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Two Problems in Homogenization of Porous Media

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1. Introduction

The main goal of this work is to present two different problems arising in Fluid Mechanics of *perforated domains* or *porous media*. The first problem concerns the compressible flow of an ideal gas through a porous media and our goal is the mathematical derivation of the Darcy's law. This is relevant in oil reservoirs, agriculture, soil infiltration, etc. The second problem deals with the incompressible flow of a fluid reacting with the exterior of many packed solid particles. This is related with absorption and adsorption phenomena in beds or towers, of interest in Chemical Engineering (separation, chemical industry, etc.,).

A common aspect to both problems is the nature of the spatial domain: the porous medium. Some examples arising in different applications can be found in the books by Bear [5], Bensoussan, Lions and Papanicolau [6], Ene and Polisevski [12], Hornung [14], Morris [25], Norman [26], Oleinik, Shmaev and Yosefian [27], and Sanchez-Palencia [28].

We shall assume some periodicity structure on the porous media. More precisely, we start by considering an open bounded set Ω of R^N, with $N = 2$ or 3, of regular boundary $\partial\Omega$. For any small $\varepsilon > 0$ we consider the perforated domain Ω_ε obtained by intersecting the $\varepsilon-$multiple of a periodic geometry with Ω: i.e., we define $Y =]0, l_1[\times]0, l_2[\times\cdot\cdot\times]0, l_N[$, a bounded regular subset $\theta \subset Y$ with $\Gamma = \partial\theta - \partial Y$ and $Y^* = Y - \overline{\theta}$. Finally, we define

$$\Omega_\varepsilon = \Omega \cap \varepsilon\theta \text{ and } \Gamma_\varepsilon = \Omega \cap \varepsilon\Gamma.$$

In the first problem the reference open set $\theta = Y_f$ will be the exterior to the solid part Y_s and we will assume that the union of all the solid parts,

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$\Omega - \overline{\Omega}_\varepsilon$, and all the fluid parts, Ω_ε, are connected (i.e. the solid and fluid parts are of one piece) which is possible when $N = 3$. In the second problem, by the contrary, we shall assume that the solid part $\Omega - \overline{\Omega_\varepsilon}$ is constituted of a sequence of nonconnected obstacles, open subsets of Ω.

In both problems the main goal is the same: to determine the *laws* (or system of partial differential equations) satisfied by the homogenized flow unknowns such as the velocity

$$\mathbf{v} = \lim_{\varepsilon \to 0} \mathbf{v}_\varepsilon,$$

the density ρ_ε, the pressure p_ε, the concentration of some chemical component u_ε, etc.

Usually, the homogenization method starts by the formal derivation of the limit problem. One starts by the *ansatz* that the unknowns functions $\rho_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon$ have an asymptotic expansion, with respect to ε, of the form

$$\mathbf{v}_\varepsilon(x,t) = \mathbf{v}_\varepsilon(x,y,t)|_{y=\frac{x}{\varepsilon}} = \mathbf{v}_0(x,y,t) + \varepsilon \mathbf{v}_1(x,y,t) + \varepsilon^2 \mathbf{v}_2(x,y,t) + \ldots \tag{1}$$

with $\mathbf{v}_i(x,y,t)$ y-periodic with respect to the $y = \frac{x}{\varepsilon}$ variable. And the same for the rest of fluid unknowns (the dependence on the two variables x, y justifies the name of *two-space method* used in the engineering literature: see, e.g. Keller [18]). The homogenized laws are obtained by using, as a first element of the analysis, that

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y.$$

We point out that some times it is required to assume a special time dependence on $\mathbf{v}^\varepsilon$ (see Section 2).

The second part, considerably more difficult, consists in to obtain a rigorous proof of the consequences obtained via formal expansions, but now without any analytical assumption of the type (1). In other words, it must be proved that there exists a $\mathbf{v}_0$ such that $\mathbf{v}^\varepsilon \to \mathbf{v}_0$, in some functional space, as $\varepsilon \to 0$ and the same for the rest of fluid variables ρ_ε and p_ε.

In Section 2 of this paper we shall present the formal derivation of the Darcy's law [10] (the flow of a liquid through a porous medium the velocity is proportional to the gradient of the pressure)

$$\mathbf{v}_0 = \frac{1}{\mu}\mathbf{K}(\rho_0 \mathbf{f} - \nabla_x p_0) \tag{2}$$

which, jointly with the homogenized density equation and a state assumption $\rho_\varepsilon = F(p_\varepsilon)$, leads to the, so called, *porous media equation*

$$\delta\frac{\partial\rho_0}{\partial t} - \mathrm{div}(\frac{1}{\mu}\mathbf{K}\rho_0\nabla_x F^{-1}(\rho_0)) + \mathrm{div}(\frac{1}{\mu}\mathbf{K}\rho_0^2\mathbf{f}) = 0. \tag{3}$$

We shall see later that the derivation of the above equation requires some special arguments: $\mathbf{v}_0$ must be replaced by $\mathbf{v}_2$, i.e. the asymptotic expansion starts with the term of ε^2, and a different macroscopic time scale must be introduced. These (unpublished) results were presented in a postgraduate course by the author in 1992 (see Díaz [11]). Notice that the above formulation include, as special case, the equation

$$w_t - \Delta w^m = 0, \tag{4}$$

(where usually $m > 1$) which is a simpler formulation very studied in the mathematical literature since it is a degenerate equation leading to finite speed of propagation properties (see, e.g. the survey Kalashnikov [17] and its references).

Equation (3) is very useful for the study of the flow since instead of several scalar unknowns (five if $N = 3$) we reduce the problem to the determination of only one, ρ_0. This fact is well known in the literature but a rigorous proof was not derived until the work by Luc Tartar [29], in 1980, by using homogenization techniques. The motivation of our interest comes from the fact that Tartar's proof deals merely with stationary incompressible fluids and so the conservation of the mass is reduced to $\rho_0 = \rho_c$ (a known constant) and

$$\mathrm{div}\,\mathbf{v}_0 = 0.$$

In that case Darcy's law (2) leads to stationary equation

$$\mathrm{div}(\frac{1}{\mu}\mathbf{K}\rho_c\nabla_x\, p_0) + \mathrm{div}(\frac{1}{\mu}\mathbf{K}\rho_c\mathbf{f}) = 0. \tag{5}$$

Notice that equation (5) is now a linear elliptic partial differential equation on p_0 and so of a very different nature to the mathematically richer nonlinear parabolic equation (3). The rigorous proof of the derivation of equation (3) seems to be far to be a mere modification of the Tartar result once that the derivation of a priori estimates, for compressible fluids, is a very delicate question (see P.-L. Lions [21]).

Section 3 is devoted to a short presentation of some of the results contained in the unpublished manuscript Conca, Díaz and Liñan [8]. It concerns the, already mentioned, second problem in which a stationary reactive fluid confined in Ω_ε, of concentration u_ε, reacts on the boundary of a porous medium, $\Omega - \overline{\Omega}_\varepsilon$, constituted by a collection of nonconnected open subsets of Ω. A very simplified version of the problem is the following

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ -\dfrac{\partial u_\varepsilon}{\partial n} = \alpha\varepsilon \left|u_\varepsilon\right|^{p-1} u_\varepsilon & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\alpha > 0$ and the exponent p (called as the *order of the reaction*) verifies that $p \in (0,1)$. In that case it is possible to give a rigorous proof of the result obtained via formal expansions and so we shall prove the following

THEOREM 1. *Assume $p \in (0,1)$, $f \in L^2(\Omega)$ and let $V_\varepsilon := \{w \in H^1(\Omega_\varepsilon) : w = 0 \text{ on } \partial\Omega\}$ and $P_\varepsilon \in L(V_\varepsilon : H_0^1(\Omega))$ be a family of extension operators (i.e. such that $(P_\varepsilon w)(x) = w(x)$ a.e. $x \in \Omega_\varepsilon$). Then $P_\varepsilon u_\varepsilon$ converges, (weakly) in $H_0^1(\Omega)$, as $\varepsilon \to 0$, to a function u_0 characterized as the unique solution of the problem*

$$\begin{cases} -\displaystyle\sum_{i,j} q_{i,j} \frac{\partial^2 u_0}{\partial x_i \partial x_j} + \alpha\delta \left|u_0\right|^{p-1} u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where

$$\delta = \frac{\text{meas}_{N-1}\, \partial\theta}{\text{meas}(Y^*)}$$

and $q_{i,j}$ are suitable constants depending of θ.

2. A MATHEMATICAL DERIVATION OF THE DARCY'S LAW.

Let $\mathbf{v}_\varepsilon$ be the velocity, ρ_ε the density and p_ε the pressure of a compressible fluid occupying the region Ω_ε. The correspondent Navier-Stokes system is formed by the *mass conservation equation*

$$\frac{\partial \rho_\varepsilon}{\partial t} + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = 0, \tag{8}$$

and the *momentum conservation equation*

$$\rho_\varepsilon(\frac{\partial \mathbf{v}_\varepsilon}{\partial t} + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon) = -\nabla p_\varepsilon + \mu\Delta\mathbf{v}_\varepsilon + \lambda\nabla(\operatorname{div}\mathbf{v}_\varepsilon) + \rho_\varepsilon\mathbf{f}. \tag{9}$$

We assume a constitutive law of the form

$$\rho_\varepsilon = F(p_\varepsilon), \tag{10}$$

where $F : R \to R$ is a strictly increasing function of class C^1. The auxiliary conditions are formed by a boundary condition

$$\mathbf{v}_\varepsilon = \mathbf{0}, \quad \text{on } \partial\Omega_\varepsilon \times (0,T)$$

and the initial conditions

$$\rho_\varepsilon(x,0) = \rho_I(x), \quad \text{on } \Omega_\varepsilon,$$
$$\mathbf{v}_\varepsilon(x,0) = \mathbf{v}_I(x), \quad \text{on } \Omega_\varepsilon,$$

where ρ_I and $\mathbf{v}_I$ are functions defined on the whole domain Ω, $\rho_I \geq 0, \rho_I \neq 0$. As mentioned at the introduction we assume a formal expansion in terms of powers of ε. In our case we introduce the variables

$$y = \frac{x}{\varepsilon} \quad \text{and} \quad \tau = \varepsilon^k t$$

and assume the *ansatz*

$$\rho_\varepsilon(x,t) = \rho_0(x,y,\tau) + \varepsilon\rho_1(x,y,\tau) + \varepsilon^2\rho_2(x,y,\tau) + \cdots |_{y=\frac{x}{\varepsilon},\tau=\varepsilon^k t}$$
$$\mathbf{v}_\varepsilon(x,t) = \varepsilon^n\ (\mathbf{v}_0(x,y,\tau) + \varepsilon\mathbf{v}_1\ (x,y,\tau) + \varepsilon^2\mathbf{v}_2(x,y,\tau) + \cdots |_{y=\frac{x}{\varepsilon},\tau=\varepsilon^k t}$$
$$p_\varepsilon(x,t) = p_0(x,y,\tau) + \varepsilon p_1(x,y,\tau) + \varepsilon^2 p_2(x,y,\tau) + \cdots |_{y=\frac{x}{\varepsilon},\tau=\varepsilon^k t}$$

with k and n to be determined later (see Remark 2 for a justification of such type of expansion). We have

$$\frac{\partial}{\partial t} = \varepsilon^k \frac{\partial}{\partial \tau},$$
$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y,$$
$$\operatorname{div} = \operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y,$$
$$\Delta = \Delta_x + \frac{2}{\varepsilon}\Delta_{xy} + \frac{1}{\varepsilon^2}\Delta_y\ (\Delta_{xy} = \sum_{i=1}^N \frac{\partial^2}{\partial x_i \partial y_i}).$$

By choosing

$$k = n = 2 \tag{11}$$

(see Remark 2 for a justification via Dimensional Analysis) the identification of the coefficients of ε^{-1} at the momentum equation leads to the condition

$$\nabla_y p_0 = 0.$$

So, from (10), "p_0 and ρ_0 are independent of y". The identification of the coefficients of ε at the momentum equation imply that

$$\mathbf{0} = -(\nabla_x p_0 + \nabla_y p_1) + \mu\Delta_y \mathbf{v}_0 + \lambda\nabla_y(\operatorname{div} \mathbf{v}_0) + \rho_0 \mathbf{f}.$$

Then, using (12) the above equation reduces to

$$\mathbf{0} = -(\nabla_x p_0 + \nabla_y p_1) + \mu\Delta_y \mathbf{v}_0 \ + \rho_0 \mathbf{f}.$$

On the other hand, since $k = n$ we get, through the conservation of the mass, by identifying the coefficients of ε^n and ε^{n-1}, that

$$\frac{\partial \rho_0}{\partial t} + \operatorname{div}_x(\rho_0 \mathbf{v}_0) + \operatorname{div}_y(\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) = 0, \tag{12}$$

and

$$\operatorname{div}_y(\rho_0 \mathbf{v}_0) = 0. \tag{13}$$

Since ρ_0 is independent of y and obviously we are interested in the case

$$\rho_0(x, \tau) \neq 0, \tag{14}$$

and as $\mathbf{v}_0$ is Y-periodic we conclude that, for fixed x and τ,

$$\operatorname{div}_y \mathbf{v}_0 = 0 \quad \text{in } \theta.$$

So, at the local level the flow is incompressible. Now, we define the *mean operator*

$$\widetilde{\bullet} = \frac{1}{|Y|} \int_Y \bullet dy$$

and extend by zero all the functions defined on θ. The main result of this section is the following

THEOREM. *Assume* (14) *and* (11). *Then*

$$\delta\frac{\partial\rho_0}{\partial\tau}+\mathrm{div}_x(\rho_0\widetilde{\mathbf{v}_0})=0, \tag{15}$$

where

$$\delta=\frac{|\theta|}{|Y|}\ \textit{(the porosity of the medium).} \tag{16}$$

Moreover, there exists a constant symmetric and positively defined matrix $\mathbf{K}$ *such that*

$$\widetilde{\mathbf{v}}_0(x,\tau)=\frac{1}{\mu}\mathbf{K}[\rho_0(x,\tau)\mathbf{f}(x,\tau)-\nabla_x p_0(x,\tau)]. \tag{17}$$

Proof. It is clear that

$$\widetilde{\rho_0}(x,\tau)=\delta\rho_0(x,\tau).$$

Now we apply the mean operator to the equation (12). We have

$$\widetilde{\mathrm{div}_y\,\mathbf{v}_1}=\frac{1}{|Y|}\int_Y \mathrm{div}_y\,\mathbf{v}_1 dy=\frac{1}{|Y|}\int_{\partial Y}\mathbf{v}_1\cdot\mathbf{n}d\sigma=0$$

since $\mathbf{v}_1=\mathbf{0}$ on Γ and $\mathbf{v}_1$ is Y-periodic. Moreover

$$\begin{aligned}\widetilde{\nabla_y\rho_1\mathbf{v}_0}&=\frac{1}{|Y|}\int_Y\nabla_y\rho_1\mathbf{v}_0 dy\\ &=\frac{1}{|Y|}[\int_Y \mathrm{div}_y(\rho_1\mathbf{v}_0)dy-\int_Y\rho_1\,\mathrm{div}_y\,\mathbf{v}_0 dy]=\frac{1}{|Y|}\int_{\partial Y}\rho_1\mathbf{v}_0\cdot\mathbf{n}d\sigma=0\end{aligned}$$

and so we get equation (15) which is the macroscopic mass conservation of the homogenized fluid. In order to show (17) we point out that $\mathbf{v}_0$ solves the Stokes problem

$$\begin{cases}-\mu\Delta_y\mathbf{v}_0=-\nabla_y p_1+\mathbf{f}^* & \text{in } \theta\times(0,\infty),\\ \mathrm{div}_y\,\mathbf{v}_0=0 & \text{in } \theta\times(0,\infty),\\ \mathbf{v}_0=\mathbf{0} & \text{on } \Gamma\times(0,\infty),\\ \mathbf{v}_0 \text{ is } Y\text{-periodic,} & \end{cases}$$

where $\mathbf{f}^*=\rho_0\mathbf{f}-\nabla_x p_0$. So, $\mathbf{v}_0$ coincides with the unique weak solution in the sense of Leray (see, for instance [30]), i.e. $\mathbf{v}_0\in V_\theta$ and

$$\mu\int_Y\nabla_y\mathbf{v}_0\cdot\nabla_y\mathbf{w}dy=\int_\theta\mathbf{f}^*\cdot\mathbf{w}dy$$

for any $\mathbf{w} \in V_\theta$ where

$$V_\theta := \{\mathbf{w} \in \mathbf{H}^1(\theta) : \operatorname{div}_y \mathbf{w} = 0, \mathbf{w} \text{ is } Y\text{-periodic and } \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}.$$

As in [28, Proposition 2.1], if for $1 \leq i \leq N$ we define $\mathbf{v}^i\ (y)$, $\mathbf{v}^i \in V_\theta$, as the solutions of the auxiliary problems

$$\int_Y \nabla_y \mathbf{v}^i \cdot \nabla_y \mathbf{w} dy = \int_\theta w_i dy$$

assumed $\mathbf{w} = \sum w_i \mathbf{e}_i$, then, by linearity, we get that

$$\mathbf{v}_0 = \frac{1}{\mu}(\rho_0 f_i - \frac{\partial p_0}{\partial x_i})\mathbf{v}^i.$$

Thus, applying the mean operator we get that

$$v_{0j} = \frac{K_{ij}}{\mu}(\rho_0 f_i - \frac{\partial p_0}{\partial x_i}).$$

The fact that the permeability matrix $\mathbf{K} = (K_{ij})$ is symmetric and positively defined follows as Proposition 2.2 of [28]. ∎

COROLLARY 1. *Under the assumptions of the above theorem and the state law* (10) *we have that* ρ_0 *satisfies the quasilinear parabolic equation*

$$\delta\frac{\partial \rho_0}{\partial \tau} - \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_0 \nabla F^{-1}(\rho_0)) + \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_0^2 \mathbf{f}) = 0. \quad (18)$$

In particular, if $\mathbf{f} = \mathbf{0}, \delta = 1/\mu$ *and* $\mathbf{K} = \mathbf{I}$ (*the identity matrix*) *then*

$$\frac{\partial \rho_0}{\partial \tau} - \Delta\varphi(\rho_0) = 0, \quad (19)$$

where φ *is the increasing function defined as*

$$\varphi(s) := \int_0^s \frac{\sigma}{F'(F^{-1}(\sigma))} d\sigma.$$

Remark 1. The special expansion could be replaced by a standard one (i.e., with terms in ε and ε^0 for the velocity and without a macroscopic time scale) by including physical parameters suitably scaled at the microscopic equations as, for instance

$$\rho_\varepsilon(\varepsilon^k \frac{\partial \mathbf{v}_\varepsilon}{\partial t} + \varepsilon^k(\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon) = -\nabla p_\varepsilon + \mu\varepsilon^n \Delta \mathbf{v}_\varepsilon + \lambda \varepsilon^n \nabla(\operatorname{div} \mathbf{v}_\varepsilon) + \rho_\varepsilon \mathbf{f}.$$

The condition (11) means that (very small) viscosities $\mu\varepsilon^2$ and $\lambda\varepsilon^2$ are in good dimensional balance with the rest of the terms of the equation. If we do not assume the condition (11) then the Darcy's law may become integro-differential (see Lions [19] and Allaire [1]), nonlinear, or it may disappear as a deterministic law (see Section 7.4 of Sanchez-Palencia [28] and Mikelic [23]). Nonlinear Darcy's laws appears in a natural way in the study of Non-Newtonian flows in porous media (see Lions and Sanchez-Palencia [20] and the survey Mikelic [24]).

Remark 2. The above special expansion and the condition (11) may be justified by using Dimensional Analysis. In order to do that let us introduce some characteristics units $L,\ t_c, T_c, p_c,\ \rho_c, v_c, V_c$ for the macroscopic length, the time in the microscopic and macroscopic flow, the pressure, the density and the velocity in the microscopic and macroscopic flow respectively. We also introduce the dimensionless variables

$$\overline{x} = \frac{x}{L},\ \overline{t} = \frac{t}{t_c},\ \overline{\tau} = \frac{\tau}{T_c},\ \overline{p} = \frac{p}{p_c},\ \overline{\rho} = \frac{\rho}{\rho_c},\ \overline{\mathbf{v}_\varepsilon} = \frac{\mathbf{v}_\varepsilon}{v_c},\ \overline{\mathbf{v}_0} = \frac{\mathbf{v}_0}{V_c}.$$

Notice that the microscopic characteristic length is then given by εL. Thus we see that the microscopic momentum conservation equation becomes

$$\begin{aligned}(\rho_c \frac{v_c}{t_c}\overline{\rho}_\varepsilon)\frac{\partial \overline{\mathbf{v}}_\varepsilon}{\partial \overline{t}} + (\rho_c \frac{v_c^2}{\varepsilon L})\overline{\rho}_\varepsilon(\overline{\mathbf{v}}_\varepsilon \cdot \nabla)\overline{\mathbf{v}}_\varepsilon) \\ = -(\frac{\delta_c p}{\varepsilon L})\nabla \overline{p}_\varepsilon + (\mu \frac{v_c}{\varepsilon^2 L^2})\Delta \overline{\mathbf{v}}_\varepsilon + (\lambda \frac{v_c}{\varepsilon^2 L^2})\nabla(\operatorname{div} \overline{\mathbf{v}}_\varepsilon) + \rho_c \overline{\rho}_\varepsilon \mathbf{f},\end{aligned} \tag{20}$$

where $\delta_c p$ denotes the characteristic pressure changes. Since the Reynolds and Reynolds-Strouhal of the microscopic flow

$$Re = \frac{\rho_c v_c \varepsilon L}{\mu}, \quad ReSt = \frac{\rho_c v_c \varepsilon^2 L^2}{t_c}$$

are very small (remember that $\varepsilon << 1$) the material time derivative terms of the equation (20) can be neglected and we get that

$$-(\mu \frac{v_c}{\varepsilon^2 L^2})\Delta \overline{\mathbf{v}}_\varepsilon - (\lambda \frac{v_c}{\varepsilon^2 L^2})\nabla(\operatorname{div} \overline{\mathbf{v}}_\varepsilon) = -(\frac{\delta_c p}{\varepsilon L})\nabla \overline{p}_\varepsilon + \rho_c \overline{\rho}_\varepsilon \mathbf{f}. \tag{21}$$

Making

$$\frac{\delta_c p}{\varepsilon L} = \frac{p_c}{L}$$

we get, identifying the parameters of (21), that

$$v_c = \frac{p_c L}{(\mu + \lambda)} \varepsilon^2 \tag{22}$$

and so the significant terms of the microscopic velocity are of order two in ε such as is implied by the special expansion and the assumption (11). On the other hand, from (22) and the expansion for $\mathbf{v}_\varepsilon$we deduce that necessarily $V_c = \frac{c}{\varepsilon^2}$ for some constant c. Then arguing as before but now for the macroscopic mass conservation equation

$$\delta \frac{\partial \rho_0}{\partial \tau} + \mathrm{div}_x(\rho_0 \widetilde{\mathbf{v}_0}) = 0$$

we deduce that

$$\frac{\rho_c}{T_c} = \frac{\rho_c V_c}{L}.$$

So, we get

$$T_c = \frac{c/L}{\varepsilon^2}$$

which justifies the change of scale $\tau = \varepsilon^2 t$ previously assumed.

Remark 3. Notice that the macroscopic scaling $\tau = \varepsilon^2 t$ means, qualitatively, that the homogenized equation is obtained for times asymptotically large in the microscopic time scale (for instance, $\tau = 1$ corresponds to $t = 1/\varepsilon^2$).

Remark 4. In the mathematical literature on the case of nonstationary incompressible flows in porous media it also used a quasilinear parabolic equation of the type (19) nevertheless its justification via homogenization theory is not clear. In the case of two miscible fluids in a porous medium the presence of nonlinear terms is due to capillary effects leading to jump pressure empirical relations of the type

$$p^1 - p^2 = G(c^1)$$

where p^i is the pressure of the $i-$fluid and c^1 is the concentration of one of the fluids (see Antontsev, Kazhikhov and Monhakhov [2], Auriault and Sanchez-Palencia [4], Gagneux and Madaune-Tort [13] and the survey Bourgat [7]). We also point out that our point of view is different to the one considered by other authors in which the homogenization process is applied to some microscopic quasilinear parabolic equations (see, e.g. Artola [3] and Damlamian [9]).

Remark 5. As already mentioned, a rigorous proof of the convergence, for the case of stationary incompressible fluids, was given in Tartar [29] (see also the survey Allaire [1] indicating some improvements but always for incompressible fluids). The main difficulty in this treatment is that is not enough to obtain a priori estimates for $(\mathbf{v}_\varepsilon/\varepsilon^2, p_\varepsilon)$ (which are independent of ε) since they concerns with a functional space, $\mathbf{H}_0^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, which varies with ε. Therefore, $(\mathbf{v}_\varepsilon, p_\varepsilon)$ needs to be extended to the whole homogenized set Ω. The extension of $\mathbf{v}_\varepsilon$ is obvious (we take the value $\mathbf{0}$ outside Ω_ε). In the case of the pressure p_ε we take the value

$$\frac{1}{|\theta|}\int_\theta p_\varepsilon.$$

We conjecture that this type of extension and the recent results of P.L. Lions [21] (see also P.L. Lions and Masmoudi [22]) will allow to get a rigorous proof of the above Theorem.

3. On the homogenized reaction between a fluid and a solid chemical specie on the walls of a porous medium

Idea of the proof of Theorem 1. By multiplying by u_ε, integrating by parts and using the monotonicity of the function $u \to |u|^{p-1}u$ we get that there exists $M > 0$, independent of ε, such that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq M, \ \forall \varepsilon > 0. \tag{23}$$

By assumption the extension operators are continuous i.e.,

$$\int_\Omega |\nabla P_\varepsilon(v)|^2\,dx \leq C\int_{\Omega_\varepsilon} |\nabla v|^2\,dx, \ \forall v \in V_\varepsilon, \ \forall \varepsilon.$$

So, there exists a subsequence (labeled as $P_{\varepsilon'}u_{\varepsilon'}$) such that $P_{\varepsilon'}u_{\varepsilon'}$ converges, (weakly) in $H_0^1(\Omega)$, as $\varepsilon' \to 0$, to a function $u_0 \in H_0^1(\Omega)$ (in fact, since problems (6) and (7) have a unique solution the above convergence holds for the complete sequence $P_\varepsilon u_\varepsilon$). It remains to show that u_0 is a weak solution of (7). To do that we define

$$\widetilde{\xi}_\varepsilon = \begin{cases} \nabla u_\varepsilon, & \text{in } \Omega_\varepsilon, \\ 0, & \text{in } \Omega - \Omega_\varepsilon. \end{cases}$$

It is clear that $\widetilde{\xi}_{\varepsilon'} \rightharpoonup \xi$, in $L^2(\Omega)$-weakly, as $\varepsilon' \to 0$, and that

$$\int_\Omega \widetilde{\xi}_{\varepsilon'} \cdot \nabla\varphi dx + \alpha\varepsilon' \int_{\Gamma_\varepsilon} |u_{\varepsilon'}|^{p-1} u_{\varepsilon'}\varphi ds = \int_\Omega \mathcal{X}_{\Omega_{\varepsilon'}} f\varphi dx, \tag{24}$$

where $\mathcal{X}_{\Omega_{\varepsilon'}}$ is the characteristic function of $\Omega_{\varepsilon'}$. By standard homogenization techniques, it is possible to show that

$$\begin{cases} \mathcal{X}_\Omega \rightharpoonup \frac{\text{meas}\,(Y^*)}{\text{meas}\,(Y)} & \text{in } L^q(\Omega) - \text{weakly}, \ \forall\ 1 \le q < \infty, \ \text{as } \varepsilon' \to 0 \\ \mathcal{X}_\Omega \rightharpoonup \frac{\text{meas}\,(Y^*)}{\text{meas}\,(Y)} & \text{in } L^\infty(\Omega) - \text{weakly-star, as } \varepsilon' \to 0. \end{cases}$$

On the other hand, thanks to the assumption $p < 1$ we get that

$$\lim_{\varepsilon'\to 0} \int_{\Gamma_\varepsilon} |u_{\varepsilon'}|^{p-1} u_{\varepsilon'}\varphi ds = \frac{\text{meas}(\partial\theta)}{\text{meas}(Y)} \int_\Omega |u_0|^{p-1} u_0\varphi dx.$$

So, at the limit

$$-\operatorname{div}\xi + \alpha\delta\,|u_0|^{p-1} u_0 = f, \text{ in } \Omega.$$

In order to obtain an expression of ξ in terms of ∇u_0, for $i = 1, .., N$, we introduce the auxiliary cellular problems

$$\begin{cases} -\Delta_y \mathcal{X}_i = 0 & \text{in } Y^*, \\ -\dfrac{\partial \mathcal{X}_i}{\partial n_y} = n_i & \text{on } \partial\theta, \\ \mathcal{X}_i \ Y^*\text{-periodic in } y \end{cases}$$

and then the functions

$$\phi_{i\varepsilon}(x) = \varepsilon[\mathcal{X}_i(\frac{x}{\varepsilon}) + y_i], \ \forall x \in \Omega_\varepsilon$$

and

$$\eta_i^\varepsilon = \nabla\phi_{i\varepsilon}.$$

It is not difficult to show that if $\widetilde{\eta_i^\varepsilon}$ denotes the corresponding extension by zero to $\Omega - \Omega_\varepsilon$, then

$$\begin{cases} \left(\widetilde{\eta_i^\varepsilon}\right)_j \rightharpoonup \frac{1}{\text{meas}(Y)}(\int_{Y^*} \frac{\partial \mathcal{X}_i}{\partial y_i} dy + \text{meas}(Y^*)\delta_{ij}) \\ \qquad := \frac{\text{meas}(Y^*)}{\text{meas}(Y)} q_{ij} \text{ in } L^2(\Omega) - \text{weakly}. \end{cases}$$

Finally, using that (after some technical arguments)

$$-\int_\Omega \xi \cdot \nabla x_i \varphi dx + \frac{\text{meas}(Y^*)}{\text{meas}(Y)} \int_\Omega q_i \cdot \nabla u_0 \varphi dx = 0$$

the conclusion follows. ∎

Remark 6. After proving the above result (as consequence of the visit of C. Conca to the Universidad Complutense de Madrid, in November of 1986), we become aware of some related results in the literature, mainly the papers U. Hornung and W. Jäger [15], [16]. In those papers, the authors consider a very general formulation which contain problem (6) as an special case but under the structural assumption $p \geq 1$. The case $p < 1$ is left there as an open problem and so the above theorem is not covered by their results.

Remark 7. The reaction term on the boundary of the particles

$$-\frac{\partial u_\varepsilon}{\partial n} = \alpha\varepsilon \left|u_\varepsilon\right|^{p-1} u_\varepsilon \quad \text{on } \Gamma_\varepsilon,$$

is a simplification of a more complicated situation. In fact, each of the particles is also a porous medium and so there is a diffusion and reaction at its interior. In the case of a spherically symmetric isolated particle it is possible to express $\partial u_\varepsilon / \partial n$ in terms of the value of u_ε on Γ_ε once that one assumes that at the interior of the particle there is a chemical reaction of the type

$$-\Delta u_\varepsilon + \alpha\varepsilon \left|u_\varepsilon\right|^{m-1} u_\varepsilon = 0, \text{ in } \theta$$

(see Vega and Liñán [31]).

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