ISSAC '94 Proceedings of the International Symposium on Symbolic and Algebraic Computation. Oxford, United Kingdom — July 20 - 22, 1994.ACM Press. New York. Pages 354 - 358. doi>10.1145/190347.190439

On the symbolic insimplification of the general 6R-manipulator kinematic equations

T. Recio.*† and M. J. González-López*†
Dpto. Matemáticas, Estadística y Computación
Universidad de Cantabria, Santander 39071, Spain

Abstract

When symbolically solving inverse kinematic problems for robot classes, we deal with computations on ideals representing these robot's geometry. Therefore, such ideals must be considered over a base field K, where the parameters of the class (and also the possible relations among them) are represented. In this framework we shall prove that the ideal corresponding to the general $6R$ manipulator is real and prime over K. The practical interest of our result is that it confirms that the usual inverse kinematic equations of this robot class do not add redundant solutions and that this ideal cannot be "factorized", establishing therefore, Kovács [7] conjecture. We prove also that this robot class has six degrees of freedom (i.e. the corresponding ideal is six-dimensional), even over the extended field K, which is the algebraic counterpart to the fact that the $6R$ manipulator is completely general. Our proof uses, as intermediate step, some dimensionality analysis of the Elbow manipulator, which is a specialization of the *6R*.

1 Introduction

Recently, a lot of attention has been devoted to the kinematic problem for the general 6R manipulator, see [11], [12], [15], [17], [18]. After twenty five years of research on this subject, starting with the work of Pieper [14] in 1.968, the complete symbolic solution to the inverse kinematic problem for this manipulator has been presented in [11] and [15]. Following this pattern in a numerical framework, [12] presented a real time algorithm for solving the inverse kinematics of the general $6R$ manipulator.

*Partially supported by CICyT PB 92/0498/C02/01.
†Partially supported by Esprit/Bra 6846 (Posso).

On the other hand, Kovács [7] has studied different triangulations of the kinematic equations system of this manipulator, searching for simplifications on the corresponding determining equations. In particular, he explicitly conjectured that the kinematic equations system should generate a prime ideal ([7], pg. 331). In fact, ([7], pg. 334) he mentions that *"extensive tests for ideal factorization were performed and no factorization was found. However the irreducibility could not be established with certainty"*. As a consequence, he comments that, with high probability, the solution presented in [11] and [15] must be optimal in some sense.

In this paper we show that, in fact, the inverse kinematic ideal of the general $6R$ manipulator is a prime ideal. Moreover, we show that this is also the best possible ideal in some other sense that was not regarded by Kovács, namely, that this is a real ideal, which roughly means that it describes the solutions of the inverse kinematic problem without redundancy. We remark that, when working in a complex number setting, primality alone implies this property, but this is not true anymore when one is interested in the real solutions (which is, usually, the case in Robotics): $\langle x^2+y^2\rangle$ is a prime ideal in $\mathbf{R}[x,y]$ but its solutions are represented in a simpler way by the real ideal $\langle x,y\rangle$. Therefore, we consider that proving both primality and reality, is required to establish the idea behind Kovács conjecture.

Moreover, we obtain these two properties for two different approaches to the concept of the inverse kinematic ideal: first, as in the formulation of Kovács, that follows [2] (see our Proposition 3.1); and second, with an enhanced version of inverse kinematic ideal that involves a more symbolic setting, in which the parameters are considered as true indeterminates and not merely nameholders for numbers (Theorem 4.2). Completing the algebraic study of these ideals, we show that the corresponding hand ideals have the expected dimension (six) for the general $6R$ manipulator (Theorem 4.1). This is achieved showing that the particular instance of the $6R$, namely the Elbow manipulator, has also this dimension for the hand ideal (Lemma 4.1) and, in order to obtain this, we use Paul solution ([13])

to the inverse kinematic problem of this robot.

2 Symbolic approaches to the inverse kinematic problem of the general 6R manipulator

Following Denavit-Hartenberg formalism ([4]) to represent the geometry of the $6R$-manipulator, a coordinate system is attached to each link in order to describe the relative positions among the links. The 4×4 homogeneous matrix relating the $i+1$ coordinate system to the i coordinate system is (considering the links numbered from the base, which is link 1):

$$A_i^* = \begin{pmatrix} c_i & -s_i\lambda_i & s_i\mu_i & a_ic_i \\ s_i & c_i\lambda_i & -c_i\mu_i & a_is_i \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; i = 1, \dots, 6$$

where s_i and c_i represent the sine and cosine, respectively, of the rotation angle at joint i; λ_i and μ_i represent the sine and cosine, respectively, of the twist angle between the axes of joints i and $i+1$; a_i is the length of link $i+1$; and d_i is the offset distance at joint i (see [12] and [15]).

Let us consider now the group $SE(3)$ of matrices of the form:

$$P = \begin{pmatrix} H & V \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & v_1 \\ h_{21} & h_{22} & h_{23} & v_2 \\ h_{31} & h_{32} & h_{33} & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the matrix H is a proper orthogonal matrix, i.e. it is an element of the proper orthogonal group $SO(3)$. An element in $SE(3)$ represents the position and orientation, with respect to some fixed universal frame, of a rigid body in the space; in particular it gives the position and orientation of the hand (last body) of the $6R$ robot. Thus, considering that the universal frame agrees with the coordinate system of link 1 we have, for any given pose P of the hand, the identity:

$$P = A_1^*A_2^*A_3^*A_4^*A_5^*A_6^*.$$

Besides these (twelve) equations, the ideal corresponding to the general $6R$ manipulator must contain the conditions expressing that each one of the considered matrices (A_i^*'s and P) belongs to $SE(3)$, which is obtained imposing that the 3×3 rotational part (i.e. the first three rows and colums of the homogeneous matrices considered) is proper orthogonal. As the rotational matrix in A_i^* is proper orthogonal when $s_i^2 + c_i^2 = 1$ and $\lambda_i^2 + \mu_i^2 = 1$, we can describe the ideal by the following (non necessarily irredundant) generators[1]:

$$a_{6R} =$$

$$\begin{gathered} \langle P - A_1^*A_2^*A_3^*A_4^*A_5^*A_6^*, \\ HH^t - I, \\ \det(H) - 1, \\ \{s_i^2 + c_i^2 - 1, \; i = 1, \dots, 6\}, \\ \{\lambda_i^2 + \mu_i^2 - 1, \; i = 1, \dots, 6\}\rangle \end{gathered}$$

where $-^t$ denotes the transpose matrix and I is the 3×3 identity matrix. In order to simplify notation we group the variables as follows:

$$\begin{aligned} P &= (h_{11}, h_{12} \dots, h_{33}, v_1, v_2, v_3), \\ S &= (s_1, \dots, s_6), \\ C &= (c_1, \dots, c_6), \\ L &= (\lambda_1, \dots, \lambda_6), \\ U &= (\mu_1, \dots, \mu_6), \\ A &= (a_1, \dots, a_6), \\ D &= (d_1, \dots, d_6). \end{aligned}$$

With this notation, a_{6R} is an ideal in

$$\mathbf{R}[P, S, C, L, U, A, D].$$

In this context, the standard symbolic approach to solving kinematics consists in the symbolic manipulation of the polynomials in the ideal a_{6R} yelding the variables S, C as a function of the variables in P, and considering that L, U, A, D can take any arbitrary real value. [2] proposes the triangulation of the system of equations using Gröbner basis computations. Based on this method there are several techniques introduced by [9], [7] (see also the recent monography [8]) and [16], consisting essentially in determining sets of parameters that make kinematic equations simpler to solve. However, these triangulation procedures may fail to give a complete answer to the posed problem (see, for example, [3] and [5]), as there can be numerical values of the variables for which the specialization of the triangular system is not triangular, or is not an equivalent system to the given one. Weispfenning construction of Comprehensive Gröbner basis ([19]) is a relevant tool to avoid such problems, as already remarked in [5].

The more general (i.e. completely symbolic) approach that we propose here is to consider the variables defining the robot class of $6R$ robots, namely L, U, A, D, and the hand variables P, as true indeterminates (independent parameters) and not merely nameholders for numbers, which implies to work in a base field where these variables are represented and also the possible relations among them always hold. To show an easy example, if we want to solve the general 2-degree equation $ax^2 + bx + c$, with the condition $a + b + c = 0$ on the coefficients, the ideal to be considered must correspond to $\mathbf{p} := \langle ax^2 + bx + c, a + b + c\rangle$ in the base field:

$$\begin{aligned} &q.f.\left(\mathbf{R}[a, b, c]/\mathbf{p} \cap \mathbf{R}[a, b, c]\right) = \\ &= q.f.\left(\mathbf{R}[a, b, c]/\langle a + b + c\rangle\right). \end{aligned}$$

[1] In particular, the equations $HH^t - I$ and $\det(H) - 1$ are redundant, as they are formal consequences of the remaining ones (see [6]).

where $(q.f.)$ denotes the quotient field. For the $6R$ case, if we denote

$$\beta := (P, L, U, A, D),$$
$$x := (S, C),$$

the base field is:

$$K := q.f.\left(\mathbf{R}[\beta]/a_{6R}^{c}\right)$$

where $a_{6R}^{c} := a_{6R} \cap \mathbf{R}[\beta]$.
In order to describe more explicitly the completely symbolic ideal, we consider the following ring isomorphisms (where $-^{e}$ represents the extension ideal to the corresponding ring):

$$\mathbf{R}[\beta, x]/a_{6R}^{ce} \simeq \left[\mathbf{R}[\beta]/a_{6R}^{c}\right][x]$$

$$\mathbf{R}[\beta, x]/a_{6R} \simeq \left[\mathbf{R}[\beta, x]/a_{6R}^{ce}\right] / \left[a_{6R}/a_{6R}^{ce}\right] \simeq$$
$$\simeq \left[\mathbf{R}[\beta]/a_{6R}^{c}\right][x]/\mathfrak{a}$$

where the last isomorphism determines the ideal $\mathfrak{a}$ and:

$$\mathfrak{a}.K[x]$$

is the ideal of the kinematic equations for the general $6R$ manipulator in a fully symbolic approach.

3 Insimplificability and dimension of kinematic ideals over the reals

Towards the goal of finding the best defining equations of the ideals that appear in robotics [7] made the following conjecture:

"Kinematic equation systems are irreducible (i.e. they generate prime ideals). In particular, all determining equations of all triangulations are irreducible."

We will prove in the two following subparagraphs that the kinematic ideal of the general 6R manipulator is a prime ideal and also that it is a real ideal, which roughly means that it describes the (real) solutions of the inverse kinematic problem without redundancy, (this is required to show Kovács conjecture in the real case, as commented in the introduction). We also compute its dimension. The same conclusions will be obtained for the Elbow manipulator (see [13] for a description of this robot).

Definition 3.1 *An ideal* $J \subset \mathbf{R}[X] := \mathbf{R}[x_1, \ldots, x_s]$ *is real if it contains all the polynomials vanishing on the real algebraic variety* $V(J) = \{x \in \mathbf{R}^s / \ p(x) = 0$ *for all* $p(X) \in J\}$.

There are several characterizations and computational issues related to this concept of real ideal (see [1]).

3.1 The kinematic ideal of the general 6R manipulator

Proposition 3.1 *The ideal* a_{6R} *(in the ring* $\mathbf{R}[P, S, C, L, U, A, D]$*) is real, prime, of dimension* 24.

Proof:
The ring homomorphism given by:

$$\begin{array}{ccc} \mathbf{R}[P, S, C, L, U, A, D] & \longrightarrow & \mathbf{R}[S, C, L, U, A, D] \\ P & \longmapsto & A_1^* \ldots A_6^* \end{array}$$

and inducing the identity over the remaining variables, is surjective and determines a ring isomorphism

$$\mathbf{R}[P, S, C, L, U, A, D]/a_{6R} \simeq$$
$$\simeq \mathbf{R}[S, C, L, U, A, D]/\mathbf{q}$$

where

$$\mathbf{q} := \langle \{s_i^2 + c_i^2 - 1, \ i = 1, \ldots, 6\}, \{\lambda_i^2 + \mu_i^2 - 1, \ i = 1, \ldots, 6\} \rangle.$$

$\mathbf{q}$ is a real and prime ideal of dimension 12, as it is the sum, with separated variables, of the real, prime and one-dimensional ideals $\langle s_i^2 + c_i^2 - 1 \rangle$, $i = 1, \ldots, 6$ and $\langle \lambda_i^2 + \mu_i^2 - 1 \rangle$, $i = 1, \ldots, 6$. This follows from some classical results on the ideal of the product of varieties: see [10] for the proof of the general fact.

With respect to the dimension, we have $\dim(a_{6R}) = 12 + 12 = 24$, where the first 12 correspond to those independent modulo $\mathbf{q}$ among S, C, L and U; and the last 12 are the twelve independent variables in A and D. ∎

3.2 The kinematic ideal of the Elbow manipulator

As mentioned above, the Elbow manipulator is a particular instance of $6R$, when the parameters of the $6R$ class are set to some values. Following [13] and using the Denavit-Hartenberg representation, the six homogeneous matrices A_i corresponding to the six joints of the Elbow manipulator are obtained giving the following values to the variables in (L, U, A, D) in the matrices A_i^*:

i=	1	2	3	4	5	6
λ_i	0	1	1	0	0	1
μ_i	1	0	0	-1	1	0
a_i	0	1	1	1	0	0
d_i	0	0	0	0	0	0

Following the same notation as in the $6R$ case, the ideal for the Elbow manipulator is:

$$a_{Elbow} = \langle P - A_1 A_2 A_3 A_4 A_5 A_6, \ HH^t - I, \ \det(H) - 1 \ \{s_i^2 + c_i^2 - 1, \ i = 1, \ldots, 6\} \rangle$$

Proposition 3.2
The ideal $a_{Elbow} \subset \mathbf{R}[P, S, C]$ *is real, prime, of dimension* 6.

Proof:
It suffices to remark that there is a ring isomorphism

$$\mathbf{R}[P,S,C]/a_{Elbow} \simeq \\ \simeq \mathbf{R}[S,C]/\langle\{s_i^2 + c_i^2 - 1,\ i = 1,\ldots,6\}\rangle$$

and the conclusion follows as in the $6R$ case. ∎

4 Insimplificability and dimension of the kinematic ideal of the $6R$ general manipulator over K

We prove now the insimplificability (reality and primality) over K of the kinematic ideal of the $6R$ manipulator, $\mathfrak{a}.K[x]$, in the general symbolic approach that we propose for this problem in Section 2, computing also its dimension. Some properties of the hand ideals (intersection of a_{6R} and a_{Elbow} with $\mathbf{R}$[P]) of the $6R$ case and of the Elbow case will be required.

Let us consider the ideal whose zeros define the set $SO(3)$ of all proper orthogonal matrices:

$$so(3) := \langle HH^t - I, \det(H) - 1\rangle \subset \\ \subset \mathbf{R}[h_{11}, h_{12}, \ldots, h_{33}]$$

where H is the 3×3-matrix with indeterminated coefficientes $(h_{11}, h_{12}, \ldots, h_{33})$. In [6] we have proved that, for arbitrary n, the ideal $so(n)$ is real and prime.

The following lemma will be used in Theorem 4.1, about the dimension of the hand ideal of the $6R$ manipulator. We remark that both results (Lemma and Theorem) could have been proved just making Gröbner basis computations, as we merely have to:

- compute the intersection of an ideal with $\mathbf{R}[P]$, and
- test the equality of two ideals.

However, after spending many hours with the computer, trying to get an answer using different software packages (COCOA, Maple, Axiom), the high number of involved variables did not allow to finish these computations. Therefore, we have tried to follow an alternative, more theoretical, proof.

Lemma 4.1 $a_{Elbow} \cap \mathbf{R}[P] = so(3).\mathbf{R}[P]$.

Proof: (Sketch)
To prove the inclusion $so(3).\mathbf{R}[P] \subset a_{Elbow} \cap \mathbf{R}[P]$ it suffices to remark that any matrix P which is the product of the six proper orthogonal matrices A_i is also proper orthogonal. This is accomplished at the ideal theoretic level using the results of [6].

To prove the other inclusion, it suffices now to show that

$$V(so(3).\mathbf{R}[P]) \subset V(a_{Elbow} \cap \mathbf{R}[P]).$$

As $V(a_{Elbow} \cap \mathbf{R}[P])$ is the Zariski closure of the projection of $V(a_{Elbow})$ on the P-variables, $\mathrm{cl}\big(\Pi_P(V(a_{Elbow}))\big)$, we are reduced to prove that there exists a Zariski-dense set $\mathcal{A}$ in $V(so(3).\mathbf{R}[P])$ such that $\mathcal{A} \subset \Pi_P(V(a_{Elbow}))$. We find this set $\mathcal{A}$ using [13], where closed formulae to determine real solutions to the inverse kinematics for the Elbow manipulator are explicitly given. A detailed look to these formulae shows that they are valid over an euclidean non empty open set (therefore, Zariski-dense) of the space of the P-variables. ∎

Obviously the A_i matrices in the Elbow manipulator are numerical specialization of the A_i^* matrices in the $6R$ one. This is translated in the commutative algebra framework, by means of the identity:

$$a_{Elbow} = \\ \langle a_{6R}, \\ \lambda_1, \lambda_2 - 1, \lambda_3 - 1, \lambda_4, \lambda_5, \lambda_6 - 1, \\ = \mu_1 - 1, \mu_2, \mu_3, \mu_4 + 1, \mu_5 - 1, \mu_6, \\ a_1, a_2 - 1, a_3 - 1, a_4 - 1, a_5, a_6, \\ d_1, d_2, d_3, d_4, d_5, d_6\rangle \cap \mathbf{R}[P,S,C]$$

Theorem 4.1 $a_{6R} \cap \mathbf{R}[P] = so(3).\mathbf{R}[P]$. *In particular, the hand ideal* $a_{6R} \cap \mathbf{R}[P]$ *of the general 6R manipulator is* 6*-dimensional.*

Proof:
We have the following chain of inclusions:

$$so(3).\mathbf{R}[P] \subset \\ a_{6R} \cap \mathbf{R}[P] \subset \\ \langle a_{6R}, \lambda_1, \mu_1 - 1, \lambda_2 - 1, \ldots, d_4, d_5, d_6\rangle \cap \mathbf{R}[P] = \\ a_{Elbow} \cap \mathbf{R}[P] = \\ so(3).\mathbf{R}[P]$$

∎

Lemma 4.2

$$\dim a_{6R}^c = \dim a_{6R} = 24.$$

Proof:
We can easily prove the inclusion

$$\langle so(3).\mathbf{R}[\beta], \{\lambda_i^2 + \mu_i^2 - 1,\ i = 1,\ldots,6\}\rangle \subset a_{6R}^c$$

and then, we have that

$$\dim a_{6R}^c \leq \\ \leq \dim\langle so(3).\mathbf{R}[\beta], \{\lambda_i^2 + \mu_i^2 - 1,\ i = 1,\ldots,6\}\rangle = \\ = 24.$$

Let $a_{Elbow'} \subset \mathbf{R}[\beta]$ be the ideal obtained when giving fixed constant values $c_i = c_{i_0}$ and $s_i = s_{i_0}$, $i = 1,\ldots,6$, $(c_{i_0}^2 + s_{i_0}^2 = 1)$ to the $6R$ general robot. As (s_i, c_i) and (λ_i, μ_i) have equivalent roles in the ideal, we have that in $\mathbf{R}[\beta]$:

$$\dim a_{Elbow'} = 24$$

Now, remark that

$$a_{6R}^c \subset \\ \subset \langle a_{6R}, \{c_i - c_{i_0}, s_i - s_{i_0}\}_{i=1,\ldots,6}\rangle \cap \mathbf{R}[\beta] = \\ = a_{Elbow'}$$

and then $24 = \dim a_{Elbow'} \leq \dim a_{6R}^c$ ∎

Theorem 4.2 *The ideal of the general 6R manipulator, $\mathfrak{a}.K[x]$, is real, prime and 0-dimensional.*

Proof:
Set $R = \mathbf{R}[\beta]/_{a^c_{6R}}$, then $\mathfrak{a}$ is an ideal in $R[x]$, $R[x]/\mathfrak{a}$ is a real integral domain of dimension 24 and $\mathfrak{a} \cap R = \{0\}$; hence setting $S = R - \{0\}$ we get:

$$K[X] = (R[X])_S,$$
$$\mathfrak{a}K[X] = \mathfrak{a}_S,$$
$$K[X]/\mathfrak{a}_{K[X]} = (R[X]/\mathfrak{a})_{\overline{S}},$$

where $\overline{S} = S + \mathfrak{a}$. Clearly, $(R[X]/\mathfrak{a})_{\overline{S}}$ is a real integral domain. Thus, $\mathfrak{a}K[X]$ is a real prime ideal of

$$\begin{aligned} &\dim \mathfrak{a}K[X] = \\ &= tr(K[X]/\mathfrak{a}_{K[X]}, K) = \\ &= tr(q.f.(R[X]/\mathfrak{a}), K) = \\ &= tr(q.f.(R[X]/\mathfrak{a}), \mathbf{R}) - tr(K, \mathbf{R}) = \\ &24 - 24 = 0 \end{aligned}$$

where tr denotes the trascendence degree. ∎

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