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LECTURES ON MODULI SPACES OF VECTOR BUNDLES ON ALGEBRAIC VARIETIES

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1. Introduction

These notes are the expanded and detailed notes of the lectures given by the author during the school entitled "School on vector bundles and low codimensional subvarieties", held at CIRM, Trento, (Italy), during the period September 11-16, 2006. In no case do I claim it is a survey on moduli spaces of vector bundles on algebraic projective varieties. Many people have made important contributions without even being mentioned here and I apologize to those whose work I made have failed to cite properly. The author gave 5 lectures of length 90 minutes each. She attempted to cover the basic facts on moduli spaces of vector bundles over smooth projective varieties. Given the extensiveness of the subject, it is not possible to go into great detail in every proof. Still, it is hoped that the material that she chose will be beneficial and illuminating for the participants, and for the reader.

Moduli spaces are one of the fundamental constructions of algebraic geometry and they arise in connection with classification problems. Roughly speaking a moduli space for a collection of objects A and an equivalence relation $\sim$ is a classification space, i.e. a space (in some sense of the word) such that each point corresponds to one, and only one, equivalence class of objects. Therefore, as a set, we define the moduli space as equivalence classes of objects $A/\sim$. In our setting the objects are algebraic objects, and because of

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this we want an algebraic structure on our classification set. Finally, we want our moduli space to be unique (up to isomorphism).

General facts on moduli spaces can be found, for instance, in [59], [60] or [66] (see also [55] and [56]). In this paper, we will restrict our attention to moduli spaces of stable vector bundles on smooth, algebraic, projective varieties. We have attempted to give an informal presentation of the main results, addressed to a general audience.

A moduli space of stable vector bundles on a smooth, algebraic variety X is a scheme whose points are in "natural bijection" to isomorphic classes of stable vector bundles on X. The phrase "natural bijection" can be given a rigorous meaning in terms of representable functors. Using Geometric Invariant Theory the moduli space can be constructed as a quotient of certain Quot-scheme by a natural group action.

Once the existence of the moduli space is established, the question arises as what can be said about its local and global structure. More precisely, what does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry? Until now, there is no a general answer to these questions and the goal of these notes is to review some of the known results which nicely reflect the general philosophy that moduli spaces inherit a lot of properties of the underlying variety; essentially when the underlying variety is a surface.

Next we outline the structure of the course. In Lecture 1, we introduce the crucial concept of stability of vector bundles over smooth projective varieties, we give a cohomological characterization of the (semi)stability and we investigate the stability of a number of vector bundles. The notion of (semi)stability is needed to ensure that the set of vector bundles one wants to parameterize is small enough to be parameterized by a scheme of finite type. In Lecture 2, we introduce the formal definition of moduli functor, fine moduli space and coarse moduli space and we recall some generalities on moduli spaces of vector bundles all of them well-known to the experts on this field. Lecture 3 deals with vector bundles on algebraic surfaces. Quite a lot is known in this case and we will review the main results some of them will illustrate how the geometry of the surface is reflected in the geometry of the moduli space. In section 5, we introduce the notion of monad. Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ can be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \oplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \oplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

Monads appeared in a wide variety of contexts within algebraic geometry, and they are very useful when we want to construct vector bundles with prescribed invariants like rank, determinant, Chern classes, etc. In Lecture 4, we will mainly study linear monads (see definition 5.3). The last lecture is devoted to moduli spaces of vector bundles on higher dimensional varieties. Very few results are known. As we will stress, the situation drastically differs and results like the smoothness and irreducibility of moduli spaces of stable vector bundles on algebraic surfaces turn to be false for moduli spaces of stable vector bundles on higher dimensional algebraic varieties. We could not resist to discuss some details that perhaps only the experts will care about, but hopefully will also introduce the non-expert reader to subtle subject. To this end, we present new results on moduli spaces of stable vector bundles on rational normal scrolls of arbitrary dimension (see [10]) with the hope to find a clue which could facilitate the study of moduli spaces of stable vector bundles on arbitrary n-dimensional varieties.

During the School Laura Costa gave two exercise sessions where she made examples and exercises and she introduced some topics that are complementary to the lectures. We collect these exercises in Section 7.

Throughout these notes I have mentioned various open problems. Some of them and further problems related to moduli spaces of vector bundles on smooth projective varieties are collected in the last section of these notes.

The author is grateful to the main organizers of this School/Workshop, Gianfranco Casnati, Claudio Fontanari, Roberto Notari and Maria Luisa Speafrico for giving her the opportunity to speak about one of her favorite subjects: Moduli spaces of vector bundles on algebraic varieties. She is also grateful to the participants for their hospitality and mathematical discussions that made for a very interesting and productive week in the lovely city of Trento.

Notation Let $(X, \mathcal{O}_X(1))$ be a polarized irreducible smooth projective variety over an algebraically closed field k of characteristic zero. Recall that the Euler characteristic of a locally free sheaf E on X is $\chi(E) := \sum_i (-1)^i h^i(X,E)$ where $h^i(X,E) = dim_k H^i(X,E)$. The Hilbert polynomial $P_E(m)$ is given by $m \to \chi(E \otimes \mathcal{O}_X(m))/rk(E)$.

2. Lecture One: Stability and its properties

In this lecture, we introduce the notion of stability of vector bundles on irreducible, smooth projective varieties and its basic properties. We give a cohomological characterization of the stability, we investigate the stability of a number of vector bundles and the restrictions that the stability imposes on the Chern classes of vector bundles.

Definition 2.1. Let X be a smooth algebraic variety. A *linear fibration* of rank r on X is an algebraic variety E and a surjective map $p : E \longrightarrow X$ such that for each $x \in X$, $p^{-1}(x)$ is a k-vector space of rank r.

Given two fibrations $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$, a morphism of varieties $f : E \longrightarrow E'$ is a map of linear fibrations if it is compatible with the projections p and p', i.e. $p'f = p$, and, for each $x \in X$, the induced map $f_x : E_x \longrightarrow E'_x$ is linear. The bundle $X \times k^r \longrightarrow X$ given by projection to the first factor is called the *trivial fibration* of rank r. For each open set $U \subset X$, we write $E_{|U}$ for the fibration $p^{-1}(U) \longrightarrow U$ given by restriction to U.

An *algebraic vector bundle* of rank r on X is a linear fibration $E \longrightarrow X$ which is locally trivial, that is, for any $x \in X$ there exists an open neighborhood U of x and an isomorphism of fibrations $\varphi : E_{|U} \longrightarrow U \times k^r$.

Definition 2.2. Let $p : E \longrightarrow X$ be a vector bundle of rank r over an algebraic variety X. We define a *regular section* of E over an open subset $U \subset X$ to be a morphism $s : U \longrightarrow E$ of algebraic varieties such that $p(s(x)) = x$ for all $x \in U$.

The set $\Gamma(U, E)$ of regular sections of E over U has a structure of module over the algebra $\mathcal{O}_X(U)$. So, we obtain a sheaf of $\mathcal{O}_X$-modules $\mathcal{E} = \mathcal{O}_X(E)$ over X, locally isomorphic to $\mathcal{O}_X^r$; i.e. a locally free sheaf of rank r.

Proposition 2.3. *The functor which associates the locally free sheaf $\mathcal{E} = \mathcal{O}_X(E)$ to a vector bundle E on X is an equivalence of categories between the category of vector bundles of rank r over X and the category of locally free sheaves of rank r on X.*

Proof. See [32]; Chapter II, Exercise 5.18. □

Remark 2.4. From now on, we will not distinguish between a vector bundle E on X and its locally free sheaf of sections.

As we said in the introduction, we would like to endow the set of vector bundles on X with a natural algebraic structure. More precisely, we would like to classify holomorphic structures on a given topological vector bundle. In general, we cannot expect that the set of all vector bundles E on X with fixed rank $r \in \mathbb{Z}$ and fixed Chern classes $c_i = c_i(E) \in H^{2i}(X, \mathbb{Z})$ have a nice natural structure of algebraic variety. The necessity of restricting to some subset of the set of all vector bundles can be understood by the following well-known example.

Example 2.5. We consider the set of vector bundles $\{\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)\}_{n\in\mathbb{Z}}$ over $\mathbb{P}^1$. Since $rk(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) = 2$ and $c_1(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) = 0$, the topological invariants are fixed but we have infinitely many points which cannot form a nice algebraic variety. Moreover, there exists a family of vector bundles $\{E_t\}_{t\in k}$ such that $E_t \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for $t \neq 0$ and $E_0 \cong \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. So the point corresponding to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ would not be a closed point in the moduli space.

The natural class of vector bundles which admits a nice natural algebraic structure comes from Mumford's Geometric Invariant Theory. The corresponding vector bundles are called stable vector bundles. However, if we want to get a projective moduli space then we need to add some non-locally free sheaves at the boundary of the moduli space. So, we need to define semistability and stability in the more general setting.

Definition 2.6. Let X be a smooth irreducible projective variety of dimension d and let H be an ample line bundle on X. For a torsion free sheaf F on X one sets

$$\mu_H(F) := \frac{c_1(F)H^{d-1}}{rk(F)}, \qquad P_F(m) := \frac{\chi(F \otimes \mathcal{O}_X(mH))}{rk(F)}.$$

The sheaf F is **μ-semistable** (resp. **GM-semistable**) with respect to the polarization H if and only if

$$\mu_H(E) \leq \mu_H(F) \qquad (\text{resp. } P_E(m) \leq P_F(m) \quad for \quad m \gg 0)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then F is **μ-stable** (resp. **GM-stable**) with respect to H.

We will simply say μ-(semi)stable (resp (semi)-stable) when there is no confusion on H. One easily checks the implications

$$\mu - stable \Rightarrow GM - stable \Rightarrow GM - semistable \Rightarrow \mu - semistable.$$

The notion of μ-stability was introduced for vector bundles on curves by Mumford and later generalized to sheaves on higher dimensional varieties by Takemoto, Gieseker, Maruyama and Simpson. The notion of stability is natural from an algebraic point of view as well as from a gauge theoretical point of view, for there is a deep relation between stability of vector bundles and existence of Hermite-Einstein metrics. This relation is known as the Kobayashi-Hitchin correspondence and was established by works of Narasimhan-Seshadri ([65]), Donaldson ([17], [18]) and Uhlenbeck-Yau ([79]).

Remark 2.7. The definition of stability depends on the choice of the ample line bundle H. The changes of the moduli space that occur when the line bundle H varies have been studied by several people in greater detail often with respect to their relation to Gauge theory and the computation of Donaldson polynomials (see, for instance, [21], [25], [72] and [73]). See next Example and Lectures 3 and 4 for more details.

Example 2.8. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a quadric surface. We denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. So, $K_X = -2\ell - 2m$, $\ell^2 = m^2 = 0$ and $\ell m = 1$. Let E be a rank 2 vector bundle on X given by a non-trivial extension:

$$0 \neq e: \qquad 0 \longrightarrow \mathcal{O}_X(\ell - 3m) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \longrightarrow 0.$$

The non-zero extension $e \in \operatorname{Ext}^1(\mathcal{O}_X(3m), \mathcal{O}_X(\ell - 3m))$ exists because

$$\operatorname{Ext}^1(\mathcal{O}_X(3m), \mathcal{O}_X(\ell - 3m)) \cong H^1(X, \mathcal{O}_X(\ell - 6m)) \cong k^{10}.$$

We easily check that $c_1(E) = \ell$ and $c_2(E) = 3$. We now consider the ample line bundles $L = \ell + 5m$ and $L' = \ell + 7m$ on X. We claim:

- E is not μ-semistable with respect to L', and
- E is μ-stable with respect to L.

In fact, E is not μ-stable with respect to L' because $\mathcal{O}_X(\ell - 3m)$ is a rank 1 subbundle of E and

$$c_1(\mathcal{O}_X(\ell - 3m))L' > \frac{c_1(E)L'}{2}.$$

Indeed, $c_1(\mathcal{O}_X(\ell - 3m))L' = (\ell - 3m)(\ell + 7m) = 4$ and $\frac{c_1(E)L'}{2} = \frac{\ell(\ell+7m)}{2} = \frac{7}{2}$.

Let us check that E is μ-stable with respect to L, i.e., for any rank 1 subbundle $\mathcal{O}_X(D)$ of E we have

$$c_1(\mathcal{O}_X(D))L < \frac{c_1(E)L}{2} = \frac{\ell(\ell + 5m)}{2} = \frac{5}{2}.$$

Since E sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_X(\ell - 3m) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \longrightarrow 0$$

we have

(a) $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(\ell - 3m)$, or
(b) $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(3m)$.

In the first case $\ell - 3m - D$ is an effective divisor. Since L is an ample line bundle on X, we have $(\ell - 3m - D)L \geq 0$ and $c_1(\mathcal{O}_X(D))L = DL \leq (\ell - 3m)L = (\ell - 3m)(\ell + 5m) = 2 < \frac{5}{2} = \frac{c_1(E)L}{2}$.

If $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(3m)$ then $3m - D$ is an effective divisor. Write $D = \alpha\ell + \beta m$. Since $3m-D$ is an effective divisor, we have $\alpha \leq 0$ and $\beta \leq 3$. On the other hand, $(\alpha, \beta) \neq (0,3)$ because the extension e does not split. Putting altogether we get $c_1(\mathcal{O}_X(D))L = DL = (\alpha\ell + \beta m)(\ell + 5m) = 5\alpha + \beta < \frac{5}{2} = \frac{c_1(E)L}{2}$ which proves what we want.

Restricting our attention to μ-semistable torsion free sheaves is not very restrictive since any torsion-free sheaf has a canonical increasing filtration with μ-semistable torsion-free quotients, the so-called *Harder-Narasimhan filtration*.

Proposition 2.9. *Let E be a torsion-free sheaf on a smooth projective variety X and let H be an ample line bundle on X. There is a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that

1. *all quotients $F_i = E_i/E_{i-1}$ are μ-semistable with respect to H, and*
2. *$\mu(F_1) > \mu(F_2) > \cdots > \mu(F_{m-1}) > \mu(F_m)$.*

This filtration is called the Harder-Narasimhan filtration *and we define $\mu_{max}(E) = \mu(F_1)$ and $\mu_{min}(E) = \mu(F_m)$.*

Proof. We consider the set $\{\mu(F) \mid F \subset E\}$. This set has a maximal element μ_{max} and the set $\{F \mid F \subset E, \mu(F) = \mu_{max}\}$ contains a torsion-free sheaf E_1 of largest rank. This element is the largest element in this set with respect to the inclusion relation and it is called the maximal destabilizing subsheaf of E.

Now we consider the maximal destabilizing subsheaf E_2' in E/E_1 and we set $E_2 = p^{-1}E_2'$ where $p : E \longrightarrow E/E_1$ is the natural projection. Reiterating this process we obtain a unique filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$$

with the required properties. □

We have the following simple properties of stability and semistability (see [69]:

1. Line bundles are always μ-stable.
2. The sum $E_1 \oplus E_2$ of two μ-semistable sheaves is μ-semistable if and only if $\mu(E_1) = \mu(E_2)$.
3. E is μ-semistable if and only if E^* is.
4. If E_1 and E_2 are μ-semistable with respect to H then $E_1 \otimes E_2$ is also μ-semistable with respect to H.
5. E is μ-semistable if and only if for any line bundle L, $E \otimes L$ is μ-semistable.
6. For rank r vector bundles with $(c_1(E)H^{d-1}, r) = 1$ the concepts of μ-stability and μ-semistability with respect to H coincides.

Notation 2.10. Let E be a rank r reflexive sheaf on $\mathbb{P}^n$. We set $E_{norm} := E(k_E)$ where k_E is the unique integer such that $c_1(E(k_E)) \in \{-r+1, \cdots, 0\}$.

For rank 2 reflexive sheaves on $\mathbb{P}^n$ we have the following useful stability criterion.

Lemma 2.11. *Let E be a rank 2 reflexive sheaf on $\mathbb{P}^n$. Then, E is μ-stable if and only if $H^0(\mathbb{P}^n, E_{norm}) = 0$. If $c_1(E)$ is even, then E is μ-semistable if and only if $H^0(\mathbb{P}^n, E_{norm}(-1)) = 0$.*

Proof. If $H^0(\mathbb{P}^n, E_{norm}) \neq 0$, then there is an injection $\mathcal{O}_{\mathbb{P}^n} \hookrightarrow E_{norm}$. Since $\mu(\mathcal{O}_{\mathbb{P}^n}) = 0 \geq \mu(E_{norm})$, we conclude that E_{norm} and thus also E is not μ-stable.

Conversely, suppose $H^0(\mathbb{P}^n, E_{norm}) = 0$. Let $F \subset E$ be a coherent sheaf of rank 1. Without lost of generality we can assume that E/F is torsion-free and hence F is a reflexive sheaf and even more a line bundle, $F = \mathcal{O}_{\mathbb{P}^n}(k)$. The inclusion $F \subset E$ defines a non-zero section s of $E(-k)$. Therefore, we must have $-k > k_E$ because we have assumed $H^0(\mathbb{P}^n, E_{norm}) = 0$ and we conclude that $\mu(F) = c_1(F) = k < -k_E \leq \mu(E)$.

In exactly the same way we can see that E is rank 2 μ-semistable reflexive sheaf if and only if $H^0(\mathbb{P}^n, E_{norm}(-1)) = 0$, provided $c_1(E)$ is even. □

The last result was generalized by Hoppe. In [36]; Lemma 2.6, he gave the following useful cohomological criterion for the stability of vector bundles.

Proposition 2.12. *Let E be a rank r locally-free sheaf on a smooth projective variety X with $Pic(X) = \mathbb{Z}$. We have:*

(a) *If $H^0(X, (\Lambda^q E)_{norm}) = 0$ for $1 \leq q \leq r-1$, then E is μ-stable.*
(b) *If $H^0(X, (\Lambda^q E)_{norm}(-1)) = 0$ for $1 \leq q \leq r-1$, then E is μ-semistable.*

Proof. (a) For a contradiction, assume that E is not μ-stable, and let F be the destabilizing reflexive sheaf of rank q, $1 \leq q \leq r-1$, with torsion-free quotient G. So, we have an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

and, moreover, $\mu(F) = \frac{c_1(F)}{rk(F)} \geq \frac{c_1(E)}{rk(E)} = \mu(E)$. The injective map $F \to E$ induces an injective map $\mathcal{O}_X(c_1(F)) = det(F) = (\wedge^q F)^{**} \hookrightarrow \wedge^q E$, determining a section in $H^0(X, \wedge^q E(-c_1(F)))$. Since $\mu(F) = \frac{c_1(F)}{rk(F)} \geq \frac{c_1(E)}{rk(E)} = \mu(E)$, it follows that $H^0(X, \wedge^q E_{norm}) \neq 0$, as desired.

(b) The second statement regarding μ-semistability is proved in exactly the same way. □

Remark 2.13. The conditions of the above proposition are not necessary. The simplest counterexamples are the nullcorrelation bundles N on $\mathbb{P}^n$ (n odd) where by a nullcorrelation bundle we mean a rank $n-1$ vector bundle N on $\mathbb{P}^n$ (n odd) defined by an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \Omega^1_{\mathbb{P}^n}(1) \longrightarrow N \longrightarrow 0.$$

N is a μ-stable (prove it!) vector bundle of rank $n-1$ on $\mathbb{P}^n$ (n odd) and $H^0(\mathbb{P}^n, (\wedge^2 N)_{norm}) \neq 0$ (in fact, $(\wedge^2 N)_{norm}$ contains $\mathcal{O}_{\mathbb{P}^n}$ as a direct summand).

Let us give some examples of stable bundles.

Example 2.14. (1) The cotangent bundle $\Omega^1_{\mathbb{P}^n}$ and the tangent bundle $T_{\mathbb{P}^n}$ are μ-stable. Since $T_{\mathbb{P}^n}$ is the dual of $\Omega^1_{\mathbb{P}^n}$, it suffices to show that $\Omega^1_{\mathbb{P}^n}$ is μ-stable. Applying Hoppe's criterion we have to see that

$$H^0(\mathbb{P}^n, (\wedge^q \Omega^1_{\mathbb{P}^n})_{norm}) = 0 \text{ for } 1 \leq q \leq rk(\Omega^1_{\mathbb{P}^n}) - 1 = n-1$$

(Proposition 2.12). But $(\wedge^q \Omega^1_{\mathbb{P}^n})_{\mathrm{norm}} = \Omega^q_{\mathbb{P}^n}(q)$ and by Bott's formula $H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(q)) = 0$ for all $1 \le q \le n-1$ which proves that $\Omega^1_{\mathbb{P}^n}$ is μ-stable.

(2) Let Y be the disjoint union of $d > 1$ lines in $\mathbb{P}^3$ and let E be the rank 2 vector bundle on $\mathbb{P}^3$ determined by a general non-zero section $s \in H^0\omega_Y(2) = \mathrm{Ext}^1(I_Y(2), \mathcal{O}_{\mathbb{P}^3})$. So, E sits in an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow E \longrightarrow I_Y(2) \longrightarrow 0$$

and $c_1(E) = 2$. Therefore, $E_{norm} = E(-1)$ and

$$H^0(\mathbb{P}^3, E_{norm}) = H^0(\mathbb{P}^3, E(-1)) = H^0(\mathbb{P}^3, I_Y(1)) = 0$$

because Y is not planar. Applying Lemma 2.11, we conclude that E is μ-stable. For a generalization of this example see Exercise 7.11.

(3) The Schwarzenberger bundle S_k over $\mathbb{P}^n$ defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{k+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+k+1} \longrightarrow S_k \longrightarrow 0 \tag{2.1}$$

is μ-stable. Indeed, we consider the long exact sequence of symmetric powers associated to the exact sequence (2.1), twisted by $\mathcal{O}_{\mathbb{P}^n}(-1)$:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-q-1)^{\binom{k+q}{q}} \to \mathcal{O}_{\mathbb{P}^n}^{n+k+1} \otimes \mathcal{O}_{\mathbb{P}^n}(-q)^{\binom{k+q-1}{q-1}} \to \cdots$$

$$\to \wedge^{q-1}\mathcal{O}_{\mathbb{P}^n}^{n+k+1} \otimes \mathcal{O}_{\mathbb{P}^n}(-2)^{k+1} \to \wedge^q \mathcal{O}_{\mathbb{P}^n}(-1)^{n+k+1} \to \wedge^q S_k(-1) \to 0\ .$$

Cutting it into short exact sequences and passing to cohomology, we obtain

$$H^0(\mathbb{P}^n, \wedge^q S_k(-1)) = 0 \text{ for } 1 \le q < n = rk(S_k).$$

On the other hand, we have $\mu(\wedge^q S_k) = q(k+1)/n > 0$, hence $(\wedge^q S_k)_{norm} = (\wedge^q S_k)(t)$ for some $t \le -1$ and we have

$$H^0((\wedge^q S_k)_{norm}) = 0 \text{ for all } 1 \le q \le n-1\ . \tag{2.2}$$

Therefore, S_k is μ-stable by Hoppe's criterion (see Proposition 2.12). For a generalization of this example see Exercise 7.12.

To end this first lecture let us consider the following question which naturally arises in our context:

Question 2.15. Let X be a smooth projective variety of dimension n. Which Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$ are realized by μ-(semi-)stable vector bundles E on X?

A partial answer to the above question is given by Bogomolov's inequality.

Theorem 2.16. *Let X be a smooth projective variety of dimension $n \ge 2$ and let H be an ample line bundle on X. Then for any rank r torsion-free μ-semistable with respect to H sheaf E on X we have*

$$\Delta(E) := (2rc_2(E) - (r-1)c_1^2(E))H^{n-2} \ge 0.$$

The class $\Delta(E) := (2rc_2(E) - (r-1)c_1^2(E))H^{n-2}$ is called the discriminant of E.

Proof. Assume first that X is a surface. Notice that the double dual E^{**} of E is still μ-semistable, and the discriminant of E and E^{**} are related by $\Delta(E) = \Delta(E^{**}) + 2r length(E^{**}/E) \geq \Delta(E^{**})$. Hence replacing E by E^{**}, if necessary, we may assume that E is a locally free sheaf on X. Since $End(E)$ is also μ-semistable, $\Delta(End(E)) = 2r^2\Delta(E)$ and $c_1(End(E)) = 0$, replacing E by $End(E)$, if necessary, we may also assume that $c_1(E) = 0$ and E is isomorphic to its dual E^*.

Let $k \gg 0$ so that $k \cdot H^2 > HK_X$ and that there is a smooth curve $C \in |kH|$. Since for any integer m the symmetric power $S^m(E)$ of a μ-semistable locally free sheaf with $c_1(E) = 0$ is also a μ-semistable locally free sheaf and μ-semistable sheaves of negative slope have no global sections we have $H^0(S^mE(-C)) = 0$. Therefore, the standard exact sequence

$$0 \longrightarrow S^mE \otimes \mathcal{O}_X(-C) \longrightarrow S^mE \longrightarrow S^mE_{|C} \longrightarrow 0$$

and Serre's duality lead to the estimates:

$$h^0(S^mE) \leq h^0(S^mE(-C)) + h^0(S^mE_{|C}) = h^0(S^mE_{|C})$$

and

$$\begin{aligned} h^2(S^mE) = h^0((S^mE)^\vee \otimes K_X) = h^0((S^mE) \otimes K_X) \\ \leq h^0(S^mE \otimes K_X(-C)) + h^0(S^mE_{|C} \otimes K_X|_C) = h^0(S^mE_{|C} \otimes K_X|_C) \end{aligned}$$

where the last equality comes from the fact that $S^m(E)$ is a μ-semistable locally free sheaf with $c_1(S^m(E)) = 0$ and $k \cdot H^2 > HK_X$. Therefore, we can bound the Euler characteristic of S^mE by

$$\chi(S^mE) \leq h^0(S^mE) + h^2(S^mE) \leq h^0(S^mE_{|C}) + h^0(S^mE_{|C} \otimes K_X|_C).$$

Considering $\pi : Y = \mathbb{P}(E|_C) \longrightarrow C$ we see that

$$h^0(S^mE_{|C}) = h^0(Y, \mathcal{O}_{\mathbb{P}(E|_C)}(m))$$

by the projection formula. Since $dim(Y) = r$, there exists a constant A such that

$$h^0(S^mE) \leq h^0(Y, \mathcal{O}_{\mathbb{P}(E|_C)}(m)) \leq A \cdot m^r$$

for all $m > 0$. Similarly, one can show that there exists a constant B such that

$$h^2(S^mE) = h^0((S^mE)^* \otimes K_X) \leq B \cdot m^r$$

for all $m > 0$. Therefore,

$$\chi(S^mE) \leq h^0(S^mE) + h^2(S^mE) \leq (A + B) \cdot m^r. \tag{2.3}$$

On the other hand, by Exercise 7.13, we have

$$\chi(S^mE) = -\frac{\Delta(E)}{2r}\frac{m^{r+1}}{(r+1)!} + \text{ terms of lower order in } m. \tag{2.4}$$

If $\Delta(E)$ were negative, this would contradict (2.3).

The case of μ-semistable torsion-free sheaves E on higher dimensional varieties follows from the case of μ-semistable torsion-free sheaves on surfaces taking into account that the restriction of E to a general complete intersection $Y := D_1 \cap \cdots \cap D_{n-2}$ with $D_i \in |aH|$ and

$a \gg 0$ is again a μ-semistable torsion-free sheaf (Theorem of Mehta and Ramamathan, see [52] and [53]) and $a^{n-2}\Delta(E) = \Delta(E|_Y)$. $\square$

Bogomolov's inequality $\Delta(E) := (2rc_2(E) - (r-1)c_1^2(E))H^{n-2} \geq 0$ if E is a μ-semistable vector bundle on a smooth projective variety was first proved for vector bundles on a surface over a characteristic zero field by Bogomolov. As we have seen the result can easily be generalized to higher dimensional smooth projective varieties over a characteristic zero field using the Mumford-Mehta-Ramanathan's restriction theorem which says that the restriction of a μ-semistable sheaf to a general hypersurface of sufficiently large degree is still μ-semistable. Bogomolov's inequality was generalized by Shepherd-Barron [76], Moriwaki [58] and Megyesi [51] to positive characteristic but only in the surface case and recently by Langer [39] in the higher dimensional case.

In [19], Douglas, Reinbacher and Yau inspired by superstring theory made the following conjecture which is a slight strengthening of Bogomolov's inequality.

Conjecture 2.17. *Let X be a simply connected surface with ample or trivial canonical line bundle. Then, the Chern classes of any stable vector bundle E on X of rank $r \geq 2$ obey*

$$2rc_2(E) - (r-1)c_1^2(E) - \frac{r^2}{12}c_2(X) \geq 0.$$

In [14], we provide two kinds of counterexamples. The first one concerning rank two vector bundles on a generic $K3$ surface X (i.e. on a generic algebraic surface X with $q(X) = 0$ and trivial canonical line bundle). The second one is devoted to rank $r \geq 3$ vector bundles on a surface X in $\mathbb{P}^3$ of degree $d \geq 7$ (and hence its canonical line bundle is ample). More precisely, we have

Proposition 2.18. *Let X be a generic $K3$ surface and H an arbitrary ample line bundle on X. For any Mukai vector $v = (r, c_1, \frac{c_1^2}{2} + r - c_2)$ such that $(r, c_1H) = 1$ and*

$$2rc_2 - (r-1)c_1^2 = 2r^2 - 2$$

there exist a μ_H-stable rank r vector bundle E on X with Mukai vector $v(E) = v$.

Proposition 2.19. *Let X be a smooth surface of degree $d \geq 7$ on $\mathbb{P}^3$. Then, there exist a rank $r \geq 3$ vector bundle F on X with Chern classes $c_1(F) = c_1$ and $c_2(F) = c_2$ verifying*

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) < 0.$$

3. Lecture Two: Moduli functor. Fine and coarse moduli spaces

In this lecture, we recall the formal definition of moduli functor, fine moduli space and coarse moduli space and we gather the results on moduli spaces of vector bundles on a smooth projective variety that are important to our study; all of them are well known to the experts.

The first step in the classification of vector bundles on smooth projective varieties is to determine which cohomology classes on a projective variety can be realized as Chern classes of vector bundles. On curves the answer is known. On surfaces the existence of vector bundles was settled by Schwarzenberger; and it remains open on higher dimensional varieties. The next step aims at a deeper understanding of the set of all vector bundles with a fixed rank and Chern classes. This naturally leads to the concept of moduli space which I will shortly recall.

Moduli spaces are one of the fundamental constructions of Algebraic Geometry and they arise in connection with classification problems. Roughly speaking, a moduli space for a collection of algebraic objects A and an equivalence relation $\backsim$ is a "space" (in some sense of the word) which parameterizes equivalence classes of objects in a "continuous way", i.e, it takes into account how the equivalence classes of objects change in one or more parameter families. In our setting, the objects are algebraic and therefore we want an algebraic structure on our classification space $A/\backsim$. Moreover, we want our moduli space to be unique up to isomorphism. Let us start with the formal definition of a moduli space.

Let $\mathcal{C}$ be a category (e.g., $\mathcal{C} = (Sch/k)$) and let $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ be a contravariant moduli functor (The precise definition of the moduli functor $\mathcal{M}$ depends on the particular classification problem we are dealing with).

Definition 3.1. We say that a moduli functor $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ is represented by an object $M \in Ob(\mathcal{C})$ if it is isomorphic to the functor of points of M, h_M, defined by $h_M(S) = \mathrm{Hom}_{\mathcal{C}}(S, M)$. The object M is called a *fine moduli space* for the moduli functor $\mathcal{M}$.

If a fine moduli space exists, it is unique up to isomorphism. Unfortunately, there are very few contravariant moduli functors for which a fine moduli space exits and it is necessary to find some weaker conditions, which nevertheless determine a unique algebraic structure on $A/\backsim$. This leads to the following definition.

Definition 3.2. We say that a moduli functor $\mathcal{M} : \mathcal{C} \longrightarrow (Sets)$ is corepresented by an object $M \in Ob(\mathcal{C})$ if there is a natural transformation $\alpha : \mathcal{M} \longrightarrow h_M$ such that $\alpha(\{pt\})$ is bijective and for any object $N \in Ob(\mathcal{C})$ and for any natural transformation $\beta : \mathcal{M} \longrightarrow h_N$ there exists a unique morphism $\varphi : M \longrightarrow N$ such that $\beta = h_\varphi \alpha$. The object M is called a *coarse moduli space* for the contravariant moduli functor $\mathcal{M}$.

Again, if a coarse moduli space exists, it is unique up to isomorphism. A fine moduli space for a given contravariant moduli functor $\mathcal{M}$ is always a coarse moduli space for this moduli functor but, in general, not vice versa. General facts on moduli spaces can be found, for instance, in [66] or [59].

Let us give an example.

Example 3.3. Let $\mathbb{P}^r$ be the r-dimensional projective space over a field k. We want to classify closed projective subschemes $X \subset \mathbb{P}^r$. Let us first recall the notion of flat family. A *flat family of closed subschemes* of $\mathbb{P}^r$ parameterized by a k-scheme S is a closed subscheme $\mathcal{X} \subset \mathbb{P}^r_S = \mathbb{P}^r \times S$ such that the morphism $\mathcal{X} \longrightarrow S$ induced by the projection $\mathbb{P}^r_S = \mathbb{P}^r \times S \longrightarrow S$ is flat. It is important to recall that flat families

$\mathbb{P}^r_S = \mathbb{P}^r \times S \supset \mathcal{X} \longrightarrow S$ of closed subschemes of $\mathbb{P}^r$ parameterized by a connected k-scheme S have all their fibres with the same Hilbert polynomial.

Therefore, we fix a polynomial $p(t) = \sum_{i=0}^{r} a_i \binom{t+r}{i} \in \mathbb{Q}[t]$ and we want to classify, modulo isomorphism, closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$. To this end, we consider the contravariant moduli functor

$$\underline{Hilb}^r_{p(t)} : (Sch/k) \longrightarrow (Sets)$$

defined by

$$\underline{Hilb}^r_{p(t)}(S) := \{ \text{ flat families } \mathcal{X} \text{ of closed subschemes of } \mathbb{P}^r \text{ with Hilbert polynomial } p(t) \text{ parameterized by } S\}.$$

In 1960, A. Grothendieck proved (See [31]; Théorème 3.2):

There exists a unique scheme $Hilb^r_{p(t)}$ which parameterizes a flat family

$$\pi : \mathcal{W} \subset \mathbb{P}^r \times Hilb^r_{p(t)} \longrightarrow Hilb^r_{p(t)}$$

of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, and having the following universal property: for every flat family $f : \mathcal{X} \subset \mathbb{P}^r_S = \mathbb{P}^r \times S \longrightarrow S$ of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, there is a unique morphism $g : S \longrightarrow Hilb^r_{p(t)}$, called the classification map for the family f, such that π induces f by base change; i.e. $\mathcal{X} = S \times_{Hilb^r_{p(t)}} \mathcal{W}$.

In the usual language of categories we say that the pair $(Hilb^r_{p(t)}, \pi)$ represents the moduli functor $\underline{Hilb}^r_{p(t)}$, π is the universal family and the classification problem for closed projective subschemes $X \subset \mathbb{P}^r$ has a fine moduli space. Once the existence of the Hilbert scheme $Hilb^r_{p(t)}$ is established we would like to know what does the Hilbert scheme look like? Is it smooth or irreducible? What is its dimension? etc. In spite of the great progress made during the last decades in the problem of studying the local and global structure of the Hilbert scheme $Hilb^r_{p(t)}$, there is no general answer to all these questions.

From now on we will deal with the problem of classifying vector bundles on smooth, irreducible, projective varieties. So, we are interested in providing the set of isomorphic classes of vector bundles on a smooth, irreducible, projective variety with a natural structure of scheme. This leads us to consider the following contravariant moduli functor:

Let X be a smooth, irreducible projective variety over an algebraically closed field k of characteristic zero. For a fixed polynomial $P \in \mathbb{Q}[z]$, we consider the contravariant moduli functor

$$\begin{aligned} \mathcal{M}^P_X(-) : (Sch/k) &\to (Sets) \\ S &\longmapsto \mathcal{M}^P_X(S), \end{aligned}$$

where $\mathcal{M}^P_X(S) = \{S-$ flat families $\mathcal{F} \longrightarrow X \times S$ of vector bundles on X all whose fibers have Hilbert polynomial $P\}/\sim$, with $\mathcal{F} \sim \mathcal{F}'$ if, and only if, $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$ for some $L \in Pic(S)$ being $p : S \times X \to S$ the natural projection. And if $f : S' \to S$ is a morphism in (Sch/k), let $\mathcal{M}^P_X(f)(-)$ be the map obtained by pulling-back sheaves via $f_X = f \times id_X$:

$$\begin{aligned} \mathcal{M}^P_X(f)(-) : \mathcal{M}^P_X(S) &\longrightarrow \mathcal{M}^P_X(S') \\ [\mathcal{F}] &\longmapsto [f^*_X \mathcal{F}]. \end{aligned}$$

Definition 3.4. A fine moduli space of vector bundles on X with Hilbert polynomial $P \in \mathbb{Q}[z]$ is a scheme M^P_X together with a family (Poincaré bundle) of vector bundles $\mathcal{U}$ on $M^P_X \times X$ such that the contravariant moduli functor $\mathcal{M}^P_X(-)$ is represented by $(M^P_X, \mathcal{U})$.

If M^P_X exists, it is unique up to isomorphism. Nevertheless, in general, the contravariant moduli functor $\mathcal{M}^P_X(-)$ is not representable. In fact, as we already pointed out before there are very few classification problems for which a fine moduli space exists. To get, at least, a coarse moduli space we must somehow restrict the class of vector bundles that we consider. What kind of vector bundles should we taken? In [46] and [47], M. Maruyama found an answer to this question: **stable** vector bundles (see Lecture One).

Definition 3.5. Let X be a smooth, irreducible, projective variety of dimension n over an algebraically closed field k of characteristic 0 and let H be an ample divisor on X. For a fixed polynomial $P \in \mathbb{Q}[z]$, we consider the contravariant moduli subfunctor $\mathcal{M}^{s,H,P}_X(-)$ of the contravariant moduli functor $\mathcal{M}^P_X(-)$:

$$\begin{aligned} \mathcal{M}^{s,H,P}_X(-) : (Sch/k) &\longrightarrow (Sets) \\ S &\longmapsto \mathcal{M}^{s,H,P}_X(S), \end{aligned}$$

where $\mathcal{M}^{s,H,P}_X(S) = \{$ S-flat families $\mathcal{F} \longrightarrow X \times S$ of vector bundles on X all whose fibers are μ-stable with respect to H and have Hilbert polynomial $P\}/\sim$, with $\mathcal{F} \sim \mathcal{F}'$ if, and only if, $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$ for some $L \in Pic(S)$, being $p : S \times X \longrightarrow S$ the natural projection.

In 1977, M. Maruyama proved:

Theorem 3.6. *The contravariant moduli functor $\mathcal{M}^{s,H,P}_X(-)$ has a coarse moduli scheme $M^{s,H,P}_X$ which is a separated scheme and locally of finite type over k. This means*

(1) *There is a natural transformation*

$$\Psi : \mathcal{M}^{s,H,P}_X(-) \longrightarrow Hom(-, M^{s,H,P}_X),$$

which is bijective for any reduced point x_0.

(2) *For every scheme N and every natural transformation $\Phi : \mathcal{M}^{s,H,P}_X(-) \longrightarrow Hom(-, N)$ there is a unique morphism $\varphi : M^{s,H,P}_X \longrightarrow N$ for which the diagram*

$$\begin{tikzcd} \mathcal{M}^{s,H,P}_X(-) \arrow[rr, "\Psi"] \arrow[dr, "\Phi"'] & & Hom(-, M^{s,H,P}_X) \arrow[dl, "\varphi_*"] \\ & Hom(-, N) & \end{tikzcd}$$

commutes.

In addition, $M^{s,H,P}_X$ decomposes into a disjoint union of schemes $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ where $n = dimX$ and $M^s_{X,H}(r; c_1, \cdots, c_{min(r,n)})$ is the moduli space of rank r μ-stable with respect to H vector bundles on X with Chern classes $(c_1, \cdots, c_{min(r,n)})$ up to numerical equivalence.

Proof. See [46]; Theorem 5.6. □

Remark 3.7. (1) If a coarse moduli space exists for a given classification problem, then it is unique (up to isomorphism). So, $M_X^{s,H,P}$ is unique (up to isomorphism).

(2) A fine moduli space for a given classification problem is always a coarse moduli space for this problem but, in general, not vice versa. In fact, there is no a priori reason why the map $\Psi(S) : \mathcal{M}_X^{s,H,P}(S) \rightarrow Hom(S, M_X^{s,H,P})$ should be bijective for varieties S other than $\{pt\}$.

It is one of the deepest problems in algebraic geometry to determine when the moduli space of μ-stable vector bundles on X, $M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)})$, is non-empty. If the underlying variety is a smooth projective curve C of genus $g \geq 2$, it is well known that the moduli space of μ-stable vector bundles of rank r and fixed determinant is smooth of dimension $(r^2-1)(g-1)$ ([37]; Corollary 4.5.5).

If the underlying variety X has dimension greater or equal to three, there are no general results which guarantee the non-emptiness of the moduli space of μ-stable vector bundles on X.

Finally, if the underlying variety is a smooth projective surface X, then the existence conditions are well known whenever X is $\mathbb{P}^2$ (see Exercise 7.15) or $\mathbb{P}^1 \times \mathbb{P}^1$ and, in general, it is known that the moduli space $M_{X,H}^s(r; c_1, c_2)$ of μ-stable with respect to H, rank r, vector bundles E on X with Chern classes $c_i(E) = c_i$ is empty if $\Delta(r; c_1, c_2) < 0$ (Bogomolov's inequality) and non-empty provided $\Delta(r; c_1, c_2) \gg 0$ (see for instance [37], [28], [46], [47] and [67]).

We refer to [37]; §4.5 for general facts on the infinitesimal structure of the moduli space $M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)})$. Let me just recall the results which are basic for us.

Proposition 3.8. *Let X be a smooth, irreducible, projective variety of dimension n and let E be a μ-stable vector bundle on X with Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$, represented by a point $[E] \in M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)})$. Then the Zariski tangent space of $M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)})$ at $[E]$ is canonically given by*

$$T_{[E]} M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)}) \cong Ext^1(E, E).$$

If $Ext^2(E, E) = 0$ then $M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)})$ is smooth at $[E]$. In general, we have the following bounds:

$$dim_k Ext^1(E, E) \geq dim_{[E]} M_{X,H}^s(r; c_1, \cdots, c_{min(r,n)})$$

$$\geq dim_k Ext^1(E, E) - dim_k Ext^2(E, E).$$

Proof. See for instance [37]; Theorem 4.5.2. □

If E is a locally free sheaf on X, then the trace map $tr : End(X) \longrightarrow \mathcal{O}_X$ induces maps $tr^i : Ext^i(E, E) \longrightarrow H^i(X, \mathcal{O}_X)$. We denote the kernel of tr^i by $Ext^i(E, E)_0$. Fix $L \in Pic(X)$ and denote by $\mathcal{M}_{X,H}^s(r; L, c_2, \cdots, c_{min(r,n)})$ the moduli space of rank r, μ-stable with respect to H vector bundles E with fixed determinant $det(E) = L \in Pic(X)$ and $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$ for $2 \leq i \leq min(r, n)$. Note that $\mathcal{M}_{X,H}^s(r; L, c_2, \cdots, c_{min(r,n)})$ is the fiber of the locally trivial fibration $M_{X,H}^s(r; c_1, c_2, \cdots, c_{min(r,n)}) \longrightarrow Pic(X)$, $E \mapsto$

$det(E)$. Hence,

$$q(X) = dimPic(X) =$$
$$dimM^s_{X,H}(r; c_1, c_2, \cdots, c_{min(r,n)}) - dim\mathcal{M}^s_{X,H}(r; L, c_2, \cdots, c_{min(r,n)}).$$

Concerning the moduli space $\mathcal{M}^s_{X,H}(r; L, c_2, \cdots, c_{min(r,n)})$, we have an analogous result on its infinitesimal structure.

Proposition 3.9. *Let X be a smooth projective variety of dimension n and let E be a μ-stable vector bundle on X with fixed $det(E) = L \in Pic(X)$ and Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$ for $2 \leq i \leq min(r, n)$. Then the Zariski tangent space of $\mathcal{M}^s_{X,H}(r; L, c_2, \cdots, c_{min(r,n)})$ at $[E]$ is canonically given by*

$$T_{[E]}\mathcal{M}^s_{X,H}(r; L, c_2, \cdots, c_{min(r,n)}) \cong Ext^1(E, E)_0.$$

If $Ext^2(E, E)_0 = 0$ then $\mathcal{M}^s_{X,H}(r; L, c_2, \cdots, c_{min(r,n)})$ is smooth at $[E]$. In general, we have the following bounds:

$$dim_k Ext^1(E, E)_0 \geq dim_{[E]}\mathcal{M}^s_{X,H}(r; L, c_2, \cdots, c_{min(r,n)})$$
$$\geq dim_k Ext^1(E, E)_0 - dim_k Ext^2(E, E)_0.$$

Proof. See for instance [37]; Theorem 4.5.4. □

• In case X is a smooth projective curve, we can make the above dimension bounds more explicit. Indeed, for any μ-stable vector bundle E on X, $Ext^2(E, E)_0$=0. Thus the moduli space is smooth and according to the above proposition its dimension is given by

$$dim_k Ext^1(E, E)_0 = -\chi(E, E) + \chi(\mathcal{O}_X) = (r^2 - 1)(g - 1)$$

where the last equality follows from Hirzebruch-Riemann-Roch's Theorem.

• In case X is a smooth surface, we can make the above dimension bounds more explicit. Indeed, for any μ-stable vector bundle $E \in \mathcal{M}^s_{X,H}(r; L, c_2)$ we have

$$dim_k Ext^1(E, E)_0 - dim_k Ext^2(E, E)_0 =$$
$$\chi(\mathcal{O}_X) - \sum_i (-1)^i ext^i(E, E) = \Delta(E) - (r^2 - 1)\chi(\mathcal{O}_X) =$$
$$2rc_2(E) - (r - 1)L^2 - (r^2 - 1)\chi(\mathcal{O}_X)$$

where the last equality follows from Hirzebruch-Riemann-Roch's Theorem. The number $2rc_2(E) - (r - 1)L^2 - (r^2 - 1)\chi(\mathcal{O}_X)$ is called the *expected dimension* of $\mathcal{M}^s_{X,H}(r; L, c_2)$.

Remark 3.10. In spite of the great progress made during the last decades in the problem of moduli spaces of μ-stable vector bundles on smooth projective varieties (essentially in the framework of the Geometric Invariant Theory by Mumford) a lot of problems remain open and for varieties of arbitrary dimension very little is known about their local and global structure. More precisely, what does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry?

See [66], [59] or [60] for the definition of categorical quotient of a variety by the action of a group and its connection with moduli problems.

We will end this second lecture with an example of moduli spaces of vector bundles.

Example 3.11. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a quadric surface. We denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. So, $K_X = -2\ell - 2m$, $\ell^2 = m^2 = 0$ and $\ell m = 1$.

We fix the ample line bundle $L = \ell + 5m$ on X and we denote by $M_{X,L}(2; \ell, 3)$ the moduli space of rank 2 vector bundles E on X with $det(E) = \ell \in Pic(X)$, $c_2(E) = 3 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, and μ-stable with respect to L. By Example 2.8, $M_{X,L}(2; \ell, 3)$ is non-empty.

Claim: $M_{X,L}(2; \ell, 3) \cong \mathbb{P}^9$

Proof of the Claim: First of all we observe that for any $E \in M_{X,L}(2; \ell, 3)$ we have $\chi E(-\ell + 3m) = 1$. In fact, $c_1E(-\ell + 3m) = -\ell + 6m$, $c_2E(-\ell + 3m) = 0$ and

$$\begin{aligned}\chi E(-\ell + 3m) &= 2 - \frac{c_1E(-\ell + 3m)K_X}{2} + \frac{c_1E(-\ell + 3m)^2 - 2c_2E(-\ell + 3m)}{2}\\ &= 2 - \frac{(-\ell + 6m)(-2\ell - 2m)}{2} + \frac{(-\ell + 6m)^2}{2} = 1.\end{aligned}$$

Since $\chi E(-\ell + 3m) = 1$, we have $h^0E(-\ell + 3m) > 0$ or $h^2E(-\ell + 3m) > 0$ and we will prove that the last inequality is not possible. Indeed, by Serre's duality, $0 < h^2E(-\ell + 3m) = h^0E^*(-\ell - 5m)$. A non-zero section $0 \neq \sigma \in H^0E^*(-\ell - 5m)$ defines an injection

$$\mathcal{O}_X(\ell + 5m) \hookrightarrow E^* \cong E(-\ell)$$

or, equivalently,

$$\mathcal{O}_X(2\ell + 5m) \hookrightarrow E.$$

Since E is μ-stable with respect to L we have

$$15 = (2\ell + 5m)(\ell + 5m) < \frac{c_1(E)L}{2} = \frac{\ell(\ell + 5m)}{2} = \frac{5}{2}$$

which is a contradiction. Therefore, $h^2E(-\ell + 3m) = 0$ and $h^0E(-\ell + 3m) > 0$.

A non-zero section $0 \neq s \in H^0E(-\ell + 3m)$ gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_X(\ell - 3m + D) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \otimes I_Z(-D) \longrightarrow 0$$

where $D = a\ell + bm$ is an effective divisor on X (hence $a, b \geq 0$) and $Z \subset X$ is a locally complete intersection 0-dimensional subscheme. Since E is μ-stable with respect to $L = \ell + 5m$, we have

$$(D + \ell - 3m)L = ((a + 1)\ell + (b - 2)m)(\ell + 5m) = 5a + 3b + 2 < \frac{c_1(E)L}{2} = \frac{5}{2}.$$

Hence, $a = b = 0$ and we conclude that any vector bundle sits into an exact sequence

$$0 \longrightarrow \mathcal{O}_X(\ell - 3m) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \otimes I_Z \longrightarrow 0$$

where $Z \subset X$ is a locally complete intersection 0-cycle of length $|Z| = c_2E(-\ell+3m) = 0$. Thus $Z = \emptyset$ and any vector bundle $E \in M_{X,L}(2; \ell, 3)$ is given by a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X(\ell - 3m) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \longrightarrow 0,$$

i.e., $E \in \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(3m), \mathcal{O}_X(\ell - 3m)))$. Since we have

$$\mathrm{Ext}^1(\mathcal{O}_X(3m), \mathcal{O}_X(\ell - 3m)) \cong H^1(X; \mathcal{O}_X(\ell - 6m)) \cong k^{10}$$

we conclude that $M_{X,L}(2;\ell,3) \cong \mathbb{P}^9$. In particular, $M_{X,L}(2;\ell,3)$ is smooth, irreducible, rational and it has dimension 9.

Remark 3.12. By [9]; Theorem 3.9, for any two ample divisors L_1 and L_2 on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, the moduli spaces $M_{X,L_1}(2;\ell,3)$ and $M_{X,L_2}(2;\ell,3)$ are birational whenever non-empty. Nevertheless, we want to point out that the birational map between $M_{X,L_1}(2;\ell,3)$ and $M_{X,L_2}(2;\ell,3)$ is not, in general, an isomorphism. See, Example 4.8, Exercise 7.14 or [10], Theorem 4.4.

Example 3.11 can be generalized and we have

Proposition 3.13. *Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a smooth quadric surface and denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. For any integer $0 < c_2 \in \mathbb{Z}$, we fix an ample divisor $L = \ell + (2c_2 - 1)m$. Then, we have*

(i) *The moduli space $M_{X,L}(2;\ell,c_2)$ is a smooth, irreducible, rational projective variety of dimension $4c_2 - 3$. Even more, $M_{X,L}(2;\ell,c_2) \cong \mathbb{P}^{4c_2-3}$.*

(ii) *For any two ample divisors L_1 and L_2 on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$, the moduli spaces $M_{X,L_1}(2;\ell,c_2)$ and $M_{X,L_2}(2;\ell,c_2)$ are birational whenever non-empty.*

Proof. (i) See [9]; Theorem 3.12 and Propositions 3.11 and 4.2.1.

(ii) See [9]; Theorem 3.9. □

4. Lecture Three: Moduli spaces of vector bundles on surfaces

Throughout this lecture X will be a smooth, irreducible, algebraic surface over an algebraically closed field of characteristic zero and we will denote by $M_{X,H}(r;L,n)$ (resp. $\overline{M}_{X,H}(r;L,n)$) the moduli space of rank r, vector bundles (resp. torsion free sheaves) E on X, μ-stable (resp. GM-semistable) with respect to an ample line bundle H with $det(E) = L \in Pic(X)$ and $c_2(E) = n \in \mathbb{Z}$. Moduli spaces for μ-stable vector bundles on smooth algebraic surfaces were constructed in the 1970's by M. Maruyama and quite a lot is known about them. In the 1980's, S. Donaldson proved that the moduli space $M_{X,H}(2;0,n)$ is generically smooth of the expected dimension provided n is large enough ([16]). As a consequence he obtained some spectacular new results on the classification of C^∞ four manifolds. Since then, many authors have studied the structure of the moduli space $M_{X,H}(r;L,n)$ from the point of view of algebraic geometry, of topology and of differential geometry; giving very pleasant connections between these areas.

Many interesting results have been proved and before recalling you some of them let me just give one example to show how the geometry of the surface is reflected in the geometry of the moduli space.

Example 4.1. Let X be a K3 surface. Then, the moduli space $M_{X,H}(r;L,n)$ is a smooth, quasi-projective variety of dimension $2rn-(r-1)L^2-2(r^2-1)$ with a symplectic structure. In addition, if $M_{X,H}(r;L,n)$ is 2-dimensional and compact then, it is isomorphic to a K3 surface isogenous to X (see [61] and [62]).

A more precise example could be the following:

Example 4.2. Let $X \subset \mathbb{P}^3$ be a general quartic hypersurface. X is a K3 surface and its Picard group is generated by the restriction, $\mathcal{O}_X(1)$, of the tautological line bundle on $\mathbb{P}^3$ to X. We have an isomorphism

$$\rho : X \cong M_{X,\mathcal{O}_X(1)}(2; \mathcal{O}_X(-1), 3)$$

which on closed points $y \in X$ is defined by $\rho(y) := F_y$ being F_y the kernel of the epimorphism $H^0(X, \mathcal{I}_y(1)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_y(1)$.

From now on, we will assume that the discriminant

$$\Delta(r; L, n) := 2rn - (r-1)L^2 \gg 0.$$

The moduli space is empty if $\Delta(r; L, n) < 0$, by Bogomolov's inequality and, on the other hand, it is non-empty if $\Delta(r; L, n) \gg 0$ (See, for instance, [47] and [26]). For small values of the discriminant $\Delta(r; L, n)$ the moduli space $M_{X,H}(r; L, n)$ of vector bundles on an algebraic surface X can look rather wild; there are many examples of moduli spaces which are not of the expected dimension, which are neither irreducible nor reduced (see, for instance, [26], [67] or [54]). This changes when the discriminant increases: the moduli spaces of vector bundles become irreducible, normal, of the expected dimension and the codimension of the singular locus increases. We have summarized these ideas in next Theorem which is one of the most important in the theory of vector bundles on an algebraic surface X.

Theorem 4.3. *Let X be a smooth, irreducible, projective surface and let H be an ample line bundle on X. If $\Delta(r; L, n) \gg 0$, then the moduli space $M_{X,H}(r; L, n)$ is a normal, generically smooth, irreducible, quasi-projective variety of dimension $2rn - (r-1)L^2 - (r^2-1)\chi(\mathcal{O}_X)$.*

Proof. Generic smoothness was first proved by S. K. Donaldson in [16] for rank 2 vector bundles with trivial determinant, and by K. Zuo in [82] for general determinants. Asymptotic irreducibility was proved for the rank 2 case by D. Gieseker and J. Li in [27], and for arbitrary ranks for D. Gieseker and J. Li in [28] and for K. O'Grady in [67]. Finally, asymptotic normality was proved by J. Li in [41]. □

Remark 4.4. In [40], A. Langer has generalized Donaldson, Gieseker, Li and O'Grady's results on generic smoothness and irreducibility of the moduli spaces of sheaves with fixed determinant and large discriminant to positive characteristic. He also shows optimal bounds for the Castelnuovo-Mumford regularity of sheaves on surfaces and he uses it to give the first general effective results on irreducibility of the moduli spaces.

Remark 4.5. For smooth, projective, anticanonical, rational surfaces (i.e. rational surfaces X whose anticanonical divisor $-K_X$ is effective) and for the rank 2 case, we can omit the hypothesis $\Delta(r; L, c_2) \gg 0$. The irreducibility and smoothness of $M_{X,H}(2; L, c_2)$ holds whenever $M_{X,H}(2; L, c_2)$ is non-empty. Indeed, assume $M_{X,H}(2; L, c_2) \neq \emptyset$. Since

$-K_X$ is effective, for any vector bundle $E \in M_{X,H}(2; L, c_2)$, we have $Ext^2(E,E)_0 = Hom(E, E \otimes K_X)_0 = 0$. Hence, $M_{X,H}(2; L, c_2)$ is smooth at $[E]$ and

$$dim_{[E]} M_{X,H}(2; L, c_2) = 4c_2 - L^2 - 3.$$

The irreducibility of $M_{X,H}(2; L, c_2)$ follows from [4]; Theorem 2.2 and [7]; Theorem 2.1.10.

Another remark should be made. As we pointed out in Remark 2.7 the definition of stability depends on the choice of the ample line bundle H and hence the following natural question arises:

Question 4.6. Let X be a smooth, irreducible, projective surface and let H and H' be two different ample line bundles on X. What is the difference between the moduli spaces $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$? Are $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$ isomorphic or, at least, birational?

It turns out that the ample cone of X has a chamber structure such that the moduli space $M_{X,H}(r; L, n)$ only depends on the chamber of H and, in general, the moduli space $M_{X,H}(r; L, n)$ changes when H crosses a wall between two chambers (see for instance, [21], [25], [72] and [73]). Let me give you an example. To this end, we need first to introduce some definitions from [72].

Definition 4.7. (See [72]; Definition I.2.1.5) Let X be a smooth, irreducible, algebraic surface over an algebraically closed field and let C_X be the ample cone in $\mathbb{R} \otimes Num(X)$. For any $\xi \in Num(X)$, we define

$$W^\xi := C_X \cap \{x \in Num(X) \otimes \mathbb{R} \ \ s.t. \ \ x \cdot \xi = 0\}.$$

W^ξ is called the *wall of type* (c_1, c_2) determined by ξ if and only if there exists $G \in Pic(X)$ with $G \equiv \xi$ such that $G + c_1$ is divisible by 2 in $Pic(X)$ and $c_1^2 - 4c_2 \leq G^2 < 0$. W^ξ is a non-empty wall of type (c_1, c_2) if there exits an ample line bundle L with $L\xi = 0$. Let $W(c_1, c_2)$ be the union of the walls of type (c_1, c_2). A chamber of type (c_1, c_2) is a connected component of $C_X \setminus W(c_1, c_2)$. A non-empty wall W^ξ separates two ample line bundles L_1 and L_2 if $\xi L_1 < 0 < \xi L_2$.

In [72], Z. Qin proves that the moduli space $M_{X,H}(r; c_1, c_2)$ only depends on the chamber of H ([72]; Corollary 2.2.2) and that the study of moduli spaces of rank two vector bundles stable with respect to an ample line bundle lying on walls may be reduced to the study of moduli spaces of rank two vector bundles stable with respect to an ample line bundle lying in chambers ([72]; Remark 2.2.6).

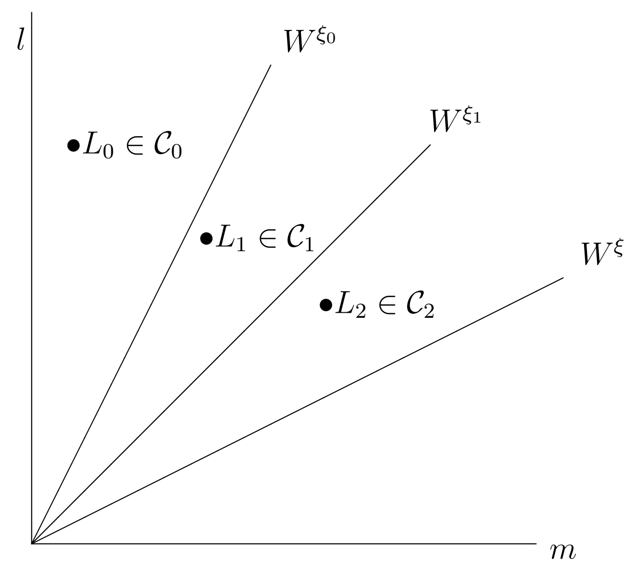
We are now ready to give an explicit example to illustrate how the moduli space $M_{X,H}(2; c_1, c_2)$ changes when the ample line bundle H crosses a wall between two chambers.

Example 4.8. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a quadric surface. We denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. Consider $\xi_0 = \ell - 6m \in Num(X)$ and $\xi_1 = \ell - 4m \in Num(X)$. It is not difficult to check that ξ_0 and ξ_1 define non-empty walls of type $(c_1, c_2) = (\ell, 3)$. In fact, $\xi_0 + \ell = 2\ell - 6m$ and $\xi_1 + \ell = 2\ell - 4m$ are both divisible by 2 in $Pic(X)$,

$\ell^2 - 4c_2 = -12 \leq \xi_0^2 = (\ell - 6m)^2 = -12 < 0$, $\ell^2 - 4c_2 = -12 \leq \xi_1^2 = (\ell - 4m)^2 = -8 < 0$, and there exist ample divisors L and L' on X such that $L\xi_0 = L'\xi_1 = 0$ (Take $L = \ell + 6m$ and $L' = \ell + 4m$).

We consider the ample line bundles $L_0 = \ell + 7m$, $L_1 = \ell + 5m$ and $L_2 = \ell + 3m$ on X. Since $L_1\xi_0 < 0 < L_0\xi_0$ and $L_2\xi_1 < 0 < L_1\xi_1$, the wall W^{ξ_0} separates L_0 and L_1 and the wall W^{ξ_1} separates L_1 and L_2. We will denote by $\mathcal{C}_0$ (resp. $\mathcal{C}_1$ and $\mathcal{C}_2$) the chamber containing L_0 (resp. L_1 and L_2).

Given an ample line bundle $L = a\ell + bm$ on X, we can represent L as a point of coordinates (a, b) in the plane. The following picture gives us an idea of the situation we are discussing:



We have:

(i) for any $L \in \mathcal{C}_0$, $M_{X,L}(2; \ell, 3) = \emptyset$,
(ii) for any $L \in \mathcal{C}_1$, $M_{X,L}(2; \ell, 3) = \mathbb{P}^9$, and
(iii) for any $L \in \mathcal{C}_2$, $M_{X,L}(2; \ell, 3)$ is a non-empty open subset of $\mathbb{P}^9$ of codimension 2.

Proof of (i). Since the moduli space $M_{X,L}(2; \ell, 3)$ only depends on the chamber $\mathcal{C}_0$ of L, it is enough to check that $M_{X,L_0}(2; \ell, 3) = \emptyset$. Assume $M_{X,L_0}(2; \ell, 3) \neq \emptyset$. By Riemann-Roch Theorem we have $\chi E(-\ell + 3m) = 1$ for any $E \in M_{X,L_0}(2; \ell, 3)$. Therefore, for any vector bundle $E \in M_{X,L_0}(2; \ell, 3)$, we have $h^0E(-\ell + 3m) > 0$ or $h^2E(-\ell + 3m) > 0$ and we will first prove that the last inequality is not possible. Indeed, by Serre's duality, $0 < h^2E(-\ell + 3m) = h^0E^*(-\ell - 5m)$. Since E is a rank two vector bundle, $E^* \cong E(-det(E))$ and hence, a non-zero section $0 \neq \sigma \in H^0E^*(-\ell - 5m) = H^0E(-2\ell - 5m)$ defines an injection

$$\mathcal{O}_X(2\ell + 5m) \hookrightarrow E.$$

Since E is μ-stable with respect to L_0, we have

$$19 = (2\ell + 5m)(\ell + 7m) < \frac{c_1(E)L_0}{2} = \frac{\ell(\ell+7m)}{2} = \frac{7}{2}$$

which is a contradiction. Therefore, $h^2E(-\ell + 3m) = 0$ and $h^0E(-\ell + 3m) > 0$.

A non-zero section $0 \neq s \in H^0E(-\ell + 3m)$ gives rise to an injective map

$$\mathcal{O}_X(\ell - 3m) \hookrightarrow E$$

and hence $\mathcal{O}_X(\ell - 3m)$ is a rank 1 subbundle of E. Since E is μ-stable with respect to L_0, we have

$$4 = (\ell - 3m)(\ell + 7m) = c_1(\mathcal{O}_X(\ell - 3m))L_0 < \frac{c_1(E)L_0}{2} = \frac{7}{2}$$

which again is a contradiction. Therefore, $M_{X,L_0}(2;\ell,3) = \emptyset$ and also $M_{X,L}(2;\ell,3) = \emptyset$ for any $L \in \mathcal{C}_0$.

Proof of (ii). See Example 3.11.

Proof of (iii). Since $M_{X,L}(2;\ell,3)$ only depends on the chamber $\mathcal{C}_2$ of L, it is enough to study $M_{X,L_2}(2;\ell,3)$ and to compare it to $M_{X,L_1}(2;\ell,3)$. To this end, we consider the open subset $\mathcal{U}$ of $M_{X,L_1}(2;\ell,3) = \mathbb{P}^9$ defined by

$$\mathcal{U} := \{E \in M_{X,L_1}(2;\ell,3) \mid H^0E(-2m) = 0\}$$

and we will see that $\mathcal{U}$ is non-empty, $dim(M_{X,L_1}(2;\ell,3) \setminus \mathcal{U}) = 2$ and $\mathcal{U} \cong M_{X,L_2}(2;\ell,3)$. Let us first prove that $dim(M_{X,L_1}(2;\ell,3) \setminus \mathcal{U}) = 2$. To this end, for any vector bundle $E \in M_{X,L_1}(2;\ell,3) \setminus \mathcal{U}$, we take a non-zero section $0 \neq s \in H^0E(-2m)$ and the associated exact sequence

$$0 \longrightarrow \mathcal{O}_X(D + 2m) \longrightarrow E \longrightarrow I_Z(\ell - 2m - D) \longrightarrow 0$$

where $D = a\ell + bm$ is an effective divisor on X (hence $a, b \geq 0$) and $Z \subset X$ is a 0-dimensional subscheme. Since E is μ-stable with respect to L_1, we have

$$(D + 2m)L_1 = (a\ell + (b+2)m)(\ell + 5m) = 5a + b + 2 < \frac{c_1(E)L_1}{2} = \frac{5}{2}.$$

Hence, $a = b = 0$ and we conclude that any vector bundle $E \in M_{X,L_1}(2;\ell,3) \setminus \mathcal{U}$ sits into an exact sequence

$$0 \longrightarrow \mathcal{O}_X(2m) \longrightarrow E \longrightarrow I_Z(\ell - 2m) \longrightarrow 0 \tag{4.1}$$

where $Z \subset X$ is a 0-dimensional subscheme of length $|Z| = c_2E(-2m) = 1$. Since the irreducible family $\mathcal{M}$ of rank 2 vector bundles on X given by an exact sequence of type (4.1) is a 2-dimensional family (check it!), we have got that $dim(M_{X,L_1}(2;\ell,3) \setminus \mathcal{U}) = 2$.

To see that $M_{X,L_2}(2;\ell,3) \cong \mathcal{U}$ we first prove that a vector bundle $E \in M_{X,L_1}(2;\ell,3)$ is μ-stable with respect to L_2 if and only if $E \in \mathcal{U}$. Obviously, if $E \in M_{X,L_1}(2;\ell,3) \setminus \mathcal{U}$ then E is not μ-stable with respect to L_2 (Indeed, $\mathcal{O}_X(2m)$ is a rank 1 subbundle of E and $c_1(\mathcal{O}_X(2m))L_2 = 2m(\ell + 3m) = 2 > \frac{c_1(E)L_2}{2} = \frac{\ell(\ell+3m)}{2} = \frac{3}{2}$). Vice versa, assume $E \in \mathcal{U}$

and let us see that E is μ-stable with respect to L_2. By Example 3.11, any vector bundle $E \in \mathcal{U} \subset M_{X,L_1}(2;\ell,3)$ sits in an exact sequence

$$e: \qquad 0 \longrightarrow \mathcal{O}_X(\ell-3m) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \longrightarrow 0.$$

Hence, for any rank 1 subbundle $\mathcal{O}_X(D)$ of E we have $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(\ell-3m)$ or $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(3m)$. In the first case $D = \ell - 3m - C$ being C an effective divisor on X. Therefore, we have $(\ell-3m-C)L_2 \leq (\ell-3m)L_2 = 0 < \frac{3}{2} = \frac{c_1(E)L_2}{2}$. In the second case, we have $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(3m)$. So, $3m-D$ is an effective divisor. Write $D = \alpha\ell+\beta m$. Since $3m-D$ is an effective divisor, we have $\alpha \leq 0$ and $\beta \leq 3$. On the other hand, $(\alpha,\beta) \neq (0,3)$ because the extension e does not split and $(\alpha,\beta) \neq (0,2)$ because $H^0E(-2m) = 0$. Putting altogether we get $c_1(\mathcal{O}_X(D))L_2 = DL_2 = (\alpha\ell+\beta m)(\ell+3m) = 3\alpha+\beta < \frac{3}{2} = \frac{c_1(E)L_2}{2}$ which proves what we want.

To finish the proof of (iii) we only need to see that $M_{X,L_2}(2;\ell,3) \subset M_{X,L_1}(2;\ell,3)$ or, equivalently, that any $E \in M_{X,L_2}(2;\ell,3)$ sits into a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X(\ell-3m) \longrightarrow E \longrightarrow \mathcal{O}_X(3m) \longrightarrow 0.$$

But this follows from the fact that E is μ-stable with respect to L_2, $\chi E(-\ell+3m) = 1$, $H^2E(-\ell+3m) = 0$ and $c_2(E(-\ell+3m)) = 0$.

The change of $M_{X,H}(r;L,n)$ when H passes through a wall between two chambers can be somehow controlled and we have (see [37]; Theorem 4.C.7)

Theorem 4.9. *Let X be a smooth, irreducible, projective surface and let H and H' be ample line bundles on X. If $\Delta(r;L,n) \gg 0$, then the moduli spaces $M_{X,H}(r;L,n)$ and $M_{X,H'}(r;L,n)$ are birational.*

For rank 2 vector bundles on Fano surfaces the hypothesis $\Delta(2;L,n) \gg 0$ can be weakened and we have:

Proposition 4.10. *Let X be a smooth Fano surface, $L \in Pic(X)$ and $c_2 \in \mathbb{Z}$. For any two ample line bundles H and H' on X, the moduli spaces $M_{X,H}(2;L,c_2)$ and $M_{X,H'}(2;L,c_2)$ are birational whenever non-empty.*

Proof. See [9]; Corollary 3.10. □

Remark 4.11. The Example 4.8 illustrates that, in general, the birational map between $M_{X,H}(2;L,c_2)$ and $M_{X,H'}(2;L,c_2)$ is not an isomorphism.

Theorem 4.9 and Proposition 4.10 imply that for many purposes we can fix the ample line bundle H; and this is what we do for studying the birational geometry of the moduli spaces $M_{X,H}(r;L,n)$. For example, we can reduce the study of the rationality of the moduli space $M_{X,H}(r;L,n)$ for any ample line bundle H to the study of the rationality of $M_{X,H}(r;L,n)$ for a suitable ample line bundle H.

Example 4.12. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a quadric surface. We denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$ and we fix the ample line bundle $L = \ell + 5m$. By Example 3.11, the moduli space $M_{X,L}(2;\ell,3) \cong \mathbb{P}^9$ and hence it is rational. Applying Proposition 4.10, we conclude that for any other ample line bundle H on X the moduli space $M_{X,H}(2;\ell,3)$ is rational whenever non-empty.

From the point of view of birational geometry the study of moduli spaces of vector bundles on algebraic surfaces discloses highly interesting features. We will now see how Serre's correspondence (see Exercise 7.8) can be used, for instance, to obtain information about the birational geometry of moduli spaces and more precisely to address the following problem stated by Nakashima in the 90's ([64]):

Problem 4.13. Let X be a K3 surface and let H be an ample line bundle on X. To determine invariants $(r, c_1, c_2, l) \in \mathbb{Z} \times Pic(X) \times \mathbb{Z}^2$ for which the moduli space $M_{X,H}(r; c_1, c_2)$ and the punctual Hilbert scheme $Hilb^l(X)$ are birational.

Recall that a K3 surface is an algebraic surface with trivial canonical bundle $K_X \cong \mathcal{O}_X$ and the vanishing irregularity $q(X) = 0$. Examples of K3 surfaces are provided by

(1) Smooth complete intersections of type $a_1, ..., a_{n-2}$ in $\mathbb{P}^n$ with $\sum_i a_i = n + 1$. In particular, smooth quartics in $\mathbb{P}^3$, smooth complete intersections of type $(2, 3)$ in $\mathbb{P}^4$, and smooth complete intersection of 3 quadrics in $\mathbb{P}^5$; and
(2) Kummer surfaces.

The first contribution to Problem 4.13 is due to K. Zuo who for the case of rank 2 vector bundles with $c_1 = 0$ proved the following result:

Proposition 4.14. *Let X be a K3 surface and H an ample line bundle on X. For all $n \in \mathbb{N}$, set $k(n) = n^2H^2 + 3$. Then, there exists a birational map*

$$\Phi : M_{X,H}(2; 0, k(n)) \longrightarrow Hilb^{2k(n)-3}(X).$$

Proof. See [83]; Theorem 1. □

Later on T. Nakashima generalized Zuo's Theorem to the triples $(r; L, c_2) = (2; L, k(n))$ where $k(n) := (n^2 + n + \frac{1}{2})L^2 + 3$ and L is an arbitrary ample line bundle ([63]). L. Costa generalized this last result to vector bundles of higher rank and she proved (see [8]; Theorem A):

Theorem 4.15. *Let X be a K3 surface, H an ample line bundle on X, $L \in Pic(X)$ and $2 \leq r \in \mathbb{Z}$. For any $0 << n \in \mathbb{N}$, set*

$$k(n) := \frac{L^2}{2} + \frac{r}{2}n^2H^2 + nLH + r + 1$$

$$l(n) := k(n) + \frac{r(r-1)}{2}n^2H^2 + (r-1)nLH.$$

Then, there exists a birational map

$$\Phi : M_{X,H}(r; L, k(n)) \longrightarrow Hilb^{l(n)}(X).$$

Sketch of the proof. First of all we consider $\mathcal{F}$ the irreducible family of isomorphism classes of rank r torsion free sheaves F on X, GM-semistable with respect to H with Chern classes $(L, k(n))$ given by a non-trivial extension

$$0 \to \mathcal{O}^{r-1} \to F(nH) \to I_Z(L + rnH) \to 0$$

where $Z \subset X$ is a 0-dimensional subscheme of length $|Z| = c_2(F(nH)) = c_2(F) + (r-1)nLH + \frac{r(n-1)}{2}n^2H^2 = k(n) + (r-1)nLH + \frac{r(n-1)}{2}n^2H^2 = l(n)$ such that $H^0I_Z(c_1+rnH) = 0$.

For $n \gg 0$, $\mathcal{F}$ is non-empty (check it!), any $E \in \mathcal{F}$ is simple (check it!) and $dim\mathcal{F} = 2\ell(n)$ (check it!). In particular, we have

$$\begin{aligned} dim\mathcal{F} = &\ dimHilb^{\ell(n)}(X) \\ = &\ 2\ell(n) \\ = &\ 2rk(n) - (r-1)L^2 - 2(r^2-1) \\ = &\ dimM_{X,H}(r;L,k(n)) = dim\overline{M}_{X,H}(r;L,k(n)). \end{aligned}$$

We give to $\mathcal{F}$ a natural structure of variety and we observe that we have a dominant map $q : \mathcal{F} \longrightarrow Hilb^{\ell(n)}(X)$. From the universal property of moduli spaces we deduce the existence of a morphism $\varphi : \mathcal{F} \to \overline{M} = \overline{M}_{X,H}(r;L,k(n))$ which is injective (check it!) and dominant. Hence, q and φ are birational maps. Composing them we get a birational map

$$\psi = q \cdot \varphi^{-1} : \overline{M}_{X,H}(r;L,k(n)) \to Hilb^{l(n)}(X).$$

Since $M_{X,H}(r;L,k(n)) \subset \overline{M}_{X,H}(r;L,k(n))$ is an open dense subset, restricting ψ to $M_{X,H}(r;L,k(n))$ we get a birational map

$$\phi = \psi_{|M_{X,H}(r;L,k(n))} : M_{X,H}(r;L,k(n)) \to Hilb^{l(n)}(X)$$

which proves what we want. □

Further birational identifications can be found in [6]. In this paper, by using a Fourier-Mukai transform for sheaves on K3 surfaces, Bruzzo and Maciocia show that for a wide class of K3 surfaces X the Hilbert scheme $Hilb^n(X)$ can be identified for all $n \geq 1$ with suitable moduli spaces of GM-stable vector bundles.

In the last part of this section we turn our attention to the study of the rationality of the moduli space $M_{X,H}(r;L,n)$. This geometric property of $M_{X,H}(r;L,n)$ nicely reflects the general philosophy that the moduli spaces inherit a lot of geometrical properties of the underlying surface. There is at present no counterexample known to the question whether the moduli spaces are always rational provided the underlying surface is rational. For $X = \mathbb{P}^2$, M. Maruyama (resp. G. Ellingsrud and S.A. Strømme) proved that if $c_1^2 - 4c_2 \not\equiv 0$ $(mod\ 8)$, then the moduli space $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ is rational ([49] and [22]). Later on, T. Maeda proved that the rationality of $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ holds for all $(c_1,c_2) \in \mathbb{Z}^2$ provided $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ is non-empty ([50]). In particular, $\overline{M}_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(2;c_1,c_2)$ has Kodaira dimension $-\infty$. For some ruled surfaces X, Z. Qin also showed that $\overline{M}_{X,H}(r;L,n)$ has Kodaira dimension $-\infty$. As for K3 surfaces X, a consequence of S. Mukai's work [61] shows that $\overline{M}_{X,H}(r;L,n)$ has Kodaira dimension 0. More recently, J. Li has proved that if X is a minimal surface of general type with reduced canonical divisor then $\overline{M}_{X,H}(r;L,n)$ is also of general type ([41]). All this indicates that the Kodaira dimension of $\overline{M}_{X,H}(r;L,n)$ is closely related to the Kodaira dimension of X and moduli spaces associated to rational surfaces should be rational. In fact, we have the following interesting problem (see [74]; Problem 2, [75]; Problem 21 and [71]; Problem 2):

Problem 4.16. Let X be a smooth, rational, projective surface. Fix $L \in Pic(X)$ and $0 \ll c_2 \in \mathbb{Z}$. Is there an ample line bundle H on X such that $M_{X,H}(r; L, c_2)$ rational?

For $r = 2$ the answer to the above question is Yes! In fact, either using the criterions of rationality for moduli spaces given by Costa and Miró-Roig in [9]; Theorem 3.12 and [11]; Theorem 3.8, or constructing irreducible families of stable vector bundles over a big enough rational basis, we have got

Theorem 4.17. *Let X be a smooth rational surface, $L \in Pic(X)$ and $n \in \mathbb{Z}$. Assume that $\Delta(2; L, n) \gg 0$. Then, there exists an ample line bundle H on X such that the moduli space $M_{X,H}(2; L, n)$ is rational.*

Proof. [11]; Theorem A. □

For arbitrary rank, despite the wealth of new techniques introduced recently, the above problem remains open. For partial results in the arbitrary rank case, the reader can see [42], [43], [44], [45], [12], [29], [77] and [81].

Although in this lecture we have missed many interesting results and we have not considered the branch of beautiful results that works for special surfaces like abelian surfaces, elliptic surfaces, ruled surfaces, $\cdots$, as well as Fourier-Mukai transformations, symplectic structures, Gauge theoretical aspects of moduli spaces; we have seen many subtle and interesting results regarding the moduli space $M_{X,L}(r; c_1, \cdots, c_{min\{r,n\}})$ when the underlying variety X has dimension $n = 2$. For moduli spaces of vector bundles on higher dimensional varieties X, no general results are known and, as we will stress in the last lecture, the situation drastically changes. Results like the smoothness and irreducibility of the moduli space of vector bundles on surfaces, turn to be false for moduli spaces of vector bundles on higher dimensional varieties.

5. Lecture Four: Monads, instanton bundles and stability

The main techniques to construct vector bundles on smooth projective varieties are Serre's construction (see Exercise 7.8), Maruyama's elementary transformations (see Exercise 7.9) and monads. With these techniques at hand, one can produce vector bundles with prescribed invariants like rank, determinant, Chern classes, etc. In this lecture, we will focus our attention in the third method. So, let us start recalling the definition and basic fact about monads.

Definition 5.1. Let X be a smooth projective variety. A *monad* on X is a complex of vector bundles:

$$M_\bullet : \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

which is exact at A and at C. The sheaf $E := Ker(\beta)/Im(\alpha)$ is called the *cohomology sheaf of the monad $M_\bullet$*. The set:

$$S = \{x \in X \mid \alpha_x \text{ is not injective}\}$$

is a subvariety called the *degeneration locus* of the monad $M_\bullet$. Note that S is also the locus where the sheaf E is not locally-free.

Clearly, the cohomology sheaf E of a monad $M_\bullet$ is always a coherent sheaf, but more can be said in particular cases. In fact, we have

Proposition 5.2. *Let E be the cohomology sheaf of a monad $M_\bullet$.*

(1) *E is locally-free if and only if the degeneration locus of $M_\bullet$ is empty;*
(2) *E is reflexive if and only if the degeneration locus of $M_\bullet$ is a subvariety of codimension at least 3;*
(3) *E is torsion-free if and only if the degeneration locus of $M_\bullet$ is a subvariety of codimension at least 2.*

Proof. Let S be the degeneration locus of the monad $M_\bullet$ associated to the sheaf E. We easily check that $\mathcal{E}xt^p(E, \mathcal{O}_X) = 0$ for $p \geq 2$ and

$$\operatorname{supp} \mathcal{E}xt^1(E, \mathcal{O}_X) = \{x \in X \mid \alpha_x \text{ is not injective }\} = S.$$

The first statement is clear; so it is now enough to argue that E is torsion-free if and only if S has codimension at least 2 and that E is reflexive if and only if S has codimension at least 3.

Recall that the m^{th}-singularity set of a coherent sheaf F on X is given by:

$$S_m(F) = \{x \in X \mid dh(F_x) \geq n - m\}$$

where n is the dimension of X and $dh(F_x)$ stands for the homological dimension of F_x as an $\mathcal{O}_x$-module:

$$dh(F_x) = d \iff \begin{cases} \operatorname{Ext}^d_{\mathcal{O}_x}(F_x, \mathcal{O}_x) \neq 0 \\ \operatorname{Ext}^p_{\mathcal{O}_x}(F_x, \mathcal{O}_x) = 0 \ \forall p > d. \end{cases}$$

In the case at hand, we have that $dh(E_x) = 1$ if $x \in S$, and $dh(E_x) = 0$ if $x \notin S$. Therefore $S_0(E) = \cdots = S_{n-2}(E) = \emptyset$, while $S_{n-1}(E) = S$. It follows that [78, Proposition 1.20]:

- if codim $S \geq 2$, then $\dim S_m(E) \leq m - 1$ for all $m < n$, hence E is a locally 1^{st}-syzygy sheaf;
- if codim $S \geq 3$, then $\dim S_m(E) \leq m - 2$ for all $m < n$, hence E is a locally 2^{nd}-syzygy sheaf.

The desired statements follow from the observation that E is torsion-free if and only if it is a locally 1^{st}-syzygy sheaf, while E is reflexive if and only if it is a locally 2^{nd}-syzygy sheaf [69, p. 148-149]. $\square$

A monad $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ has a so-called display: this is a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $K := Ker(\beta)$ and $Q := Coker(\alpha)$. From the display one easily deduces that if a coherent sheaf E on X is the cohomology sheaf of a monad $M_\bullet$, then

(i) $rk(E) = rk(B) - rk(A) - rk(C)$, and
(ii) $c_t(E) = c_t(B)c_t(A)^{-1}c_t(C)^{-1}$.

Monads were first introduced by Horrocks who showed that all vector bundles E on $\mathbb{P}^3$ can be obtained as the cohomology bundle of a monad of the following kind:

$$0 \longrightarrow \oplus_i \mathcal{O}_{\mathbb{P}^3}(a_i) \longrightarrow \oplus_j \mathcal{O}_{\mathbb{P}^3}(b_j) \longrightarrow \oplus_n \mathcal{O}_{\mathbb{P}^3}(c_n) \longrightarrow 0.$$

Monads appeared in a wide variety of contexts within algebraic-geometry, like the construction of locally free sheaves on $\mathbb{P}^n$, the classification of space curves in $\mathbb{P}^3$ and surfaces in $\mathbb{P}^4$. In this lecture, we will focus our attention on the so-called *linear monads* defined as follows:

Definition 5.3. Let X be a nonsingular projective variety over an algebraically closed field of characteristic zero and let $\mathcal{L}$ denote a very ample invertible sheaf. Given finite-dimensional k-vector spaces V, W and U, a *linear monad* on X is the short complex of sheaves

$$M_\bullet \ : \ 0 \to V \otimes \mathcal{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathcal{L} \to 0 \tag{5.1}$$

which is exact on the first and last terms, i.e. $\alpha \in \mathrm{Hom}(V, W) \otimes H^0\mathcal{L}$ is injective while $\beta \in \mathrm{Hom}(W, U) \otimes H^0\mathcal{L}$ is surjective.

Definition 5.4. A torsion-free sheaf E on X is said to be a *linear sheaf* on X if it can be represented as the cohomology sheaf of a linear monad.

Note that if E is the cohomology sheaf of a linear monad as in (5.1), then:

$$rk(E) = w - v - u \text{ and } c_1(E) = (v - u) \cdot \ell$$

where $w = \dim W$, $v = \dim V$, $u = \dim U$ and $\ell = c_1(\mathcal{L})$.

These somewhat mysterious definitions are made natural once we recall that Manin and Drinfield proved that mathematical instanton bundles E on $\mathbb{P}^3$ with quantum number k correspond to linear monads of the following type

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0$$

with $\dim V = \dim U = k$ and $\dim W = 2k+2$.

Using Penrose-Ward correspondence, Atiyah-Drinfield-Hitchin-Manin constructed all self-dual solutions of the Yang-Mills equations over S^4 and proved the existence of a one-to-one correspondence between linear monads

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0$$

with dim$V$ =dim$U = k$ and dim$W = 2k+2$, and $SL(2,\mathbb{C})$ instantons on $\mathbb{R}^4$ with charge k. More general, we define

Definition 5.5. An instanton bundle on $\mathbb{P}^{2n+1}$ with quantum number k is a rank $2n$ vector bundle E on $\mathbb{P}^{2n+1}$ satisfying the following properties

(i) $c_t(E) = \frac{1}{(1-t^2)^k}$,
(ii) E has natural cohomology in the range $-2n-1 \leq j \leq 0$, i.e., for every j in that range at most one of the cohomology groups $H^q(\mathbb{P}^{2n+1}, E(j))$ is non-zero,
(iii) E has trivial splitting type, i.e., for a general line $l \subset \mathbb{P}^{2n+1}$ we have $E_{|l} \cong {\mathcal{O}_l}^{2n}$.

The existence of instanton bundles on $\mathbb{P}^{2n+1}$ was given by Okonek and Spindler in [68] answering a question posed by Salamon. We have

Proposition 5.6. *Any instanton bundle E on $\mathbb{P}^{2n+1}$ with quantum number k is a linear vector bundle; more precisely, E is the cohomology bundle of a linear monad*

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^{2n+1}} \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(1) \longrightarrow 0$$

with dim$V$ =dim$U = k$ and dim$W = 2k+2n$.

Conversely, a linear vector bundle arising as the cohomology bundle of a linear monad

$$0 \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^{2n+1}} \longrightarrow U \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(1) \longrightarrow 0$$

with dim$V$ =dim$U = k$ and dim$W = 2k+2n$ is an instanton bundle provided it has trivial splitting type.

Proof. It follows from Beilinson's spectral sequence. □

In [3], Proposition 2.11, Ancona and Ottaviani proved that any instanton bundle E on $\mathbb{P}^{2n+1}$ with quantum number k is simple (See Exercise 7.16). Moreover, if $k=1$ or $n=1$ then E is stable and the stability is left as an open problem when $k \geq 2$ and $n \geq 3$. The existence of the moduli space $MI_{\mathbb{P}^{2n+1}}(k)$ of instanton bundles on $\mathbb{P}^{2n+1}$ with quantum number k was established by Okonek and Spindler in [68]; Theorem 2.6. Determining the irreducibility and smoothness of $MI_{\mathbb{P}^{2n+1}}(k)$ is a long standing question far of being solved, see [15] for a recent survey on the topic and next lecture for the proof of the non-smoothness of the moduli space $MI_{\mathbb{P}^{2n+1}}(k)$ for $k \geq 3$ and $n \geq 2$.

The existence and classification of linear monads on $\mathbb{P}^n$ was given by Fløystad in ([23]). He proved

Proposition 5.7. *Let $n \geq 1$. There exist monads on $\mathbb{P}^n$ whose entries are linear maps, i.e. linear monads*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

if and only if at least one of the following conditions holds:

(1) $b \geq 2c+n-1$ *and* $b \geq a+c$.

(2) $b \geq a + c + n$.

If so, there actually exists a linear monad with the map α degenerating in expected codimension $b - a - c + 1$.

As an application of Beilinson's spectral sequence, we will give a cohomological characterization of linear sheaves on $\mathbb{P}^n$. Let $n \geq 1$. Fix integers a, b and c such that

(1) $b \geq 2c + n - 1$ and $b \geq a + c$, or

(2) $b \geq a + c + n$.

We have:

Proposition 5.8. *Let E be a rank $b-a-c$ torsion free sheaf on $\mathbb{P}^n$ with Chern polynomial $c_t(E) = \frac{1}{(1-t)^a(1+t)^c}$. It holds:*

(1) If $b < c(n+1)$ and E has natural cohomology in the range $-n \leq j \leq 0$, then E is the cohomology sheaf of a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$

(2) If E is the cohomology sheaf of a linear monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0$$

and $H^0(\mathbb{P}^n, E) = 0$, then E has natural cohomology in the range $-n \leq j \leq 0$.

Proof. (1) Since E is a rank $b - a - c$ torsion free sheaf on $\mathbb{P}^n$ with Chern polynomial $c_t(E) = \frac{1}{(1-t)^a(1+t)^c}$, using Riemann-Roch formula we get

$$\chi E(t) = \begin{cases} b - c(n+1) & \text{if} \quad t = 0 \\ -c & \text{if} \quad t = -1 \\ 0 & \text{if} \quad -n+1 \leq t \leq -2 \\ (-1)^{n+1}a & \text{if} \quad t = -n. \end{cases} \tag{5.2}$$

On the other hand, for any coherent sheaf F on $\mathbb{P}^n$, we have the Beilinson spectral sequence with E_1-term

$$E_1^{pq} = H^q(\mathbb{P}^n, F(p)) \otimes \Omega^{-p}(-p) \tag{5.3}$$

situated in the square $-n \leq p \leq 0$, $0 \leq q \leq n$ which converges to

$$E_\infty^i = \begin{cases} F \text{ for } i = 0 \\ 0 \text{ for } i \neq 0 \end{cases}.$$

Using the degeneration of this spectral sequence, the values of the Euler characteristic $\chi E(t)$ given in (5.2) together with the fact that E has natural cohomology in the range $-n \leq j \leq 0$, we deduce that E is the cohomology bundle of the monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \otimes U \longrightarrow \Omega^1(1) \otimes V \longrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes W \longrightarrow 0 \tag{5.4}$$

where $U = H^{n-1}(\mathbb{P}^n, E(-n))$, $V = H^1(\mathbb{P}^n, E(-1))$ and $W = H^1(\mathbb{P}^n, E)$ are k-vector spaces of dimension $\chi E(-n) = a$, $-\chi E(-1) = c$ and $-\chi E = c(n+1) - b$, respectively.

Tensoring the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

with V and combining it with the exact sequence (5.4) we get the following commutative diagram ($\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$, $H^iF = H^i(\mathbb{P}^n, F)$):

(5.5)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1)\otimes H^{n-1}E(-n) & \longrightarrow & \Omega^1_{\mathbb{P}^n}(1)\otimes H^1E(-1) & \longrightarrow & \mathcal{O}\otimes H^1E & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}\otimes H^1(\Omega^1_{\mathbb{P}^n}(1)\otimes E(-1)) & \longrightarrow & \mathcal{O}\otimes H^0\mathcal{O}(1)\otimes H^1E(-1) & \longrightarrow & \mathcal{O}\otimes H^1E & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}(1)\otimes H^1E(-1) & = & \mathcal{O}(1)\otimes H^1E(-1) & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $H^1(\Omega^1_{\mathbb{P}^n}(1)\otimes E(-1))$ is a k-vector space of dimension b. Since the first row and the first column are monads with the same cohomology we get that E is the cohomology bundle of a special monad of the following type
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}^b_{\mathbb{P}^n} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$

(2) Assume that E is the cohomology bundle of a special monad
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\alpha} \mathcal{O}^b_{\mathbb{P}^n} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0.$$
Considering the cohomological long exact sequences associated to the exact sequences
$$0 \longrightarrow K := \ker(\beta) \longrightarrow \mathcal{O}^b_{\mathbb{P}^n} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c \longrightarrow 0 \quad \text{and} \tag{5.6}$$
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \longrightarrow K \longrightarrow E \longrightarrow 0, \tag{5.7}$$
we get
$$\begin{cases} H^i(\mathbb{P}^n, E(t)) = 0 & \text{for all } t \text{ and } 2 \le i \le n-2,\\ H^{n-1}(\mathbb{P}^n, E(t)) = 0 & \text{for all } t \ge -n+1,\\ H^i(\mathbb{P}^n, E(-n)) = 0 & \text{for all } i \ne n-1,\\ H^i(\mathbb{P}^n, E(-1)) = 0 & \text{for all } i \ne 1,\\ H^i(\mathbb{P}^n, E) = 0 & \text{for all } i \ne 0, 1.\end{cases} \tag{5.8}$$
These equalities together with the assumption $H^0(\mathbb{P}^n, E) = 0$, prove that E has natural cohomology in the range $-n \le j \le 0$. □

Following the ideas developed by Fløystead in [23], it is not difficult to determine when there exists a monad on a hyperquadric $Q_n \subset \mathbb{P}^{n+1}$ whose maps are linear forms. Indeed, we have

Proposition 5.9. *Let $Q_n \subset \mathbb{P}^{n+1}$ be a hyperquadric, $n \ge 3$. There exist monads on Q_n whose entries are linear maps, i.e. linear monads*
$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}^b_{Q_n} \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0$$
if and only if at least one of the following conditions holds:

(1) $b \ge 2c + n - 1$ *and* $b \ge a + c$.

(2) $b \geq a+c+n$.

If so, there actually exists a linear monad with the map α degenerating in expected codimension $b-a-c+1$.

Proof. First, let us prove the existence part. Without lost of generality we may assume that Q_n is the quadric hypersurface in $\mathbb{P}^{n+1} = Proj(k[x_0, x_1, \cdots, x_{n+1}])$ defined by $f(x_0, \cdots, x_{n+1}) = x_0^2 + x_1^2 + \cdots + x_{n+1}^2$. By [23]; Main Theorem, if $b \geq 2c+n$ and $b \geq a+c$ or $b \geq a+c+n+1$ then there exist linear monads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^a \xrightarrow{\alpha} \mathcal{O}^b_{\mathbb{P}^{n+1}} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^{n+1}}(1)^c \longrightarrow 0 \tag{5.9}$$

with the map α degenerating in the expected codimension $b-a-c+1$. So restricting a general monad (5.9) to Q_n we get a linear monad

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}^b_{Q_n} \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0$$

with the map α degenerating in the expected codimension $b-a-c+1$. So, it is enough to consider the cases

(a) $b = 2c+n-1$ and $b \geq a+c$.
(b) $b = a+c+n$.

(a)Assume $b = 2c+n-1$ and $b \geq a+c$. We distinguish two subcases

(a1) $b = 2c+n-1$ and $b = a+c$.
(a2) $b = 2c+n-1$ and $b > a+c$.

(a1) $b = 2c+n-1$ and $b = a+c$. Set $n_1 = \frac{n-1}{2}$ if n is odd and $n_1 = \frac{n-2}{2}$ if n is even. Consider the $(n_1+c) \times c$, $(n-1-n_1+c) \times c$, $(n-1+c) \times (n-1-n_1+c)$ and $(n-1+c) \times (n_1+c)$ matrices

$$A_1 = \begin{pmatrix} x_0 & x_1 & \dots & \dots & x_{n_1} & 0 & 0 & \dots & \dots & 0 \\ 0 & x_0 & x_1 & \dots & \dots & x_{n_1} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_0 & x_1 & \dots & \dots & x_{n_1} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n & 0 & 0 & \dots & \dots & 0 \\ 0 & x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n \end{pmatrix}$$

$$A_3 = \begin{pmatrix} x_0 & x_1 & \dots & \dots & x_{n_1} & 0 & 0 & \dots & \dots & 0 \\ 0 & x_0 & x_1 & \dots & \dots & x_{n_1} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_0 & x_1 & \dots & \dots & x_{n_1} \end{pmatrix}$$

$$A_4 = \begin{pmatrix} x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n & 0 & 0 & \dots & \dots & 0 \\ 0 & x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & x_{n_1+1} & x_{n_1+2} & \dots & \dots & x_n \end{pmatrix}.$$

Define the complex

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{Q_n}^b \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0 \tag{5.10}$$

where β is the map given by the matrix $B = (A_1 \quad A_2)$ and α is the map given by

$$A = \begin{pmatrix} A_4 \\ -A_3 \end{pmatrix}.$$

It is not difficult to see that α degenerates in codimension $b - c - a + 1 = 1$.

(a2) If $b = 2c + n - 1$ and $b > a + c$, we consider a sufficiently general injection $\phi : \mathcal{O}_{Q_n}(-1)^a \longrightarrow \mathcal{O}_{Q_n}(-1)^{b-c}$ and the composition $\alpha\phi : \mathcal{O}_{Q_n}(-1)^a \longrightarrow \mathcal{O}_{Q_n}^{2c+n-1}$ where α is the map appearing in (5.10). Since for ϕ general enough, $\alpha\phi$ degenerates in codimension $b - c - a + 1$ we get the special monad

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\alpha\phi} \mathcal{O}_{Q_n}^{2c+n-1} \xrightarrow{\beta} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0$$

we were looking for.

(b) $b = a + c + n$. We may assume that $b < 2c + n - 1$ since otherwise we are in one of the cases already covered. Since $b = a + c + n$ and $b < 2c + n - 1$ we get $a \leq c - 1$. Thus $b = a + c + n \geq 2a + n - 1$ and there exists a special monad

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^c \xrightarrow{\rho} \mathcal{O}_{Q_n}^b \xrightarrow{\eta} \mathcal{O}_{Q_n}(1)^a \longrightarrow 0$$

with the map ρ degenerating in the expected codimension $b - a - c + 1 \geq n + 1$ and so ρ does not degenerates. Dualizing we get a linear monad

$$0 \longrightarrow \mathcal{O}_{Q_n}(-1)^a \xrightarrow{\eta^*} \mathcal{O}_{Q_n}^b \xrightarrow{\rho^*} \mathcal{O}_{Q_n}(1)^c \longrightarrow 0$$

with the map η^* not degenerating and we are done.

Pursuing the ideas developed by Fløystead in [23] and essentially changing the role of $\mathbb{P}^n$ by Q_n we get that the numerical conditions on a, b, c and n are indeed necessary. □

Linear monads and instanton sheaves have been extensively studied for the case $X = \mathbb{P}^n$ during the past 30 years, see for instance [38, 69] and the references therein. In a recent preprint, the authors have initiated the study of linear monads over smooth quadric hypersurfaces Q_n within $\mathbb{P}^{n+1}$ ($n \geq 3$) [13]. They have asked whether every linear locally free sheaf of rank $n - 1$ over Q_n is μ-stable (in the sense of Mumford-Takemoto) [13, Question 5.1]. More general, they ask:

Question 5.10. Is any linear sheaf on a cyclic variety μ-stable or at least μ-semistable?

The main goal of the last part of this section is to give a partial answer to this question showing that linear locally-free sheaves with $c_1 = 0$ and rank $r \leq 2n - 1$ on an n-dimensional smooth projective variety with cyclic Picard group are μ-semistable. Furthermore, we also show that the bound on the rank is sharp by providing examples of rank $2n$ linear locally free sheaves with $c_1 = 0$ on $\mathbb{P}^n$ which are not μ-semistable.

Proposition 5.11. *Let X be a smooth projective cyclic (i.e. $Pic(X) = \mathbb{Z}$) variety of dimension n and let E be a linear sheaf on X associated to the linear monad*

$$0 \to \mathcal{O}_X(-1)^a \to \mathcal{O}_X^b \xrightarrow{\beta} \mathcal{O}_X(1)^c \to 0 \ . \tag{5.11}$$

Assume that $\omega_X \cong \mathcal{O}_X(\lambda)$ for some integer $\lambda < 0$. Then, we have:

(1) $H^0(E(k)) = H^0(E^*(k)) = 0$ for all $k \leq -1$,
(2) $H^1(E(k)) = 0$ for all $k \leq -2$,
(3) $H^i(E(k)) = 0$ for all k and $2 \leq i \leq n-2$,
(4) $H^{n-1}(E(k)) = 0$ for all $k \geq \lambda + 2$,
(5) $H^n(E(k)) = 0$ for all $k \geq \lambda + 1$,
and if E is locally-free:
(6) $H^n(E^*(k)) = 0$ for all $k \geq \lambda + 1$.

Proof. The crucial observation is that by Kodaira Vanishing Theorem we have

$$H^i(\mathcal{O}_X(k)) = 0 \text{ for all } i < n \text{ and } k \leq -1; \text{ and}$$

$$H^i(\mathcal{O}_X(k) \otimes \omega_X) = 0 \text{ for all } i > 0 \text{ and } k \geq 1.$$

By Serre's duality $H^i(X, \mathcal{O}_X(k) \otimes \omega_X) \cong H^{n-i}(X, \mathcal{O}_X(-k))$. So, we conclude that

$$H^0(\mathcal{O}_X(k)) = 0 \text{ for all } k \leq -1,$$

$$H^i(\mathcal{O}_X(k)) = 0 \text{ for all } k \text{ and } 1 \leq i \leq n-1, \text{ and}$$

$$H^n(\mathcal{O}_X(k)) = 0 \text{ for all } k \geq \lambda + 1.$$

Set $\mathcal{K} = \ker \beta$; it is a locally-free sheaf of rank $b - c$ fitting into the sequences:

$$0 \to \mathcal{K}(k) \to \mathcal{O}_X(k)^b \xrightarrow{\beta} \mathcal{O}_X(k+1)^c \to 0 \text{ and} \tag{5.12}$$

$$0 \to \mathcal{O}_X(k-1)^a \xrightarrow{\alpha} \mathcal{K}(k) \to E(k) \to 0 \ . \tag{5.13}$$

Passing to cohomology, the exact sequence (5.12) yields:

$$H^i(X, \mathcal{K}(t)) = 0 \text{ for all } t \text{ and } 2 \leq i \leq n-1 \ ,$$

$$H^n(X, \mathcal{K}(k)) = 0 \text{ for } t \geq \lambda + 1 \ ,$$

$$H^0(X, \mathcal{K}(k)) = 0 \text{ for } t \leq -1 \ ,$$

$$H^1(X, \mathcal{K}(t)) = 0 \text{ for } t \leq -2 \ .$$

Passing to cohomology, the exact sequence (5.13) yields:

$$H^0(E(k)) = 0 \text{ for all } k \leq -1 \ ,$$

$$H^1(E(k)) = 0 \text{ for all } k \leq -2 \ ,$$

$$H^i(E(k)) = 0 \text{ for all } k \text{ and } 2 \leq i \leq n-2 \ ,$$

$$H^{n-1}(E(k)) = 0 \text{ for all } k \geq \lambda + 2 \ ,$$

$$H^n(E(k)) = 0 \text{ for all } k \geq \lambda + 1 \ .$$

Dualizing sequences (5.12) and (5.13), we obtain:

$$0 \to \mathcal{O}_X(-k-1)^c \xrightarrow{\beta^*} \mathcal{O}_X(-k)^b \to \mathcal{K}^*(-k) \to 0 \text{ and} \tag{5.14}$$

$$0 \to E^*(-k) \to \mathcal{K}^*(-k) \xrightarrow{\alpha^*} \mathcal{O}_X(-k+1)^a \to \mathcal{E}xt^1(E(k), \mathcal{O}_X) \to 0 \ . \tag{5.15}$$

Again, passing to cohomology, (5.15) forces $H^0(E^*(k)) \subseteq H^0(\mathcal{K}^*(k))$ for all k, while (5.14) implies $H^0(\mathcal{K}^*(k)) = 0$ for $k \leq -1$.

Finally, if E is locally-free, we have $H^n(E^*(k)) = 0$ for all $k \geq \lambda + 1$, by Serre's duality. □

Finally, we have:

Theorem 5.12. *Every rank $r \leq 2n-1$ linear locally-free sheaf with $c_1 = 0$ on a cyclic variety X of dimension n is μ-semistable.*

Proof. We argue that every rank $r \leq 2n-1$ linear locally-free sheaf with $c_1 = 0$ on an n-dimensional cyclic variety X satisfies Hoppe's criterion (see Proposition 2.12).

Indeed, let E be a rank r linear locally-free sheaf with $c_1 = 0$ on X. Thus, E can be represented as the cohomology of the linear monad

$$0 \to \mathcal{O}_X(-1)^c \to \mathcal{O}_X^{r+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^c \to 0 \ .$$

Consider the short exact sequences

$$0 \to K \to \mathcal{O}_X^{r+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^c \to 0 \ , \tag{5.16}$$

$$0 \to \mathcal{O}_X(-1)^c \xrightarrow{\alpha} K \to E \to 0 \tag{5.17}$$

and take the long exact sequence of exterior powers associated to the sheaf sequence (5.16), twisted by $\mathcal{O}(-1)$. We have:

$$0 \to \wedge^q K(-1) \to \wedge^q(\mathcal{O}^{r+2c})(-1) \to \cdots \ .$$

Thus $H^0(\wedge^q K(-1)) = 0$ for $1 \leq q \leq r + c$.

Now consider the long exact sequence of symmetric powers associated to the sheaf sequence (5.17), twisted by $\mathcal{O}(-1)$:

$$\begin{gathered} 0 \to \mathcal{O}(-q-1)^{\binom{c+q-1}{q}} \to K \otimes \mathcal{O}(-q)^{\binom{c+q-2}{q-1}} \to \cdots \\ \to \wedge^{q-1} K \otimes \mathcal{O}(-2)^c \to \wedge^q K(-1) \to \wedge^q E(-1) \to 0 \ . \end{gathered}$$

Cutting into short exact sequences and passing to cohomology, we obtain

$$H^0(\wedge^p E(-1)) = 0 \text{ for } 1 \leq p \leq n-1 \ . \tag{5.18}$$

Since E is a linear locally-free sheaf with $c_1 = 0$, the dual E^* is also a linear locally free sheaf with $c_1 = 0$ on X, so

$$H^0(\wedge^q(E^*)(-1)) = 0 \text{ for } 1 \leq q \leq n-1 \ . \tag{5.19}$$

But $\wedge^p(E) \simeq \wedge^{r-p}(E^*)$, since $\det(E) = \mathcal{O}_X$; it follows that:

$$\begin{aligned} H^0(\wedge^p E(-1)) = H^0(\wedge^{r-p}(E^*)(-1)) = 0 \quad &\text{for} \quad 1 \leq r-p \leq n-1 \\ &\Longrightarrow \quad r-n+1 \leq p \leq r-1 \ . \end{aligned} \tag{5.20}$$

Together, (5.19) and (5.20) imply that if E is a rank $r \leq 2n-1$ linear locally-free sheaf with $c_1 = 0$, then:

$$H^0(\wedge^p E(-1)) = 0 \text{ for } 1 \leq p \leq 2n-2 \tag{5.21}$$

hence E is μ-semistable by Hoppe's criterion. □

We will end this section with an example which illustrates that the upper bound in the rank given in Theorem 5.12 is sharp, in the sense that there are rank $2n$ linear locally-free sheaves with $c_1 = 0$ on certain n-dimensional cyclic varieties which are not μ-semistable. To prove it we first need to provide the following useful cohomological characterization of linear sheaves on projective spaces.

Proposition 5.13. *Let F be a torsion-free sheaf on $\mathbb{P}^n$. F is a linear sheaf if and only if the following cohomological conditions hold:*

- *for $n \geq 2$, $H^0(F(-1)) = 0$ and $H^n(F(-n)) = 0$;*
- *for $n \geq 3$, $H^1(F(k)) = 0$ for $k \leq -2$ and $H^{n-1}(F(k)) = 0$ for $k \geq -n+1$;*
- *for $n \geq 4$, $H^p(F(k)) = 0$ for $2 \leq p \leq n-2$ and all k.*

Proof. The fact that linear sheaves satisfy the cohomological conditions above is a consequence of Proposition 5.11.

For the converse statement, first note that $H^0(F(-1)) = 0$ implies that $H^0(F(k)) = 0$ for $k \leq -1$, while $H^n(F(-n)) = 0$ implies that $H^n(F(k)) = 0$ for $k \geq -n$.
Claim:

$$H^q(F(-1) \otimes \Omega^{-p}(-p)) = 0 \text{ for } q \neq 1 \text{ and for } q = 1,\ p \leq -3\ . \tag{5.22}$$

Proof of the Claim: It follows from repeated use of the exact sequence

$$\begin{aligned} H^q(E(k))^m \to H^q(E(k+1) \otimes \Omega^{-p-1}_{\mathbb{P}^n}(-p-1)) \to \\ \to H^{q+1}(E(k) \otimes \Omega^{-p}_{\mathbb{P}^n}(-p)) \to H^{q+1}(E(k))^m \end{aligned} \tag{5.23}$$

associated with Euler sequence for p-forms on $\mathbb{P}^n$ twisted by $E(k)$:

$$0 \to E(k) \otimes \Omega^{-p}_{\mathbb{P}^n}(-p) \to E(k)^m \to E(k) \otimes \Omega^{-p-1}_{\mathbb{P}^n}(-p) \to 0\ , \tag{5.24}$$

where $q = 0, \ldots, n$, $p = 0, -1, \ldots, -n$ and $m = \begin{pmatrix} n+1 \\ -p \end{pmatrix}$ (Prove it !).

Now the key ingredient is the *Beilinson spectral sequence* [69]: for any coherent sheaf F on $\mathbb{P}^n$, there exists a spectral sequence $\{E_r^{p,q}\}$ whose E_1-term is given by ($q = 0, \ldots, n$ and $p = 0, -1, \ldots, -n$):

$$E_1^{p,q} = H^q(F \otimes \Omega^{-p}_{\mathbb{P}^n}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

which converges to

$$E^i = \begin{cases} F\ , \text{ if } p+q=0 \\ 0 \text{ otherwise} \end{cases}.$$

Applying the Beilinson spectral sequence to $F(-1)$, it then follows that the Beilinson spectral sequence degenerates at the E_2-term, so that the monad

$$\begin{aligned} 0 \ \to\ & H^1(F(-1) \otimes \Omega^2_{\mathbb{P}^n}(2)) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \to \\ \to\ & H^1(F(-1) \otimes \Omega^1_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to H^1(F(-1)) \otimes \mathcal{O}_{\mathbb{P}^n} \to 0 \end{aligned} \tag{5.25}$$

has $F(-1)$ as its cohomology. Tensoring (5.25) by $\mathcal{O}_{\mathbb{P}^n}(1)$, we conclude that F is the cohomology of a linear monad, as desired. □

We are finally ready to construct rank $2n$ linear locally-free sheaves with $c_1 = 0$ on $\mathbb{P}^n$ which are not μ-semistable.

Example 5.14. Let $X = \mathbb{P}^n$, $n \geq 4$. By Fløystad's theorem [23], there is a linear monad:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^2 \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{n+3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \to 0 \tag{5.26}$$

whose cohomology F is a locally-free sheaf of rank n on $\mathbb{P}^n$ and $c_1(F) = 1$.

Dualizing we get a linear monad:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\beta^*} \mathcal{O}_{\mathbb{P}^n}^{n+3} \xrightarrow{\alpha^*} \mathcal{O}_{\mathbb{P}^n}(1)^2 \to 0$$

whose cohomology is F^*, hence it is a locally-free linear sheaf of rank n on $\mathbb{P}^n$ and $c_1(F^*) = -1$.

Take an extension E of F^* by F:

$$0 \to F \to E \to F^* \to 0.$$

Such extensions are classified by $\mathrm{Ext}^1(F^*, F) = H^1(F \otimes F)$. We claim that there are non-trivial extensions of F^* by F. Indeed, we consider the exact sequences

$$0 \to K = Ker(\beta) \to \mathcal{O}_{\mathbb{P}^n}^{n+3} \xrightarrow{\beta} (1) \to 0\ , \tag{5.27}$$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^2 \to K \to F \to 0 \tag{5.28}$$

associated to the linear monad (5.26). We apply the exact covariant functor $\cdot \otimes F$ to the exact sequences (5.27) and (5.28) and we obtain the exact sequences

$$0 \to K \otimes F \to F^{n+3} \to F(1) \to 0\ ,$$

$$0 \to F(-1)^2 \to K \otimes F \to F \otimes F \to 0\ .$$

Using Proposition 5.11, we obtain $H^i(K \otimes F) = H^i(F \otimes F) = 0$ for all $i \geq 3$. Hence, $\chi(F \otimes F) = h^0(F \otimes F) - h^1(F \otimes F) + h^2(F \otimes F)$. On the other hand,

$$\chi(F \otimes F) = \chi(K \otimes F) - 2\chi(F(-1)) =$$

$$(n+3)\chi(F) - \chi(F(1)) - 2\chi(F(-1)) = 8 - \frac{n^2}{2} - \frac{n}{2} < 0\ ,\ \text{if } n \geq 4\ .$$

Thus if $n \geq 4$, we must have $h^1(F \otimes F) > 0$, hence there are non-trivial extensions of F^* by F.

Using the cohomological criterion given in Proposition 5.13, it is easy to see that the extension of linear sheaves is also a linear sheaf. Moreover, $c_1(E) = 0$. So, E is a rank $2n$ linear locally-free sheaf on $\mathbb{P}^n$, $n \geq 4$, with $(c_1, c_2) = (0, 3)$ which is not μ-semistable because $F \subset E$ and $\mu(F) = \frac{1}{n} > 0 = \mu(E)$.

For $X = \mathbb{P}^n$, $2 \leq n \leq 3$, arguing as above, we can construct a rank $2n$ linear locally-free sheaves with $c_1 = 0$ which are not μ-semistable as a non-trivial extension E of F^* by F, where F is a linear sheaf represented as the cohomology of the linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^4 \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{n+7} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^3 \to 0.$$

6. Lecture Five: Moduli spaces of vector bundles on higher dimensional varieties

Let X be a smooth, projective, n-dimensional variety over an algebraically closed field of characteristic 0 and we denote by $M_{X,L}(r;c_1,\cdots,c_{min\{r,n\}})$ the moduli space of rank r, vector bundles E on X, μ-stable with respect to an ample line bundle L with fixed Chern classes $c_i(E)=c_i\in H^{2i}(X,\mathbb{Z})$.

It was a major result in the theory of vector bundles on an algebraic surface S the proof that the moduli space $M_{S,L}(r;c_1,c_2)$ of rank r vector bundles E on S, μ-stable with respect to a fixed ample line bundle L and with given Chern classes $c_i\in H^{2i}(S,\mathbb{Z})$ is irreducible and smooth provided $c_2\gg 0$. The result is not true for higher dimensional varieties and it is rather common the existence of moduli spaces of stable vector bundles on X which are neither irreducible nor smooth. Indeed, in [20] (resp. [2]), L. Ein (resp. V. Ancona and G. Ottaviani) proved that the minimal number of irreducible components of the moduli space of rank 2 (resp. rank 3) stable vector bundles on $\mathbb{P}^3$ (resp. $\mathbb{P}^5$) with fixed c_1 and c_2 going to infinity grows to ∞. Inspired in Ein's result we have proved

Theorem 6.1. *Let X be a smooth projective 3-fold, $c_1,H\in Pic(X)$ with H ample and $d\in\mathbb{Z}$. Assume that there exist integers $a\neq 0$ and b such that $ac_1\equiv bH$. Let $M_{X,H}(c_1,d)$ be the moduli space of rank 2 vector bundles E on X, μ-stable with respect to H with $det(E)=c_1$ and $c_2(E)H=d$ and let $m(d)$ be the number of irreducible components of $M_{X,H}(c_1,d)$. Then $liminf_{d\to\infty}m(d)=+\infty$.*

Proof. See [5]; Theorem 0.1. □

As examples of singular moduli spaces of vector bundles on higher dimensional varieties we have the moduli spaces of mathematical instanton bundles over $\mathbb{P}^{2n+1}$ $(n\geq 2)$ with second Chern class $c_2=k$ $(k\geq 3)$.

Theorem 6.2. *For all integers $k\geq 3$ and $n\geq 2$, the moduli spaces, $MI_{2n+1}(k)$, of mathematical instanton bundles over $\mathbb{P}^{2n+1}$ with second Chern class $c_2=k$ are singular.*

Idea of the proof. Let E be a mathematical instanton bundle on $\mathbb{P}^{2n+1}$ with second Chern class k. Then, E is the cohomology bundle of a monad of the following type (Proposition 5.6):

$$0\longrightarrow\mathcal{O}(-1)^k\xrightarrow{A}\mathcal{O}^{2n+2k}\xrightarrow{B}\mathcal{O}(1)^k\longrightarrow 0.$$

We fix coordinates $x_0,x_1,\cdots,x_n,y_0,y_1,\cdots,y_n$ on $\mathbb{P}^{2n+1}$ and we denote by E_u the mathematical instanton bundle defined as the cohomology bundle of the monad (Proposition 5.6):

$$0\longrightarrow\mathcal{O}(-1)^k\xrightarrow{A_u}\mathcal{O}^{2n+2k}\xrightarrow{B}\mathcal{O}(1)^k\longrightarrow 0$$

where

$$B^t := \begin{pmatrix} x_0 & x_1 & \dots & \dots & x_n & & & & y_0 & y_1 & \dots & \dots & y_n & & \\ & x_0 & x_1 & \dots & \dots & x_n & & & & y_0 & y_1 & \dots & \dots & y_n & \\ & & \ddots & \ddots & & & \ddots & & & & \ddots & \ddots & & & \ddots \\ & & & x_0 & x_1 & \dots & \dots & x_n & & & & y_0 & y_1 & \dots & \dots & y_n \end{pmatrix}$$

and

$$A_u := \begin{pmatrix} & & & y_n & \dots & \dots & y_1 & y_0 & & & & -x_n & \dots & \dots & -x_1 & -x_0 \\ & & y_n & \dots & \dots & y_1 & y_0 & & & & -x_n & \dots & \dots & -x_1 & -x_0 & \\ & & & & & & & & & & & & & & & \\ M_n & \dots & \dots & M_1 & M_0 & \dots & \dots & M & N_n & \dots & \dots & N_1 & N_0 & \dots & \dots & N \end{pmatrix}$$

with $M_i = (1-u)y_i + uy_{i-1}$ for $i = 1, \cdots n$, $M_0 = (1-u)y_0$, $M = uy_n$, $N_i = (u-1)x_i - ux_{i-1}$ for $i = 1, \cdots, n$, $N_0 = (u-1)x_0$ and $N = -ux_n$.

E_0 is a special symplectic instanton bundle on $\mathbb{P}^{2n+1}$. The Zariski tangent space to $MI_{2n+1}(k)$ at the point corresponding to E_0 is isomorphic to the vector space $Ext^1(E_0, E_0)$ and the obstructions to extending an infinitesimal deformation lie in $Ext^2(E_0, E_0)$. From [70]; Theorem 4.1, we get

$$dimExt^1(E_0, E_0) = 4k(3n-1) + (2n-5)(2n-1)$$

and, by [57]; Lemma 3.2, we have

$$dimExt^1(E_{u=1}, E_{u=1}) < 4k(3n-1) + (2n-5)(2n-1) = dimExt^1(E_0, E_0).$$

Hence, we conclude that the moduli space $MI_{2n+1}(k)$ is singular at the point corresponding to E_0, which proves what we want.

See [57]; Theorem 3.1 for the details. $\square$

Nevertheless, we will see that for a $(d+1)$-dimensional, rational, normal scroll X and for suitable choice of $c_i \in H^{2i}(X, \mathbb{Z})$, $i = 1, 2$, and a fixed ample line bundle $L = L(c_1, c_2)$ the moduli space $M_{X,L}(2; c_1, c_2)$ is a smooth, irreducible, rational, projective variety. To prove this, we need to fix some more notation.

Take $\mathcal{E} := \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $0 = a_0 \leq a_1 \leq ... \leq a_d$ and $a_d > 0$. Let

$$X := \mathbb{P}(\mathcal{E}) = Proj(Sym(\mathcal{E})) \xrightarrow{\pi} \mathbb{P}^1$$

be the projectived vector bundle and let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle. $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defines a birational map

$$X := \mathbb{P}(\mathcal{E}) \xrightarrow{f} \mathbb{P}^N,$$

where $N = d + \sum_{i=0}^{d} a_i$. The image of f is a variety Y of dimension $d+1$ and minimal degree $(deg(Y) = \sum_{i=0}^{d} a_i)$ called **rational normal scroll**. By abusing, we shall also call to X rational normal scroll.

Let H be the class in $Pic(X)$ associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on X and let F be the fiber of π. We have

$$Pic(X) \cong \mathbb{Z}^2 \cong < H, F > \quad \text{with} \quad H^{d+1} = \sum_{i=0}^{d} a_i; \quad H^d F = 1; \quad F^2 = 0.$$

Moreover, a divisor $L = aH + bF$ on X is ample if and only if $a > 0$ and $b > 0$. In the following Lemma we will compute the cohomology groups of line bundles on a $\mathbb{P}^d$-bundle $X := \mathbb{P}(\mathcal{E})$ where $\mathcal{E} := \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $0 = a_0 \leq a_1 \leq ... \leq a_d$ and $a_d > 0$.

Lemma 6.3. *With the above notation we have*

$$H^i(X, \mathcal{O}_X(aH+bF)) = \begin{cases} 0 & \text{if } -d-1 < a < 0 \\ H^i(\mathbb{P}^1, S^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(b)) & \text{if } a \geq 0 \\ H^{d+1-i}(\mathbb{P}^1, S^{-d-1-a}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(-b+s-2)) & \text{if } a \leq -d-1 \end{cases}$$

where $s := \sum_{i=0}^{d} a_i$.

Proof. By the projection formula we have

$$R^i\pi_*\mathcal{O}_X(aH + bF) = R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(b)$$

being $R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $0 < i < d$ and all $a \in \mathbb{Z}$ and $R^d\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $a > -d-1$. Moreover, using the Base Change Theorem we get $R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(a) = 0$ for $i \geq d+1$.

Since, $R^i\pi_*\mathcal{O}_X(aH + bF) = 0$ for $i > 0$ and $a > -d-1$, by the degeneration of the Leray Spectral sequence

$$H^i(\mathbb{P}^1, R^j\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bF)) \Longrightarrow H^{i+j}(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH + bF))$$

we obtain

$$H^i(X, \mathcal{O}_X(aH + bF)) = H^i(\mathbb{P}^1, \pi_*\mathcal{O}_X(aH + bF)) \text{ for all } a > -d-1$$

with $\pi_*\mathcal{O}_X(aH + bF) = S^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^1}(b)$ if $a \geq 0$ and 0 otherwise. The case $a \leq -d-1$ follows from the case $a \geq 0$ and Serre's duality. Hence, the Lemma is proved. $\square$

Let E be a rank 2 vector bundle on a $(d+1)$-dimensional rational normal scroll X. Since $H^2(X, \mathbb{Z})$ is generated by the classes H and F, and $H^4(X, \mathbb{Z})$ is generated by the classes HF and H^2; the Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$, $i = 1, 2$ of E may be written as $c_1(E) = aH + bF$ and $c_2(E) = xH^2 + yHF$ with $a, b, x, y \in \mathbb{Z}$. Moreover, since a rank 2 vector bundle E on X is μ-stable with respect to an ample line bundle L if, and only if, $E \otimes \mathcal{O}_X(D)$ is μ-stable with respect to L for any divisor $D \in Pic(X)$, we may assume without loss of generality that $c_1(E)$ is one of the following: 0, H, F or $H + F$.

From now on, X will be a $(d+1)$-dimensional, rational, normal scroll. We will compute the dimension and prove the irreducibility, smoothness and rationality of the moduli spaces $M_L(2; c_1, c_2)$ of rank 2 vector bundles E on X with certain Chern classes c_1 and c_2; and μ-stable with respect to an ample line bundle L closely related to c_2. We want to stress that the ample line bundle L that we choose strongly depends on c_2 and our results turn to be untrue if we fix c_1, L and c_2L^{d-1} goes to infinity. Indeed, for $d = 2$ and fixed L, the minimal number of irreducible components of the moduli space $M_L(2; c_1, c_2)$ of rank

2, μ-stable vector bundles with respect to L with fixed c_1 and c_2L going to infinity grows to infinity (it follows from Theorem 6.1).

Our approach will be to write μ-stable with respect to L, rank 2 vector bundles E on X, as an extension of two line bundles. A well known result for vector bundles on curves is that any vector bundle of rank $r \geq 2$, can be written as an extension of lower rank vector bundles. For higher dimensional varieties we may not be able to get such a nice result. (For instance, it is not true for vector bundles on $X = \mathbb{P}^n$). However it turns to be true for certain μ-stable with respect to L, rank 2 vector bundles E on rational normal scrolls. In fact, using this idea we construct big enough families of μ-stable with respect to L, rank 2 vector bundles E on rational normal scrolls.

Construction 6.4. Let X be a $(d+1)$-dimensional, rational, normal scroll, c_2 an integer such that $c_2 > (H^{d+1} + d + 2)/2$ and $\epsilon \in \{0, 1\}$. We construct a rank 2 vector bundle E on X as a non-trivial extension

$$e: \quad 0 \longrightarrow \mathcal{O}_X(H - c_2F) \longrightarrow E \longrightarrow \mathcal{O}_X((c_2 + \epsilon)F) \longrightarrow 0. \tag{6.1}$$

We shall call $\mathcal{F}$ the irreducible family of rank two vector bundles constructed in this way.

Lemma 6.5. *With the above notation, let $L = dH + bF$ be an ample divisor on X with $b = 2c_2 - H^{d+1} - (1 - \epsilon)$. For any $E \in \mathcal{F}$, we have:*

(a) *$H^0E(-(c_2 + \epsilon)F) = 0$.*
(b) *E is a rank two, L-stable vector bundle on X with $c_1(E) = H + \epsilon F$ and $c_2(E) = (c_2 + \epsilon)HF$.*
(c) *$dim\mathcal{F} = h^1\mathcal{O}_X(H - (2c_2 + \epsilon)F) - 1$.*

Proof. (a) Using Lemma 6.3 and the hypothesis $c_2 > (H^{d+1} + d + 2)/2$, we get $H^0\mathcal{O}_X(H - (2c_2 + \epsilon)F) = 0$. Now we consider the exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{O}_X(H - (2c_2 + \epsilon)F)) \longrightarrow H^0(E(-(c_2 + \epsilon)F)) \longrightarrow H^0(\mathcal{O}_X) \xrightarrow{\delta}$$

$$H^1(\mathcal{O}_X(H - (2c_2 + \epsilon)F)) \longrightarrow H^1(E(-(c_2 + \epsilon)F)) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow \cdots$$

associated to the exact sequence (6.1). Since

$$H^1(\mathcal{O}_X(H - (2c_2 + \epsilon)F)) = Ext^1(\mathcal{O}_X(c_2 + \epsilon)F, \mathcal{O}_X(H - c_2F)),$$

the map

$$\delta : H^0(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X(H - (2c_2 + \epsilon)F))$$

given by $\delta(1) = e$ is an injection. This fact together with $H^0(\mathcal{O}_X(H - (2c_2 + \epsilon)F)) = 0$ gives us $H^0(E(-(c_2 + \epsilon)F)) = 0$.

(b) It is easy to see that for any $E \in \mathcal{F}$, $c_1(E) = H + \epsilon F$ and $c_2(E) = (c_2 + \epsilon)HF$. Let us see that E is L-stable, i.e., for any rank 1 subbundle $\mathcal{O}_X(D)$ of $E \in \mathcal{F}$ we obtain $DL^d < \frac{c_1(E)L^d}{2}$. For any subbundle $\mathcal{O}_X(D)$ of E we have

$$(1) \quad \mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(H - c_2F) \quad \text{or} \quad (2) \quad \mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X((c_2 + \epsilon)F).$$

In the first case, $D \equiv H - c_2F - C$ with C numerically equivalent to an effective divisor. Hence,

$$\begin{aligned} DL^d = (H - c_2F - C)L^d &\leq (H - c_2F)L^d = d^d(H^{d+1} + b - c_2) \\ &< \tfrac{d^d(H^{d+1}+b+\epsilon)}{2} = \tfrac{c_1(E)L^d}{2}. \end{aligned}$$

Assume $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X((c_2+\epsilon)F)$. From (a) we have $H^0E(-(c_2+\epsilon)F) = 0$. Therefore, $D \equiv (c_2 + \epsilon)F - C'$ with $C' \equiv nH + mF$ numerically equivalent to a non-zero effective divisor. Hence,

$$\begin{aligned} DL^d = ((c_2 + \epsilon)F - C')L^d &= ((c_2 + \epsilon)F - nH - mF)L^d \\ &= d^d(c_2 + \epsilon - 2nc_2 + n(1 - \epsilon) - m) \\ &< \tfrac{c_1(E)L^d}{2} = \tfrac{d^d(2c_2+2\epsilon-1)}{2} \end{aligned}$$

if, and only if, $-4nc_2 + 2n(1 - \epsilon) - 2m < -1$. Since C' is numerically equivalent to a non-zero effective divisor, we have $-m \leq n(H^{d+1} + d)$ and $n > 0$ or $n = 0$ and $m > 0$. By hypothesis $c_2 > \frac{H^{d+1}+d}{2} + 1$, therefore $-4nc_2 + 2n(1 - \epsilon) - 2m < -1$ and E is L-stable.

(c) $dim\mathcal{F} = dimExt^1(\mathcal{O}_X((c_2+\epsilon)F), \mathcal{O}_X(H-c_2F))-1 = h^1O_X(H-(2c_2+\epsilon)F)-1$. □

Remark 6.6. The existence of big enough families of indecomposable rank 2 vector bundles over $\mathbb{P}^d$-bundles of arbitrary dimension faces up to Hartshorne's conjecture on the non-existence of indecomposable rank 2 vector bundles over projective spaces $\mathbb{P}^n$, $n \geq 6$ (see [33]).

Now we are ready to state the main result of this section.

Theorem 6.7. *Let X be a $(d+1)$-dimensional, rational, normal scroll and c_2 an integer such that $c_2 > (H^{d+1} + d + 2)/2$. We fix the ample divisor $L = dH + bF$ on X with $b = 2c_2 - H^{d+1} - (1 - \epsilon)$ and $\epsilon = 0, 1$. Then $M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d+2)$.*

Sketch of the Proof. We divide proof in several steps.

<u>1^{st} *step*</u>: We prove that any vector bundle $E \in M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ sits in an exact sequence of the following type

$$0 \to \mathcal{O}_X(H - c_2F) \to E \to \mathcal{O}_X((c_2 + \epsilon)F) \to 0.$$

The key point for proving the 1st step is the fact that any rank two vector bundle $E \in M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$, $E(-H + c_2F)$ has a section whose scheme of zeros has codimension ≥ 2 ([10]; Proposition 2.8). Take a non-zero section $0 \neq s \in H^0E(-H+c_2F)$. Since,

$$c_2E(-H + c_2F) = c_2E + (H + \epsilon F)(-H + c_2F) + (-H + c_2F)^2 = 0,$$

the section s defines an exact sequence

$$0 \to \mathcal{O}_X(H - c_2F) \to E \to \mathcal{O}_X((c_2 + \epsilon)F) \to 0$$

which proves step 1.

<u>2^{nd} *step*</u>: For any vector bundle $E \in M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$, we compute the dimension of the Zariski tangent space of $M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ at the point $[E]$ and

we get

$$dimT_{[E]}M_L(2; H+\epsilon F,(c_2+\epsilon)HF) = 2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d+2).$$

In fact, by deformation theory $T_{[E]}M_L(2; H+\epsilon F,(c_2+\epsilon)HF) \cong Ext^1(E,E)$. Let us compute $dimExt^1(E,E)$. We have seen that any $E \in M_L(2; H+\epsilon F,(c_2+\epsilon)HF)$ sits in an extension

$$e: \qquad 0 \longrightarrow \mathcal{O}_X(H-c_2F) \longrightarrow E \longrightarrow \mathcal{O}_X((c_2+\epsilon)F) \longrightarrow 0.$$

Applying $Hom(.,E)$ to the exact sequence (6.1) we get

$$\begin{aligned} (6.2) \qquad & 0 \longrightarrow H^0(E(-(c_2+\epsilon)F)) \longrightarrow Hom(E,E) \longrightarrow H^0(E(c_2F-H)) \longrightarrow \\ & H^1(E(-(c_2+\epsilon)F)) \longrightarrow Ext^1(E,E) \longrightarrow H^1(E(c_2F-H)) \longrightarrow \cdots . \end{aligned}$$

Now, We consider the exact cohomology sequence

$$\begin{aligned} & 0 \longrightarrow H^0(\mathcal{O}_X(H-(2c_2+\epsilon)F)) \longrightarrow H^0(E(-(c_2+\epsilon)F)) \longrightarrow H^0(\mathcal{O}_X) \xrightarrow{\delta} \\ & H^1(\mathcal{O}_X(H-(2c_2+\epsilon)F)) \longrightarrow H^1(E(-(c_2+\epsilon)F)) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow \cdots \end{aligned}$$

associated to the exact sequence (6.1). By Lemma 6.5, $H^0(E(-(c_2+\epsilon)F)) = 0$. So, using the fact that $H^1\mathcal{O}_X = 0$, we get $H^1(E(-(c_2+\epsilon)F)) = H^1(\mathcal{O}_X(H-(2c_2+\epsilon)F)) - 1$.

On the other hand, the exact cohomology sequence

$$\begin{aligned} & 0 \longrightarrow H^0\mathcal{O}_X \longrightarrow H^0(E(c_2F-H) \longrightarrow H^0(\mathcal{O}_X((2c_2+\epsilon)F-H)) \longrightarrow \\ & H^1\mathcal{O}_X \longrightarrow H^1(E(c_2F-H)) \longrightarrow H^1(\mathcal{O}_X((2c_2+\epsilon)F-H)) \longrightarrow \cdots \end{aligned}$$

associated to the exact sequence e, gives us $h^0(E(c_2F-H) = 1$ and $H^1(E(c_2F-H)) = 0$.

Putting altogether, an taking into account that E is L-stable (and therefore simple), we get

$$\begin{aligned} dimExt^1(E,E) &= h^1E(-(c_2+\epsilon)F) - h^0E(c_2F-H) + dimHom(E,E) \\ &= h^1O_X(H-(2c_2+\epsilon)F) - 1 \\ &= 2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d+2). \end{aligned}$$

<u>3^{rd} *step*</u>: We prove that

$$M_L(2; H+\epsilon F,(c_2+\epsilon)HF) \cong \mathbb{P}(Ext^1(\mathcal{O}_X((c_2+\epsilon)F), \mathcal{O}_X(H-c_2F))).$$

Using Lemma 6.5 and the universal property of the moduli space $M_L(2; H+\epsilon F,(c_2+\epsilon)HF)$, we deduce the existence of a morphism

$$\phi : \mathcal{F} \longrightarrow M_L(2; H+\epsilon F,(c_2+\epsilon)HF)$$

which is an injection. In fact, assume that there are two non-trivial extensions

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(H-c_2F) \xrightarrow{\alpha_1} E \xrightarrow{\beta_1} \mathcal{O}_X((c_2+\epsilon)F) \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_X(H-c_2F) \xrightarrow{\alpha_2} E \xrightarrow{\beta_2} \mathcal{O}_X((c_2+\epsilon)F) \longrightarrow 0. \end{aligned}$$

Since $Hom(\mathcal{O}_X(H-c_2F), \mathcal{O}_X((c_2+\epsilon)F)) = H^0\mathcal{O}_X(-H+(2c_2+\epsilon)F) = 0$ (Lemma 6.3), we have $\beta_2 \circ \alpha_1 = \beta_1 \circ \alpha_2 = 0$. So, there exists $\lambda \in Aut(\mathcal{O}_X(H-c_2F)) \cong k$ such that $\alpha_2 = \alpha_1 \circ \lambda$. Therefore, ϕ is an injection. By the first step of the proof, ϕ is surjective.

In addition, $dim\mathcal{F} = dimT_{[E]}M_L(2; H + \epsilon F, (c_2 + \epsilon)HF) = 2(d + 1)c_2 - H^{d+1} + \epsilon(d + 1) - (d + 2)$. Therefore, it follows that $M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ is smooth at any point E of $M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ and, moreover,

$$M_L(2; H + \epsilon F, (c_2 + \epsilon)HF) \cong \mathbb{P}(Ext^1(\mathcal{O}_X((c_2+)F, \mathcal{O}_X(H - c_2F)).$$

It follows from the last isomorphism that the moduli space $M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d + 2)$ which proves what we want. □

Analogously we prove

Theorem 6.8. *Let X be a $(d + 1)$-dimensional, rational, normal scroll and c_2 an integer such that $c_2 > H^{d+1} + d + 1$. We fix the ample divisor $L = dH + bF$ on X with $b = c_2 - H^{d+1} - (1 - \epsilon)$ and $\epsilon = 0, 1$. Then $M_L(2; \epsilon F, (2c_2 + \epsilon)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(\epsilon d + 1)c_2 - H^{d+1} + 2(\epsilon - 1) - \epsilon(d + 2)H^{d+1}$.*

Remark 6.9. Using Bogomolov's inequality $(4c_2(E) - c_1(E)^2) \cdot H^{d-1} > 0$, we can see that the hypothesis $2c_2 > (H^{d+1} + d + 2)$ (resp. $c_2 > H^{d+1} + d + 2$) when $c_1(E) = H + \epsilon F$ (resp. $c_1(E) = \epsilon F$) with $\epsilon \in \{0, 1\}$ is not too restrictive.

Theorems 6.7 and 6.8 reflect nicely the general philosophy that (at least for suitable choice of the Chern classes and the ample line bundle) the geometry of the underlying variety and of the moduli spaces are intimately related. We hope that phenomena of this sort will be true for other high dimensional varieties.

7. Exercises

During the School Laura Costa will give two exercise sessions where she will make examples and exercises and she will introduce some topics that are complementary to the standard lectures. In this section, we collect these exercise sessions.

Exercise 7.1. Show that $c_i(T_{\mathbb{P}^n}(-1)) = 1$ for all $1 \leq n$. (Hint: Use Euler sequence).

Exercise 7.2. Compute $c_t(\Omega^1_{\mathbb{P}^n}(2))$ and show that $c_n(\Omega^1_{\mathbb{P}^n}(2)) = 0$ if and only if n is odd.

Exercise 7.3. Let E be a vector bundle of rank r on $\mathbb{P}^n$. Define the discriminant of E by $\Delta(E) := 2rc_2E - (r-1)c_1^2(E)$. Show that $\Delta(E \otimes L) = \Delta(E)$ for any line bundle L on $\mathbb{P}^n$.

Exercise 7.4. Let X be a smooth irreducible surface and $Z \subset X$ a 0-dimensional subscheme of length l. Consider D_1 and D_2 divisors on X and let E be the rank 2 vector bundle on X given by

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow E \longrightarrow I_Z(D_2) \longrightarrow 0.$$

Prove that $c_1(E) = D_1 + D_2$ and $c_2(E) = D_1D_2 + l$.

Exercise 7.5. (a) Let E be a vector bundle of rank r on a smooth projective variety. Show that $\mu(S^tE) = t\mu(E)$ and $\mu(\wedge^tE) = t\mu(E)$.

(b) Let E and F be vector bundles on a smooth projective variety X of rank e and f, respectively. Show that $c_1(E \otimes F) = fc_1(E) + ec_1(F)$.

Exercise 7.6. ($Char(k) = 0$) Let X be a smooth, irreducible, projective variety and let H be an ample divisor on X. Show that if E is a torsion free sheaf μ-semistable with respect to H then $\mathcal{E}nd(E)$, $\wedge^r(E)$ and $S^r(E)$ are again μ-semistable with respect to H.

Is it true for locally free-sheaves $\mu - stable$ with respect H?

Exercise 7.7. A rank r vector bundle on $\mathbb{P}^n$ is called simple if $\text{Hom}(E, E) = k$. A first example for a simple vector bundle is the tangent bundle of $\mathbb{P}^n$ (Prove it!). Show that μ-stable bundles are simple. Show that any simple rank 2 vector bundle on $\mathbb{P}^n$ is μ-stable.

Is it true for vector bundles of higher rank?

Exercise 7.8. (Serre's correspondence)

(a) Let X be a smooth irreducible projective surface, $Z = \{p_1, \cdots, p_s\} \subset X$ a 0-dimensional subscheme of distinct reduced points and L, L' line bundles on X. Then, there exist a rank two vector bundle E on X given by a non-trivial extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow I_Z \otimes L' \longrightarrow 0$$

if and only if every section of $L^{-1} \otimes L' \otimes K_X$ which vanishes at all but one of the p_i vanishes at the remaining point as well. Notice that in this case, Z is the zero locus of a non-zero global section of $E \otimes L^{-1}$ (For details see [24]; Section 2, Theorem 12).

(b) Let X be a smooth irreducible projetive variety of dimension $n \geq 3$, $Y \subset X$ a codimension two subscheme and L a line bundle on X such that $H^1(X, L^{-1}) = H^2(X, L^{-1}) = 0$. Then, Y occurs as the scheme of zeros of a section of a rank two vector bundle E on X with $det(E) \cong L$ if and only if Y is a locally complete intersection and $\omega_Y \cong \omega_X \otimes \mathcal{O}_Y \otimes L$.

In such case, E sits in an exact sequence

$$0 \longrightarrow L^* \longrightarrow E^* \longrightarrow I_Y \longrightarrow 0.$$

(See [34]; Theorem 1.1 and Remark 1.1.1 for the rank two case and [80] for a generalization to the arbitrary rank case).

Exercise 7.9. (Elementary transformations) Let E be a rank r vector bundle on a smooth irreducible algebraic variety and $Z \subset X$ a hypersurface. Denote by $i : Z \hookrightarrow X$ the embedding and let F be a quotient bundle of the restriction i^*E. Show that there exists a rank r vector bundle G on X given by the exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow i_*F \longrightarrow 0.$$

The vector bundle G is called the Elementary Transformation of E along (Z, F).

Exercise 7.10. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a quadric surface and denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. Let $Z \subset X$ a 0-dimensional scheme of length 6 and such that $H^0(I_Z(2l+m)) = 0$. Consider E a rank 2 vector bundle on X given by a non-trivial extension

$$0 \longrightarrow \mathcal{O}_X(-l) \longrightarrow E \longrightarrow I_Z(l+m) \longrightarrow 0.$$

Prove:
(a) $c_1(E) = m$ and $c_2(E) = 5$.
(b) $h^0(E(l)) = 1$.
(c) E is simple

Exercise 7.11. Let E be a rank 2 vector bundle on $\mathbb{P}^3$ associated to a curve $Y \subset \mathbb{P}^3$, so we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-c_1(E)) \longrightarrow E(-c_1(E)) \longrightarrow I_Y \longrightarrow 0.$$

Show that E is μ-stable (resp. μ-semistable) if and only if $c_1(E) > 0$ (resp. $c_1(E) \geq 0$) and Y is not contained in any surface of degree $\leq \frac{c_1(E)}{2}$ (resp. $< \frac{c_1(E)}{2}$).

Exercise 7.12. Fix integers $a_1 \geq a_2 \geq \cdots \geq a_k$ and $b_1 \geq b_2 \geq \cdots \geq b_{n+k}$ such that $a_1 < b_{n+1}$, $a_2 < b_{n+2}$, $\cdots$, $a_k < b_{n+k}$. Let E be a rank n vector bundle on $\mathbb{P}^n$ sitting in an exact sequence

$$0 \longrightarrow \oplus_{i=1}^k \mathcal{O}_{\mathbb{P}^n}(a_i) \longrightarrow \oplus_{j=1}^{n+k} \mathcal{O}_{\mathbb{P}^n}(b_j) \longrightarrow E \longrightarrow 0.$$

Show that the following conditions are equivalent
(a) E is μ-stable,
(b) $H^0(\mathbb{P}^n, E(m)) = 0$ for $m \leq -\mu(E)$,
(c) $b_1 < \mu(E) = (\sum_{j=1}^{n+k} b_j - \sum_{i=1}^k a_i)/n$,
(d) $H^0(\mathbb{P}^n, \wedge^q E(m)) = 0$ for all $m \leq -q\mu(E)$, $1 \leq q \leq n-1$.

Exercise 7.13. Let E be a rank r vector bundle on a smooth irreducible surface X with $c_1(E) = 0$. Using the Hirzebruch-Riemman-Roch Theorem prove that

$$\chi(S^n E) = -\frac{\Delta(E)}{2r} \frac{n^{r+1}}{(r+1)!} + \text{ terms of lower order in } n.$$

Exercise 7.14. Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a quadric surface and denote by ℓ and m the standard basis of $Pic(X) \cong \mathbb{Z}^2$. Consider the ample line bundles $L_0 = \ell + 11m$, $L_1 = \ell + 9m$ and $L_2 = \ell + 7m$ on X. Prove:
(i) $M_{X,L_0}(2; \ell, 5) = \emptyset$,
(ii) $M_{X,L_1}(2; \ell, 5) = \mathbb{P}^{17}$, and
(iii) $M_{X,L_2}(2; \ell, 5)$ is a non-empty open subset of $\mathbb{P}^{17}$ of codimension 2.

Exercise 7.15. Prove the following statements:
(a) There exists a rank 2 stable vector bundle E on $\mathbb{P}^2$ with $c_1(E) = -1$ and $c_2(E) = c_2$ if and only if $c_2 \geq 1$.
(b) There exists a rank 2 stable vector bundle E on $\mathbb{P}^2$ with $c_1(E) = 0$ and $c_2(E) = c_2$ if and only if $c_2 \geq 2$.

Exercise 7.16. Prove that any instanton bundle E on $\mathbb{P}^{2n+1}$ is simple.
(Hint: Let $0 \longrightarrow \mathcal{O}(-1)^k \xrightarrow{\alpha} \mathcal{O}^{2n+2k} \xrightarrow{\beta} \mathcal{O}(1)^k \longrightarrow 0$ be the monad associated to E. Apply Hoppe's criterion to see that $K = Ker(\beta)$ is stable and hence simple. Then prove that $h^0(E \otimes E^*) \leq h^0(K \otimes K^*)$).

8. Open problems

In this section we collect the open problems that were mentioned in the preceding sections, and add some more.

1. Is $M_{\mathbb{P}^2}(r; c_1, c_2)$ rational?
2. What is the maximal dimension of a complete subvariety $S \subset M_{\mathbb{P}^2}(2; c_1, c_2)$?
3. Are the moduli spaces $MI_{\mathbb{P}^3}(k)$ non-singular, irreducible and rational?
4. Investigate the behavior of (semi-)stable vector bundles on $\mathbb{P}^n$ under restriction to hyperplanes.
5. Estimates of Chern classes of rank r, μ-semistable vector bundles on $\mathbb{P}^n$. Even more, which Chern classes $c_i \in H^{2i}(X, \mathbb{Z})$ are realized by rank r, μ-(semi-)stable vector bundles E on a smooth projective variety X?
6. Let E be a vector bundle on $\mathbb{P}^n$ such that for some k, $0 < k < n$, $h^i E(*) = 0$ for $0 < i < n$ with $i \neq k$, $H^k E(*) \neq 0$. Does it follows that $rk(E) \geq \binom{n}{k}$?
7. Let X be a smooth, irreducible, projective surface and let H be an ample divisor on X. Denote by $M_{X,H}(r; L, n)$ (resp. $\overline{M}_{X,H}(r; L, n)$) the moduli space of rank r, vector bundles (resp. torsion free sheaves) E on X, μ-stable (resp. GM-semistable) with respect to H with $det(E) = L \in Pic(X)$ and $c_2(E) = n \in \mathbb{Z}$.
   (a) Assume that X is of general type. Is $M_{X,H}(r; L, n)$ of general type?.
   (b) Assume $Kod(X) = 1$. Is it true that $dim M_{X,H}(r; L, n) = 2Kod(M_{X,H}(r; L, n))$?.
8. The bigger $n - r$ is the more difficult it becomes to find rank r vector bundles on $\mathbb{P}^n$. For which n and r are there indecomposable rank r vector bundles on $\mathbb{P}^n$?
9. Investigate the stability of a mathematical instanton bundle E on $\mathbb{P}^{2n+1}$ with quantum number k (It remains an open problem when $k \geq 2$ and $n \geq 3$).

Appendix A. Chern classes

In this appendix we will outline the theory of Chern classes. All the details can be found in [30] and [35].

Let X be a smooth irreducible projective variety of dimension n. By a cycle on X we mean an element of the free abelian group generated by all closed irreducible subvarieties of X. If all the components of a cycle have codimension r, we say that the cycle has codimension r. It is possible to introduce an equivalence relation, called *rational equivalence*, on codimension r-cycles which coincides with linear equivalence for $r = 1$. The group of equivalence classes is denoted by $A^r(X)$. There exists an intersection pairing $A^r(X) \times A^s(X) \to A^{r+s}(X)$ making

$$A(X) := \oplus_{i=0}^{n} A^i(X)$$

into a commutative graded ring, called the Chow ring of X.

Example A.1. If $X = \mathbb{P}^n$, then

$$A(X) = \mathbb{Z}[h]/(h^{n+1})$$

where $h \in A^1(X)$ is the class of a hyperplane.

If E is a vector bundle on X, we can define certain classes, called the *Chern classes of* E: $c_s(E) \in A^s(X)$, $0 \leq s \leq n$ with $c_0(E) = 1$ and $c_s(E) = 0$ for $s > rank(E)$.

By the *total Chern class* of E we mean

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) \in A(X)$$

and by the *Chern polynomial* we mean

$$c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + \cdots + c_n(E)t^n.$$

One can show that this theory of Chern classes is uniquely determined by the following three properties:

(C1) If $\mathcal{L}$ is a line bundle on X, then it is of the form $\mathcal{L} = \mathcal{O}_X(D)$ for some divisor D on X and then $c_1(\mathcal{L})$ is the class of D in $A^1(X)$.

(C2) If

$$0 \to E \to F \to G \to 0$$

is an exact sequence of vector bundles then $c_t(F) = c_t(E) \cdot c_t(G)$.

(C3) If $f : X \to Y$ is a morphism and E a vector bundle on Y then

$$c(f^*E) = f^*c(E).$$

There are also formalisms for computing the Chern classes of exterior powers, tensor product, etc.

Example A.2. If $X = \mathbb{P}^n$, then $A(X) = \mathbb{Z}[h]/(h^{n+1}) \cong \mathbb{Z}$ and we can view the Chern classes of a vector bundle E over $\mathbb{P}^n$ as integers.

Example A.3. If E is a coherent sheaf of rank $r \geq 0$ on a non-singular projective variety X of dimension n and L is a line bundle on X then

$$c_k(E \otimes L) = \sum_{i=0}^{s} \binom{r-i}{s-i} c_i(E) L^{s-i}.$$

Riemann-Roch's theorem allows to express the Euler characteristic of a coherent sheaf in terms of its Chern classes. In particular, we have the following useful formula

Theorem A.4. *Let E be a torsion free sheaf of rank $r \geq 0$ on a smooth projective surface X. Let c_1 and c_2 be the Chern classes of E. Then,*

$$\chi(E) = \sum_{i=0}^{2}(-1)^i dim H^i(X, E) = r(1 + p_a(X)) - \frac{c_1 K}{2} + \frac{c_1^2 - 2c_2}{2}$$

where K denotes the class of the canonical line bundle on X.

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