A NOTE ON BRANCHED CYCLIC COVERINGS OF SPHERES

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By a *represented link* (nudo coloreado) is meant a link L of, say, μ components in the 3-dimensional sphere S^3, together with a representation $\omega\colon \pi(S^3 - L) \longrightarrow \mathcal{S}_d$ of the link group into the symmetric group of permutations of the d symbols $0, 1, \ldots, d-1$. To each represented link (L, ω) there is uniquely associated an orientable closed 3-dimensional manifold $M = M(L, \omega)$, the covering of S^3 branched over L that is determined by the representation ω. The manifold $M(L, \omega)$ is connected if and only if ω is transitive.

Study of the function $\Phi\colon (L, \omega) \longrightarrow M(L, \omega)$ seems to be an interesting problem, and, in view of a theorem of Alexander [1] that states that any orientable closed 3-dimensional manifold is $M(L, \omega)$ for some link L and representation ω, it might lead to new insights on the problem of classifying these manifolds.

It was indicated by Alexander [1] and proved in detail by Montesinos [6, 7] (cf. also [3]) that $\Phi^{-1}(M)$ always contains at least one (L, ω) for which ω represents each meridian of L by an appropriate transposition; let us call such representations *simple*.

It was shown in [5] that an alteration of a certain type D (cf. [4], p. 199) replaces a transitive simply represented link (L, ω) by another transitive simply represented link (L', ω') which is such that $d = d'$ and $M(L, \omega) = M(L', \omega')$. It may happen that a link that is not split is thereby replaced by a link L' that splits; when this happens $L' = L_1 \cup L_2$, where L_1 is contained in a 3-cell whose complementary 3-cell contains L_2, the transitive simple representation

$$\omega'\colon \pi(S^3 - L') \longrightarrow \mathcal{S}_d$$

induces transitive simple representations

$$\omega_1\colon \pi(S^3 - L_1) \longrightarrow \mathcal{S}_{d_1},$$

and

$$\omega_2\colon \pi(S^3 - L_2) \longrightarrow \mathcal{S}_{d_2},$$

where

$$2 \leqslant d_1 \leqslant d, \quad 2 \leqslant d_2 \leqslant d, \quad d_1 + d_2 \geqslant d + 1,$$

and $M(L', \omega')$ decomposes into the connected sum of $M(L_1, \omega_1)$ and $M(L_2, \omega_2)$ and $d_1 + d_2 - d - 1$ copies of $S^2 \times S^1$. Hence if $M(L, \omega)$ is irreducible then $d_1 + d_2 = d + 1$, $M(L_2, \omega_2)$, say, is a 3-sphere, and

$$M(L_1, \omega_1) = M(L, \omega).$$

Of course $d_1 < d$. Since such a splitting is not possible unless $d > 2$, it is natural to make the following conjecture:

CONJECTURE A ([5, V.6.1, p. 113]).—*Every transitive simply represented link (L, ω) can be transformed by operations D into a represented link (L^*, ω^*) that splits into represented links $(L_1, \omega_1), \ldots, (L_m, \omega_m)$ such that* $d_i \leqslant 2$, $i = 1, \ldots, m$. Such a represented link (L, ω) was called *separable* by Montesinos ([5, p. 113]).

Since $M(L, \omega)$ is S^3 when $d = 1$, and $M(L, \omega)$ is a cyclic branched covering of S^3 when $d = 2$, conjecture A is complimented by the following conjecture:

CONJECTURE B ([5, I.1.1, p. 9]).—*A cyclic branched covering of degree 2 of S^3 is never simply connected, except in the trivial case $L = 0$.*

As observed in [5], conjectures A and B together imply the Poincaré conjecture (cf. also [4, p. 199]). It was shown in [5, p. 9] that Poincaré's conjecture implies conjecture B; however Poincaré's conjecture does not imply conjecture A.

The principal object of this note is to prove a theorem about cyclic branched covering spaces of homology (*) spheres, that shows in particular that conjecture A is not true as stated. I shall then discuss

(*) All homology considered is with integral coefficients.

the possibility of modifying conjecture A, and finally I shall indicate a reformulation of conjecture B.

A branched covering of an n-dimensional (combinatorial) manifold N is an n-dimensional manifold M together with a (non-degenerate) simplicial mapping e of M onto N that maps the star of any $(n-1)$-dimensional simplex of M topologically onto its image (cf. [2]). The branch set L is a pure $(n-2)$-dimensional subcomplex of N. (It should be noted that given an $(n-2)$-dimensional subcomplex L of N and a representation in $\mathcal{S}_d$ of $\pi(\mathrm{N}-\mathrm{L})$ the branched covering M that is determined might not be a manifold; in order to ensure that M be a manifold it is however sufficient to assume that L is a locally flat submanifold. I shall deal with this and related questions in another place.)

THEOREM 1.—*If an* n-*dimensional connected orientable closed manifold M is a* d-*fold cyclic branched covering of a homology sphere N, then there is an autohomeomorphism* t *of M such that, for* $k = 1, 2, \ldots, n-1$, *the matrix* $\mathbf{T}$, *of the induced automorphism*

$$t_*\colon \mathrm{H}_k(\mathrm{M}) \longrightarrow \mathrm{H}_k(\mathrm{M})$$

satisfies the equation

$$\mathbf{B}+\mathbf{T}+\mathbf{T}^2+\ldots+\mathbf{T}^{d-1}=0.$$

PROOF.—Let σ and τ range over the simplexes of N of dimension $k+1$ and k respectively. For each σ choose a $(k+1)$-dimensional simplex $\bar{\sigma}$ of M that projects onto σ, and for each τ choose a k-dimensional simplex $\bar{\tau}$ of M that projects onto τ. Thus

$$e^{-1}(\sigma) = \cup_{i=0}^{d-1} t^i\,\bar{\sigma}$$

and

$$e^{-1}(\tau) = \cup_{i=0}^{d-1} t^i\,\bar{\tau}.$$

Notice that if σ or τ lies in the branch set these simplexes of M are not all distinct. This, however, does not affect the following argument.

Consider an arbitrary k-dimensional cycle

$$\bar{\beta} = \Sigma_\tau \Sigma_{i=0}^{d-1} g(t^i\,\bar{\tau})\, t^i\,\bar{\tau}$$

of M. Then

$$\Sigma_{j=0}^{d-1} t^j\,\bar{\beta} = \Sigma_\tau \Sigma_{i,j=0}^{d-1} g(t^i\,\bar{\tau})\, t^{i+j}\,\bar{\tau} = \Sigma_\tau \Sigma_{h=0}^{d-1} g(\tau)\, t^h\,\bar{\tau},$$

where

$$g(\tau) = \Sigma_{i=0}^{d-1} g(t^i\,\bar{\tau}),$$

and

$$\beta = e_*(\bar{\beta}) = \Sigma_\tau\, g(\tau)\,\tau.$$

Since N is a homology sphere there is a $(k+1)$-dimensional chain

$$\alpha = \Sigma_\sigma f(\sigma)\,\sigma$$

of N such that $\partial\alpha = \beta$. Let $\varepsilon(\sigma, \tau)$ denote the incidence number of σ and τ. Thus $\partial\sigma = \Sigma_\tau\, \varepsilon(\sigma,\tau)\,\tau$. Hence

$$\beta = \Sigma_\sigma f(\sigma)\,\partial\sigma = \Sigma_{\sigma,\tau} f(\sigma)\,\varepsilon(\sigma,\tau)\,\tau$$

and consequently

$$g(\tau) = \Sigma_\sigma f(\sigma)\,\varepsilon(\sigma,\tau).$$

Now let us consider the transfer of α, the chain

$$\bar{\alpha} = \Sigma_\sigma f(\sigma)\, \Sigma_{i=0}^{d-1} t^i\,\bar{\sigma}.$$

Its boundary is

$$\partial\bar{\alpha} = \Sigma_\sigma f(\sigma)\,\partial\,(\Sigma_{i=0}^{d-1} t^i\,\bar{\sigma}) = \Sigma_{\sigma,\tau} f(\sigma)\,\varepsilon(\sigma,\tau)\,\Sigma_{h=0}^{d-1} t^h\,\bar{\tau} =$$

$$= \Sigma_\tau\, g(\tau)\, \Sigma_{h=0}^{d-1} t^h\,\bar{\tau} = \Sigma_{j=0}^{d-1} t^j\,\bar{\beta}.$$

This shows that $\bar{\beta} + t\,\bar{\beta} + \ldots + t^{d-1}\,\bar{\beta} \sim 0$ in M, and hence that

$$\mathbf{E} + \mathbf{T} + \ldots + \mathbf{T}^{d-1} = 0.$$

COROLLARY. — *If* $n \geqslant 3$ *the n-dimensional torus* $S^1 \times S^1 \times \ldots \times S^1$ *cannot be a 2-fold cyclic branched covering of* S^n.

PROOF.—If $\gamma_1, \ldots, \gamma_n$ is a basis for $H_1\,(S^1 \times S^1 \times \ldots \times S^1)$ then according to the theorem, there is an autohomeomorphism t of M for which the induced automorphism t_* of H_1 must be the automorphism $\gamma_i \longrightarrow -\gamma_i$, $i = 1, \ldots, n$. Since $S^1 \times S^1 \times \ldots \times S^1$ is an aspherical space it follows from a standard Hurewicz theorem that t is homotopic to the mapping

$$(z_1, z_2, \ldots, z_n) \longrightarrow (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n),$$

where S^1 denote the subset $\{z \mid |z| = 1\}$ of the complex plane. But this implies that, for $1 \leqslant k \leqslant n-1$, the matrix of the automorphism $t_*: H_k \longrightarrow H_k$ is $(-1)^k\,\mathbf{E}$, and, for $n \geqslant 3$, this contradicts the theorem.

Now for the represented link (L, ω) of figure 1, the manifold

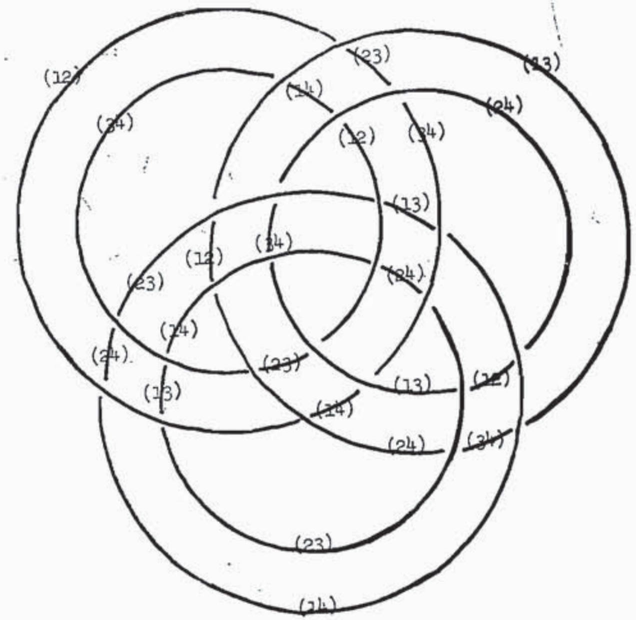

Fig. 1.

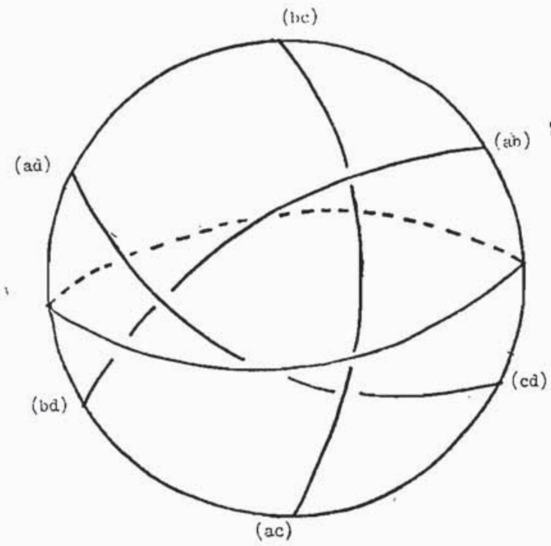

Fig. 2.

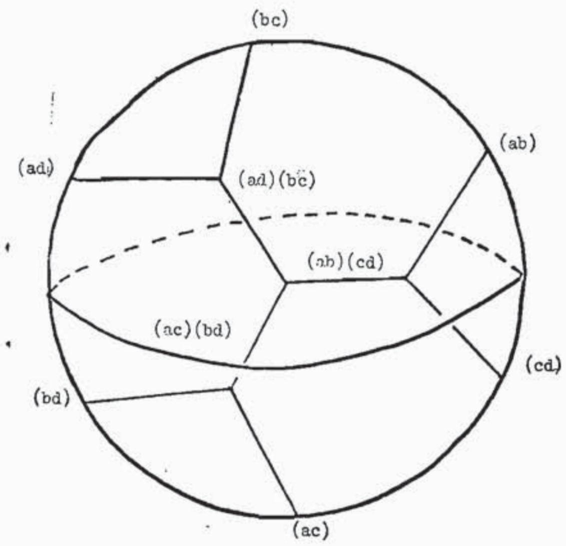

Fig. 3.

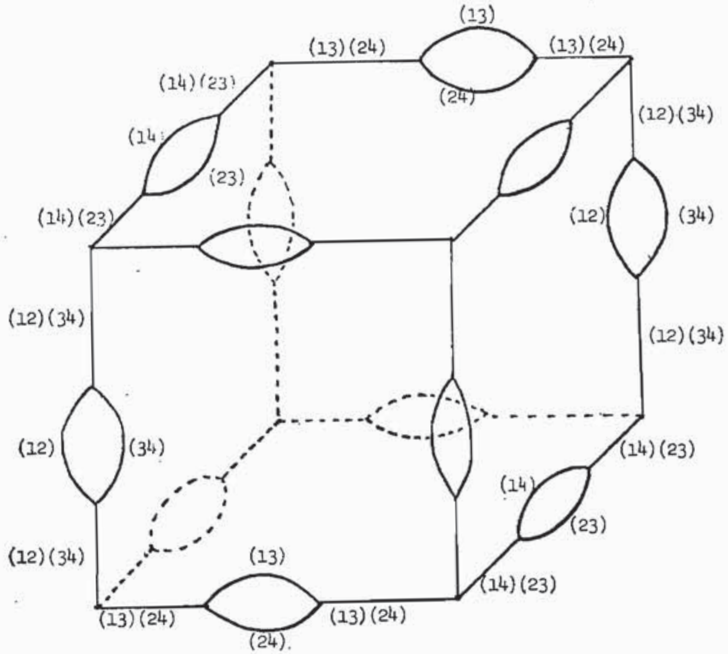

Fig. 4.

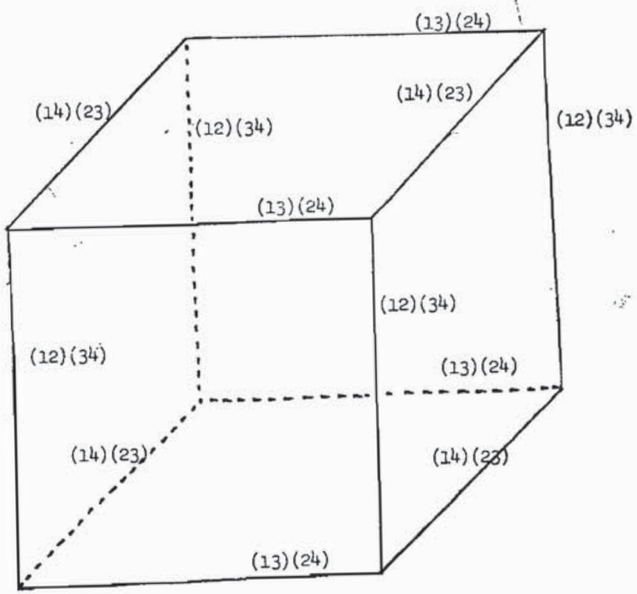

Fig. 5.

M (L, ω) has the same fundamental group as the torus $S^1 \times S^1 \times S^1$. That it is, in fact, $S^1 \times S^1 \times S^1$ has been demonstrated to be by J. M. Montesinos. A sketch of the proof is as follows: Since over a 3-cell branched as shown in figure 2 is a 3-cell, and the same is true over a 3-cell branched as shown in figure 3, the manifold M (L, ω) is not altered by any modification of (L, ω) that replaces figure 2 by figure 3 in an appropriate 3-cell. After modifying figure 1 in this manner at eight places, figure 4 is obtained and this can be replaced by figure 5. From figure 5 one can exhibit M (L, ω) as $S^1 \times S^1 \times S^1$ conveniently subdivided into eight cubes. (I think that a *direct* proof that this represented link is not separable would be very difficult, if not impossible.)

Of course to prove the Poincaré conjecture it would not be necessary for conjecture A to apply to every represented link (L, ω). It would be good enough if it applied at least to those represented links (L, ω) for which M (L, ω) is simply connected. Thus we may replace conjecture A by the following weaker conjecture:

CONJECTURE A'. — *Every transitive simply represented link (L, ω) for which $\pi(M(L, \omega)) = 1$ can be transformed by operations D into a represented link (L^*, ω^*) that splits into represented links*

$$(L_1, \omega_1), \ldots, (L_m, \omega_m)$$

such that $d_i \leqslant 2$, $i = 1, \ldots, m$.

Of course there is no obvious way in which Poincaré's conjecture implies conjecture A' either. This is because of the *possibility* of the existence of a transitive simply represented link (L, ω) that is not separable even though $M(L, \omega) = S^3$.

Now let us consider the question of whether a 2-fold cyclic branched covering of S^3 can be simply connected if L is not a single unknotted curve.

In the case of a 2-fold cyclic covering $e: M \longrightarrow S^3$ of S^3 branched over L, the group $\pi(M - K)$, where $K = e^{-1}(L)$, is mapped by e_* onto a subgroup of $\pi(N - L)$ of index 2, and a group isomorphic to $\pi(M)$ can be obtained by adjoining to this group $e_* \pi(M - K)$ the relations $x^2 = 1$ that set equal to 1 the squares of the meridians. Now this is the same thing as adjoining these relations directly to the group $\pi(N - L)$ and then, afterwards, taking the subgroup of index 2 of the resulting homomorph of $\pi(N - L)$. Hence $\pi(M) = 1$ if and only if the group π (N — L)/$\langle x^2 = 1\rangle$ is cyclic of order 2. Since abelianization maps the group π (N — L)/$\langle x^2 = 1\rangle$ onto the direct product of μ cyclic groups of order 2, π (N — L)/$\langle x^2 = 1\rangle$ cannot possibly be cyclic of order 2 unless $\mu = 1$, i. e. unless L is a single connected knot. Thus we may reformulate conjecture B as follows:

Conjecture B′.—If the group G of a knot is non-abelian then the quotient group G/$\langle x^2 = 1\rangle$ obtained by setting $=1$ the square of a meridian is also non-abelian.

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