X,T)| + |\nabla_X w(X,T)| \\ &\le \frac{C}{r^{2/p+2/q}} + C \\ &\le \frac{C}{r^{1-\alpha}} \end{aligned}$$

where we assumed $\alpha_0 \le 1 - 2/p - 2/q$. Thus, (4.7) is then proved.

We now turn to the proof of Theorem D.

PROOF OF THEOREM D. The case $2 \le p < 2 + \varepsilon$ is contained in Corollary 2.15.

For $1 < p < 2$. the existence follows from Lemma 3.6 and interpolation. To show the uniqueness, let $\vec{u}$ be a solution of (0.1) in Ω_T, $\partial\vec{u}/\partial\nu = \vec{0}$, $\vec{u}\big|_{t=0} = \vec{0}$ and $\|(\nabla\vec{u})^*\|_p < \infty$. First, we assume that Ω is a smooth domain. Given $\vec{F} \in C_0^\infty(\Omega_T)$. It is well known that, there exists $\vec{v} \in C^\infty(\overline{\Omega}_T)$, such that

$$\begin{cases} \dfrac{\partial\vec{u}}{\partial t} + \mu\,\Delta\vec{v} + (\lambda+\mu)\nabla(\operatorname{div}\vec{v}) = \vec{F}, \\ \qquad\qquad \dfrac{\partial\vec{v}}{\partial\nu}\Big|_{\Sigma_T} = \vec{0}, \\ \qquad\qquad \vec{v}\big|_{t=T} = \vec{0}. \end{cases}$$

It then follows from integration by parts that

$$\int_0^T \int_\Omega \vec{u}\cdot\vec{F} = \iint_{\Sigma_T} \vec{u}\cdot\frac{\partial\vec{v}}{\partial\nu} - \iint_{\Sigma_T} \frac{\partial\vec{u}}{\partial\nu}\cdot\vec{v} = \vec{0}.$$

Since $\vec{F}$ is arbitary, we have $\vec{u} \equiv \vec{0}$ on Ω_T.

Next, if Ω is a general Lipschitz domain, let $\{\Omega_j\}$ be a sequence of smooth domains approximating Ω such that $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ ([V], Theorem 1.12). Let $\Sigma_T^j = \partial\Omega_j \times (0,T)$. By the existence and uniqueness results on $\Omega_i \times (0,T)$, we have

$$\|(\nabla \vec{u})^*\|_{L^p(\Sigma_T^j)} \leq C \|\frac{\partial \vec{u}}{\partial \nu}\|_{L^p(\Sigma_T^j)} \to 0 \quad \text{as } j \to \infty .$$

Hence $\vec{u} \equiv 0$ on Ω_T. The proof is complete.

To establish Theorem C, we need the following lemma.

Lemma 4.8. *Let $\vec{u}$ be a solution of* (0.1) *in Ω_T. Assume that*

$$\|(\nabla \vec{u})^*\|_{L^{p,q}(\Sigma_T)} < \infty \quad \textit{for some} \quad 1 < p, q < \infty .$$

Then

$$\|(\vec{u})^*\|_{L^{\bar{p},\bar{q}}(\Sigma_T)} < \infty$$

where $1 < \bar{p}, \bar{q} < \infty$, $1 + 2/\bar{p} + 2/\bar{q} = 2/p + 2/q$ *and* $0 < 1/p - 1/\bar{p} < 1/2$, $0 < 1/q - 1/\bar{q} < 1/2$.

PROOF. By interior estimates, we have

$$(\vec{u})^*(P,t) \leq C(\vec{u}) + C \iint_{\Sigma_T} \frac{(\nabla \vec{u})^*\,(Q,s)}{(|Q-P| + |t-s|^{1/2})^3} dQds$$

where $C(\vec{u}) < \infty$ is a constant depending on $\vec{u}$. From this, the lemma follows from Minkowski's inequality and the theorem on fractional integrals in one dimension.

We are now ready to prove Theorem C.

PROOF OF THEOREM C. The case $2 \leq p < 2 + \varepsilon$ is contained in Corollary 2.16. For $1 < p < 2$, as before, the existence follows from Lemma 3.11 and interpolation. To see uniqueness, let $\vec{u}$ be a solution of (0.1) in Ω_T, $\vec{u}|_{\Sigma_T} = \vec{0}$, $\vec{u}|_{t=0} = \vec{0}$ and $\|(\nabla \vec{u})^*\|_{L^p(\Sigma_T)} < \infty$. Let $\vec{g} \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$. By Corollary 2.16, there exists $\vec{v}$ satisfying (0.1) in Ω_T. $\vec{v}|_{\Sigma_T} = \vec{g}|_{\Sigma_T}$, $\vec{v}|_{t=0} = \vec{0}$ and $\|(\nabla \vec{v})^*\|_{L^{p^*,q^*}(\Sigma_T)} < \infty$ for $p^* \in [2, 2 + \varepsilon_2)$, $q^* \in [2, \infty)$. Choose p^*, q^*, such that $2/p^* + 2/q^* = 5 - 4/p$ (we may assume that p close to 1). Then, by Lemma 4.8, $\|(\vec{u})^*\|_{L^{(p^*)',(q^*)'}(\Sigma_T)} <$

∞ where $1/p^* + 1/(p^*)' = 1$ and $1/q^* + 1/(q^*)' = 1$. It then follows from integration by parts and a limiting argument that

$$\int_0^T \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu}(Q,t)\vec{g}(Q,T-t)\,dQ\,dt = \int_0^T \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu}(Q,t)\vec{v}(Q,T-t)dQdt$$
$$= \int_0^T \int_{\partial\Omega} \vec{u}(Q,t)\frac{\partial \vec{v}}{\partial \nu}(Q,T-t)dQdt = \vec{0}\,.$$

So $\partial\vec{u}/\partial\nu = \vec{0}$ on Σ_T. Therefore, by the uniqueness for traction problem (Theorem D), $\vec{u} \equiv \vec{0}$ in Σ_T. The proof is finished.

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Un problema de valores propios para un sistema parabólico periódico y aplicaciones

Beatriz Elena Villa

1. Introducción y preliminares.

En este artículo estudiaremos el sistema lineal parabólico de valores propios

$$(1.1)\qquad \begin{cases} \tilde{L}[\vec{u}\,] = \lambda\gamma\,\vec{u} & \text{en } \Omega\times\mathbb{R}, \\ \vec{u}(x,t) = \vec{u}(x,t+\ell) & \text{en } \bar{\Omega}\times\mathbb{R}, \\ \vec{u}(x,t)\equiv 0 & \text{en } \partial\Omega\times\mathbb{R}, \end{cases}$$

donde $\vec{u} = \mathrm{col}(u^1,\ldots,u^m)$, ℓ es un número real positivo fijo, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ es un dominio acotado cuya frontera $\partial\Omega$ es una variedad de clase $C^{2+\alpha}$, $\alpha \in (0,1)$, $\tilde{L}[\vec{u}\,] = \mathrm{col}(L[u^1],\ldots,L[u^m])$, L es un operador lineal de segundo orden uniformemente parabólico en $\Omega\times\mathbb{R}$ y $\gamma(x,t) = [\gamma_{ij}(x,t)]$ es una matriz $m\times m$ con componentes $\gamma_{ij}(x,t)$ continuas en $\overline{\Omega}\times\mathbb{R}$, ℓ-periódicas en la variable t, (m es un entero positivo) y $\lambda\in\mathbb{R}$.

Los resultados sobre (1.1) se aplicarán al estudio del sistema

$$
(1.2)\quad \begin{cases} \widetilde{L}[\vec{u}](x,t) = H(T(\vec{u}),x,t)\,\vec{u}(x,t) & \text{en } \Omega\times\mathbb{R}, \\ L[T](x,t) + C(x,t)\,T(\vec{u},x,t) & \\ \qquad = \sum_{j=1}^{m} G_j(T(\vec{u}),x,t)\,u^j(x,t) & \text{en } \Omega\times\mathbb{R}, \\ u^i(x,t) = u^i(x,t+\ell)\,,\ T(x,t)\equiv T(x,t+\ell) & \text{en } \bar{\Omega}\times\mathbb{R}, \\ T(x,t) = 0 = u^i(x,t) & \text{en } \partial\Omega\times\mathbb{R}, \qquad i=1,\dots,m\,, \end{cases}
$$

donde $H(s,x,t) = [H_{ij}(s,x,t)]$ es una matriz $m\times m$, las funciones G_j, H_{ij} son continuas en $\mathbb{R}_+\times\overline{\Omega}\times\mathbb{R}$, ℓ-periódicas en la variable t y $C(x,t)$ es continua en $\overline{\Omega}\times\mathbb{R}$ y ℓ-periódica en t.

Los resultados referentes a (1.1) tienen muchas aplicaciones en la Teoría de Bifurcación.

Se conocen varias publicaciones sobre el estudio del caso escalar de (1.1) ($m=1$). Para L elíptico y autoadjunto por Manes-Micheletti [10] y por Hess-Kato para operadores no necesariamente autoadjuntos [4]. El caso parabólico fue estudiado por Laser en [7], ($\gamma=1, m=1$).

El sistema (1.2) es un modelo que describe el fenómeno de reacción difusión en el interior de un reactor nuclear de fisión. Ha sido estudiado para el caso elíptico con resultados parciales por Anthony Leung y Gen-Shun Chen en [9] y [8]. En este artículo demostraremos la existencia de soluciones clásicas positivas de este sistema, con hipótesis que se especificarán en esta misma sección. Para mayor información sobre el significado físico de las funciones que aparecen en (1.2), veánse los artículos antes mencionados.

En este artículo J denotará al conjunto $J=\{1,\dots,m\}$ y $\sigma\in(\alpha,1)$. Se introducen los siguientes espacios de Banach sobre los reales

$$
\mathbb{F} = \{u\in C^{\alpha,\alpha/2}(\overline{\Omega}\times\mathbb{R}):\quad u(x,t) \text{ es } \ell\text{-periódica en } t\}\,,
$$

con la norma

$$
\|\cdot\|_{\mathbb{F}} = \|\cdot\|_{C^{\alpha,\alpha/2}(\overline{\Omega}\times[0,\ell])} = \|\cdot\|_{\alpha}^{\overline{\Omega}\times[0,\ell]}\,,
$$

$$
\mathbb{E} = \{u\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times\mathbb{R}):\quad u(x,t+\ell)=u(x,t) \quad \text{en } \overline{\Omega}\times\mathbb{R},\quad u\equiv 0 \qquad \text{en } \partial\Omega\times\mathbb{R}\}\,,
$$

con la norma

$$\|\cdot\|_{\mathbb{E}} = \|\cdot\|_{C^{2+\alpha,1+(\alpha/2)}(\overline{\Omega}\times[0,\ell])} = \|\cdot\|_{2+\alpha}^{\overline{\Omega}\times[0,\ell]},$$

$$\widetilde{\mathbb{F}} = \{u \in C^{1+\sigma,\sigma/2}(\overline{\Omega}\times\mathbb{R}): \quad u(x,t+\ell) = u(x,t)\},$$

con la norma

$$\|\cdot\|_{\widetilde{\mathbb{F}}} = \|\cdot\|_{C^{1+\sigma,\sigma/2}(\overline{\Omega}\times[0,\ell])} = \|\cdot\|_{1+\sigma}^{\overline{\Omega}\times[0,\ell]}.$$

También usaremos los espacios: $\mathbb{E}^m = \prod_{i=1}^m E_i$, $E_i = \mathbb{E}$, $\mathbb{F}^m = \prod_{i=1}^m F_i$, $F_i = \mathbb{F}$; y $\widehat{\mathbb{E}} = \{\vec{u} \in \widetilde{\mathbb{F}}^m : \ \vec{u} \equiv 0 \text{ en } \partial\Omega\times\mathbb{R}\}$, con las respectivas normas:

$$\|\vec{u}\|_{\mathbb{E}^m} = \max\{\|u^i\|_{\mathbb{E}}\,,\ i \in J\},$$
$$\|\vec{u}\|_{\mathbb{F}^m} = \max\{\|u^i\|_{\mathbb{F}}\,,\ i \in J\},$$

y

$$\|\vec{u}\|_{\widehat{\mathbb{E}}} = \max\{\|u^i\|_{\widetilde{\mathbb{F}}}\,,\ i \in J\},$$

para todo $\vec{u} = \text{col}(u^1,\ldots,u^m)$. Para mayor información acerca de las funciones Hölder-continuas veáse [7] y [3].

En $\widehat{\mathbb{E}}$ consideramos el orden natural dado por el cono positivo $\mathbb{P} = \{\vec{u} = \text{col}(u^1,\cdots,u^m) \in \widehat{\mathbb{E}},\ u^i(x,t) \geq 0 \text{ en } \Omega\times\mathbb{R},\ i \in J\}$. Obsérvese que el interior de $\mathbb{P}$, $\text{Int}\,\mathbb{P}$, es no vacío.

Un par $(\lambda,\vec{u})$ es *solución clásica* de (1.1) si $(\lambda,\vec{u}) \in \mathbb{R}_+ \times \mathbb{E}^m$ y satisface la ecuación (1.1). En forma análoga se definen las soluciones clásicas $(T,\vec{u}) \in \mathbb{E}^{m+1}$ de (1.2).

Estamos interesados en la existencia de soluciones clásicas de (1.1) y (1.2) con componentes positivas.

En lo que sigue L denotará un operador de la forma

$$L[u] = \frac{\partial u}{\partial t} - \Big(\sum_{i=1}^N\sum_{j=1}^N \overline{a}_{ij}D_{ij}u + \sum_{i=1}^N \overline{b}_i D_i u + \overline{c}_0 u\Big), \tag{1.3}$$

donde los coeficientes $\overline{a}_{ij}(\cdot)$, $\overline{b}_i(\cdot)$, $\overline{c}_0(\cdot)$ pertenecen a $\mathbb{F}$, $\overline{a}_{ij} = \overline{a}_{ji}$ y $\overline{c}_0 \leq 0$ en $\overline{\Omega}\times\mathbb{R}$, $\overline{a}_{ij} \in C^{1,0}(\overline{\Omega}\times\mathbb{R})$.

Todos los resultados de este artículo son válidos si L es uniformemente elíptico en Ω y las funciones involucradas en (1.1) y (1.2) son independientes de t. Es decir, si L es un operador de la forma

$$L[u] = -\Big(\sum_{i=1}^N\sum_{j=1}^N \widetilde{a}_{ij}D_{ij}u + \sum_{i=1}^N \widetilde{b}_i D_i u + \widetilde{c}_0 u\Big)$$

donde las funciones $\widetilde{a}_{ij}$, $\widetilde{b}_i$, $\widetilde{c}_0$, pertenecen a $C^\alpha(\Omega)$, $\widetilde{a}_{ij} = \widetilde{a}_{ji}$ y $\widetilde{c}_0 \leq 0$ en Ω, $\widetilde{a}_{ij} \in C^1(\overline{\Omega})$ y

$$\sum_{i,j=1}^{n} \widetilde{a}_{ij}(x)\,\xi_i\xi_j \geq M \sum_{i=1}^{N} \xi_i^2\,,$$

para todo $x \in \Omega$, $(\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ y algún $M > 0$.

El primer resultado que presentaremos es un Teorema de Comparación, el cual es muy útil en el estudio de sistemas parabólicos y elípticos y jugará un papel muy importante en este trabajo.

Teorema 1.1. (Teorema de comparación). *Supongamos que* $\vec{u}$, $\vec{v} \in C^{2,1}(\Omega \times \mathbb{R}, \mathbb{R}^m) \bigcap C^{1,0}(\overline{\Omega} \times \mathbb{R}, \mathbb{R}^m)$, $\vec{u} \not\equiv \vec{0}$, *satisfacen*

$$\text{(1.4)} \qquad \begin{cases} L[u^i] = \sum_{j=1}^m p_{ij} u^j & \text{en } \Omega \times \mathbb{R}, \quad i \in J\,, \\ \vec{u}(x, t+\ell) = \vec{u}(x,t)\,, \\ \vec{u}\,|_{\partial\Omega \times \mathbb{R}} \equiv \vec{0}\,, \end{cases}$$

$$\text{(1.5)} \qquad \begin{cases} L[v^i] \geq \sum_{j=1}^m q_{ij} v^j & \text{en } \Omega \times \mathbb{R}\,, \\ v^i(x, t+\ell) = v^i(x,t)\,, & i \in J\,, \end{cases}$$

donde p_{ij} *y* q_{ij} *son funciones acotadas en* $\overline{\Omega} \times \mathbb{R}$, ℓ*—periódicas en* t. *Supongamos que* $v^i \geq 0$, $v^i \not\equiv 0$ *en* $\Omega \times \mathbb{R}$, $q_{ij} \geq p_{ij}$ *en* $\overline{\Omega} \times \mathbb{R}$, $i, j \in J$ *y* $q_{ij}(x,t)$, $p_{ij}(x,t) \geq 0$ *en* $\overline{\Omega} \times \mathbb{R}$, *para* $i \neq j$. *Entonces existen* $\delta \in \mathbb{R}$ *y* $s \in J$ *tales que* $v^s \equiv \delta u^s$, $v^j - \delta u^j \geq 0$, *y* $p_{sj} \equiv q_{sj}$ *en* $\overline{\Omega} \times \mathbb{R}$, *para todo* $j \in J$. $(\delta = \min\{\sup\{c_i :\ v^i - c_i u^i > 0 \text{ en } \Omega \times \mathbb{R}\} :\ i \in J\})$.

Corolario 1.1. *Si suponemos las mismas hipótesis del Teorema* 1.1 *y suponemos que alguna componente de* $\vec{u}$ *toma algún valor positivo, entonces el resultado del Teorema* 1.1 *sigue siendo válido, aun cambiando la igualdad en* (1.4) *por "inferior o igual".*

Adoptaremos las siguientes hipótesis sobre la matriz γ

(A1) Existe una biyección θ del conjunto $\{1, \ldots, m\}$ en sí mismo tal que

a) $\gamma_{\theta(1)k} \not\equiv 0$, $k = 1, \ldots, m$.

b) Para todo $j \geq 2$ existe $i \in \{1, \ldots, j-1\}$ tal que $\gamma_{\theta(j)\theta(i)} \not\equiv 0$.

(A2) Las componentes de γ pertenecen a $\mathbb{F}$ y $\gamma_{ij} \geq 0$ en $\overline{\Omega} \times \mathbb{R}$ si $i \neq j$; $i, j = 1, \ldots, m$.

(A3) Existe una función continua $h : [0, \ell] \to \Omega$ tal que $h(0) = h(\ell)$ y para algún k,

$$\int_0^\ell \gamma_{kk}(h(s), s)\, ds > 0\,. \tag{1.6}$$

OBSERVACIÓN 1.1. Por la condición (A1), si $\vec{u} \in \mathbb{P}$ es solución de (1.1) para $\lambda > 0$ y una de las componentes de $\vec{u}$ es positiva en $\Omega \times \mathbb{R}$, entonces $\vec{u} \in \operatorname{Int} \mathbb{P}$. La condición (A1) se satisface por ejemplo si todas las componentes de la fila i y la columna i de γ son no nulas. Por el contrario, es de notar que si todas las componentes de dos filas de γ son nulas, excepto los elementos de la diagonal, los cuales suponemos mayores o iguales a cero y diferentes entre sí, entonces el problema (1.1) no tiene solución positiva en $\Omega \times \mathbb{R}$, para $\lambda > 0$.

El resultado principal referente a (1.1) está contenido en el siguiente teorema.

Teorema 1.2. *Si γ satisface* (A1)-(A3), *entonces existe* $(\lambda_1, \vec{u}_1) \in \mathbb{R}_+ \times \mathbb{E}^m$, $\lambda_1 > 0$, $\vec{u}_1 = \operatorname{col}(u_1^1, \ldots, u_1^m)$ *tal que*

(i) $(\lambda_1, \vec{u}_1)$ *satisface* (1.1).

(ii) $u_1^j > 0$ *en* $\Omega \times \mathbb{R}$, *y* $\partial u_1^j / \partial \vec{n} < 0$ *en* $\partial\Omega \times \mathbb{R}$, $j = 1, \ldots, m$. ($\partial/\partial \vec{n}$ *denota diferenciación en la dirección de la normal exterior*),

(iii) λ_1 *es único y simple.*

Sobre las funciones H_{ij}, G_i y C adoptaremos las siguientes hipótesis:

(B1) $H_{ij}(T(\cdot), \cdot) \in \mathbb{F}$ si $T \in \mathbb{F}$, H_{ij} es uniformemente acotada en $\mathbb{R}_+ \times \Omega \times [0, \ell]$ y la matriz $H(0, \cdot)$ satisface (A1)-(A3).

(B2) a) $C \in \mathbb{F}$, las funciones $G_i(s, x, t)$ y $\partial G_i(s, x, t)/\partial s$ son continuas, $G_i(T(\cdot), \cdot)$ y $\partial G_i(T(\cdot), \cdot)/\partial s \in \mathbb{F}$ si $T \in \mathbb{F}$ y $G_i(\cdot, x, t)$ es de-

creciente en $\mathbb{R}_+$, para todo $(x,t) \in \overline{\Omega} \times [0,\ell]$, $i = 1,\dots,m$.

b) Existen m funciones $g_i \in \mathbb{F}$ tales que $0 < g_i(x,t) \le G_i(T,x,t)$ para todo $(T,x,t) \in \mathbb{R}_+ \times \overline{\Omega} \times [0,\ell]$, $i = 1,\dots,m$.

Sea $\widehat{\lambda_0}$ el valor propio de (1.1) para $\gamma \equiv H(0,\cdot)$. En las hipótesis siguientes consideramos dos posibilidades para $\widehat{\lambda_0}$:

(B3) a) $\widehat{\lambda_0} > 1$ y existen dos números reales a_0, T_0 tales que $a_0 \ge \widehat{\lambda}_0$ y $H_{ij}(T,x,t) \ge a_0\, H_{ij}(0,x,t)$, para todo $T \ge T_0$ y $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

b) Para algún $k \in \{1,\dots,m\}$, $H_{kk}(0,\cdot)$ satisface (A3) y $H_{kk}(T,x,t) \ge H_{kk}(0,x,t)$, para todo $(x,t) \in \Omega \times \mathbb{R}$ y $T > 0$.

(B4) a) $0 < \widehat{\lambda_0} < 1$ y existen dos números reales a_0^* y T_0^* tales que $0 < a_0^* < \widehat{\lambda_0}$ y $H_{ij}(T,x,t) < a_0^*\, H_{ij}(0,x,t)$, para todo $T \ge T_0^*$ y $(x,t) \in \overline{\Omega} \times \mathbb{R}$.

b) Existe un conjunto compacto $D \subset \Omega$ tal que $H_{ij}(T,x,t) = 0$ para todo $T \ge 0$, $t \in \mathbb{R}$ y $x \in (\overline{\Omega}/D)$.

En algunas situaciones supondremos que Ω satisface la condición:

(C^*) Existe una sucesión $\{D_k\}_{k=1}^{\infty}$ de subdominios de $\mathbb{R}^N$, con frontera de clase $C^{2+\alpha}$ tales que $\overline{D}_k \subset D_{k+1} \subset \overline{D}_{k+1} \subset \Omega$, para todo k y $\bigcup_{k=1}^{\infty} D_k = \Omega$.

Teorema 1.3. a) *Supongamos que se satisfacen* (B1)-(B3) *y que* Ω *satisface* (C^*). *Entonces* (1.2) *admite una solución* $(T,\vec{u}_0) \in \mathbb{E}^{m+1}$, $\vec{u}_0 = \mathrm{col}(u_0^1,\dots,u_0^m)$ *tal que* $T(x,t) > 0$ *y* $u_0^i(x,t) > 0$ *en* $\Omega \times \mathbb{R}$, $i = 1,\dots,m$.

b) *Si se satisfacen* (B1), (B2) y (B4) *obtenemos el mismo resultado de la parte* a).

OBSERVACIÓN 1.2. Todos los resultados permanecen válidos si suponemos que $\widetilde{L}(\vec{u}) = (L_1(u^1),\dots,L_m(u^m))$, $\vec{u} = \mathrm{col}(u^1,\dots,u^m)$, donde L_k es uniformemente parabólico en $\Omega \times \mathbb{R}$ para $k = 1,\dots,m$ y tiene la forma

$$L_k[u] = \frac{\partial u}{\partial t} - \Big(\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^k D_{ij} u + \sum_{i=1}^{N} b_i^k D_i u + c_0^k u \Big),$$

y sus coeficientes satisfacen las condiciones antes impuestas para el operador L.

2. Resultados principales.

En esta sección demostraremos los teoremas 1.1 y 1.2. Respecto al sistema (1.1), consideraremos primero, en la Proposición 1, el caso especial en el cual γ satisface (A1) y (A2) y las componentes de su diagonal son mayores o iguales que cero y no nulas.

En la demostración del Teorema 1.1 utilizaremos el siguiente lema:

Lema 2.1. *Si* $u, v \in C^{2,1}(\Omega\times\mathbb{R})\bigcap C^{1,0}(\overline{\Omega}\times\mathbb{R})$, $v(x,t) > 0$, $u(x,t+l) \equiv u(x,t)$, $v(x,t+\ell) \equiv v(x,t)$, *en* $\overline{\Omega}\times\mathbb{R}$, $u(x,t) \equiv 0$ *en* $\partial\Omega\times\mathbb{R}$ *y además* $\partial v\,(\overline{x},\overline{t})/\partial\vec{n} < 0$ *para todo* $(\overline{x},\overline{t}) \in \partial\Omega\times\mathbb{R}$, *para el cual* $v(\overline{x},\overline{t}) = 0$, *entonces existe un número real* $a > 0$ *tal que* $v(x,t) > a\,u(x,t)$ *para todo* $(x,t) \in \Omega\times\mathbb{R}$.

DEMOSTRACIÓN. Supongamos que el lema no es cierto. Entonces podemos construir una sucesión $\{(c_n,\, x_n,\, t_n)\}_{n=1}^{\infty}$, con $c_n > 0$ y $(x_n,t_n) \in \overline{\Omega}\times\mathbb{R}$ tales que $c_n \to 0$ y $(v-c_nu)(x_n,t_n) = 0$, para todo entero positivo n. Sin pérdida de generalidad suponemos que $(x_n,t_n) \to (\overline{x},\overline{t})$, para un $(\overline{x},\overline{t}) \in \overline{\Omega}\times[0,\ell]$. Además $0 = \lim_{n\to\infty}(v-c_nu)(x_n,t_n) = v(\overline{x},\overline{t})$, implica que $(\overline{x},\,\overline{t}) \in \partial\Omega\times\mathbb{R}$. Sea z_n el punto de $\partial\Omega$ más cercano a x_n. De nuestras hipótesis sobre u y v y del Teorema del Valor Medio obtenemos

$$\begin{aligned} 0 &\le (v-c_nu)(z_n,t_n) - (v-c_nu)(x_n,t_n) \\ &= \langle(\nabla_x v - c_n\nabla_x u)(\xi_n,t_n),\ z_n - x_n\rangle\,, \end{aligned}$$

donde $\xi_n \in \Omega$ está en el segmento que une x_n con z_n, $\langle\cdot,\cdot\rangle$ es el producto interno en $\mathbb{R}^N$ y ∇_x es el gradiente respecto a x. Entonces,

$$\langle(\nabla_x v - c_n\nabla_x u)(\xi_n,t_n),\ \ \vec{n}(z_n,t_n)\rangle \ge 0\,.$$

$\big(\vec{n}(z_n,t_n)$ es la normal unitaria hacia el exterior a $\partial\Omega\times\mathbb{R}$ en $(z_n,t_n)\big)$. Si hacemos tender n a infinito en la última desigualdad obtenemos $\partial v\,(\overline{x},\overline{t})/\partial\vec{n} \ge 0$, lo cual contradice una de nuestras hipótesis.

DEMOSTRACIÓN DEL TEOREMA 1.1. Sea $k > 0$ una constante suficientemente grande tal que $p_{ii}+k > 0$ y $q_{ii}+k > 0$ en $\overline{\Omega}\times\mathbb{R}$, $i = 1,\ldots,m$.

De las hipótesis, obtenemos

$$L[v^i] + k\,v^i = (q_{ii} + k)v^i + \sum_{\substack{j=1 \\ j\neq i}}^{m} q_{ij}\,v^j \geq 0 \qquad \text{en } \Omega \times \mathbb{R},$$

$v^i \geq 0$ y $v^i \not\equiv 0$ en $\overline{\Omega} \times \mathbb{R}$. Entonces, por el Principio del Máximo para ecuaciones parabólicas [12], tenemos que

$$v^i > 0 \ \text{ en } \Omega \times \mathbb{R}, \qquad \text{y} \qquad \frac{\partial v^i}{\partial \vec{n}}(\overline{x}, \overline{t}) < 0 \tag{2.1}$$

si $v^i(\overline{x}, \overline{t}) = 0$, y $(\overline{x}, \overline{t}) \in \partial\Omega \times \mathbb{R}$.

Sea $\delta_i = \sup\{a_i : v^i - a_i u^i > 0 \text{ en } \Omega \times \mathbb{R}\}$. Por (2.1) y el Lema 2.1, $0 < \delta_i \leq \infty$, $i = 1, \ldots, m$. Por otra parte podemos suponer sin pérdida de generalidad que alguna de las componentes de $\vec{u}$ toma un valor positivo. Entonces, si $\delta_j \equiv \min\{\delta_1, \ldots, \delta_m\}$, se tiene que

$$0 < \delta_j < \infty \quad \text{y} \quad v^i - \delta_j u^i \geq 0 \quad \text{en } \Omega \times \mathbb{R}, \quad i = 1, \ldots, m.$$

Más aún, probaremos que $v^j - \delta_j u^j \equiv 0$ en $\overline{\Omega} \times \mathbb{R}$. Por hipótesis tenemos

$$\begin{aligned}(L + k)(v^j - \delta_j u^j) \geq{}& (k + p_{jj})(v^j - \delta_j u^j) \\ &+ \sum_{\substack{i=1 \\ i\neq j}}^{m} p_{ji}(v^i - \delta_j u^i) \\ &+ \sum_{i=1}^{m} (q_{ji} - p_{ji})v^i \geq 0 \qquad \text{en } \Omega \times \mathbb{R}.\end{aligned} \tag{2.2}$$

Por esta desigualdad, el Lema 2.1 y la definición de δ_j, existe un punto $(x_0, t_0) \in \Omega \times \mathbb{R}$ tal que $(v^j - \delta_j u^j)(x_0, t_0) = 0$. Finalmente por (2.2) y el Principio del Máximo, concluímos que $v^j - \delta_j u^j \equiv 0$ y $p_{ji} = q_{ji}$ en $\Omega \times \mathbb{R}$, para $i = 1, \ldots, m$.

El próximo lema se utilizará en lo que sigue.

Lema 2.2. *Para cada $f \in \mathbb{F}$, existe una única función $u \in \mathbb{E}$ tal que $L[u] \equiv f$ en $\Omega \times \mathbb{R}$. Además existen dos constantes k_1, $k_2 > 0$ tales que*

a) $\|u\|_{\mathbb{E}} \leq k_1 \|L[u]\|_{\mathbb{F}}$, *para todo* $u \in \mathbb{E}$,

b) $\|u\|_{C^{1+\sigma,\sigma/2}(\overline{\Omega}\times[0,\ell])} \leq k_2 \|L[u]\|_{\infty}$ *para todo* $u \in \mathbb{E}$,

(k_2 *depende de* $\sigma \in (\alpha, 1)$) (*cf.* [11], [13])

En la demostración del Teorema 1.2 utilizaremos la siguiente proposición.

Proposición 2.1. *Si la matriz* γ *satisface* (A1) y (A2) *y* $\gamma_{ii}(x,t) \geq 0$ *y* $\gamma_{ii} \not\equiv 0$ *en* $\overline{\Omega} \times \mathbb{R}$ *para todo* $i = 1, \ldots, m$, *entonces existe* $(\lambda_0, \vec{u}_0) \in \mathbb{R}_+ \times \mathbb{E}^m$ *tal que* λ_0 *es único y*

i) $(\lambda_0, \vec{u}_0)$ *es solución de* (1.1),

ii) $\vec{u}_0 \in \mathbb{E}^m \cap \operatorname{Int} \mathbb{P}$ *y* $\partial u_0^j / \partial \vec{n} < 0$ *en* $\partial\Omega \times \mathbb{R}$, $j = 1, \ldots, m$.

iii) λ_0 *es un valor propio simple.*

DEMOSTRACIÓN. Sea $S : \mathbb{F}^m \to \mathbb{F}^m$ el operador lineal completamente continuo y positivo definido así:

$$S(\operatorname{col}(u^1, \ldots, u^m)) = \operatorname{col}\Big(J^* L^{-1}\Big(\sum_{j=1}^{m} \gamma_{1j} u^j \Big), \ldots, J^* L^{-1}\Big(\sum_{j=1}^{m} \gamma_{mj} u^j \Big) \Big)$$

donde $J^* : \mathbb{E} \to \mathbb{F}$ es la inyección natural, la cual es compacta, [3].

Sean $\vec{z} = \operatorname{col}(L^{-1}(1), \ldots, L^{-1}(1))$ y $\vec{v} = (\widetilde{L})^{-1}(\gamma \vec{z}) = S(\vec{z})$. Con el mismo argumento usado en el Teorema 1.1, obtenemos

$$\begin{aligned} &S(\vec{z}) - \delta \vec{z} \in \mathbb{P}, \\ &\delta = \min\{\sup\{c_i : \ v^i - c_i z^i > 0 \text{ en } \Omega \times \mathbb{R}\} : \ i \in J\}. \end{aligned} \tag{2.3}$$

Por (2.3) y un resultado muy conocido de la teoría de operadores positivos (*cf.* [5, Teorema 2.5]), existen $\vec{u}_0 \in \mathbb{P}$, $\vec{u}_0 \not\equiv \vec{0}$ y un número real $\lambda_1 > \delta$ tales que $S(\vec{u}_0) = \lambda_1 \vec{u}_0$, esto es, $\widetilde{L}[\vec{u}_0] = \lambda_0 \gamma \vec{u}_0$, $(\lambda_0 = 1/\lambda_1)$ y $(\lambda_0, \vec{u}_0)$ satisface (i) y (ii).

La unicidad de λ_0 se sigue directamente del Teorema 1.1. Utilizaremos también este teorema para demostrar que λ_0 es simple. Sea $\vec{w} = \text{col}(w^1, \ldots, w^m)$, $\vec{w} \not\equiv \vec{0}$ tal que $\widetilde{L}[\vec{w}] = \lambda_0 \gamma \vec{w}$. Podemos suponer que alguna componente de $\vec{w}$ toma un valor positivo en $\Omega \times \mathbb{R}$. Entonces, por el Teorema 1.1, existe $\delta^* > 0$ tal que $u_0^j - \delta^* w^j \equiv 0$, para ciertos $j \in J$ y $u_0^i - \delta^* w^i \geq 0$ en $\Omega \times \mathbb{R}$, para todo $i \in J$. Del Principio del Máximo y la Observación 1.1 se deduce que $\vec{u}_0 - \delta^* \vec{w} \equiv \vec{0}$ en $\Omega \times \mathbb{R}$.

Para demostrar que la multiplicidad algebraica de λ_0 es uno, supongamos que $\vec{v}_0 = \text{col}(v_0^1, \ldots, v_0^m) \in \widehat{\mathbb{E}}$, $\vec{v}_0 \not\equiv \vec{0}$, satisface

$$(S - \frac{1}{\lambda_0} I)^2 \vec{v}_0 = 0 .$$

(I es la matriz identidad en $\widehat{\mathbb{E}}$).

Entonces $(S - \lambda_1 I)\vec{v}_0 = -k^* \vec{u}_0$, para una constante k^* que podemos suponer positiva. Sea $\widehat{\delta} > 0$ suficientemente grande tal que

$$w_0^i = \widehat{\delta} u_0^i + v_0^i > 0 \qquad \text{en } \Omega \times \mathbb{R}, \text{ para todo } i \in J .$$

Por un simple cálculo se obtiene

$$L[w_0^i] \geq \lambda_0 \sum_{j=1}^{m} \gamma_{ij} w_0^j \qquad \text{en } \Omega \times \mathbb{R}, \ i \in J .$$

Entonces, por el Teorema 1.1, existen $s \in J$ y $a \in \mathbb{R}_+$ tales que

$$w_0^s = a u_0^s, \quad w_0^j - a u_0^j \geq 0 \qquad \text{en } \Omega \times \mathbb{R}, \ j = 1, \ldots, m . \tag{2.4}$$

Por la identidad

$$0 = L(w_0^s - a u_0^s) = \lambda_0 \sum_{j=1}^{m} \gamma_{sj} (w_0^j + (\lambda_0 k^* - a) u_0^j) \qquad \text{en } \Omega \times \mathbb{R},$$

y (2.4), existe un punto $(x_0, t_0) \in \Omega \times \mathbb{R}$ tal que

$$(w_0^s + (\lambda_0 k^* - a) u_0^s)(x_0, t_0) = 0 .$$

Puesto que

$$L[w_0^s + (\lambda_0 k^* - a) u_0^s] = \lambda_0 k^* (\lambda_0 k^* - a) u_0^s \geq 0 \qquad \text{en } \Omega \times \mathbb{R},$$

por el Principio del Máximo obtenemos que $w_0^s + (\lambda_0 k^* - a)u_0^s \equiv 0$ en $\Omega \times \mathbb{R}$, y por (2.4), $k^* = 0$.

En el resto de esta sección supondremos que γ satisface (A1)-(A3) y $k \in \{1, \dots, m\}$ es el entero positivo tal que γ_{kk} satisface (1.6). Denotaremos con $\gamma^+ \equiv [\gamma_{ij}^+]$ y $\gamma^- \equiv [\gamma_{ij}^-]$. Es claro que si $i \neq j$, $\gamma_{ij}^+ = \gamma_{ij}$ y γ^- es una matriz diagonal. Para cada $s \in (0,1]$, denotaremos con $\rho_s = [\rho_{ij}^s]$, donde

$$\rho_{ij}^s = s\gamma_{ij}^+ \qquad \text{si } i \neq j \text{ ó } i = j \neq k, \qquad \rho_{kk}^s = \gamma_{kk}^+ .$$

Por la Proposición 2.1, para cada $s \in (0,1]$, $\lambda > 0$ existe $(B_s(\lambda), \vec{u}_{s\lambda}) \in \mathbb{R}_+ \times (\mathbb{E}^m \bigcap \operatorname{Int} \mathbb{P})$ tal que

$$\widetilde{L}[\vec{u}_{s\lambda}] + \lambda\gamma^-[\vec{u}_{s\lambda}] = B_s(\lambda)\rho_s[\vec{u}_{s\lambda}] \qquad \text{en } \Omega \times \mathbb{R}. \tag{2.5}$$

Claramente, (1.1) tiene solución si y sólo si $B_1(\lambda) = \lambda$ para algún $\lambda > 0$. Por el Teorema 1.1, para s fijo, $B_s(\lambda)$ es estrictamente creciente en $\mathbb{R}_+$ y, para λ fijo, $B_s(\lambda)$ es estrictamente decreciente.

El siguiente resultado se utilizará en el resto de esta sección.

Lema 2.3. *Si γ satisface* (A1)-(A3), *entonces $B_s(\lambda)$ es continua en* $(0,1] \times \mathbb{R}_+$.

DEMOSTRACIÓN. Sean $a, b > 0$, $\lambda^* \in [0,a]$, $s^* \in [b,1]$ y $\{(s_n, \lambda_n)\}_{n=1}^{\infty}$ una sucesión en $[b,1] \times [0,a]$ tal que $(s_n, \lambda_n) \to (s^*, \lambda^*)$. Podemos suponer que $B_{s_n}(\lambda_n) \to d \in [0, B_b(a)]$. Para cada n, existe $\vec{u}_n = \operatorname{col}(u_n^1, \dots, u_n^m) \in \mathbb{E}^m \bigcap \operatorname{Int} \mathbb{P}$ tal que

$$\|\vec{u}_n\|_\infty \equiv \max\{\|u_n^i\|_\infty : \ i = 1, \dots, m\} = 1$$

y

$$(\widetilde{L})[\vec{u}_n] + \lambda_n\gamma^-\vec{u}_n = B_{s_n}(\lambda_n)\rho_{s_n}\vec{u}_n \quad \text{en } \Omega \times \mathbb{R}.$$

Por el Lema 2.2 la sucesión $\{u_n^i\}$ es acotada en $\widetilde{\mathbb{F}}$ y puesto que la inyección natural $J^* : \widetilde{\mathbb{F}} \to \mathbb{F}$ es compacta, (para una subsucesión) $u_n^i \to v^i \in \mathbb{F}$, en la norma de $\mathbb{F}$, $i = 1, \dots, m$, $\vec{v} = \operatorname{col}(v^1, \dots, v^m)$, $\|\vec{v}\|_\infty = 1$. Entonces

$$\begin{aligned} \vec{v} &= \lim_{n\to\infty} \vec{u}_n = \lim_{n\to\infty} (\widetilde{L})^{-1}(-\lambda_n\gamma^-\vec{u}_n + B_{s_n}(\lambda_n)\rho_{s_n}\vec{u}_n) \\ &= (\widetilde{L})^{-1}(-\lambda^*\gamma^-\vec{v} + d\rho_{s^*}\vec{v}). \end{aligned} \tag{2.6}$$

Finalmente para (s^*, λ^*), existe $\vec{u}_{s^*\lambda^*} \in \mathbb{E}^m \bigcap \operatorname{Int} \mathbb{P}$ tal que

$$(\widetilde{L})(\vec{u}_{s^*\lambda^*}) = -\lambda^* \gamma^- \vec{u}_{s^*\lambda^*} + B_{s^*}(\lambda^*)\rho_{s^*}\vec{u}_{s^*\lambda^*} \quad \text{en } \Omega \times \mathbb{R}\,. \tag{2.7}$$

De (2.6), (2.7) y el Teorema 1.1 se deduce que $B_{s^*}(\lambda^*) = d$.

Lema 2.4. *Para s fijo, $B_s(\lambda)$ tiene por lo menos un punto fijo.*

DEMOSTRACIÓN. Sea $s \in (0,1]$ fijo. Puesto que $B_s(\lambda)$ es continua y $B_s(0) > 0$, es suficiente probar que $B_s(\lambda^*) < \lambda^*$ para algún $\lambda^* > 0$.

Sea $G \in \mathbb{R}^N$ una bola abierta con centro 0, tal que $y + h(r) \in \Omega$ para todo $y \in G$ y $r \in [0, \ell]$ y

$$\int_0^\ell \gamma_{kk}(y + h(r), r)\, ds > 0$$

para todo $y \in G$.

Sea $\phi : \mathbb{R}^N \to [0,1]$, $\phi \not\equiv 0$ una función de clase C^∞ con soporte compacto contenido en G. Es claro que

$$\int_{G\times[0,\ell]} \gamma_{kk}(y + h(r), r)\phi^2 > 0\,. \tag{2.8}$$

Introducimos la siguiente notación

$$\begin{aligned}
h(r) &= (h_1(r), \ldots, h_N(r))\,,\\
a_{ij}(y,r) &= \overline{a}_{ij}(y + h(r), r)\,,\\
b_i(y,r) &= \overline{b}_i(y + h(r), r)\,,\\
c_0(y,r) &= \overline{c}_0(y + h(r), r)\,,\\
\overline{\psi}^j_\lambda(x,t) &= \log(u^j_{s\lambda}(x,t))\,,\\
\psi^j_\lambda(y,r) &= \overline{\psi}^j_\lambda(y + h(r), r)\,,\\
D_i\psi^k_\lambda &= D_{y_i}\psi^k_\lambda\,,\\
D_{ij}\psi^k_\lambda &= D_{y_iy_j}\psi^k_\lambda\,.
\end{aligned}$$

Remplazando $\overline{\psi}^k_\lambda$ y sus derivadas en (2.5) obtenemos

$$\frac{\partial}{\partial t}\overline{\psi}^k_\lambda(x,t) - \sum_{i,j=1}^N \overline{a}_{ij} D_{x_ix_j}\overline{\psi}^k_\lambda - \sum_{i,j=1}^N \overline{a}_{ij} D_{x_i}\overline{\psi}^k_\lambda D_{x_j}\overline{\psi}^k_\lambda -$$

$$-\sum_{i=1}^{N} \overline{b}_i D_{x_i} \overline{\psi}_\lambda^k \geq \overline{c}_0(x,t) - \lambda \overline{\gamma}_{kk}(x,t) + B_s(\lambda)\, \gamma_{kk}^+(x,t)\,.$$

Cambiando de variables, multiplicando por ϕ^2 e integrando sobre $D = G \times [0,\ell]$, obtenemos

$$\begin{aligned} \int_D \Big(\frac{\partial \psi_\lambda}{\partial r} &- \sum_{i=1}^{N} D_i \psi_\lambda^k h_i' - \sum_{i,j=1}^{N} a_{ij} D_{ij} \psi_\lambda^k \\ &- \sum_{i,j=1}^{N} a_{ij} D_i \psi_\lambda^k D_j \psi_\lambda^k - \sum_{i=1}^{N} b_i D_i \psi_\lambda^k \Big) \phi^2 \\ &\geq \int_D [c_0 - \lambda\, \gamma_{kk}^-(y + h(r), r) + B_s(\lambda)\, \gamma_{kk}^+(y + h(r), r)]\, \phi^2. \end{aligned} \tag{2.9}$$

Puesto que $\int_D \dfrac{\partial \psi_\lambda^k}{\partial r} \phi^2\, dr\, dy = 0$, integrando por partes obtenemos de (2.9)

$$\begin{aligned} &\int_D \left(B_s(\lambda) \gamma_{kk}^+ - \lambda \gamma_{kk}^- \right)(y + h(r), r) \phi^2(y, r) \\ &\qquad \leq - \int_D \left(\langle \phi \nabla_y \psi_\lambda^k,\ A(\phi \nabla_y \psi_\lambda^k) + \vec{d} \rangle + c_0 \phi^2 \right), \end{aligned} \tag{2.10}$$

donde $\vec{d} = (d_1, \ldots, d_m)$, con

$$d_j = -\sum_{i=1}^{N} (D_i a_{ij} \phi + 2 a_{ij} D_i \phi) + b_j \phi + h_j' \phi\,,$$

A es la matriz $A = [a_{ij}]$, ∇_y es el gradiente respecto a y, y $\langle \cdot, \cdot \rangle$ el producto escalar en $\mathbb{R}^N$. Observemos que $\vec{d}$ es independiente de ψ_λ^k.

Por ser A simétrica y L uniformemente parabólico, tenemos que para todo vector $\vec{B}$

$$\begin{aligned} \langle \vec{B},\ A\vec{B} + \vec{d} \rangle &= \langle \vec{B} + \frac{1}{2} A^{-1} \vec{d},\ A(\vec{B} + \frac{1}{2} A^{-1} \vec{d}) \rangle \\ &\quad - \frac{1}{4} \langle A^{-1} \vec{d},\ \vec{d} \rangle \geq -\frac{1}{4} \langle A^{-1} \vec{d},\ \vec{d} \rangle\,. \end{aligned}$$

Entonces, de (2.10), obtenemos

$$B_s(\lambda) \leq a + \frac{\lambda \int_D \gamma_{kk}^-(y+h(r),r)\phi^2}{\int_D \gamma_{kk}^+(y+h(r),r)\phi^2}\,, \tag{2.11}$$

donde

$$a = \frac{\int_D [\frac{1}{4}\langle A^{-1}\vec{d},\vec{d}\rangle - c_0\phi^2]}{\int_D \gamma_{kk}^+(y+h(r),r)\phi^2}\,.$$

De (2.8), deducimos que existe λ^* tal que para todo $\lambda > \lambda^*$,

$$\frac{\lambda(\int_D \gamma_{kk}(y+h(r),r)\phi^2)}{\int_D \gamma_{kk}^+(y+h(r),r)\phi^2} > a\,,$$

y por (2.11), $B_s(\lambda) < \lambda$ para todo $\lambda \geq \lambda^*$, y por tanto, $B_s(\lambda)$ tiene un punto fijo menor que λ^*.

Del lema anterior se desprende la existencia de soluciones de (1.1). El resto de esta sección lo dedicamos a probar la unicidad de los valores propios de (1.1) con función propia con componentes positivas. Para cada $s \in (0,1]$ denotaremos con $\lambda_1(s)$ el primer punto fijo de $B_s(\lambda)$.

Lema 2.5. *Si γ satisface* (A1)-(A3), *entonces $\lambda_1(s)$ es estrictamente decreciente y continua.*

DEMOSTRACIÓN. Supongamos que $s_1 < s_2$. Del Teorema 1.1 deducimos que $B_{s_2}(\lambda) < B_{s_1}(\lambda)$ para todo $\lambda > 0$. En particular para $\lambda_1 = \lambda_1(s_1)$ y $\lambda_2 = \lambda_1(s_2)$ se tiene que $\lambda_2 < B_{s_2}(\lambda_1) < B_{s_1}(\lambda_1) = \lambda_1$.

Supongamos ahora que $\overline{s} \in (0,1]$, $\{s_n\}_{n=1}^{\infty}$ es una sucesión en $(0,1]$, $s_n \to \overline{s}$, y $\lambda_1(s_n) \to d$. Para cada n, existe $\vec{u}_{s_n} \in \mathbb{E}^m \cap \operatorname{Int}\mathbb{P}$ tal que $\|\vec{u}_{s_n}\|_\infty = 1$, y $\widetilde{L}[\vec{u}_{s_n}] + \lambda_1(s_n)\gamma^-\vec{u}_{s_n} = \lambda_1(s_n)\rho_{s_n}\vec{u}_{s_n}$ en $\Omega \times \mathbb{R}$. Con el mismo argumento usado en el Lema 2.3 obtenemos una función $\vec{v_0} \in \mathbb{E}^m \cap \operatorname{Int}\mathbb{P}$, tal que $\|\vec{v_0}\|_\infty = 1$, y

$$\widetilde{L}[\vec{v_0}] + d\gamma^-\vec{v_0} = d\rho_s\text{-}\vec{v_0} \qquad \text{en } \Omega \times \mathbb{R}\,. \tag{2.12}$$

De (2.12) es claro que $d \geq \lambda_1(\overline{s})$. Si $d \neq \lambda_1(\overline{s})$, $B_{\overline{s}}(\lambda)$ tendría un segundo punto fijo $\lambda_0 > \lambda_1(\overline{s})$. Sea $\overline{\overline{\lambda}} \in (\lambda_1(\overline{s}), \lambda_0)$, es claro que $B_{\overline{s}}(\overline{\overline{\lambda}}) < \overline{\overline{\lambda}}$ y puesto que $\lim_{n\to\infty} B_{s_n}(\overline{\overline{\lambda}}) = B_{\overline{s}}(\overline{\overline{\lambda}})$, existe un entero positivo N_0 tal que $B_{s_n}(\overline{\overline{\lambda}}) < \overline{\overline{\lambda}} < \lambda_0$ para todo $n \geq N_0$. Consecuentemente, $\lambda_1(s_n) < \overline{\overline{\lambda}} < \lambda_0$, y entonces, $d = \lambda_1(\overline{s}) < \overline{\overline{\lambda}} < \lambda_0$.

Lema 2.6. *Sean $p \in \mathbb{F}$ y $\xi(t) = \max_{\Omega} p(x,t)$. El problema*

$$L[u] = \lambda p u \quad \text{en } \Omega \times \mathbb{R},\ u \in \mathbb{E},\ u > 0,\ \text{en } \Omega \times \mathbb{R},\ \lambda > 0, \tag{2.13}$$

tiene solución si $\int_0^\ell \xi(t)\, dt > 0$ *y no tiene solución si* $\int_0^\ell \xi(t)\, dt = 0$.

DEMOSTRACIÓN. Si $\int_0^\ell \xi(t)\, dt > 0$, la existencia de soluciones de (2.13) es consecuencia directa del Lema 2.4, puesto que es posible construir una función $\sigma : \mathbb{R} \to \Omega$ de clase C^2 tal que $\int_0^\ell p(\sigma(t), t)\, dt > 0$.

Supongamos que $\int_0^\ell \xi(t)\, dt = 0$ y que existe $(\overline{\lambda}, u_{\overline{\lambda}})$ solución de (2.13), $\overline{\lambda} > 0$, $u_{\overline{\lambda}} > 0$. Sea $\phi_0 \in \mathbb{E}$, $\|\phi_0\|_\infty = 1$, la solución del problema

$$L[\phi] = \lambda_0 \phi \qquad \text{en } \Omega \times \mathbb{R},\ \lambda_0 > 0,\ \phi > 0,\ \phi \in \mathbb{E},\ \|\phi\|_\infty = 1,$$

(*cf.* [6]). Sea

$$v_{\overline{\lambda}} = \phi(x,t) e^{\overline{\lambda} \int_0^t \xi(s)\, ds},$$

entonces

$$L[v_{\overline{\lambda}}] = \lambda_0 v_{\overline{\lambda}} + \overline{\lambda}\, \xi(t) v_{\overline{\lambda}}.$$

Puesto que $\xi(t) \geq p(x,t)$, el Teorema 1.1 implica que $\overline{\lambda}\, \xi(t) + \lambda_0 = \overline{\lambda}\, p(x,t)$. Esta contradicción demuestra el lema.

Denotemos con $\overline{\xi}(t) = \max\{\gamma_{kk}(x,t) :\ x \in \Omega\}$. Por (1.6),

$$\int_0^\ell \overline{\xi}(t) dt > 0.$$

Sea $s^* \in (0,1)$, tal que

$$s^* \int_0^\ell \overline{\xi}^{+}(t)\, dt = \int_0^\ell \overline{\xi}(t)\, dt.$$

Para cada $s \in (0,1]$, $\lambda > 0$ existen $u_\lambda \in \mathbb{E}, u_\lambda > 0$ y $\overline{B}_s(\lambda) > 0$ tales que

$$L[u_\lambda] + \lambda\gamma_{kk}^- u_\lambda = \overline{B}_s(\lambda)(s\gamma_{kk}^+)u_\lambda \qquad \text{en } \Omega \times \mathbb{R}\,.$$

Por el Lema anterior si $s \in (s^*, 1]$, $\overline{B}_s(\lambda)$ tiene por lo menos un punto fijo, el primero de estos será denotado $\overline{\lambda}_1(s)$. Como en el Lema 2.5 se prueba que $\overline{\lambda}_1(s)$ es estrictamente decreciente y continua en $(s^*, 1]$.

Lema 2.7. $\overline{\lambda}_1(s) \to \infty$ *cuando* $s \to s^*$ *por la derecha.*

DEMOSTRACIÓN. Supongamos que $\overline{\lambda}_1(s) \to d < \infty$ cuando $s \to s^*$, $s > s^*$. Sea $u_s \in \mathbb{E}$, $u_s > 0$ en $\Omega \times \mathbb{R}$, $\|u_s\|_\infty = 1$, tal que

$$L[u_s] + \overline{\lambda}_1(s)\,\gamma_{kk}^- u_s = \overline{\lambda}_1(s)\,s\,\gamma_{kk}^+ u_s \qquad \text{en } \Omega \times \mathbb{R}\,.$$

Con el mismo argumento del Lema 2.3, existe $v \in \mathbb{E}$, $v > 0$ en $\Omega \times \mathbb{R}$ tal que

$$L[v] + d\gamma_{kk}^- v = ds^*\gamma_{kk}^+ v \qquad \text{en } \Omega \times \mathbb{R}\,,$$

lo cual contradice el lema anterior.

Lema 2.8. *Si* $(\lambda, \vec{u}) \in (\mathbb{R}_+ - \{0\}) \times (\mathbb{E}^m \cap \operatorname{Int} \mathbb{P})$ *satisface* (1.1), *entonces* $\lambda \leq \overline{\lambda}_1(1)$.

DEMOSTRACIÓN. Sea $\vec{u} = \operatorname{col}(u^1, \ldots, u^m)$, es claro que

$$L[u^k] \geq \lambda[\gamma_{kk}^+ - \gamma_{kk}^-]u^k\,.$$

Si suponemos que $\lambda > \overline{\lambda}_1(1)$, por el lema anterior, existen $s \in (s^*, 1)$ y $u_s \in \mathbb{E}$, $u_s > 0$, tales que $\lambda = \overline{\lambda}_1(s)$ y

$$L[u_s] = \lambda[\gamma_{kk}^+ - \gamma_{kk}^-]u_s \qquad \text{en } \Omega \times \mathbb{R}\,.$$

Entonces, por el Teorema 1.1, obtenemos una contradicción.

Lema 2.9. $\lambda_1(s) \to \overline{d} \geq \overline{\lambda}_1(1)$ *si* $s \to 0$, $(\overline{d} \leq \infty)$.

DEMOSTRACIÓN. Suponiendo que $\lambda_1(s) \to \overline{d} < \infty$, cuando $s \to 0$, como en los lemas 2.7 y 2.3, obtenemos una función $\vec{v} \in \mathbb{E}^m \cap \operatorname{Int} \mathbb{P}$,

$\vec{v} = \mathrm{col}(v^1, \ldots, v^m)$, tal que $L[v^k] = d\gamma_{kk}v^k$ en $\Omega \times \mathbb{R}$. Entonces $\overline{d} \geq \overline{\lambda}_1(1)$.

DEMOSTRACIÓN DEL TEOREMA 1.2. El Lema 2.4 y el Principio del Máximo implican la existencia de $(\lambda, \vec{u}_1) \in \mathbb{R}_+ \times \mathbb{E}^m \cap \operatorname{Int} \mathbb{P}$, el cual satisface i) y ii), $\lambda_1 = \lambda_1(1)$.

Sea $(\lambda, \vec{u}_\lambda) \in \mathbb{R}_+ \times \mathbb{E}^m \cap \operatorname{Int} \mathbb{P}$, una solución de (1.1), con $\lambda > \lambda_1(1)$. Por el Lema 2.8, $\lambda \leq \overline{\lambda}_1(1)$, y por los Lemas 2.5 y 2.9, existen $s \in (0,1)$ y $\vec{u}_s \in \mathbb{E}^m \cap \operatorname{Int} \mathbb{P}$ tales que $\lambda = \lambda_1(s)$, y

$$\widetilde{L}[\vec{u}_s] = \lambda[s\gamma^+ - \gamma^-]\vec{u}_s \qquad \text{en } \Omega \times \mathbb{R}.$$

Entonces, del Teorema 1.1, obtenemos que $s\gamma^+ = \gamma^+$. Esta contradicción demuestra la unicidad de λ_1. La simplicidad de λ_1 es consecuencia de un resultado de Teoría de Perturbaciones Analíticas debido a Kato, (*cf.* [4, Lema 8.a]).

3. Aplicaciones.

En esta sección aplicaremos los teoremas 1.1 y 1.2 al estudio del sistema (1.2). En la demostración del Teorema 1.3 utilizaremos los siguientes lemas.

Lema 3.1. *Si* $u \in C^{2+\alpha,1+(\alpha/2)}(\overline{D}_k \times \mathbb{R})$, $f \in C^{\alpha,\alpha/2}(\overline{D}_k \times \mathbb{R})$, $f(x,t+\ell) \equiv f(x,t)$, $u(x,t+\ell) \equiv u(x,t)$ *y* $L[u] \equiv f$ *en* $D_k \times \mathbb{R}$, *entonces*

$$\|u\|_{2+\alpha}^{D_k \times [0,\ell]} \leq k \left[\|u\|_{\alpha+1}^{\partial D_k \times [0,\ell]} + \|u\|_{\alpha}^{D_k \times [0,\ell]} + \|f\|_{\alpha}^{D_k \times [0,\ell]} \right] \tag{3.1}$$

(D_k *está dado en* (C^*)).

DEMOSTRACIÓN. Sea $\phi \in C^\infty(\mathbb{R})$, tal que $\phi(\ell/2) = 0$, $\phi(t) = 1$ para $t \geq \ell$, $0 \leq \phi \leq 1$ en $\mathbb{R}$. Es claro que

$$L[\phi u] = \phi' u + \phi f \qquad \text{en } D_k \times \mathbb{R},$$

para todo k, y

$$\|u\|_{2+\alpha}^{D_k \times [0,\ell]} = \|u\|_{2+\alpha}^{D_k \times [\ell,2\ell]} = \|\phi\, u\|_{2+\alpha}^{D_k \times [\ell,2\ell]} \leq \|\phi\, u\|_{2+\alpha}^{D_k \times [\ell/2,2\ell]}.$$

Entonces, a partir de una desigualdad clásica en la teoría de ecuaciones parabólicas (*cf.* [6, Cap IV, Teorema 5.3]), obtenemos

$$\begin{aligned}\|\phi u\|_{2+\alpha}^{D_k\times[\ell/2,\ell]} &\le c_1\left(\|\phi u\|_{1+\alpha}^{\partial D_k\times[\ell/2,\ell]} + \|\phi f+\phi' u\|_{\alpha}^{D_k\times[\ell/2,2\ell]}\right)\\ &\le k_3\left(\|u\|_{1+\alpha}^{\partial D_k\times[0,\ell]} + \|u\|_{\alpha}^{D_k\times[0,\ell]} + \|f\|_{\alpha}^{D_k\times[0,\ell]}\right).\end{aligned}$$

OBSERVACIÓN 3.1. Sean $\widehat{\mathbb{F}} = \{\hat{u} \in C(\overline{\Omega}\times\mathbb{R}) : \text{existe una sucesión } \{u_n\} \text{ en } \mathbb{F} \text{ tal que } \|u_n-\hat{u}\|_\infty \to 0\}$, $\|\cdot\|_{\widehat{\mathbb{F}}} = \|\cdot\|_\infty$, y $\widetilde{\mathbb{F}}$ como en los preliminares.

Por el Lema 2.2, el operador $L^{-1}:\mathbb{F}\to\mathbb{E}$ puede ser extendido por continuidad al operador continuo y positivo $\widehat{L}^{-1}:\widehat{\mathbb{F}}\to\widetilde{\mathbb{F}}$, (se define $\widehat{L}^{-1}(\hat{u})=\hat{v}$ si y sólo si $\hat{v}=\lim_{n\to\infty}(L^{-1}(u_n))$ en la $\widetilde{\mathbb{F}}$-norma, donde $\{u_n\}$ está en $\mathbb{F}$ y $\|u_n-\hat{u}\|_\infty\to 0$)

El siguiente Lema es muy conocido (*cf.* [1]).

Lema 3.2. *Sea $g(x,t,u)$ una función continua definida de $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}$ en $\mathbb{R}$, ℓ-periódica en t, tal que $\partial g/\partial u$ es continua y $g(\cdot,\cdot,v)$ y $\partial g(\cdot,\cdot,v)/\partial u$ pertenecen a $\mathbb{F}$, para todo $v\in\mathbb{F}$. Si existen dos funciones $v,w\in\mathbb{F}\cap C^{2,o}(\overline{\Omega}\times\mathbb{R})$ tales que*

$$L[v]\ge g(\cdot,\cdot,v), \qquad L[w]\le g(\cdot,\cdot,w) \tag{3.2}$$

$w\le v$ en $\Omega\times\mathbb{R}$, $v\ge 0\ge w$ en $\partial\Omega\times\mathbb{R}$, entonces existen dos soluciones $u^*,u_*\in\mathbb{E}$ del problema

$$L[u]=g(\cdot,\cdot,u) \qquad \text{en } \Omega\times\mathbb{R},\ u\in\mathbb{E} \tag{3.3}$$

tales que

i) *$w\le u_*\le u^*\le v$ en $\overline{\Omega}\times\mathbb{R}$, y*

ii) *si $\tilde{u}$ es solución de (3.3) y $w\le\tilde{u}\le v$ en $\Omega\times\mathbb{R}$, entonces $u_*\le\tilde{u}\le u^*$ en $\Omega\times\mathbb{R}$.*

(Dos funciones v, w que satisfacen (3.2) se llaman supersoluciones y subsoluciones de (3.3) respectivamente).

Lema 3.3. *Si se satisface* (B2), *para cada* $\vec{u} = \mathrm{col}(u^1, \dots, u^m) \in \mathbb{P}$, *existe una única solución* $T(\vec{u}) \in \mathbb{E}$, $T(\vec{u}) \geq 0$ *en* $\Omega \times \mathbb{R}$ *del problema*

$$L[T(\vec{u})] = -C(x,t)\,T(\vec{u})(x,t) + \sum_{j=1}^{m} G_j(T(\vec{u}), x, t)u^j \quad \text{en } \Omega \times \mathbb{R}. \tag{3.4}$$

DEMOSTRACIÓN. Sean $\vec{u} = \mathrm{col}(u^1, \dots, u^m) \in \mathbb{P}$, $\underline{T}(x,t) \equiv 0$ y $\overline{T}(x,t) \equiv k$ en $\Omega \times \mathbb{R}$, con k suficientemente grande tal que

$$\begin{aligned} L[\overline{T}] = -\bar{c}_0 k \geq 0 \geq & -C(x,t)\overline{T}(x,t) \\ & + \sum_{j=1}^{m} G_j(\overline{T}, x, t)u^j \qquad \text{en } \Omega \times \mathbb{R}. \end{aligned} \tag{3.5}$$

$\underline{T}$ y $\overline{T}$ son una subsolución y supersolución de (3.4) respectivamente. Entonces, por el lema anterior, existe una solución $T \in \mathbb{E}$, $T \geq 0$ de (3.4).

La unicidad de la solución también es consecuencia del lema anterior. Supongamos que T_1 y T_2 son dos soluciones de (3.4). Tomemos $T(x,t) = k \geq T_1$, $T_2 \geq 0$ en $\overline{\Omega} \times \mathbb{R}$, y $\underline{T}(x,t) \equiv 0$. Por el Lema 3.2, existen dos soluciones T^*, T_* de (3.4) tales que $\underline{T} \leq T_* \leq T_i \leq T^* \leq \overline{T}$ en $\overline{\Omega} \times \mathbb{R}$, $i = 1, 2$. Puesto que

$$L[T_* - T^*] = C(\cdot)(T^* - T_*) + \sum_{j=1}^{m} (G_j(T_*, \cdot) - G_j(T^*, \cdot))u^j \geq 0$$

en $\Omega \times \mathbb{R}$, el Principio del Máximo implica que $T_* = T^*$.

El problema (1.2) es equivalente a

$$\vec{u} = \mathbb{K}[\vec{u}] + G[\vec{u}], \quad \vec{u} \in \widehat{\mathbb{E}} \tag{3.6}$$

donde

$$\mathbb{K}[\vec{u}] = (\widetilde{L})^{-1}(H(0,\cdot)\vec{u})$$

y

$$G[\vec{u}] = (\widetilde{L})^{-1}([H(T(\vec{u}),\cdot) - H(0,\cdot)]\vec{u}).$$

Consideraremos el problema auxiliar

$$[I - \lambda\mathbb{K}]\vec{u} + \lambda(-G[\vec{u}]) = 0, \quad (\lambda, \vec{u}) \in \mathbb{R}_+ \times \widehat{\mathbb{E}}. \tag{3.7}$$

Por el Teorema 1.2, $\mathbb{K}$ tiene un valor característico $\widehat{\lambda}_0$ con función propia correspondiente $\vec{\psi}_0 = \mathrm{col}(\psi_0^1, \ldots, \psi_0^m) \in \mathbb{E}^m$, $(\widehat{\lambda}_0, \vec{\psi}_0)$ satisface (ii) y (iii) del mismo teorema . Puesto que

$$\frac{\|-\lambda G(\vec{u})\|_{\widehat{\mathbb{E}}}}{\|\vec{u}\|_{\widehat{\mathbb{E}}}} \leq \frac{\lambda k_2}{[\vec{u}]_\infty} \|[H(T(\vec{u}),\cdot) - H(0,\cdot)]\vec{u}\|_\infty \to 0, \text{ si } \|\vec{u}\|_{\widehat{\mathbb{E}}} \to 0,$$

por un resultado de la Teoría de Bifurcación (*cf.* [2, Teorema 29.1]), obtenemos el siguiente lema:

Lema 3.4. *Si se satisfacen* (B1)-(B2), *entonces* $(\widehat{\lambda}_0, \vec{0})$ *es un punto de bifurcación por soluciones positivas de* (3.7) *y la componente conexa* $\mathcal{C}$ *de* $\overline{M} \subseteq \mathbb{R}_+ \times \widehat{\mathbb{E}}$, *la cual contiene a* $(\widehat{\lambda}_0, \vec{0})$ *es no acotada, donde* $M = \{(\lambda, \vec{u}) \in \mathbb{R}_+ \times \mathbb{P} :\ \vec{u} \not\equiv \vec{0}$ *y* $(\lambda, \vec{u})$ *es solución de* (3.7)$\}$.

La bifurcación es por soluciones positivas debido a que $\vec{\psi}_0$ pertenece al interior de $\mathbb{P}$.

Si Ω satisface (C^*), por el Teorema 1.2, para cada entero positivo k, existen $d_k > 0$ y $\vec{\phi}_k = \mathrm{col}(\phi_k^1, \ldots, \phi_k^m)$, $\phi_k^i \in C^{2+\alpha, 1+(\alpha/2)}(\overline{D}_k \times \mathbb{R})$ tales que $(d_k, \vec{\phi}_k)$ satisface (ii) y (iii) del mismo teorema, $\|\vec{\phi}_k\|_{1+\sigma}^{\overline{D}_k \times [0,\ell]} = 1$, $\vec{\phi}_k \equiv \vec{0}$ en $\partial\Omega \times \mathbb{R}$, y

$$\begin{aligned} &\widetilde{L}(\vec{\phi}_k) = d_k H(0,\cdot)[\vec{\phi}_k] \quad \text{en } D_k \times \mathbb{R}, \\ &\text{y} \quad \vec{\phi}_k(x, t+\ell) = \vec{\phi}_k(x,t), \end{aligned} \tag{3.8}$$

$$\left(\sigma \in (\alpha, 1) \text{ fijo, } \|\vec{\phi}_k\|_{1+\alpha}^{\overline{D}_k \times [0,\ell]} = \max\{\|\phi_k^i\|_{1+\alpha}^{\overline{D}_k \times [0,\ell]},\ i = 1, \ldots, m\}\right).$$

Lema 3.5. *Si* $H(0,\cdot)$ *satisface* (A1)-(A3) *y* Ω *satisface* (C^*), *entonces* $d_k > d_{k+1}$ *para todo entero positivo* k, *y* $d_k \to \widehat{\lambda_0}$.

DEMOSTRACIÓN. Por el Teorema 1.1, $d_1 > \ldots > d_k > d_{k+1} > \ldots > \widehat{\lambda}_0$ y por lo tanto $d_k \to \widehat{\lambda}_1$, para cierto $\widehat{\lambda}_1 \geq \widehat{\lambda}_0$. Denotaremos con $\widetilde{\phi}_k^i$ la extensión por cero de ϕ_k^i a $\overline{\Omega} \times \mathbb{R}$, $\widetilde{\phi}_k = \mathrm{col}(\widetilde{\phi}_k^1, \ldots, \widetilde{\phi}_k^m)$. Por ser compacta la inclusión

$$J^* : C^{\sigma, \sigma/2}(\overline{\Omega} \times \mathbb{R}) \to C^{\alpha, \alpha/2}(\overline{\Omega} \times \mathbb{R}),$$

(para una subsucesión), $\widetilde{\phi}_k \to \vec{\psi}$ en $\mathbb{F}^m$, para un $\vec{\psi} \in \mathbb{F}^m$.

Para todo entero positivo s fijo, si $k \geq s$,

$$1 \geq \|\vec{\phi}_k\|_{1+\sigma}^{\overline{D}_s \times [0,\ell]}.$$

Entonces (para una subsucesión) $\vec{\phi}_k \to \vec{\psi}$ en la $\|\cdot\|_{1+\alpha}^{\overline{D}_s \times [0,\ell]}$-norma y $\vec{\psi} \in C^{1+\alpha,(\alpha/2)}(\overline{D}_s \times [0,\ell])$. Puesto que

$$\widetilde{L}(\vec{\phi}_k - \vec{\phi}_n) = d_k H(0,\cdot)[\vec{\phi}_k - \vec{\phi}_n] + (d_k - d_n)\, H(0,\cdot)\, \vec{\phi}_n$$

end $D_s \times \mathbb{R}$, k, $n \geq s$, utilizando la desigualdad (3.1), obtenemos que

$$\|\phi_k^i - \phi_n^i\|_{2+\alpha}^{\overline{D}_s \times [0,\ell]} \to 0, \qquad \text{cuando } n, k \to \infty.$$

En consecuencia $\phi_k^i \to \psi^i$ en $C^{2+\alpha,1+(\alpha/2)}(\overline{D}_s \times [0,\ell])$ y $\vec{\psi} \not\equiv \vec{0}$. Tomando límite en (3.8) obtenemos

$$\widetilde{L}(\vec{\psi}) = \widehat{\lambda}_1 H(0,\cdot)\vec{\psi} \qquad \text{en } D_s \times [0,l], \text{ para todo } s \geq 1.$$

Por lo tanto

$$\widetilde{L}[\vec{\psi}] = \widehat{\lambda}_1 H(0,\cdot)\vec{\psi} \qquad \text{en } \Omega \times \mathbb{R}, \ \vec{\psi} \in \mathbb{P}, \ \vec{\psi} \not\equiv \vec{0}.$$

Finalmente, por el Teorema 1.1, concluimos que $\widehat{\lambda}_1 = \widehat{\lambda}_0$.

OBSERVACIÓN 3.2. Si $H(0,\cdot)$ satisface (A1)-(A3) y (B3) b), entonces siguiendo el mismo razonamiento del Lema 2.4, podemos probar que el conjunto

$$\Lambda = \{\lambda > 0 : \text{existe } \vec{u}_\lambda \in \mathbb{E}^m \cap \mathbb{P}, \ \vec{u}_\lambda \not\equiv \vec{0} \text{ tal que } (\lambda, \vec{u}) \text{ es solución de (3.7)}\}$$

es acotado.

DEMOSTRACIÓN DEL TEOREMA 1.3. Para probar la parte a) demostraremos que si se satisfacen (B1)-(B3) y Ω satisface (C^*), entonces existe $\vec{u} \in \mathbb{P}$ tal que $(1, \vec{u}) \in M$.

Por la Observación 3.2, el conjunto $\{\lambda : (\lambda, \vec{u}) \in \mathcal{C}\}$ es acotado. Por el Lema 3.4, existe una sucesión $\{(\lambda_n, \vec{u}_n)\}$ en M tal que $\|\vec{u}_n\|_{\widehat{\mathbb{E}}} \to \infty$, y $\lambda_n \to \overline{\lambda}$, para cierto $\overline{\lambda} \in \mathbb{R}$, $\vec{u}_n = \text{col}(u_n^1, \ldots, u_n^m)$ y

$$\widetilde{L}[\vec{u}_n] = \lambda_n[H(T(\vec{u}_n),\cdot)]\, \vec{u}_n \qquad \text{en } \Omega \times \mathbb{R}. \tag{3.9}$$

Probaremos primero que $T(u_n) \to \infty$, uniformemente sobre subconjuntos compactos de $\Omega \times \mathbb{R}$. Definimos las sucesiones

$$z_n = \frac{T(\vec{u}_n)}{\|\vec{u}_n\|_{\widehat{\mathbb{E}}}} \quad \text{y} \quad \vec{w}_n = \frac{\vec{u}_n}{\|\vec{u}_n\|_{\widehat{\mathbb{E}}}}, \quad w_n = \operatorname{col}(w_n^1, \cdots, w_n^n).$$

Dividiendo (3.9) por $\|\vec{u}_n\|_{\widehat{\mathbb{E}}}$, por el Lema 2.2, obtenemos que (para una subsucesión) $\vec{w}_n \to \vec{w}$ en $\widehat{\mathbb{E}}$, para un $\vec{w} \in \widehat{\mathbb{E}}$. Análogamente, por ser acotado el lado derecho de

$$L[z_n] = -Cz_n + \sum_{j=1}^{m} G_j(T(\vec{u}_n), \cdot)\, w_n^j \qquad \text{en } \Omega \times \mathbb{R},$$

(para una subsucesión) $z_n \to z \in \widetilde{\mathbb{F}}$, para un $z \in \widetilde{\mathbb{F}}$. Para cada $\vec{w}_n$ (respectivamente, para $\vec{w}$) existe $v_n \in \mathbb{E}$ (respectivamente, $v \in \mathbb{E}$) tal que

$$L[v_n] = -Cv_n + \sum_{j=1}^{m} g_j w_n^j, \quad v_n > 0, \qquad \text{en } \Omega \times \mathbb{R},$$

(respectivamente, $L[v] = -Cv + \sum_{j=1}^{m} g_j w^j$, $v > 0$, en $\Omega \times \mathbb{R}$). Es fácil ver que (para una subsucesión), $v_n \to v$ en $\widetilde{\mathbb{F}}$, $v_n \le z_n$, y por lo tanto, $z \ge v$ en $\Omega \times \mathbb{R}$. Puesto que

$$z_n = \frac{T(\vec{u}_n)}{\|\vec{u}_n\|_{\widehat{\mathbb{E}}}},$$

concluimos que

$$T(\vec{u}) \to \infty \quad \text{si} \quad n \to \infty, \tag{3.10}$$

uniformemente sobre conjuntos compactos de $\Omega \times \mathbb{R}$.

Por el Lema 3.5, existe un entero r tal que $a_0 > d_r > \widehat{\lambda}_0$, y por (3.10) y (B3), existe n_0 tal que si $n \ge n_0$, entonces

$$H_{ij}\big(T(\vec{u}_n)(x,t), x, t\big) \ge a_0\, H_{ij}(0,x,t) \qquad \text{en } \overline{D}_r \times [0,\ell]. \tag{3.11}$$

Puesto que

$$\widetilde{L}[\vec{u}_n] = \lambda_n H(T(\vec{u}_n), \cdot)\, \vec{u}_n, \qquad \text{en } D_r[0,\ell],$$

y

$$\widetilde{L}[\vec{\phi}_r] = d_r H(0,\cdot)\vec{\phi}_r\,, \qquad \text{en } D_r \times [0,\ell]\,, \qquad \vec{\phi}_r \equiv \vec{0}\,, \qquad \text{en } D_r[0,\ell]\,,$$

(3.11) y el Teorema 1.1 implican que $\lambda_n \leq d_r/a_0 < 1$, para todo $n \geq n_0$. De la conexión de $\mathcal{C}$ se sigue el lema.

La parte b) se demuestra en forma similar.

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A wavelet characterization for weighted Hardy Spaces

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Abstract. In this paper, we give a wavelet area integral characterization for weighted Hardy spaces $H^p(\omega)$, $0 < p < \infty$, with $\omega \in A_\infty$. Our wavelet characterization establishes the identification between $H^p(\omega)$ and $T_2^p(\omega)$, the weighted discrete tent space, for $0 < p < \infty$ and $\omega \in A_\infty$. This allows us to use all the results of tent spaces for weighted Hardy spaces. In particular, we obtain the isomorphism between $H^p(\omega)$ and the dual space of $H^{p'}(\omega)$, where $1 < p < \infty$ and $1/p + 1/p' = 1$, and the wavelet and the Carleson measure characterizations of BMO_ω. Moreover, we obtain interpolation between A_∞-weighted Hardy spaces $H^{p_1}(\omega)$ and $H^{p_2}(\omega)$, $1 \leq p_1 < p_2 < \infty$.

1. Introduction.

In this paper, we give a wavelet area integral characterization for weighted Hardy spaces $H^p(\omega)$, $0 < p < \infty$, with $\omega \in A_\infty$. Coifman and Meyer had earlier given a wavelet characterization for H^1, [9]. Our proof differs from [9], in that it follows from two good-λ inequalities between the non-tangential maximal function and the area integral function with respect to some wavelets. At the same time, our wavelet characterization establishes the identification between $H^p(\omega)$ and $H_0^p(\omega)$, the weighted discrete tent space, for $0 < p < \infty$ and $\omega \in A_\infty$. This allows us to use all the results of tent spaces for weighted Hardy spaces. In particular, we obtain the isomorphism between $H^p(\omega)$ and the dual

space of $H^{p'}(\omega)$, where $1 < p < \infty$ and $1/p + 1/p' = 1$, and the wavelet and the Carleson measure characterizations of BMO_ω. Moreover, we obtain interpolation between A_∞-weighted Hardy spaces $H^{p_1}(\omega)$ and $H^{p_2}(\omega)$, $1 \le p_1 < p_2 < \infty$.

In Section 2, we will give the two good-λ inequalities and their proofs. In Section 3, we will give the wavelet characterization of weighted Hardy spaces and its corollaries.

2. Good-λ Inequalities.

A dyadic multiscale analysis of $L^2(\mathbb{R}^d)$ with respect to lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ is defined as an increasing sequence V_j of closed subspaces of $L^2(\mathbb{R}^d)$ with the following four properties [7]:

(1) $\bigcap V_j = \{0\}$, $\bigcup V_j$ is dense in $L^2(\mathbb{R}^d)$,
(2) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,
(3) for every $f \in V_0$ and every $\gamma \in \mathbb{Z}^d$, we have $f(x-\gamma) \in V_0$,
(4) there exist two constants $C_2 > C_1 > 0$ and a function $g \in V_0$ such that V_0 is the closed linear span of $g(x-\gamma)$, $\gamma \in \mathbb{Z}^d$ and

$$C_1\Big(\sum_{\gamma\in\mathbb{Z}^d} |\alpha_\gamma|^2\Big)^{1/2} \le \Big\|\sum_{\gamma\in\mathbb{Z}^d} \alpha_\gamma g(x-\gamma)\Big\|_2 \le C_2\Big(\sum_{\gamma\in\mathbb{Z}^d} |\alpha_\gamma|^2\Big)^{1/2}.$$

Denoting by W_j the orthogonal complement of V_j in V_{j+1}. There are $2^d - 1$ functions ψ_m, $1 \le m < 2^d$, such that $\psi_m(x-\gamma)$, $\gamma \in \mathbb{Z}^d$, $1 \le m < 2^d$ form an orthonormal basis of W_0, [7]. These functions ψ_m, $1 \le m < 2^d$ are called analyzing wavelets if they satisfy certain decay and moment vanishing conditions.

I. Daubechies discussed the existence of compactly supported wavelets in [4]. In fact, she showed that for any $n \in \mathbb{Z}^+$, there is a collection of functions $\{\psi^\varepsilon,\ \phi :\ \varepsilon = 1, 2, \ldots, 2^d - 1\}$ on $\mathbb{R}^d$ such that for some dyadic multiscale analysis $\{V_j\}$, $\phi \in V_0$ satisfies the property (4) and ψ^ε, $\varepsilon = 1, 2, \ldots, 2^d - 1$, $\in W_0$ are the wavelets corresponding to $\{V_j\}$. Moreover, they have the following properties

a) $\psi^\varepsilon \in C^1$,
b) ψ^ε is compactly supported, say, for some integer $m \ge 1$, supp $\psi^\varepsilon \subset [-m, m]^d$,

c) The collection $\{2^{jd/2}\psi^\varepsilon(2^j x-\gamma):\ j\in\mathbb{Z},\ \gamma\in\mathbb{Z}^d,$ and $\varepsilon=1,2,\dots,2^d-1\}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$,

d) $\int \psi^\varepsilon(x)x^k\,dx=0,$ for $k=0,1,\dots,n,$

e) ϕ is continuous and compactly supported, say, supp $\phi\subset[0,l]^d$ for some integer l,

f) For every $1\le\varepsilon<2^d$, $\psi^\varepsilon(x)$ is a finite linear combination of $\{\phi(x-\gamma),\ \gamma\in\mathbb{Z}^d\}$, *i.e.* there exist $m_\varepsilon\in\mathbb{Z}$ and $b_\gamma^\varepsilon\in\mathbb{R}$, $-m_\varepsilon\le\gamma\le m_\varepsilon$, such that

$$\psi^\varepsilon(x)=\sum_{\gamma=-m_\varepsilon}^{m_\varepsilon} b_\gamma^\varepsilon\phi(x-\gamma),$$

g) $\int \phi\,dx\neq 0\,.$

In this paper, we work with this collection of functions.

Let $B=\prod_{i=1}^{d}(\frac{\alpha_i}{2^\nu},\frac{\alpha_i+1}{2^\nu})$ be a dyadic cube in $\mathbb{R}^d$. We write

$$\psi_B(x)=2^{\nu d/2}\psi(2^\nu x-\alpha),\qquad \text{where } \alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)\in\mathbb{Z}^d$$

and

$$2kB=\prod_{i=1}^{d}(\frac{\alpha_i-k}{2^\nu},\frac{\alpha_i+k}{2^\nu}).$$

By property c), any testing function f can be written as

$$f(x)=\sum_{\varepsilon}\sum_{B\text{ dyadic}} a_B^\varepsilon\psi_B^\varepsilon(x)\,,$$

where $a_B^\varepsilon=\langle f,\psi_B^\varepsilon\rangle$. Setting

$$N_{2k}f(x)=\sup_{\substack{Q\text{ dyadic}\\ 2kQ\ni x}}|\langle f,\phi_Q\rangle|\,|Q|^{-1/2}$$

$$Nf(x)=\sup_{\substack{Q\text{ dyadic}\\ Q\ni x}}|\langle f,\phi_Q\rangle|\,|Q|^{-1/2}$$

$$S_{2k}f(x)=\Big(\sum_{\varepsilon}\sum_{\substack{B\text{ dyadic}\\ 2kB\ni x}}|a_B^\varepsilon|^2|B|^{-1}\Big)^{1/2}$$

$$D_{2k}f(x) = \sup_{\varepsilon} \sup_{2kB\ni x} |\langle f, \psi_B^\varepsilon\rangle |B|^{-1/2}|$$
$$= \sup_{\varepsilon} \sup_{2kB\ni x} |a_B^\varepsilon |B|^{-1/2}|\,.$$

We have

Theorem 2.1. *There exist constants $r_0 > 0$, and $k \in \mathbb{Z}^+$, such that for any test function f, which is a finite linear combination of $\{\psi_B^\varepsilon :$ B dyadic, $\varepsilon = 1, 2, \ldots, 2^d - 1\}$, for any $\lambda > 0$, and $0 < r < r_0$,*

$$|\{x :\ Nf(x) > 3\lambda,\ g_r^*(x) \le 1/2\}| \le C\, r^2 |\{x :\ Nf(x) > \lambda\}|\,,$$

where $g_r = \chi_{\{S_{2k}f > r\lambda\}}$ and g_r^* is the Hardy-Littlewood maximal function of g_r, C is a constant.

Theorem 2.2. *There exist constants $C > 0$, $\delta_0 > 0$, and $k \in \mathbb{Z}^+$, such that for any $0 < \delta < \delta_0$, for any $\lambda > 0$, and for any test function f which is a finite linear combination of $\{\psi_B^\varepsilon :$ B dyadic, $\varepsilon = 1, 2, \ldots, 2^d - 1\}$,*

$$|\{x :\ S_k f(x) > 2\lambda,\ N_4 f(x) \le \delta\lambda,\ D_{2k}f(x) \le \delta\lambda\}|$$
$$\le C\,\delta^2 |\{x :\ S_{2k}f(x) > \lambda\}|\,.$$

For simplicity, we only prove it for the one-dimensional case. The argument can be extended directly to higher dimensions.

In the following, we denote ψ^1 by ψ. All cubes Q, Q_1, B, etc. are dyadic cubes. All C's are constants, they need not to be equal in each appearance.

Lemma 2.3. *If $|B| \le |Q|$ or $2\,l\,Q \cap 2\,m\,B = \emptyset$, $\langle \psi_B, \phi_Q\rangle = 0$.*

The proof is trivial. $|B| = 2^{-i}$, $|Q| = 2^{-j}$, for some $i, j \in \mathbb{Z}$. We have $\psi_B \in W_i$, $\phi_Q \in V_j$. When $|B| \le |Q|$, *i.e.* $j \le i$, $V_j \subseteq V_i$. By definition, $W_i \subset V_{i+1}$ is the orthogonal complement of V_i in V_{i+1}. Therefore, $V_j \perp W_i$, which implies $\langle \psi_B, \phi_Q\rangle = 0$. On the other hand, because supp $\psi_B \subset 2\,m\,B$, supp $\phi_Q \subset 2\,l\,Q$, the condition $2\,l\,Q \cap 2\,m\,B = \emptyset$ implies supp $\psi_B \cap$ supp $\phi_Q = \emptyset$. This proves Lemma 2.3.

We first prove Theorem 2.1. Taking

$$f = \sum a_B \psi_B\,,$$

where only finite number of a_B is nonvanishing. For any $x \in \{x : Nf(x) > \lambda\}$, there is a dyadic cube $Q \ni x$, such that

$$|\langle f, \phi_Q \rangle| \, |Q|^{-1/2} > \lambda \, .$$

This implies $Q \subset \{x : Nf(x) > \lambda\}$. Therefore $\{Nf > \lambda\}$ is a union of a collection $\mathcal{Q}_1$ of dyadic cubes. Meanwhile,

$$\begin{aligned} |\langle f, \phi_Q \rangle |Q|^{-1/2}| &\leq \sum |a_B| \, |\langle \psi_B, \phi_Q \rangle| \, |Q|^{-1/2} \\ &\leq C \sum |a_B| \, |Q|^{-1/2} \longrightarrow 0 \, , \quad \text{when } |Q| \longrightarrow +\infty \, . \end{aligned}$$

So we can pick up a collection $\mathcal{Q}$ of maximal dyadic cubes out of $\mathcal{Q}_1$, and

$$\{Nf > \lambda\} = \bigcup_{Q \in \mathcal{Q}} Q$$

is a disjoint union. Theorem 2.1 follows from the following Lemma. Taking $k = m + 2\,l$ in the definition of g_r ,

Lemma 2.4. *There exists a constant $r_0 > 0$, such that for any $0 < r < r_0$, and any $Q \in \mathcal{Q}$,*

$$\cdot \quad |\{x \in Q : \, Nf(x) > 3\lambda, \; g_r^*(x) \leq 1/2\}| \leq C \, r^2 |Q| \, .$$

Setting

$$E = \{Nf > 3\lambda, \; g_r^* \leq 1/2\} \cap Q \, ,$$

without loss of generality, we suppose $|E| \neq 0$. Otherwise the proof of Lemma 2.4 will be done. Taking $Q \in \mathcal{Q}$, we have

$$|\langle f, \phi_Q \rangle |Q|^{-1/2}| > \lambda$$

and for any $Q_1 \supseteq Q^*$, where Q^* is the father dyadic cube of Q,

$$|\langle f, \phi_{Q_1} \rangle |Q_1|^{-1/2}| \leq \lambda \, .$$

Then for any $x \in E$,

$$\begin{aligned} Nf(x) &= \sup_{Q_1 \ni x} |\langle f, \phi_{Q_1} \rangle |Q_1|^{-1/2}| \\ &= \sup_{\substack{Q_1 \subseteq Q \\ Q_1 \ni x}} |\langle f, \phi_{Q_1} \rangle |Q_1|^{-1/2}| \, . \end{aligned} \tag{2.5}$$

Now for

$$f_Q = \sum_{Q \subset 2kB} a_B \psi_B ,$$

we estimate

$$|\langle f_Q, \phi_{Q_1}\rangle |Q_1|^{-1/2}|, \qquad \text{for } Q_1 \subseteq Q .$$

Lemma 2.6. *For $Q_1 \subseteq Q$,*

$$|\langle f_Q, \phi_{Q_1}\rangle |Q_1|^{-1/2}| \le \lambda + C \inf_{x \in Q} S_{2k} f(x) .$$

PROOF.

$$\langle f_Q, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle = \sum_{Q \subset 2kB} a_B \langle \psi_B, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle .$$

Suppose

$$Q_1 = \left(\frac{\alpha}{2^\nu}, \frac{\alpha+1}{2^\nu}\right), \ B = \left(\frac{\beta}{2^\mu}, \frac{\beta+1}{2^\mu}\right), \text{ and } Q^* = \left(\frac{\gamma}{2^\iota}, \frac{\gamma+1}{2^\iota}\right) .$$

Then $Q_1 \subset Q^*$ implies $\iota < \nu$ and $\left|\frac{\gamma}{2^\iota} - \frac{\alpha}{2^\nu}\right| \le \frac{1}{2^\iota}$, $Q \subset 2kB$ implies $|Q| \le 2k|B|$, then $2^{-\iota} \le 4\,k\,2^{-\mu}$, *i.e.* $\mu \le \iota + \log_2 4k$. Because $\psi \in C^1$, and ψ is compactly supported,

$$\begin{aligned}
&|B|^{1/2}|\langle \psi_B, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle| \\
&\quad = |\int \psi(2^\mu x - \beta)(2^\nu \phi(2^\nu x - \alpha) - 2^\iota \phi(2^\iota x - \gamma))\, dx| \\
&\quad = |\int \psi(2^\mu x - \beta)\phi(2^\nu x - \alpha)\, d(2^\nu x - \alpha) \\
&\qquad - \int \psi(2^\mu x - \beta)\phi(2^\iota x - \gamma)\, d(2^\iota x - \gamma)| \\
&\quad = |\int (\psi(2^{\mu-\nu} x + 2^{\mu-\nu}\alpha - \beta) \\
&\qquad - \psi(2^{\mu-\iota} x + 2^{\mu-\iota}\gamma - \beta))\phi(x)\, dx| \\
&\quad \le \int_0^l \|\phi\|_\infty \|\psi'\|_\infty |2^{\mu-\nu} x + 2^{\mu-\nu}\alpha - 2^{\mu-\iota} x - 2^{\mu-\iota}\gamma|\, dx
\end{aligned}$$

$$\leq C\,2^{\mu-\iota}.$$

Therefore,

$$\begin{aligned}|\langle f_Q, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle| &\leq \sum_{Q\subset 2kB} |a_B||B|^{-1/2}\,C\,2^{\mu-\iota}\\ &\leq C \inf_{x\in Q} S_{2k}f(x)\Big(\sum_{\mu=-\infty}^{\iota+\log_2 4k}(2^{\mu-\iota})^2\Big)^{1/2}\\ &\leq C \inf_{x\in Q} S_{2k}f(x)\,.\end{aligned}$$

Because $|E| \neq 0$, there exists $x \in E \subset Q$, such that $g_r^*(x) \leq 1/2$, which implies $S_{2k}f(x) \leq r\lambda$. Taking $r_0 = 1/C$, where C is the constant appeared in the last inequality, by Lemma 2.3, we have

$$(2.7)\quad |\langle f_Q, \phi_{Q_1}\rangle|Q_1|^{-1/2}| \leq \lambda + Cr\lambda \leq 2\lambda\,, \qquad \text{for all } 0 < r \leq r_0\,.$$

Now setting

$$\begin{aligned}E_1 &= \{x:\ g_r^*(x) \leq 1/2\} \cap 2\,kQ\,,\\ E_2 &= \{x:\ S_{2k}f(x) \leq r\lambda\} \cap 10\,kQ\,,\end{aligned}$$

and

$$U_i = V \cap \cup_{x\in E_i}\{B:\ 2kB \ni x\},$$

where $i = 1, 2$,

$$V = \{B:\ B \subset 2kQ,\ |B| < |Q| \text{ and } Q \not\subset 2kB\}\,.$$

Setting

$$f_1 = \sum_{B\in U_1} a_B\psi_B\,,$$

we prove

Lemma 2.8. *For any $x \in E$, $Nf_1(x) \geq \lambda$.*

PROOF. Setting

$$V_{Q_1} = \{B:\ |Q_1| < |B|,\ \text{and } 2^l Q_1 \cap 2^m B \neq \emptyset\}\,,$$

we have $V_{Q_1} \subset \{B:\ Q_1 \subset 2kB\}$. Taking $x \in Q_1 \subset Q$, by Lemma 2.3,

$$\begin{aligned}\langle f, \phi_{Q_1}\rangle |Q_1|^{-1/2} &= \sum_{\substack{x\in 2kB\\ B\in V_{Q_1}}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &= \sum_{\substack{x\in 2kB\\ B\in V_{Q_1}\setminus\{B:\ Q\subset 2kB\}}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &\quad + \sum_{\substack{x\in 2kB\\ Q\subset 2kB}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}.\end{aligned}$$

It is easy to check that

$$V_{Q_1} \setminus \{B:\ Q \subset 2kB\} \subset V\,.$$

Therefore,

$$\begin{aligned}\langle f, \phi_{Q_1}\rangle |Q_1|^{-1/2} &= \sum_{\substack{x\in 2kB\\ B\in V}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &\quad + \sum_{\substack{x\in 2kB\\ Q\subset 2kB}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}.\end{aligned}$$

By (2.5) and (2.7),

$$\sup_{x\in Q_1\subset Q} \Big| \sum_{\substack{x\in 2kB\\ B\in V}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}\Big| \geq \lambda\,,$$

for any $x \in E$. Because

$$\langle f_1, \phi_{Q_1}\rangle |Q_1|^{-1/2} = \sum_{\substack{x\in 2kB\\ B\in U_1}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}\,,$$

and for any $x \in E \subset E_1$,

$$\{B:\ B\in U_1,\ x\in 2kB\} = \{B:\ B\in V,\ x\in 2kB\}\,,$$

we have

$$Nf_1(x) = \sup_{Q_1\ni x} |\langle f_1, \phi_{Q_1}\rangle |Q_1|^{-1/2}| \geq \lambda\,, \qquad \text{for all } x\in E\,.$$

Now we can start to prove Lemma 2.4. Because

$$f_1 = \sum_{B\in U_1} a_B\psi_B\ ,$$

$$S_{2k}f_1(x) \le S_{2k}f(x), \qquad \text{for all } x\in\mathbb{R}.$$

So

$$\int_{E_2}(S_{2k}f_1(x))^2\,dx \le \int_{E_2}(S_{2k}f(x))^2\,dx \le C\,r^2\,\lambda^2|Q|,$$

and

$$\begin{aligned}\int_{E_2}(S_{2k}f_1(x))^2\,dx &= \int_{E_2}\sum_{\substack{x\in 2kB\\ B\in U_1}}|a_B|^2|B|^{-1}\,dx\\ &= \sum_{B\in U_1}|a_B|^2|B|^{-1}|2kB\cap E_2|.\end{aligned}$$

For any $B\in U_1$, there exists $x\in E_1$, such that $B\in V$ and $2kB\ni x$, therefore $2kB\subset 10\,kQ$, so,

$$2kB\cap\{x:\ S_{2k}f(x)\le r\lambda\} = 2kB\cap E_2\ .$$

The fact $x\in E_1$ implies $g_r^*(x)\le 1/2$, then

$$|2kB, S_{2k}f > r\lambda| \le \frac{1}{2}\,|2kB|.$$

Therefore we have

$$|2kB, S_{2k}f(x)\le r\lambda| = |2kB\cap E_2| \ge \frac{1}{2}\,|2kB|.$$

Consequently,

$$\begin{aligned}\int(S_{2k}f_1(x))^2\,dx &= \sum_{B\in U_1}|a_B|^2|B|^{-1}|2kB|\\ &\le 2\int_{E_2}(S_{2k}f_1(x))^2\,dx\\ &\le C\,r^2\,\lambda^2|Q|.\end{aligned}$$

On the other hand, for any $g = \sum c_B \psi_B$, we have

$$\int (S_{2k} g)^2 \, dx = \int \sum_{x \in 2kB} |c_B|^2 |B|^{-1} \, dx$$
$$= C \sum |c_B|^2 = C \int |g|^2 \, dx .$$

Because $Nf \leq f^*$, where f^* is the Hardy-Littlewood maximal function of f, we have

$$\lambda^2 |E| \leq \int_E (N f_1(x))^2 \, dx \leq C \int f_1^2(x) \, dx$$
$$= C \int (S_{2k} f_1(x))^2 \, dx \leq C r^2 \lambda^2 |Q| .$$

So

$$|E| \leq C r^2 |Q| \qquad \text{for } 0 < r < r_0 .$$

This completes the proof of Lemma 2.4 and then Theorem 2.1.

To prove Theorem 2.2, we rewrite $S_{2k} f$ as

$$S_{2k} f(x) = \sup_{\substack{Q \ni x \\ Q \text{ dyadic}}} \left(\sum_{Q \subset 2kB} |a_B|^2 |B|^{-1} \right)^{1/2} .$$

This equality holds for a.e. $x \in \mathbb{R}$. Therefore

$$\{x : \ S_{2k} f(x) > \lambda\} = \bigcup_{Q \in \Re_1} Q,$$

where

$$\Re_1 = \{Q : \ Q \text{ dyadic}, \ \left(\sum_{Q \subset 2kB} |a_B|^2 |B|^{-1} \right)^{1/2} > \lambda \} .$$

Because $\sum_{Q \subset 2kB} |a_B|^2 |B|^{-1} \longrightarrow 0$, as $|Q| \longrightarrow +\infty$, we can pick up a set $\Re$ of maximal dyadic cubes out of $\Re_1$, and

$$\{x : \ S_{2k} f(x) > \lambda\} = \bigcup_{Q \in \Re} Q .$$

Theorem 2.2 follows whenever we prove the following lemma. Taking $k = 8m + 8l$,

Lemma 2.9. *For any* $Q \in \Re$,

$$|\{x \in Q : \ S_k f(x) > 2\lambda,\, N_4 f(x) \le \delta\lambda,\, D_{2k} f(x) \le \delta\lambda\}| \le C\,\delta^2\,|Q|\,.$$

For any $Q \in \Re$, we have

$$\begin{aligned} &\Big(\sum_{Q \subset 2kB} |a_B|^2 |B|^{-1}\Big)^{1/2} > \lambda\,, \quad \text{and} \\ &\Big(\sum_{Q^* \subset 2kB} |a_B|^2 |B|^{-1}\Big)^{1/2} \le \lambda\,, \end{aligned} \tag{2.10}$$

where Q^* is the father dyadic cube of Q. Setting

$$V = \{B : \ B \text{ dyadic},\ B \subset 2kQ^*,\ |B| < |Q^*| \text{ and } Q^* \not\subset 2kB\}$$

and

$$f_V = \sum_{B \in V} a_B \psi_B\,,$$

and setting

$$E = \{x \in Q : \ S_k f(x) > 2\lambda,\, N_4 f(x) \le \delta\lambda,\, D_{2k} f(x) \le \delta\lambda\},$$

we have

Lemma 2.11. *For any* $x \in E$, $\quad S_k f_V(x) > \lambda\,.$

PROOF. Taking $x \in E$, it is easy to check that for any $Q_1 \ni x$,

$$\{B : kB \supset Q_1,\, B \in V\} \cup \{B : 2kB \supset Q^*, kB \supset Q_1\} \supset \{B : kB \supset Q_1\}.$$

Therefore we have

$$\sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} + \sum_{\substack{2kB \supset Q^* \\ kB \supset Q_1}} |a_B|^2 |B|^{-1} \ge \sum_{kB \supset Q_1} |a_B|^2 |B|^{-1}.$$

Because $S_k f(x) > 2\lambda$, *i.e.*

$$\sup_{Q_1 \ni x} \sum_{kB \supset Q_1} |a_B|^2 |B|^{-1} > 4\lambda^2,$$

there exists $Q_1 \ni x$, such that

$$\sum_{kB \supset Q_1} |a_B|^2 |B|^{-1} > 4\lambda^2.$$

By (2.10),

$$\begin{aligned} 4\lambda^2 &< \sum_{kB \supset Q_1} |a_B|^2 |B|^{-1} \\ &\leq \sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} + \sum_{\substack{2kB \supset Q^* \\ kB \supset Q_1}} |a_B|^2 |B|^{-1} \\ &\leq \sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} + \lambda^2. \end{aligned}$$

Consequently,

$$\sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} \geq 3\lambda^2,$$

and then

$$\sup_{Q_1 \ni x} \sum_{\substack{kB \supset Q_1 \\ B \in V}} |a_B|^2 |B|^{-1} > \lambda^2,$$

i.e.

$$S_k f_V(x) > \lambda.$$

Lemma 2.12. *There exists a constant $C > 0$, such that for any $x \in E$, $N_4 f_V(x) < C\,\delta\,\lambda$.*

PROOF. Taking $x \in E$. Because $N_4 f(x) \leq \delta\lambda$,

$$|\langle f, \phi_{Q^*} \rangle |Q^*|^{-1/2}| = |\cdot \sum_{2kB \supset Q^*} a_B \langle \psi_B, \phi_{Q^*} \rangle |Q^*|^{-1/2}| \leq \delta\lambda.$$

Now for any Q_1, such that $4Q_1 \ni x$, and $|Q_1| \le |Q^*|$, checking as for the case in Lemma 2.8, we have

$$\begin{aligned}\langle f, \phi_{Q_1}\rangle |Q_1|^{-1/2} &= \sum_{B\in V} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &\quad + \sum_{2kB\supset Q^*} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2},\end{aligned} \tag{2.13}$$

and by a same argument as that for Lemma 2.6,

$$\begin{aligned}&\Big|\sum_{2kB\supset Q^*} a_B\langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} - \sum_{2kB\supset Q^*} a_B\langle \psi_B, \phi_{Q^*}\rangle |Q^*|^{-1/2}\Big| \\ &\le \sum_{2kB\supset Q^*} |a_B|\,|B|^{-1/2}|B|^{1/2}|\langle \psi_B, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{Q^*}|Q^*|^{-1/2}\rangle| \\ &\le C\sum_{2kB\supset Q^*} |a_B|\,|B|^{-1/2} \\ &\le C\,D_{2k}f(x) \le C\delta\lambda\,.\end{aligned}$$

Therefore,

$$\Big|\sum_{2kB\supset Q^*} a_B\langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}\Big| \le \delta\lambda + C\delta\lambda\,.$$

From (2.13)

$$\Big|\sum_{B\in V} a_B\langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}\Big| \le 2\delta\lambda + C\delta\lambda\,.$$

For Q_1 such that $4Q_1 \ni x$ and $|Q_1| > |Q^*|$, because $x \in 4Q_1 \cap Q^*$, $Q^* \subset 4Q_1$. Therefore $Q_1 \subset 2kB/4$ implies that $Q^* \subset 2kB$. Meanwhile,

$$\begin{aligned}\langle f_V, \phi_{Q_1}\rangle |Q_1|^{-1/2} &= \sum_{\substack{B\in V\\ Q_1\subset 2kB/4}} a_B\langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &= \sum_{\substack{B\in V\\ Q^*\subset 2kB}} a_B\langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}.\end{aligned}$$

Then the definition of V gives

$$\langle f_V, \phi_{Q_1}\rangle |Q_1|^{-1/2} = 0\,.$$

This proves that $N_4 f_V(x) < (2+C)\,\delta\,\lambda$, for $x \in E$.

From Lemma 2.11 and Lemma 2.12,

$$E \subset \{x \in Q : N_4 f_V(x) \le C\delta\lambda,\ S_k f_V(x) > \lambda,\ D_{2k} f(x) \le \delta\lambda\}.$$

Obviously we have supp $Nf_V \subset \alpha Q$, for some large constant α. Setting

$$E_1 = \alpha Q \cap \{N_4 f_V \le C\delta\lambda,\ D_{2k} f \le \delta\lambda\}$$

and

$$W = V \cap \cup_{x \in E_1}\{B : 2kB \ni x\},$$

defining

$$f_W = \sum_{B \in W} a_B \psi_B,$$

we have $E_1 \supset E$, and for any $x \in E$,

$$\begin{aligned} S_k f_W(x) &= \Big(\sum_{\substack{kB \ni x \\ B \in W}} |a_B|^2 |B|^{-1}\Big)^{1/2} \\ &= \Big(\sum_{\substack{kB \ni x \\ B \in V}} |a_B|^2 |B|^{-1}\Big)^{1/2} = S_k f_V(x) > \lambda. \end{aligned}$$

For any $x \in E_1$,

$$\begin{aligned} N_4 f_W(x) &= \sup_{4Q_1 \ni x} \Big| \sum_{\substack{B \in W \\ 2kB \ni x}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}\Big| \\ &= \sup_{4Q_1 \ni x} \Big| \sum_{\substack{B \in V \\ 2kB \ni x}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}\Big| \\ &= N_4 f_V(x) \le C\delta\lambda. \end{aligned}$$

And also for $x \notin \alpha Q$, we have

$$Nf_W(x) = 0.$$

Lemma 2.14. *There exists a constant $C > 0$, such that for any $x \in \mathbb{R}$, $Nf_W(x) \le C\delta\lambda$.*

PROOF. We need only prove for $x \in \alpha Q \cap E_1^c$. Setting

$$\begin{aligned}\Omega &= \alpha Q \cap E_1^c \\ &= \alpha Q \cap (\{x : N_4 f_V(x) > C\delta\lambda\} \cup \{x : D_{2k} f(x) > \delta\lambda\}).\end{aligned}$$

Then Ω is an open set. Therefore Ω is a union of a collection $\Im$ of disjoint open intervals. (In higher dimensional case, we use Whitney decomposition.) Taking $I \in \Im$, there exist at most two dyadic cubes C_1 and C_2, such that $|C_1| = |C_2| \sim |I|$ and $I \subset \overline{C_1} \cup \overline{C_2} \subset 4C_1$, $4C_1 \cap E_1 \neq \emptyset$. Now for any $x \in I$, setting

$$\mathcal{B} = \{B : B \in W, 2kB \ni x\},$$

if $\mathcal{B} = \emptyset$,

$$\begin{aligned}Nf_W(x) &= \sup_{Q_1 \ni x} \langle f_W, \phi_{Q_1}\rangle |Q_1|^{-1/2} \\ &= \sup_{Q_1 \ni x} \sum_{\substack{2kB \ni x \\ B \in W}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2} = 0.\end{aligned}$$

In case $\mathcal{B} \neq \emptyset$, taking $B_1 \in \mathcal{B}$ such that

$$|B_1| \leq 2 \inf_{\mathcal{B}} |B|,$$

then for any $B \in \mathcal{B}$, the fact that $x \in 2kB_1 \cap 2kB$ implies that

$$2kB_1 \subset 12\,kB.$$

By the definition of W, there is a $y \in E_1$, such that $2kB_1 \ni y$. Therefore we have $2kB_1 \cap \partial I \neq \emptyset$. And taking a dyadic cube $\mathbf{B}$ out of $2kB$, which has the same size as B_1, $\mathbf{B} \subset 2kB_1$, such that $\overline{\mathbf{B}} \cap \partial I \neq \emptyset$. Then $\mathbf{B} \subset 12k\,B$ for any $B \in \mathcal{B}$, *i.e.*

$$\mathcal{B} = \{B \in \mathcal{B} : \mathbf{B} \subset 12\,kB\}$$

and also $E_1 \cap 4\mathbf{B} \neq \emptyset$. Now for $Q_1 \ni x$, consider

$$\sum_{B \in W} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}.$$

1) If $4Q_1 \cap E_1 \neq \emptyset$, from $N_4 f_W(x_0) < C\delta\lambda$ for any $x_0 \in E_1$,

$$|\sum_{B\in W} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}| < C\delta\lambda\,.$$

2) If $4Q_1 \cap E_1 = \emptyset$, $\operatorname{dist}\{x, E_1\} \geq |Q_1|$. Because $4k\mathbf{B} \ni x$ and $4k\mathbf{B} \ni y$, $y \in E_1$, we have $4k|B_1| \geq |Q_1|$. And $4k\mathbf{B} \cap Q_1 \neq \emptyset$ implies $Q_1 \subset 12k\mathbf{B}$. Therefore, by 1) and by a same argument as that for Lemma 2.6,

$$\begin{aligned}
|\sum_{\substack{B\in W\\ 2kB\ni x}} a_B \langle \psi_B, \phi_{Q_1}\rangle |Q_1|^{-1/2}| &\leq |\sum_{\substack{B\in W\\ 12kB\supset \mathbf{B}}} a_B < \psi_B, \phi_{\mathbf{B}} > |\mathbf{B}|^{-1/2}| \\
&\quad + |\sum_{\substack{B\in W\\ 12kB\supset \mathbf{B}}} a_B \langle \psi_B, \phi_{Q_1}|Q_1|^{-1/2} - \phi_{\mathbf{B}}|\mathbf{B}|^{-1/2}\rangle| \\
&\leq C\delta\lambda + C\sum_{B\in W} |a_B|\,|B|^{-1/2} \\
&\leq C\delta\lambda\,.
\end{aligned}$$

This proves Lemma 2.14.

Now we can prove Lemma 2.9. Because $S_k f_W(x) > \lambda$ for $x \in E$, $N f_W(x) < C\delta\lambda$ for $x \in \alpha Q$ and $\operatorname{supp} N f_W \subset \alpha Q$, we have

$$\lambda^2 m(E) \leq \int_E (S_k f_W(x))^2\,dx \leq C\int (N f_W)^2\,dx \leq C\delta^2\lambda^2 m(Q)\,.$$

So,

$$m(E) \leq C\delta^2 m(Q)\,.$$

This proves Lemma 2.9 and then Theorem 2.2.

By the property f) it is easy to check that

$$D_{2k} f(x) \leq C N_{4k} f(x) \qquad \text{a.e. } x\,,$$

where $k = 8m + 8l$. And by a similar argument as that in [3], we can prove that

$$\|S_{2k} f\|_{L^p(\omega)} \approx \|S_{2l} f\|_{L^p(\omega)}\,,$$

for $0 < p < \infty$, $\omega \in A_\infty$ and $k, l \in \mathbb{Z}^+$. Also

$$|\{N_{2k} f > \lambda\}| \leq C\,|\{N_2 f > \lambda\}|\,.$$

Then, as a direct consequence of Theorem 2.1 and Theorem 2.2, we have

Corollary 2.15. *For any $\omega \in A_\infty$ and $0 < p < \infty$, there exist constants C_1 and C_2, such that for any test function f, which is a linear combination of $\{\psi_B^\varepsilon,\ B$ dyadic, $\varepsilon = 1, 2, \ldots, 2^d - 1\}$,*

$$C_1 \|N_2 f\|_{L^p(\omega)} \leq \|S_2 f\|_{L^p(\omega)} \leq C_2 \|N_2 f\|_{L^p(\omega)} .$$

REMARK. Because of property c), Corollary 2.15 is true for any f with $\|S_2 f\|_{L^p(\omega)} < \infty$.

The Main Results.

In this section, we will give the wavelet area integral characterization of the weighted Hardy spaces $H^p(\omega)$, $0 < p < \infty$, with $\omega \in A_\infty$, which establishes the identification between $H^p(\omega)$ and $H_0^p(\omega)$, the weighted discrete tent space. Therefore, a series of corollaries parallel to those of tent spaces follows [3]. Because most of the proofs are almost the same as those in [3], we omit them. For simplicity, we only discuss the one-dimensional case.

In Section 2, we proved that for $0 < p < \infty$, $\omega \in A_\infty$,

$$\|N_2 f\|_{L^p(\omega)} \approx \|S_2 f\|_{L^p(\omega)} .$$

Define

$$H_0^p(\omega) = \{f : N_2 f \in L^p(\omega)\} = \{f : S_2 f \in L^p(\omega)\},$$

with

$$\|f\|_{H_0^p(\omega)} = \|N_2 f\|_{L^p(\omega)} .$$

And for $H_0^p(\omega)$, $0 < p \leq 1$, we define an atom of $H_0^p(\omega)$ to be a function a which satisfies that for some cube R,

(A1) $$a = \sum_{\substack{I \subset R \\ I \text{ dyadic}}} a_I \psi_I$$

(A2) $$\sum_{I \subset R} |a_I|^2 \frac{\omega(I)}{|I|} \leq \omega(R)^{1-2/p}.$$

Because for a an atom of $H_0^p(\omega)$,

$$\begin{aligned}\int |a(x)|^2\omega(x)\,dx &\le \int |N_2a(x)|^2\omega(x)\,dx\\ &\le C\int |S_2a(x)|^2\omega(x)\,dx\\ &\le C\sum_{I\subset R}|a_I|^2\frac{\omega(I)}{|I|}\\ &\le C\omega(R)^{1-2/p},\end{aligned}$$

an atom of $H_0^p(\omega)$ is also an atom of $H^p(\omega)$. The space $H_0^p(\omega)$ can be viewed as a weighted discrete tent space. Therefore, using the same argument as in [3], we can get the following lemma.

Lemma 3.1. *Suppose $f \in H_0^p(\omega)$, with $0 < p \le 1$ and $\omega \in A_\infty$. Then $f = \sum_{j=1}^\infty \lambda_j a_j$, with a_j $H_0^p(\omega)$-atoms, $\lambda_j \in \mathbb{C}$ and*

$$\sum |\lambda_j|^p \le C\,\|f\|^p_{H_0^p(\omega)}\,.$$

Now for any $f \in H_0^p(\omega)$, $0 < p \le 1$, with $f = \sum \lambda_j a_j$ being its atomic decomposition,

$$\begin{aligned}\|f\|^p_{H^p(\omega)} = \|\sum \lambda_j a_j\|^p_{H^p(\omega)} &\le \sum |\lambda_j|^p \|a_j\|^p_{H^p(\omega)}\\ &\le C\sum |\lambda_j|^p \le C\,\|f\|^p_{H_0^p(\omega)}\,.\end{aligned}$$

Therefore, $H_0^p(\omega) \subset H^p(\omega)$ and $\|f\|_{H^p(\omega)} \le C\,\|f\|_{H_0^p(\omega)}$. We want to prove that for $f \in H_0^p(\omega)$, with $0 < p < \infty$, $\omega \in A_\infty$, there exists a constant $C > 0$, such that $\|f\|_{H^p(\omega)} \le C\,\|f\|_{H_0^p(\omega)}$. Setting

$$Mf(x) = \sup_{\Gamma(x)} |f * \phi_t(y)|\,,$$

where $\Gamma(x) = \{(y,t) : |y - x| < t\}$. And suppose $\omega \in A_{p_0}$ for some $p_0 > 1$, then we have

$$\|Mf\|_{L^{p_0}(\omega)} \le C\,\|f\|_{L^{p_0}(\omega)} \le C\,\|f\|_{H_0^{p_0}(\omega)}$$

and also

$$\|Mf\|_{L^1(\omega)} = \|f\|_{H^1(\omega)} \le C\,\|f\|_{H_0^1(\omega)}\,.$$

By interpolation, we obtain

$$\|Mf\|_{L^p(\omega)} = \|f\|_{H^p(\omega)} \le C\,\|f\|_{H_0^p(\omega)}\,, \qquad \text{for } 1 \le p \le p_0\,.$$

Because $A_{p_0} \subset A_q$ for $p_0 < q$, $\omega \in A_q$ for any $q > p_0$. Thus

$$\|f\|_{H^p(\omega)} \le C\,\|f\|_{H_0^p(\omega)}\,, \qquad \text{for } 1 \le p < \infty\,,$$

and then $H_0^p(\omega) \subset H^p(\omega)$ for $0 < p < \infty$. On the other hand,

$$N_2 f(x) = \sup_{\substack{2Q \ni x \\ Q \text{ dyadic}}} |\langle f, \phi_Q\rangle|\,|Q|^{-1/2} \le \sup_{\Gamma(x)} |f * \phi_t(y)| = Mf(x)\,.$$

Therefore,

$$\|f\|_{H_0^p(\omega)} = \|N_2 f\|_{L^p(\omega)} \le \|Mf\|_{L^p(\omega)} = \|f\|_{H^p(\omega)}\,,$$

for $0 < p < \infty$, $\omega \in A_\infty$. Then we have proved

Theorem 3.2. *For $0 < p < \infty$, $\omega \in A_\infty$, $H^p(\omega) = H_0^p(\omega) = \{f : S_2 f \in L^p(\omega)\}$, with*

$$\|f\|_{H^p(\omega)} \approx \|S_2 f\|_{L^p(\omega)}\,.$$

Theorem 3.2 establishes the identification between $H^p(\omega)$ and $H_0^p(\omega)$, a discrete tent space. Therefore, all the properties of tent spaces can be applied to the weighted Hardy spaces $H^p(\omega)$. Especially, we have the following consequences.

Corollary 3.3. *$[H^{p_0}(\omega), H^{p_1}(\omega)]_\theta = H^p(\omega)$, where $1 \le p_0 < p < p_1 < \infty$ with $1/p = (1-\theta)1/p_0 + \theta/p_1$ and $[\cdot,\cdot]_\theta$ is the complex method of interpolation described in* [2].

For $f = \sum f_I \psi_I$, where $f_I = \int f\psi_I\,dx$, define

$$c(f)(x) = \sup_{B \ni x} \left(\frac{1}{\omega(B)} \sum_{I \subset B} |a_I|^2 \frac{\omega(I)}{|I|}\right)^{1/2}$$

and

$$H_0^\infty(\omega) = \{f : c(f) \in L^\infty\}\,.$$

We have the following duality result.

Corollary 3.4.

1. *The following inequality holds, whenever* $f \in H^1(\omega)$ *and* $g \in H_0^\infty(\omega)$

$$\sum_{I \text{ dyadic}} |f_I g_I| \frac{\omega(I)}{|I|} \leq C \int S_2 f(x)\, c(g)(x)\, \omega(x)\, dx\,,$$

where $f = \sum f_I \psi_I$, $g = \sum g_I \psi_I$.

2. *The pairing*

$$\langle f, g \rangle_\omega \longmapsto \sum f_I g_I \frac{\omega(I)}{|I|}$$

realizes $H_0^\infty(\omega)$ *as equivalent to the Banach space dual of* $H^1(\omega)$.

3. *Suppose* $1 < p < \infty$, *then the dual space of* $H^p(\omega)$ *is* $H^{p'}(\omega)$, *with* $1/p + 1/p' = 1$. *More precisely, the pairing*

$$\langle f, g \rangle_\omega = \sum f_I g_I \frac{\omega(I)}{|I|}$$

realizes $H^{p'}(\omega)$ *as equivalent with the dual space of* $H^p(\omega)$.

We have known that $\mathrm{BMO}_\omega = (H^1(\omega))^*$ realized by the pairing

$$(f, g) = \int f g\, dx = \sum f_I g_I\,.$$

Therefore, we can get as a consequence of the last corollary the following wavelet and also Carleson measure characterization of BMO_ω.

Theorem 3.5.

$$\mathrm{BMO}_\omega = \{f \,:\, f = \sum f_I \psi_I\,,\ \sup_{B \text{ ball}} \frac{1}{\omega(B)} \sum_{\substack{I \subset B \\ I \text{ dyadic}}} |a_I|^2 \frac{|I|}{\omega(I)} < \infty\}\,.$$

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Régularité des bases d'ondelettes et mesures ergodiques

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Résumé. Nous reprenons la construction des bases orthonormées d'ondelettes à partir des filtres miroirs en quadrature telle qu'elle apparaît dans [4]. Nous montrons que leur régularité est liée à une mesure invariante pour la transformation $\omega \to 2\omega$ mod-2π. Cette méthode permet d'obtenir le facteur exact qui relie asymptotiquement la régularité des ondelettes construites dans [4] à la taille de leur support.

Introduction.

Une étape importante de la construction d'analyses multi-échelle basée sur une "fonction d'échelle" ϕ (voir [4],[6],[7]) est l'étude de la régularité de ϕ. Cette fonction ϕ est déterminée par une équation fonctionnelle, et sa régularité est reliée à l'ordre de grandeur d'un produit infini exprimant sa transformée de Fourier.

Nous montrons dans ce travail comment l'étude de ce produit infini peut être interprétée en terme de mesure invariante pour un système dynamique (ici la transformation $\omega \to 2\omega$ mod-2π). Dans le cas particulier des bases orthonormées d'ondelettes construites dans [4], cette méthode permet d'évaluer exactement le comportement de la transformée de Fourier de ϕ, et par là d'obtenir le facteur qui relie asymp-

totiquement la régularité de ces ondelettes à la taille de leur support.

Les résultats obtenus ici précisent les estimations obtenues dans [2]. D'autres méthodes d'évaluation de la régularité des fonctions d'échelle ont été développées ([5],[3]).

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1. Rappels sur la construction d'analyses multi-échelle.

Depuis les travaux de Haar, on sait construire des bases orthonormées de $L^2(\mathbb{R})$ de la forme

$$\{\psi_k^j\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}} \tag{1}$$

engendrées par une fonction "mère" ψ, à laquelle on applique des translations entières et des changements d'échelle. Le système de Haar est engendré par la fonction discontinue

$$\psi(x) = \chi_{]0,1/2]} - \chi_{]1/2,1]}\,. \tag{2}$$

Au cours des dernières années, la construction de bases plus régulières de cette forme a été systématisée par Y. Meyer [7], S. Mallat [6] et I. Daubechies [4] dans le cadre de la théorie des ondelettes. Dans tous les cas, on part d'une analyse multi-résolution, c'est-à-dire d'une suite d'espaces emboités $\{V_j\}_{j\in\mathbb{Z}}$ formant une approximation de l'espace $L^2(\mathbb{R})$,

$$\{0\} \to \ldots \subset V_j \subset V_{j+1} \ldots \to L^2(\mathbb{R}) \tag{3}$$

telle que l'on ait

$$f(\cdot) \in V_j \Longleftrightarrow f(2\,\cdot) \in V_{j+1} \Longleftrightarrow f(2^{-j}\cdot) \in V_0\,, \tag{4}$$

et telle que V_0 soit engendré par une famille orthonormée de la forme $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$, obtenue par translations entières d'une fonction ϕ, appelée fonction d'échelle.

La fonction d'échelle satisfait alors une équation fonctionnelle qui exprime l'inclusion de V_0 dans V_1

$$\phi(x) = 2\sum_{n\in\mathbb{Z}} h_n \phi(2x-n)\,. \tag{5}$$

En définissant

$$\psi(x) = 2\sum_{n\in\mathbb{Z}} (-1)^n h_{1-n}\phi(2x-n)\,, \tag{6}$$

on obtient une ondelette: la famille $\{\psi(\cdot - k)\}$ est orthonormée et engendre le complément orthogonal W_0 de V_0 dans V_1. Par translations entières et changement d'échelle, la famille $\{\psi_k^j\}_{j,k\in\mathbb{Z}}$ définie par (1) à partir de ψ fournit une base hilbertienne de l'espace $L^2(\mathbb{R})$ tout entier.

L'équation (5) fait apparaître la suite des coefficients $\{h_n\}_{n\in\mathbb{Z}}$. Ceux-ci permettent de construire la fonction d'échelle ϕ en utilisant la série de Fourier m_0 associée à $\{h_n\}_{n\in\mathbb{Z}}$

$$m_0(\omega) = \sum_{n\in\mathbb{Z}} h_n e^{-in\omega}\,. \tag{7}$$

On a, en notant $\widehat{\phi}$ la transformée de Fourier de ϕ,

$$\widehat{\phi}(2\omega) = m_0(\omega)\,\widehat{\phi}(\omega)\,, \tag{8}$$

et on peut écrire, si m_0 est régulière et égale à 1 en 0,

$$\widehat{\phi}(\omega) = \prod_{k=1}^{\infty} m_0(2^{-k}\omega)\,. \tag{9}$$

Dans la pratique, on part de la donnée du filtre discret h, de fonction de transfert m_0, qui apparaît dans l'algorithme rapide de calcul des coefficients du développement en ondelettes. On engendre par la formule (9) une fonction d'échelle ϕ. Une condition nécessaire d'orthonormalité de la famille $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ est la relation

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1\,, \tag{10}$$

qui signifie que m_0 est un filtre miroir en quadrature au sens de [1].

Nous allons maintenant chercher à caractériser la régularité de ϕ (celle de ψ s'en déduisant), à partir des propriétés de m_0.

2. Un critère fréquenciel de régularité.

Définissons, pour $\alpha \in \mathbb{R}_+$, dans la classe des fonctions intégrables sur $\mathbb{R}$, trois sous-espaces fonctionnels:

$$C^\alpha = \{\phi : \text{ höldérienne d'exposant } \alpha\},$$
$$\mathcal{R}^\alpha = \{\phi : (1+|\omega|^\alpha)|\widehat{\phi}(\omega)| \in L^1(\mathbb{R})\},$$
$$\mathcal{D}^\alpha = \{\phi : (1+|\omega|^\alpha)|\widehat{\phi}(\omega)| \in L^\infty(\mathbb{R})\}.$$

On a clairement, pour $\varepsilon > 0$, $\mathcal{D}^{\alpha+1+\varepsilon} \subset \mathcal{R}^\alpha \subset C^\alpha$. D'autre part, si la suite $\{h_n\}_{n\in\mathbb{Z}}$ est finie, ϕ est à support compact, et l'on a également l'inclusion $C^\alpha \subset \mathcal{D}^\alpha$.

Si m_0 est une fonction régulière, 2π-périodique, satisfaisant (10) et valant 1 à l'origine, on peut opérer la factorisation

$$m_0(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^N p(\omega), \tag{11}$$

où N est le degré d'annulation de m_0 en π. On a alors, d'après (9),

$$\begin{aligned} |\widehat{\phi}(\omega)| &= \Big|\prod_{k=1}^{\infty} (\cos(2^{-k-1}\omega))^N p(2^{-k}\omega)\Big| \\ &= \Big|\frac{2\sin(\omega/2)}{\omega}\Big|^N \prod_{k=1}^{\infty} |p(2^{-k}\omega)|. \end{aligned} \tag{12}$$

Posons $u(\omega) = |p(\omega)|$, et

$$u_n(\omega) = \prod_{k=1}^{n} u(2^{-k}\omega).$$

Pour contrôler la croissance du deuxième facteur $\Pi_1(\omega) = \prod_1^\infty p(2^{-k}\omega)$ dans (12), introduisons les suites (B_n) et (b_n) définies par

$$B_n = \sup_{\omega\in\mathbb{R}} u_n(\omega), \quad b_n = \frac{\log B_n}{n \log 2}, \quad n \geq 1. \tag{13}$$

La suite (b_n) converge, par sous-additivité, vers la quantité $b = \inf_{n\geq 1} b_n$ que nous appellerons "exposant critique" du filtre $m_0(\omega)$.

Les deux résultats suivants [2] permettent de préciser la régularité de ϕ à partir de ces données.

Théorème 1. *Si $\alpha < N - b$, alors ϕ est dans $\mathcal{D}^\alpha$, $\mathcal{R}^{\alpha-1}$ et $C^{\alpha-1}$.*

PREUVE. Il suffit de montrer que ces propriétés sont vérifiées si $\alpha < N - b_n$, pour un n quelconque.

Notons $[x]$ la partie entière d'un réel x. Écrivons le produit $\Pi_1(\omega)$ sous la forme

$$\Pi_1(\omega) = \prod_{k=1}^{[\log_2 \omega]} p(2^{-k}\omega) \prod_{k=[\log_2 \omega]+1}^{\infty} p(2^{-k}\omega)\,. \tag{14}$$

Puisque $p(\omega)$ est régulière et vaut 1 à l'origine, on peut majorer le deuxième facteur par une constante. En divisant le premier facteur en paquets de taille n, on obtient

$$|\Pi_1(\omega)| \leq C\,(1 + |\omega|)^{b_n}. \tag{15}$$

D'après (12) et (15), ϕ est dans $\mathcal{D}^\alpha$ si α est inférieur à $N - b_n$.

Dans le théorème suivant, nous ferons l'hypothèse

$$|p(\pi)| > |p(0)| = 1\,, \tag{16}$$

toujours vérifiée dans la pratique. Nous utiliserons également l'orthonormalité de la famille $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ qui se traduit par l'identité

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi}(\omega + 2k\pi)|^2 = 1\,. \tag{17}$$

Théorème 2. *Sous les conditions* (16) *et* (17), *si $\alpha > N - b$, alors ϕ n'est pas dans $\mathcal{D}^\alpha$. En particulier, si ϕ est à support compact, elle n'est pas höldérienne d'ordre α.*

PREUVE. On considère la suite (ω_n) des points tels que

$$u_n(\omega_n) = B_n = \sup_\omega |\prod_{k=1}^{n} p(2^{-k}\omega)|\,.$$

L'hypothèse (16) permet de montrer que $2^{-n}|\omega_n|$ reste supérieur à une constante $C_1 > 0$. D'autre part, il est possible de choisir ω_n tel que $|\Pi_1(\omega_n)|$ reste supérieur à une constante $C_2 > 0$. En effet, l'identité (17) permet de supposer $|\widehat{\phi}(\omega_n)| > C_3$, quitte à remplacer ω_n par $\omega_n + 2k\pi$.

On a par conséquent

$$|\Pi_1(2^n\omega_n)| \geq C_4\,(1 + |2^n\omega_n)|)^{b_n}\,. \tag{18}$$

La suite $2^n\omega_n$ semble convenir pour montrer que ϕ n'est pas dans $\mathcal{D}^\alpha$ si $\alpha > N - b$. Il convient cependant d'éviter l'annulation du premier facteur, en $\sin\omega$, dans (12). Pour cela, on remarque (en utilisant la convergence de la suite (b_n)), que la dérivée $u_n'(\omega)$ peut être majorée selon: $|u_n'(\omega)| \leq CB_n$. Ceci permet d'affirmer que (18) est vérifiée sur un intervalle de taille constante entourant $2^n\omega_n$. On évite ainsi les zéros de $\sin(\omega/2)$ et le théorème est démontré.

Notons qu'une variante de la démonstration précédente peut être basée sur les résultats de la Section 3, qui permettent de remplacer la suite (ω_n) par un point fixé ω_0 générique pour une mesure invariante par la transformation $T : \omega \to 2\omega$ mod-2π; il est alors possible de s'affranchir des conditions (16) et (17). Néanmoins, la méthode précédente couvre le cas d'application que nous avons en vue (Section 4).

Les deux résultats précédents montrent l'intérêt de l'exposant critique b qu'il convient d'estimer aussi précisément que possible. Nous utilisons à cet effet des méthodes de théorie ergodique appliquées à la transformation T.

3. Exposant critique et mesures invariantes.

Dans cette section, nous relions l'étude d'un produit infini de la forme $\prod_1^\infty u(\omega/2^n)$ à un problème de mesure invariante en théorie ergodique. Reformulons d'abord le problème sous une forme plus générale, d'abord pour une fonction u régulière positive quelconque, puis pour une transformation continue sur un compact.

Soit u une fonction continue périodique, positive ou nulle, de période 2π, lipschitzienne en 0, telle que $u(0) = 1$. Considérons, comme

dans la Section 2, les produits

$$u_n(\omega) = \prod_{k=1}^{n} u(\omega/2^k)\,, \tag{19}$$

$$w(\omega) = \lim_n u_n(\omega) = \prod_{j=1}^{\infty} u(\omega/2^j)\,. \tag{20}$$

Nous appellerons exposant uniforme du produit w le nombre α_∞ défini par

$$\alpha_\infty = \inf\{\alpha : \text{ il existe } M \text{ vérifiant } w(\omega) \leq M(1+|\omega|)^\alpha\}\,. \tag{21}$$

Comme précédemment, la suite $(b_n)_{n\in\mathbb{Z}}$ définie par

$$b_n = \frac{1}{n} \sup_\omega \log_2 u_n(\omega)$$

converge vers une limite b égale à $\inf_n b_n$.

Nous allons examiner la relation entre b et l'exposant uniforme défini par (21). On a

$$w(\omega) = w(2^{-n}\omega)\, u_n(\omega)\,,$$

où $n = n(\omega)$ est tel que $2^{-n}|\omega| \in]\pi/2, \pi]$. Soit $\varepsilon > 0$. Il existe n_0 tel que, pour $n \geq n_0$, on ait $b_n < b + \varepsilon$. Soit A tel que, pour $|\omega| \geq A$, on ait $n(\omega) \geq n_0$. En majorant w sur le compact $[-A, A]$, on obtient la majoration

$$w(\omega) \leq C''\, |\omega|^{b+\varepsilon}\,.$$

D'où la majoration $\alpha_\infty \leq b$.

La majoration inverse $b \leq \alpha_\infty$ résulte immédiatement de l'inégalité

$$u_n(2^n\omega) w(\omega) \leq M\, 2^{n\alpha} (1+|\omega|)^\alpha\,,$$

dans le cas où la fonction u ne s'annule pas. Dans le cas général, on peut encore conclure en utilisant les résultats de la suite de cette section (Remarque 2).

De cette façon, l'étude de l'exposant uniforme du produit infini w est ramenée à celle de b, et nous devons tenter d'évaluer, pour une fonction u donnée, cette quantité. En tenant compte de la périodicité de u, on se ramène à l'étude d'un "produit ergodique", en observant que

$$\sup_{\omega} u_n(\omega) = \sup_{\omega} u_n(2^n\omega) = \sup_{\omega} \prod_{1}^{n} u(2^k\omega)\,.$$

Ceci conduit donc à étudier, dans le cas particulier de la transformation $\omega \to 2\omega$ mod-2π, le comportement de la borne supérieure d'un produit ergodique. Dans le paragraphe suivant, nous allons montrer, dans le cadre général des transformations, que le comportement asymptotique de la borne supérieure d'un produit ergodique est déterminé par une mesure invariante.

Mesure invariante maximisant l'intégrale d'une fonction.

Considérons la situation générale d'une transformation $T : x \to Tx$, continue d'un espace métrique compact dans lui-même. Notons $\mathcal{I}$ le convexe compact formé des mesures de probabilité sur X invariante par T. Fixons une fonction g continue sur X. Nous notons $\mu(g)$ l'intégrale $\int g(x)\, d\mu(x)$. Les exemples discutés dans la Section 2 correspondent à $g = \log_2 |p|$ et $T : x \to 2x$ mod-2π.

Comme précédemment, la sous-additivité de la suite $(\sup_x \sum_0^{n-1} g(T^k x))_{n\in\mathbb{N}}$ implique la convergence vers une limite, notée $\beta(g)$, de la suite $(\beta_n)_{n\in\mathbb{Z}}$ définie par

$$\beta_n = \frac{1}{n} \sup_x \sum_0^{n-1} g(T^k x)\,.$$

Théorème 3. *Pour toute mesure $\mu \in \mathcal{I}$, on a $\beta(g) \geq \mu(g)$. La borne supérieure $\beta(g)$ est atteinte pour une mesure invariante ergodique μ_0 et il existe x_0 dans X tel que*

$$\lim_n \frac{1}{n} \sum_0^{n-1} g(T^k x_0) = \beta(g) = \mu_0(g)\,.$$

PREUVE. Si μ est une mesure invariante, on a

$$\mu(g) = \int \frac{1}{n} \sum_0^{n-1} g(T^k x)\, d\mu(x) \leq \beta_n\,,$$

d'où l'inégalité $\mu(g) \leq \beta(g)$.

Soit (x_n) une suite dans X telle que

$$\frac{1}{n}\sum_{k=0}^{n-1} g(T^k x_n) = \beta_n\,.$$

Par le procédé diagonal, on peut construire une suite (n_j) telle que, pour toute fonction f continue sur X, la suite

$$\frac{1}{n_j}\sum_{k=0}^{n_j-1} f(T^k x_{n_j})$$

converge vers une limite notée $\mu_0(f)$. La forme linéaire $f \to \mu_0(f)$ définit une mesure invariante telle que $\mu_0(g) = \beta(g)$. Si μ_0 n'était pas ergodique, les mesures ergodiques intervenant dans sa décomposition vérifieraient encore cette égalité. On peut donc supposer μ_0 ergodique. L'existence d'un point x_0 tel que

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} g(T^k x_0) = \beta(g)$$

résulte alors du théorème ergodique de Birkhoff appliqué à μ_0.

REMARQUES. 1) On note que la mesure μ_0 maximisant l'intégrale de g vérifie la condition

$$\lim_n \Big(\int \prod_1^n \exp(g(T^k x))d\mu_0(x)\Big)^{1/n} = \exp(\beta(g))\,,$$

et on peut penser que cette mesure maximisante, pour une classe convenable de fonctions g et de transformations T, est concentrée sur un très petit sous-ensemble de X.

Nous allons voir que, pour une famille de fonctions $g = \log_2 |p|$ fournies par la construction des ondelettes orthogonales donnée par I. Daubechies [4], μ_0 est une mesure portée par des points périodiques pour la transformation $T : x \to 2x \bmod 2\pi$. Une question est de savoir

si cette propriété est vérifiée plus généralement pour des transformations dilatantes du type de T et pour une classe assez large de fonctions g.

2) Il est possible d'étendre les résultats précédents au cas d'une fonction g qui est le logarithme d'une fonction u régulière ≥ 0, u pouvant s'annuler.

4. Etude d'un exemple: les ondelettes orthogonales à support compact.

Nous considérons dans ce qui suit la transformation $T : x \to 2x \bmod 2\pi$, et montrons que, dans le cas de la famille des ondelettes orthogonales construites par I. Daubechies, on a exactement $\beta(\log_2(|p|)) = \log_2(|p(2\pi/3)|)$.

Le résultat suivant ramène cette démonstration à la preuve d'une inégalité portant sur $\log_2 |p|$. Dans la preuve du Lemme 1, la notation $[a, b[$ désigne l'arc du cercle unité décrit du point a vers le point b suivant le sens trigonométrique.

Lemme 1. *Soit μ une mesure de probabilité invariante par T sur le cercle unité identifié à $[0, 2\pi[$. Si g est une fonction continue telle que, l'on ait*

$$\begin{cases} g(x) + g(2x) \leq 2\,, & \text{sur } [2\pi/3, 4\pi/3]\,, \\ g(x) \leq 1\,, & \text{sur } [0, 2\pi/3], \text{ et sur } [4\pi/3, 2\pi] \end{cases}$$

alors, on a $\mu(g) \leq 1$.

PREUVE. Nous avons

$$\mu(g) = \int_{[4\pi/3, 2\pi/3[} g(x)\, d\mu - \int_{[2\pi/3, 4\pi/3[} g(2x)\, d\mu + \int_{[2\pi/3, 4\pi/3[} (g(x) + g(2x))\, d\mu\,.$$

On a $T^{-1}([4\pi/3, 2\pi/3[) = [5\pi/3, \pi/3[\cup [2\pi/3, 4\pi/3[$. En utilisant l'invariance de la mesure, on peut donc écrire

$$\int_{[4\pi/3, 2\pi/3[} g(x)\, d\mu = \int_{[5\pi/3, \pi/3[} g(2x)\, d\mu + \int_{[2\pi/3, 4\pi/3[} g(2x)\, d\mu\,.$$

D'où

$$\mu(g) = \int_{[2\pi/3,4\pi/3[} (g(x) + g(2x))\, d\mu + \int_{[5\pi/3,\pi/3[} g(2x)\, d\mu\,.$$

Comme $x \in [5\pi/3, \pi/3[$ implique $g(2x) \leq 1$, en utilisant

$$\mu([2\pi/3, 4\pi/3[) = \mu(T^{-1}([2\pi/3, 4\pi/3[)) = \mu([\pi/3, 2\pi/3[\cup[4\pi/3, 5\pi/3[)\,,$$

on obtient

$$\begin{aligned}\mu(g) \leq\ & \mu([2\pi/3, 4\pi/3[) \\ & + \mu([\pi/3, 2\pi/3[\cup[4\pi/3, 5\pi/3[) \\ & + \mu([5\pi/3, \pi/3[) = 1\,.\end{aligned}$$

Construction explicite des ondelettes régulières à support compact.

Rappelons d'abord la méthode de construction d'ondelettes orthogonales de I. Daubechies [4]. La discussion de la Section 2 a montré que la construction d'une fonction ϕ à support compact engendrant une base d'ondelettes orthogonales s'obtient à partir d'un polynôme trigonométrique m_0 vérifiant la relation (10). Nous avons écrit m_0 sous la forme

$$m_0(\omega) = \left(\frac{1 + e^{i\omega}}{2}\right)^N p(\omega)\,, \tag{22}$$

où N est le degré d'annulation de m_0 en π.

Le polynôme $|p(\omega)|^2$ peut s'écrire comme un polynôme P en $\sin(\omega/2)^2$. En posant $y = \cos(\omega/2)^2$, l'équation (10) devient

$$y^N P(1-y) + (1-y)^N P(y) = 1\,, \tag{23}$$

avec $P \geq 0$ sur $[0, 1]$.

Inversement, à partir d'un polynôme P vérifiant ces conditions, on peut construire un polynôme trigonométrique p et donc une solution m_0 de (10) de la forme (22).

On est ainsi conduit à construire un polynôme P solution de l'équation (23). Un calcul simple montre qu'une solution particulière P_N de (23), de degré $< N$, est donnée par

$$P_N(y) = \sum_{k=0}^{N-1} C_{N-1+k}^{k} y^k . \tag{24}$$

Avec la condition de degré, la solution est unique, les coefficients du polynôme cherché étant déterminés complètement par une formule de récurrence. On obtient la solution générale de (23) en ajoutant à P_N un polynôme R de degré $\geq N$ de la forme $R(y) = y^N R_1(y)$, où $R_1(1/2 - y)$ est un polynôme impair. Nous nous limiterons à l'étude des polynômes P_N.

Calcul de l'exposant critique pour les polynômes P_N.

Dans les sections précédentes, nous avons ramené l'étude de la régularité de la fonction ϕ à la détermination de l'exposant critique associé au facteur $p(\omega)$, et montré que cet exposant est fourni par $\mu_0(\log_2 |p|)$, où μ_0 est une mesure invariante par la transformation $\omega \to 2\omega \bmod 2\pi$, rendant maximum l'intégrale de $\log_2 |p|$. Pour cette famille d'exemples, la mesure μ_0 peut être déterminée explicitement

Théorème 4. *La mesure μ_0 rendant maximum l'intégrale de $\log_2 |p|$ est la mesure portée par les points périodiques $2\pi/3, 4\pi/3$.*

PREUVE. Par le changement de variable $x = \sin(\omega/2)^2$, il revient au même de considérer l'intervalle $[0,1]$, et de faire opérer la transformation $h : x \to h(x) = 4x(1-x)$ sur le polynôme P_N. (Cette transformation est un facteur de T dans l'identification des couples de points symétriques par rapport à l'axe des x, et le couple de points périodiques $2\pi/3$ et $4\pi/3$ correspond au point fixe $3/4$ de la transformation h.)

Dans le Lemme 2 suivant, nous établissons l'inégalité

$$P_N(x)P_N(h(x)) \leq P_N(3/4)^2, \quad \text{pour } x \in [3/4, 1]. \tag{25}$$

Comme $P_N(y) \leq P_N(3/4)$ sur $[0, 3/4]$, l'inégalité (25) assure que le Lemme 1 peut être appliqué. Pour toute mesure invariante μ, on a donc

$$\int \log_2 |p| \, d\mu \leq \frac{1}{2} \log_2 P_N(3/4),$$

et ce majorant est atteint quand μ est la mesure portée par les points périodiques $2\pi/3$ et $4\pi/3$.

On en déduit que l'exposant critique cherché est égal à $(1/2)\log_2 P_N(3/4)$.

Corollaire. *Soit $\alpha(N)$ l'exposant de Hölder de la fonction ϕ_N associée au polynôme $P_N(y)$. On a l'estimation asymptotique:*

$$\lim_{N\to+\infty} \frac{\alpha(N)}{N} = 1 - \frac{1}{2}\log_2 3 \simeq 0.2075\,. \tag{26}$$

PREUVE. Lorsque N tend vers l'infini, il est clair que $\alpha(N)$ est équivalent à $N - b(N)$, où $b(N)$ est l'exposant critique associé à P_N, c'est à dire $(1/2)\log_2(P_N(3/4))$. Une estimation du terme dominant de (24) conduit alors à la relation

$$\lim_{N\to+\infty} \frac{b(N)}{N} = \frac{1}{2}\log_2 3\,.$$

Il reste donc à démontrer le résultat suivant:

Lemme 2. *Les polynômes P_N définis par* (24) *vérifient l'inégalité*

$$P_N(x)P_N(h(x)) \le P_N(3/4)^2, \quad \textit{pour } x \in [3/4, 1]. \tag{25}$$

PREUVE. a) Nous cherchons d'abord des conditions sur un polynome Q à coefficients positifs pour que

$$Q(x)Q(h(x)) \le Q(3/4)^2, \quad \text{pour } x \in [3/4, 1]\,. \tag{26}$$

Pour établir (25), nous raisonnons par récurrence sur le degré n de Q. Soit $Q(x) = R(x) + a_n x^n$. Il suffit d'établir que chacun des quatre termes du développement

$$\begin{aligned} Q(x)Q(h(x)) =& R(x)R(h(x)) + a_n^2[xh(x)]^n \\ &+ a_n h(x)^n R(x) + a_n x^n R(h(x))\,, \end{aligned}$$

est majoré, sur $[3/4, 1]$, par sa valeur en $3/4$.

Cette propriété est vérifiée par hypothèse de récurrence pour le premier facteur et elle résulte d'un calcul élémentaire pour les deuxième et troisième termes. Il reste donc à vérifier l'inégalité

$$x^n R(h(x)) \leq (\frac{3}{4})^n R(3/4), \quad \text{sur } [3/4, 1]. \tag{27}$$

Cette condition est assurée si les coefficients du polynôme Q croissent assez vite. L'inégalité (27) résulte en effet des conditions suivantes

$$\begin{cases} \dfrac{a_k}{a_{k-1}} \geq \dfrac{2}{3}\dfrac{n}{k}, & \text{pour } k = 1, 2, ..., n-2, \\ \dfrac{a_{n-1}}{a_{n-2}} \geq \dfrac{4}{3}\dfrac{n}{n-2} \end{cases} \tag{28}$$

b) Soit maintenant P_N donné par (24)

$$P_N(x) = \sum_{k=0}^{N-1} a_k y^k = \sum_{k=0}^{N-1} C_{N-1+k}^k \, y^k .$$

Pour $N < 7$ une vérification numérique directe prouve l'inégalité (25). Pour $N \geq 7$, les inégalités (28) sont vérifiées par les coefficients des polynômes $Q_n(x) = \sum_{k=0}^{n-1} a_k x^k$, pour tout n tel que $3 \leq n \leq N-1$.

Enfin, pour initialiser le raisonnement par récurrence du point a), on vérifie directement l'inégalité (25), pour $\sum_{k=0}^{2} a_k x^k$, sous la condition $N \geq 7$.

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Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and aplications

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Abstract. In this paper we mainly prove weighted Poincaré inequalities for vector fields satisfying Hörmander's condition. A crucial part here is that we are able to get a pointwise estimate for any function over any metric ball controlled by a fractional integral of certain maximal function. The Sobolev type inequalities are also derived. As applications of these weighted inequalities, we will show the local regularity of weak solutions for certain classes of strongly degenerate differential operators formed by vector fields.

Introduction.

Let $X_1, \ldots, X_m$ be real C^∞ vector fields satisfying Hörmander's condition. Since Hörmander [H] proved the hypoellipticity of

$$L = X_0 + \sum_{i=1}^{m} X_j^2 ,$$

many authors have studied the behavior of solutions to the general class of differential operators

$$\mathcal{L} = \sum_{i=1}^{m} X_j^2 + \sum_{i,j=1}^{m} f_{ij}[X_i, X_j] + \sum_{j=1}^{m} f_j X_j + f_0$$

with some conditions of smoothness imposed on f_{ij}, f_j, f_0. In [Fo], [FeP], [FeS], [FS], [JS], [NSW], [RS], [Sa], a complete description of solutions to several particular cases of $\mathcal{L}$ have been studied. A simpler proof of the results of [FeS] can be found in [Ch1]. An extensive study has been made for the analytic hypoellipticity for operators formed by vector fields. We refer the reader to [Ch2] and a lot of references therein.

In [RS], Rothschild and Stein linked these vector fields with the left invariant vector fields on certain nilpotent Lie groups. Simplification of this approach was given by Hörmander-Melin [HM]. In [NSW], Nagel, Stein and Wainger studied the equivalence of several metrics and pseudometrics and obtained a volume estimate for metric balls. In [J], Jerison proved a unweighted Poincaré inequality.

This paper consists of two parts. One of the main parts of this paper is to prove weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition. As a byproduct, we also derive a unweighted Poincaré inequality with better exponent on the left side of the inequality than that in [J]. Several estimates given by Jerison in [J] play an important role here. A very crucial point in this present paper is that we are able to obtain a pointwise estimate for any smooth function over the metric ball controlled by a fractional integral of certain maximal function. The work of Jerison [J] is very helpful here in order to get this pointwise estimate. The Sawyer-Wheeden condition for weighted inequality simplifies the proof of our main theorems. All these will be presented in Part I of the paper.

Various authors have studied the weighted Poincaré and Sobolev inequalities for the special case $X_i = \partial/\partial x_i$, we refer the reader to the works by Fabes, Kenig and Serapioni [FKS] and Chanillo and Wheeden [CW1] and references therein. A related situation was also considered by Franchi and Lanconelli [FrL] and Franchi and Serapioni [FrS] for nonsmooth vector fields.

The second part of the paper is to apply the weighted Poincaré and Sobolev inequalities to study the local regularity of certain classes of degenerate differential operators formed by vector fields. Precisely speaking, we will show the Harnack inequalities of the following two

type of differential operators

$$L = \sum_{i,j=1}^{m} X_i^*(a_{ij}(x)X_j)$$

and

$$\mathcal{L} = -\sum_{i,j=1}^{m} X_i(a_{ij}(x)X_j)$$

where X_i^* is the adjoint of X_i and the coefficient matrix $A = (a_{ij}(x))$ is symmetric and satisfies the following:

$$c^{-1}w(x)\ |\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(x)\ \xi_i\xi_j \leq c\ w(x)\ |\xi|^2$$

for some weight $w \in A_2(\Omega)$. Several fairly general mean value inequalities for subsolutions to L and $\mathcal{L}$ will be derived also. As an immediate consequence, the Hölder continuity of the solutions to L and $\mathcal{L}$ will be obtained. All these results will be presented in Part II of this paper. We also remark out here that when the matrix $A = (a_{ij})$ satisfies

$$w(x)\ |\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \leq v(x)\ |\xi|^2$$

for certain pair of weights v and w, we can also show the Harnack inequality by obvious modifications of the proof in the present work.

Part I: Weighted Poincaré-Sobolev inequalities for vector fields satisfying Hörmander's condition.

1. The main theorems of Part I.

Let Ω be a bounded, open and path-connected domain in $\mathbb{R}^d$, and let $X_1, X_2, \ldots, X_m$ be a collection of C^∞ real vector fields defined in a neighbourhood of the closure $\overline{\Omega}$ of Ω. For a multi-index $\alpha = (i_1, i_2, \ldots, i_k)$, denote by X_α the commutator

$$[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]], \ldots,]$$

of length $|\alpha| = k$. Throughout this paper we assume that the vector fields satisfy Hörmander's condition: there exists some positive integer s such that $\{X_\alpha\}_{|\alpha|\le s}$ span the tangent space of $\mathbb{R}^d$ at each point of Ω. Now let

$$\begin{aligned} X^{(1)} &= \{X_1, X_2, \dots, X_m\}, X^{(2)} \\ &= \{[X_1, X_2], \dots, [X_{m-1}, X_m]\}, \dots, \text{ etc.} \end{aligned}$$

so that the components of $X^{(k)}$ are the commutators of length k. Let $Y_1, Y_2, \dots, Y_q$ be some enumeration of $X^{(1)}, \dots, X^{(s)}$. If Y_i is an element of $X^{(j)}$, we say Y_i has formal degree $d(Y_i) = j$. For simplicity, we denote $d(Y_i)$ by d_i. Now we define the metric associated to the vector fields as in [NSW].

Let $C(\delta)$ denote the class of absolutely continuous mappings $\phi : [0,1] \to \Omega$ which almost everywhere satisfy the differential equation

$$\phi'(t) = \sum_{j=1}^{q} a_j(t)\, Y_j(\phi(t))$$

with

$$|a_j(t)| < \delta^{d_j} \ .$$

Then define

$$\varrho(x,y) = \inf\{\delta > 0 : \text{ there exists } \phi \in C(\delta) \text{ with } \phi(0) = x,\ \phi(1) = y\}$$

We then define the metric ball

$$B(x,\delta) = \{y \in \Omega :\ \varrho(x,y) < \delta\}\,.$$

In [NSW], Nagel, Stein and Wainger proved the *doubling property* of Lebesgue measure with respect to the metric balls defined as above, *i.e.*

$$|B(x, 2\,\delta)| \le C\ |B(x,\delta)|$$

where C is a constant independent of $x \in \Omega$ with $B(x, 2\,\delta) \subset \Omega$. Thus (Ω, ϱ) is a homogeneous metric space in the sense of Coifman and Weiss, [CoW].

Let $w(x) \ge 0$ be a locally integrable weight function defined in Ω, and $1 < p < \infty$. We say w is a doubling weight if $w(2B) \le Cw(B)$ with C independent of the balls $B \subset \Omega$. We say $w \in A_p(\Omega)$ if

$$\left[\int_B w(x)dx\right]\left[\int_B w(x)^{-1/(p-1)}dx\right]^{p-1} \le C_w\ |B|^p\ ,$$

for all metric balls $B \subset \Omega$. For an example of A_p weights on the Heisenberg group, see Subsection 2.3. We shall use the notation $w(E)$ to denote $\int_E w(x)dx$ and $|E|$ to denote the Lebesgue measure of set E, and $\rho(B)$ to denote the radius of the ball B throughout this paper.

Our main results are

Theorem A. 1) (Poincaré) *Assume that* $E \subset\subset \Omega$, $1 < p < q < \infty$, w_1, w_2 *are two doubling weights satisfying* $w_1 \in A_p(\Omega)$, $w_1 \leq w_2$ *and the following condition*

$$\frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)}\right)^{1/q} \leq C_1 \left(\frac{w_1(I)}{w_1(B)}\right)^{1/p}$$

for all metric balls $I \subset cB \subset \Omega$. *Then there are positive constants* C, r_0 *and some* q_0 *with* $p < q_0 < q$ *such that for any metric ball* $B = B(x,r) \subset \Omega$, $x \in E$, *and any* $f \in C^\infty(\overline{B})$, *the following inequality holds*

$$\left(\frac{1}{w_2(B)} \int_B |f - f_B|^{q_0} w_2\right)^{1/q_0} \leq C\, r \left(\frac{1}{w_1(B)} \int_B \sum_{i=1}^m |X_i f|^p w_1\right)^{1/p}$$

provided $0 < r < r_0$, *where* C, r_0 *depend only on the constants associated to the weights,* E, *and* Ω, *and* f_B *may be taken to be*

$$f_B = \frac{1}{w_2(B)} \int_B f w_2 \,.$$

2) (Sobolev) *If we assume* $f \in C_0^\infty(B)$, *then we can take* $f_B = 0$ *in the above inequality.*

When $w_1 = w_2 = w \in A_p$, we can have precise exponent on the left side of the above inequalities.

Theorem B. 1) (Poincaré) *Let* $w \in A_p(\Omega)$, $1 < p < \infty$, *and* $E \subset\subset \Omega$. *Then there exist constants* $r_0 > 0$, $C > 0$, $Q \geq 2$, *such that for any metric ball* $B = B(x,r) \subset \Omega$, $x \in E$, *and any* $f \in C^\infty(\overline{B})$, *the following inequality holds*

$$\left(\frac{1}{w(B)} \int_B |f - f_B|^q w\right)^{1/q} \leq C\, r \left(\frac{1}{w(B)} \int_B \sum_{i=1}^m |X_i f|^p w\right)^{1/p}$$

provided $0 < r < r_0$, $q = kp$, $1 \le k < Q/(Q-1) + \delta_p$, $\delta_p > 0$, *where* C, r_0 *depend only on the* A_p *constant of the weight* w, E, Ω, *and* δ_p *only depends on* p *and the* A_p *constant of* w, f_B *may be taken to be either*

$$\frac{1}{|B|}\int_B f \qquad or \qquad \frac{1}{w(B)}\int_B fw\,.$$

2) (Sobolev) *If we assume* $f \in C_0^\infty(B)$, *then* f_B *can be taken to be* 0 *in the above inequality.*

In the special case $w = 1$, the above theorems can be improved further as follows

Theorem C. 1) (Poincaré) *Let* $E \subset\subset \Omega$, $1 < p < Q$, *then there exist constants* $r_0 > 0$, $C > 0$, *such that for any metric balls* $B = B(x,r) \subset \Omega$, $x \in E$, *and any* $f \in C^\infty(\overline{B})$, *the following inequality holds*

$$\left(\frac{1}{|B|}\int_B |f - f_B|^q\right)^{1/q} \le C\, r \left(\frac{1}{|B|}\int_B \sum_{i=1}^m |X_i f|^p\right)^{1/p}$$

provided $0 < r < r_0$, $1 \le q < pQ/(Q-p)$, *where* C, r_0 *depend only on* E, Ω, f_B *may be taken to be*

$$f_B = \frac{1}{|B|}\int_B f\,.$$

2) (Sobolev) *If we assume* $f \in C_0^\infty(B)$, *then the above inequality hold for all* $1 \le q \le pQ/(Q-p)$ *and* $f_B = 0$.

REMARK 1. The above Q is actually the homogeneous dimension of the graded nilpotent group generated by the left invariant vector fields corresponding to the lifted vector fields $\{\tilde{X}_i\}$ of $\{X_i\}$. We will define Q precisely later.

REMARK 2. From the proof of Theorem A, we see that other versions of the weighted inequalities are available under weaker assumptions (*e.g.*, removing $w_1 \le w_2$). Moreover, the Sobolev inequality in Theorem C is probably already known.

As applications of the present article, we will prove the Harnack inequality and local regularity of certain class of degenerate second

order differential operators of vector fields (see Part II of the present paper and [L1], [L2]). In Part II, we study the degenerate differential operators

$$L = \sum_{i,j=1}^{m} X_i^*(a_{ij}(x)X_j)$$

and

$$\mathcal{L} = -\sum_{i,j=1}^{m} X_i(a_{ij}(x)X_j)$$

where X_i^* is the adjoint of X_i and the coefficient matrix $A = (a_{ij}(x))$ is symmetric and satisfies the following

$$c^{-1}\, w(x)\, |\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(x)\, \xi_i \xi_j \leq c\, w(x)\, |\xi|^2$$

for some weight $w \in A_2(\Omega)$. The existence and size estimation of fundamental solution to L will be treated (see [L1]). In [L2], we show the Harnack inequality of the degenerate Schrödinger operator of the form

$$\sum_{i,j=1}^{m} X_i^*(a_{ij}(x)X_j) + V$$

for certain potential V satisfying an analogue of the Kato-Stummel condition. The matrix $A = (a_{ij}(x))$ here satisfies the same degenerate condition as above.

The following is an outline of the proof of the Poincaré inequality in Theorem A. We first prove a version of the Poincaré inequalities on the homogeneous metric spaces $(\tilde{\Omega}, \tilde{\varrho})$ associated to the lifted vector fields $\{\tilde{X}_i\}$ with extra variables. The important technique here is to obtain a pointwise estimate for $|f(\xi) - C_B|$ for $\xi \in B$ with some constant C_B, where $B \subset \tilde{\Omega}$ is any metric ball. It is then possible to get control $|f(\xi) - C_B|$ on the ball B by using the fractional integrals of certain maximal function. This is the essential difference between our proof and the one given by Jerison [J] in the unweighted case. In order to derive pointwise estimates, we first obtain a pointwise estimate on the graded nilpotent Lie group. The proof uses the scaling on the group, *i.e.* group translations and dilations. In the case of the homogeneous space $(\tilde{\Omega}, \tilde{\varrho})$ associated to the lifted vector fields, we will adapt this estimate on the group. Once we have the pointwise estimate, we can apply the Sawyer-Wheeden condition for weighted inequality on homogeneous

spaces [SW] together with the A_p-theory introduced by Calderón in [Ca]. This leads us to a variant of the Poincaré inequality $(\tilde{\Omega}, \tilde{\varrho})$, which is

$$\begin{aligned} &\left(\frac{1}{w_2(B)}\int_B |f-f_B|^q w_2\right)^{1/q} \\ &\qquad \le C\, r\left(\frac{1}{w_1(B)}\int_{cB}\left(\sum_{i=1}^m |\tilde{X}_i f| + |f|\right)^p w_1\right)^{1/p} \end{aligned} \tag{1.1}$$

for some constants $C > 0$ and $c \ge 1$ independent of f, B.

From the above inequality (1.1), we can easily obtain a variant of the Poincaré inequality on the homogeneous space (Ω, ϱ) associated to the original vector fields $X_1, \ldots, X_m$. Finally, by employing an argument based on Whitney's decomposition in [K], and also in [J] we can get the Poincaré inequality.

The proof of weighted Sobolev's inequality in Theorem A is much easier. The pointwise estimate of the function f with compact support in a ball can be easily obtained by using the size estimate for the fundamental solution to the operator defined by taking the sum of squares of vector fields (see [NSW] or [Sa]). The remainder of the proof proceeds as in the Poincaré inequality case except that we do not need Whitney's decomposition. The proofs of theorems B and C are similar to those of A and B.

We will adopt the following notations throughout this paper. Let $B = B_r = B(x, r)$, then we use cB, B_{cr} or $B(x, cr)$ as the ball centered at x, and with radius cr. C, c and $C_1, C_2, \ldots$ will denote generic constants and may differ at different occurences.

2. Preliminaries.

In the last section, we give a definition about the metric (Ω, ϱ) defined by vector fields $X_1, \ldots, X_m$. We now give another equivalent definition of metric defined in [NSW], we will use this alternate definition whenever necessary. We assume $X_1, \ldots, X_m$ are C^∞ vector fields in $\Omega \subset \mathbb{R}^N$ satisfying Hörmander's condition and define Y_j and $d_j = d(Y_j)$ as in the introduction.

We first introduce a simplification of notation. Let $x \in E \subset \Omega$ and $I = (i_1, i_2, \ldots, i_N)$ be fixed. We shall relabel the vector fields $\{Y_{i_j}\}_{1 \le j \le q}$ by setting $U_j = Y_{i_j}, 1 \le j \le N$, and by letting V_j, $1 \le$

$j \leq q - N$, be some enumeration of the remaining vector fields. Let $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$, we define

$$B_I(x, \delta) = \{y \in \Omega : y = \exp\big(\sum_{j=1}^{N} u_j \cdot U_j\big)(x), \text{ with } |u_j| < \delta^{d(U_j)}\}\,.$$

Thus $B_I(x, \delta)$ is exactly the image of the box $\{u \in \mathbb{R}^N : |u_j| < \delta^{d(U_j)}\} = Q(\delta)$ under the exponential map.

We denote $d(I) = d_{i_1} + \cdots + d_{i_N}$ where $d_{i_j} = d(Y_{i_j})$ is as in [NSW].

2.1. Analysis on nilpotent Lie groups.

We now review some definitions and useful results associated with graded, nilpotent Lie groups following Folland [Fo], Rothschild and Stein [RS], and Jerison [J].

Let $\mathcal{G}$ be a finite-dimensional, graded, nilpotent Lie algebra. Assume

$$\mathcal{G} = \oplus_{i=1}^{s} V_i\,,$$

and $[V_i, V_j] \subset V_{i+j}$ for $i + j \leq s$, $[V_i, V_j] = 0$ for $i + j > s$. Let $Y_1, \ldots, Y_m$ be a basis for V_1 and suppose that $Y_1, \ldots, Y_m$ generate $\mathcal{G}$ as a Lie algebra. Thus we can choose a basis $\{Y_{ij}\}$, for $1 \leq j \leq s$, $1 \leq i \leq m_j$ for V_j consisting of vectors of the form Y_α for some multi-indices α of length j. In particular, $Y_{i1} = Y_i$, $i = 1, \ldots, m$ and $m = m_1$.

Let G be the simply connected Lie group associated to $\mathcal{G}$. Since the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to G, for each $g \in G$, there is $y = (y_{ij}) \in \mathbb{R}^N$, $1 \leq i \leq m_j$, $1 \leq j \leq s$, $N \doteq \sum_{j=1}^{s} m_j$, such that

$$g = \exp(\sum y_{ij} Y_{ij})\,.$$

Thus we define a homogeneous norm function $|\cdot|$ on G by

$$|g| = (\sum |y_{ij}|^{2s!/j})^{1/2s!}\,.$$

Let δ_t be a dilation on G defined by

$$\delta_t y = (t^j y_{ij})_{1 \leq i \leq m_j, 1 \leq j \leq s}$$

for each $t > 0$. It is easy to see that δ_t is an automorphism of G for each $t > 0$. Lebesgue measure dy is the bi-invariant Haar measure of

G and the Jacobian of δ_t, $J\delta_t$, is equal to t^Q, where $Q = \sum_{j=1}^{s} jm_j$ is called the *homogeneous dimension* of G, which is usually greater than $\dim G = N$. We note that the homogeneous norm $|\cdot|$ on G satisfies the following

(i) $|u| \geq 0$ for $u \in G$ and $|u| = 0$ if and only if $u = 0$,
(ii) $u \to |u|$ is continuous on G and smooth on $G\backslash\{0\}$,
(iii) $|\delta_t(u)| = t\,|u|$,
(iv) $|uv| \leq C(|u| + |v|)$,
(v) $C_1\|u\| \leq |u| \leq C_2\,\|u\|^{1/s}$ for $|u| \leq 1$ and $\|\cdot\|$ is the usual Euclidean norm on G.

For $f \in C^\infty(G)$, and $g \in C_0^\infty(G)$, we define the convolution on G by

$$(f * g)(x) = \int f(xy^{-1})g(y)dy = \int f(y)g(y^{-1}x)dy\,.$$

Denote

$$I_t f(x) = t^{-Q} f(\delta_{t^{-1}}\, x)\,.$$

The following lemma is due to Jerison [J].

Lemma 2.1. (Jerison) a) *For every multi-index α, there exist differential operators $D_{i\alpha}$, $1 \leq i \leq m$, such that*

$$Y_\alpha(f * \phi) = \sum_{i=1}^{m} (Y_i f) * D_{i\alpha}\phi$$

for all $f \in C^\infty(G)$ and $\phi \in C_0^\infty(G)$.

b) *There exist differential operators $D^{(i)}, 1 \leq i \leq m$, such that for any $\phi \in C_0^\infty(G)$, the function $\phi^{(i)} = D^{(i)}\phi$ satisfy*

$$(\frac{\partial}{\partial t})(f * I_t\phi) = \sum_{i=1}^{m} (Y_i f) * I_t\phi^{(i)}\,.$$

Now let $\varrho : G \times G \to \mathbb{R}^+$ be defined by $\varrho(x,y) = |xy^{-1}|$, the homogeneous norm of xy^{-1}. We denote by $B(x,r) = \{y \in G : \varrho(x,y) < r\}$, the ball centered at x and with radius r. We note that (G,ϱ) is a homogeneous metric space in the sense of Coifman and Weiss [CoW].

2.2. Lifting of vector fields.

In this subsection we recall some results of Rothschild and Stein [RS].

Theorem 2.2. (Rothschild-Stein) *Let $X_1, \dots, X_m$ be C^∞ vector fields on a C^∞ manifold Ω of dimension d such that the commutators of length $\leq s$ span the tangent space at $\xi \in \Omega$. Then in terms of new variables, $t_{d+1}, t_{d+2}, \dots, t_N$, there exist smooth functions $\lambda_{kl}(\eta, t)$ defined in a neighbourhood $\tilde{U}$ of $\tilde{\xi} = (\xi, 0) \in \Omega \times \mathbb{R}^{N-d} = \tilde{\Omega}$ such that the vector fields $\tilde{X}_k$ given by*

$$\tilde{X}_k = X_k + \sum_{l=d+1}^{N} \lambda_{kl}(\eta, t) \frac{\partial}{\partial t_l}$$

are free up to step s at every point in $\tilde{U}$, i.e. $\{\tilde{X}_i\}$ and their commutators have no linear relations except that $[\tilde{X}_i, \tilde{X}_j] = -[\tilde{X}_j, \tilde{X}_i]$ and the Jacobian identity.

Theorem 2.3. (Rothschild-Stein). *Let $\tilde{X}_1, \dots, \tilde{X}_m$ be vector fields on a manifold $\tilde{\Omega}$, $\tilde{\xi}_0 \in \tilde{\Omega}$ such that*

a) *Commutators of length $\leq s$ span the tangent space.*
b) *$\{\tilde{X}_k\}$ is free up to step s at $\tilde{\xi}_0$.*

Choose $\{\tilde{X}_{jk}\}$, commutators of length $\leq s$, determining a system of canonical coordinates (u_{jk}) around $\tilde{\xi}_0$ by

$$(u_{jk}) \leftrightarrow \exp(\sum u_{jk} \tilde{X}_{jk})(\tilde{\xi}_0).$$

Let G be the free Lie group of step s on m generators and $\mathcal{G}$ be its Lie algebra. Then there is a basis $\{Y_{jk}\}$ of $\mathcal{G}$ and neighbourhoods $\tilde{V}$ of $\tilde{\xi}_0 \in \tilde{\Omega}$ and U of $0 \in G$ with the following propertities

(i) *There exists a mapping Θ on $\tilde{V} \times \tilde{V}$ to U such that*

$$\Theta(\tilde{\xi}, \tilde{\eta}) = \exp(\sum u_{jk} Y_{jk}) \in U$$

where $\tilde{\eta} = \exp(\sum u_{jk} \tilde{X}_{jk})(\tilde{\xi})$.

(ii) *For each fixed* $\tilde{\xi}$, *the mapping*

$$\tilde{\eta} \to \Theta_{\tilde{\xi}}(\tilde{\eta}) = \Theta(\tilde{\xi}, \tilde{\eta}) = (u_{jk})$$

is a coordinate chart for $\tilde{V}$ *centered at* $\tilde{\xi}$.

(iii) *Furthermore,*

$$(\Theta_{\tilde{\xi}})_*(\tilde{X}_\alpha) = Y_\alpha + E_\alpha^{\tilde{\xi}}$$

where $E_\alpha^{\tilde{\xi}}$ *is a differential operator of local degree* $\leq |\alpha| - 1$.

Define

$$\tilde{\varrho}(\tilde{\xi}, \tilde{\eta}) = |\Theta(\tilde{\xi}, \tilde{\eta})| \, .$$

Thus from [RS], we can see that $\tilde{\varrho} : \tilde{\Omega} \times \tilde{\Omega} \to \mathbb{R}^+$ is a pseudometric on $\tilde{\Omega}$ (we shall shrink $\tilde{\Omega}$ if necessary). We note that the Lebesgue measure of the ball $|\tilde{B}(\tilde{\xi}, r)| \approx r^Q$, where Q is the homogeneous dimension of G, and $\tilde{B}(\tilde{\xi}, r)$ is the metric ball in $(\tilde{\Omega}, \tilde{\varrho})$. Thus $(\tilde{\Omega}, \tilde{\varrho})$ is a homogeneous space in the sense of [CoW].

2.3. A_p weights on homogeneous spaces.

In this subsection, we are going to review the theory of A_p weights on homogeneous spaces introduced by Calderón [Ca]. We will also discuss the relation between A_p weights on (Ω, ϱ) and $(\tilde{\Omega}, \tilde{\varrho})$.

Let (X, ϱ) be the homogeneous space and μ the doubling Borel measure such that

$$\varrho(x, z) \leq K \, (\varrho(x, y) + \varrho(y, z))$$

and

$$0 < \mu(B(x, 2\, r)) \leq A \; \mu(B(x, r)) < \infty$$

for $x, y, z \in X$, $r > 0$. A and K are called the *metric constants* of (X, ϱ). We note that both Vitali and Whitney type covering lemmas hold as proved in [CoW]. We say the weight function $w(x) > 0$, belongs to class $A_p = A_p(X)$ if

$$\left[\int_B w(x) d\mu\right] \left[\int_B w(x)^{-1/(p-1)} d\mu\right]^{p-1} \leq C_w \, |B|^p, \qquad \text{for } 1 < p < \infty$$

or

$$\int_B w(x)dx \leq C_w \ |B| \ \operatorname*{ess\,inf}_{x\in B} \ w(x)\,, \qquad \text{for } p=1$$

for all balls B, where $|B| = \mu(B)$. We call C_w the A_p constant of w. We refer to [Ca] for proofs of the theorems that follow.

Theorem 2.4. *If $w \in A_p$, $p > 1$, then $w \in A_{p_0}$ for some $1 < p_0 < p$ and p_0 depends on p, C_w, A and K.*

Theorem 2.5. *If $w \in A_p$, then $w \in A_\infty$, i.e. for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $E \subset B$ is any subset with $|E| \leq \delta|B|$, then $w(E) \leq \varepsilon\, w(B)$.*

We define the Hardy-Littlewood Maximal function in homogeneous spaces as follows

$$Mf(x) = \sup \frac{1}{\mu(B)} \int_B |f(y)|d\mu\,,$$

the above supremum is taken over all balls B centered at x. We then have

Theorem 2.6. *If $w \in A_p$, $p > 1$, then $\|Mf\|_{L^p_w} \leq C \ \|f\|_{L^p_w}$.*

We now discuss the relations between A_p weights on metric spaces (Ω, ϱ) and $(\tilde{\Omega}, \tilde{\varrho})$. As before we denote $\tilde{\Omega} = \Omega \times \mathbb{R}^l$, $l = N - d$, $N = \dim G$. We also denote by $\tilde{B}_r(\tilde{\xi})$ the ball of radius r and centered at $\tilde{\xi}$ in $\tilde{\Omega}$ for the metric associated with vector fields $\tilde{X}_1, \ldots, \tilde{X}_m$, where $\tilde{\xi} = (\xi, t) \in \tilde{\Omega}$, $\xi \in \Omega$, $t \in \mathbb{R}^l$.

Now let $w(\xi)$ be a function defined in Ω, we define a new function in $\tilde{\Omega}$ by

$$\tilde{w}(\tilde{\xi}) = \tilde{w}(\xi, t) \equiv w(\xi)$$

for $\tilde{\xi} = (\xi, t) \in \tilde{\Omega}$, $\xi \in \Omega$.

Lemma 2.7. *If $w(\xi) \in A_p(\Omega)$, then $\tilde{w}(\tilde{\xi}) \in A_p(\tilde{\Omega})$, for $p \geq 1$.*

PROOF. Let $\tilde{B} = \tilde{B}(\tilde{\xi}, r) \subset \tilde{\Omega}$, $\tilde{\xi} = (\xi, t)$, $\xi \in \Omega$, then $\tilde{B} \subset B \times \mathbb{R}^l$ for $B = B(\xi, r)$. We consider the case $p > 1$ first. By Lemma 4.4 in [J], we have

$$\int_{\tilde{B}} \tilde{w}(\tilde{\xi})d\tilde{\xi} = \int_{\tilde{B}} w(\xi)\chi_{\tilde{B}}\, d\xi dt$$

$$\leq \int_B w(\xi)\left(\int_{\mathbb{R}^l} \chi_{\tilde{B}}\, dt\right) d\xi$$
$$\leq C\frac{|\tilde{B}|}{|B|}\int_B w(\xi)d\xi\,.$$

Similarly,

$$\left(\int_{\tilde{B}} \tilde{w}(\tilde{\xi})^{-1/(p-1)}d\tilde{\xi}\right)^{p-1} \leq C\left(\frac{|\tilde{B}|}{|B|}\right)^{p-1}\left(\int_B w(\xi)^{-1/(p-1)}d\xi\right)^{p-1}$$

Since $w(\xi) \in A_p(\Omega)$, then

$$\left(\int_B w(\xi)d\xi\right)\left(\int_B w(\xi)^{-1/(p-1)}d\xi\right)^{p-1} \leq C\,.$$

Thus,

$$\left(\int_{\tilde{B}} \tilde{w}(\tilde{\xi})d\tilde{\xi}\right)\left(\int_{\tilde{B}} \tilde{w}(\tilde{\xi})^{-1/(p-1)}d\tilde{\xi}\right)^{p-1} \leq C\ |\tilde{B}|^p\,.$$

This shows $\tilde{w}(\tilde{\xi}) \in A_p(\tilde{\Omega})$. For the case $p = 1$, we note

$$\int_{\tilde{B}} \tilde{w}(\tilde{\xi})d\tilde{\xi} \leq C\left(\frac{|\tilde{B}|}{|B|}\right)\int_B w(\xi)d\xi$$
$$\leq C\ |\tilde{B}|\ \operatorname*{ess\,inf}_{\xi\in B}\ w(\xi)$$
$$\leq C\ |\tilde{B}|\ \operatorname*{ess\,inf}_{\tilde{\xi}\in\tilde{B}}\ \tilde{w}(\tilde{\xi})\,.$$

The last inequality is because $\tilde{\xi} \in \tilde{B}$ implies $\xi \in B$. Thus $\tilde{w} \in A_1(\tilde{\Omega})$.

We conclude this section by giving an example of A_p weights on the Heisenberg group. Let H^d be the Heisenberg group (of degree d), *i.e.* the nilpotent Lie group whose underlying manifold is $\mathbb{C}^d \times \mathbb{R}$ with coordinates $(z_1, \ldots, z_d, t) = (z, t)$ and whose group law is

$$(z,t)\cdot(z',t') = (z+z', t+t'+2\ \mathrm{Im}(z\cdot\overline{z}'))$$

where $z\cdot\overline{z}' = \sum_{i=1}^d z_j\,\overline{z}'_j$.

We define the group dilations on H^d by $\delta_r(z,t) = (rz, r^2t)$ which satisfy

$$\delta_r\left((z,t)\cdot(z',t')\right) = \delta_r(z,t)\delta_r(z',t')\,,$$

and we also define the *homogeneous norm* ρ by

$$\rho(z,t) = \left(|z|^4 + |t|^2\right)^{1/4}, \qquad \text{with } |z|^2 = z \cdot z\,.$$

Thus the function ρ satisfies $\rho\left(\delta_r(z,t)\right) = r\rho(z,t)$. We define the distance between two points $u = (z_1, t_1)$ and $v = (z_2, t_2)$ by $\rho(u,v) = \rho\left(u^{-1} \cdot v\right)$. Given $u_0 = (z_0, t_0) \in H^d$, the Heisenberg ball centered at u_0 with radius r is given by

$$B_r(u_0) = \{u = (z,t) \in H^d : \ \rho(u, u_0) < r\}.$$

Let $z = x + iy$. Then, $x_1, \ldots, x_d, y_1, \ldots, y_d, t$ are real coordinates on H^d. Set

$$X_j = \frac{\partial}{\partial x_j} + 2\, y_j \frac{\partial}{\partial t}\,, \quad Y_j = \frac{\partial}{\partial y_j} - 2\, x_j \frac{\partial}{\partial t}\,, \qquad j = 1, \ldots, d\,.$$

Then $X_1, \ldots, X_d, Y_1, \ldots, Y_d$ generate the Lie algebra of H^d. These vector fields satisfy Hörmander's condition of step 2. The homogeneous dimension of H^d is $Q = 2d + 2$.

We now let $w(z,t) = \rho(z,t)^\alpha$, then it is easy to show that w is locally integrable on H^d for $\alpha > -Q$. We can also check that there are constants c_1, c_2 independent of $u_0 = (z_0, t_0) \in H^d$ and r such that

$$c_1 \left(\rho(u_0) + r\right)^\alpha r^Q \le \int_{B_r(u_0)} w(u)du \le c_2 \left(\rho(u_0) + r\right)^\alpha r^Q\,.$$

Thus $w(u) = w(z,t) = \left(|z|^4 + t^2\right)^{1/4}$ is doubling if $\alpha > -Q$ and $w \in A_p(H^d)$ if and only if $-Q < \alpha < Q(p-1)$ for $p > 1$. Furthermore, we can also show $w \in A_1(H^d)$ if and only if $-Q < \alpha \le 0$.

3. Pointwise estimate on the homogeneous space $(\tilde{\Omega}, \tilde{\varrho})$ with extra variables.

We first prove the pointwise estimate $|f(y) - C_B|$ for $f \in C^\infty(B)$ on the graded nilpotent Lie group G defined in Subsection 2.1. We are going to use the same notation as used there.

Let (G, ϱ) be the homogeneous metric space with Lebesgue measure as its doubling measure defined in Subsection 2.1, *i.e.* $\varrho(x,y) = |x^{-1}y|$, where $|\cdot|$ is the homogeneous norm on G. Let $B = B(x,r) = \{y \in$

$G : \varrho(x,y) < r\}$ be the ball in G and $Y_1, \ldots, Y_m$ be the left invariant vector fields which generates the Lie algebra $\mathcal{G}$ of G. Then we have the following

Lemma 3.1. *Given any ball $B = B(x,r) \subset G$, then there exist positive constants C_1, C_2 and C_3, such that for any $f \in C^\infty(\overline{C_3 B})$, the following holds*

$$|f(y) - C_2| \le C_1 \int_{C_3 B} \frac{\sum_{i=1}^m |Y_i f(z)|}{|z^{-1}y|^{Q-1}}\, dz$$

for any $y \in B$, where C_1 and C_3 are independent of y, B, f.

PROOF. *Step 1.* We first show the lemma for the special case $B = B_1 = B(0,1) = \{y \in G : |y| \le 1\}$.

Let $\phi \in C_0^\infty(G)$, $\phi \ge 0$, $\operatorname{supp}\phi \subset B_1$, and denote $I_t\phi(x) = t^{-Q}\phi(\delta_{t^{-1}}x)$. Thus by Lemma 2.1 b), we have for $y \in B_1$

$$\begin{aligned}
|(f*\phi)(y) - f(y)| &= \left| \int_0^1 \frac{\partial}{\partial t}(f * I_t\phi)(y)\, dt \right| \\
&= \left| \int_0^1 \sum_{i=1}^m [(Y_i f) * I_t\phi^{(i)}](y)\, dt \right| \\
&\le C \sum_{i=1}^m \int_0^1 \int_G \left|(Y_i f)(z) I_t\phi^{(i)}(z^{-1}y)\right| dz dt \\
&\le C \sum_{i=1}^m \int_0^1 \frac{1}{t^Q} \int_{|z^{-1}y| \le t} \left|(Y_i f)(z)\right| dz dt \\
&\le C \sum_{i=1}^m \int_{|z^{-1}y| \le 1} \left|(Y_i f)(z)\right| \int_{|z^{-1}y|}^\infty \frac{1}{t^Q}\, dt \\
&\le C \int_{|z^{-1}y| \le 1} \frac{\sum_{i=1}^m |(Y_i f)(z)|}{|z^{-1}y|^{Q-1}}\, dz \\
&\le C \int_{|z| \le c} \frac{\sum_{i=1}^m |(Y_i f)(z)}{|z^{-1}y|^{Q-1}}\, dz\,.
\end{aligned}$$

The last inequality above is due to $|z| = |yy^{-1}z| \le c\,(|y| + |y^{-1}z|) \le c$.

On the other hand, we note $B_1 \subset \tilde{B} = \{y : \|y\| \le c\}$, the Euclidean ball in G centered at the origin and with radius c. We denote by ∇ the usual Euclidean gradient. We also recall $\dim G = N$. Then by Lemma

2.1 a) and the well-known result in [FKS], we have

$$\begin{aligned}\left|(f*\phi)(y)-\frac{1}{|\tilde{B}|}\int_{\tilde{B}}(f*\phi)(z)\,dz\right| &\\ &\le C\int_{\tilde{B}}\frac{|\nabla(f*\phi)(z)|}{\|z-y\|^{N-1}}\,dz\\ &\le C\sum_{|\alpha|\le s}\int_{\tilde{B}}\frac{|Y_\alpha(f*\phi)(z)|}{\|z-y\|^{N-1}}\,dz\\ &\le C\sum_{|\alpha|\le s}\sum_{i=1}^{m}\int_{\tilde{B}}\frac{|(Y_if)*(D_{i\alpha}\phi)(z)|}{\|z-y\|^{N-1}}\,dz\,.\end{aligned}$$

But,

$$\begin{aligned}(Y_if)*(D_{i\alpha}\phi)(z)&=\int_G(Y_if)(w)(D_{i\alpha}\phi)(w^{-1}z)\,dw\\ &\le C\int_{|w^{-1}z|\le 1}|(Y_if)(w)|\,dw\,.\end{aligned}$$

Now because $|w^{-1}z|\le 1$ and $|z|\le c$ implies $|w|\le c$, thus the integral above is bounded by

$$C\int_{|w|\le c}|(Y_if)(w)|\,dw\,.$$

Thus we obtain

$$\begin{aligned}\left|(f*\phi)(y)-\frac{1}{|\tilde{B}|}\int_{\tilde{B}}(f*\phi)(z)dz\right| &\\ &\le C\sum_{i=1}^{m}\int_{\tilde{B}}\int_{|w|\le c}\frac{|(Y_if)(w)|}{\|z-y\|^{N-1}}\,dwdz\\ &= C\sum_{i=1}^{m}\int_{|w|\le c}\left(\int_{\tilde{B}}\frac{dz}{\|z-y\|^{N-1}}\right)|Y_if(w)|\,dw\\ &\le C\sum_{i=1}^{m}\int_{|w|\le c}|(Y_if)(w)|\,dw\end{aligned}$$

since

$$\int_{\tilde{B}}\frac{dz}{\|z-y\|^{N-1}}$$

is uniformly bounded for $y \in B_1 \subset \tilde{B}$. Note further $|w^{-1}y| \leq c(|w^{-1}|+|y|) \leq c$, the integral above is bounded by

$$C\sum_{i=1}^{m}\int_{|w|\leq c}\frac{|(Y_if)(w)|}{|w^{-1}y|^{Q-1}}\,dw\,.$$

Therefore, we have proved

$$\left|f(y)-\frac{1}{|\tilde{B}|}\int_{\tilde{B}}(f*\phi)(z)\,dz\right| \leq C\sum_{i=1}^{m}\int_{|w|\leq c}\frac{|(Y_if)(w)|}{|w^{-1}y|^{Q-1}}dw$$

for $y \in B_1$.

Step 2. For the case $B = B_r = B(0,r) = \{y : |y| < r\}$, we define the function $f_r(w) = f(\delta_r w)$, thus by the result in Step 1, we will be done for this case.

Step 3. Now we consider the general case $B = B(x,r) = \{y \in G : |x^{-1}y| \leq r\}$. We define a function $f_x(z) = f(xz)$ for $|z| \leq r$. By Step 2, the lemma will follow.

Thus this completes the proof of Lemma 3.1.

We now are going to derive the pointwise estimate for $|f(\xi) - C_B|$ over the metric ball B in $(\tilde{\Omega}, \tilde{\varrho})$, where $(\tilde{\Omega}, \tilde{\varrho})$ is the homogeneous space with the metric associated with the lifted vector fields $\tilde{X}_1, \ldots, \tilde{X}_m$. We will set Mf to be the Hardy-Littlewood maximal function for f in the space $(\tilde{\Omega}, \tilde{\rho})$ (see Section 2). Throughout this section, and Section 4, we drop the tildes from $\tilde{X}_i$, $\tilde{\varrho}$, $\tilde{\Omega}$, $\tilde{\xi}$, etc. The main result of this section is the following

Lemma 3.2. *Given any ball $B(\xi_0, r)$, $\xi_0 \in E \subset\subset \Omega$, any function $f \in C^\infty(\overline{cB})$, there exist constansts C, c and C_B such that for any $\xi \in B$*

$$|f(\xi) - C_B| \leq C\int_{cB}\frac{M[(\sum_{i=1}^{m}|X_if| + |f|)\chi_{cB}](\eta)}{\varrho(\xi,\eta)^{Q-1}}\,d\eta$$

where C and c are independent of f, B and ξ.

Now we let ϱ_r be the metric defined by vector fields $rX_1, \ldots, rX_m$, and denote by $B_r(t)$ the ball of radius t relative to this metric. Then

the volume of the ball $|B_r(t)| \approx (r\,t)^Q$. We will need the following lemma from [J].

Lemma 3.3. *Let $\phi \in C_0^\infty(G)$, $\operatorname{supp}\phi \subset \{y : |y| < 1\}$. For $h(\xi)$, $\xi \in \mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^l$, $r > 0$, define*

$$M_t h(\xi) = \int h(\eta) I_t \phi(\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q}\, d\eta\,.$$

Then,

$$\text{(i)} \qquad \frac{\partial}{\partial t} M_t h(\xi) = \sum_{i=1}^{m} \int r X_i h(\eta) I_t \phi^{(i)}(\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q}\, d\eta + \int h(\eta) K_t^r(\xi, \eta) r^{-Q}\, d\eta$$

and $|K_t^r(\xi, \eta)| \le c\,r\,t^{-Q}$, $\operatorname{supp}\{K_t^r\} \subset \{(\xi, \eta) :\ \varrho_r(\xi, \eta) \le c\,t\}$.

$$\text{(ii)} \qquad r^{|\alpha|} X_\alpha M_1 h(\xi) = \sum_{i=1}^{m} \int r X_i h(\eta) \phi_{i\alpha}(\delta_r^{-1} \circ \Theta(\eta, \xi)) r^{-Q}\, d\eta + \int h(\eta) F_\alpha^r(\xi, \eta) r^{-Q}\, d\eta$$

and $|F_\alpha^r(\xi, \eta)| \le c\,r$, $\operatorname{supp}\{F_\alpha^r\} \subset \{(\xi, \eta) :\ \varrho_r(\xi, \eta) \le c\}$. (The functions $\phi^{(i)} = D^{(i)}\phi$, $\phi_{i\alpha} = D_{i\alpha}\phi$ for some differential operators $D_{i\alpha}$, $D^{(i)}$ as defined in Lemma 2.1.

PROOF OF LEMMA 3.2. Define $h(\xi)$ by $f(\xi) = g(\xi)h(\xi)$, where $g(\xi) = \tilde{g}(\xi, 0)$, $\tilde{g} \in C^\infty$ is defined by the pull-back

$$\left((\delta_r^{-1} \circ \Theta_\xi)^{-1}\right)^* (r^{-Q} d\eta) = \tilde{g}(\xi, \delta_r y) dy$$

and $\tilde{g}$ is bounded above and also bounded below from 0 on $B(\xi, r) \times \{y : |y| < 1\}$ for ξ in a compact subset of Ω (see [J] also). Then

$$\text{(3.4)} \qquad \sum_{i=1}^{m} |X_i h(\xi)| \le C \left(\sum_{i=1}^{m} |X_i f(\xi)| + |f(\xi)|\right).$$

Since $\lim_{t \to 0} M_t h(\xi) = h(\xi) g(\xi) = f(\xi)$, we have

$$M_1 h(\xi) - f(\xi) = \int_0^1 \frac{\partial}{\partial t} M_t h(\xi)\, dt$$

$$
\begin{aligned}
&= \int_0^1 \int \sum_{i=1}^m r X_i h(\eta) I_t \phi^{(i)} \left(\delta_r^{-1} \circ \Theta(\eta,\xi)\right) r^{-Q}\, d\eta dt \\
&\quad + \int_0^1 \int h(\eta) K_t^r(\xi,\eta) r^{-Q}\, d\eta dt \\
&= I + II\,,
\end{aligned}
$$

and we have

$$
\begin{aligned}
|I| &\le C \int_0^1 \int_{\varrho(\xi,\eta)\le rt} \sum_{i=1}^m r |X_i h(\eta)| t^{-Q} r^{-Q}\, d\eta dt \\
&\le C \int_{\varrho(\xi,\eta)\le r} \sum_{i=1}^m |X_i h(\eta)| \int_{t\ge \varrho(\xi,\eta)/r} (tr)^{-Q} r\, dt d\eta \\
&\le C \int_{\varrho(\xi,\eta)\le r} \sum_{i=1}^m |X_i h(\eta)| \varrho(\xi,\eta)^{-Q+1}\, d\eta\,.
\end{aligned}
$$

By (3.4), the inequality above is

$$
\begin{aligned}
&\le \int_{\varrho(\xi,\eta)\le r} \frac{\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)|}{\varrho(\xi,\eta)^{Q-1}}\, d\eta \\
&\le C \int_{B(\xi_0,cr)} \frac{\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)|}{\varrho(\xi,\eta)^{Q-1}}\, d\eta\,.
\end{aligned}
$$

Since $|K_t^r(\xi,\eta)| \le C\, r\, t^{-Q}$, $\operatorname{supp}\{K_t^r\} \subset \{(\xi,\eta) :\ \varrho_r(\xi,\eta) \le ct\}$, we obtain

$$
|II| \le C \int_0^1 \int_{\varrho(\xi,\eta)\le crt} h(\eta) r t^{-Q} r^{-Q} d\eta dt \le C \int_{B(\xi_0,Cr)} \frac{|f(\eta)|}{\varrho(\xi,\eta)^{Q-1}}\, d\eta\,.
$$

Hence,

$$
|M_1 h(\xi) - f(\xi)| \le C \int_{B(\xi_0,Cr)} \frac{\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)|}{\varrho(\xi,\eta)^{Q-1}}\, d\eta
$$

for $\xi \in B(\xi_0, r)$. Since any function is bounded above almost everywhere by its maximal function, thus

$$
|M_1(\xi) - f(\xi)| \le C \int_{B(\xi_0,Cr)} \frac{M\left[\left(\sum_{i=1}^m |X_i f| + |f|\right)\chi_{cB}\right](\eta)}{\varrho(\xi,\eta)^{Q-1}}\, d\eta\,. \tag{3.5}
$$

Next, we want to estimate $|M_1 h(\xi) - C_B|$ over the ball $B = B(\xi_0, r)$ for some constant C_B which is independent of $\xi \in B$. The basic idea is to apply the corresponding estimate on the group obtained in the last section.

Let $F(y) = M_1 h\left((\delta_r^{-1} \circ \Theta_{\xi_0})^{-1}(y)\right)$ for $|y| < 1$, where $\delta_r^{-1} \circ \Theta_{\xi_0}(\xi) = y$. It is easy to see that $\delta_r^{-1} \circ \Theta_{\xi_0}(B(\xi_0, r))$ is comparable to the unit ball $\{y : |y| < 1\} \subset G$. Set $\delta_r^{-1} \circ \Theta_{\xi_0}(\eta) = z$. Thus by Lemma 3.1, we have

$$|F(y) - C_1| \leq C \int_{|z| \leq c} \frac{\sum_{i=1}^m |Y_i F(z)|}{|z^{-1}y|^{Q-1}}\, dz\ .$$

Now we note that $|z^{-1}y| \approx r^{-1}\varrho(\xi, \eta)$ and that

$$\left(\delta_r^{-1} \circ \Theta_{\xi_0}\right)_* (rX_i) = Y_i + rE_i^{\xi_0, r}$$

where $E_i^{\xi_0, r}$ is a vector field on G of weight ≤ 0 whose coefficients are C^p norm bounded for some p and $r \leq r_0$ for some r_0 (see [FeS]). Thus

$$Y_i F(z) = rX_i M_1 h(\eta) - rE_i^{\xi_0, r} F(z)\,.$$

We note that

$$\begin{aligned} X_i M_1 h(\eta) &= X_i \int h(\zeta)\phi\left(\delta_r^{-1} \circ \Theta(\zeta, \eta)\right) r^{-Q}\, d\zeta \\ &= \int X_i^* h(\zeta)\phi\left(\delta_r^{-1} \circ \Theta(\zeta, \eta)\right) r^{-Q}\, d\zeta\ , \end{aligned}$$

where X_i^* is the adjoint of X_i with respect to $d\zeta$. Since X_i^* differs of $-X_i$ by a bounded function, we then have

$$|X_i^* h(\zeta)| \leq C\ (|X_i h(\zeta)| + |h(\zeta)|)\,.$$

Now,

$$\varrho(\zeta, \xi_0) \leq K(\varrho(\zeta, \eta) + \varrho(\eta, \xi_0)) \leq c\, r\,.$$

Thus

$$\begin{aligned} |X_i M_1 h(\eta)| &\leq C r^{-Q} \int_{\varrho(\zeta, \eta) \leq r} (|X_i h(\zeta)| + |h(\zeta)|)\chi_{cB}(\zeta)\, d\zeta \\ &\leq CM[(|X_i h| + |h|)\chi_{cB}](\eta)\,. \end{aligned}$$

On the other hand, if we note $z = \delta_r^{-1} \circ \Theta_{\xi_0}(\eta)$, then

$$E_i^{\xi_0, r} F(z) = E_i^{\xi_0, r} M_1 h(\eta) = \int h(\zeta) E_i^{\xi_0, r} \phi\left(\delta_r^{-1} \circ \Theta(\zeta, \eta)\right) r^{-Q}\, d\zeta\,.$$

Keeping in mind that the above derivatives $E_i^{\xi_0,r}$ are taken with respect to z and η is a function of z, and also recalling $E_i^{\xi_0,r}$ is a vector field on G of weight ≤ 0, then $|E_i^{\xi_0,r}\phi\left(\delta_r^{-1}\circ\Theta(\zeta,\eta)\right)|$ is bounded for all $r\leq r_0$. Thus,

$$|E_i^{\xi_0,r}M_1h(\eta)|\leq C\,r^{-Q}\int_{\varrho(\zeta,\eta)\leq r}h(\zeta)\,d\zeta\leq C\,M[h\chi_{cB}](\eta)\,.$$

Hence,

$$|Y_iF(z)|=|(rX_i-rE_i^{\xi_0,r})M_1h(\eta)|\leq C\,r\,M[(|X_ih|+|h|)\chi_{cB}](\eta)\,.$$

Note

$$|z^{-1}y|^{Q-1}\approx\varrho(\xi,\eta)^{Q-1}r^{-Q+1}$$

and

$$\left((\delta_r^{-1}\circ\Theta_{\xi_0})^{-1}\right)^*(r^{-Q}d\eta)=\tilde{g}(\xi_0,\delta_rz)\,dz\,.$$

Further, $|z|\leq c$ implies $\varrho(\eta,\xi_0)\leq cr$. Thus

$$\begin{aligned}|M_1h(\xi)-C_1|&\leq C\int_{\varrho(\xi_0,\eta)\leq cr}\frac{r\sum_{i=1}^m M[(|X_ih|+|h|)\chi_{cB}](\eta)}{r^{-Q+1}\varrho(\xi,\eta)^{-Q+1}}r^{-Q}\,d\eta\\&\leq C\int_{\varrho(\xi_0,\eta)\leq cr}\frac{\sum_{i=1}^m M[(|X_ih|+|h|)\chi_{cB}](\eta)}{\varrho(\xi,\eta)^{Q-1}}\,d\eta\\&\leq C\int_{\varrho(\xi_0,\eta)\leq cr}\frac{M[(\sum_{i=1}^m|X_ih|+|h|)\chi_{cB}](\eta)}{\varrho(\xi,\eta)^{Q-1}}\,d\eta\,.\end{aligned}$$

Thus by (3.4) again,

$$|M_1h(\xi)-C_1|\leq C\int_{\varrho(\xi_0,\eta)\leq cr}\frac{M[(\sum_{i=1}^m|X_if|+|f|)\chi_{cB}](\eta)}{\varrho(\xi,\eta)^{Q-1}}\,d\eta\,.\tag{3.6}$$

Now,

$$|f(\xi)-C_1|\leq|M_1h(\xi)-f(\xi)|+|M_1h(\xi)-C_1|$$

Applying (3.5) to the first term and (3.6) to the second, we get our lemma.

4. A variant of the Poincaré inequalities on homogeneous spaces with extra variables

In this section, we are going to prove a variant of the Poincaré inequalities on the homogeneous space $(\tilde{\Omega}, \tilde{\varrho})$. The proofs adapt the Sawyer-Wheeden condition for the weighted inequality of fractional integrals on homogeneous spaces [SW]. This proof is much simplier than that in the author's thesis [L1]. As in last section, we will drop the tildes. We first state these results.

Theorem 4.1. 1) *Let $1 < p < q < \infty$ and assume $w_1 \in A_p(\Omega)$, w_2 a doubling weight satisfying the following condition*

$$\frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)}\right)^{1/q} \leq C \left(\frac{w_1(I)}{w_1(B)}\right)^{1/p} \tag{4.2}$$

for all metric balls I and B with $I \subset cB \subset \Omega$, where $\rho(B)$ is the radius of the ball B and c is as in Lemma 3.2. Then there is a positive constant C such that for any balls $B = B(\xi_0, r)$ with $cB \subset \Omega$ and $f \in C^\infty(\overline{cB})$, the following inequality holds

$$\left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2\right)^{1/q} \leq Cr \left(\frac{1}{w_1(B)} \int_{cB} \left(\sum_{i=1}^m |X_i f| + |f|\right)^p w_1\right)^{1/p} \tag{4.3}$$

where f_B may be taken to be

$$f_B = \frac{1}{w_2(B)} \int_B f w_2 .$$

2) *In the case $w_1 = w_2 = w \in A_p$ and $p > 1$, the q in (4.3) can be taken as $q = kp$ for $1 \leq k \leq Q/(Q-1) + \delta_p$ for some $\delta_p > 0$ and f_B can be taken as*

$$\frac{1}{|B|} \int_B f \quad \text{or} \quad \frac{1}{w(B)} \int_B f w .$$

3) *In the case $w_1 = w_2 = 1$ and $1 < p < Q$, the above q can be taken as $1 \leq q \leq pQ/(Q-p)$ and f_B can be taken as*

$$f_B = \frac{1}{|B|} \int_B f .$$

PROOF OF THEOREM 4.1. We define the fractional integral

$$Tf(\xi) = \int_\Omega f(\eta) K(\xi, \eta)\, d\eta$$

where $K(\xi, \eta) = 1/\varrho(\xi, \eta)^{Q-1}$.

Given a metric ball $B \subset \Omega$, we will show for certain pair of weights $\tilde{v}$ and $\tilde{w}$, that

$$\|Tf\|_{L^q(B,\tilde{v})} \leq C \, \|f\|_{L^p(cB,\tilde{w})} \tag{4.4}$$

for all $f \geq 0$ and $\operatorname{supp} f \subset cB$, where

$$\|f\|_{L^p(S,\tilde{v})} = \left(\int_S f^p \tilde{v} \right)^{1/p}$$

and $p < q$. By Theorem 3 in [SW], if $\tilde{v}, \tilde{w}$ are doubling weights and the following condition holds

$$\phi(I) \left(\int_I \tilde{v}(\xi)\, d\xi \right)^{1/q} \left(\int_I \tilde{w}^{1-p'}(\xi)\, d\xi \right)^{1/p'} \leq C \,, \tag{4.5}$$

for all metric balls $I \subset cB \subset \Omega$, then (4.4) holds. $\phi(I)$ in the above expression is defined to be

$$\phi(I) = \sup\{K(\xi, \eta) : \ \xi, \eta \in I, \ \varrho(\xi, \eta) \geq \frac{1}{2}\rho(I)\} \approx \frac{1}{|I|^{1-1/Q}}$$

where $\rho(I)$ is the radius of the ball I and $|I| \approx \rho(I)^Q$.

We let

$$\tilde{v} = \frac{w_2}{w_2(B)} \quad \text{and} \quad \tilde{w} = \frac{\sigma(B)^{p/p'} \rho(B)^p}{|B|^p}\, w_1 \,,$$

where w_1 and w_2 are doubling weights, $\sigma(\xi) = w_1(\xi)^{1-p'}$, $1/p + 1/p' = 1$. Then (4.5) is equivalent to

$$\frac{\rho(I)}{\rho(B)} \frac{|B|}{|I|} \left(\frac{w_2(I)}{w_2(B)} \right)^{1/q} \left(\frac{\sigma(I)}{\sigma(B)} \right)^{1/p'} \leq C \,. \tag{4.6}$$

Thus by (4.4) and Lemma 3.2, we have

$$\left(\frac{1}{w_2(B)}\int_B |f-C_B|^q w_2\,\right)^{1/q} \le C(|B|^{-1}w_1(B)^{1/p}\sigma(B)^{1/p'})\rho(B)$$

$$\cdot\left[\frac{1}{w_1(B)}\int_{cB}\left(M\Big(\sum_{i=1}^m |X_i f|+|f|\Big)\chi_{cB}\right)^p w_1\right]^{1/p} \tag{4.7}$$

for all $f \in C^\infty(\overline{cB})$. If assume $w_1 \in A_p$, then $|B|^{-1}w_1(B)^{1/p}\sigma(B)^{1/p'} \le C$. Furthermore, by Theorem 2.6 the expression on the right side of (4.7) is bounded by

$$C\rho(B)\left(\frac{1}{w_1(B)}\int_{cB}\left(\sum_{i=1}^m |X_i f|+|f|\right)^p w_1\right)^{1/p}.$$

We also note that (4.2) is equivalent to (4.6) when $w_1 \in A_p$. This shows the first part of Theorem 4.1.

When $w_1 = w_2 = w \in A_p(\Omega)$, we note that (4.2) is equivalent to

$$\frac{\rho(I)}{\rho(B)}\left(\frac{w(I)}{w(B)}\right)^{1/q} \le C\left(\frac{w(I)}{w(B)}\right)^{1/p} \tag{4.8}$$

Since $|B| \approx \rho(B)^Q$, $|I| \approx \rho(I)^Q$ and $A_p \subset D_p$ (doubling of order p), *i.e.*

$$w(B) \le C\left(\frac{|B|}{|I|}\right)^p w(I), \quad \text{for any } I \subset cB \subset \Omega$$

then the assumption that $w \in A_p(\Omega)$ implies that (4.8) holds if q is chosen so that $(Q(1/p-1/q))^{-1} = p$, *i.e.* $q = pQ/(Q-1)$. Actually, since $w \in A_p$ implies $w \in A_{p_0}$ for some $p_0 < p$ by Theorem 2.4, it follows that q can be chosen slightly larger than $pQ/(Q-1)$. Thus the second part of Theorem 4.1 follows by Hölder's inequality.

When $w_1 = w_2 = 1$, (4.8) is further equivalent to

$$\frac{\rho(I)}{\rho(B)}\left(\frac{|I|}{|B|}\right)^{1/q} \le C\left(\frac{|I|}{|B|}\right)^{1/p}. \tag{4.9}$$

We note that $|B| \approx \rho(B)^Q$ and $\rho(I) \le c\,\rho(B)$, thus (4.9) holds if $p < q \le pQ/(Q-p)$, which implies the third part of Theorem 4.1 for $p < q \le pQ/(Q-p)$. For $1 \le q \le p$, the result follows by Hölder's inequality.

5. A variant of Sobolev inequalities on the original spaces (Ω, ϱ) .

The main purpose of this section is to show the variant of Sobolev inequalities on the homogeneous space (Ω, ϱ) associated to the original vector fields $X_1, \ldots, X_m$. We start with the pointwise estimate for any function $f \in C_0^\infty(B)$, and $B \subset \Omega$.

Lemma 5.1. *Given any ball $B = B(\xi_0, r)$, $\xi_0 \in E \subset\subset \Omega$, any $f \in C_0^\infty(B)$, there exists a constant $C > 0$ independent of f, B, $\xi \in B$ such that for $\xi \in B$,*

$$|f(\xi)| \leq C \int_B \frac{\varrho(\xi, \eta)}{|B(\eta, \varrho(\xi, \eta))|} \Big(\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)| \Big) \, d\eta$$

PROOF OF LEMMA 5.1. Let us consider the second order differential operator $L = \sum_{i=1}^m X_i^2$. By the result in [NSW], there is a unique fundamental solution $G(\xi, \eta)$ to L with

$$|X_i G(\xi, \eta)| \leq C \, \frac{\varrho(\xi, \eta)}{|B(\eta, \varrho(\xi, \eta))|} \, .$$

Thus for every $f \in C_0^\infty(B)$, we have

$$f(\xi) = \int \sum_{i=1}^m X_i^2 G(\xi, \eta) f(\eta) \, d\eta = \int \sum_{i=1}^m X_i G(\xi, \eta) X_i^* f(\eta) \, d\eta \, .$$

Note that X_i^* is the adjoint of X_i with respect to η and that

$$|X_i^* f(\eta)| \leq C \, (|X_i f(\eta)| + |f(\eta)|) \, .$$

Hence we have the following

$$|f(\xi)| \leq C \int_B \frac{\varrho(\xi, \eta)}{|B(\eta, \varrho(\xi, \eta))|} \Big(\sum_{i=1}^m |X_i f(\eta)| + |f(\eta)| \Big) \, .$$

Then the lemma follows.

By the above lemma we can show the following theorems.

Theorem 5.2. 1) *Let $1 < p < q < \infty$ and assume $w_1 \in A_p(\Omega)$, w_2 a doubling weight satisfying the following condition*

$$\frac{\rho(I)}{\rho(B)}\left(\frac{w_2(I)}{w_2(B)}\right)^{1/q} \le C\left(\frac{w_1(I)}{w_1(B)}\right)^{1/p} \tag{5.3}$$

for all metric balls I and B with $I \subset cB \subset \Omega$, where $\rho(B)$ is the radius of the ball B. Then there are positive constants C and r_0 such that for any ball $B = B(\xi_0, r) \subset \Omega$, $r \le r_0$, and any $f \in C_0^\infty(B)$, the following inequality holds

$$\left(\frac{1}{w_2(B)}\int_B |f|^q w_2\right)^{1/q} \le C\, r\left(\frac{1}{w_1(B)}\int_B \left(\sum_{i=1}^m |X_i f| + |f|\right)^p w_1\right)^{1/p}.$$

2) *In the case that $w_1 = w_2 = w \in A_p$ and $p > 1$, the above q can be replaced by $q = kp$ with $1 \le k \le Q/(Q-1) + \delta_p$ for some $\delta_p > 0$.*

3) *In the case that $w_1 = w_2 = 1$ and $1 < p < Q$, the above q ranges $1 \le q \le pQ/(Q-p)$.*

PROOF OF THEOREM 5.2. The proof is basically similar to the one in Section 4. Thus we adapt the same notations of $\tilde{v}$, $\tilde{w}$, I, B, $\sigma(\xi)$ and just indicates the substantial difference. We define the fractional integral

$$Tf(\xi) = \int_\Omega f(\eta) K(\xi, \eta)\, d\eta$$

where $K(\xi, \eta) = \varrho(\xi, \eta)/|B(\eta, \varrho(\xi, \eta))|$.

We note here $\phi(I)$ will be

$$\phi(I) = \sup\{K(\xi, \eta) :\ \xi, \eta \in I,\ \varrho(\xi, \eta) \ge \frac{1}{2}\,\rho(I)\} \approx \frac{\rho(I)}{|I|}$$

where $\rho(I)$ is the radius of the ball I.

But the Sawyer-Wheeden condition here (inequality (4.5) in Section 4) is still equivalent to

$$\frac{\rho(I)}{\rho(B)}\frac{|B|}{|I|}\left(\frac{w_2(I)}{w_2(B)}\right)^{1/q}\left(\frac{\sigma(I)}{\sigma(B)}\right)^{1/p'} \le C\,, \tag{5.4}$$

for all balls $I \subset cB$. Under this condition we have by Lemma 5.1 and Theorem 3 in [SW],

$$\left(\frac{1}{w_2(B)}\int_B |f|^q w_2\right)^{1/q} \le C\big(|B|^{-1} w_1(B)^{1/p}\sigma(B)^{1/p'}\big)\rho(B) \cdot \left(\frac{1}{w_1(B)}\int_B \left(\sum_{i=1}^m |X_i f| + |f|\right)^p w_1\right)^{1/p} \tag{5.5}$$

for all $f \in C_0^\infty(B)$. If assume $w_1 \in A_p$, then $|B|^{-1} w_1(B)^{1/p}\sigma(B)^{1/p'} \le C$. We also note that (5.3) is equivalent to (5.4) when $w_1 \in A_p$. This shows the first case of Theorem 5.2.

When $w_1 = w_2 = w \in A_p(\Omega)$, (5.3) is equivalent to

$$\frac{\rho(I)}{\rho(B)}\left(\frac{w(I)}{w(B)}\right)^{1/q} \le C\left(\frac{w(I)}{w(B)}\right)^{1/p}. \tag{5.6}$$

We note when the ball B is small, say $\rho(B) \le r_0$, there is some N-tuple J such that for any ball $I \subset cB \subset \Omega$ we have

$$|I| \approx \rho(I)^{d(J)}, \qquad |B| \approx \rho(B)^{d(J)},$$

where $d(J) \le Q$ is the length of the commutator X_J (see Section 2 or see [NSW]). Since $A_p \subset D_p$ (doubling of order p), *i.e.*

$$w(B) \le C\left(\frac{|B|}{|I|}\right)^p w(I), \quad \text{for any } I \subset cB \subset \Omega,$$

then the assumption that $w \in A_p(\Omega)$ implies that (5.6) holds if q is chosen so that $(d(J)(1/p - 1/q))^{-1} = p$, *i.e.* $q = (d(J)p/(d(J) - 1))$. Actually, since $w \in A_p$ implies $w \in A_{p_0}$ for some $p_0 < p$ by Theorem 2.4, it follows that q can be chosen slightly larger than $d(J)p/(d(J)-1)$. By noticing that

$$\frac{Qp}{Q-1} \le \frac{d(J)p}{d(J)-1}, \quad \text{since } d(J) \le Q,$$

thus case 2 of the Theorem 5.2 follows.

When $w_1 = w_2 = 1$, (5.2) is further equivalent to

$$\frac{\rho(I)}{\rho(B)}\left(\frac{|I|}{|B|}\right)^{1/q} \le C\left(\frac{|I|}{|B|}\right)^{1/p}. \tag{5.7}$$

We note again that $|B| \approx \rho(B)^{d(J)}$, $|I| \approx \rho(I)^{d(J)}$, and $\rho(I) \leq c\,\rho(B)$, thus (5.7) holds if $d(J)(1/q - 1/p) + 1 \geq 0$. But

$$Q(\frac{1}{q} - \frac{1}{p}) + 1 \leq d(J)(\frac{1}{q} - \frac{1}{p}) + 1\,, \quad \text{since } p < q \text{ and } d(J) \leq Q\,,$$

and $Q(1/q - 1/p) + 1 \geq 0$ for $1 \leq q \leq pQ/(Q-p)$, thus the third case of Theorem 5.2 follows for $p < q \leq pQ/(Q-P)$. For $1 \leq q \leq p$ the result follows by Hölder's inequality.

6. Proof of the main theorems.

This section is devoted to the proof of the main theorems in this paper, theorems A, B and C. The proof of Poincaré inequalities is based on the variants of Poincaré type inequalities on the homogeneous space $(\tilde{\Omega}, \tilde{\varrho})$ proved in Section 4. The basic technique is an argument by Kohn (see [K]). This method was further developed and applied to the proof of unweighted Poincaré inequality for vector fields by Jerison (see [J]). We first prove the following variants of Poincaré inequalities on the homogeneous space (Ω, ϱ) associated to the original vector fields $X_1, \ldots, X_m$.

Lemma 6.1. 1) *Let $1 < p < q < \infty$ and assume $w_1 \in A_p(\Omega)$, w_2 a doubling weight satisfying the following condition*

$$\frac{\rho(I)}{\rho(B)} \left(\frac{w_2(I)}{w_2(B)}\right)^{1/q} \leq C \left(\frac{w_1(I)}{w_1(B)}\right)^{1/p} \tag{6.2}$$

for all metric balls I and B with $I \subset cB \subset \Omega$, where $\rho(B)$ is the radius of the ball B. Let $E \subset\subset \Omega$, then there is a positive constant C such that for any balls $B = B(\xi, r)$ with $\xi \in E$, $cB \subset \Omega$ and $f \in C^\infty(\overline{cB})$, the following inequality holds

$$\left(\frac{1}{w_2(B)} \int_B |f - f_B|^q w_2\right)^{1/q}$$
$$\leq C\, r \left(\frac{1}{w_1(B)} \int_{cB} \Big(\sum_{i=1}^m |X_i f| + |f|\Big)^p w_1\right)^{1/p}$$

where f_B may be taken to be

$$f_B = \frac{1}{w_2(B)} \int_B f w_2\,.$$

2) *In the case that* $w_1 = w_2 = w \in A_p$, *the above* q *in the inequality can be taken as* $q = kp$ *with* $1 \le k \le Q/(Q-1) + \delta_p$ *for some* $\delta_p > 0$, *and* f_B *can be taken as*

$$\frac{1}{|B|}\int_B f \quad or \quad \frac{1}{|B|}\int_B fw\,.$$

3) *In the case that* $w_1 = w_2 = 1$ *and* $1 < p < Q$, *the* q *can be taken as* $1 \le q \le pQ/(Q-p)$ *and* f_B *can be taken as*

$$f_B = \frac{1}{|B|}\int_B f\,.$$

PROOF OF LEMMA 6.1. Let $B = B(\xi_0, r)$ and $\tilde{B} = \tilde{B}((\xi_0, 0), r)$. Define $\tilde{w}_i(\tilde{\xi}) = \tilde{w}(\xi, t) \equiv w(\xi)$ $(i = 1, 2)$ as in Subsection 2.3, then by Lemma 2.7, $\tilde{w}_1 \in A_p(\tilde{\Omega})$. It is also easy to check that (4.2) holds for $\tilde{w}_1$ and $\tilde{w}_2$. We also define $\tilde{f}(\tilde{\xi}) = \tilde{f}(\xi, t) \equiv f(\xi)$. Then $\tilde{f}$ is a function on $B(\xi, r)\times\mathbb{R}^l$ that is independent of $t \in \mathbb{R}^l$ and $\tilde{X}_i\tilde{f} = X_i f$. Consequently, by Theorem 4.1, there are constants $C = C(\tilde{f}, \tilde{B})$ and $c \ge 1$, such that

$$\begin{aligned}\Big(\frac{1}{\tilde{w}_2(\tilde{B}((\xi_0,0),r))}\int_{\tilde{B}((\xi_0,0),r)} |\tilde{f}(\tilde{\xi}) - C|^q \tilde{w}_2(\tilde{\xi})\,d\tilde{\xi}\Big)^{1/q}\\ \le C\,r\Big(\frac{1}{\tilde{w}_1(\tilde{B}((\xi_0,0),cr))}\\ \cdot\int_{\tilde{B}((\xi_0,0),cr)}\Big(\sum_{i=1}^m |\tilde{X}_i\tilde{f}(\tilde{\xi})| + |\tilde{f}(\tilde{\xi})|\Big)^p \tilde{w}_1(\tilde{\xi})\,d\tilde{\xi}\Big)^{1/p}.\end{aligned}$$

Note $B(\xi_0, r)\times\{0\} \subset \tilde{B}((\xi_0, 0), r) \subset B(\xi_0, r)\times\mathbb{R}^l$ and $d\tilde{\xi} = d\xi dt$, also note $\tilde{w}_i\Big(\tilde{B}((\xi_0,0),r)\Big) \approx |\tilde{B}|w_i(B)/|B|$ $(i = 1, 2)$, thus integration with respect to t and an application of Lemma 4.4 in [J] yields the following

$$\begin{aligned}\Big(\frac{1}{w_2(B(\xi_0,r))}\int_{B(\xi_0,r)} |f(\xi) - C|^q w_2(\xi)d\xi\Big)^{1/q}\\ \le C\,r\Big(\frac{1}{w_1(B(\xi_0,cr))}\\ \cdot\int_{B(\xi_0,cr)}\Big(\sum_{i=1}^m |X_i f(\xi)| + |f(\xi)|\Big)^p w_1(\xi)\,d\xi\Big)^{1/p}\end{aligned}$$

for some constant $c \geq 1$ independent of f and B. Then the first part of the lemma follows.

By using the corresponding cases in Theorem 4.1, we can prove the other two cases of the lemma in the same way.

6.1. A Whitney decomposition.

In this subsection we will recall the Whitney decomposition in metric spaces (Ω, ϱ). We first give an alternate definition of the metric on Ω. An admissible path γ is a Lipschitz curve $\gamma : [a, b] \to \Omega$ such that there exist functions c_i, $a \leq t \leq b$, with $\sum_{i=1}^m c_i(t)^2 \leq 1$ and $\gamma'(t) = \sum_{i=1}^m c_i(t) X_i(\gamma(t))$ for almost every $t \in [a, b]$. We define a metric associated to $X_1, \ldots, X_m$ by $\varrho(\xi, \eta) = \min\{b \geq 0 : \text{there exists } \gamma : [0, b] \to \Omega$ such that $\gamma(0) = \xi, \gamma(b) = \eta\}$. Obviously, this metric is equivalent to those defined before. Throughout this subsection and the remaining of this section, we always use this definition of the metric.

For a ball $B = B(\xi, r)$, we denote $B' = B(\xi, 2r), B'' = B(\xi, 4r)$ and $B^* = B(\xi, 10r)$. We shall also denote the radius of B by $\rho(B)$. We also recall the doubling condition

$$|B(\xi, 2r)| \leq A\,|B(\xi, r)|, \quad \xi \in E \subset\subset \Omega\,. \tag{6.3}$$

Now we recall some results proved by Jerison [J].

Lemma 6.4. (Whitney decomposition). *Let $E_1 = B(\xi_1, r_1)$, then there is a pairwise disjoint family of balls $\mathcal{F}$ and a constant M depending only on the doubling constant in* (6.3) *such that*

(i) $E_1 = \bigcup_{B \in \mathcal{F}} B'$,

(ii) $B \in \mathcal{F}$ *implies* $10^2 \rho(B) \leq \rho(B, \partial E_1) \leq 10^3 \rho(B)$,

where $\rho(B, \partial E_1)$ is the distance in the metric ϱ from B to ∂E_1.

(iii) $\#\{B \in \mathcal{F} : \eta \in B^*\} \leq M$.

($\#S$ is the number of elements in the set S).

For $B \in \mathcal{F}$, define γ_B as an admissible path from the center of B to ξ_1 (the center of E_1) of length $\leq r_1$. Denote the subset of E_1 defined by the image of γ_B by γ_B as well. This path may not be unique, but will be fixed throughout this subsection and next one. Denote $\mathcal{F}(B) = \{A \in \mathcal{F} : A' \bigcap \gamma_B \neq \emptyset\}$. The following has been proved by Jerison [J].

Lemma 6.5. (Jerison) *Let $B \in \mathcal{F}$, then*

(i) *There are no elements of $\mathcal{F}(B)$ of radius less than $10^{-2}\rho(B)$.*

(ii) *For any r, $\#\{A \in \mathcal{F}(B) : r \le \rho(A) \le 2r\} \le M$, a constant depending only on the metric of Ω.*

(iii) $\#\mathcal{F}(B) \le C \log(r_1/\rho(B))$.

Lemma 6.6. *Let w be a doubling weight in (Ω, ϱ), then there exists some $\varepsilon > 0$ such that for any $A \in \mathcal{F}$ and any $r > 0$,*

$$\sum_{\substack{B \in A(\mathcal{F}) \\ r \le \rho(B) \le 2r}} w(B) \le C \left(\frac{\rho(B)}{\rho(A)}\right)^{\varepsilon} w(A),$$

where $A(\mathcal{F}) = \{B \in \mathcal{F} : A \in \mathcal{F}(B)\}$.

PROOF. The proof follows by using the argument for the unweighted case in [J] and in view of the doubling condition on w.

We also need the following lemma.

Lemma 6.7. *Let $w \in D_\mu(\Omega)$, i.e. doubling of order μ, and $q < 1$. For $c_1 > 0$, there exists $c_2 = c_2(c_1, w, q)$ such that for every pair of balls B_1, B_2 in the metric space (Ω, ϱ) satisfying $|B_2| \le c_1 |B_1 \cap B_2|$ and any f such that $\int_{B_j} |f - f_{B_j}|^q w \le A$ for $j = 1, 2$, we have*

$$\int_{B_1 \bigcup B_2} |f - f_{B_1}|^q w \le c_2 A.$$

The proof is easy and we omit it here.

By applying Lemma 6.7 and a well-known covering lemma argument, we can show the following

Lemma 6.8. 1) *Let w_1, w_2, p, q be as in* (6.2). *Given any $\xi \in E \subset\subset \Omega$, $r > 0$ with $B_{4r} = B(\xi, 4r) \subset \Omega$ and $f \in C^\infty(\overline{B_{2r}})$, we have*

$$\left(\frac{1}{w_2(B_r)} \int_{B_r} |f - f_{B_r}|^q w_2\right)^{1/q}$$
$$\le Cr \left(\frac{1}{w_1(B_r)} \int_{B_{2r}} \left(\sum_{i=1}^{m} |X_i f| + |f|\right)^p w_1\right)^{1/p}$$

where $B_r = B(\xi, r)$, $B_{2r} = B(\xi, 2r)$ *and* C *independent of* B *and* f.

2) *In the case that* $w_1 = w_2 = w \in A_p$ *and* $p > 1$, *the* q *in the above inequality can be replaced by* $q = kp$ *with* $1 \le k \le Q/(Q-1) + \delta_p$ *for some* $\delta_p > 0$.

3) *In the case that* $w_1 = w_2 = 1$ *and* $1 < p < Q$, *the above inequality holds for* $1 \le q \le pQ/(Q-p)$.

The proof is easy and well-known and we refer the reader to the final subsection for a more crucial argument.

6.2. Proof of the main theorems.

We first like to remark out that the Sobolev inequalities in Theorem B and C follow immediately from Theorem 5.2. Indeed, we note that

$$\left(\frac{1}{w(B)}\int_B |f|^p w\right)^{1/p} \le \left(\frac{1}{w(B)}\int_B |f|^q w\right)^{1/q} \quad \text{for } p \le q\,.$$

Thus by Lemma 5.2 and Minkowski's inequality, Sobolev's inequality in Theorem B follows if

$$p \le q \le \left(\frac{Q}{Q-1} + \delta_p\right) p$$

and $r = \rho(B)$ small. For $q \le p$, it follows from Hölder's inequality. We can argue similarly to show Sobolev's inequality in Theorem C. We now turn to prove Theorem A, and Poincaré's inequalities in theorems B and C. It should be pointed out here that an analogue of Lemma 6.9 below has been proved by J. Fernándes [Fer] in their context. This kind of argument seems well-known now and one may also find it in the various places in the paper by C. Gutiérrez and R. Wheeden [GW]. We need such a lemma here to show the Poincaré inequality for the two weights case. We modify the proof given in [Fer] or [GW]. We recall again that $\Omega \subset \mathbb{R}^N$.

Lemma 6.9. *Assume* (6.2) *holds for doubling weights* w_1, w_2 *and* $1 < p < q < \infty$. *Then there are* $r_0 > 0$, $\alpha > 0$ *and some* q_0 *with* $p < q_0 < q$ *such that*

$$\left(\frac{\rho(I)}{\rho(B)}\right)^{p-\alpha}\left(\frac{w_2(I)}{w_2(B)}\right)^{p/q_0} \le C\,\frac{w_1(I)}{w_1(B)}\,,$$

for all metric balls $I \subset cB \subset \Omega$ *provided* $\rho(B) \le r_0$.

PROOF. Since $w_2 \in D_\mu$ (doubling of order μ), then $w_2 \in RD_\nu$ (reverse doubling of order ν). Thus we have for all $I \subset cB \subset \Omega$,

$$\frac{w_2(I)}{w_2(B)} \ge C \left(\frac{|I|}{|B|}\right)^{\mu},$$

and

$$\frac{w_2(I)}{w_2(B)} \le C \left(\frac{|I|}{|B|}\right)^{\nu}.$$

By Theorem 7 in [NSW], there is an N-tuple J, depending on B, such that $|I| \approx \rho(I)^{d(J)}$, $|B| \approx \rho(B)^{d(J)}$ for all $I \subset cB \subset \Omega$ provided $\rho(B) \le r_0$ for some $r_0 > 0$. The above $d(J)$ is the length of the commutator X_J (see Section 2) and $N \le d(J) \le Q$.

If we select $\alpha < \nu(1 - p/q)N$, then $\alpha < \nu(1 - p/q)d(J)$. By (6.2), we have

$$\begin{aligned}
\frac{w_2(I)}{w_2(B)} &= \left(\frac{w_2(I)}{w_2(B)}\right)^{p/q} \left(\frac{w_2(I)}{w_2(B)}\right)^{1-p/q} \\
&\le C \left(\frac{\rho(B)}{\rho(I)}\right)^{p-\alpha} \frac{w_1(I)}{w_1(B)} \left(\frac{w_2(I)}{w_2(B)}\right)^{1-p/q} \left(\frac{\rho(B)}{\rho(I)}\right)^{\alpha} \\
&\le C \left(\frac{\rho(B)}{\rho(I)}\right)^{p-\alpha} \frac{w_1(I)}{w_1(B)} \left(\frac{\rho(I)}{\rho(B)}\right)^{\nu(1-p/q)d(J)-\alpha} \\
&\le C \left(\frac{\rho(B)}{\rho(I)}\right)^{p-\alpha} \frac{w_1(I)}{w_1(B)} \left(\frac{w_2(I)}{w_2(B)}\right)^{\nu(1-p/q)/\mu-\alpha/\mu d(J)}
\end{aligned}$$

Thus,

$$\left(\frac{w_2(I)}{w_2(B)}\right)^{1-\nu(1-p/q)\mu+\alpha\mu d(J)} \le C \left(\frac{\rho(B)}{\rho(I)}\right)^{p-\alpha} \frac{w_1(I)}{w_1(B)}.$$

Therefore, by noticing $w_2(I) \le c\, w_2(B)$ and $\alpha < \nu(1 - p/q)N$, Lemma 6.9 will follow if we select $p < q_0 < q$ satisfying

$$1 - \frac{\nu}{\mu}(1 - \frac{p}{q}) + \frac{\alpha}{\mu N} < \frac{p}{q_0},$$

which implies

$$1 - \frac{\nu}{\mu}(1 - \frac{p}{q}) + \frac{\alpha}{\mu\, d(J)} < \frac{p}{q_0}.$$

As an easy consequence, we have

Corollary 6.10. *Assume* (6.2) *holds for doubling weights* w_1, w_2 *and* $1 < p < q < \infty$. *Then there are* $r_0 > 0$, $\alpha > 0$ *and some* q_0 *with* $p < q_0 < q$ *such that*

$$\left(\log \frac{\rho(B)}{\rho(I)}\right)^{q_0-1} \left(\frac{\rho(I)}{\rho(B)}\right)^{q_0} \left(\frac{w_2(I)}{w_2(B)}\right) \leq C \left(\frac{w_1(I)}{w_1(B)}\right)^{q_0/p},$$

for all metric balls $I \subset cB \subset \Omega$ *provided* $\rho(B) \leq r_0$.

We like to remark out that we do not know how big q_0 above can be. The only information is that q_0 is some number less than q. However, when $w_1 = w_2 = w \in A_p$ or $w_1 = w_2 = 1$, we will be able to know the precise range of q_0. We first show the following two lemmas.

Lemma 6.11. *Assume* $w \in A_p(\Omega)$, $p > 1$. *Given any two balls* I *and* B *with* $I \subset cB \subset \Omega$, *and* $\delta_1 = \rho(I)$, $\delta_2 = \rho(B)$, *we have the following*

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \leq C$$

provided that $p \leq q \leq pQ/(Q-1)+\delta_p$, *for some* $\delta_p > 0$, *and* $\rho(B) \leq r_0$ *for some* $r_0 > 0$.

Lemma 6.12. *Assume* $p > 1$, *then for any two balls* I *and* B *as in Lemma* 6.11, *we have the following*

$$\left(\frac{\delta_1}{\delta_2}\right) \left(\frac{|I|}{|B|}\right)^{1/q-1/p} \leq C$$

provided that $1 \leq q \leq pQ/(Q-p)$ *and* $\rho(B) \leq r_0$ *for some* $r_0 > 0$.

PROOF OF LEMMA 6.11. Since $w \in A_p(\Omega)$ implies $w \in A_{p_0}(\Omega)$ for some $p_0 < p$, and noting that $I \subset cB$, then we have

$$\left(\frac{w(B)}{w(I)}\right) \leq c \left(\frac{|B|}{|I|}\right)^{p_0}.$$

Thus an easy calculation shows by noting that $|I| \leq c\,|B|$,

$$\left(\frac{|I|}{|B|}\right)^{1/Q} \left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \leq C \tag{6.13}$$

provided that $p \le q \le pQ/(Q-1) + \delta_p$. Reasoning as before, there is an N-tuple J such that for $\delta_2 < r_0$,

$$|I| \approx \delta_1^{d(J)}, \quad |B| \approx \delta_2^{d(J)} .$$

Hence,

$$\left(\frac{|I|}{|B|}\right)^{1/Q} \approx \left(\frac{\delta_1}{\delta_2}\right)^{d(J)/Q} .$$

Since $\delta_1 < c\,\delta_2$, $d(J) \le Q$, we obtain

$$\left(\frac{\delta_1}{\delta_2}\right) \le C \left(\frac{\delta_1}{\delta_2}\right)^{d(J)/Q} .$$

Therefore, by (6.13), it completes the proof.

The proof of Lemma 6.12 is similar and we do not present it here. Now we prove the following two corollaries which will be needed when we prove the Poincaré inequalities for the case of equal weights and unweighted case.

Corollary 6.14. *Let δ_1, δ_2, I, B, w be as above, then*

$$\left(\frac{\delta_1}{\delta_2}\right)\left(\log\frac{\delta_2}{\delta_1}\right)^{1-1/q}\left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \le C$$

provided $p \le q < pQ/(Q-1) + \delta_p$ for the same δ_p in Lemma 6.11.

PROOF. It is sufficient to show that

$$\left(\frac{\delta_1}{\delta_2}\right)\left(\frac{w(I)}{w(B)}\right)^{1/q_0-1/p} \le C_1$$

implies

$$\left(\frac{\delta_1}{\delta_2}\right)\left(\log\frac{\delta_2}{\delta_1}\right)^{1-1/q}\left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \le C_2$$

for $q < q_0$.

Since $q < q_0$, there is some $\varepsilon > 0$ such that

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{p} + \varepsilon .$$

We note since $w \in A_p \subset A_\infty$, then by Theorem 2.5 there is $\eta > 0$ such that

$$\frac{w(I)}{w(B)} \leq C \left(\frac{|I|}{|B|}\right)^{\eta} \leq C \left(\frac{\delta_1}{\delta_2}\right)^{d(J)\eta}.$$

Then,

$$\begin{aligned}
&\left(\frac{\delta_1}{\delta_2}\right)\left(\log\frac{\delta_2}{\delta_1}\right)^{1-1/q}\left(\frac{w(I)}{w(B)}\right)^{1/q-1/p} \\
&\quad = \left(\frac{\delta_1}{\delta_2}\right)\left(\log\frac{\delta_2}{\delta_1}\right)^{1-1/q}\left(\frac{w(I)}{w(B)}\right)^{1/q_0-1/p}\left(\frac{w(I)}{w(B)}\right)^{\varepsilon} \\
&\quad \leq C_1\left(\log\frac{\delta_2}{\delta_1}\right)^{1-1/q}\left(\frac{\delta_1}{\delta_2}\right)^{\varepsilon\eta d(J)} \leq C_2
\end{aligned}$$

since $\delta_1 < c\,\delta_2$.

Corollary 6.15. *Let δ_1, δ_2, I, B, w be as above, then*

$$\left(\frac{\delta_1}{\delta_2}\right)\left(\log\frac{\delta_2}{\delta_1}\right)^{1-1/q}\left(\frac{|I|}{|B|}\right)^{1/q-1/p} \leq C$$

provided $1 \leq q < pQ/(Q-p)$.

The proof of Corollary 6.15 is similar to that of Corollary 6.14. We recall that the inequality in Lemma 6.8 is equivalent to

$$\begin{aligned}
\int_B |f - f_B|^q w_2 &\leq C\rho(B)^q w_2(B) w_1(B)^{-q/p} \\
&\quad \left(\int_{2B} \Big(\sum_{i=1}^m |X_i f| + |f|\Big)^p w_1\right)^{q/p}
\end{aligned} \tag{6.16}$$

for $p < q$ and w_1, w_2 satisfying (6.2).

Now we are ready to prove the following Poincaré inequality for two weights (*i.e.* Theorem A).

Theorem 6.17. *Let $1 < p < q$, w_1, w_2 satisfy* (6.2), *$w_1 \in A_p$, and $E_1 = B(\xi_1, r_1)$ be as in Lemma* 6.4. *We also assume here that $w_1 \leq w_2$.*

Then there are constants $C > 0$, $r_0 > 0$ and some q_0 as in Lemma 6.9 such that for any $f \in C^\infty(\overline{E_1})$,

$$\left(\frac{1}{w_2(E_1)}\int_{E_1}|f-f_{E_1}|^{q_0}w_2\right)^{1/q_0} \leq C\rho(E_1)\left(\frac{1}{w_1(E_1)}\int_{E_1}\sum_{i=1}^{m}|X_if|^p w_1\right)^{1/p}$$

provided $\rho(E_1) \leq r_0$, where C and r_0 are independent of the ball E_1 and the function f.

PROOF. Choose $B_0 \in \mathcal{F}$ such that $\xi_1 \in B_0'$. Write $f_0 = f_{B_0''}$, and order elements of $\mathcal{F}(B) = \{A_1, A_2, \ldots, A_l\}$ such that $A_1 = B, \ldots, A_l = B_0$ and $A_k' \bigcap A_{k+1}' \neq \emptyset$, all k. Note that $2A_k'' \subset A_k^*$ and (6.16) also holds by replacing q by q_0, then apply (6.16) to $f - f_0$ yields

$$\int_{A_k''}|f-f_{A_k''}|^{q_0}w_2$$
$$\leq C\rho(A_k)^{q_0}w_2(A_k)w_1(A_k)^{-q_0/p}\left(\int_{A_k^*}(\sum_{i=1}^{m}|X_if|+|f-f_0|^p)w_1\right)^{q_0/p}$$

Moreover, A_k, A_{k+1} have comparable radii and volumes and $A_k' \bigcap A_{k+1}'$ contains a ball of comparable radius to those of A_k and A_{k+1}. Thus,

$$\begin{aligned}\int_{A_k''\cup A_{k+1}''}&|f_{A_k''}-f_{A_{k+1}''}|^{q_0}w_2\\ &\leq C\int_{A_k''\cup A_{k+1}''}|f-f_{A_k''}|^{q_0}w_2+\int_{A_k''\cup A_{k+1}''}|f-f_{A_{k+1}''}|^{q_0}w_2\\ &\leq C\rho(A_k)^{q_0}w_2(A_k)w_1(A_k)^{-q_0/p}\\ &\quad\cdot\left(\int_{A_k^*\cup A_{k+1}^*}\left(\sum_{i=1}^{m}|X_if|+|f-f_0|\right)^p w_1\right)^{q_0/p}.\end{aligned}$$

Hence,

$$\int_{B'}|f-f_0|^{q_0}w_2=\int_{B'}\left|f-f_{B''}+\sum_{k=1}^{l-1}(f_{A_k''}-f_{A_{k+1}''})\right|^{q_0}w_2$$

$$\leq l^{q_0-1}\left(\int_{B'}|f-f_{B''}|^{q_0}w_2+\sum_{k=1}^{l-1}\int_{B'}|f_{A_k''}-f_{A_{k+1}''}|^{q_0}w_2\right)$$

$$\leq C\,l^{q_0-1}\sum_{k=1}^{l}\rho(A_k)^{q_0}w_2(A_k)w_1(A_k)^{-q_0/p}\frac{w_2(B)}{w_2(A_k)}$$

$$\cdot\left(\int_{A_k^*}\left(\sum_{i=1}^{m}|X_if|+|f-f_0|\right)^p w_1\right)^{q/p}.$$

Summing over $B\in\mathcal{F}$,

$$\int_{E_1}|f-f_0|^{q_0}w_2\leq\sum_{B\in\mathcal{F}}\int_{B'}|f-f_0|^{q_0}w_2$$

$$\leq C\sum_{B\in\mathcal{F}}\sum_{A\in\mathcal{F}(B)}(\#\mathcal{F}(B))^{q_0-1}\rho(A)^{q_0}\frac{w_2(B)}{w_2(A)}w_2(A)w_1(A)^{-q_0/p}$$

$$\cdot\left(\int_{A^*}\left(\sum_{i=1}^{m}|X_if|+|f-f_0|\right)^p w_1\right)^{q_0/p}.$$

By Lemma 6.5 and 6.6,

$$\sum_{B\in A(\mathcal{F})}(\#\mathcal{F}(B))^{q_0-1}w_2^{\mathfrak{k}}(B)$$

$$=\sum_{k=1}^{\infty}\sum_{\substack{B\in A(\mathcal{F})\\ 2^{-k}\rho(A)\leq\rho(B)/100\leq 2^{-k+1}\rho(A)}}(\#\mathcal{F}(B))^{q_0-1}w_2(B)$$

$$\leq C\sum_{k=1}^{\infty}\left(k+\log\left(\frac{r_1}{\rho(A)}\right)\right)^{q_0-1}2^{-\varepsilon k}w_2(A)$$

$$\leq C\left(\log\frac{r_1}{\rho(A)}\right)^{q_0-1}w_2(A)\,.$$

By Corollary 6.10

$$\rho(A)^{q_0}w_2(A)w_1(A)^{-q_0/p}\left(\log\frac{r_1}{\rho(A)}\right)^{q_0-1}$$

$$=\left(\log\frac{r_1}{\rho(A)}\right)^{q_0-1}\frac{w_2(A)}{w_2(E_1)}\left(\frac{w_1(A)}{w_1(E_1)}\right)^{-q_0/p}.$$

$$\cdot \left(\frac{\rho(A)}{\rho(E_1)}\right)^{q_0} \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p}$$
$$\leq C\rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p}.$$

Thus,

$$\int_{E_1} |f - f_0|^{q_0} w_2$$
$$\leq \sum_{A\in\mathcal{F}} \rho(A)^{q_0} w_2(A) w_1(A)^{-q_0/p} \left(\log \frac{r_1}{\rho(A)}\right)^{q_0-1}$$
$$\cdot \left(\int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0|\right)^p w_1\right)^{q_0/p}$$
$$\leq C \sum_{A\in\mathcal{F}} \rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p}$$
$$\cdot \left(\int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0|\right)^p w_1\right)^{q_0/p}$$
$$\leq C\rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p}$$
$$\cdot \left(\sum_{A\in\mathcal{F}} \int_{A^*} \left(\sum_{i=1}^m |X_i f| + |f - f_0|\right)^p w_1\right)^{q_0/p},$$

since $q_0/p \geq 1$

$$\leq C\rho(E_1)^{q_0} w_2(E_1) w_1(E_1)^{-q_0/p}$$
$$\cdot \left(\int_{E_1} \left(\sum_{i=1}^m |X_i f| + |f - f_0|\right)^p w_1\right)^{q_0/p}$$

The last inequality above is because all A^*, for $A \in \mathcal{F}$, have bounded overlap by Lemma 6.4. Thus by Minkowski's inequality

$$\left(\frac{1}{w_2(E_1)} \int_{E_1} |f - f_0|^{q_0} w_2\right)^{1/q_0}$$
$$\leq C\rho(E_1) \left(\frac{1}{w_1(E_1)} \left(\sum_{i=1}^m |X_i f|\right)^p w_1\right)^{1/p}$$
$$+ C\rho(E_1) \left(\frac{1}{w_1(E_1)} \int_{E_1} |f - f_0|^p w_1\right)^{1/p}.$$

We recall that (6.9) implies

$$\left(\frac{\rho(E_1)}{\rho(D)}\right)^{1-\alpha/p}\left(\frac{w_2(E_1)}{w_2(D)}\right)^{1/q_0} \le C\left(\frac{w_1(E_1)}{w_1(D)}\right)^{1/p} \le C\left(\frac{w_1(E_1)}{w_1(D)}\right)^{1/q_0},$$

for any $E_1 \subset c\,D \subset \Omega$. Since the center ξ_1 of the ball $E_1 = B(\xi_1, r_1)$ is in $E \subset\subset \Omega$, we can always pick the ball $D \subset \Omega$ such that $\rho(D) \ge c_1$. We note $w_1 \in A_p(\Omega)$, then

$$\frac{1}{|D|^p}\int_D w_1 \approx \left(\int_D w^{-1/(p-1)}\right)^{-(p-1)} \ge C\,,$$

since

$$\left(\int_D w^{-1/(p-1)}\right)^{p-1} \le \left(\int_\Omega w^{-1/(p-1)}\right)^{p-1}.$$

Thus $w_1(D) \ge c_2$. The above c_1 and c_2 are independent of the ball E_1 but only dependent on E, Ω and w_1. We also note that

$$\rho(D) \le c_3, \quad w_2(D) \le w_2(\Omega).$$

Thus

$$\rho(E_1)^{1-\alpha/p} w_2(E_1)^{1/q_0} \le C\; w_1(E_1)^{1/q_0}\,,$$

this is equivalent to

$$\rho(E_1)^{1-\alpha/p}\left(\frac{w_2(E_1)}{w_1(E_1)}\right)^{1/q_0} \le C\,,$$

and this further implies that

$$\rho(E_1)\left(\frac{w_2(E_1)}{w_1(E_1)}\right)^{1/q_0} \le C\rho(E_1)^{\alpha/p}.$$

By noting that $w_1 \le w_2$ and $p \le q_0$, and applying the Hölder inequality, we have

$$\left(\frac{1}{w_1(E_1)}\int_{E_1}|f-f_0|^p w_1\right)^{1/p}$$
$$\le \left(\frac{1}{w_1(E_1)}\int_{E_1}|f-f_0|^{q_0} w_1\right)^{1/q_0}$$

$$\leq \left(\frac{1}{w_2(E_1)}\int_{E_1}|f-f_0|^{q_0}w_2\right)^{1/q_0}\left(\frac{w_2(E_1)}{w_1(E_1)}\right)^{1/q_0}.$$

Therefore,

$$\rho(E_1)\left(\frac{1}{w_1(E_1)}\int_{E_1}|f-f_0|^{p}w_1\right)^{1/q_0}$$
$$\leq C\rho(E_1)^{\alpha/p}\left(\frac{1}{w_2(E_1)}\int_{E_1}|f-f_0|^{q_0}w_2\right)^{1/q_0}.$$

Thus if $\rho(E_1)$ is small enough, say $C\rho(E_1)^{\alpha/p}\leq 1/2$, we have

$$\left(\frac{1}{w_2(E_1)}\int_{E_1}|f-f_0|^{q}w_2\right)^{1/q}$$
$$\leq C\rho(E_1)\left(\frac{1}{w_1(E_1)}\int_{E_1}\left(\sum_{i=1}^{m}|X_if|\right)^{p}w_1\right)^{1/p}$$

The proof of Sobolev's inequalities is even simpler since the integrals on both sides of the inequality in Lemma 5.2 are over the equal ball and thus we do not need the above covering lemma argument. The proof of Poincare's inequalities in theorems B and C is similar to the proof given above if we apply corollaries 6.14 and 6.15. The detailed proof can be found in [L3].

REMARK 3. Since Jerison-Kohn's argument only works for metrics, the Poincaré inequalities in theorems A, B and C only hold for metric balls B. If the balls B are pseudometric balls, such as $B_I(x,\delta)$, the Poincaré inequality will have the following form

$$\left(\frac{1}{w_2(B)}\int_{B}|f-f_B|^{q}w_2\right)^{1/q}$$
$$\leq C\rho(B)\left(\frac{1}{w_1(B)}\int_{cB}\left(\sum_{i=1}^{m}|X_if|\right)^{p}w_1\right)^{1/p}$$

for some constant $c\geq 1$.

On the other hand, the Sobolev inequalities hold for both metrics and pseudometrics.

REMARK 4. The reason for not having the Poincaré inequality for $q = pQ/(Q-p)$ in Theorem C is due to Corollary 6.15. We can see this when we do the above covering lemma argument. It is hoped that this inequality should be true for $q = pQ/(Q-p)$.

Part II: The local regularity of solutions to certain classes of degenerate differential operators formed by vector fields.

This part studies Harnack's inequalities and several general mean-value inequalities for the solutions to the following degenerate differential equations

$$L = \sum_{i,j=1}^{m} X_i^*(a_{ij}(x)X_j)$$

and

$$\mathcal{L} = -\sum_{i,j=1}^{m} X_i(a_{ij}(x)X_j)$$

where $X_1, \ldots, X_m$ are vector fields satisfying Hörmander's condition, and X_i^* is the adjoint of X_i. The coefficient matrix $A = (a_{ij}(x))$ is symmetric and satisfies

$$c^{-1}w(x)\,|\xi|^2 \leq \langle A\xi, \xi\rangle \leq c\,w(x)\,|\xi|^2, \quad \xi \in \mathbb{R}^m$$

where $w(x) \in A_2(\Omega)$ is as defined in Part I. The Hölder continuity of the solutions will also be derived.

7. The main theorems of Part II.

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and connected domain, and let $X_1, \ldots, X_m$ be C^∞ real vector fields, in a neighbourhood of $\overline{\Omega}$, satisfying Hörmander condition. Let X_i^* be the adjoint of X_i, we will consider the differential operators

$$L = \sum_{i,j=1}^{m} X_i^*(x)(a_{ij}(x)X_j(x)),$$

and

$$\mathcal{L} = -\sum_{i,j=1}^{m} X_i(x)(a_{ij}(x)X_j(x))$$

where the coefficients a_{ij} are measurable, real-valued functions whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies

$$c^{-1}w(x)\,|\xi|^2 \le \langle A\xi, \xi\rangle \le c\; w(x)\,|\xi|^2\,, \tag{7.1}$$

$\langle\cdot,\cdot\rangle$ denotes the usual dot product, and $w \in A_2(\Omega)$ is a nonnegative weight function as defined in Part I.

Our main aim here will be to derive the mean value and Harnack inequalities for suitably defined weak solutions of L and $\mathcal{L}$. Many authors have studied such equations, we refer the reader to the following work: [De], [EP], [K], [M], [N], [GT], [FKS], [CW2], [St]. The situation considered there was for the case when $X_i = \partial/\partial x_i$ and $m = N$. The Harnack inequality for differential operator $\mathcal{L}$ when the matrix $A = (a_{ij}(x))$ is the identity one follows from the work of fundamental solutions by Sánchez-Calle [Sa], Nagel-Stein-Wainger [NSW] or Sánchez-Calle and Jerison [JS]. A related situation was considered in [Bon], [FS] and [FrL].

We now recall the Poincaré and Sobolev inequalities in the particular case $p = 2$. By Theorem B and the Remark 3 in Part I for the metric ball $B_I \subset \Omega$ we have

Poincaré Inequality. *Let $w \in A_2(\Omega)$ and $E \subset\subset \Omega$, then there exist constants $r_0 > 0$, $C > 0$, $\delta > 0$, $a > 1$, such that for any metric ball $B = B_I(x,r)$ with $aB = B_I(x,ar) \subset \Omega$, $x \in E$, and function $f \in C^\infty(B)$, the following inequality holds*

$$\begin{aligned}&\left(\frac{1}{w(B)}\int_B |f - f_B|^q w(y)dy\right)^{1/q}\\ &\qquad \le Cr\left(\frac{1}{w(B)}\int_{aB}\sum_{i=1}^m |X_i f|^2 w(y)dy\right)^{1/2}\end{aligned} \tag{7.2}$$

provided $0 < r < r_0$, $q = 2k$, $1 \le k \le Q/(Q-1) + \delta$, where C, r_0 depend only on the A_2 constant of the weight w, E, Ω, and f_B may be taken to be either

$$\frac{1}{|B|}\int_B f(y)dy \quad \text{or} \quad \frac{1}{w(B)}\int_B f(y)w(y)dy\,.$$

By Theorem B in Part I again, for any metric ball $B \in \Omega$ we also have

Sobolev Inequality. *Let $w \in A_2(\Omega), E \subset\subset \Omega$, then there exist constants $r_0 > 0$, $C > 0$, $\delta > 0$, such that for any metric ball $B = B(x,r) \subset \Omega$, $x \in E$ as above, and any function $f \in C_0^\infty(B)$, the following inequality holds*

$$\left(\frac{1}{w(B)}\int_B |f|^q w(y)dy\right)^{1/q} \leq Cr\left(\frac{1}{w(B)}\int_B \sum_{i=1}^m |X_i f|^2 w(y)dy\right)^{1/2} \tag{7.3}$$

provided $q = 2k$, $1 \leq k \leq Q/(Q-1)+\delta$, $0 < r < r_0$, where $C > 0$, $r_0 > 0$ only depend on the A_2 constant of the weight w, E, Ω.

REMARK 5. The above constant $Q \geq 2$ is the homogeneous dimension of the graded nilpotent group G corresponding to the lifted vector fields $\{\tilde{X}_i\}$ of $\{X_i\}$ (see Part I).

Throughout this part, w is always assumed to be in $A_2(\Omega)$, and all integrals are Lebesgue integrals. We also use the notations $Xf = (X_1 f, \ldots, X_m f)$, $|Xf| = \left(\sum_{i=1}^m |X_i f|^2\right)^{1/2}$. We also fix $q > 2$ on the left sides of inequalities (7.2) and (7.3) and denote $q = 2\sigma$ for some $\sigma > 1$. For the definitions of solutions, subsolutions and supersolutions, and the associated function $\tilde{u}$ to the function u, see Section 8.

Now we state the main theorems in this part.

Theorem 7.4. (Mean-Value inequality) *Given $B = B_I(x_0, h) \subset \Omega$. Let $u \in H(B)$ be a nonnegative solution of $Lu = 0$, and $\tilde{u}$ be the function in L_w^2 associated to u. Then there exist positive constants c and d depending only on the parameters in (7.1) and (7.3) such that for any ball $B = B_I(x,h) \subset \Omega$, and for $1/2 \leq \alpha < 1$ and $-\infty < p < \infty$, we have*

$$\operatorname*{ess\,sup}_{\alpha B} \tilde{u}^p \leq \frac{c}{(1-\alpha)^d}(1+|p|)^{2\sigma/(\sigma-1)}\frac{1}{w(B)}\int_B \tilde{u}^p w\,.$$

Theorem 7.5. (Mean-Value inequality) *Let u be a subsolution of $Lu = 0$ belonging to $H(B)$. Let $\tilde{u}$ be the function in L_w^2 associated with u.*

Then there exist constants c and d such that for $1/2 \leq \alpha < 1$ and $0 < p < \infty$,

$$(\operatorname*{ess\,sup}_{\alpha B} \tilde{u}^+)^p \leq C \frac{1}{w(B)} \int_B (\tilde{u}^+)^p w \ ,$$

where $C \leq c/(1-\alpha)^d$ if $p \geq 2$ and $\leq c^{\log(3/p)}/(1-\alpha)^d$ if $0 < p < 2$, and c, d are independent of u and B.

Corollary 7.6. *If in the theorem above u is a solution, then the conclusion holds with $\tilde{u}^+$ replaced by $|\tilde{u}|$, i.e. for $0 < p < \infty$, $0 < \alpha < 1$,*

$$\operatorname*{ess\,sup}_{\alpha B} |\tilde{u}|^p \leq C \frac{1}{w(B)} \int_B |\tilde{u}|^p w \ ,$$

where C is as in the Theorem 7.5 above.

Theorem 7.7. (Harnack inequality) *Let u be nonnegative solution of $Lu = 0$ belonging to $H(2B)$ for a ball $B = B(x,r)$ with $2B \subset \Omega, r \leq r_0$ for some $r_0 > 0$. Let $\tilde{u}$ be the function in L^2_w associated with u, then*

$$\operatorname*{ess\,sup}_{B} \tilde{u} \leq c \operatorname*{ess\,inf}_{B} \tilde{u} \ ,$$

with c independent of u and B.

As an easy consequence of Theorem 7.7 and following the proof of continuity of solutions given by Moser [M], we can also obtain the Hölder continuity of solutions to L.

Theorem 7.8. (Hölder continuity of solutions) *Let u be a weak solution to L. Let $B = B(x_0, r)$, $r \leq r_0$ with $B \subset \Omega$. Then for $\tilde{u}$, and some $\alpha > 0$, and $x, y \in B(x_0, r/2)$, and any $p > 0$,*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c \left(\frac{\varrho(x,y)}{r}\right)^{\alpha} \left(\frac{1}{w(B(x_0,r))} \int_{B(x_0,r)} |\tilde{u}|^p w\right)^{1/p}$$

with $c = c(p)$ independent of u and B.

The Mean-value and Harnack inequality for the operator $\mathcal{L}$ will also be proved in this part. Since the statement is the same, we do not present here (see Section 10).

We conclude this section by proving the existence of a cut-off function relative to certain metric balls. Before we do so, we first review some basic results in [NSW].

Let $Y_1, \ldots, Y_q$ be an enumeration of the vector fields $X_1, \ldots, X_m$ and its commutators. Given any N-tuple $I = (i_1, \ldots, i_N)$, let $U_j = Y_{i_j}, 1 \le j \le N$ and $\{V_j\}_{j=1}^{q-N}$ be the remaining $q - N$ vector fields. Let $d(U_j), d(V_j)$ be the degrees of vector fields U_j and V_j respectively, *i.e.* the length of their commutators. For $u \in \mathbb{R}^N, v \in \mathbb{R}^{q-N}$, set

$$z = \exp(vV) = \exp\Big(\sum_{j=1}^{q-N} v_j V_j\Big)(x)$$

$$\Phi_v(u) = \exp(uU + vV)(x)$$

and define

$$B_I(x, z, \delta) = \{y \in \Omega : y = \exp(uU + vV)(x) \text{ with } |u_j| < \delta^{d(U_j)}\} .$$

In the sequel, we denote $d_j = d(U_j), 1 \le j \le N$. We also denote $B_I(x, z, \delta)$ by $B_I(x, \delta)$ when $v = 0 \in \mathbb{R}^{q-N}$. Then by [NSW], one has the following theorem.

Theorem 7.9. (Nagel-Stein-Wainger) *Let $E \subset\subset \Omega$ be compact. Then there exist $0 < \eta_2 < \eta_1 < 1$, $\delta_0 > 0$ such that if $x \in E$, then there exists an N-tuple $I = (i_1, \ldots, i_N)$ such that for any $\delta < \delta_0$, the following properties hold*

(i) $B(x, \eta_2\delta) \subset B_I(x, z, \eta_1\delta) \subset B(x, \delta)$.

(ii) *There is an open neighbourhood of $y = \Phi_v(u)$ on which Φ_v has an inverse map $\Psi = (\Psi_1, \ldots, \Psi_N)$ so that locally $\Psi_j(\Phi_v(u)) = u_j$. Then we can regard $\Psi_1, \ldots, \Psi_N$ as coordinate functions near y. Moreover,*

$$|U_l(\Psi_k)| \le c\,\delta^{d(U_k)-d(U_l)} = c\,\delta^{d_k - d_l} \tag{7.10}$$

for $y \in B_I(x, z, \delta)$, where $U_j = Y_{i_j}$.

(iii) *Let $X_i = \sum_{l=1}^N a_i^l U_l$, then*

$$|a_i^l| \le c\,\delta^{d_l - 1} \tag{7.11}$$

for $y \in B(x, \eta_2\delta)$ and $0 < \delta < \delta_0$.

The inclusions in (i) were proved in Theorem 7 in [NSW], while (ii) and (iii) were proved in Theorem 6 in [NSW].

We now prove the following lemma concerning the existence of cut-off functions.

Lemma 7.12. *For any $x \in E \subset\subset \Omega$, $0 < r_1 < r_2 < \delta_0$, and I as chosen in the above theorem, then there is a $\phi \in C_0^\infty(B_I(x, r_2))$ such that*

$$\phi \equiv 1 \quad \text{on } B_I(x, r_1),$$

and

$$|X\phi| = \sum_{i=1}^{N} |X_i \phi| \leq \frac{c}{r_2 - r_1}$$

where c is independent of the balls B but only depends on $E \subset\subset \Omega$.

PROOF. Let $\phi_j(t)$ be 1 for $|t| < (r_1/r_2)^{d_j}$, and 0 for $|t| > 1$, $\phi_j \in C_0^\infty$ and $|\phi_j'| \leq c/(1 - (r_1/r_2)^{d_j})$, $1 \leq j \leq N$, where $d_j = d(U_j), U_j = Y_{i_j}, I = (i_1, \ldots, i_N)$ as before. Now let

$$\phi(y) = \prod_{j=1}^{N} \phi_j\Big(\frac{\Psi_j(y)}{r_2^{d_j}}\Big)$$

where $y = \exp\big(\sum_{j=1}^{N} u_j U_j\big)(x)$, $u_j = \Psi_j(y)$. Then by (7.10), $|U_l(\Psi_j)| \leq C\, r_2^{d_j - d_l}$ on $B_I(x, r_2)$. Let $X_i = \sum_{l=1}^{N} a_i^l U_l$, then by (7.11),

$$|a_i^l| \leq C\, r_2^{d_l - 1} \qquad \text{on } B_I(x, r_2)$$

Now it is easy to verify $\phi(y)$ satisfying the following

(i) $\phi(y) \equiv 1$ on $B_I(x, r_1) = \{y \in \Omega : y = \exp(\sum_{j=1}^{N} u_j U_j)(x), |u_j| < r_1^{d(U_j)}\}$.

(ii) $\phi \equiv 0$ outside $B_I(x, r_2) = \{y \in \Omega : y = \exp(\sum_{j=1}^{N} u_j U_j)(x), |u_j| < r_2^{d(U_j)}\}$.

(iii) $|X_i \phi(y)| \leq \dfrac{C}{r_2 - r_1}$.

The assertions (i) and (ii) are trivial. To prove (iii), note

$$U_l \phi(y) = \sum_{k=1}^{N} \Big(\prod_{j \neq k} \phi_j\Big)(U_l \phi_k) = \sum_{k=1}^{N} \Big(\prod_{j \neq k} \phi_j\Big)\, \phi_k'\Big(\frac{\Psi_k}{r_2^{d_k}}\Big) \frac{1}{r_2^{d_k}}\, U_l(\Psi_k)\,.$$

Thus,

$$|U_l\phi(y)| \le \sum_{k=1}^{N} \frac{1}{1-(r_1/r_2)^{d_k}} \frac{1}{r_2^{d_k}} r_2^{d_k-d_l} = C\sum_{k=1}^{N} \frac{r_2^{d_k}}{r_2^{d_k}-r_1^{d_k}} r_2^{-d_l}.$$

Hence

$$\begin{aligned} |X_i\phi| &\le \sum_{l=1}^{N} |a_i^l||U_l\phi| \\ &\le C\sum_{l=1}^{N} r_2^{d_l-1} \sum_{k=1}^{N} \frac{r_2^{d_k}}{r_2^{d_k}-r_1^{d_k}} r_2^{-d_l} \qquad (7.13) \\ &= CN\sum_{k=1}^{N} \frac{r_2^{d_k-1}}{r_2^{d_k}-r_1^{d_k}}. \end{aligned}$$

We note,

$$r_2^{d_k} - r_1^{d_k} = (r_2-r_1)\sum_{j=1}^{d_k-1} r_2^j\, r_1^{d_k-j-1} \ge (r_2-r_1)\, r_2^{d_k-1}.$$

Substituting this inequality into the expression (7.13), we see that (7.13) is bounded by $C(r_2-r_1)^{-1}$. This proves the lemma.

The rest of this part is organized as follows: In Section 8, we first define the Hilbert spaces with respect to the vector fields and also define the weak solutions to $Lu = 0$. Harnack's inequality is proved for nonnegative solutions for the operator L. In Section 9, Mean-value inequalities will be stated for general solutions and subsolutions to $Lu = 0$ and Hölder's continuity will be derived. In Section 10, the Harnack inequality for the operator $\mathcal{L}$ is proved.

8. Harnack's inequality for solutions to $L = \sum_{i,j=1}^{m} X_i^*(a_{ij}X_j)$.

We begin this section by introducing the space $H(\Omega)$. For $u \in \mathrm{Lip}_1(\overline{\Omega})$, define

$$\|u\|^2 = \int_\Omega \langle AXu, Xu\rangle + \int_\Omega u^2 w \qquad (8.1)$$

where $Xu = (X_1u, \dots, X_mu)$, $w \in A_2(\Omega, \varrho)$. Note

$$\|u\|^2 \sim \int_\Omega |Xu|^2 w + \int_\Omega u^2 w \tag{8.2}$$

where $|Xu| = (\sum_{i=1}^m |X_iu|)^{1/2}$. We also define

$$a(u, \phi) = \int_\Omega \langle AXu, X\phi \rangle + \int_\Omega u\,\phi\,w\,. \tag{8.3}$$

for $u, \phi \in \mathrm{Lip}_1(\overline{\Omega})$.

Using the fact that

$$|\langle Ax, y\rangle| \le \langle Ax, x\rangle^{1/2} \langle Ay, y\rangle^{1/2}$$

we can show that $a(u, \phi)$ is an inner product on $\mathrm{Lip}_1(\overline{\Omega})$, so that $\|\cdot\|$ is a norm on $\mathrm{Lip}_1(\overline{\Omega})$. We also note for any $\varepsilon > 0$,

$$|\langle Ax, y\rangle| \le \frac{\varepsilon}{2}\langle Ax, x\rangle + \frac{1}{2\varepsilon}\langle Ay, y\rangle \tag{8.4}$$

Let $H = H(\Omega)$ be the completion of $\mathrm{Lip}_1(\overline{\Omega})$ under $\|\cdot\|$, *i.e.* H is formed by adjoining to $\mathrm{Lip}_1(\overline{\Omega})$ elements $u = \{u_k\}, u_k \in \mathrm{Lip}_1(\overline{\Omega})$, which are Cauchy sequences with respect to $\|\cdot\|$. $\mathrm{Lip}_1(\overline{\Omega})$ is contained in H by considering $\{u_k\}$ with all $u_k = u \in \mathrm{Lip}_1(\overline{\Omega})$. If given $u = \{u_k\}$ and $\phi = \{\phi_k\}$ in H, it is easy to see that $a(u_k, \phi_k)$ is convergent, and we can define

$$a(u, \phi) = \lim_{k\to\infty} a(u_k, \phi_k), \quad \|u\| = \lim_{k\to\infty} \|u_k\|\,.$$

Thus, H becomes a Hilbert space with inner product $a(u, \phi)$ and norm $\|u\| = a(u, u)^{1/2}$.

We note from (8.2) that if $u \in H, u = \{u_k\}$, then both $\{Xu_k\}$ and $\{u_k\}$ are Cauchy sequences in $L^2_w(\Omega)$. Consequently, there exist $\tilde{u} \in L^2_w$ and a vector $\overrightarrow{\alpha} \in L^2_w$ such that $u_k \to \tilde{u}$ in L^2_w and $Xu_k \to \overrightarrow{\alpha}$ in L^2_w. We also note that any equivalent sequence gives rise to the same $\tilde{u}$ and $\overrightarrow{\alpha}$ by (8.2). We shall refer to $\tilde{u}$ as the function in L^2_w associated with u.

Now define

$$a_0(u, \phi) = \int_\Omega \langle AXu, X\phi\rangle \qquad \text{for } u, \phi \in \mathrm{Lip}_1(\overline{\Omega})\,,$$

$$\|u\|_0 = \int_\Omega \langle AXu, Xu\rangle \qquad \text{on } \mathrm{Lip}_1(\overline{\Omega})\,.$$

Then

$$\|u\|_0 \sim \int_\Omega |Xu|^2 w\,.$$

If $u = \{u_k\}$ and $\phi = \{\phi_k\}, u, \phi \in H$, let $a_0(u,\phi) = \lim_{k\to\infty} a_0(u_k, \phi_k)$. Then

$$|a_0(u,\phi)| \le a_0(u,u)^{1/2}\ a_0(\phi,\phi)^{1/2}$$

and

$$a(u,\phi) = a_0(u,\phi) + \int_\Omega \tilde{u}\,\tilde{\phi}\,w\,.$$

Although $\|\cdot\|_0$ is not a norm on $\mathrm{Lip}_1(\overline{\Omega})$, it is still easy to check $\|\cdot\|_0$ is a norm on $\mathrm{Lip}_0(\Omega)$ since $X_1, \ldots, X_m$ satisfy Hörmander condition. ($X_i u = 0$ implies $X_\alpha u \equiv 0$ for any commutator X_α and thus $\partial u/\partial x_i = 0$ for each i, thus u is constant and thus $u \equiv 0$.)

Let $H_0 = H_0(\Omega)$ be the completion of $\mathrm{Lip}_0(\Omega)$ under $\|\cdot\|_0$. Thus $a_0(u,u)^{1/2}$ is a norm on H_0, and we still denote $\|u\|_0 = a_0(u,u)^{1/2}$ for $u \in H_0$.

Let $u \in H$, u is said to be $u \ge 0$, if $u_k \ge 0$ for all k for some $\{u_k\}$ representing u. Note that if $\tilde{u}$ is the function in L^2_w asssociated with u, then $\tilde{u} \ge 0$ a.e. if $u \ge 0$. Now we can give the definition of solution to L.

Definition 8.5. *$u \in H(\Omega)$ is said to be a solution of $Lu = \sum_{i=1}^m X_i^*(a_{ij}X_j) = 0$ if for any $\phi \in H_0(\Omega)$, we have $a_0(u,\phi) = 0$. $u \in H(\Omega)$ is said to be a subsolution if*

$$a_0(u,\phi) \le 0, \quad \textit{for all } \phi \ge 0, \quad \phi \in H_0(\Omega)\,.$$

u is said to be a supersolution if

$$a_0(u,\phi) \ge 0, \quad \textit{for all } \phi \ge 0, \quad \phi \in H_0(\Omega)\,.$$

For fixed $\Psi \in H$, $-a_0(\Psi, \cdot)$ is a continuous linear functional on H_0. Thus by Riesz representation theorem, there is a unique $F \in H_0$ such that $a_0(F,\phi) = -a_0(\Psi,\phi)$ for all $\phi \in H_0$. Thus, $u = F + \Psi$ will solve uniquely the Dirichlet problem $Lu = 0$ with $u = \Psi$ on $\partial\Omega$ in the sense that $u - \Psi \in H_0$ and $a_0(u,\phi) = 0$ for all $\phi \in H_0$.

Lemma 8.6. *Given $B = B_I(x_0, h) \subset \Omega$. Let $u \in H(B)$ be a non-negative subsolution of $Lu = 0$, and $\tilde{u}$ be the function in L^2_w associated to u. Then there exist positive constants c and d depending only the parameters in* (7.1) *and* (7.3) *such that for $1/2 \le \alpha < 1$ and $p \ge 2$,*

$$(\operatorname*{ess\,sup}_{\alpha B} \tilde{u})^p \le \frac{c}{(1-\alpha)^d} \frac{1}{w(B)} \int_B \tilde{u}^p w \, .$$

PROOF. For $\beta \ge 1$ and $0 < M < \infty$, define $H_M(t) = t^\beta$ for $t \in [0, M]$ and $H_M(t) = M^\beta + \beta M^{\beta-1}(t - M)$ for $t > M$. Note $H'_M(t)$ exists for all t and bounded for each fixed M since $H'_M(t) = \beta t^{\beta-1}$ for $t \le M$ and equals to $\beta M^{\beta-1}$ for $t \ge M$. Let $u = \{u_k\}, u_k \in \mathrm{Lip}_1(\overline{B}), u_k \ge 0$, $\|u_k - u_j\| \to 0$, and for each fixed M, define

$$\phi_k(x) = \eta^2(x) \int_0^{u_k} H'_M(t)^2 dt \, ,$$

for $\eta \in C_0^\infty(B)$ to be chosen later.

Now $\phi_k \in \mathrm{Lip}_0(B)$ and $\phi_k \ge 0$. Further,

$$X\phi_k = \eta^2 H'_M(u_k)^2 X u_k + 2\, \eta X \eta \int_0^{u_k} H'_M(t)^2 dt \, . \tag{8.7}$$

Computation now shows that,

$$\begin{aligned} \|\phi_k\|_0^2 = & \int \langle AX\phi_k, X\phi_k \rangle \\ = & \int (\eta^2 H'_M(u_k))^2 \langle AXu_k, Xu_k \rangle \\ & + 4 \int \eta^3 H'_M(u_k)^2 \left(\int_0^{u_k} H'_M(t)^2 dt \right) \langle AXu_k, X\eta \rangle \\ & + 4 \int \eta^2 \left(\int_0^{u_k} H'_M(t)^2 dt \right)^2 \langle AX\eta, X\eta \rangle \\ & = I + II + III \, . \end{aligned}$$

Then we can show as in [CW2] that each of I, II and III is bounded and thus $\|\phi_k\|_0$ is bounded. Then there exists a weakly convergent sequence, $\phi_{k_j} \to \phi \in H$. Since $u \in H$, and $|a_0(u, \phi)| \le a_0(u, u)^{1/2} a_0(\phi, \phi)^{1/2} \le \|u\| \, \|\phi\|_0$, it follows that $a_0(u, \cdot)$ is a continuous linear functional on H_0. Thus

$$a_0(u, \phi) = \lim a_0(u, \phi_{k_j}) = \lim a_0(u_{k_j}, \phi_{k_j}) \, .$$

Since u is a subsolution, $a_0(u,\phi) \le 0$, and to continue further, we distinguish the following two cases $\lim a_0(u_{k_j},\phi_{k_j}) = 0$ and $\lim a_0(u_{k_j},\phi_{k_j}) < 0$. We will set $a_0(u_{k_j},\phi_{k_j}) = \delta_{k_j}$. For simplicity, we drop the subscripts for u_{k_j}, ϕ_{k_j} and δ_{k_j}. From (8.7) and noticing that $H'_M(u)Xu = XH_M(u)$, we obtain

$$
\begin{aligned}
\delta = a(u,\phi) = & \int \langle AXu, Xu\rangle \eta^2 H'_M(u)^2 \\
& + \int \langle AXu, X\eta\rangle 2\eta \int_0^u H'_M(t)^2 dt \\
= & \int \langle AXH_M(u), XH_M(u)\rangle \eta^2 \\
& + 2\int \langle AXu, X\eta\rangle \eta \int_0^u H'_M(t)^2 dt .
\end{aligned} \tag{8.8}
$$

Since $\int_0^u H'_M(t)^2 dt \le u\,(H'_M(u))^2$, thus

$$
\begin{aligned}
& \int \langle AXH_M(u), XH_M(u)\rangle \eta^2 \\
& \quad = -2\int \langle AXu, X\eta\rangle \eta \int_0^u H'_M(t)^2 dt + \delta \\
& \quad \le 2\int |\langle AXH_M(u), u\,(H'_M(u))^2 X\eta\rangle| + \delta .
\end{aligned}
$$

By (8.4), the expression above is

$$
\le \varepsilon \int \langle AXH_M(u), XH_M(u)\rangle \eta^2 + \frac{1}{\varepsilon}\int \langle AX\eta, X\eta\rangle u^2 H'_M(u)^2 + \delta .
$$

Taking $\varepsilon = 1/2$, we have

$$
\int \langle AXH_M(u), XH_M(u)\rangle \eta^2 \le 4\int \langle AX\eta, X\eta\rangle u^2 H'_M(u)^2 + 2\delta .
$$

Thus

$$
\int |XH_M(u)|^2 \eta^2 w \le c\int |X\eta|^2 u^2 H'_M(u)^2 w + c\,\delta . \tag{8.9}
$$

In the case $\lim \delta_{k_j} < 0$, we simply drop the term δ above. Proceeding further note,

$$
\begin{aligned}
|X(\eta H_M(u))|^2 w &= |(X\eta)H_M(u) + \eta XH_M(u)|^2 w \\
&\le 2\,(|X\eta|^2 H_M(u)^2 w + \eta^2 |XH_M(u)|^2 w) .
\end{aligned} \tag{8.10}
$$

Hence by (8.9) and (8.10) and using the estimate $H_M(u) \le uH'_M(u)$,we have

$$\begin{aligned}\int |X(\eta H_M(u))|^2 w &\le c\int |X\eta|^2 u^2 H'_M(u)^2 w \\ &\quad + c\int |X\eta|^2 H_M(u)^2 w + c\,|\delta| \\ &\le c\int |X\eta|^2 u^2 H'_M(u)^2 w + c\,|\delta|\,.\end{aligned} \tag{8.11}$$

Now recall that h is the radius of the ball B, given $1/2 \le s < t < 1$, taking $0 \le \eta \le 1, \eta = 1$ on sB, $\eta = 0$ outside tB, and $|X\eta| \le c/(t-s)h$. Thus by the Sobolev inequality (7.3) and together with (8.11),

$$\begin{aligned}&\left(\frac{1}{w(tB)}\int_{tB} |\eta H_M(u)|^q w\right)^{1/q} \\ &\le c\,t\,h\left(\frac{1}{w(tB)}\int_{tB} |X(\eta H_M(u)|^2 w\right)^{1/2} \\ &\le c\,t\,h\frac{1}{w(tB)^{1/2}}\left(\int_{tB} |X\eta|^2 u^2 H'_M(u)^2 w + |\delta|\right)^{1/2} \\ &\le c\,t\,h\frac{1}{w(tB)^{1/2}}\left(\int_{tB} |X\eta|^2 u^2 H'_M(u)^2 w\right)^{1/2} \\ &\quad + c\,t\,h\frac{1}{w(tB)^{1/2}}|\delta|^{1/2} \\ &\le \frac{c\,t\,h}{w(tB)^{1/2}}\frac{1}{(t-s)h}\left(\int_{tB} u^2 H'_M(u)^2 w\right)^{1/2} \\ &\quad + \frac{c\,t\,h}{w(tB)^{1/2}}|\delta|^{1/2}\,.\end{aligned} \tag{8.12}$$

So,

$$\begin{aligned}&\left(\frac{1}{w(sB)}\int_{sB} |H_M(u)|^q w\right)^{1/q} \\ &\le \frac{c\,t}{t-s}\left(\frac{1}{w(tB)}\int_{tB} u^2 H'_M(u)^2 w\right)^{1/2} + \frac{c\,t\,h}{w(tB)^{1/2}}|\delta|^{1/2}\,.\end{aligned} \tag{8.13}$$

Recalling that $u = u_{k_j} \to \tilde{u} \in L^2_w$ and $\delta = \delta_{k_j} \to 0$, and using a further subsequence if necessary, we may assume that $u_{k_j} \to \tilde{u}$ a.e. Now

$$|uH'_M(u) - \tilde{u}H'_M(\tilde{u})|^2 \le 2\left(|u-\tilde{u}|^2|H'_M(u)|^2 + |\tilde{u}|^2|H'_M(u) - H'_M(\tilde{u})|^2\right).$$

Thus $u_{k_j} H'_M u_{k_j} \to \tilde{u} H'_M(\tilde{u})$ in L^2_w. Consequently, as $j \to \infty$ the right side of (8.13) converges to

$$\frac{ct}{t-s}\left(\frac{1}{w(tB)}\int_{tB}\tilde{u}^2 H'_M(\tilde{u})^2 w\right)^{1/2}$$

Since $H_M(u_{k_j}) \to H_M(\tilde{u})$ a.e., by applying Fatou's lemma on the left side of (8.13), we obtain

$$\left(\frac{1}{w(sB)}\int_{sB}|H_M(\tilde{u})|^q w\right)^{1/q} \leq \frac{ct}{t-s}\left(\frac{1}{w(tB)}\int_{tB}\tilde{u}^2 H'_M(\tilde{u})^2 w\right)^{1/2}. \tag{8.14}$$

Now letting $M \to \infty$, using $\tilde{u}^\beta \chi_{\{\tilde{u}\leq M\}} \leq H_M(\tilde{u})$ for the left hand of (8.14), and $\tilde{u} H'_M(\tilde{u}) \leq \tilde{u}\beta\tilde{u}^{\beta-1} = \beta\tilde{u}^\beta$ for the right hand of (8.14), we obtain

$$\left(\frac{1}{w(sB)}\int_{sB}\tilde{u}^{q\beta} w\right)^{1/q} \leq \frac{ct\beta}{t-s}\left(\frac{1}{w(tB)}\int_{tB}(\tilde{u}^{2\beta})w\right)^{1/2}.$$

Applying Moser's iteration argument to the inequality above, we get

$$\operatorname*{ess\,sup}_{\alpha B}\tilde{u} \leq \frac{c}{(1-\alpha)^d}\left(\frac{1}{w(B)}\int_B \tilde{u}w\right)^{1/p}$$

for $p \geq 2$ (see [CW2]).

We now show the following mean-value inequality for nonnegative solutions when $-\infty < p < \infty$, *i.e.* Theorem 7.4 stated in the introduction.

Theorem 8.15. *With the same notation and hypothesis as in Lemma* 8.6, *except that now* u *is a nonnegative solution and* $-\infty < p < \infty$, *we have*

$$\operatorname*{ess\,sup}_{\alpha B}\tilde{u}^p \leq \frac{c}{(1-\alpha)^d}(1+|p|)^{2\sigma/(\sigma-1)}\frac{1}{w(B)}\int_B \tilde{u}^p w.$$

PROOF. The case $p \geq 2$ follows from Lemma 8.6. So it is enough to consider $-\infty < p < 2$. Let $u = \{u_k\}, u_k \in \mathrm{Lip}_1(\overline{B}), u_k \geq 0, \|u_k - u_j\| \to 0$. By considering $u_k + \varepsilon_0, \varepsilon_0 > 0$, and letting $\varepsilon_0 \to 0$ at the end, we

may assume that $u_k \geq \varepsilon_0 > 0$. Let η be as in Lemma 8.6, $\eta \geq 0$, $\operatorname{supp}\eta \subset B, -\infty < \beta \leq 1$, and define $\phi_k = \eta^2 u_k^\beta$, then as checked in [CW2], we can show that $\|\phi_k\|_0$ is bounded for all k. Now pick $\phi_{k_j} \to \phi \in H_0$, then $a_0(u_{k_j}, \phi_{k_j}) \to a_0(u, \phi) = 0$, that is,

$$\int \langle AXu_{k_j}, X\phi_{k_j}\rangle = \delta_{k_j} \to 0 .$$

Note that

$$X\phi_{k_j} = \eta^2 \beta u^{\beta-1} Xu_{k_j} + 2\,\eta(X\eta)u_{k_j}^\beta$$

and for $\beta \neq -1$,

$$X(u_{k_j}^{(\beta+1)/2}) = \frac{\beta+1}{2} u_{k_j}^{(\beta-1)/2} Xu_{k_j}$$

If we drop the subscripts, we have

$$\begin{aligned}
\delta = & \int \langle AXu, X\phi\rangle \\
= & \int \langle AXu, \eta^2 \beta u^{\beta-1} Xu\rangle + \int \langle AXu, 2\eta(X\eta)u^\beta\rangle \\
= & \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle \frac{2\beta}{\beta+1}\eta^2 \\
& + \int \langle AX(u^{(\beta+1)/2}), X\eta\rangle \Big(\frac{2}{\beta+1}\Big) 2\,\eta u^{(\beta+1)/2} .
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{\beta}{\beta+1} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle \eta^2 \\
& \qquad = \frac{\beta+1}{4}\delta - \int \langle AX(u^{(\beta+1)/2}), X\eta\rangle u^{(\beta+1)/2}\eta .
\end{aligned}$$

Applying (8.4) with $\varepsilon = |\beta/(\beta+1)|$, $-\infty < \beta \leq 1$, $\beta \neq 0, -1$, we have

$$\begin{aligned}
& \frac{|\beta|}{|\beta+1|} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle \eta^2 \\
& \qquad \leq \frac{|\beta+1|}{4}|\delta| + \frac{\varepsilon}{2} \int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle \eta^2 \\
& \qquad\quad + \frac{1}{2\varepsilon} \int \langle AX\eta, X\eta\rangle u^{\beta+1}
\end{aligned}$$

$$= \frac{|\beta+1|}{4}|\delta| + \frac{|\beta|}{2|\beta+1|}\int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle \eta^2 + \frac{|\beta+1|}{2|\beta|}\int \langle AX\eta, X\eta\rangle u^{\beta+1}.$$

Thus,

$$\frac{|\beta|}{|\beta+1|}\int \langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle \eta^2 \leq c\,\frac{|\beta+1|}{|\beta|}\int |X\eta|^2 u^{\beta+1} w + \frac{|\beta+1|}{4}|\delta|\,. \tag{8.16}$$

Now note

$$|X(\eta u^{(\beta+1)/2})|^2 w \leq 2\,(|X\eta|^2 u^{\beta+1} + \eta^2 |X u^{(\beta+1)/2})|^2\, w\,. \tag{8.17}$$

Thus from (8.16) and (8.17) we get

$$\int |X(\eta u^{(\beta+1)/2})|^2 w \leq c\left[\left(\frac{|\beta+1|}{|\beta|}\right)^2 + 1\right]\int |X\eta|^2 u^{\beta+1} w + \frac{|\beta+1|}{4}|\delta|\,.$$

Taking $\eta = 1$ on sB, $0 \leq \eta \leq 1$, supp $\eta \subset tB, |X\eta| \leq c/((t-s)h), 1/2 \leq s < t < 1$, we have

$$\int_{tB} |X(\eta u^{(\beta+1)/2})|^2 w \leq c\left(\frac{|\beta+1|^2}{|\beta|} + 1\right)\int_{tB} |X\eta|^2 u^{\beta+1} w + \frac{|\beta+1|}{4}|\delta|\,.$$

By Sobolev's inequality (7.3),

$$\begin{aligned}\left(\frac{1}{w(tB)}\int_{tB} |\eta u^{(\beta+1)/2}|^q w\right)^{1/q} &\leq c\,t\,h\left(\frac{1}{w(tB)}\int_{tB} |X(\eta u^{(\beta+1)/2})|^2 w\right)^{1/2} \\ &\leq \frac{c\,t\,h}{w(tB)^{1/2}}\left\{\left[\left(\frac{|\beta+1|}{|\beta|}\right)^2 + 1\right]\right. \\ &\qquad \left.\cdot\,\frac{1}{(t-s)^2 h^2}\int_{tB} u^{\beta+1} w + \frac{|\beta+1|}{4}\delta\right\}^{1/2}\end{aligned}$$

Letting $k_j \to \infty$,

$$\left(\frac{1}{w(sB)}\int_{sB}|\tilde{u}|^{(\beta+1)/2q}\, w\right)^{1/q}$$
$$\leq \frac{ct}{t-s}\left(\frac{|\beta+1|}{|\beta|}+1\right)\left(\frac{1}{w(tB)}\int_{tB}|\tilde{u}|^{\beta+1}w\right)^{1/2}.$$

Using Moser's iteration argument (see [GT] or [CW2]), the lemma follows.

Lemma 8.18. *Let u be a nonnegative solution such that $u \geq \varepsilon$ for some $\varepsilon > 0$. For $1/2 \leq \alpha < 1$, define $k = k(\alpha, \tilde{u})$ by*

$$\log k = \frac{1}{w(\alpha B)}\int_{\alpha B}(\log \tilde{u})w$$

then for $\lambda > 0$,

$$w(\{x \in \alpha B : |\log \frac{\tilde{u}}{k}| > \lambda\}) \leq \frac{c}{(1-\alpha)\lambda}w(\alpha B)\,.$$

PROOF. Let $\eta \equiv 1$ on αB_a and supp $\eta \subset B_a, |X\eta| \leq c/a(1-\alpha)h$, h is the radius of B, $B_a = aB$, and a is the constant given in (7.2). Let $u = \{u_k\}, u_k \geq \varepsilon > 0$, and let $\phi_k = \eta^2/u_k$, then

$$X\phi_k = -\eta^2 u_k^{-2} X u_k + 2\eta u_k^{-1} X\eta\,.$$

It is easy to check that $\|\phi\|_0 = \int\langle AX\phi_k, X\phi_k\rangle$ is bounded for all k. Thus there is a subsequence $\phi_{k_j} \to \phi$ weakly in H_0, *i.e.*

$$\delta_{k_j} = \int\langle AXu_{k_j}, X\phi_{k_j}\rangle \to 0\,.$$

If we drop the subscripts,

$$\delta = \int\langle AXu, X\phi\rangle = -\int\langle AXu, Xu\rangle u^{-2}\eta^2$$
$$+ 2\int\langle AXu, X\eta\rangle u^{-1}\eta\,.$$

Thus,

$$\int\langle AXu, Xu\rangle u^{-2}\eta^2 = 2\int\langle AXu, X\eta\rangle u^{-1}\eta - \delta\,.$$

From this we get

$$\begin{aligned}\int &\langle AX(\log u), X(\log u)\rangle \eta^2 \\ &\leq 2\int \langle AX(\log u), X(\log u)\rangle^{1/2} \langle AX\eta, X\eta\rangle^{1/2}\eta + |\delta| \\ &\leq \varepsilon \int \langle AX(\log u), X(\log u)\rangle \eta^2 + \frac{1}{\varepsilon}\int \langle AX\eta, X\eta\rangle + |\delta| \,.\end{aligned}$$

Taking $\varepsilon = 1/2$,

$$\int \langle AX(\log u), X(\log u)\rangle \eta^2 \leq 4\int \langle AX\eta, X\eta\rangle + 2\,|\delta| \,.$$

Thus,

$$\int |X(\log u)|^2 w\eta^2 \leq c\int |X\eta|^2 w + c\,|\delta| \,.$$

Then,

$$\int_{\alpha a B} |X(\log u)|^2 w \leq \frac{c}{a^2(1-\alpha)^2 h^2}\int_{aB} w + c\,|\delta| \,. \tag{8.19}$$

By Poincaré's inequality (7.2), for q replaced by 2, and together with (8.19),

$$\begin{aligned}&\left(\frac{1}{w(\alpha B)}\int_{\alpha B} |\log u - a v_{\alpha B}\log u|^2 w\right)^{1/2} \\ &\qquad \leq \frac{c\,\alpha\, h}{w(a\alpha B)^{1/2}}\left[\frac{1}{(1-\alpha)ah}\left(\int_{a\alpha B} w\right)^{1/2} + |\delta|^{1/2}\right] \,.\end{aligned}$$

Note $a > 1$ is a constant, $1 > \alpha > 1/2$, by the doubling property of w,

$$\int_{\alpha B} |\log u - a v_{\alpha B}\log u|^2 w \leq \left(\frac{c}{1-\alpha} + c\,|\delta|\right) w(a\alpha B)^{1/2} \,.$$

Recalling $u = u_{k_j}, \delta = \delta_{k_j}$, taking limits and noticing that $u_k \geq \varepsilon > 0$ implies $\tilde{u} \geq \varepsilon > 0$.

Moreover, $|\log u_{k_j} - \log \tilde{u}| \leq (|u_{k_j} - \tilde{u}|)/\varepsilon$. Thus $\log u_{k_j} \to \log \tilde{u}$ in L^2_w since $u_{k_j} \to \tilde{u}$ in L^2_w. Therefore,

$$a\, v_{\alpha B}(\log u_{k_j}) = \frac{1}{w(\alpha B)}\int_{\alpha B} (\log u_{k_j}) w \to \log k$$

and

$$\frac{1}{w(\alpha B)}\int_{\alpha B}|\log\tilde{u}-\log k|^2 w \le \frac{c}{(1-\alpha)^2}\frac{1}{w(B)^{1/2}}.$$

Thus by Chebishev's inequality, for $\lambda > 0$,

$$w\left(\{x\in\alpha B : |\log\frac{\tilde{u}}{k}| > \lambda\}\right) \le \frac{1}{\lambda}\int_{\alpha B}|\log\frac{\tilde{u}}{k}|w$$
$$\le \frac{1}{\lambda}\left(\int_{\alpha B}|\log\frac{\tilde{u}}{k}|^2 w\right)^{1/2} w(\alpha B)^{1/2} \le \frac{c}{\lambda(1-\alpha)}w(\alpha B).$$

The lemma that follows is a variant of Bombieri's lemma for homogeneous spaces.

Lemma 8.20. *If $f \ge 0$, w a doubling measure, $B = B_I(x,r)$ and there are constants c, d such that*

$$\text{(i)} \qquad \operatorname*{ess\,sup}_{sB} f^p \le \frac{c}{(t-s)^d}\frac{1}{w(tB)}\int_{tB} f^p w \quad \textit{for } 0 < p < \frac{1}{\mu},$$

$$\text{(ii)} \qquad w(\{\alpha B : \log f > \lambda\}) \le \frac{c\,\mu}{\lambda}w(B).$$

Then there exist constants C, D such that

$$\operatorname*{ess\,sup}_{B} f \le \exp\left(\frac{C}{(1-\alpha)^d}\right).$$

The proof of the above in the ball $B = B_I$ is exactly as the original Bombieri's lemma (see [Bom]).

Finally we can derive the following Harnack's inequality, *i.e.* Theorem 7.7, by applying Theorem 8.15, Lemma 8.18 and Lemma 8.20. Since the proof is easy, we omit it here.

Theorem 8.21. (Harnack inequality) *Let $w \in A_2$, and let u be non-negative solution of $Lu = 0$ belonging to $H(2B)$ for a ball $B = B(x,r)$ with $2B \subset \Omega$, $r \le r_0$ for some $r_0 > 0$. Let $\tilde{u}$ be the function in L^2_w associated with u, then*

$$\operatorname*{ess\,sup}_{B} \tilde{u} \le c \operatorname*{ess\,inf}_{B} \tilde{u},$$

with c independent of u and B.

9. Mean value inequalities for general solutions and subsolutions, and Hölder's continuity.

In the last section we proved the mean value inequalities for non-negative subsolutions to L. The mean value inequalities for general solutions and subsolutions can also be obtained by the similar method. Since the proof is arguing as in [CW2] for the case $X_i = \partial/\partial x_i$ and $m = N$, we do not present the proof. Thus in this section, we state the following results first.

Theorem 9.1. (Mean-value inequality) *Let $w \in A_2$ and u be a subsolution of $Lu = 0$ belonging to $H(B)$. Let $\tilde{u}$ be the function in L^2_w associated with u. Then there exist constants c and d depending only the parameters in* (7.1) *and* (7.3) *such that for $1/2 \le \alpha < 1$ and $0 < p < \infty$,*

$$(\operatorname*{ess\,sup}_{\alpha B} \tilde{u}^+)^p \le C \frac{1}{w(B)} \int_B (\tilde{u}^+)^p w \ ,$$

where $C \le c/(1-\alpha)^d$ if $p \ge 2$ and $\le c^{\log(3/p)}/(1-\alpha)^d$ if $0 < p < 2$.

Corollary 9.2. *If in Theorem* 9.1 *u is a solution, then the conclusion holds with $\tilde{u}^+$ replaced by $|\tilde{u}|$, i.e. for $0 < p < \infty$, $0 < \alpha < 1$,*

$$\operatorname*{ess\,sup}_{\alpha B} |\tilde{u}|^p \le C \frac{1}{w(B)} \int_B |\tilde{u}|^p w \ ,$$

where C as before.

Theorem 9.1, which is stated in Section 7 as Theorem 7.5, can be proved as in [CW2] or [GT].

Corollary 9.2, which is stated in Section 7 as Corollary 7.6, is just an easy consequence.

We now make the following additional assumption

(*) If u is an element of $H(B)$ whose associated function $\tilde{u}$ satisfies $\tilde{u} \ge 0$ a.e. in B, then $u \ge 0$, *i.e.* u is the limit in $H(B)$ of a sequence $\{u_k\}$ with $u_k \in \mathrm{Lip}(B)$ and $u_k \ge 0$ in B.

We define the oscillation over $B(x,h)$ of a bounded $\tilde{u}$ by

$$\operatorname{osc}(x,h) = \operatorname*{osc}_{\tilde{u}}(x,h) = \operatorname*{ess\,sup}_{B(x,h)} \tilde{u} - \operatorname*{ess\,inf}_{B(x,h)} \tilde{u} \ .$$

As a consequence of Corollary 9.2 we will obtain the Hölder continuity of solutions to L, *i.e.* Theorem 7.8. We will follow the method of Moser [M].

Theorem 9.3. (Hölder continuity of solutions) *Let u be a weak solution to L. Let $B = B(x_0, r)$, $r \leq r_0$ with $B \subset \Omega$. Then for $\tilde{u}$, and some $\alpha > 0$, we have for any $x, y \in B(x_0, r/2)$, and any $p > 0$,*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c \left(\frac{\varrho(x,y)}{r}\right)^{\alpha} \left(\frac{1}{w(B(x_0,r))} \int_{B(x_0,r)} |\tilde{u}|^p w\right)^{1/p}$$

with $c = c(p)$ independent of u and B.

PROOF. By Corollary 9.2, the function $\tilde{u}$ associated to u is essentially bounded on each ball B with $2B \subset \Omega$. Let $B_h = B(x_0, h)$ and let M and m be respectively the essential supremum and infimum of $\tilde{u}$ over $B_{h/2}$, and let M' and m' be these bounds over B_h. Since $M' - \tilde{u}$ and $\tilde{u} - m'$ are nonnegative a.e. in B_h, it follows from the assumption (*) that $M' - u \geq 0$ and $u - m' \geq 0$. Since $M' - u$ and $u - m'$ are solutions in $H(B)$, Harnack's inequality (8.12) implies that

$$M' - m \leq c\,(M' - M), \qquad M - m' \leq c\,(m - m')\,.$$

Adding the above two inequalities gives rise to

$$\operatorname{osc}(x_0, \frac{h}{2}) \leq \frac{c-1}{c+1} \operatorname{osc}(x, h)\,. \tag{9.4}$$

Since $c - 1 < c + 1$, by iterating (9), and applying (9.2) and in view of the metric ϱ, the Hölder inequality of $\tilde{u}$ will follow.

10. Harnack's inequality for solutions to $\mathcal{L} = -\sum_{i,j=1}^{m} X_i(a_{ij}X_j)$.

In this section, we will prove a Harnack inequality for the operator $\mathcal{L} = -\sum_{i,j=1}^{m} X_i(a_{i,j}X_j)$. The difference between $\mathcal{L}$ and L is that there are lower order terms in $\mathcal{L}$. So it is a little more complicated to treat the operator $\mathcal{L}$. However, the method used above does apply to $\mathcal{L}$. Hence we will be as brief as possible and just point out the essential difference. Throughout this section, we make the following hypothesis: Ω is small

enough so that the Poincaré and Sobolev inequalities hold on some ball B_0 containing Ω.

We note that any $u \in \mathrm{Lip}_0(\Omega)$ can be extended to a function in $\mathrm{Lip}_0(B_0)$ by setting $u = 0$ outside Ω. From the Sobolev inequality on the ball B_0, it follows that $H_0 \subset H$, and it is also possible to associate with each $u \in H_0$ a pair $(\tilde{u}, \vec{\alpha})$ such that if $u = \{u_k\}$, then $u_k \to \tilde{u}$ in L^2_w (even L^q_w) and $Xu_k \to \vec{\alpha}$ in L^2_w.

We now introduce several notations and definitions. Throughout this section, we will denote by X_i^* the adjoint of $-X_i$, $X^*u = (X_1^*u, \ldots, X_m^*u)$, and $X_i^* = X_i + b_i$, where

$$b_i = \sum_{j=1}^{n} \frac{\partial c_{ij}}{\partial x_j} \qquad \text{if } X_i = \sum_{j=1}^{n} c_{ij} \frac{\partial}{\partial x_j} .$$

We also let $\vec{b} = (b_1, \ldots, b_m)$. Then there is a constant $\Lambda > 0$ such that $\sum_{j=1}^{m} b_j^2 \le \Lambda$.

Definition. *$u \in H(\Omega)$ is said to be a solution to $\mathcal{L}$ if*

$$\int_\Omega \langle AXu, X^*\phi \rangle = 0, \qquad \textit{for every } \phi \in H_0(\Omega) ;$$

$u \in H(\Omega)$ is said to be a subsolution to $\mathcal{L}$ if

$$\int_\Omega \langle AXu, X^*\phi \rangle \le 0, \qquad \textit{for every } \phi \ge 0, \phi \in H_0(\Omega) ;$$

$u \in H(\Omega)$ is said to be a supersolution if $-u$ is a subsolution. We also define

$$a_0^*(u, \phi) = \int_\Omega \langle AXu, X^*\phi \rangle .$$

We note that

$$a_0^*(u, \phi) = a_0(u, \phi) + \int_\Omega \langle AXu, \vec{b}\phi \rangle .$$

Since $a(u, \phi) \le a(u, u)^{1/2}\ a(\phi, \phi)^{1/2}$,

$$\left| \int_\Omega \langle AXu, \vec{b}\phi \rangle \right| \le \int_\Omega \langle AXu, Xu \rangle^{1/2} \langle A\vec{b}\phi, \vec{b}\phi \rangle^{1/2}$$

$$\leq \|u\| C(\Lambda) \left(\int_\Omega \phi^2 w \right)^{1/2}$$
$$\leq C(\Lambda) \|u\|_0 \left(\int_\Omega |X\phi|^2 w \right)$$
$$\leq C(\Lambda) \|u\| \|\phi\|_0 .$$

Then we can see that $a_0^*(u, \cdot)$ is a continuous linear functional on H_0 for fixed $u \in H$.

Lemma 10.1. *Given $B = B_I(x_0, h) \subset \Omega$. Let $u \in H(B)$ be non-negative subsolution to $\mathcal{L}u=0$, and $\tilde{u}$ be a function in L^2_w associated to u. Then there exist positive constants c and d depending only on the parameters in (7.1) and (7.3) such that for $1/2 \leq \alpha < 1$ and $p \geq 2$,*

$$(\operatorname*{ess\,sup}_{\alpha B} \tilde{u})^p \leq \frac{c}{(1-\alpha)^d} \frac{1}{w(B)} \int_B \tilde{u}^p w .$$

PROOF. For $\beta \geq 1$ and $0 < M < \infty$, define $H_M(t)$ as in Section 8. Let $u = u_k$, $u_k \in \mathrm{Lip}_1(\overline{B})$, $u_k \geq 0, \|u_k - u_j\| \to 0$, define also

$$\phi_k = \eta^2 \int_0^{u_k} H'_M(t)^2 dt, \qquad \text{for } \eta \in C_0^\infty(B) \text{ to be chosen later,}$$

then $\phi_k \in \mathrm{Lip}_0(B)$ and $\phi_k \geq 0$. As in Section 8, $\|\phi_k\|_0$ is bounded for all k and fixed M. Then there is a subsequence ϕ_{j_k} weakly convergent to ϕ in H_0. Thus $a_0^*(u, \phi_{k_j}) \to a_0(u, \phi)$. Let $\delta_j = a_0^*(u_{k_j}, \phi_{k_j})$ and drop the subscripts, then

$$\begin{aligned} \delta = a_0^*(u, \phi) = & \int \langle AXu, Xu \rangle \eta^2 H'_M(u)^2 \\ & + 2 \int \langle AXu, X\eta \rangle \eta \Big(\int_0^u H'_M(t)^2 dt \Big) \\ & + \int \langle AXu, \vec{b} \rangle \eta^2 \Big(\int_0^u H'_M(t)^2 dt \Big) . \end{aligned}$$

So,

$$\begin{aligned} 0 &\leq \int \langle AXH_M(u), XH_M(u) \rangle \eta^2 \\ &= -2 \int \langle AXu, X\eta \rangle \eta \Big(\int_0^u H'_M(t)^2 dt \Big) \end{aligned}$$

$$
\begin{aligned}
&\quad - \int \langle AXu, \vec{b}\rangle \eta^2 \Big(\int_0^u H'_M(t)^2 dt \Big) + \delta \\
&\le 2 \int |\langle AXu, X\eta\rangle \eta u H'_M(u)^2 \\
&\quad + \int |\langle AXu, \vec{b}\rangle| \eta^2 u H'_M(u)^2 + |\delta| \\
&= 2 \int |\langle AXH_M(u), uH'_M(u) X\eta\rangle| \\
&\quad + \int |\langle AXH_M(u), uH'_M(u)\vec{b}\rangle| \eta^2 + |\delta| \\
&\le \varepsilon \int \langle AXH_M(u), XH_M(u)\rangle \eta^2 \\
&\quad + \frac{1}{\varepsilon} \int \langle AX\eta, X\eta\rangle u^2 H'_M(u)^2 \\
&\quad + \frac{\varepsilon}{2} \int \langle AXH_M(u), XH_M(u)\rangle \eta^2 \\
&\quad + \frac{1}{2\varepsilon} \int \langle A\vec{b}, \vec{b}\rangle u^2 H'_M(u)^2 \eta^2 + |\delta| \,.
\end{aligned}
$$

Taking $\varepsilon = 1/2$,

$$
\begin{aligned}
\int \langle AXH_M(u), XH_M(u)\rangle \eta^2 \le C \int |X\eta|^2 u^2 H'_M(u)^2 w \\
+ C \int u^2 H'_M(u)^2 \eta^2 w + |\delta| \,.
\end{aligned}
$$

Note that

$$
|X(\eta H_M(u))|^2 \le 2 \left(|X\eta|^2 H_M(u)^2 w + \eta^2 |XH_M(u)|^2 w \right) .
$$

Thus by noticing that $H_M(u) \le u\, H'_M(u)$, we have

$$
\int |X(\eta H_M(u)|^2 w \le C \int |X\eta|^2 u^2 H'_M(u)^2 w + C \int u^2 H'_M(u)^2 \eta^2 w + |\delta| \,.
$$

Taking $\eta \ge 0, \eta \equiv 1$ on sB, $|X\eta| \le c/(t-s)h$, supp $\eta \subset tB$, for $1/2 \le s < t < 1$, then by Sobolev inequality (7.3)

$$
\left(\frac{1}{w(tB)} \int_{tB} |\eta H_M(u)|^q w \right)^{1/q}
$$

$$\leq C\,t\,h\left(\frac{1}{w(tB)}\int_{tB}|X(\eta H_M(u))|^2 w\right)^{1/2}.$$

Therefore

$$\begin{aligned}
&\left(\frac{1}{w(sB)}\int_{sB}|H_M(u)|^q\right)^{1/q}\\
&\quad\leq \frac{C\,t\,h}{w(sB)^{1/2}}\left(\frac{1}{(t-s)^2h^2}\int_{tB}u^2H'_M(u)^2w\right.\\
&\qquad\left.+\int_{tB}u^2H'_M(u)^2w+C\,|\delta|\right)^{1/2}\\
&\quad\leq \frac{C}{t-s}\left(\frac{1}{w(sB)}\int_{tB}u^2H'_M(u)^2w\right)^{1/2}\\
&\qquad+\left(\frac{Ch}{w(tB)}\int_{tB}u^2H'_M(u)^2w\right)^{1/2}+\frac{Ch}{w(sB)^{1/2}}|\delta|^{1/2}.
\end{aligned}$$

Letting $k_j\to\infty$, and noticing $h\leq r_0$, $t-s\leq 1/2$, we obtain

$$\left(\frac{1}{w(sB)}\int_{sB}|H_M(\tilde{u})|^q\right)^{1/q}\leq\frac{C}{t-s}\left(\frac{1}{w(tB)}\int_{tB}(\tilde{u}H'_M(\tilde{u}))^2w\right)^{1/2}.$$

The remaining details of the proof are the same as that in Section 8.

Lemma 10.2. *Let u be a nonnegative solution and $-\infty<p<\infty$, then*

$$(\operatorname*{ess\,sup}_{\alpha B}\tilde{u})^p\leq\frac{c}{(1-\alpha)^d}(1+|p|)^{2\sigma/(\sigma-1)}\frac{1}{w(B)}\int_B\tilde{u}^p w$$

where c and d are two constants only depending on the parameters in (7.1) *and* (7.3).

PROOF. The case $p\geq 2$ follows by Lemma 10.1. Now consider $-\infty<p<2$. Let $u=\{u_k\}$, $u_k\in \mathrm{Lip}_1(\overline{B})$, $u_k\geq 0$, $\|u_k-u_j\|\to 0$. By considering $u_k+\varepsilon_0$, $\varepsilon_0>0$ and letting $\varepsilon_0\to 0$ at the end, we may assume $u_k\geq\varepsilon_0>0$. Define $\phi_k=\eta^2u_k^\beta$, $-\infty<\beta\leq 1$, supp $\eta\subset B$, then $\|\phi_k\|_0$ is bounded in k. So we can pick a subsequence $\phi_{k_j}\to\phi$ in H_0, and then

$$a_0^*(u_{K_j},\phi_{k_j})\to a_0^*(u,\phi)=0\,.$$

Note if we drop the subscripts,

$$X^*\phi_{k_j} = X\phi_{k_j} + \vec{b}\phi_{k_j} = \eta^2\beta u^{\beta-1}X\eta + 2\,\eta X\eta u^\beta + \vec{b}\eta^2 u^\beta$$

and

$$X(u^{(\beta+1)/2}) = \frac{\beta+1}{2}u^{(\beta-1)/2}Xu\ .$$

Then for $\beta \neq -1$, we have

$$\begin{aligned}
\delta &= \int\langle AXu, X^*\phi\rangle \\
&= \int\langle AXu, X\phi\rangle + \int\langle AXu, \vec{b}\phi\rangle \\
&= \int\langle AXu, Xu\rangle\eta^2\beta u^{\beta-1} + 2\int\langle AXu, X\eta\rangle\eta u^\beta + \int\langle AXu, \vec{b}\rangle\eta^2 u^\beta \\
&= \int\langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle\Big(\frac{2}{\beta+1}\Big)^2\beta\eta^2 \\
&\quad + 2\int\langle AX(u^{(\beta+1)/2}), X\eta\rangle\frac{2}{\beta+1}\eta u^{(\beta+1)/2} \\
&\quad + \int\langle AX(u^{(\beta+1)/2}), \vec{b}\rangle\frac{2}{\beta+1}u^{(\beta+1)/2}\eta^2\ .
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\beta}{\beta+1}\int\langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle\eta^2 &= \frac{\beta+1}{4}\ \delta \\
&\quad - \int\langle AX(u^{(\beta+1)/2}), X\eta\rangle u^{(\beta+1)/2}\eta \\
&\quad - \frac{1}{2}\int\langle AX(u^{(\beta+1)/2})\vec{b}\rangle u^{(\beta+1)/2}\eta^2\ .
\end{aligned}$$

Applying (8.4) with $\varepsilon = |\beta/(\beta+1)|, -\infty < \beta \leq 1, \beta \neq 0, -1$, then

$$\begin{aligned}
&\int|\langle AX(u^{(\beta+1)/2}), \vec{b}\rangle|u^{(\beta+1)/2}\eta^2 \\
&\qquad \leq \frac{\varepsilon}{2}\int\langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle\eta^2 + \frac{1}{2\varepsilon}\int\langle A\vec{b}, \vec{b}\rangle u^{\beta+1}\eta^2
\end{aligned}$$

and

$$\int|\langle AX(u^{(\beta+1)/2}), X\eta\rangle|u^{(\beta+1)/2}\eta \leq \frac{\varepsilon}{2}\int\langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle\eta^2$$

$$+\frac{1}{2\varepsilon}\int\langle AX\eta, X\eta\rangle u^{\beta+1}\,.$$

Thus

$$\frac{1}{4}\frac{|\beta|}{|\beta+1|}\int\langle AX(u^{(\beta+1)/2}), X(u^{(\beta+1)/2})\rangle\eta^2 \leq C\,\frac{|\beta+1|}{|\beta|}\left(\int|X\eta|^2u^{\beta+1}w+\int u^{\beta+1}\eta^2 w\right)$$

by noticing that $|\langle A\vec{b},\vec{b}\rangle|$ is bounded above. For $1/2\leq s<t<1$, we pick $\eta\in C_0^\infty(tB)$ as before, then

$$\frac{|\beta|}{|\beta+1|}\int|Xu^{(\beta+1)/2}|^2\eta^2 w \leq C\,\frac{|\beta+1|}{|\beta|}\frac{1}{(t-s)^2h^2}\int_{tB}u^{\beta+1}w + C\,\frac{|\beta+1|}{|\beta|}\int_{tB}u^{\beta+1}w + C\,\frac{|\beta+1|^2}{|\beta|}\,|\delta|\,.$$

If we note that $h\leq r_0, t-s\leq 1/2$, then

$$\int|X(u^{(\beta+1)/2})|^2\eta^2 w \leq \frac{C}{(t-s)^2h^2}\frac{|\beta+1|^2}{|\beta|^2}\int_{tB}u^{\beta+1}w + C_\beta\,|\delta|\,.$$

Note that

$$|X(u^{(\beta+1)/2})\eta|^2 w \leq 2\,(|X\eta|^2u^{\beta+1}+\eta^2|X(u^{(\beta+1)/2})|)w$$

and by Sobolev's inequality

$$\begin{aligned}
&\left(\frac{1}{w(sB)}\int_{sB}|X(u^{(\beta+1)/2})|^2w\right)^{1/2}\\
&\quad\leq\left(\frac{1}{w(sB)}\int_{tB}|\eta u^{(\beta+1)/2}|^q\right)^{1/q}\\
&\quad\leq t\,h\left(\frac{1}{w(tB)}\int_{tB}|X(\eta u^{(\beta+1)/2}|^2w\right)^{1/2}\\
&\quad\leq t\,h\,\frac{1}{w(tB)^{1/2}}\left[\left(\frac{|\beta+1|}{|\beta|}\right)^2\frac{1}{(t-s)^2h^2}\int_{tB}u^{\beta+1}w+C_\beta\,|\delta|\right]^{1/2}\\
&\quad\leq\frac{C}{t-s}\frac{1}{w(sB)^{1/2}}\frac{|\beta+1|}{|\beta|}\left(\int_{tB}u^{\beta+1}w\right)^{1/2}+\frac{C_\beta}{w(tB)^{1/2}}\,|\delta|^{1/2}\,.
\end{aligned}$$

Recall that $u = u_{k_j}, \delta = \delta_j, \delta_j \to 0$, let $k_j \to \infty$ and proceeding as in the proof of Theorem 8.15, we are done.

Lemma 10.3. *Let u be a nonnegative solution to $\mathcal{L}$ such that $u \geq \varepsilon$ for some $\varepsilon > 0$. For $1/2 \leq \alpha < 1$, define $k = k(\alpha, \tilde{u})$ by*

$$\log k = \frac{1}{w(\alpha B)} \int_{\alpha B} (\log \tilde{u}) w \, .$$

Then for $\lambda > 0$,

$$w(\{x \in \alpha B : |\log \frac{\tilde{u}}{k}| > \lambda\}) \leq \frac{c}{(1-\alpha)\lambda} \, w(\alpha B) \, .$$

PROOF. Let η, ϕ_k as in Lemma 8.18. Then

$$X^* \phi_k = -\eta^2 u_k^{-1} X u_k + 2 \, \eta u_k^{-1} X \eta + \vec{b} \phi_k \, .$$

As in Lemma 8.18, $\|\phi_k\|_0$ is bounded for all k, and there is $\phi \in H_0$ such that $\phi_{k_j} \to \phi$ in H_0. Thus

$$\delta_{k_j} = a_0^*(u_{k_j}, \phi_{k_j}) \to a_0^*(u, \phi) = 0$$

we drop the subscripts again,

$$\int \langle AXu, Xu \rangle u^{-2} \eta^2 = 2 \int \langle AXu, X\eta \rangle u^{-1} \eta + \int \langle AXu, \vec{b} u^{-1} \eta^2 \rangle - \delta \, .$$

Thus

$$\begin{aligned} &\int \langle AX(\log u), X(\log u) \rangle \eta^2 \\ &\quad \leq 2 \int \langle AX(\log u), X(\log u) \rangle^{1/2} \langle AX\eta, X\eta \rangle^{1/2} \eta \\ &\qquad + \int \langle AX(\log u), X(\log u) \rangle^{1/2} \langle AX\vec{b}, \vec{b} \rangle^{1/2} \eta^2 + |\delta| \\ &\quad \leq \varepsilon \int \langle AX(\log u), X(\log u) \rangle \eta^2 + \frac{1}{\varepsilon} \int \langle AX\eta, X\eta \rangle \end{aligned}$$

$$+\frac{\varepsilon}{2}\int\langle AX(\log u),X(\log u)\rangle\eta^2+\frac{1}{2\varepsilon}\int\langle A\vec{b},\vec{b}\rangle\eta^2+|\delta|\,.$$

Taking $\varepsilon=1/2$, we obtain

$$\frac{1}{4}\int\langle AX(\log u),X(\log u)\rangle\eta^2\le 2\int\langle AX\eta,X\eta\rangle + \frac{1}{2}\int\langle A\vec{b},\vec{b}\rangle\eta^2+|\delta|\,.$$

Then

$$\int|X(\log u)|^2\,w\,\eta^2\le C\int|X\eta|^2w+C\int\eta^2w+C\,|\delta| \le\frac{C}{a^2(t-s)^2h^2}\int_B w+C\int_B w+C\,|\delta|\,.$$

This follows that

$$\int_{\alpha B}|X(\log u)|^2w\le\frac{C}{(1-\alpha)^2h^2}\;w(B)+|\delta|$$

by noticing that $h\le r_0$ and $t-s\le 1/2$. By Poincaré's inequality with q replaced by 2,

$$\left(\frac{1}{w(\alpha B)}\int_{a\alpha B}|\log u-av_{\alpha B}\log u|^2w\right)^{1/2} \le C\alpha h\left(\frac{1}{w(\alpha B)}\int_{\alpha B}|X(\log u)|^2w\right)^{1/2}\,.$$

The remaining details of the proof are the same as that in the proof of Lemma 8.18.

We now derive Harnack's inequality by applying Bombieri's lemma.

Theorem 10.4. (Harnack inequality) *Let u be a nonnegative solution to $\mathcal{L}u=0$ belonging to $H(2B)$ for a ball $B=B_I(x_0,r)$ with $x_0\in E\subset\subset\Omega$ and $2B\subset\Omega, r\le r_0$. Let $\tilde{u}$ be the function in L^2_w associated to u. Then there are constants C depending only on the parameters in* (7.1), (7.2) *and* (7.3) *such that*

$$\operatorname*{ess\,sup}_B\;\tilde{u}\le C\operatorname*{ess\,inf}_B\;\tilde{u}\,.$$

REMARK. It is easy to see the mean value inequalities for the general solutions and subsolutions to $\mathcal{L}$ also hold. Since the proof is also a modification of that in the case of the operator L, we do not present the details.

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Sur la géometrie symplectique de l'espace des géodésiques d'une variété à courbure négative

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Introduction.

Soit N une variété complète, munie d'une métrique à courbure sectionnelle négative de volume fini. On définit le *spectre marqué des longueurs* de la métrique m de la façon suivante. Soit $\mathcal{C}$ l'ensemble des classes de conjugaison du groupe fondamental de N: le spectre marqué $\mathcal{L}(m)$ de la métrique m est l'élément $(l_m(\gamma))_{\gamma\in\mathcal{C}}$ du produit direct $\mathbb{R}^{\mathcal{C}}$, où $l_m(\gamma)$ désigne la longueur de la géodésique de la métrique m représentant l'élément γ.

Il est conjecturé dans [BK] que cette application $\mathcal{L}$, définie sur l'espace des métriques à courbure négative à isotopie près, est injective (nous dirons que deux métriques sont *isotopes* si elles sont isométriques par un difféomorphisme homotope à l'identité). On montre dans [O1] que c'est bien le cas lorsque la dimension de la variété est 2 (voir aussi [C],[F] et [M]).

Ce problème est à rapprocher du problème suivant. Soit D le disque compact de dimension n. Si D est muni d'une métrique riemannienne de courbure sectionnelle négative m, on peut considérer la fonction dis-

tance associée à cette métrique et aussi la restriction de cette fonction distance au bord ∂D, restriction que nous noterons d_m. Par analogie avec le problème précédent sur l'injectivité de l'application $\mathcal{L}$, on conjecture que l'application $m \to d_m$ est injective.

Nous allons considérer dans cet article ces deux problèmes du point de vue de la géométrie symplectique, le résultat principal étant que la donnée de l'application $\mathcal{L}$ détermine dans un certain sens la structure symplectique sur l'espace des géodésiques de la métrique m.

Dans la première section, on définit à partir de la structure symplectique canonique sur l'espace des géodésiques d'un disque D muni d'une métrique à courbure négative un *birapport* pour les quadruplets sur le bord ∂D.

Dans la section suivante, on généralise cette définition au cas des variétés de courbure négative, et simplement connexes. Dans le cas où une telle variété est le revêtement universel d'une variété compacte munie d'une métrique m de courbure négative, nous montrerons que la donnée de ce birapport est équivalente à la donnée du spectre marqué $\mathcal{L}(m)$.

Nous analyserons ensuite la topologie des ensembles de niveau de certaines fonctions définies à partir de ce birapport (Section 3), les liens entre ce birapport et la structure quasiconforme sur la sphère à l'infini (Section 4).

1. La mesure symplectique sur l'espace des géodésiques d'une boule compacte.

Soit m une métrique C^∞ sans points conjugués sur la boule compacte B de dimension n et supposons pour simplifier que le bord de cette boule est strictement convexe pour cette métrique. *L'espace des géodésiques orientées* de cette métrique dans l'intérieur de la boule B, que nous noterons $\mathcal{G}_m$ s'identifie au complémentaire de la diagonale Δ dans le produit $\partial B \times \partial B$. L'identification est définie par l'application qui à une géodésique associe ses deux points d'intersection (le point passé et le point futur) avec le bord ∂B. Cette identification est différentiable.

Le fibré tangent unitaire de cette boule porte une 2-forme fermée canonique , la forme de Liouville que nous noterons ω et qui est définie par le Wronskien de deux champs de Jacobi (*cf.* [K, Section 3]). Cette forme est non-dégénérée sur les sous-variétés transverses au flot. La

forme induite sur ces sous-variétés est une forme symplectique, invariante par le flot géodésique: c'est donc une forme symplectique sur *l'espace des géodésiques.*

Comment s'exprime la forme symplectique ω lorsqu'on paramètre les géodésiques par $\partial B \times \partial B - \Delta$? Soient x et y deux points distincts de la sphère ∂B. Soient (x_i) (respectivement (y_i)) un système de coordonnées différentiables sur ∂B au point x (respectivement, au point y). Notons finalement $\tau(a, b)$ la distance riemannienne, pour la métrique de la boule B entre les points a et b de ∂B. On a alors

Proposition 1.1. *Dans les coordonnées ci-dessus, la forme symplectique ω est donnée par la formule suivante*

$$\omega = \sum \frac{\partial^2 \tau}{\partial x_i \partial y_j} \, dx_i \wedge dy_j \ .$$

PREUVE. Rappelons d'abord la définition de la forme ω. Soit $t \to \gamma(t)$ une géodésique de B, d'extrémités $x = \gamma(0)$ et $y = \gamma(l)$. Alors, l'espace tangent à $\mathcal{G}_m$ au point γ s'identifie aux champs de Jacobi J le long de γ tels que $J(0)$ et $J(l)$ sont orthogonaux à la géodésique au point x et au point y (puisque la métrique m est sans points conjugués, ceci détermine bien un espace de champs de Jacobi le long de γ de dimension $2(n-1)$). Alors, si J_1 et J_2 sont deux tels champs de Jacobi, la forme ω évaluée sur (J_1, J_2) vaut: $\langle J_1(0), J_2'(0)\rangle - \langle J_2(0), J_1'(0)\rangle$ (pour ceci, *cf.* [K, Section 3]).

Remarquons maintenant que les deux formes considérées ont chacune comme sous-espaces lagrangiens, les espaces de géodésiques qui passent respectivement par le point x ou par le point y: ces sous-espaces sont caractérisés, en termes de champs de Jacobi, comme ceux dont la valeur en x ou en y est nulle. Donc, il suffit de vérifier que les deux formes considérées sont égales, lorsqu'on les évalue sur un couple de deux vecteurs respectivement contenus dans l'un de ces sous-espaces.

Si le point x reste fixe et si le point $y(s')$ décrit une courbe passant par y avec un vecteur vitesse w, le champ de Jacobi $J(w)$ associé à la variation correspondante de géodésiques $xy(s')$ est nul au point $\gamma(0) = x$ et vaut $\bar{w}$ au point $\gamma(l) = y$, où $\bar{w}$ est la composante du vecteur w sur l'orthogonal du vecteur vitesse à la géodésique xy au point y. Considérons une variation analogue de la géodésique $x(s)y$, lorsque le point $x(s)$ décrit une courbe passant par le point x avec vitesse v. Le champ de Jacobi $J(v)$ correspondant a pour valeur $\bar{v}$ en 0 et s'annulle

en l. Alors, un calcul basé sur la formule de la variation première donne

$$\frac{\partial^2 \tau(x(s), y(s'))}{\partial s \partial s'} = \langle w, J'(v)(l) \rangle .$$

Mais le dernier terme est aussi égal à $\omega(J(w), J(v))$, puisque la dérivée $J'(v)(l)$ est orthogonale au vecteur vitesse de la géodésique xy au point y. Ceci démontre la proposition.

Ainsi, la fonction distance sur la boule détermine la structure symplectique sur l'espace des géodésiques (orientées) $\mathcal{G}_m$.

Notons maintenant I et J deux intervalles disjoints plongés différentiablement dans le bord ∂D. On peut considérer la famille de géodésiques de $\mathcal{G}_m$ dont la première extrémité est dans I et dont la seconde extrémité est dans J. C'est un carré différentiablement plongé dans $\mathcal{G}_m$ sur lequel on pourra intégrer la forme ω, une fois que les intervalles I et J sont orientés, disons I de a vers b et J de c vers d. Ces intervalles peuvent être inclus dans des axes de coordonnées (x_i) et (y_j) respectifs. Le calcul de l'intégrale $\omega(I \times J)$ donne immédiatement

$$\omega(I \times J) = \tau(b, d) + \tau(a, c) - \tau(b, c) - \tau(a, d) .$$

On notera que cette valeur ne dépend que des extrémités et de l'orientation des intervalles considérés: ceci traduit le caractère *fermé* de la forme ω ainsi que le caractère *lagrangien* des sous-variétés $x \times (\partial B - x)$.

D'autre part, en dimension 2, la formule ci-dessus coïncide avec la formule qui exprimait, en dimension 2, la mesure de Liouville à partir de la fonction distance (*cf.* [O2]). Ceci n'est pas étonnant puisqu'en dimension 2, la mesure de Liouville sur l'espace des géodésiques n'est autre que la valeur absolue de la forme symplectique.

Définition. *Nous appelerons le birapport $\mathcal{B}(a, b, c, d)$ des quatre points (a, b, c, d) pour la métrique m la mesure pour la forme symplectique ω d'un carré $I \times J$, où I et J sont des intervalles comme ci-dessus joignant respectivement a b et c d.*

Donc le birapport est un nombre réel. Cette définition qui peut sembler abusive sera justifiée dans la section suivante par un calcul dans l'espace hyperbolique $\mathbb{H}^n$. Notons toutefois les propriétés de symétrie communes suivantes

$$\mathcal{B}(a, b, c, d) = -\mathcal{B}(b, a, c, d) ,$$

$$\mathcal{B}(a,b,c,d) = \mathcal{B}(c,d,a,b)\,,$$
$$\mathcal{B}(a,b,c,d) + \mathcal{B}(a,b,d,e) = \mathcal{B}(a,b,c,e)\,.$$

Remarquons aussi finalement que la donnée du birapport de la métrique m permet de reconstruire la fonction distance τ sur le bord, de la façon suivante

$$2\,\tau(a,c) = -\lim_{(b,d)\to(a,c)} \mathcal{B}(b,d,a,c)\,.$$

2. Le birapport symplectique sur la sphère à l'infini d'une variété de courbure négative.

Dans cette section, nous allons appliquer le calcul précédent dans une situation non-compacte, celle d'une variété complète, simplement connexe; mais une hypothèse portant uniquement sur l'absence de points conjugués sera insuffisante pour compenser la non-compacité et nous considèrerons uniquement des variétés de courbure sectionnelle négative.

Soit M une variété simplement connexe munie d'une métrique m, C^∞, complète et de courbure sectionnelle strictement négative. Notons $\bar{M} = M \cup \partial M$, la variété M compactifiée par sa sphère à l'infini ∂M.

L'espace des géodésiques de M est une variété C^∞, qui est *homéomorphe* au complémentaire de la diagonale dans $\partial M \times \partial M$.

La métrique m définit une forme symplectique sur cet espace. Soient a, b, c, d quatre points deux à deux distincts de ∂M; soient γ et γ' les géodésiques joignant respectivement les points a b et les points c d. Considérons finalement l'ensemble des géodésiques de ∂M dont une extrémité est dans γ' et l'autre est dans γ, ces géodésiques étant orientées de γ' vers γ. Cette famille de géodésiques est un carré différentiable: il s'identifie en effet différentiablement à $\gamma' \times \gamma$ différentiablement, car on ne considère que des points à distance finie. Ce carré peut être épuisé par les carrés $[c_i, d_i] \times [a_i, b_i]$ où les quadruplets (a_i, b_i, c_i, d_i) approximent le quadruplet (a, b, c, d) dans $\bar{M}$.

Si nous montrons que pour n'importe quelle suite de points de ce type, la suite de mesures $\omega([c_i, d_i] \times [a_i, b_i])$ (dont la valeur a été explicitée dans la section précédente, puisque toutes les géodésiques qui y interviennent sont contenues dans un même compact) converge nous aurons établi que le carré $\gamma \times \gamma'$ a une mesure finie pour la forme ω. En fait nous allons montrer que ceci est bien le cas, même si les

paires de points (a_i, b_i), (c_i, d_i) ne sont pas situés respectivement sur les géodésiques γ ou γ'.

Lemme 2.1. *Soient (a_i, b_i, c_i, d_i) une suite de quadruplets de M qui tend dans $\bar{M}$ vers le quadruplets (a, b, c, d) formé de points de $\bar{M}$ deux à deux distincts. Alors la limite de $\tau(a_i, c_i) + \tau(b_i, d_i) - \tau(a_i, d_i) - \tau(b_i, c_i)$ existe. C'est une fonction continue sur l'espace des quadruplets de la sphère à l'infini que nous noterons $\mathcal{B}(a, b, c, d)$.*

PREUVE. Soient H_a, H_b, H_c et H_d quatre horosphères deux à deux disjointes centrées respectivement aux points a, b, c et d. Les quatre géodésiques de M joignant respectivement a à c, b à d, a à d et b à c, intersectent le complémentaire M_0 dans M des quatre horoboules délimitées par ces horosphères en quatre segments géodésiques de longueur finie que nous noterons respectivement $l(ac)$, $l(bd)$, $l(ad)$ et $l(bc)$. Posons

$$\mathcal{B}(a, b, c, d) = l(ac) + l(bd) - l(ad) - l(bc).$$

On remarque que le nombre défini ci-dessus est indépendant des horosphères choisies.

Alors, dès que l'indice i est suffisamment grand, l'intersection des géodésiques a_ic_i, b_id_i, a_id_i et b_ic_i avec M_0 contient quatre segments géodésiques I_1, I_2, I_3, I_4 de plus en plus proches des quatre segments géodésiques ac, bd, ad et bc. Ceci parce que les bouts dans la sphère à l'infini des géodésiques contenant les segments a_ic_i, b_id_i, a_id_i et b_ic_i convergent respectivement vers a, b, c, d. Donc la trace de ces géodésiques sur tout compact de M_0 converge vers la trace de ac, bd, ad et de bc.

Il nous suffit de montrer que les contributions des segments des géodésiques a_ic_i, b_id_i, a_id_i et b_ic_i, complémentaires de I_1, I_2, I_3, I_4 est nulle à la limite, c'est-à-dire que pour chacun des sommets a_i, b_i, c_i, d_i la contribution des termes qui aboutissent en ce sommet est nulle à la limite. Ceci découle du fait que dans les variétés de courbure strictement négative les sphères passant par un point donné convergent vers l'horosphère passant par le même point lorsque leur centre converge vers un point à l'infini.

Nous avons donc obtenu une expression explicite pour la limite. Maintenant, le fait que les feuilletages stables et instables du flot géodésique sont continus entraîne que la fonction ainsi définie est continue.

REMARQUE. En fait, si la courbure de la variété M est pincée entre deux constantes strictement négatives, il est connu que le feuilletage stable

est hölderien. Donc, si on munit la sphère à l'infini de coordonnées différentiables en l'identifiant avec la sphère tangente unité en un point de N, l'application $\mathcal{B}$ est hölderienne.

D'autre part, la fonction $\mathcal{B}$ peut être définie de la même façon que précédemment pour les quadruplets de points distincts de $\bar{M}$. L'argument précédent dit en fait que cette fonction est continue sur l'espace des quadruplets distincts de $\bar{M}$.

Définition. *La fonction $\mathcal{B}$ définie ci-dessus sur l'espace de quadruplets de points distincts (a, b, c, d) de $\bar{M}$ sera appelée le birapport symplectique de ces quatre points.*

Considérons, pour justifier cette définition le cas de l'espace hyperbolique de courbure constante -1, $\mathbb{H}^n$. Alors, le birapport est défini pour quatre points, de la façon suivante. Ces quatre points sont toujours contenus dans un espace hyperbolique de dimension 2 ou 3: le birapport de ces quatre points est défini comme le birapport des ces quatre points sur la sphère à l'infini de cet espace. C'est un nombre complexe qui est un invariant du groupe des isométries de $\mathbb{H}^n$ agissant sur les quadruplets. Le logarithme du module de ce nombre n'est autre que le birapport $\mathcal{B}$ que nous avons défini précédemment.

Considérons le cas de quatre points distincts $0, \infty, 1, z$. Dans un premier temps supposons que le point z est de module différent de 1. L'homothétie γ de rapport z laisse invariante la géodésique $0, \infty$. Soit $F =]a, b]$ est un intervalle semi-ouvert contenu dans cette géodésique, domaine fondamental pour l'action de γ sur la géodésique $]0, \infty]$. Alors le birapport $(0, \infty, z, 1)$ a pour module le double de la longueur hyperbolique de l'intervalle F. D'autre part, d'après l'additivité de $\mathcal{B}$, on a

$$\mathcal{B}(0, \infty, z, 1) = \sum_{n \in \mathbb{Z}} \mathcal{B}(\gamma^n(a), \gamma^n(b), z, 1)\,.$$

Or, par invariance de $\mathcal{B}$ sous l'action du groupe d'isométries de $\mathbb{H}^n$, on a

$$\mathcal{B}(\gamma^n(a), \gamma^n(b), z, 1) = \mathcal{B}(a, b, \gamma^{-n}(z), \gamma^{-n}(1))\,.$$

Donc, $\mathcal{B}(0, \infty, z, 1) = \mathcal{B}(a, b, 0, \infty)$, c'est-à-dire le double de la longueur de l'intervalle F. Cette formule se prolonge par continuité au cas où z est de module 1.

Soit maintenant N une variété munie d'une métrique complète m de courbure négative: notons M son revêtement universel. Le birap-

port défini précédemment sur la sphère à l'infini ∂M est invariant sous l'action du groupe d'isométries $\pi_1(N)$.

L'action du groupe $\pi_1(N)$ sur la sphère à l'infini ∂M a un *ensemble limite* $L = L_m(G)$ que l'on peut définir comme le plus petit fermé non-vide invariant par $G = \pi_1(N)$. Nous nous placerons dans la situation où les éléments de $\pi_1(M)$ sont tous hyperboliques. Dans beaucoup de cas intéressants, cet ensemble limite peut être décrit uniquement en termes du groupe fondamental de N (*cf.* [Gr]). C'est le cas par exemple pour une variété hyperbolique complète N contenant un convexe compact tel que l'inclusion soit une équivalence d'homotopie: une telle variété sera dite *géométriquement finie*. On sait alors que l'ensemble limite est homéomorphe au complété de Gromov du groupe et le birapport que nous allons construire définira donc, par restriction, un birapport sur ce complété.

Théorème 2.2. *Soit G le groupe fondamental d'une variété N géométriquement finie. La restriction du birapport symplectique à l'espace des quadruplets de l'ensemble limite $L = L_m(G)$ du groupe $G = \pi_1(N)$ détermine et est déterminée par le spectre marqué de la métrique m.*

PREUVE. Commençons par montrer que le spectre marqué détermine le birapport $\mathcal{B}$ restreint à l'espace des quadruplets de points distincts de l'ensemble limite L. Il nous suffit de montrer que ce birapport est déterminé sur un ensemble dense. Puisque la variété N est géométriquement finie, l'action de $\pi_1(N)$ sur l'espace des paires de points distincts de L a une orbite dense.

Nous allons montrer que le birapport est déterminé à partir du spectre sur l'ensemble des quadruplets extrémités de deux translatés (distincts) de cette paire. Pour cela, considérons une suite de points (a_i, b_i, c_i, d_i) sur ces deux translatés qui approchent des quatre points à l'infini a, b, c, d. On peut choisir ces quatre points de sorte que les projections de leurs vecteurs tangents en leurs extrémités dans la variété N soient arbitrairement proches. Ainsi les projections des segments a_ic_i, b_id_i, a_id_i et b_ic_i, pourront être refermées en des géodésiques faisant un petit angle entr'elles. La longueur de la géodésique fermée (qui est une donnée du spectre marqué) qui leur est respectivement homotope peut être choisie arbitrairement proche des longueurs de ces segments.

Ainsi le birapport des quatre points apparaît comme une limite de différences symétriques de longueurs de géodésiques fermées (*cf.* [O1]).

Pour la réciproque, il suffit de raisonner comme dans le calcul de

$\mathcal{B}$ dans l'espace hyperbolique $\mathbb{H}^n$. Soit γ un élément du groupe fondamental de N, donc agissant sur M comme une isométrie hyperbolique; soient γ^-, γ^+ ses deux points fixes dans la sphère à l'infini (donc contenus dans L). Soit z un point de l'ensemble limite, différent de ces deux points. Alors le birapport $\mathcal{B}(\gamma^-, \gamma^+, \gamma(z), z)$ est exactement le double de la longueur de la distance de translation de l'élément γ sur son axe, c'est-à-dire $2\,l_m(\gamma)$. Ainsi, le spectre marqué de la métrique m est déterminé par le birapport $\mathcal{B}$ restreint à l'espace des quadruplets de l'ensemble limite L.

Ceci termine la démonstration du Théorème 2.2.

3. La topologie de la sphère à l'infini.

Soit M une variété simplement connexe dont la courbure sectionnelle est majorée par une constante strictement négative. On a défini dans la section précédente le birapport $\mathcal{B}$, qui coïncide, dans le cas de l'espace $\mathbb{H}^n$ avec le logarithme du module du birapport usuel. En particulier, si on fixe trois points distincts a, b et c, l'ensemble $S_{a,b}(c)$ formé des points $z \in \partial M$ tels que $\mathcal{B}(a, b, c, z) = 0$ est une sphère passant par le point c, en fait le bord à l'infini du sous-espace hyperbolique $\mathbb{H}^{n-1}$ orthogonal à la géodésique $a\,b$ passant par la projection de c sur cette géodésique. Plus précisément, l'ensemble $D_{a,b}(c)$ des points de M tels que $\mathcal{B}(a, b, c, z) = 0$ est l'hyperplan géodésique précédemment décrit; son adhérence dans $\bar{M}$ est homéomorphe à un disque compact de frontière $S_{a,b}(c)$. Nous allons montrer dans cette section que c'est toujours le cas.

Ce résultat est à rapprocher de [Im H] où il est montré que si a et b appartiennent à M, l'ensemble $D_{a,b}(c)$ a une fermeture homéomorphe à un disque fermé. Les hypothèses du théorème de H.C. Im Hof sont que la courbure de M est négative ou nulle, ce qui suffit pour définir $\mathcal{B}(a, b, c, d)$ lorsque deux au plus des points a, b,c et d sont sur ∂M. Le théorème suivant est l'analogue du résultat principal de [Im H] où les points a et b sont sur ∂M.

Proposition 3.1. *Soient trois points distincts de la sphère à l'infini de la variété M. L'ensemble $D_{a,b}(c) \cup S_{a,b}(c)$ est homéomorphe à une boule fermée de dimension $n-1$.*

PREUVE. Sur la géodésique $a\,b$, il existe un unique point p tel que

$\mathcal{B}(a,b,c,p) = 0$. Nous cherchons à déterminer l'ensemble $D_{a,b}(p)$. Soient H^- et H^+ les horosphères centrées respectivement en a et en b qui passent par p. L'ensemble $D_{a,b}(p)$ est exactement l'ensemble des z qui sont équidistants des horosphères H^+ et H^-. Notons d^+ la fonction distance à H^+ et d^- la fonction distance à H^-. Chacune de ces fonctions a pour gradient le vecteur unitaire définissant la géodésique négativement asymptote à b ou à a. La fonction $d^+ - d^-$ a pour gradient la différence de ces deux champs et il est facile de voir que cette différence est partout non nulle dans M. Donc $D_{a,b}(p)$ est une sous-variété de M. La fonction $e = (d^+ + d^-)/2$ a un champ gradient partout tangent à $D_{a,b}(p)$; sur cette sous-variété, le champ grad(e) s'annulle une seule fois: au point p. On montre comme dans [Im H, Proposition 2.6] que la singularité de e est de type Morse en ce point. Donc $D_{a,b}(p)$ est difféomorphe à $\mathbb{R}^{n-1}$.

Nous allons maintenant construire un difféomorphisme explicite de $D_{a,b}(p)$ avec $\mathbb{R}^{n-1}$ qui se prolongera en un homéomorphisme de $\bar{D}_{a,b}(p)$ avec le disque fermé de dimension $n-1$. Pour un nombre réel strictement positif t, définissons S_t comme l'ensemble des points z de $D_{a,b}(p)$ tels que $e(z) = t$. Considérons le champ de vecteurs

$$Z = \frac{\text{grad}(e)}{\|\text{grad}(e)\|^2} .$$

Ce champ est non singulier en dehors du point p. Le flot ϕ_t est défini sur tout $\mathbb{R}^+$ et envoit la sphère S_τ sur la sphère $S_{\tau+t}$ (*cf.* [Im H, Proposition 3.1]). Soit τ un nombre réel strictement positif, fixé un fois pour toutes.

Lemme 3.2. *La famille d'applications $\phi_t : S_\tau \to \bar{M}$ converge uniformément vers une application ϕ_∞ de S_τ vers ∂M qui est un homéomorphisme sur son image.*

PREUVE. Tout, d'abord $d^+(\phi_t(m)) = t + \tau$, pour tout $m \in S_\tau$; donc $d(\phi_t(m), p)$ tend uniformément vers ∞ sur S_τ.

Notons maintenant π la projection orthogonale sur l'horospère H^+. Pour montrer la convergence uniforme de la suite ϕ_t, nous allons montrer que la suite des projections $\pi \circ \phi_t(m)$ sur l'horosphère H^+ convergent uniformément. Pour cela, il suffit de montrer que la longueur des chemins $\pi \circ \phi_{t'}(m)$ pour $t' \in [t, \infty[$ tend uniformément vers 0 si t tend vers ∞. On raisonne en comparant avec l'espace hyperbolique $\mathbb{H}^n$ de courbure -1 en supposant, pour simplifier que la courbure sectionnelle de M est inférieure à -1.

La différentielle $d\pi$ de la projection π au point $\phi_t(m)$ est la composée de la projection orthogonale ρ sur l'espace orthogonal du vecteur $\mathrm{grad}(d^+)$ et de la différentielle de la projection orthogonale ψ de l'horosphère passant par $\phi_t(m)$ sur l'horosphère H^+.

La norme de la projection $d\psi$ est inférieure, par les théorèmes de comparaison à la norme de la projection correspondante dans l'espace de courbure constante -1, où elle vaut $e^{-(t+\tau)}$.

Il suffit maintenant de majorer la norme de $\rho(Z(\phi_t(m)))$; la norme du vecteur $Z(\phi_t(m))$ est $1/\cos\omega$, si on désigne par 2ω l'angle en $\phi_t(m)$ du triangle de sommets a, b et $\phi_t(m)$. Cet angle est majoré par l'angle d'un triangle de comparaison dans l'espace $\mathbb{H}^n$ dont tous les côtés ont une longueur plus grande que t; en particulier, la norme $\|\rho(Z(\phi_{t'}(m)))\|$ est uniformément majorée pour $t' \geq t$.

Une intégration facile montre alors que la longueur des chemins projetés $\pi \circ \phi_{t'}(m)$ pour $t' \in [t, \infty[$ tend uniformément vers 0 si t tend vers ∞. Donc, les applications $\phi_t(m)$ convergent uniformément sur S_τ vers une application continue à valeurs dans ∂M.

Pour terminer la démonstration du Lemme 3.2, nous allons maintenant montrer que l'application obtenue ϕ_∞ est injective.

Affirmation 3.3. *Soit $g_m(t)$ la géodésique issue de p dans la direction $\phi_\infty(m)$, paramétrée par longueurs d'arcs; alors, la distance $d(\phi_t(m), g_m(t))$ est uniformément bornée en m et en $t \in \mathbb{R}^+$.*

PREUVE. En effet, notons $H^+_{t+\tau}$ l'horosphère centrée en b, passant par le point $\phi_t(m)$: cette horosphère est donc à distance $t+\tau$ de H^+; soit $\pi_{t+\tau}$ la projection orthogonale sur l'horosphère $H^+_{t+\tau}$. Notons finalement $\gamma_m(t)$ la géodésique entre b et $\phi_\infty(m)$ dont l'origine est choisie sur H^+. La distance $d(\phi_t(m), \gamma_m(t+\tau))$ est inférieure à la longueur de la courbe $\pi_{t+\tau} \circ \phi_{t'}(m)$, pour $t' \in [t, \infty[$. L'argument utilisé précédemment pour établir la convergence uniforme des applications ϕ_t montre aussi que cette longueur est uniformément bornée. Il suffit maintenant d'observer que la distance $d(g_m(0), \gamma_m(0))$ est uniformément bornée sur S_τ, de sorte que, par convexité de la fonction distance, $d(g_m(t), \gamma_m(t))$ est aussi uniformément bornée en m pour $t \in \mathbb{R}^+$. D'où l'affirmation, par l'inégalité triangulaire.

Pour terminer la démonstration de l'injectivité de l'application ϕ_∞, il nous suffit de montrer grâce à l'affirmation précédente que la distance $c(t) = d(\phi_t(m_1), \phi_t(m_2))$ tend vers l'infini avec t si m_1 et m_2 sont

deux points distincts de S_τ. Notons $\beta(s)$ la géodésique paramétrée à vitesse unité qui réalise cette distance entre les points $\phi_t(m_1) = \mu_1(t)$ et $\phi_t(m_2) = \mu_2(t)$. Par la formule de la variation première, on a

$$c'(t) = \langle -\beta'(0), \mu_1'(t) \rangle + \langle -\beta'(c(t)), \mu_2'(t) \rangle .$$

Soit $\theta_i^+(t)$ (respectivement $\theta_i^-(t)$) l'angle en $\mu_i(t)$ du triangle de sommets b, $\mu_1(t)$ et $\mu_2(t)$ (respectivement a, $\mu_1(t)$ et $\mu_2(t)$). On a

$$c'(t) \geq \frac{1}{2}(\cos\theta_1^+(t) + \cos\theta_1^-(t) + \cos\theta_2^+(t) + \cos\theta_2^-(t)) .$$

Maintenant, chacun des angles θ_i^ε est inférieur à l'angle correspondant dans l'espace $\mathbb{H}^n$, c'est-à-dire à l'angle θ non nul d'un triangle isocèle centré en un point idéal et dont le côté de longueur finie a pour longueur $c(t)$. L'angle θ vérifie

$$\sin\theta = \frac{1}{\cosh(c(t)/2)} .$$

Donc, $\cos\theta_i^\varepsilon \geq \tanh(c(t)/2)$. Finalement, la fonction $c(t)$ vérifie l'inéquation différentielle

$$c'(t) \geq 2 \tanh\frac{c(t)}{2} .$$

Comme $c(0) = d(m_1, m_2)$ n'est pas nul, on a bien que $c(t)$ tend vers l'infini avec t.

Donc l'adhérence de $D_{a,b}(c)$ dans $\bar{M}$ intersecte ∂M en une sphère topologique de dimension $n-2$. Il nous suffit de vérifier, pour montrer la Proposition 3.1 que cette adhérence est *exactement* $S_{a,b}(c)$.

Raisonnons par l'absurde, et soit $z \in S_{a,b}(c) - \phi_\infty(S_\tau)$. La sphère $\phi_\infty(S_\tau)$ disconnecte ∂M en deux composantes dont l'une contient z; supposons par exemple que cette composante contient aussi le point a. Considérons la projection orthogonale π sur l'horosphère H^+, déjà introduite dans la démonstration du Lemme 3.2.

Affirmation 3.4. *La restriction de π à l'adhérence de $D_{a,b}(p)$ dans ∂M est un homéomorphisme sur son image.*

PREUVE. Montrons que cette projection est injective dans $D_{a,b}(c)$. Soit $t \to \gamma(t)$ une géodésique asymptote négativement à b qui intersecte H^+

en $\gamma(0)$; nous voulons voir que la fonction $d^-(t) - t$ s'annulle au plus une fois: ceci est clair puisque la dérivée de cette fonction a un signe constant.

L'argument précédent montre aussi que pour une géodésique $\gamma(t)$ passant par un point de $\pi(D_{a,b}(c))$ la valeur de $\mathcal{B}(a, b, c, \gamma(\infty))$ sera non nulle. Comme le point z est extrémité $\gamma(\infty)$ d'une géodésique de ce type, on a donc une contradiction.

Donc $S_{a,b}(c)$ est une sphère topologique, bord du disque topologique $\bar{D}_{a,b}(c)$.

4. Le birapport symplectique et la structure quasi-conforme sur la sphère à l'infini.

Soient M et M' deux variétés simplement connexes à courbure négative; considérons deux actions libres proprement discontinues d'un groupe G sur M et sur M', telles que les variétés quotients admettent un coeur convexe compact (variétés *géométriquement finies*). On a un homéomorphisme canonique, ϕ entre les ensembles limites $L_m(G)$ et $L_{m'}(G)$ qui commutte aux actions de G. On a vu que la restriction du birapport $\mathcal{B}$ à l'ensemble limite du groupe $L_m(G)$ était entièrement déterminé par le spectre marqué $\mathcal{L}_m(G)$: si $\mathcal{L}_m(G) = \mathcal{L}_{m'}(G)$, l'homéomorphisme ϕ preserve le birapport $\mathcal{B}$.

Définition. *On définit le rapport de m' sur m comme la plus grande constante K telle que $\mathcal{L}_{m'}(G) \geq K\,\mathcal{L}_m(G)$, c'est-à-dire la plus grande constante K telle que*

$$\forall g \in G, \quad l_{m'}(g) \geq K\, l_m(g)\,.$$

Nous allons maintenant étudier la régularité que cette propriété entraîne pour l'homéomorphisme ϕ.

Considérons d'abord l'autre birapport $\mathcal{P}(a, b, c, d)$ sur les quadruplets de la sphère à l'infini d'une variété simplement connexe de courbure strictement négative M défini comme suit: soient c' et d' les projections orthogonales de c et de d sur la géodésique $a\,b$. On définit $\mathcal{P}(a, b, c, d)$ comme la distance orientée de d' vers c' le long de la géodésique $a\,b$.

Lemme 4.1. *Il existe une constante C, dépendant uniquement de la borne supérieure de la courbure de M telle que, pour tout quadruplet de points distincts de ∂M, on a $|\mathcal{B}(a,b,c,d) - \mathcal{P}(a,b,c,d)| \leq C$.*

PREUVE. Dans une variété simplement connexe dont la courbure sectionnelle est majorée par une constante strictement négative $-a^2$, la longueur de l'hypothénuse d'un triangle rectangle (dont les côtés sont longs), diffère de la somme des longueurs des côtés adjacents à l'angle droit par une constante qui ne dépend que de a.

Nous allons maintenant montrer

Proposition 4.2. *Supposons que $\mathcal{L}_{m'}(G) \geq K\,\mathcal{L}_m(G)$; alors, il existe une constante C telle que, pour tout quadruplet de points distincts, les birapports $\mathcal{B}$ et $\mathcal{B}'$ sur $L(G)$ respectivement associés aux métriques m et m' vérifient*

$$|\mathcal{B}'(a,b,c,d)| \geq K\,|\mathcal{B}(a,b,c,d)| + C\,.$$

PREUVE. Soit $T^1_{C(N)}$ le fermé invariant du flot géodésique sur le fibré unitaire de la variété N, quotient de M par G, formé des géodésiques contenu dans le coeur compact $C(N)$; dans le revêtement universel, le relevé de ce fermé invariant dans $T^1(M)$ est exactement l'ensemble des vecteurs unitaires tangents aux géodésiques dont les deux extrémités appartiennent à l'ensemble limite de $G = \pi_1(N)$. Il existe (*cf.* [Gr]) un homéomorphisme de $T^1_{C(N)}$ vers $T^1_{C(N')}$ qui préserve les orbites du flot géodésique.

Soient maintenant quatre points a, b, c et d de l'ensemble limite du groupe G agissant sur M. Soit π_{ab} le projection orthogonale sur la géodésique $a\,b$. D'après le lemme précédent, les projections $c' = \pi_{ab}(c)$ et $d' = \pi_{ab}(d)$ de c et d sont à une distance qui diffère de $\mathcal{B}(a,b,c,d)$ par une quantité bornée par une quantié $c(m)$ qui ne dépend que de la borne supérieure sur la courbure de la métrique m.

Puisque la restriction du flot géodésique à $T^1_{C(N)}$ contient une orbite dense, il existe une constante $c(m)$ telle qu'on puisse refermer la géodésique $c'\,d'$ par un segment de longueur inférieur à $c(m)$; c'est-à-dire qu'il existe un arc géodésique $d''c''$, dont le relevé dans le fibré unitaire est contenu dans $T^1_{C(N)}$, et dont les extrémités sont C^1-proches des extrémités de $c'\,d'$. Il existe alors une géodésique fermée γ ε-proche de $c'd' \cup d''c''$. Si on relève dans le fibré unitaire $T^1(N)$ on obtient une réunion de deux orbites $\tilde{c'd'} \cup \tilde{d''c''}$ proche du relevé $\tilde{\gamma}$.

L'image par ψ de la réunion des deux orbites $c'\tilde{d'} \cup d''\tilde{c''}$ est, d'après l'uniforme continuité de ψ, ε'-proche de l'orbite $\tilde{\gamma}$ dans la variété $T^1_{C(N')}$. En particulier, la distance le long de l'orbite $\psi(ab)$ entre les points $\psi(c')$ et $\psi(d')$ diffère en valeur absolue de $l_{m'}(\gamma)$ de la longueur de $\psi(d'')\psi(c'')$. Or cette dernière longueur est bornée par une quantité c_1 qui ne dépend que de ψ et de $c(m)$, par compacité de $C(N)$ et puisque la distance entre d'' et c'' est inférieure à $c(m)$.

Il nous reste à voir que la distance entre $\psi(d'')$ et $\psi(c'')$ ne diffère de $\mathcal{B}'(a,b,c,d)$ que par une quantité bornée.

Pour cela, désignons par π' la projection orthogonale dans la variété M' sur la géodésique $a\,b$; soit encore $\tilde{\pi}_{ab}(c)$ le vecteur unitaire tangent à la géodésique $a\,b$ basée en $\pi_{ab}(c)$. Notons encore $\partial^3 G$ l'ensemble des triplés de points distincts de $L_m(G)$: on rappelle que l'action de G sur $\partial^3 G$ est proprement discontinue et cocompacte (*cf.* [Gr]). Considérons alors la fonction F sur ∂G

$$(a,b,c) \to d(\tilde{\pi}_{ab}(c), \psi(c'))\,.$$

Alors la fonction F sur $\partial^3 G$ est G-équivariante. Comme elle est continue, et que G agit de façon cocompacte, elle est bornée en valeur absolue par une constante c_2. Donc

$$|\mathcal{B}'(a,b,c,d) - \mathcal{B}'(a,b,\psi(c'),\psi(d'))| \leq 2\,c_2\,.$$

Finalement, en additionnant toutes les inégalités obtenues, on obtient la Proposition 4.2.

REMARQUE. Dans l'esprit de la Proposition 4.2, on peut se poser la question suivante. Soient M et M' deux variétés de courbure négative (non géométriquement finies) ayant le même groupe fondamental G et dont les spectres marqués vérifient $\mathcal{L}(m) \leq K\mathcal{L}(m')$ et $\mathcal{L}(m') \leq K'\mathcal{L}(m)$ pour des constantes K et K'; les ensembles limites de ces variétés sont-ils homéomorphes par une application transformant le birapport symplectique comme dans les conclusions de la Proposition 4.2.? Il semble qu'un tel résultat n'est pas connu, même pour des variétés de dimension 3 de courbure constante ayant le type d'homotopie d'une surface fermée.

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Sur l'existence des analyses multi-résolutions en théorie des ondelettes

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Résumé. On montre qu'une base d'ondelettes $(\psi_{j,k})$ de $L^2(\mathbb{R})$ avec une fonction mère ψ höldérienne à support compact provient nécessairement d'une analyse multi-résolution. La fonction-père φ a alors la même régularité que la fonction ψ et peut être choisie à support compact.

Introduction.

Une base d'ondelettes est une base hilbertienne de $L^2(\mathbb{R})$ $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ engendrée par translations et dilatations dyadiques à partir d'une seule fonction ψ

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\,, \qquad j,k \in \mathbb{Z}\,. \tag{1}$$

La fonction ψ est appelée la fonction mère de la base $(\psi_{j,k})$. On définit alors l'espace W_j comme le sous-espace vectoriel fermé de L^2 engendré par les $(\psi_{j,k})_{k\in\mathbb{Z}}$ et l'espace V_j comme $V_j = \oplus_{p<j} W_p$. Les espaces V_j vérifient alors

$$V_j \subset V_{j+1},\ \bigcap_{j\in\mathbb{Z}} V_j = \{0\} \text{ et } \bigcup_{j\in\mathbb{Z}} V_j \text{ est dense dans } L^2(\mathbb{R})\,, \tag{2.1}$$

$$f(x) \in V_j \text{ si et seulement si } f(2x) \in V_{j+1}\,, \tag{2.2}$$

(2.3) $f(x) \in V_0$ si et seulement si $f(x-1) \in V_0$.

Lorsqu'une suite d'espaces $(V_j)_{j\in\mathbb{Z}}$ fermés dans $L^2(\mathbb{R})$ vérifie (2.1) à (2.3) et que de plus

(2.4) V_0 a une base orthonormée de la forme $\varphi(x-k)$, $k \in \mathbb{Z}$,

cette suite est appelée une analyse multi-résolution de $L^2(\mathbb{R})$. A toute analyse multi-résolution on peut associer une base d'ondelettes $(\psi_{j,k})$ telle que V_j soit le sous-espace fermé de $L^2(\mathbb{R})$ engendré par les $\psi_{p,k}$, $p < j, k \in \mathbb{Z}$. La fonction φ est appelée fonction-père de la base d'ondelettes.

La notion de fonction-père et d'analyse multi-résolution permet de construire à partir de $(\psi_{j,k})$ une base hilbertienne de $L^2(\mathbb{R}^n)$ associée aux dilatations dyadiques [6] et a permis l'introduction par S. Mallat de la transformation en ondelettes rapide [7]. Enfin, elle a permis à I. Daubechies de construire des bases d'ondelettes régulières à support compact [1].

Le but de cet article est alors de démontrer que toute base d'ondelettes höldériennes à support compact provient d'une analyse multi-résolution sous-jacente, de sorte que l'approche de S. Mallat et I. Daubechies permet bien de construire toutes les bases concernées.

NOTATIONS.

$$\langle f, g\rangle = \int f(x)\bar{g}(x)\,dx\,,$$

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi}\,dx\,,$$

$$\chi_E(x) = \begin{cases} 1 & \text{si } x \in E\,, \\ 0 & \text{si } x \notin E\,. \end{cases}$$

1. Les trois transformations en ondelettes.

La *transformation en ondelettes continue* a été introduite par A. Grosmann et J. Morlet au début des années 80, *cf.* [2]. Une *ondelette* est une fonction g à valeurs réelles, de carré intégrable, non identiquement nulle et telle que

(3) $$C_g = \int_0^\infty |\hat{g}(\xi)|^2 \frac{d\xi}{\xi} < +\infty\,.$$

La transformation en ondelettes est alors définie pour $f \in L^2(\mathbb{R})$ par

$$f \to W_f(a,b) = \langle f, g_{(a,b)} \rangle \tag{4.1}$$

où

$$g_{(a,b)}(x) = \frac{1}{\sqrt{a}}\, g\Big(\frac{x-b}{a}\Big) \qquad (a > 0,\ b \in \mathbb{R}).$$

On a alors la formule de reconstruction

$$f = \frac{1}{C_g} \iint_{a>0, b\in\mathbb{R}} \langle f, g_{(a,b)} \rangle g_{(a,b)} \frac{da}{a^2} db \tag{4.2}$$

et la formule de Plancherel

$$\|f\|_2^2 = \frac{1}{C_g} \iint_{a>0, b\in\mathbb{R}} |\langle f, g_{(a,b)} \rangle|^2 \frac{da}{a^2} db\,. \tag{4.3}$$

L'un des intérêts principaux de cette transformation est son "invariance" par dilatation-translation. Un autre intérêt majeur est que, à a fixé, la transformation $f \to W_f(a,b)$ s'apparente à une convolution (avec la fonction $g(-x/a)/\sqrt{a}$). Enfin la transformation en ondelettes continues transforme le signal $f(x)$ en l'information redondante $W_f(a,b)$, la corrélation entre les coefficients $W_f(a,b)$ s'exprimant à l'aide d'un noyau-reproduisant.

La *transformation en ondelettes orthogonales* a été introduite par Y. Meyer en 1985, [6]. Il s'agit de donner une version discrète de la transformation de Grosmann et Morlet où la redondance soit complètement éliminée. Une *base d'ondelettes* est une base orthonormée de $L^2(\mathbb{R})$ de la forme $(\psi_{j,k})$, $j, k \in \mathbb{Z}$ où

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\,, \qquad j,\ k \in \mathbb{Z} \tag{5}$$

avec ψ une fonction de carré intégrable à valeurs réelles. La transformation en ondelettes orthogonales est alors définie par

$$f \to \langle f, \psi_{j,k} \rangle\,, \qquad j,\ k \in \mathbb{Z} \tag{6.1}$$

avec la formule de reconstruction

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k} \tag{6.2}$$

et la formule de Plancherel

$$\|f\|_2^2 = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2 . \tag{6.3}$$

Cette fois-ci, l'information donnée par les $\psi_{j,k}$ n'est plus redondante; le prix à payer est que cette transformation n'est plus invariante par translation.

Proposition 1. *Si $(\psi_{j,k})$ est une base d'ondelettes, alors ψ est une ondelette.*

DÉMONSTRATION. Il s'agit de vérifier que $\int_0^\infty |\hat{\psi}(\xi)|^2 d\xi/\xi < \infty$. Pour cela, on note Q_j le projecteur

$$Q_j f = \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k} . \tag{7}$$

Alors

$$\widehat{Q_j f}(\xi) = \left(\sum_{k\in\mathbb{Z}} [\int f(x)\psi(2^j x - k)dx] e^{-ik\xi/2^j}\right) \hat{\psi}(\frac{\xi}{2^j})$$

et, par la formule sommatoire de Poisson,

$$\widehat{Q_j f}(\xi) = \left(\sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi 2^j)\overline{\hat{\psi}}(\frac{\xi}{2^j} + 2k\pi)\right) \hat{\psi}(\frac{\xi}{2^j}) .$$

Si $\hat{f}$ est portée par $[0, 2\pi]$, alors

$$\chi_{[0,2\pi]}(\xi) \sum_{j\geq 0} \widehat{Q_j f}(\xi) = \hat{f}(\xi) \sum_{j\geq 0} |\hat{\psi}(\frac{\xi}{2^j})|^2 .$$

Comme $\|\sum_{j\geq 0} Q_j f\|_2 \leq \|f\|_2$, on en conclut que sur $[0, 2\pi]$, $\sum_{j\geq 0} |\hat{\psi}(\xi/2^j)|^2 \leq 1$, d'où

$$\int_0^{2\pi} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = \int_\pi^{2\pi} \sum_{j\geq 0} |\hat{\psi}(\frac{\xi}{2^j})|^2 \frac{d\xi}{\xi} \leq \log 2$$

et donc $C_\psi < +\infty$. (En fait, les équations de Y. Meyer dans [6] montrent que

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(\frac{\xi}{2^j})|^2 = 1$$

et donc que $C_\psi = \log 2$). La Proposition 1 est donc démontrée.

On note W_j le sous-espace vectoriel fermé de $L^2(\mathbb{R})$ engendré par les $\psi_{j,k}$, $k \in \mathbb{Z}$, de sorte que Q_j est le projecteur orthogonal de L^2 sur W_j. On note $V_j = \oplus_{p<j} W_p$ et P_j le projecteur orthogonal sur V_j .

Lemme 1. *V_0 est invariant par translation entière.*

En effet, $V_0^\perp = \oplus_{j\geq 0} W_j$ et chacun des espaces W_j est invariant par translation entière (puisque, si $j \geq 0$, $\psi_{j,k}(x+1) = \psi_{j,k-2^j}(x)$).

Proposition 2.

(i) *Soit $(\psi_{j,k})_j, k \in \mathbb{Z}$ une base d'ondelettes de $L^2(\mathbb{R})$. On définit la transformation en ondelettes pseudo-continue par*

$$f \to \langle f, \psi_{[j,k]} \rangle \tag{8.1}$$

où

$$\psi_{[j,k]}(x) = 2^{j/2}\psi(2^j(x-k)), \qquad j \leq -1,\ k \in \mathbb{Z}.$$

Alors on a l'inégalité

$$\sum_{j\leq -1} \sum_{k\in\mathbb{Z}} 2^j |\langle f, \psi_{[j,k]} \rangle|^2 \leq \|P_0 f\|_2^2 . \tag{8.2}$$

(ii) *On suppose de plus que ψ est höldérienne d'exposant ε pour un $\varepsilon > 0$ et que le module*

$$\omega(x) = \sup_{|h|\leq 1} \frac{|\psi(x+h) - \psi(x)|}{|h|^\varepsilon}$$

vérifie $\forall k \in \mathbb{N}$, $x^k\omega(x) \in L^2$; alors on a la formule de reconstruction

$$P_0 f = \sum_{j\leq -1} \sum_{k\in\mathbb{Z}} 2^j \langle f, \psi_{[j,k]} \rangle \psi_{[j,k]} \tag{8.3}$$

et la formule de Plancherel

$$\|P_0 f\|_2^2 = \sum_{j\leq -1}\sum_{k\in\mathbb{Z}} 2^j \|\langle f, \psi_{[j,k]}\rangle\|_2^2 \,. \tag{8.4}$$

Avant de démontrer cette proposition, faisons quelques commentaires sur cette troisième transformation en ondelettes. Cette transformation est "invariante" par translation entière: $\langle f(x+1), \psi_{[j,k]}\rangle = \langle f, \psi_{[j,k+1]}\rangle$. Elle n'est en fait définie que sur V_0, puisque sur $V_0^\perp$ elle est identiquement nulle (comme $\psi_{j,0} \in V_0$ pour $j \leq -1$, on voit que $\psi_{[j,k]} = \psi_{j,0}(x-k) \in V_0$). Lorsqu'on est dans le cadre d'une analyse multi-résolution (c'est-à-dire que V_0 a une base orthonormée de la forme $\varphi(x-k)$, $k \in \mathbb{Z}$), alors à j fixé $\langle f, \psi_{[j,k]}\rangle$ se calcule par un filtrage numérique à partir des coefficients $\langle f, \varphi(x-k)\rangle$ (le filtre ayant pour réponse impulsionnelle $(\langle \varphi, \psi_{[j,k]}\rangle)_{k\in\mathbb{Z}}$). Lorsque $f \in V_0$, la transformation de f en ondelettes orthogonales correspond à un sous-échantillonnage de la transformation pseudo-continue pour décorréler l'information redondante fournie par cette dernière. La transformation pseudo-continue est donc l'analogue de la transformation continue lorsqu'on restreint l'analyse aux fonctions de V_0.

DÉMONSTRATION. On note τ_p la translation $\tau_p f(x) = f(x-p)$. Il est clair que

$$\|\tau_{-p}(\sum_{j=-N}^{-1} Q_j)\tau_p f\|_2 \;\leq\; \|P_0 f\|_2$$

et donc que

$$\Big\|\frac{1}{2^N}\sum_{p=1}^{2^N} \tau_{-p}(\sum_{j=-N}^{-1} Q_j)\tau_p f\Big\|_2 \;\leq\; \|P_0 f\|_2 \,,$$

or on a

$$\begin{aligned}
\sum_{p=1}^{2^N} \tau_{-p} Q_j \tau_p f &= \sum_{p=1}^{2^N}\sum_{k\in\mathbb{Z}} \langle \tau_p f, \psi_{j,k}\rangle\, \tau_{-p}(\psi_{j,k}) \\
&= \sum_{p=1}^{2^N}\sum_{k\in\mathbb{Z}} \langle f, \psi_{[j,2^{-j}k-p]}\rangle\, \psi_{[j,2^{-j}k-p]} \\
&= 2^{N+j}\sum_{\ell\in\mathbb{Z}} \langle f, \psi_{[j,\ell]}\rangle\, \psi_{[j,\ell]}
\end{aligned}$$

si $-N \leq j \leq -1$. On obtient alors

$$\| \sum_{-N}^{-1} 2^j \sum_k \langle f, \psi_{[j,k]} \rangle \, \psi_{[j,k]} \|_2 \leq \|P_0 f\|_2$$

et, par produit scalaire avec $P_0 f$

$$\sum_{j=-N}^{-1} \sum_{k \in \mathbb{Z}} 2^j |\langle f, \psi_{[j,k]} \rangle|^2 \leq \|P_0 f\|_2^2 .$$

On a donc prouvé le point i).

Supposons maintenant que ψ soit höldérienne avec un module d'höldérianité ω à décroissante rapide. Pour vérifier que l'on a bien

$$\|P_0 f\|_2^2 = \sum_{j \leq -1} \sum_{k \in \mathbb{Z}} 2^j |\langle f, \psi_{[j,k]} \rangle|^2 ,$$

il suffit de le vérifier sur un sous-espace dense de L^2. On va donc montrer le résultat pour $f \in \mathcal{S}$ (classe de Schwartz) et $\hat{f}$ nulle au voisinage de 0.

Remarquons d'abord que, puisque ψ est höldérienne et de carré intégrable, alors ψ est bornée (et tend vers 0 à l'infini). En effet supposons que ψ ne tende pas vers 0 en $+\infty$; alors il existe $\alpha > 0$ et (x_n) telle que $x_n \to +\infty$ et $|\psi(x_n)| \geq \alpha$; par ailleurs $|\psi(x_n + h) - \psi(x_n)| \leq C|h|^\varepsilon$; d'où $|\psi(x)| \geq \alpha/2$ si $|x - x_n| \leq \gamma = (\alpha/2C)^{1/\varepsilon}$ et

$$\int |\psi|^2 dx \geq \frac{\alpha}{2} \Big| \bigcup_n [x_n - \gamma, x_n + \gamma] \Big| = +\infty .$$

D'où $\lim_{|x| \to \infty} \psi(x) = 0$. De plus, $x^k \psi \in L^2$ pour tout $k \in \mathbb{N}$. En effet, on a

$$\begin{aligned} &\|\chi_{[0,+\infty[}(x) x^k \psi\|_2 \\ &\qquad = \|\chi_{[0,+\infty[}(x) x^k \sum_{p=0}^{+\infty} \psi(x+p) - \psi(x+p+1)\|_2 \\ &\qquad \leq \sum_{p=0}^{+\infty} \|\chi_{[0,+\infty[}(x) x^k \omega(x+p+1)\|_2 \end{aligned}$$

$$\leq \sum_{p=0}^{+\infty} \|\chi_{[0,+\infty[}(x)\left(\frac{x}{x+p+1}\right)^k \cdot \frac{1}{(1+p)^2}(x+p+1)^{k+2}\omega(x+p+1)\|_2$$

$$\leq \|x^{k+2}\omega(x)\|_2 \sum_{p=0}^{\infty} \frac{1}{(1+p)^2},$$

et de même pour $\|\chi_{]-\infty,0]}(x)x^k\psi(x)\|_2$ en écrivant $\psi(x) = \sum_{p=0}^{\infty}\psi(x-p) - \psi(x-p-1)$. On va alors estimer $\|\tau_p Q_j \tau_{-p} f\|_2$ pour $j \leq -1$ et $f \in \mathcal{S}$ avec $\hat{f}$ nulle au voisinage de 0. On a

$$\|\tau_p Q_j \tau_{-p} f\|_2^2 = \sum_{k\in\mathbb{Z}} |\langle f(x+p), \psi(2^j x - k)\rangle|^2 2^j$$

$$= \sum_{k\in\mathbb{Z}} |\int f(x+p)(\psi(2^j x - k) - \psi(-2^j p - k))dx|^2 2^j.$$

Or

$$|\psi(x+h) - \psi(x)| \leq |h|^\varepsilon \omega(x) + |h|^\varepsilon(|\psi(x+h)| + |\psi(x)|)$$

(en distinguant $|h| \leq 1$ et $|h| \geq 1$), d'où

$$\|\tau_p Q_j \tau_{-p} f\|_2^2$$

$$\leq 2\Big(\sum_{k\in\mathbb{Z}} 2^j \left|\int |f(x+p)|2^{j\varepsilon}|x+p|^\varepsilon \omega(2^j x - k)dx\right|^2$$

$$+ \sum_{k\in\mathbb{Z}} 2^j \left|\int |f(x+p)|2^{j\varepsilon}|x+p|^\varepsilon(|\psi(2^j x - k)|\right.$$

$$\left. + |\psi(-2^j p - k)|)dx\right|^2\Big) = 2(I_1 + I_2).$$

Le second terme de la somme se contrôle par Cauchy-Schwarz en

$$I_2 \leq 2\Big(\int |f(x+p)||x+p|^\varepsilon dx\Big) 2^{j(1+2\varepsilon)} \int |f(x+p)||x+p|^\varepsilon$$

$$\cdot \Big(\sum_{k\in\mathbb{Z}} |\psi(2^j x - k)|^2 + |\psi(-2^j p - k)|^2\Big)dx.$$

Or, si $\theta \in \mathbb{R}$, on a

$$|\psi(\theta - k)|^2 \leq 2(|\psi(x)|^2 + \omega(x)^2) \qquad \text{si} \quad x - \theta \in [k, k+1]$$

et donc

$$|\psi(\theta-k)|^2 \leq 2\int_{k+\theta}^{k+1+\theta} \big(|\psi(x)|^2 + \omega(x)^2\big)\,dx$$

et enfin

$$\sum_{k\in\mathbb{Z}} |\psi(\theta-k)|^2 \leq 2\,(\|\psi\|_2^2 + \|\omega\|_2^2)$$

et

$$I_2 \leq 8\,(\|\psi\|_2^2 + \|\omega\|_2^2)\,2^{j(1+2\varepsilon)}\||x|^\varepsilon f\|_1^2.$$

I_1 se majore par Cauchy-Schwartz en

$$\begin{aligned} I_1 &\leq \sum_{k\in\mathbb{Z}} 2^j \int |f(x+p)|^2|x+p|^{2\varepsilon}2^{2j\varepsilon}(|2^jx-k|+1)^{-2}dx \\ &\qquad \cdot \int (|2^jx-k|+1)^2\omega^2(2^jx-k)\,dx \\ &\leq 2^{2j\varepsilon}\|(1+|x|)\omega\|_2^2 \\ &\qquad \cdot \int |f(x+p)|^2|x+p|^{2\varepsilon} \sum_{k\in\mathbb{Z}} \frac{1}{(|2^jx-k|+1)^2}\,dx \\ &\leq 2\Big(\sum_1^\infty \frac{1}{k^2}\Big)\|(1+|x|)\omega\|_2^2\ \||x|^\varepsilon f\|_2^2\ 2^{2j\varepsilon}\,. \end{aligned}$$

Au total, pour $j \leq -1$,

$$\|\tau_p Q_j \tau_{-p} f\|_2 \leq 2^{j\varepsilon} C\big(\||x|^\varepsilon f\|_2 + \||x|^\varepsilon f\|_1\big)$$

pour une constante C indépendante de f et de p.

On écrit alors $P_0 f = \tau_p P_0 \tau_{-p} f$ (par invariance de V_0 par translation entière) et donc

$$\begin{aligned} P_0 f &= \frac{1}{2^N}\sum_{p=1}^{2^N} \tau_p\Big(\sum_{j=-N}^{-1} Q_j\Big)\tau_{-p}f + \frac{1}{2^N}\sum_{p=1}^{2^N}\tau_p\Big(\sum_{j=-\infty}^{-N} Q_j\Big)\tau_{-p}f \\ &= \sum_{j=-N}^{-1}\sum_{k\in\mathbb{Z}} 2^j\langle f, \psi_{[j,k]}\rangle\,\psi_{[j,k]} + R_N \end{aligned}$$

avec

$$\begin{aligned} \|R_N\|_2 &\leq \frac{1}{2^N}\sum_{p=1}^{2^N}\sum_{j=-\infty}^{-N} C\,2^{j\varepsilon}\big(\||x|^\varepsilon f\|_2 + \||x|^\varepsilon f\|_1\big) \\ &\leq C'\,2^{-N\varepsilon}\big(\||x|^\varepsilon f\|_2 + \||x|^\varepsilon f\|_1\big)\,. \end{aligned}$$

On a bien $\|R_N\|_2 \to 0$ pour $N \to +\infty$. La Proposition 2 est donc démontrée.

2. Existence de fonctions-pères.

Nous allons montrer dans cette section que, sous la seule hypothèse que les zéros de $\hat{\psi}$ sont tous d'ordre fini, la base d'ondelettes $\psi_{j,k}$ admet une fonction-père, c'est-à-dire une fonction $\varphi \in V_0$ telle que les $\varphi(x-k)$ forment une base orthonormée de V_0. Cette hypothèse est vérifiée dès que $\hat{\psi}$ est analytique, en particulier dès que ψ est à décroissance exponentielle ($e^{\varepsilon|x|}\psi \in L^2$ pour un $\varepsilon > 0$).

Théorème 1.

a) *Soit $(\psi_{j,k})$ une base d'ondelettes de $L^2(\mathbb{R})$ telle que*

i) *ψ est höldérienne d'exposant ε pour un $\varepsilon > 0$ et le module*

$$\omega(x) = \sup_{|h|\leq 1} \frac{|\psi(x) - \psi(x+h)|}{|h|^\varepsilon}$$

vérifie $\forall k \in \mathbb{N}$, $x^k\omega(x) \in L^2$,

ii) *les zéros de $\hat{\psi}$ sont tous d'ordre fini.*

Alors il existe $\varphi \in V_0$ telle que les $\varphi(x-k)$, $k \in \mathbb{Z}$, forment une base orthonormée de V_0 .

b) *φ a la même régularité que ψ si ψ est de classe C^k (respectivement, $C^{k+\varepsilon}$ avec $\varepsilon \in]0,1[$) et $\psi^{(k)}$ est à décroissance rapide (respectivement, le module*

$$\omega_k(x) = \sup_{|h|\leq 1} \frac{|\psi^{(k)}(x) - \psi^{(k)}(x+h)|}{|h|^\varepsilon}$$

est à décroissance rapide) (c'est-à-dire que $x^p\psi^{(k)} \in L^\infty$ pour tout $p \in \mathbb{N}$ (respectivement, $x^p\omega_k \in L^2$)), alors φ est de classe C^k (respectivement, $C^{k+\varepsilon}$) et on peut choisir φ avec $\varphi^{(k)}$ à décroissance rapide (respectivement, avec son k-ième module à décroissance rapide).

c) *Si dans le point* b), *on remplace partout dans les hypothèses sur ψ la décroissance rapide par la décroissance exponentielle, le même résultat est valable pour φ à décroissance exponentielle.*

DÉMONSTRATION DE THÉORÈME 1.

Lemme 2. $\hat{\psi}(\xi) = \sum_{j\geq 0} \hat{\psi}(2^j\xi)\big(\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi+4k\pi)\overline{\hat{\psi}}(2^j(\xi+4k\pi))\big)$.

En effet,

$$\psi(\frac{x}{2}) = P_0\left(\psi(\frac{x}{2})\right) = \sum_{j\leq -1}\sum_{k\in\mathbb{Z}} 2^j\langle\psi(\frac{x}{2}),\psi_{[j,k]}\rangle\,\psi_{[j,k]}$$

et donc, par la formule sommatoire de Poisson

$$\begin{aligned}2\hat{\psi}(2\xi) &= \sum_{j\leq -1}\sum_{k\in\mathbb{Z}} 2^j\int \psi(\frac{x}{2})\psi(2^j(x-k))dx\,e^{-ik\xi}\,\hat{\psi}(\frac{\xi}{2^j})\\ &= \sum_{j\leq -1}\Big(\sum_{k\in\mathbb{Z}} 2\hat{\psi}(2\xi+4k\pi)\overline{\hat{\psi}}(2^{-j}(\xi+2k\pi))\Big)\hat{\psi}(\frac{\xi}{2^j}).\end{aligned}$$

Cela se récrit en la formule du Lemme 2 en posant $\xi' = 2\xi$ et $j' = -1-j$.

Le lemme s'applique aussi à $\xi+4k\pi$ et donne

$$\hat{\psi}(\xi+4k\pi) = \sum_{j\geq 0}\hat{\psi}(2^j(\xi+4k\pi))\sum_{p\in\mathbb{Z}}\hat{\psi}(\xi+4p\pi)\overline{\hat{\psi}}(2^j(\xi+4p\pi))$$

$$\begin{aligned}|\hat{\psi}(\xi+4k\pi)|^2 &= \sum_{j\geq 0}\overline{\hat{\psi}}(\xi+4k\pi)\hat{\psi}(2^j(\xi+4k\pi))\\ &\quad\cdot\sum_{p\in\mathbb{Z}}\hat{\psi}(\xi+4p\pi)\overline{\hat{\psi}}(2^j(\xi+4p\pi))\end{aligned}$$

$$\text{(9.1)}\qquad \sum_{k\in\mathbb{Z}}|\hat{\psi}(\xi+4k\pi)|^2 = \sum_{j\geq 0}\left|\sum_{k\in\mathbb{Z}}\hat{\psi}(\xi+4k\pi)\overline{\hat{\psi}}(2^j(\xi+4k\pi))\right|^2.$$

La formule de Cauchy-Schwartz donne alors

$$\text{(9.2)}\qquad \sum_{k\in\mathbb{Z}}|\hat{\psi}(\xi+4k\pi)|^2 \leq \sum_{k\in\mathbb{Z}}|\hat{\psi}(\xi+4k\pi)|^2\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}|\hat{\psi}(2^j(\xi+4k\pi)|^2$$

avec égalité si et seulement si pour tout $j\geq 0$ les vecteurs $(\hat{\psi}(\xi+4k\pi))_{k\in\mathbb{Z}}$ et $(\hat{\psi}(2^j(\xi+4k\pi)))_{k\in\mathbb{Z}}$ sont liés.

Lemme 3. $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 4k\pi)|^2$ *ne s'annule qu'en un nombre fini de points modulo* 4π .

En effet, par hypothèse, $x^k\psi \in L^2$ pour tout k. Alors la série $\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 4k\pi)|^2$ converge en tout point ξ vers une fonction C^∞ (voir [4] ou [8] pour une démonstration de ce lemme). Cette fonction est 4π-périodique. Or ses zéros sont d'ordre fini: si $\hat{\psi}(\xi) = 0$ et que ξ est un zéro d'ordre p, comme $\hat{\psi}$ est C^∞, on a

$$\hat{\psi}(\xi + \eta) \sim \frac{\hat{\psi}^{(p)}(\xi)}{p!}\,\eta^p$$

quand $\eta \to 0$ et donc

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + \eta + 4k\pi)|^2 \;\geq\; |\hat{\psi}(\xi + \eta)|^2 \;\geq\; \gamma(\xi)\frac{|\eta|^p}{p!}$$

pour η suffisamment petit et un $\gamma(\xi) > 0$. Les zéros sont donc isolés et donc en nombre fini modulo 4π.

La relation (9.2) donne alors que $\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+4k\pi))|^2 \geq 1$ en dehors d'un nombre fini de points sur $[0, 4\pi]$. Or on a

$$\begin{aligned}\frac{1}{4\pi}\int_0^{4\pi} \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 4k\pi))|^2 d\xi &= \frac{1}{4\pi}\sum_{j\geq 0}\int |\hat{\psi}(2^j\xi)|^2 d\xi \\ &= \frac{1}{4\pi}\sum_{j\geq 0} 2\pi 2^{-j} = 1\end{aligned}$$

et donc $\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 4k\pi)|^2 = 1$ presque partout.

Pour presque tout ξ, les vecteurs $(\hat{\psi}(2^j(\xi + 4k\pi)))_{k\in\mathbb{Z}}$ $(j \geq 0)$ sont donc proportionnels au vecteur $(\hat{\psi}(\xi + 4k\pi))_{k\in\mathbb{Z}}$. C'est le point essentiel de la démonstration. Le reste est purement technique.

La fonction $\sum |\hat{\psi}(\xi + 4k\pi)|^2$ peut s'écrire $P(\cos(\xi/2))U(\xi/2)$ où P est un polynôme, positif sur $[-1, 1]$, à coefficients réels et U une fonction C^∞, 2π-périodique strictement positive et paire: si ξ_0 est un zéro de $\sum |\hat{\psi}(\xi + 4k\pi)|^2$, alors il est d'ordre pair et on peut donc factoriser un terme de la forme $|1 - e^{i(\xi-\xi_0)/2}|^{2p}$; par ailleurs si $\xi_0 \notin 2\pi\mathbb{Z}$, alors $-\xi_0$ est un autre zéro de $\sum |\hat{\psi}(\xi + 4k\pi)|^2$ de même ordre que ξ_0 (puisque

$|\hat{\psi}(\xi)| = |\hat{\psi}(-\xi)|$, ψ étant à valeurs réelles); on peut donc factoriser également $|1-e^{i(\xi+\xi_0)/2}|^{2p}$. En factorisant ainsi tous les zéros on obtient

$$\sum |\hat{\psi}(\xi+4k\pi)|^2 = |1-e^{i\xi/2}|^{2p(0)}|1+e^{i\xi/2}|^{2p(2\pi)} \cdot \prod_{\xi_0\in F} \left|(e^{i\xi/2}-e^{i\xi_0/2})(e^{i\xi/2}-e^{-i\xi_0/2})\right|^{2p(\xi_0)} U(\frac{\xi}{2}).$$

On pose alors

$$\hat{\varphi}(\xi) = \hat{\psi}(2\xi) \frac{1}{\sqrt{U(\xi)}(1-e^{i\xi})^{p(0)}(1+e^{i\xi})^{p(2\pi)}} \cdot \frac{1}{\prod_{\xi_0\in F}\left[(e^{i\xi}-e^{i\xi_0/2})(e^{i\xi}-e^{-i\xi_0/2})\right]^{p(\xi_0)}}. \tag{10}$$

$\hat{\varphi}$ est définie presque partout (en dehors d'un nombre fini de points modulo 4π) et

$$|\hat{\varphi}(\xi)|^2 = \frac{|\hat{\psi}(2\xi)|^2}{\sum_{k\in\mathbb{Z}}|\hat{\psi}(2\xi+4k\pi)|^2}.$$

Il est alors immédiat que $\sum |\hat{\varphi}(\xi+2k\pi)|^2 = 1$ p.p.. Donc $\varphi \in L^2$ et les $\varphi(x-k)$ sont orthonormées. De plus φ est à valeurs réelles car $\overline{\hat{\varphi}(\xi)} = \hat{\varphi}(-\xi)$. Enfin les $\varphi(x-k)$ engendrent V_0, car presque partout on a

$$\hat{\psi}(2^{1+j}\xi) = \frac{\sum \hat{\psi}(2^j(2\xi+4k\pi))\overline{\hat{\psi}}(2\xi+4k\pi)}{\sum|\hat{\psi}(2\xi+4k\pi)|^2}\hat{\psi}(2\xi)$$

$$= \left(\sum \hat{\psi}(2^j(2\xi+4k\pi))\hat{\varphi}(\xi+2k\pi)\right)\hat{\varphi}(\xi), \quad \text{pour } j \geq 0$$

et donc les $\varphi(x-k)$ engendrent W_j pour $j \leq -1$. Pour conclure, il reste à vérifier que $\varphi \in V_0$.

Lemme 4. *Si $g \in V_0$ est telle que $\forall k \in \mathbb{N}$, $x^k g \in L^2$ et, pour un $\xi_0 \in \mathbb{R}$, $\sum |\hat{g}(\xi_0+2k\pi)|^2 = 0$ alors*

$$\hat{\gamma}(\xi) = \frac{\hat{g}(\xi)}{e^{i\xi}-e^{i\xi_0}} \in L^2,$$

$\gamma \in V_0$ *et* $\forall k \in \mathbb{N}$, $x^k\gamma \in L^2$.

En effet $\sum |\hat{g}(\xi+2k\pi)|^2$ est C^∞ ; on peut donc factoriser $|e^{i\xi}-e^{i\xi_0}|^2$ de

$$\sum |\hat{g}(\xi+2k\pi)|^2 \quad \text{et} \quad \int_0^{2\pi} \frac{\sum |\hat{g}(\xi+2k\pi)|^2}{|e^{i\xi}-e^{i\xi_0}|^2} d\xi < +\infty ,$$

d'où $\gamma \in L^2$. Maintenant

$$\sum_{k\in\mathbb{Z}} g(x-k)e^{-i\xi_0(x-k)} = \sum \hat{g}(\xi_0+2k\pi) = 0 ,$$

on pose alors $\theta(x) = \sum_{k<0} g(x-k)e^{ik\xi_0}$. Alors $\theta(x) = -\sum_{k\geq 0} g(x-k)e^{ik\xi_0}$ et on en conclut que $\forall p \in \mathbb{N}$, $x^p\theta \in L^2$. De plus, on a

$$\theta(x-1) = \sum_{k<0} g(x-k-1)e^{ik\xi_0} = e^{-i\xi_0}(g(x)+\theta(x)) ,$$

d'où

$$\hat{\theta}(\xi) = \frac{\hat{g}(\xi)}{e^{i\xi_0}e^{-i\xi}-1} = -e^{i\xi}\,\hat{\gamma}(\xi) .$$

Il suffit donc de montrer que $\theta \in V_0$. Or

$$\theta_N = \sum_{-N\leq k<0} g(x-k)e^{ik\xi_0} \in V_0 ,$$

θ_N converge vers θ dans $\mathcal{D}'$ et

$$\|\theta_N\|_2 = \frac{1}{\sqrt{2\pi}} \|\hat{g}(\xi)\frac{e^{iN(\xi-\xi_0)}-1}{e^{i(\xi-\xi_0)}-1}\|_2 \leq 2\,\|\theta\|_2 .$$

On en conclut que θ_N converge vers θ dans L^2-faible et donc que θ appartient à l'adhérence faible de V_0 (convexe fermé de L^2) et donc à V_0.

On obtient donc que γ définie par $\hat{\gamma} = \sqrt{U(\xi)}\,\hat{\varphi}$ vérifie que $\gamma \in V_0$ et $\forall k \in \mathbb{N}$, $x^k\gamma \in L^2$. Or $\varphi = \sum \alpha_k\gamma(x-k)$, où les α_k sont à décroissance rapide (coefficients de Fourier de la fonction C^∞ $1/\sqrt{U(\xi)}$).

La même démonstration donne le contrôle de la régularité de φ en fonction de celle de ψ, et le contrôle exponentiel de la localisation de φ en fonction de celui de ψ (les fonctions intervenant étant alors analytiques sur un voisinage tubulaire de $\mathbb{R}$ dans $\mathbb{C}$).

Par ailleurs si $\gamma \in V_0$ alors $\gamma = \sum_{k\in\mathbb{Z}} a_k\varphi(x-k)$ avec $(a_k) \in \ell^2$; γ a alors la même régularité que φ (et donc que ψ), puisque la suite (a_k) est bornée et que la module de continuité de φ est à décroissance rapide. Le théorème est donc démontré.

3. Ondelettes à support compact.

Théorème 2. *Soit $(\psi_{j,k})$ une base d'ondelettes de $L^2(\mathbb{R})$ telle que ψ est à support compact et ψ est höldérienne d'exposante pour un $\varepsilon > 0$. Alors il existe $\varphi \in V_0$ à valeurs réelles telle que φ est à support compact et que les $\varphi(x-k)$, $k \in \mathbb{Z}$, forment une base orthonormée de V_0 .*

La fonction φ a la même régularité que ψ (φ est C^k si ψ est C^k, φ est $C^{k+\varepsilon}$ si ψ est $C^{k+\varepsilon}$, ...).

Au vu du théorème précédent, il s'agit seulement que V_0 admet une base orthonormée $\varphi(x-k)$, $k \in \mathbb{Z}$, avec φ à support compact. Or nous savons déjà qu'il admet une base $\theta(x-k)$, $k \in \mathbb{Z}$, avec θ à décroissance rapide. Nous sommes donc dans le cadre d'une analyse multi-résolution (au sens de S. Mallat). Cette analyse multi-résolution contient des éléments à support compact non triviaux ($\psi(x/2) \in V_0$, est à support compact et est non identiquement nulle). Nous ferons alors appel à la propositon suivante

Proposition 3. *Si (V_j) est une analyse multi-résolution de $L^2(\mathbb{R})$ telle que V_0 a une base $\theta(x-k)$, $k \in \mathbb{Z}$, à valeurs réelles et telle que V_0 contienne des fonctions à support compact non nulle, alors*

i) *Il existe une unique fonction $\varphi \in V_0$ à valeurs réelles et à support compact telle que*

- $0 = \inf \operatorname{supp} \varphi \quad et \quad \hat{\varphi}(0) = 1$,
- *les $\varphi(x-k)$, $k \in \mathbb{Z}$, forment une base de Riesz de V_0 ,*
- *toute fonction $\theta \in V_0$ à support compact s'écrit comme une combinaison linéaire finie des $\varphi(x-k)$;*

ii) *On note $N = \sup \operatorname{supp} \varphi$, ($N \in \mathbb{N}^*$), alors les restrictions à $[0,1]$ des fonctions $\varphi(x+k)$, $0 \leq k \leq N-1$, sont linéairement indépendantes.*

Le point i) a été démontré dans [4] et le point ii) est démontré de manière très simple dans [5].

Nous pouvons maintenant démontrer le théorème. On considère

la fonction φ définie dans la proposition ci-dessus. Alors il existe $h \in L^2_{\mathbb{R}}([0,1])$ telle que $\langle h, \varphi(x-k)\rangle = \delta_k$. Si $f \in V_0$, on a

$$f = \sum_{k\in\mathbb{Z}} \langle f, h(x-k)\rangle\, \varphi(x-k) = \sum_{k\in\mathbb{Z}} \langle f, P_0(h(x-k))\rangle\, \varphi(x-k)\,.$$

La base duale des $\varphi(x-k)$ est donc donnée par les $P_0(h(x-k)) = (P_0h)(x-k)$. Or $P_0h = h - \sum_{j\geq 0} Q_j h$. Si ψ a son support dans $[-M, M]$, $Q_j h$ a son support dans $\left[-2M/2^j, 1+2M/2^j\right]$, et donc $\operatorname{supp} P_0h \subset [-2M, 2M+1]$. P_0h vérifie donc que les $P_0h(x-h)$ forment une base de Riesz de V_0, que P_0h est à valeurs réelles et à support compact et que toute fonction $\theta \in V_0$ à support compact s'écrit comme une combinaison linéaire finie des $P_0h(x-k)$ (puisque la base duale des $P_0h(x-h)$ est la base des $\varphi(x-k)$). Par unicité de la fonction φ, on a $P_0h = \lambda_0\varphi(x-k_0)$; comme $\langle P_0h, \varphi(x-k)\rangle = \delta_{k,0}$, on voit que nécessairement $k_0 = 0$; la famille $\varphi(x-k)$ est donc orthogonale; d'où

$$\|\varphi\|_2^2 = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = |\hat{\varphi}(0)|^2 + \sum_{k\in\mathbb{Z}} |\hat{\varphi}(2k\pi)|^2$$

or $\hat{\varphi}(2k\pi) = 0$ pour $k \neq 0$ (*cf.* [4] par exemple) et donc $\|\varphi\|_2 = 1$. Les $\varphi(x-k)$ forment bien une base orthonormée de V_0 .

4. Contre-exemple et conjecture.

Pour mémoire, je rappelle l'exemple de base d'ondelettes $(\psi_{j,k})_{j,k\in\mathbb{Z}}$ de $L^2(\mathbb{R})$ donné dans [3] et qui ne provient pas d'une analyse multi-résolution.

La fonction ψ est donnée par

$$\hat{\psi} = \chi_{[-8\pi/7,-4\pi/7]} + \chi_{[4\pi/7,6\pi/7]} + \chi_{[24\pi/7,32\pi/7]}\,.$$

On pose

$$E = [-8\pi/7, -4\pi/7] \cup [4\pi/7, 6\pi/7] \cup [24\pi/7, 32\pi/7]\,.$$

On a les identités

$$\sum_{j\in\mathbb{Z}} \chi_E(2^j\xi) = 1 \ \text{ p.p.} \quad \text{et} \quad \sum_{k\in\mathbb{Z}} \chi_E(\xi + 2k\pi) = 1\,.$$

Il est alors immédiat que les $\psi_{j,k}$ forment une base orthonormée de $L^2(\mathbb{R})$

$$\hat{f}(\xi) = \sum_{j\in\mathbb{Z}} \hat{f}(\xi)\chi_E(2^j\xi) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \hat{f}(\xi + \frac{2k\pi}{2^j})\chi_E(2^j\xi + 2k\pi)\chi_E(2^j\xi)$$

puisque

$$\chi_E(2^j\xi)\,\chi_E(2^j\xi + 2k\pi) = \delta_{k,0}\,\chi_E(2^j\xi)\,,$$

on obtient donc

$$f = \sum_j \sum_k \langle f, \psi_{j,k}\rangle \psi_{j,k}\,,$$

l'orthogonalité étant évidente, la famille $\psi_{j,k}$ est bien une base d'ondelettes.

Si V_0 avait une base orthonormée $\varphi(x-k)$, on aurait nécessairement $|\hat{\varphi}(\xi)|^2 - |\hat{\varphi}(2\xi)|^2 = |\hat{\psi}(2\xi)|^2$, et donc $|\hat{\varphi}(\xi)|^2 = \sum_{j\geq 1} |\hat{\psi}(2^j\xi)|^2$. Ce qui donne

$$|\hat{\varphi}(\xi)|^2 = \chi_{[-4\pi/7,4\pi/7]} + \chi_{[6\pi/7,8\pi/7]} + \chi_{[12\pi/7,16\pi/7]}\,.$$

Mais alors on n'a pas $\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1$ p.p., ce qui contredit l'orthonormalité des $\varphi(x-k)$, $k \in \mathbb{Z}$. Ce qui bloque est donc que l'on n'a pas dans ce cas

$$\sum_{j\geq 0}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 4k\pi))|^2 = 1 \quad \text{p.p.}\,,$$

ce qui était le point central dans notre démonstration.

Une conjecture naturelle, au vu de ce contre-exemple et de notre démonstration, est qu'il existe $\psi \in \mathcal{S}$ telle que les $(\psi_{j,k})$ soient une base d'ondelettes et telle que cette base ne dérive pas d'une analyse multi-résolution.

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